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Modified multiple time scale method for solving strongly nonlinear damped forced vibration systems

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A B S T R A C T

In this paper, modified multiple time scale (MTS) method is employed to solve strongly nonlinear forced vibration systems. The first-order approximation is only considered in order to avoid complexity. The formulations and the determination of the solution procedure are very easy and straightforward. The classical multiple time scale (MS) and multiple scales Lindstedt-Poincare method (MSLP) do not give desire result for the strongly damped forced vibration systems with strong damping effects. The main aim of this paper is to remove these limitations. Two examples are considered to illustrate the effectiveness and convenience of the present procedure. The approximate external frequencies and the corresponding approximate solutions are determined by the present method. The results give good coincidence with corresponding numerical solution (considered to be exact) and also provide better result than other existing results. For weak nonlinearities with weak damping effect, the absolute relative error measures (first-order approximate external frequency) in this paper is only 0.07% when amplitude $A = 1.5$, while the relative error gives MSLP method is surprisingly 28.81%. Furthermore, for strong nonlinearities with strong damping effect, the absolute relative error found in this article is only 0.02%, whereas the relative error obtained by MSLP method is 24.18%. Therefore, the present method is not only valid for weakly nonlinear damped forced systems, but also gives better result for strongly nonlinear systems with both small and strong damping effect.

Introduction

Krylov-Bogoliubov-Mitropolskii (KBM) [1,2] method, Struble's technique [3], multiple time scale method [4] have been widely used to determine periodic solution of a second order weakly nonlinear systems for free vibration with small damping effect. Popov [5] extended KBM method to a similar system with strong damping effect. Bojadziev [6] utilized Popov's technique to a damped forced vibration systems. Latter, Bojadziev [7] extended the two-time-scale method to the second order differential systems with strong damping effect. Shamsul [8] presented a unified KBM method for solving damped oscillation systems and it has been shown that the results are identical to that obtained by [1]. Shamsul et al. [9] also generalized the general Struble's technique to such differential systems. Nagy and Balachandran [10] used perturbation method to investigate jump phenomenon of weakly nonlinear systems with small damping effect. Hassan [11] shown that KBM method [1,2] is equivalent to the MST method [4,12] for a small damping effect. Recently, Azad et al. [13] developed a general multiple-time-scale method for free vibrations to obtain n-th order weakly nonlinear systems and also indicated that the solutions are identical to those obtained by [9]. Pakdemirli et al. [14] used multiple scales Lindstedt-Poincare (MSLP) to obtain approximate solution for strongly nonlinear damped forced vibration systems for small damping effects. The classical MS method results for weak nonlinearities are almost same that the results obtained by MSLP method [14]. He [15] developed homotopy perturbation method to solve strongly un-damped nonlinear systems. The solution obtained in [14] is valid for strong nonlinear problem with small damping effect. In general, homotopy perturbation method is invalid when the damping and external forces act. Li et al. [16–20] have studied the free vibration system. Another author, Xiang [21] has studied damped oscillatory system without forced vibrations. Recently, Akbari et al. [22] have been investigated nonlinear vibration system.

Some analytical methods such as perturbation method, homotopy perturbation method, MS, MSLP method do not cover in all cases. It has been already mentioned that the perturbation method and MS method are valid only for weak nonlinearities, but these

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Nomenclature

ω_0	Natural frequency	Φ	Periodic function
k	Linear damping coefficient	ε	A large or small parameter
ν	The external frequency	φ	Phase variable
a	Amplitude	ψ	Phase variable chosen in our method
E	An external force		

methods are invalid for strong nonlinearities. On the other hand, the MSLP method is invalid for strong damping effect.

In the present study, a modified MTS method [13] has been used to investigate nonlinear forced vibration systems. The main advantage of the present method is that it covers all the cases: weak nonlinearities with small damping effect, weak nonlinearities with strong damping effect, and strong nonlinearities with strong damping effect. The method is very simple and also provides better result than other existing result.

The method

Let us consider a second order time dependent nonlinear non-autonomous differential system

$$\ddot{x} + 2k\dot{x} + \omega_0^2 x = \varepsilon f(x, \dot{x}) + \varepsilon E\Phi(\nu t), \tag{1}$$

where over dot denotes the derivatives with respect to t, $\omega_0 \geqslant 0$, k, ν are constants, ε denotes either small or large parameter, ω_0 is a natural frequency, $f(x, \dot{x})$ is a nonlinear function such that $f(-x, -\dot{x}) = -f(x, \dot{x})$, E is an external force and $\Phi(\nu t)$ is a periodic function.

A first approximate solution of Eq. (1) is chosen in the form [13]

$$x(t, \varepsilon) = a_1(t) + a_2(t) + \varepsilon u_1(a_1, a_2) + \cdots. \tag{2}$$

Herein, a_1 and a_2, represent rather than the amplitude and phase, variables. The variables a_1 and a_2 depend on the several time $t_0, t_1, t_2, \ldots$, where $t = t_0 + \varepsilon t_1 + \varepsilon^2 t_2 + \ldots$.

For unperturbed cases *i.e.*, when $\varepsilon = 0$, Eq. (1) has two eigen-values $\lambda_1 = -k + i\omega$, $\lambda_2 = -k - i\omega$, where $\omega = \sqrt{\omega_0^2 - k^2}$, $k < \omega$.

When $\varepsilon \neq 0$, Eq. (1) can be rewrite as

$$(D - \lambda_1)(D - \lambda_2)x = \varepsilon f(x, \dot{x}) + \varepsilon E\Phi(\nu t) \tag{3}$$

Now, we denote some notations and a relation [13] as

$$D = D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \cdots, \quad D_k() = \frac{d()}{dt_k}, \quad k = 0, 1, 2, \ldots,$$

$$D_0 a_1 = \lambda_1 a_1, \quad D_0 a_2 = \lambda_2 a_2.$$

Substituting Eq. (2) into Eq. (3) and equating the coefficient of ε, we obtain

$$(D_0 - \lambda_2)(D_1 a_1) + (D_0 - \lambda_1)(D_1 a_2) + (D_0 - \lambda_1)(D_0 - \lambda_2)(u_1) + \cdots = f + E\Phi(\nu t). \tag{4}$$

Herein, the nonlinear function f can be expanded in a Fourier series as $f = \sum_{m_1=0,\, m_2=0}^{\infty,\, \infty} F_{m_1,m_2} e^{(m_1\lambda_1 + m_2\lambda_2)t}$ and the unknown functions u_1 can be found in terms of the variables a_1, a_2 and t, under the restriction that u_1 excludes the terms $F_{m_1,m_2} e^{(m_1\lambda_1+m_2\lambda_2)t}$ of f where, $m_1 - m_2 \neq \pm 1$. On the other hand, $D_1 a_1$ and $D_1 a_2$ respectively, contain these terms where $m_1 - m_2 = 1$ and $m_1 - m_2 = -1$. This assumption keeps u_1 free from secular terms, *i.e.*, $t\cos t$, $t\sin t$. Again, the external force terms contain $D_1 a_1$ and $D_1 a_2$ respectively but u_1 excludes the external force term.

Now, equating the coefficient of ε from Eq. (4) and then separating into three parts for $D_1 a_1$, $D_1 a_2$ and u_1 as (see article [9])

$$(D_0 - \lambda_2)(D_1 a_1) = \sum_{m_1=0,\, m_2=0}^{\infty,\infty} F_{m_1,m_2} e^{(m_1\lambda_1+m_2\lambda_2)t} + Ee^{i\nu t_0}/2, \; m_1 - m_2 = 1, \tag{5}$$

$$(D_0 - \lambda_1)(D_1 a_2) = \sum_{m_1=0,\, m_2=0}^{\infty,\infty} F_{m_1,m_2} e^{(m_1\lambda_1+m_2\lambda_2)t} + Ee^{-i\nu t_0}/2, \; m_1 - m_2 = -1 \tag{6}$$

$$(D_0 - \lambda_1)(D_0 - \lambda_2)u_1 = \sum_{m_1=0,\, m_2=0}^{\infty,\infty} F_{m_1,m_2} e^{(m_1\lambda_1+m_2\lambda_2)t}, \; m_1 - m_2 \neq \pm 1. \tag{7}$$

Transforming $a_1 = ae^{i\varphi}/2$, $a_2 = ae^{-i\varphi}/2$, Eqs. (5) and (6) are transformed to amplitude and phase equations. On the other hand, this transformation represents u_1 in a usual form (*i.e.*, amplitude and phase form). Thus the determination of the first approximate solution is clear.

Examples

Example 1

Let us consider the forced vibrations of the damped duffing oscillator

$$\ddot{x} + 2k\dot{x} + \omega_0^2 x = -\varepsilon x^3 + \varepsilon E\cos \nu t, \tag{8}$$

where ε be either small or large parameter, ω_0 is a natural frequency, $\omega_0 \geqslant 0$, $k < \omega_0$.

When $\varepsilon = 0$, Eq. (8) has two eigen-values $\lambda_1 = -k + i\omega$, $\lambda_2 = -k - i\omega$ and $\omega = \sqrt{\omega_0^2 - k^2}$.

When $\varepsilon \neq 0$, then the first approximate solution of Eq. (8) is found of the form

$$x = a_1(t) + a_2(t) + \varepsilon u_1 + \cdots \tag{9}$$

and the function $f = -x^3$ can be expanded in the form

$$f = -x^3 = -[a_1^3 + 3a_1^2 a_2 + 3a_1 a_2^2 + a_2^3 + 3\varepsilon(a_1 + a_2)^2 u_1 + \cdots] \tag{10}$$

Now, applying the separation rule (discussed in Section "The method") on the Eq. (A.1), we obtain the following equations for $D_1 a_1$, $D_1 a_2$ and u_1 as

$$(D_0 - \lambda_2)(D_1 a_1) = -3a_1^2 a_2 + Ee^{i\nu t_0}/2, \tag{11}$$

$$(D_0 - \lambda_1)(D_1 a_2) = -3a_1 a_2^2 + Ee^{-i\nu t_0}/2, \tag{12}$$

$$(D_0 - \lambda_1)(D_0 - \lambda_2)u_1 = -(a_1^3 + a_2^3). \tag{13}$$

First, we consider that the particular solution without external term $(Ee^{i\nu t_0}/2)$ of Eq. (11) be $D_1 a_1 = l_1 a_1^2 a_2$. Since $D_0 a_1 = \lambda_1 a_1$ and $D_0 a_2 = \lambda_2 a_2$, then from Eq. (11) with out external term, we obtain

$$(D_0 - \lambda_2)(l_1 a_1^2 a_2) = 2\lambda_1 l_1 a_1^2 a_2 = -3a_1^2 a_2. \quad (14)$$

From Eq. (14), we obtain

$$l_1 = -3/(2\lambda_1).$$

Now, solving Eq. (11) with external term, we obtain

$$D_1 a_1 = -3a_1^2 a_2/(2\lambda_1) + E e^{i\nu t_0}/(2(i\nu - \lambda_2)). \quad (15)$$

The variational equations can be written as follows

$$\dot{a}_1 = Da_1 = (D_0 + \varepsilon D_1 + \cdots)a_1 = \lambda_1 a_1 + \varepsilon D_1 a_1 + O(\varepsilon^2). \quad (16)$$

Neglecting the second and higher order of ε from Eq. (16) and then also using Eq. (15), we obtain

$$\dot{a}_1 = (-k + i\omega)a_1 + \varepsilon[3(k + i\omega)a_1^2 a_2/(2(k^2 + \omega^2)) + E(k - i(\nu + \omega))e^{i\nu t_0}/(2(k^2 + (\nu + \omega)^2)] \quad (17)$$

Separating the real and imaginary parts on both sides of Eq. (A.2), we obtain

$$\dot{a} = -ka + 3\varepsilon a^3 k/(8(k^2 + \omega^2)) + \varepsilon E(k\cos\psi - (\nu + \omega) \times \sin\psi)/(k^2 + (\nu + \omega)^2), \quad (18)$$

$$a\dot{\varphi} = \omega a + 3\varepsilon a^3 \omega/(8(k^2 + \omega^2)) + \varepsilon E(-k\sin\psi - (\nu + \omega) \times \cos\psi)/(k^2 + (\nu + \omega)^2). \quad (19)$$

For the steady-state $\dot{a} = 0$ and $\dot{\varphi} = \nu$, then Eqs. (18) and (19) become

$$a(k - 3\varepsilon a^2 k/(8(k^2 + \omega^2))) = \varepsilon E(k\cos\psi - (\nu + \omega)\sin\psi)/(k^2 + (\nu + \omega)^2), \quad (20)$$

$$a(\nu - \omega - 3\varepsilon a^2 \omega/(8(k^2 + \omega^2))) = \varepsilon E(-k\sin\psi - (\nu + \omega)\cos\psi)/(k^2 + (\nu + \omega)^2). \quad (21)$$

Herein, Eqs. (20) and (21) represent the resonance curve in the plane (ν, a).

The particular solution of Eq. (13) is

$$u_1 = c_1 a_1^3 + c_2 a_2^3, \quad (22)$$

where $c_1 = \frac{-1}{2\lambda_1(3\lambda_1 - \lambda_2)}$, $c_2 = \frac{-1}{2\lambda_2(3\lambda_2 - \lambda_1)}$.

Substituting $a_1 = ae^{i\varphi}/2$, $a_2 = ae^{-i\varphi}/2$, $\varphi = \psi + \nu t_0$ and eliminating second and higher order terms of ε from Eq. (9), we obtain the first approximate solution of Eq. (8) as

$$x(t) = a\cos(\nu t_0 + \psi) + \varepsilon u_1, \quad (23)$$

where u_1 is given in Eq. (A.4).

Example 2

Let us consider the Van der Pol equation with linear damping and an external force

$$\ddot{x} + 2k\dot{x} + \omega_0^2 x = \varepsilon(1 - x^2)\dot{x} + \varepsilon E\sin\nu t, \quad k < \omega_0. \quad (24)$$

When $\varepsilon = 0$, Eq. (24) has two eigen-values $\lambda_1 = -k + i\omega$, $\lambda_2 = -k - i\omega$ and $\omega = \sqrt{\omega_0^2 - k^2}$.

When $\varepsilon \neq 0$, then the first approximate solution of Eq. (24) is found of the form

$$x = a_1 + a_2 + \varepsilon u_1 + \cdots \quad (25)$$

Now, applying the separation rule (discussed in Section "The method") on the Eq. (A.6), we obtain the following equations for $D_1 a_1$, $D_1 a_2$ and u_1 as

$$(D_0 - \lambda_2)(D_1 a_1) = \lambda_1 a_1 - (2\lambda_1 + \lambda_2)a_1^2 a_2 + E e^{i\nu t_0}/(2i), \quad (26)$$

$$(D_0 - \lambda_1)(D_1 a_2) = \lambda_2 a_2 - (2\lambda_2 + \lambda_1)a_1 a_2^2 - E e^{-i\nu t_0}/(2i), \quad (27)$$

$$(D_0 - \lambda_1)(D_0 - \lambda_2)u_1 = -(\lambda_1 a_1^3 + \lambda_2 a_2^3). \quad (28)$$

According to the Eq. (15), we have obtained the solution of Eq. (26) as

$$D_1 a_1 = \lambda_1 a_1/(\lambda_1 - \lambda_2) - (2\lambda_1 + \lambda_2)a_1^2 a_2/(2\lambda_1) + E e^{i\nu t_0}/(2i(i\nu - \lambda_2)). \quad (29)$$

According to the Eq. (16), the variational equations as

$$\dot{a}_1 = \lambda_1 a_1 + \varepsilon D_1 a_1 + \cdots. \quad (30)$$

Neglecting the second and higher order of ε from Eq. (30) and then also using Eq. (29), we obtain

$$\dot{a}_1 = (-2k\omega + \varepsilon\omega + i(\varepsilon k + 2\omega^2))/(2\omega))a_1 - \varepsilon(3k^2 + \omega^2 + 2ik\omega)a_1^2 a_2/(2(k^2 + \omega^2)) - i\varepsilon E(k - i(\nu + \omega))e^{i\nu t_0}/ (2(k^2 + (\nu + \omega)^2) \quad (31)$$

Separating the real and imaginary parts on both sides of Eq. (A.7), we obtain

$$\dot{a} = (-2k\omega + \varepsilon\omega)a/(2\omega) - \varepsilon a^3(3k^2 + \omega^2)/(8(k^2 + \omega^2)) - \varepsilon E((\nu + \omega)\cos\psi + k\sin\psi)/(k^2 + (\nu + \omega)^2), \quad (32)$$

$$a\dot{\varphi} = (2\omega^2 + \varepsilon k)a/(2\omega) - 2\varepsilon a^3 k\omega/(8(k^2 + \omega^2)) - \varepsilon E(k\cos\psi - (\nu + \omega)\sin\psi)/(k^2 + (\nu + \omega)^2), \quad (33)$$

where $\psi = \varphi - \nu t_0$.

The particular solution of Eq. (28) is

$$u_1 = d_1 a_1^3 + d_2 a_2^3, \quad (34)$$

where $d_1 = \frac{-1}{2(3\lambda_1 - \lambda_2)}$, $d_2 = \frac{-1}{2(3\lambda_2 - \lambda_1)}$.

Substituting $a_1 = ae^{i\varphi}/2$, $a_2 = ae^{-i\varphi}/2$, $\varphi = \psi + \nu t_0$ and eliminating second and higher order terms of ε from Eq. (9), we obtain the first approximate solution of Eq. (24) as

$$x(t) = a\cos(\nu t_0 + \psi) + \varepsilon u_1, \quad (35)$$

where u_1 is given in Eq. (A.9).

Results and discussions

Nonlinear oscillations problems are important in the physical science, mechanical structures and other kind of mathematical sciences. Most of real systems are modeled by nonlinear differential equations which are important issues in mechanical structures, mathematical physics and engineering. The Duffing oscillator with damping effect in presence of an external force is a common model for nonlinear phenomena in science and engineering. The interest in this system lies in the variety of physical phenomena that it models, such as the rolling motion of a ship, and the fact that it is isomorphic with other systems of importance in physics and engineering (e.g. Josephson junction oscillator and Foucault pendulum). Particularly interesting is the response of the Duffing oscillator to a harmonic excitation in the presence of viscous damping, which has been found to exhibit, among other features, hysteretic and chaotic behaviors. On the other hand, Balthazar Van der Pol (1889–1959) was a Dutch electrical engineer who initiated modern experimental dynamics in the laboratory during the 1920s and 1930s. He, first, introduced his (now famous) equation in order to describe triode oscillations in electrical circuits [23,24]. The mathematical model for the system, a well known second order ordinary differential equation with cubic nonlinearity, is the Van der Pol equation. The Van der Pol oscillator is a classical example

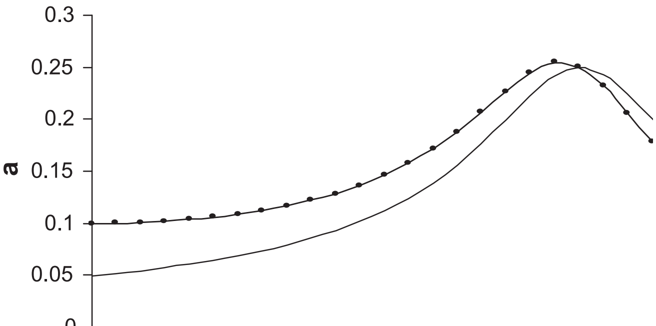

Fig. 1a. Comparison of frequency response curves of Eq. (8) obtained by the present method (represented by circles) and MSLP method (represented by dashes line) when $\varepsilon = 0.1$, $k = 0.2$, $E = 1$, $\omega_0 = 1$. Corresponding numerical simulations have been presented (denoted by solid line) to compare with present and MSLP methods.

of self-oscillatory system and is now considered as very useful mathematical model that can be used in much more complicated and modified systems.

In this paper, a simple analytical technique (based on the modified multiple time scale method) has been presented to obtain the approximate solutions of such nonlinear damped forced systems. The method is valid for the all the cases: weak nonlinearities with small damping effect, weak nonlinearities with strong damping effect, and strong nonlinearities with strong damping effect. Recently, forced vibration of Duffing equation with small damping effect has been investigated in MSLP method [14]. For the very both small nonlinearities and small damping effects, the results of present method, the classical MS method and MSLP method are almost identical. It has already mentioned that the perturbation methods [1–10] are valid only for small nonlinearities. The MSLP method [14] does not provide desire result with the numerical solution when the small nonlinearities with strong damping effect act. On the other hand, MSLP method is invalid to investigate the Van der Pol equation in presence of an external force. In this situation, the present method has been successfully applied to such equation (Van der Pol equation) and the limitation of [14] has been removed.

Frequency response relations as well as solutions of Eq. (8) are obtained by present method and MSLP [14] method and have been compared with help of numerical results (considered to be exact). In our solution, damping term is involved in the zeroth-order solution; but the perturbed equations of MSLP method include damping effect. Thus the solution of MSLP method deviates swiftly from the numerical solution increasing with damping effect.

First of all, we have been determined the frequency response curves of the damped forced Duffing oscillator (by Eq. (8)) obtained by present and MSLP methods when $\varepsilon = 0.1, k = 0.2$, $E = 1, \omega_0 = 1$; $\varepsilon = 1, k = 0.3$, $E = 0.10, \omega_0 = 1$ and respectively presented in Fig. 1(a) and 1(b). In a similar way, we have been determined the frequency response curves of the Van der Pol

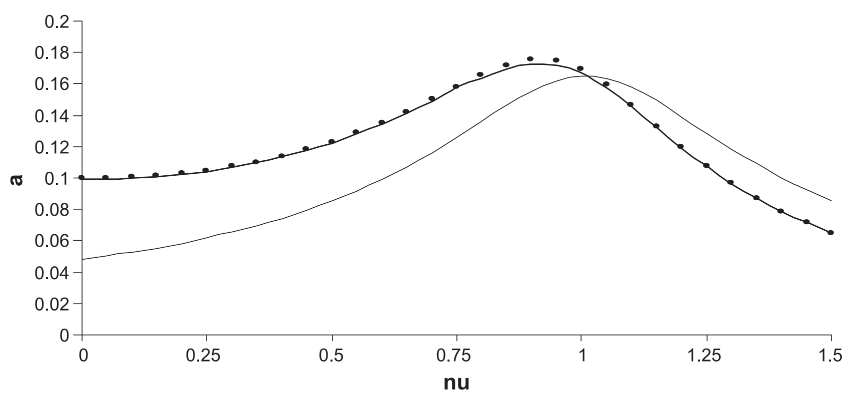

Fig. 1b. Comparison of frequency response curves of Eq. (8) obtained by the present method (represented by circles) and MSLP method (represented by dashes line) when $\varepsilon = 1$, $k = 0.3$, $E = 0.10$, $\omega_0 = 1$. Corresponding numerical results have been presented (denoted by solid line) to compare with present and MSLP methods.

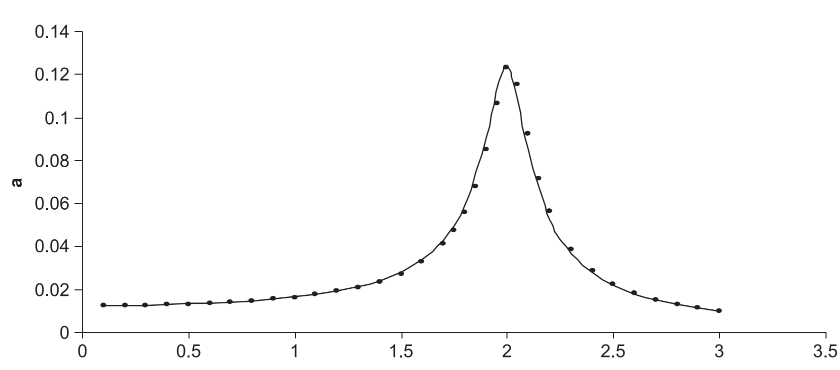

Fig. 2a. Comparison of frequency response curves of Eq. (24) obtained by the present method (represented by circles) when $\varepsilon = 0.5$, $k = 0.35$, $E = 0.1$, $\omega_0 = 2$. Corresponding numerical results have been presented (denoted by solid line) to compare with present method.

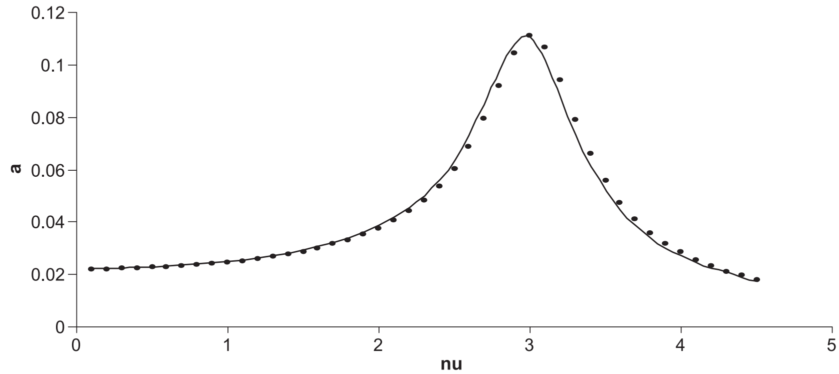

Fig. 2b. Comparison of frequency response curves of Eq. (24) obtained by the present method (represented by circles) when $\varepsilon = 1$, $k = 0.8$, $E = 0.20$, $\omega_0 = 3$. Corresponding numerical results have been presented (denoted by solid line) to compare with present method.

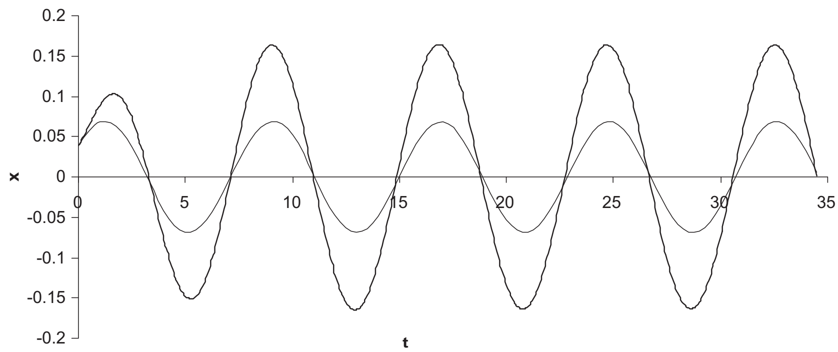

Fig. 3a. MSLP Method solution of Eq. (8) has been presented (denoting by dashes line) when $\varepsilon = 1, k = 0.3$, $E = 0.10, \omega_0 = 1, v = 0.8$ with initial conditions $[x(0) = 0.0385131, \dot{x}(0) = 0.0458238]$. Corresponding numerical solution has been presented (denoted by solid line) to compare with MSLP method.

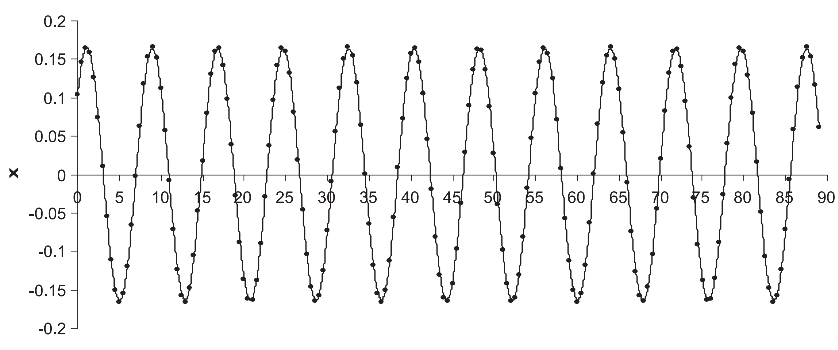

Fig. 3b. Present method solution of Eq. (8) has been presented (denoting by circles) when $\varepsilon = 1, k = 0.3$, $E = 0.10, \omega_0 = 1, v = 0.8$ with initial conditions $[x(0) = 0.103221, \dot{x}(0) = 0.103639]$. Corresponding numerical solution has been presented (denoted by solid line) to compare with present method.

equation (by Eq. (24)) obtained by present method when $\varepsilon = 0.5, k = 0.35, E = 0.1, \omega_0 = 2$; $\varepsilon = 1, k = 0.8, E = 0.20, \omega_0 = 3$ and presented in Fig. 2(a) and 2(b) respectively.

Next, the approximate solution of Eq. (8) has been determined by MSLP and present methods when $\varepsilon = 1, k = 0.3, E = 0.10$, $\omega_0 = 1, v = 0.8$ with initial conditions $[x(0) = 0.0385131$, $\dot{x}(0) = 0.0458238]$; $[x(0) = 0.103221, \dot{x}(0) = 0.103639]$ and respectively presented in Fig. 3(a) and 3(b). In a similar way, we have been determined the approximate solution of Eq. (24) obtained by present method when $\varepsilon = 0.5, k = 0.35, E = 0.1, \omega_0 = 2, v = 2$ with initial conditions $[x(0) = -0.120247, \dot{x}(0) = 0.0534403]$ and presented in Fig. 4(a). Furthermore, we have also determined the

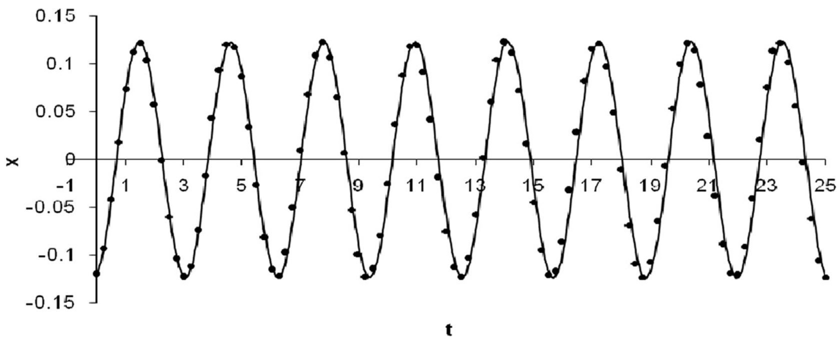

Fig. 4a. Present method solution of Eq. (24) has been presented (denoting by circles) when $\varepsilon = 0.5, k = 0.35, E = 0.1, \omega_0 = 2, v = 2$ with initial conditions $[x(0) = -0.120247, \dot{x}(0) = 0.0534403]$. Corresponding numerical solution has been presented (denoted by solid line) to compare with present method.

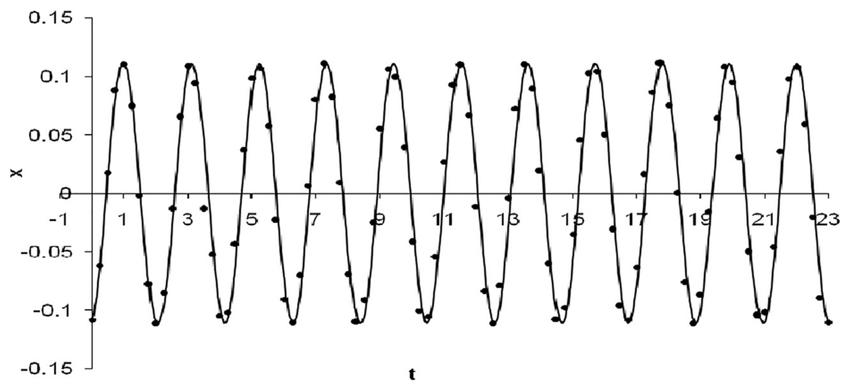

Fig. 4b. Present method solution of Eq. (24) has been presented (Denoting by circles) when $\varepsilon = 1, k = 0.8, E = 0.20, \omega_0 = 3, v = 3$ with initial conditions $[x(0) = -0.108026, \dot{x}(0) = 0.0761716]$. Corresponding numerical solution has been presented (denoted by solid line) to compare with present method.

Table 1a
Comparison of the approximate frequencies obtained by present method (Eq. (8)) with the exact external frequency v_e and other existing frequencies (those are obtained by MSLP method) when $\varepsilon = 0.1, k = 0.2, E = 1, \omega_0 = 1$.

A	v_e	MSLP $Er(\%)$	Present study $Er(\%)$
0.0	0.099900	0.049020 50.93	0.099966 0.07
0.1	0.100821	0.054220 46.22	0.100889 0.07
0.2	0.103680	0.060616 41.54	0.10376 0.08
0.3	0.108800	0.068652 36.90	0.108887 0.08
0.4	0.116900	0.079011 32.41	0.116861 0.03
0.6	0.146000	0.111647 23.53	0.146091 0.06
0.8	0.206600	0.17605 14.79	0.206985 0.19
1.0	0.250000	0.24942 0.24	0.25086 0.34
1.2	0.153900	0.177089 15.07	0.153969 0.05
1.4	0.090000	0.11185 24.28	0.090034 0.038
1.5	0.072100	0.092869 28.81	0.072147 0.07

Table 1b
Comparison of the approximate frequencies obtained by present method (Eq. (8)) with the exact frequency v_e and other existing frequencies (those are obtained by MSLP method) when $\varepsilon = 1, k = 0.3, E = 0.10, \omega_0 = 1$.

A	v_e	MSLP $Er(\%)$	Present study $Er(\%)$
0.0	0.099029	0.047813 51.72	0.099695 0.67
0.1	0.099810	0.052596 47.30	0.100488 0.68
0.2	0.102200	0.058354 42.90	0.102971 0.76
0.3	0.106700	0.065421 38.69	0.107323 0.58
0.4	0.113500	0.074178 34.65	0.113854 0.31
0.6	0.134300	0.099051 26.25	0.135109 0.60
0.8	0.163500	0.136247 16.67	0.165738 1.37
1.0	0.166800	0.164898 1.15	0.169222 1.45
1.2	0.119400	0.13921 16.69	0.119902 0.42
1.4	0.078700	0.100228 27.36	0.07872 0.03
1.5	0.065100	0.085862 31.89	0.065086 0.02

Where $Er(\%)$ denotes the absolute percentage error.

approximation solutions of Eq. (24) obtained by present method when $\varepsilon = 1, k = 0.8, E = 0.20, \omega_0 = 3, v = 3$ with initial conditions; $[x(0) = -0.108026, \dot{x}(0) = 0.0761716]$ and presented in Fig. 4(b).

Moreover, to check the accuracy of the present method, we have calculated the approximate external frequencies of Eq. (8)

Table 2a
Comparison of the approximate solution of Eq. (8) obtained by MSLP method with the corresponding numerical solution obtained by fourth-order Runge-Kutta method when $\varepsilon = 1, k = 0.3, E = 0.10, \omega_0 = 1, v = 0.8$ with initial conditions $[x(0) = 0.0385131, \dot{x}(0) = 0.0458238]$.

t	x_{nu}	MSLP solution	$Er(\%)$
0.0	0.038513	0.0385131	0.00
0.5	0.064158	0.0577838	9.94
1.0	0.088661	0.0679336	23.38
1.5	0.102298	0.0673484	34.16
2.0	0.097908	0.0561213	42.68
2.5	0.072936	0.0360371	50.59
3.0	0.030049	0.010275	65.81
3.5	−0.023622	−0.0171044	27.59
4.0	−0.078108	−0.0417913	46.50
4.5	−0.122647	−0.05989	51.17
5.0	−0.147919	−0.0685347	53.67
5.5	−0.14817	−0.0663478	55.22
6.0	−0.122569	−0.0536785	56.21

Table 2b
Comparison of the approximate solution of Eq. (8) obtained by present method with the corresponding numerical solution obtained by fourth-order Runge-Kutta method when $\varepsilon = 1, k = 0.3, E = 0.10, \omega_0 = 1, v = 0.8$ with initial conditions $[x(0) = 0.103221, \dot{x}(0) = 0.103639]$.

t	x_{nu}	Present solution	$Er(\%)$
0.0	0.103221	0.103221	0.00
0.5	0.145444	0.145604	0.11
1.0	0.164446	0.165061	0.37
1.5	0.15722	0.18338	0.71
2.0	0.125185	0.126469	1.03
2.5	0.07374	0.074645	1.23
3.0	0.011051	0.0111942	1.30
3.5	−0.053263	−0.0539224	1.24
4.0	−0.10932	−0.11061	1.18
4.5	−0.148244	−0.149997	1.18
5.0	−0.163658	−0.165736	1.27
5.5	−0.153048	−0.155172	1.39
6.0	−0.118296	−0.119975	1.42

by the present method for some particular large values of a and compared with numerical solution together with other existing solution (those solution obtained by [14]) when $\varepsilon = 0.1, k = 0.2, E = 1, \omega_0 = 1$; $\varepsilon = 1, k = 0.3, E = 0.10, \omega_0 = 1$ which have been presented in Table 1(a) and 1(b) respectively. In addition, we have determined the approximate solution of the same Eq. (8) by the present method when $\varepsilon = 1, k = 0.3, E = 0.10, \omega_0 = 1, v = 0.8$ with initial conditions $[x(0) = 0.0385131, \dot{x}(0) = 0.0458238]$ and all results with corresponding numerical solutions have been presented in Table 2(a). Similarly, we have calculated the approximate solution of the same Eq. (8) by the MSLP method when $\varepsilon = 1, k = 0.3, E = 0.10, \omega_0 = 1, v = 0.8$ with initial conditions $[x(0) = 0.103221, \dot{x}(0) = 0.103639]$ and all results with corresponding numerical solutions have been presented in Table 2(b). The errors of each table have been calculated.

From the figures, we observe that the frequency response curves and approximate solutions of Eq. (8) determined by the MSLP method deviate from numerical solution (by fourth-order Runge-Kutta formula) for both weak and strong nonlinearities with both small and large damping effect. In contrast, the similar results obtained by the present method are nicely close to the numerical result. Moreover, from the tables, we see that the absolute relative error found (first-order approximate external frequency) in this paper is only 0.07% when $\varepsilon = 0.1, k = 0.2, E = 1, \omega_0 = 1, A = 1.5$, while the relative error obtained by MSLP method is highly 28.81%. Furthermore, the relative error measures in this paper is only 0.02% for $\varepsilon = 1.0, k = 0.3, E = 0.10, \omega_0 = 1, A = 1.5$, whereas the MSLP method gives 24.18%.

Thus, the present method gives better result than other existing result for both weak and strong nonlinearities with both small and large damping effect.

Conclusion

Based on a MST method, a simple analytical technique has been presented to investigate nonlinear damped forced systems. The MSLP method is only valid for small damping effect. On the other hand, the perturbation method is valid only for small nonlinearities. In this article, the limitations of MSLP and perturbation methods have been eliminated. The method is very powerful for solving nonlinear forced vibration systems.

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Appendix 1

According to the Eq. (4) and using Eq. (10), Eq. (8) becomes

$$(D_0 - \lambda_2)(D_1 a_1) + (D_0 - \lambda_1)(D_1 a_2) + (D_0 - \lambda_1)(D_0 - \lambda_2)u_1 = -(a_1^3 + 3a_1^2 a_2 + 3a_1 a_2^2 + a_2^3) + E(e^{ivt_0} + e^{-ivt_0})/2. \quad (A.1)$$

By transforming $a_1 = ae^{i\varphi}/2$, $a_2 = ae^{-i\varphi}/2$, Eq. (17) becomes

$$\dot{a} + ia\dot{\varphi} = (-k + i\omega)a + \varepsilon[3a^3(k + i\omega)/(8(k^2 + \omega^2)) + E(k - i(v + \omega))(\cos\psi - i\sin\psi)/((k^2 + (v + \omega)^2)], \quad (A.2)$$

where $\psi = \varphi - vt_0$.

Substituting the values of $\lambda_1 = -k + i\omega$, $\lambda_2 = -k - i\omega$, $a_1 = ae^{i\varphi}/2$ and $a_2 = ae^{-i\varphi}/2$ into Eq. (22), we obtain

$$u_1 = -a^3 \times \frac{(k^2 - 2\omega^2)(e^{3i\varphi} + e^{-3i\varphi}) + 3ik\omega(e^{3i\varphi} - e^{-3i\varphi})}{32(k^4 + 5k^2\omega^2 + 4\omega^4)} \quad (A.3)$$

Again, substituting $\omega = \sqrt{\omega_0^2 - k^2}$ and $\varphi = \psi + vt_0$ into Eq. (A.3), we obtain

$$u_1 = a^3 \times \frac{(3k^2 - 2\omega_0^2)\cos 3(\psi + vt_0) - 3k\omega \sin 3(\psi + vt_0)}{16\omega_0^2(3k^2 - 4\omega_0^2)}. \quad (A.4)$$

Appendix 2

Using Eq. (25), the function $f = (1 - x^2)\dot{x}$ can be expanded in the form

$$\begin{aligned} f &= [(1 - a_1^2 - 2a_1a_2 - a_2^2 + \cdots)(D_0 + \varepsilon D_1 + \cdots)(a_1 + a_2 + \cdots)] \\ &= [(1 - a_1^2 - 2a_1a_2 - a_2^2 + \cdots)(\lambda_1 a_1 + \lambda_2 a_2 + \cdots)] \\ &= [(\lambda_1 - (2\lambda_1 + \lambda_2)a_1a_2)a_1 + (\lambda_2 - (2\lambda_2 + \lambda_1)a_1a_2)a_2 \\ &\quad - (\lambda_1 a_1^3 + \lambda_2 a_2^3) + \cdots] \end{aligned} \quad (A.5)$$

According to the Eq. (4) and using Eq. (A.5), Eq. (24) becomes

$$\begin{aligned} &(D_0 - \lambda_2)(D_1 a_1) + (D_0 - \lambda_1)(D_1 a_2) + (D_0 - \lambda_1)(D_0 - \lambda_2)u_1 \\ &= (\lambda_1 - (2\lambda_1 + \lambda_2)a_1a_2)a_1 + (\lambda_2 - (2\lambda_2 + \lambda_1)a_1a_2)a_2 \\ &\quad - (\lambda_1 a_1^3 + \lambda_2 a_2^3) + E(e^{ivt_0} - e^{-ivt_0})/(2i). \end{aligned} \quad (A.6)$$

Transforming $a_1 = ae^{i\varphi}/2$, $a_2 = ae^{-i\varphi}/2$, Eq. (31) becomes

$$\begin{aligned} \dot{a} + ia\dot{\varphi} &= (-2k\omega + \varepsilon\omega + i(2\omega^2 + \varepsilon k)a/(2\omega) - \varepsilon a^3(3k^2 \\ &\quad + \omega^2 + 2ik\omega)/(8(k^2 + \omega^2)) - \varepsilon E((v + \omega) \\ &\quad + ik)e^{i(vt_0 - \varphi)}/(k^2 + (v + \omega)^2) \end{aligned} \quad (A.7)$$

Substituting the values of $\lambda_1 = -k + i\omega$, $\lambda_2 = -k - i\omega$, $a_1 = ae^{i\varphi}/2$ and $a_2 = ae^{-i\varphi}/2$ into Eq. (34), we obtain

$$u_1 = a^3 \times \frac{k(e^{3i\varphi} + e^{-3i\varphi}) + 2i\omega(e^{3i\varphi} - e^{-3i\varphi})}{32(k^2 + 4\omega^2)}. \quad (A.8)$$

Again, substituting $\omega = \sqrt{\omega_0^2 - k^2}$ and $\varphi = \psi + vt_0$ into Eq. (A.8), we obtain

$$u_1 = a^3 \times \frac{k\cos 3(\psi + vt_0) - 2\omega \sin 3(\psi + vt_0)}{16(4\omega_0^2 - 3k^2)}. \quad (A.9)$$

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