Results in Physics 8 (2018) 559–568

Contents lists available at ScienceDirect

Results in Physics

journal homepage: www.journals.elsevier.com/results-in-physics

New integrable problems in a rigid body dynamics with cubic integral in velocities

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ARTICLE INFO

Article history:
Received 25 September 2017
Received in revised form 10 December 2017
Accepted 16 December 2017
Available online 29 December 2017

Keywords:
Integrability
Cubic integral
Rigid body dynamics

ABSTRACT

We introduce a new family of the 2D integrable mechanical system possessing an additional integral of the third degree in velocities. This system contains 20 arbitrary parameters. We also clarify that the majority of the previous systems with a cubic integral can be reconstructed from it as a special version for certain values of those parameters. The applications of this system are extended to include the problem of motion of a particle and rigid body about its fixed point. We announce new integrable problems describing the motion of a particle in the plane, pseudosphere, and surfaces of variable curvature. We also present a new integrable problem in a rigid body dynamics and this problem generalizes some of the previous results for Sokolov-Tsiganov, Yehia, Stretensky, and Goriachev.

Introduction

From about 150 years or so, Bertrand was interested in searching for the structure of forces acting on the motion of a particle in the Euclidean plane which guarantees the existence of an additional integral of a certain form. His accomplishment was slightly in solving this problem for simple forms of the integral such as a polynomial up to the third degree and a fractional function in which the numerator and denominator are linear in velocity variables [1,2]. He was followed by Darboux who studied the construction of an integrable system having a quadratic integral and solved the problem completely [3] (also, see [4]). This direction of research was studied by several authors and it is called in literature *a direct method for obtaining the second invariant*. The majority of integrable mechanical systems describing the motion of a particle in the plane having a polynomial additional integral in the velocities with degree ranging up to six were collected in Hietarienta's review [5]. Other systems were mainly presented by trials to insert new arbitrary parameters to the structure of previous knowing results [6] or by some changes in the methodology [7]. Other types of integrable systems which have a configuration space involving a large numbers of arbitrary parameters were introduced and they were classified and interpreted physically by Gaussian curvature. For instance, when Gaussian curvature vanishes (equals a negative value), the configuration space becomes Euclidean plane (Pseudosphere) (see, e.g.[8]).

The model of a rigid body acts a good example in the integrability problems owing to its applications in diverse branches of science such as astronomy and physics (see, e.g., [9–11]). Integrable problems concerning the rigid body dynamics and their generalizations to a gyrostat were classified into general and conditional integrable problems according to their validity on an arbitrary level of a cyclic integral or on a fixed level (usually zero-level) of it. The general integrable problems were tabulated in small tables (see, e.g., [12–14]) and some other problems were added (see, for example, [15]). The famous integrable problems bearing the names of Goriachev- Chaplygin [12] are the first example of conditional integrable problems and they were followed by numerous generalizations (see, for example, [16–18]). It is worthy notice that the first integrable mechanical system that possesses an additional polynomial integral of a third degree in the velocities is Goriachev's case. This case was generalized in several works such as [23–25]. A part of our interest in current work is to present a new generalization of this case and its knowing generalizations.

Equations of motion of a rigid body

The general motion of a rigid body rotating around its fixed point O due to the effect of a potential (velocity-independent) and gyroscopic (velocity-dependent) forces is described by the Lagrangian [26,27]

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https://doi.org/10.1016/j.rinp.2017.12.050
2211-3797/

$$L = \frac{1}{2}\boldsymbol{\omega}\cdot\boldsymbol{\omega}\mathbf{I} + \mathbf{l}\cdot\boldsymbol{\omega} - V, \quad (1)$$

where ω is the angular velocity and $\mathbf{I} = diag(A, B, C)$ is the matrix of principal inertia about the fixed point O. The equations of motion take the form

$$\dot{\boldsymbol{\omega}}\mathbf{I} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\mathbf{I} + \mu) = \boldsymbol{\gamma}\times\frac{\partial V}{\partial\boldsymbol{\gamma}}, \quad \dot{\boldsymbol{\gamma}} + \boldsymbol{\omega}\times\boldsymbol{\gamma} = \mathbf{0}, \quad (2)$$

where $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ is a unite vector that is fixed upward in the space while μ takes the form

$$\mu = \frac{\partial}{\partial\boldsymbol{\gamma}}(\mathbf{l}\cdot\boldsymbol{\gamma}) - (\frac{\partial}{\partial\boldsymbol{\gamma}}\cdot\mathbf{l})\boldsymbol{\gamma}. \quad (3)$$

The Eqs. (2) have three integrals of motion which are called in literatures as classical integrals. They are:

$$\text{Jacobi-integral}: \quad I_1 = \frac{1}{2}\boldsymbol{\omega}\cdot\boldsymbol{\omega}\mathbf{I} + V = h, \quad (4)$$

$$\text{Geometric integral}: \quad I_2 = \boldsymbol{\gamma}\cdot\boldsymbol{\gamma} = 1, \quad (5)$$

$$\text{Area integral}: \quad I_3 = (\boldsymbol{\omega} I + l)\cdot\boldsymbol{\gamma} = f, \quad (6)$$

where h and f are arbitrary parameters characterizing the values of Jacobi and area integrals, respectively.

Due to Jacobi theorem on a last integrating factor [12], the equations of motion (2) are completely integrable, or integrable in short, if it possesses the fourth first integral of motion that is independent on those (4)–(6). This means, one additional integral of motion is required to prove the integrability and moreover, it can be utilized to find the explicit solution of the equations of motion. In general, the problem of motion of a rigid body has six degrees of freedom: three of them for transition motion and the others for the rotational motion. As a result of the body has a fixed point, it only rotates about a certain axis passing through this point. So, it has three degrees of freedom. Its position is always determined by the Eulerian angles θ, φ and ψ. Moreover, the variable ψ is cyclic variable. Therefore, we can utilize the Routh procedure to eliminate this variable and the problem becomes two-dimensional. Consequently, it is described by the Routhian

$$R = \frac{1}{2}\left[\frac{\dot{\gamma}_3^{\,2}}{1-\gamma_3^2} + \frac{C(1-\gamma_3^2)}{D}\dot{\varphi}^2\right] + \frac{fC\gamma_3 + Al_3(1-\gamma_3^2)}{AD}\dot{\varphi} - \frac{1}{A}\left(V + \frac{(f - l_3\gamma_3^2)^2}{2D}\right), \quad (7)$$

where $D = A - (A - C)\gamma_3^2$. The new results are usually given by the two functions V and $\boldsymbol{\mu}$. This is due to, they are invariant under all possible gauge transformations. This means, when we add the gauge term $\frac{dK(\gamma_1,\gamma_2,\gamma_3)}{dt}$ to the Lagrangian (1), the linear terms in velocity will be altered while the vector μ and the potential V are not changed. And so, the equations of the motion remain unchanged. Furthermore, this illustrates the novelty of our results.

Formulation of the problem

The method for constructing two-dimensional integrable systems (not necessarily plane) admitting an additional polynomial integral in the generalized velocities was presented in [28] and it was developed in later works (see, e.g., [29]). It also was applied to build various integrable systems with additional integral quadratic (see, e.g., [30,31]), cubic (see, e.g., [23,24]) and quartic (see, e.g., [8,15–21]). The application of this method is only confined to the two-dimensional mechanical systems. Numerous examples are described by this type such as the problem of motion of a particle on a smooth surface (fixing or rotating) under the action of the variety of forces or their dimensions are reduced by using a Routh procedure to a two-dimensional mechanical systems such as the problem of motion having n degrees of freedom with $n-2$ are cyclic. Also, one of the most significant examples is the motion of a rigid body-gyrostat that rotates about its fixed point under the action of the results of potential and gyroscopic forces allowing to a cyclic variable to exist. In general, the two-dimensional mechanical systems are described by the Lagrangian

$$L = \frac{1}{2}(a_{11}\dot{q}_1^{\,2} + 2a_{12}\dot{q}_1\dot{q}_2 + a_{22}\dot{q}_2^{\,2}) + a_1\dot{q}_1 + a_2\dot{q}_2 - V, \quad (8)$$

where $a_{ij}, a_i, i, j = 1, 2$ and V are six functions in both variables q_1, q_2 and dots denote the derivative with respect to the time t. As a result of those functions rely on the generalized coordinates q_1, q_2, the system (8) accommodates Riemannian two-dimensional manifolds as possible configuration spaces. And this permits the application to several problems in the dynamics including the motion of a rigid body about a fixed point, the motion of a particle on a fixed smooth curved surface and the motion on pseudo-sphere. From another point, the linear terms in the velocities characterize the gyroscopic forces which do not produce work through the motion. This force appears due to some aspects such as the body carries some charged components moving in a stationary magnetic field and the utilization of Routh procedure to eliminate cyclic coordinates. It is denoted by the vector (a_1, a_2) and so, it is named a vector potential while V refers to the scalar potential. By virtue of Birkhoff's theorem [32], the system (8) can always be referred to some isometric coordinates ξ, η(say) and the Lagrangian (8) admits the form

$$L = \frac{\Lambda}{2}(\dot{\xi}^2 + \dot{\eta}^2) + l_1(\xi,\eta)\dot{\xi} + l_2(\xi,\eta)\dot{\eta} - V, \quad (9)$$

where Λ is a function in the two variables ξ, η. This Lagrangian has a Jacobi integral

$$I_1 = \frac{\Lambda}{2}(\dot{\xi}^2 + \dot{\eta}^2) + V = h, \quad (10)$$

where h is an arbitrary constant. The Lagrangian (9) is completely integrable if it has an additional integral I_2 that is independent on the Jacobi-integral (10) and thus, the solution of the equations of the motion is reduced to a number of quadratures and to the inversion of certain integrals. This is always ensured by Liouville theorem of the equivalent Hamiltonian systems (see, e.g., [33]). Applying the time transformation

$$dt = \Lambda d\tau, \quad (11)$$

to the Lagrangian (9), we get

$$L = \frac{1}{2}(\xi'^2 + \eta'^2) + l_1\xi' + l_2\eta' + U, \quad (12)$$

where $U = \Lambda(h - V)$ and dashes refer to differentiation with respect to τ. The Lagrangian equations corresponding to the Lagrangian (12) are

$$\xi'' + \Omega\eta' = \frac{\partial U}{\partial\xi}, \qquad \eta'' - \Omega\xi' = \frac{\partial U}{\partial\eta}, \quad (13)$$

where $\Omega = \frac{\partial l_1}{\partial\eta} - \frac{\partial l_2}{\partial\xi}$. Eqs. (13) have a Jacobi integral

$$I_1 = \frac{1}{2}(\xi'^2 + \eta'^2) - U = 0. \quad (14)$$

Notice that, the Jacobi constant h for the original system (8) enters linearly as a parameter in the force function U. The additional integral which requires to ensure the integrability of the Lagrangian system (12) is assumed to be cubic in the generalized velocities. Following [28], it takes the form

$$I_2 = \xi'^3 + P_2\xi'^2 + Q_2\xi'\eta' + P_1\xi' + Q_1\eta' + R, \quad (15)$$

where $P_i, Q_i, i, j = 1, 2$ and R are functions in both variables ξ and η. Differentiating the integral I_2 with respect to τ and using the Jacobi integral (14) to eliminate the even powers of η', we obtain

$$\frac{\partial P_2}{\partial \xi} - \frac{\partial Q_2}{\partial \eta} = 0, \quad (16)$$

$$\frac{\partial Q_2}{\partial \xi} + \frac{\partial P_2}{\partial \eta} - 3\Omega = 0, \quad (17)$$

$$\frac{\partial P_1}{\partial \xi} - \frac{\partial Q_1}{\partial \eta} + 2\Omega Q_2 + 3\frac{\partial U}{\partial \xi} = 0, \quad (18)$$

$$\frac{\partial P_1}{\partial \eta} + \frac{\partial Q_1}{\partial \xi} - 2\Omega P_2 = 0, \quad (19)$$

$$2U\left[\frac{\partial Q_1}{\partial \eta} - \Omega Q_2\right] + P_1\frac{\partial U}{\partial \xi} + Q_1\frac{\partial U}{\partial \eta} = 0, \quad (20)$$

$$\frac{\partial R}{\partial \xi} + Q_2\frac{\partial U}{\partial \eta} + 2P_2\frac{\partial U}{\partial \xi} + 2\frac{\partial Q_2}{\partial \eta}U + Q_1\Omega = 0, \quad (21)$$

$$\frac{\partial R}{\partial \eta} + Q_2\frac{\partial U}{\partial \xi} - P_1\Omega = 0. \quad (22)$$

The Eqs. (16)–(22) are seven nonlinear partial differential equations in seven unknowns. The two Eqs. (16) and (17) imply to

$$P_2 = \kappa\frac{\partial \Psi}{\partial \eta}, \quad Q_2 = \kappa\frac{\partial \Psi}{\partial \xi}, \quad \Omega = \frac{\kappa}{3}\nabla^2\Psi, \quad (23)$$

where Ψ is an arbitrary function in the two variables ξ and η while κ is an arbitrary constant. Inserting the expressions (23) into the two Eqs. (18) and (19), we get

$$P_1 = \frac{\partial^2\Phi}{\partial^2\xi} + \frac{\kappa^2}{3}\left(\left(\frac{\partial \Psi}{\partial \eta}\right)^2 - \left(\frac{\partial \Psi^2}{\partial \xi}\right)\right),$$

$$Q_1 = -\frac{\partial^2\Phi}{\partial\eta\partial\xi} + \frac{2\kappa^2}{3}\frac{\partial \Psi}{\partial \xi}\frac{\partial \Psi}{\partial \eta}, \quad U = -\frac{1}{3}\nabla^2\Phi, \quad (24)$$

where Φ is an arbitrary function in the two variables ξ and η. Taking all obtained results into the two Eqs. (21) and (22), we can write the function R -up to an additive constant- in the form

$$R = \frac{-\kappa}{9}\int\left[-3\Psi_\xi\nabla^2\Phi_\xi + [\kappa^2(\Psi_\xi^2 - \Psi_\eta^2) - 3\Phi_{\xi\xi}]\nabla^2\Psi\right] \times d\eta + \frac{\kappa}{3}\int\left[2\Psi_\eta\nabla^2\Phi_\xi + \Psi_\xi\nabla^2\Phi_\eta + 2\Psi_{\xi\eta}\nabla^2\Phi + \nabla^2\Psi(\Phi_{\xi\eta} - \frac{2\kappa^2}{3}\Psi_\xi\Psi_\eta)\right]_0 d\xi, \quad (25)$$

where $[\cdot]_0$ means that the expression in the bracket is computed for η taking an arbitrary constant value η_0 (say). Notice, the following compatibility condition

$$\frac{\partial}{\partial \eta}\left(\frac{\partial R}{\partial \xi}\right) = \frac{\partial}{\partial \xi}\left(\frac{\partial R}{\partial \eta}\right), \quad (26)$$

must be satisfied. Inserting all obtained results into the two Eqs. (20) and (26), we have

$$\kappa^2[(\Psi_\xi^2 - \Psi_\eta^2)\nabla^2\Phi_\xi - 2\Psi_\xi\Psi_\eta\nabla^2\Phi_\eta + 2(\Psi_\xi\psi_{\xi\xi} - 2\Psi_\eta\Psi_{\eta\xi} - \Psi_{\eta\eta}\Psi_\xi)\nabla^2\Phi] + 3[\Phi_{\xi\eta}\nabla^2\Phi_\eta - \Phi_{\xi\xi}\nabla^2\Phi_\xi + 2\Phi_{\xi\eta\eta}\nabla^2\Phi] = 0, \quad (27)$$

and

$$\kappa^3[(\Psi_\xi^2 - \Psi_\eta^2)\nabla^2\Psi_\xi - 2\Psi_\xi\Psi\eta\nabla^2\Psi_\eta + 2\nabla^2\Psi(\Psi_\xi(\Psi_{\xi\xi} - \Psi_{\eta\eta}) - 2\Psi_\eta\Psi_{\eta\xi}] + 3\kappa[\Psi_\xi(\Phi_{\eta\eta\eta\eta} - \Phi_{\xi\xi\xi\xi}) + 2\Psi_\eta\nabla^2\Phi_{\eta\xi} + (\Psi_{\eta\eta} - 2\Psi_{\xi\xi})\Phi_{\xi\xi\xi\xi} + (\Phi_{\xi\xi} + 2\Phi_{\eta\eta})\Psi_{\eta\eta\xi} + 3\Psi_{\xi\eta}\nabla^2\Phi_\eta + 3\Psi_{\eta\eta}\Phi_{\eta\eta\xi} - \Phi_{\xi\xi}\Psi_{\xi\xi\xi} + \Phi_{\eta\xi}\nabla^2\Psi_\eta] = 0, \quad (28)$$

Taking all obtained results into our considerations, we can formulate the following.

Theorem 1. *The 2D time-irreversible Lagrangian*

$$L = \frac{1}{2}(\xi'^2 + \eta'^2) + \kappa(\Psi_\eta\xi' - \Psi_\xi\eta') - \frac{1}{3}\nabla^2\Phi, \quad (29)$$

describes an integrable mechanical system on a zero-level of Jacobi-integral

$$I_1 = \frac{1}{2}(\xi'^2 + \eta'^2) + \frac{1}{3}\nabla^2\Phi = 0. \quad (30)$$

Its conditional cubic integral is

$$I_2 = \xi'^3 + \kappa(\Psi_\eta\xi'^2 + \Psi_\xi\xi'\eta') + [\Phi_{\xi\xi} + \frac{\kappa^2}{3}(\Psi_\eta^2 - \Psi_\xi^2)]\xi' - [\Phi_{\eta\xi} - \frac{2\kappa^2}{3}\Psi_\xi\Psi_\eta]\eta' - \frac{\kappa}{9}\int\left[-3\Psi_\xi\nabla^2\Phi_\xi + [\kappa^2(\Psi_\xi^2 - \Psi_\eta^2) - 3\Phi\xi\xi]\nabla^2\Psi\right]d\eta + \frac{\kappa}{3}\int\left[2\Psi_\eta\nabla^2\Phi_\xi + \Psi_\xi\nabla^2\Phi_\eta + 2\Psi_{\xi\eta}\nabla^2\Phi + \nabla^2\Psi(\Phi_{\xi\eta} - \frac{2\kappa^2}{3}\Psi_\xi\Psi_\eta)\right]_0 d\xi, \quad (31)$$

where the two functions Ψ and Φ satisfy the two nonlinear partial differential Eqs. (27) and (28).

Notice that, the integrable system involving in Theorem 1 is a conditionally integrable due to it is only valid on a zero-level of Jacobi integral. It is worth notice that the problem of construction 2D time- irreversible integrable mechanical system is reduced to solve the two nonlinear partial differential Eqs. (27) and (28). This reduction is introduced here for the first time. It is worth notice that, when κ vanishes, the problem under consideration becomes time-reversible, i.e., it is invariant under a time transformation $\tau \to -\tau$. It is evident that the Theorem 1 is reduced to the results obtained in [23] and can be summarized in the following.

Theorem 2. *A 2D time-reversible mechanical system that is characterized by the Lagrangian*

$$L = \frac{1}{2}(\xi'^2 + \eta'^2) - \frac{1}{3}\nabla^2\Phi, \quad (32)$$

is integrable on a zero level of energy-integral

$$I_1 = \frac{1}{2}(\xi'^2 + \eta'^2) + \frac{1}{3}\nabla^2\Phi, \quad (33)$$

and the conditional additional cubic integral has the form

$$I_2 = \xi'^3 + \Phi_{\xi\xi}\xi' - \Phi_{\eta\xi}\eta', \quad (34)$$

where the function Φ is a solution of the following partial differential equation

$$\Phi_{\xi\eta}\nabla^2\Phi_\eta - \Phi_{\xi\xi}\nabla^2\Phi_\xi + 2\Phi_{\xi\eta\eta}\nabla^2\Phi = 0. \quad (35)$$

A family of 2D-integrable system

A conditional integrable system

Now, we search for a solution of the two Eqs. (27) and (28). This solution is acceptable if it generates new integrable systems generalizing the previous results and can also be utilized to construct new integrable model as we see later in the applications. According to previous results, we can use the two variables p, q instead of ξ, η, respectively, through the following point transformation

$$\xi = \int \frac{dp}{\sqrt{a_3p^3 + a_2p^2 + a_1p + a_0}},$$
$$\eta = \int \frac{\sqrt{b_4q^4 + b_3q^3 + b_2q^2 + b_1q + b_0}}{c_3q^3 + c_2q^2 + c_1q + c_0}dq, \tag{36}$$

where a_i, b_i, c_i are arbitrary constants and the two functions Ψ and Φ are assumed to have the following forms

$$\begin{aligned}\Psi(p,q) =& f_1(q)\left[d_1\sqrt{a_3p^3 + a_2p^2 + a_1p + a_0} + d_2p\right]\\ &+ f_3(q)\left[d_3p^2 + d_4p\sqrt{a_3p^3 + a_2p^2 + a_1p + a_0}\right]\\ &+ f_0(q),\\ \Phi(p,q) =& f_5(q)\left[d_1\sqrt{a_3p^3 + a_2p^2 + a_1p + a_0} + d_2p\right]\\ &+ f_6(q)\left[d_3p^2 + d_4p\sqrt{a_3p^3 + a_2p^2 + a_1p + a_0}\right]\\ &+ f_4(q)\end{aligned} \tag{37}$$

where $f_i(q)$ are arbitrary functions. In what follows we will use Maple program to perform the calculations due to their complexity. Preforming the point transformation (36), inserting the two expressions (37) into the two Eqs. (27), (28)equating the coefficients of the variable p to zero, we obtain a system of ordinary differential equations containing the functions $f_i(q)$. Unfortunately, this system can not be written in a suitable size and its solution after tedious computations gives

$$\begin{aligned}L =& \frac{1}{2}\left[\frac{\acute{p}^2}{P} + \frac{G\acute{q}^2}{4F^2}\right] - \left[e_3\sqrt{\mu F^3}(A_1\sqrt{\mu P} + A_2P^*) - \frac{C_2}{2}\right.\\ &\times(12e_3\mu F - F^\circ F^{\circ\circ}) + C_1F^\circ]\frac{\acute{p}}{G\sqrt{P}} + \frac{\sqrt{F^3}}{G}[2A_4\sqrt{P}\\ &+ A_5P^*] + \frac{e_3\sqrt{\mu F^3}}{2G}\left[3C_3 + \frac{1}{G}(2C_1F^\circ + C_2(F^\circ F^{\circ\circ}\right.\\ &-12\mu e_3C_2F))][A_1\sqrt{\mu P} + A_2P^*] + \frac{\mu e_3^2F^3}{16G^2}[16A_1A_2P^*\\ &\left.\times\sqrt{\mu P} + (A_1^2 - 4A_2^2)(b^2 + 4c\mu - 2P^{*^2})\right]\\ &+ \frac{\mu e_3^2(A_1^2 + 4A_2^2)(b^2 + 4c\mu)F^3}{16G^2},\end{aligned} \tag{38}$$

where e_i, A_i, C_i, μ are arbitrary constants that are inserted instead of the original ones for simplicity and the following two functions are introduced owing to the appropriateness,

$$F(q) = \mu(e_3q^3 + e_1q + e_0) + e_2q^2, P(p) = -\mu p^2 + bp + c. \tag{39}$$

Moreover, we denote by circle $()\circ$ and asterisk $()^*$ differentiation with respect q and p, respectively. We introduce the notation

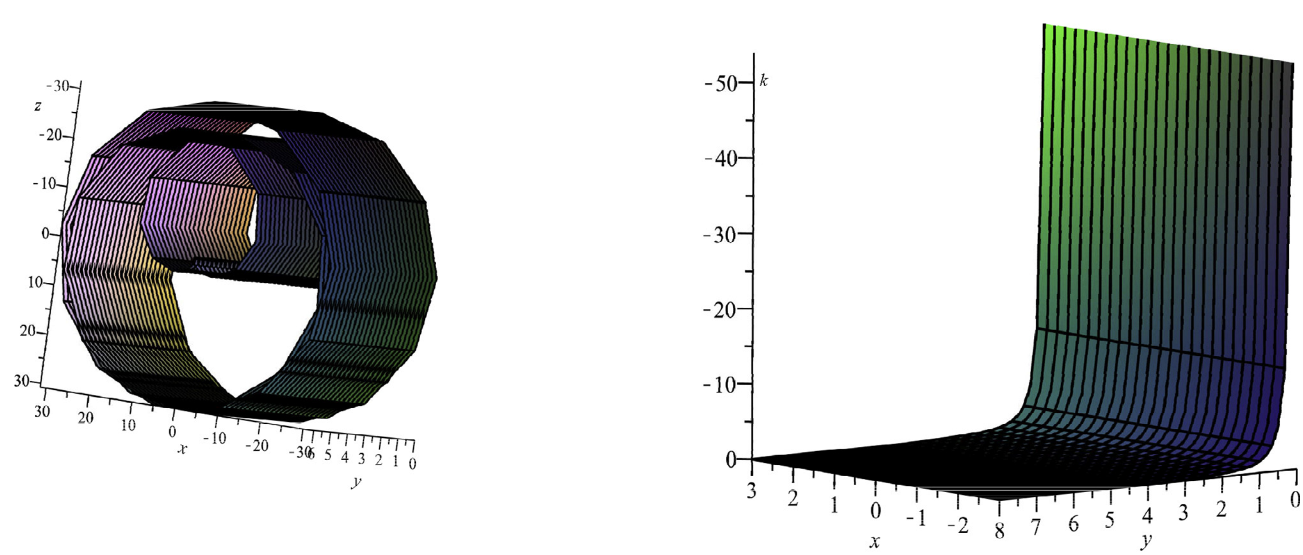
$$\begin{aligned}G =& -\mu[3e_3^2q^4 + 6e_1e_3q^2 + 12e_0e_3q - e_1^2] - 4e_2(e_3q^3 + e_0)\\ =& \frac{1}{\mu}[F^{\circ^2} - 2FF^{\circ\circ}].\end{aligned} \tag{40}$$

The Jacobi integral for this system is

$$\begin{aligned}I_1 =& \frac{1}{2}\left[\frac{\acute{p}^2}{P} + \frac{G\acute{q}^2}{4F^2}\right] - \frac{\sqrt{F^3}}{G}[2A_4\sqrt{P} + A_5P^*]\\ &+ [e_3\sqrt{\mu F^3}(A_1\sqrt{\mu P} + A_2P^*) - \frac{e_3\sqrt{\mu F^3}}{2G}\\ &\times\left[3C_3 + \frac{1}{G}(2C_1F^\circ + C_2(F^\circ F^{\circ\circ} - 12\mu e_3C_2F))\right]\\ &\times[A_1\sqrt{\mu P} + A_2P^*] - \frac{\mu e_3^2F^3}{16G^2}[16A_1A_2\\ &\times P^*\sqrt{\mu P} + (A_1^2 - 4A_2^2)(b^2 + 4c\mu - 2P^{*^2})\Big]\\ &- \frac{\mu e_3^2(A_1^2 + 4A_2^2)(b^2 + 4c\mu)F^3}{16G^2} = 0,\end{aligned} \tag{41}$$

The conditional cubic integral is

$$\begin{aligned}I_2 =& I_2(p, q, \acute{p}, \acute{q}, \mu, b, c, e_0, e_1, e_2, e_3, C_1C_2, C_3, A_1, A_2\\ &, A_3, A_4, A_5)\\ =& \frac{e_3\acute{p}^3}{\sqrt{P^3}} - \frac{e_3}{4PG}\left[\frac{(A_1\mu\sqrt{P} + A_2\sqrt{\mu}P^*)(12e_3F^2 + GF^\circ)}{\sqrt{F}}\right.\\ &-6(C_2(12\mu e_3F - F^\circ F^{\circ\circ}) - 2C_1F^\circ)]\acute{p}^2 - \frac{e_3G}{16\sqrt{PF^3}}\\ &[4A_2\mu\sqrt{\mu P} - a_1\mu P^*]\acute{q}\acute{p} + \frac{\acute{p}}{8G^2\sqrt{P}}\left\{\frac{-\mu e_3}{\sqrt{F}}(\mu A_1\sqrt{P}\right.\\ &+ \sqrt{\mu}A_2P^*)(-3A_3F^\circ G^2 - 2C_1G^2 + C_2(288e_3^2F^3\\ &+ 24e_3GFF^\circ - G^2F^{\circ\circ})) + 6e_3(\mu G + 2FF^{\circ\circ})(C_2F^{\circ\circ} + 2C_1)^2\\ &+ 4e_3\sqrt{F}(\mu A_1\sqrt{P} + A_2\sqrt{\mu}P^*)(6e_3FF^\circ\end{aligned}$$



(a) $\alpha = 0.001, \beta = 7$ (b) $\alpha = 0.001, \beta = 7$

Fig. 1. (a) The revolution surface characterized by the two Eqs. (80). (b) The corresponding Gaussian curvature for the revolution surface given by the two Eqs. (80).

$$
\begin{aligned}
&+GF^{\circ\circ})(C_2F^{\circ\circ}+2C_1)-e_3^2\mu F(6e_3F^2+GF^{\circ})(A_1^2\\
&\times(P^{*^2}-b^2-4c\mu)-4A_2^2P^{*^2})+24e_3\mu\times(A_1^2+4A_2^2)\\
&\times(b^2+4c\mu)G^2+192e_3A_1A_2F\sqrt{\mu P}P^*(6e_3F^2\\
&+GF^{\circ})]+144e_3^2\mu FC_2\times(C_2(6e_3\mu F-F^{\circ}F^{\circ\circ})\\
&-2C_1F^{\circ})-24e_3\mu p_0G^2+\frac{2G^2F^{\circ}}{\sqrt{F}}(2A_4\sqrt{P}+A_5P^*)\Big\}\\
&+\Big\{\frac{1}{32\sqrt{F^3}}[4G(2\mu A_5\sqrt{P}-A_4P^*)-e_3\mu(4A_2\sqrt{\mu P}\\
&-A_1P^*)(C_2(12e_3\mu F-F^{\circ}F^{\circ\circ})-2C_1F^{\circ}-3A_3\mu G)]\\
&-\frac{e_3^2\mu}{16}(\sqrt{\mu}A_1A_2(2P^{*^2}-b^2-4c\mu)-\mu P^*\sqrt{P}(A_1^2\\
&-4A_2^2)\Big\}\acute{q}+\frac{3e_3p_0}{2G}\{4e_3\sqrt{\mu^3F^3}(A_2P^*+A_1\sqrt{\mu P})\\
&+(C_2(\mu F^{\circ}F^{\circ\circ}-12e_3\mu^2F)+2C_1\mu F^{\circ})\}\\
&-\frac{3\mu^2e_3^2(3A_3F^{\circ}+2C_1)}{96G}\{F[A_1^2(b^2+4c\mu-P^{*2})\\
&+4A_2^2P^{*^2}]-96C_2[\mu A_1\sqrt{P}+\sqrt{\mu}P^*]\}-\frac{e_3F^{\circ 3}}{2G^3}(C_2F^{\circ\circ}+2C_1)^3-\frac{e_3}{G^3}\left(\frac{C_2F^{\circ\circ}}{2}+C_1\right)^4\\
&\times\Big\{-18\mu C_2e_3FF^{\circ^2}+\sqrt{F}F^{\circ\circ}(\sqrt{\mu}A_2P^*+\mu A_1\sqrt{P})\Big\}+\Big\{-\frac{\mu e_3}{G^3}[e_3F^2
\end{aligned}
\tag{42}
$$

$$
\begin{aligned}
&\times(3e_3FF^{\circ}+GF^{\circ\circ})(A_1^2(P^{*^2}-b^2-4c\mu)-4A_2^2P^{*^2})\\
&+\frac{G^2F^{\circ}F^{\circ\circ}(A_1^2+4A_2^2)(b^2+4c\mu)}{48}-\frac{2A_1A_2e_3^2F\sqrt{\mu^3P}P^*}{G^3}\\
&\times[3e_3FF^{\circ}+GF^{\circ\circ}]-\frac{F^{\circ\circ}\sqrt{F}}{2G}[A_5P^*+2A_4\sqrt{P}]\\
&-\frac{3e_3A_3\sqrt{\mu^3F}F^{\circ\circ}}{2G}[A_1\sqrt{\mu P}+A_2P^*]+\frac{72e_3^3C_2F^{\circ}\sqrt{\mu^3F^5}}{G^3}\\
&\times[A_1\sqrt{\mu P}+A_2P^*]-\frac{3A_1e_3^2\sqrt{PF^3}}{G^2}(2C_1-3C_2F^{\circ\circ})\Big\}\\
&\times(\frac{C_2}{2}F^{\circ\circ}+C_1)+\frac{e_3F(b^2+4c\mu-P^{*^2})}{8G^3}\times\{e_3\mu^2[6e_3F\\
&\times(6E_3F^2+GF^{\circ})A_1^2+A_2^2G^2F^{\circ\circ}]-G^2A_1A_4F^{\circ}\}\\
&+\frac{e_3(b^2+4c\mu)}{384}\times(A_1^2+4A_2^2)\{\mu^2(8\mu e_3C_2-3A_3F^{\circ\circ})\\
&-\frac{4F^{\circ\circ}\sqrt{\mu^3F^3}}{G}(A_2P^*+A_1\sqrt{\mu P})\}-\frac{3e_3^3\mu^2A_1A_2^2\sqrt{F^5P}}{16G^3}\\
&\times(4e_3F^2+GF^{\circ})(4P^{*^2}+b^2+4c\mu)-\frac{\mu^2e_3^3A_1^3\sqrt{PF^5}}{64G^3}\\
&\times(4e_3F^2+GF^{\circ})(7(b^2+4c\mu)-4P^{*^2})+\frac{e_3^3\mu\sqrt{F^3}}{320G^3}\\
&\times(4e_3F^2+GF^{\circ})-640\sqrt{\mu}A_2^3\frac{P^{*^3}}{\sqrt{F}}\}-\frac{F^{\circ\circ}}{96}(b^2+4c\mu)\\
&\times(2\sqrt{\mu}A_2A_5+A_1A_4)+\frac{e_3}{32G}\Big\{-8\sqrt{P\mu^5F}P^*A_1A_2(C_2F^{\circ\circ}+2C_1)\\
&+48e_3C_2^2\sqrt{F\mu^5}F^{\circ\circ}(A_2P^*+A_1\sqrt{\mu P})+C_2\\
&\times[48\mu\sqrt{F}F^{\circ}(2A_4\sqrt{P}+A_5P^*)+A_1^2e_3\mu^2FF^{\circ\circ}P^{*^2}]\\
&-8A_2\sqrt{\mu}FF^{\circ}P^*(2e_3A_1A_3\mu^2\sqrt{P}+A_5P^*)-8FF^{\circ}P^*(\mu A_1\\
&\times A_5+2A_2A_4\sqrt{\mu})\sqrt{P}\Big\}-\frac{3e_3^2A_2\sqrt{\mu^3F^3}P^*}{2G^2}[4C_1^2-4C_2C_1\\
&\times F^{\circ\circ}+3C_2^2(4e_3\mu F^{\circ}-F^{\circ\circ^2})]-\frac{6e_3^2C_2}{G^3}\Big\{e_3\Big[-A_2P^*F^4\mu^2\\
&\times(2A_1\sqrt{\mu P}+A_2P^*)+12C_2\sqrt{\mu^5F^7}(3A_2P^*+2\mu A_1\sqrt{P})\Big]\\
&-9\mu^2F^2(4e_3\mu F-F^{\circ}F^{\circ\circ})C_2^2+18C_2C_1\mu F^{\circ}F^2\Big\}.
\end{aligned}
\tag{43}
$$

A general integrable system:

The Lagrangian (38) and its two first integrals (41) and (43) characterize a conditional integrable system owing to it is only valid on a zero level of Jacobi integral (41). Therefore, we are going to preform the inverse of time transformation (11) (for a detailed for the computations of this method, see, e.g., [34]). Introducing new arbitrary parameters N_i, n_i by the relations

$$
\begin{aligned}
&A_4=N_4+n_4h,\quad A_5=N_5+n_5h,\quad C_3=N_6+n_6h,\\
&A_3=N_7+n_7h,\quad C_1=N_8+n_8h,
\end{aligned}
\tag{44}
$$

and performing the change of independent variable to the actual time parametrization by using the relation

$$
d\tau=\frac{dt}{\Lambda}
\tag{45}
$$

where

$$
\Lambda=\frac{\sqrt{F^3}}{2G}\left[2(2n_4\sqrt{P}+n_5P^*)+3\sqrt{\mu}e_3n_6(A_1\sqrt{\mu P}+A_2P^*)\right]-\frac{F}{G}[n_7+n_8F^{\circ\circ}],
\tag{46}
$$

we arrive at the new Lagrangian

$$
\begin{aligned}
L=&\frac{\Lambda}{2}\left[\frac{\dot{p}^2}{P}+\frac{G\dot{q}^2}{4F^2}\right]-\Big[e_3\sqrt{\mu F^3}(A_1\sqrt{\mu P}+A_2P^*)\\
&-\frac{C_2}{2}(12e_3\mu F-F^{\circ}F^{\circ\circ})+N_8F^{\circ}\Big]\frac{\dot{p}}{G\sqrt{P}}\\
&+\frac{1}{\Lambda}\Big\{\frac{\sqrt{F^3}}{G}[2N_4\sqrt{P}+N_5P^*]+\frac{e_3\sqrt{\mu F^3}}{2G}[3N_6\\
&+\frac{1}{G}(2N_8F^{\circ}+C_2(F^{\circ}F^{\circ\circ}-12\mu e_3C_2F))\Big][A_1\sqrt{\mu P}\\
&+A_2P^*]+\frac{\mu e_3^2F^3}{16G^2}\Big[16A_1A_2P^*\sqrt{\mu P}+(A_1^2-4A_2^2)(b^2\\
&+4c\mu-2P^{*^2})\Big]+\frac{\mu e_3^2(A_1^2+4A_2^2)(b^2+4c\mu)F^3}{16G^2}\Big\}+h,
\end{aligned}
\tag{47}
$$

and its unconditional Jacobi integral takes the form

$$
\begin{aligned}
L=&\frac{\Lambda}{2}\left[\frac{\dot{p}^2}{P}+\frac{G\dot{q}^2}{4F^2}\right]-\frac{1}{\Lambda}\Big\{\frac{\sqrt{F^3}}{G}[2N_4\sqrt{P}\\
&+\frac{e_3\sqrt{\mu F^3}}{2G}[3N_6+N_5P^*]+\frac{1}{G}(2N_8F^{\circ}+C_2(F^{\circ}F^{\circ\circ}\\
&-12\mu e_3C_2F))][A_1\sqrt{\mu P}+A_2P^*]+\frac{\mu e_3^2F^3}{16G^2}\Big[16A_1A_2\\
&\times P^*\sqrt{\mu P}+(A_1^2-4A_2^2)(b^2\\
&+4c\mu-2P^{*^2})\Big]+\frac{\mu e_3^2(A_1^2+4A_2^2)(b^2+4c\mu)F^3}{16G^2}\Big\},=h.
\end{aligned}
\tag{48}
$$

Its unconditional additional cubic integral can be formulated by replacing $(p', q')\to(\Lambda\dot{p},\Lambda\dot{q})$ and taking into account (44), we have

$$
\begin{aligned}
I_2=&I_2(p,q,\Lambda\dot{p},\Lambda\dot{q},\mu,b,c,e_0,e_1,e_2,e_3,N_8+n_8h,C_2,N_6\\
&+n_6h,A_1,A_2,N_7+n_7h,N_4+n_4h,N_5+n_5h).
\end{aligned}
\tag{49}
$$

Indeed the appearance of an arbitrary parameter h in the Lagrangian (47) is unimportant and can be disregarded. The same arbitrary constant h is now interpreted as the value of the Jacobi integral. We should eliminate it from the additional cubic integral by using the unconditional Jacobi integral (48) and this sometimes makes a change in the degree of the additional integral. The Lagrangian system (47)–(49) characterizes *a new family of integrable 2D time-irreversible mechanical system* or in some times, it is named *a multi-parameters 2D time-irreversible integrable mechanical system*. It contains 20 arbitrary parameters. They are

$$e_0,e_1,e_2,e_3,\mu,b,c,A_1,A_2,C_2,N_4,n_4,N_5,n_5,N_6,n_6,N_7,n_7,N_8,n_8$$

This system is new. It also generates a large class of 2*D* integrable systems including all the previous results. For instance, the 2*D* time-irreversible integrable system that was introduced by Yehia [23] can be obtained as a special case from our results by setting $n_4 = n_5 = n_6 = A_1, A_2 = e_2 = c = 0, c = -\mu = 1, p = \cos x$. Until now, the full physical interpretation of this system is unknown. Nevertheless, one of an essential advantage of this system is the structure of a configuration manifold including a large set of free parameters. This structure extends the range of its applicability to numerous problems such as the problem of motion in the Euclidean plane, the hyperbolic plane and distinct varieties of curved two-dimensional manifolds (for example, the problem of rigid body dynamics).

Applications

Applications in the dynamics of a particle:

The presence of extra-parameters in the Lagrangian (47) can be employed to construct new integrable problems that generalize and unify previous results. In what follows we will only write the Lagrangian and the additional integral for each integrable case owing to the Jacobi integral can be constructed immediately from the Lagrangian.

Case 1. The first case can be constructed by setting $\mu = 0$ in the Lagrangian (47) and its unconditional cubic integral (49), we get after some manipulations:

$$L_1 = \frac{1}{2}(a_1 + a_2 q + a_3 q x)\left[\frac{q^2}{\alpha q^3 + \beta}\dot{x}^2 + \frac{\dot{q}^2}{a_0^2 q^2}\right] + \frac{c_1 q}{\alpha q^3 + \beta}\dot{x} + \frac{c_1^2}{2(\alpha q^3 + \beta)(a_1 + a_2 q + a_3 q x)} + \frac{b_1 + b_2 q + b_3 q x}{a_1 + a_2 q + a_3 q x} + h, \quad (50)$$

and its additional cubic integral

$$\begin{aligned} I_2 =& \alpha q^6 \left(\frac{a_3 q x + a_2 q + a_1}{\alpha q^3 + \beta}\right)^3 \dot{x}^3 + \frac{3\alpha c_1 q^5}{(\alpha q^3 + \beta)} \\ &\times \left(\frac{a_3 q x + a_2 q + a_1}{\alpha q^3 + \beta}\right)^2 \dot{x}^2 - \frac{q^2(a_3 q x + a_2 q + a_1)}{\alpha q^3 + \beta} \\ &\times \left[2(b_2 + b_3 x) - \frac{3\alpha c_1^2 q^2}{(\alpha q^3 + \beta)^2}\right]\dot{x} + \frac{2b_3}{a_0^2 q}(a_3 q x \\ &+ a_2 q + a_1)\dot{q} - \frac{4c_1 q}{\alpha q^3 + \beta}\left\{\frac{b_2 + b_3 x}{2} \right. \\ &\left. - \frac{\alpha q^2 c_1^2}{4(\alpha q^3 + \beta)^2}\right\} + 2h(a_3 q x + a_2 q + a_1) \\ &\times \left\{\frac{a_3}{a_0^2 q}\dot{q} - \frac{q(a_3 x + a_2)}{\alpha q^3 + \beta}\left[q\dot{x} + \frac{c_1}{a_3 q x + a_2 q + a_1}\right]\right\}. \end{aligned} \quad (51)$$

where $a_1, a_2, a_3, \alpha, \beta, b_1, b_2, b_3, a_0$ and c_1 are new parameters, introduced instead of the original parameters for convenience. We should note also the variable x is introduced instead of the variable p through the expression $x - x_0 = \int \frac{dp}{\sqrt{bp+c}}, x_0$ is an arbitrary constant. This system is a new integrable problem. It contains ten arbitrary parameters $a_1, a_2, a_3, \alpha, \beta, b_1, b_2, b_3, a_0$ and c_1. It also generalizes the case obtained by Yehia [23] by adding one free parameter $c_1 = 0$ which turns on the irreversible term. In the present time, we are not able to give a mechanical interpretation for the full system. So that, it is more suitable to calculate Gaussian curvature

$$\begin{aligned}\Im =& \frac{1}{4(\alpha q^3 + \beta)^2[\alpha q(a_3 x + a_2) + a_1]^3} \times [3\alpha^2 \beta q^2(7\alpha q^3 - 2\beta)(a_2 + a_3 x)^2 \\ &- 3\alpha a_1(\alpha^2 q^6 - 13\alpha\beta q^3 + 4\beta^2)(a_2 + a_3 x) + 2\alpha q^3[\alpha^2(q^6 + 3\alpha\beta q^3 \\ &+ 3\beta^2)a_3^2 + 11 a_1^2 \beta] - (a_1^2 q^6 - 2a_3^2 \beta^3)\alpha^2 - 4a_1^2\beta^2]. \end{aligned} \quad (52)$$

The following two special cases may indicate the richness of this system.

Case 1.a: The first system is obtained from Eq. (50) by setting $a_2 = a_3 = 0, a_1 = 1$ and $q = e^{a_0 y}$. It has the Lagrangian

$$L = \frac{1}{2}\left[\dot{y}^2 + \frac{\dot{x}^2}{\alpha e^{a_0 y} + \beta e^{-2a_0 y}}\right] + \frac{c_1 \dot{x}}{\alpha e^{2a_0 y} + \beta e^{-a_0 y}} + e^{a_0 y}[b_2 + b_3 x] + \frac{c_1^2}{2[\alpha e^{3a_0 y} + \beta]}, \quad (53)$$

and admits the unconditional second integral

$$\begin{aligned} I_2 =& \left(\frac{\dot{x}}{\alpha e^{a_0 y} + \beta e^{-2a_0 y}}\right)^3 + \frac{3c_1 e^{5a_0 y}}{(\alpha e^{3a_0 y} + \beta)^3}\dot{x}^2 \\ &- \frac{e^{2a_0 y}}{\alpha}\left\{\frac{2(b_2 + b_3 x)}{\alpha e^{3a_0 y} + \beta} - \frac{3\alpha c_1^2 e^{2a_0 y}}{(\alpha e^{3a_0 y} + \beta)^3}\right\}\dot{x} \\ &- \frac{c_1 e^{a_0 y}}{\alpha}\left\{\frac{2(b_2 + b_3 x)}{\alpha e^{3a_0 y} + \beta} - \frac{\alpha c_1^2 e^{2a_0 y}}{(\alpha e^{3a_0 y} + \beta)^3}\right\} + \frac{2b_3}{\alpha a_0}\dot{y}. \end{aligned} \quad (54)$$

It is worth notice that in this special case when $\beta = 0$, the system (53) and (54) reduces to

$$L = \frac{1}{2}\left[\dot{y}^2 + \frac{\dot{x}^2}{\alpha e^{a_0 y}}\right] + \frac{c_1}{\alpha e^{2ay}}\dot{x} + e^{a_0 y}[b_2 + b_3 x] + \frac{c_1^2}{2\alpha e^{3a_0 y}}, \quad (55)$$

$$I_2 = \frac{\dot{x}^3}{e^{3a_0 y}} + 3c_1\frac{\dot{x}^2}{e^{-5a_0 y}} + e^{-5a_0 y} \times \left\{3c_1^2 - \frac{2\alpha(b_2 + b_3 x)}{e^{-4a_0 y}}\right\}\dot{x} + \frac{2\alpha^2 b_3}{a_0}\dot{y} + c_1 e^{-6a_0 y}\left(c_1^2 - \frac{2\alpha(b_2 + b_3 x)}{e^{-4a_0 y}}\right). \quad (56)$$

Inserting these values of parameters in (52), we found the line element of the last system has constant negative Gaussian curvature ($\Im = -\frac{1}{4}$). This system can be interpreted as an integrable case of motion on the pseudo-sphere.

Case 1.b: The second system corresponds to the case $a_1 = a_3 = 0, a_2 = 1, a_0 = 2$ and $q = y^2$. The final result for this case is

$$L = \frac{1}{2}\left[\frac{y^6}{\alpha y^6 + \beta}\dot{x}^2 + \dot{y}^2\right] + \frac{c_1 y^2}{\alpha y^6 + \beta}\dot{x} + \frac{b_1}{y^2} + b_2 x + \frac{c_1^2}{2y^2(\alpha y^6 + \beta)}, \quad (57)$$

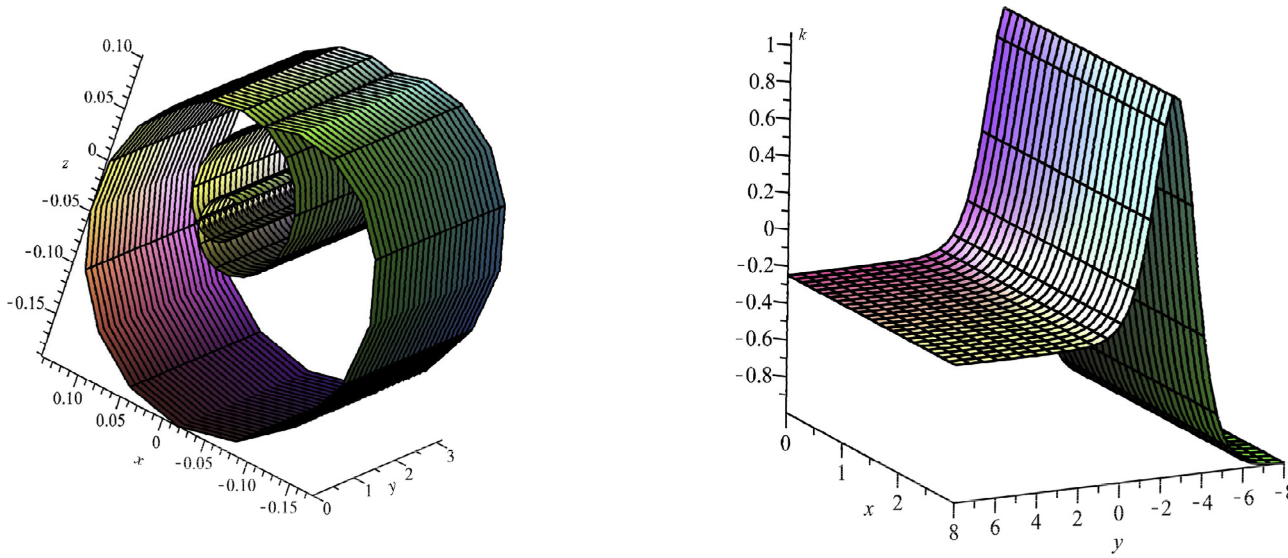
and

$$\begin{aligned} I_2 =& \frac{\beta y^{12}}{(\alpha y^6 + \beta)^3}\dot{x}^3 - \frac{c_1 y^8(2\alpha y^6 - \beta)}{(\alpha y^6 + \beta)^3}\dot{x}^2 + \dot{x}\dot{y}^2 \\ &\times \frac{y^6}{\alpha y^6 + \beta} + \frac{c_1 y^2}{\alpha y^6 + \beta}\dot{y}^2 - b_2 y\dot{y} - \frac{y^4}{(\alpha y^6 + \beta)^3} \\ &\times [c_1^2(4\alpha y^6 + \beta) + 2b_1(\alpha y^6 + \beta)^2]\dot{x} \\ &- \frac{c_1}{(\alpha y^6 + \beta)^3}[c_1^2(2\alpha y^6 + \beta) + 2b_1(\alpha y^6 + \beta)^2]. \end{aligned} \quad (58)$$

This system is also new integrable system. The Gaussian curvature (52) of this configuration space becomes

$$\Im = \frac{3\beta}{\alpha y^2}\left(\frac{7\alpha y^6 - 2\beta}{\alpha y^6 + \beta}\right). \quad (59)$$

The Gaussian curvature (59) vanishes if $\beta = 0$ and thus, the system represents the motion of a particle in the plane xy under the action of two scaler and vector potential. This system is new. It generalizes the Holt supper-integrable separable system by inserting the parameter c_1 [35]. In other words, the Holt system remains integrable in the presence of a gyroscopic force that is determined by the vector potential $(\frac{c_1 y^2}{\alpha y^6 + \beta}, 0)$. But it losses its separability.



(a) $\alpha = 50, \beta = 1$. (b) $\alpha = 50, \beta = 1$.

Fig. 2. (a) The revolution surface characterized by the two Eqs. (76) and (77). (b) The corresponding Gaussian curvature for the revolution surface given by the two Eqs. (76) and (77).

Case 2. The second case can be constructed by setting $e_3 = 0$ in the Lagrangian (47) and (49), we get after some computations

$$\begin{aligned} L =&\frac{1}{2}\Big[a_1(e_2q^2+e_1q+e_0) - (e_2q^2+e_1q+e_0)^{\frac{3}{2}} \\ &\times\{a_3(b-2\mu p)+2a_2\sqrt{-\mu p^2+bp+c}\}\Big] \\ &\times\Big(\frac{\dot{p}^2}{-\mu p^2+bp+c}+\frac{e_1^2-4e_0e_2}{4\mu[e_2q^2+e_1q+e_3]^2}\dot{q}^2\Big) \\ &-(a_4+\sqrt{e_2q^2+e_1q+e_0}\{a_6(b-2\mu p)+2a_5 \\ &\times\sqrt{-\mu p^2+bp+c}\})/(a_1-\sqrt{e_2q^2+e_1q+e_0} \\ &\times\{a_3(b-2\mu p)-2a_2\sqrt{-\mu p^2+bp+c}\}) \\ &+\frac{c_1(e_1+2e_2q)\dot{p}}{\sqrt{-\mu p^2+bp+c}}+h. \end{aligned} \tag{60}$$

Its complementary integral:

$$\begin{aligned} I_2 =&((2e_2q+e_1)\Big[2(a_2h+a_5)\sqrt{-\mu p^2+bp+c} \\ &-(a_3h+a_6)(2\mu p-b)\Big]\Big)\dot{p}/(4\sqrt{-\mu p^2+bp+c} \\ &\times\sqrt{e_2q^2+e_1q+e_0})+((e_1^2-4e_2e_0) \\ &\times\Big[2(a_3h+a_6)\sqrt{-\mu p^2+bp+c}-(a_2h+a_5) \\ &\times(2\mu p-b)\Big])/(\mu(e_2q^2+e_1q+e_0)^{\frac{3}{2}})\dot{q} \\ &+c_2e_2\sqrt{e_2q^2+e_1q+e_0}\{2(a_2h+a_5) \\ &\times\sqrt{-\mu p^2+bp+c}+(a_3h+a_6)(2\mu p-b)\Big\}, \end{aligned} \tag{61}$$

where a_i and c_1 are arbitrary constants, used instead of the original parameters for suitability. It should be noted that the additional cubic integral is reduced to a linear integral in velocities. The system (60) and (61) characterizes a new integrable problem which contains fourteen arbitrary parameters. For the special case $b = 0, c = \mu$, the Lagrangian (60) and additional integral (61) take the form

$$\begin{aligned} L =&\frac{1}{2}\Big[\frac{a_1\cos\sqrt{\mu}x+a_2\sin\sqrt{\mu}x}{\cosh^3\sqrt{\mu}y}+\frac{a_3}{\cosh^2\sqrt{\mu}y}\Big]\times(\dot{x}^2+\dot{y}^2) \\ &+c_1\tanh\sqrt{\mu}y\dot{x} \\ &+\frac{b_1\cos\sqrt{\mu}x+b_2\sin\sqrt{\mu}x+b_3\cosh\sqrt{\mu}y}{a_1\cos\sqrt{\mu}x+a_2\sin\sqrt{\mu}x+a_3\cosh\sqrt{\mu}y}+h \end{aligned} \tag{62}$$

where a_i, b_i $(i = 1, 2, 3)$ and c_1 are arbitrary constants while the two variables p and q are replaced by the two variables x and y through the point transformation

$$\begin{aligned} x =&\frac{1}{\sqrt{\mu}}\arcsin(p) \\ y =&\frac{1}{2}\sqrt{\frac{e_1^2-4e_0e_2}{\mu}}\int\frac{dq}{e_2q^2+e_1q+e_0} \end{aligned} \tag{63}$$

Its complementary integral

$$\begin{aligned} I_2 =&\Big(\frac{a_1\cos\sqrt{\mu}x+a_2\sin\sqrt{\mu}x}{\cosh^3\sqrt{\mu}y}+\frac{a_3}{\cosh^2\sqrt{\mu}y}\Big) \\ &\times[\sinh\sqrt{\mu}y(b_1\cos\sqrt{\mu}x+b_2\sin\sqrt{\mu}x)\dot{x} \\ &-\cosh\sqrt{\mu}y(b_1\sin\sqrt{\mu}x-b_2\cos\sqrt{\mu}x)]\dot{y} \\ &-\frac{c_1[b_1\cos\sqrt{\mu}x+b_2\sin\sqrt{\mu}x]}{\cosh\sqrt{\mu}y} \\ &+h\Big(\frac{a_1\cos\sqrt{\mu}x+a_2\sin\sqrt{\mu}x}{\cosh^3\sqrt{\mu}y}+\frac{a_3}{\cosh^2\sqrt{\mu}y}\Big) \\ &\times\Big[\sinh\sqrt{\mu}y(a_1\cos\sqrt{\mu}x+a_2\sin\sqrt{\mu}x)\dot{x}. \\ &-\cosh\sqrt{\mu}y(a_1\sin\sqrt{\mu}x-a_2\cos\sqrt{\mu}x)\dot{y} \\ &-\frac{c_1[a_1\cos\sqrt{\mu}x+a_2\sin\sqrt{\mu}x]}{\cosh\sqrt{\mu}y}\Big], \end{aligned} \tag{64}$$

where h is the numerical value of Jacobi integral and it should be eliminated by using the Jacobi-integral expression. This makes the degree of the additional integral turns on a cubic in the generalized velocities again. This problem is new integrable problem. It includes eight arbitrary parameters.

Application in a rigid body dynamic

Identifing the line element corresponding the two Lagrangians (7) and (47), we find they are identical if $A = 4C$, $\mu = c = 1$, $b = e_2 = 0$, $e_3 = 2$, $e_1 = -\frac{3}{2}$, $e_0 = -\frac{1}{2}$, $n_7 = -\frac{2}{3}$, $n_8 = -\frac{1}{36}$. Inserting those values in the Lagrangian (47) and using new parameters instead of the original ones, we get

$$\begin{aligned} L =& \frac{1}{2}\left(\frac{\dot{\gamma_3}^2}{1-\gamma_3^2} + \frac{1-\gamma_3^2}{4-3\gamma_3^2}\dot{\varphi}^2\right) + \left(k + c\gamma_1 + d\gamma_2 + \frac{\nu}{\gamma_1^2}(3\gamma_2^2+1)\right)\frac{1-\gamma_3^2}{4-3\gamma_3^2}\dot{\varphi} - \frac{1}{4}\left[a\gamma_1 + b\gamma_2 + \frac{\lambda}{\gamma_3^2}\right. \\ &+ k(c\gamma_1 + d\gamma_2) + \frac{1}{2}(c\gamma_2 - d\gamma_1)^2 + \frac{\nu(2\nu - k\gamma_3^2)}{\gamma_1^2} \\ &- \frac{\nu}{\gamma_1^2}(3\gamma_1^2 + \gamma_3^2)(c\gamma_1 + d\gamma_2) - \frac{\nu^2(1-\gamma_2^2)(1+3\gamma_2^2)}{2\gamma_1^4} \\ &\left.+ \frac{\gamma_3^2}{2[4-3\gamma_3^2]}\left(k + c\gamma_1 + d\gamma_2 + \frac{\nu}{\gamma_1^2}(3\gamma_2^2+1)\right)^2\right]. \qquad (65) \end{aligned}$$

Its Jacobi integrals admits the form

$$\begin{aligned} I_1 =& \frac{1}{2}\left(\frac{\dot{\gamma_3}^2}{1-\gamma_3^2} + \frac{1-\gamma_3^2}{4-3\gamma_3^2}\dot{\varphi}^2\right) + \frac{1}{4}\left[a\gamma_1 + b\gamma_2 + \frac{\lambda}{\gamma_3^2}\right. \\ &+ k(c\gamma_1 + d\gamma_2) + \frac{1}{2}(c\gamma_2 - d\gamma_1)^2 + \frac{\nu(2\nu - k\gamma_3^2)}{\gamma_1^2} \\ &- \frac{\nu}{\gamma_1^2}(3\gamma_1^2 + \gamma_3^2)(c\gamma_1 + d\gamma_2) - \frac{\nu^2(1-\gamma_2^2)(1+3\gamma_2^2)}{2\gamma_1^4} \\ &\left.+ \frac{\gamma_3^2}{2[4-3\gamma_3^2]}\left(k + c\gamma_1 + d\gamma_2 + \frac{\nu}{\gamma_1^2}(3\gamma_2^2+1)\right)^2\right]. = h. \qquad (66) \end{aligned}$$

The cubic additional integral can be written in form

$$\begin{aligned} I_2 =& \frac{4(1-\gamma_3^2)^3}{(3\gamma_3^2-4)^3}\dot{\varphi}^3 + 2\frac{\sqrt{(1-\gamma_3^2)^3}}{(3\gamma_3^2-4)^3}[(c\sin\varphi + d\cos\varphi) \\ &(3\gamma_3^4 - 6\gamma_3^2 + 4) - 2k\sqrt{1-\gamma_3^2}] - \frac{2\gamma_3\sqrt{1-\gamma_3^2}}{3\gamma_3 - 4} \\ &\times [d\sin\varphi - c\cos\varphi]\dot{\varphi}\dot{\gamma_3} + \frac{\dot{\varphi}}{3\gamma_3^2 - 4}[4(3\gamma_3^4 - 6\gamma_3^2 + 2) \\ &\times [(c^2 - d^2)\cos 2\varphi - 2cd\sin 2\varphi] - \sqrt{1-\gamma_3^2}[(dk(9\gamma_3^6 \\ &- 32\gamma_3^4 + 40\gamma_3^2 - 16) + b(3\gamma_3^2 - 2)(3\gamma_3^2 - 4)^2)\cos\varphi \\ &+ (kc(9\gamma_3^6 - 32\gamma_3^4 + 40\gamma_3^2 - 16) + a(3\gamma_3^2 - 2) \\ &\times (3\gamma_3^2 - 4)^2(\sin\varphi + \frac{h}{2}(9\gamma_3^6 - 33\gamma_3^4 + 40\gamma_3^2 - 16) \\ &- \frac{1-\gamma_3^2}{4}(8(c^2+d^2)(3\gamma_3^4(\gamma_3^2 - 2) + 4) + k^2\gamma_3^2(3\gamma_3^2 \\ &- 8)] + \frac{\dot{\gamma_3}}{3\gamma_3^2 - 4}[2\gamma_3(1-\gamma_3^2)((c^2 - d^2)\sin 2\varphi \\ &+ 2cd\cos 2\varphi) + \frac{\gamma_3}{\sqrt{1-\gamma_3^2}}(((ck + 3c)\gamma_3^2 - 2kc \\ &- 4a(\cos\varphi) - ((dk + 3b)\gamma_3^2 - 2kd - 4b)\sin\varphi] \\ &+ \frac{\sqrt{(1-\gamma_3^2)^3}(5\gamma_3^4 - 10\gamma_3^2 + 4))}{(3\gamma_3^2 - 4)^3}[d(3c^2 - d^2) \\ &\times \cos 3\varphi + c(c^2 - 3d^2)\sin 3\varphi] + \frac{(1-\gamma_3^2)(3\gamma_3^2 - 2)}{3\gamma_3^2 - 4} \\ &[(ca - bd)\cos 2\varphi - (cd - bd)\sin 2\varphi] - \frac{k(2-\gamma_3^2)(1-\gamma_3^2)}{(3\gamma_3^2 - 4)} \\ &\times (9\gamma_3^4 - 22\gamma_3^2 + 12)[(c^2 - d^2)\cos 2\varphi - 2cd\sin 2\varphi] \\ &- \frac{6k\sqrt{1-\gamma_3^2}}{3\gamma_3^2 - 4}(a\sin\varphi + b\cos\varphi) + \frac{\sqrt{1-\gamma_3^2}}{2(3\gamma_3 - 4)^3} \\ &\times [c\sin\varphi + \cos\varphi] \\ &\times [h(\gamma_3^2 - 1)(3\gamma_3^2 - 4)^2 + (3\gamma_3^6 + 48\gamma_3^2 - 23\gamma_3^4 - 32)k^2 \\ &+ 4(\gamma_3^2 - 1)(3\gamma_3^6 - 3\gamma_3^4 - 10\gamma_3^2 + 12)(c^2 + d^2)] \\ &- \frac{\gamma_3^2 - 2}{8(\gamma_3^2 - 4)^3}[8k(7\gamma_3^4 - 18\gamma_3^2 + 12)(c^2 + d^2) \\ &+ 2(3\gamma_3^2 - 4)^2[4(ca + bd) + kh]]. \qquad (67) \end{aligned}$$

It is worth noting that the two constants ν and λ appearing in the Lagrangian (65) do not exist in the additional cubic integral (67). They can be inserted into the additional integral (67) by replacing the Jacobi-constant h by its expression (66). This is a new integrable problem in a rigid body dynamic. It is more convenient to address it in terms of the traditional Euler-Poisson variables to clarify that is new and to make the comparisons with previous results easier. This is summarized in the following.

Theorem 3. *If a rigid body with principal inertia matrix* $\mathbf{I} = \mathrm{diag}(4C, 4C, C)$ *rotates about its fixed point under the influence of potential and gyroscopic forces taking the form*

$$\begin{aligned} V = C&\left[a\gamma_1 + b\gamma_2 + \frac{\lambda}{\gamma_3^2} + k(c\gamma_1 + d\gamma_2) + \frac{1}{2}(c\gamma_2 - d\gamma_1)^2\right. \\ &\left.+ \frac{\nu(2\nu - k\gamma_3^2)}{\gamma_1^2} - \frac{\nu}{\gamma_1^2}(3\gamma_1^2 + \gamma_3^2)(c\gamma_1 + d\gamma_2) - \frac{\nu^2(1-\gamma_2^2)(1+3\gamma_2^2)}{2\gamma_1^4}\right], \qquad (68) \end{aligned}$$

$$\begin{aligned} \boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3) = C&\left(c\gamma_3 - \frac{2\nu\gamma_3(1+3\gamma_2^2)}{\gamma_1^3}, d\gamma_3 + \frac{6\nu\gamma_2\gamma_3}{\gamma_1^2}, k + c\gamma_1\right. \\ &\left.+ d\gamma_2 + \frac{\nu(1+3\gamma_2^2)}{\gamma_1^2}\right). \qquad (69) \end{aligned}$$

Or, equivalently,

$$\mathbf{l} = (l_1, l_2, l_3) = C\left(0, 0, k + c\gamma_1 + d\gamma_2 + \frac{\nu}{\gamma_1^2}(3\gamma_2^2 + 1)\right), \qquad (70)$$

where a, b, c, d, k, λ *and* ν *are free parameters. Then the Euler-Poisson Eqs. (2) with the two expressions (68) and (69) are integrable on a zero-level of a cyclic integral*

$$I_3 = 4(p\gamma_1 + q\gamma_2) + \left(r + k + c\gamma_1 + d\gamma_2 + \frac{\nu}{\gamma_1^2}(1 + 3\gamma_2^2)\right)\gamma_3 = 0. \qquad (71)$$

Its complementary cubic integral is

$$\begin{aligned} I =& (r + c\gamma_1 + d\gamma_2 + 3\nu - k)\left\{\left(p + \frac{c\gamma_3}{2}\right)^2 + \left(q + \frac{d\gamma_3}{2}\right)^2 + \frac{\lambda}{2\gamma_3^2}\right\} + \nu\left[\frac{\lambda}{2\gamma_1^2} + \frac{\gamma_3^2}{\gamma_1^2}(p^2 + q^2)\right] \\ &- \gamma_3\left\{a\left(p + \frac{c}{2}\gamma_3\right) + b(q + \frac{d}{2}\gamma_3)\right\} + \frac{\nu^3\gamma_3^2}{2\gamma_1^6}(3\gamma_1^2 + \gamma_3^2) \\ &\times (\gamma_1^2 - 2\gamma_2^2 - \gamma_3^2) + \frac{\nu^2\gamma_3^2}{\gamma_1^4}[(\gamma_2^2 - 2\gamma_1^2)(k - c\gamma_1 - d\gamma_2) \\ &- r] + \frac{\nu\gamma_3^2}{4\gamma_1^2}[(c^2 + d^2)\gamma_3^2 + 4(cp + dq)\gamma_3 + 2[(c\gamma_1 + d\gamma_2)^2 \\ &- 2(c\gamma_1 + d\gamma_2)k + k^2 - 2(a\gamma_1 + b\gamma_2) - r^2]]. \qquad (72) \end{aligned}$$

This case is a conditional integrable problem owing to it is only correct on a zero level of cyclic integral. To avoid the inscrutability, the comparison with previous results are tabulated in the Table 1.

Table 1
Comparison with previous results.

Case 1	Conditions	Author-Reference	Year
1.	$\nu = \lambda = 0$	Sokolov and Tsiganov [36]	2002
2.	$c = d = \nu = 0$	Yehia [23]	2002
3.	$\lambda = c = d = 0$	Stretensky [25]	1963
4.	$k = c = d = 0$	Goriachev [37]	1915
5.	$k = \lambda = c = d = 0$	Goriachev [22]	1900

Discussion

It is well known that the two families of 2D integrable mechanical systems with complementary cubic integral in velocities that were introduced by Yehia in [23] and Yehia and Elmandouh in [24] were only known until this time. These studies assumed that the potential and gyroscopic forces admit certain formulas that were inserted in the Eqs. (13)–(22) to find the coefficients of the additional integral. In the present study, we do not postulate any structure for these forces and solve the problem by reducing the Eqs. (13)–(22) to two nonlinear partial differential Eqs. (27), (28) which contain two unknown functions instead of seven unknown functions. Therefore our results contain these previous results as special cases due to it contains extra parameters that extend the range of applications to diverse problems as we see above in the applications. Although we are not able to interpret our system physically, some applications including the dynamics of a particle and a rigid body are studied to illustrate its richness with physical applications. Now we are going to study the physical interpretation for the obtained results in the dynamics of a particle. As we know the Gaussian curvatures play an important rule in classification and the interpretation of such problems. If the Gaussian curvature is zero, positive value and negative value, the problem will describe motion of a particle in the Euclidean plane, standard sphere and Pseudosphere, respectively. In some times the Gaussian curvature is variable, i.e., depends on the generalized coordinates and so the surface on which the motion takes place is not know. Now we are going to find the conditions on the parameters making the configuration space corresponding to such problems to be Riemannian.

Now, we consider the case 1.a. The metric corresponding the Lagrangian (53) takes the form

$$dS^2 = dy^2 + \frac{1}{f(y)}dx^2, \tag{73}$$

where $f(y)$ is given by

$$f(y) = \alpha e^{a_0 y} + \beta e^{-2a_0 y}. \tag{74}$$

The metric (73) is Riemannian if $f(y) > 0$ and Pseudo-Riemannian if $f(y) < 0$ while the coefficient of dx^2 is infinite if $f(y) = 0$ and we drop this case from our consideration since it requires further investigation. It is clear that the sign of $f(y)$ does not rely on the parameter a_0 and so, we can put $a_0 = 1$. Thus the conditions on the two parameters α and β for which the metric (73) to be Riemannian or Pseudo Riemannian are collected and summarized in Table 2 Thus for the first three cases in Table 2, we can find the Riemannian metric corresponding to the last system on a certain surface of revolution. If we postulate $(r(y), \theta, z(y))$ is a point on this surface in cylindrical coordinates, we get

$$dr^2 + r^2 d\theta^2 + dz^2 = dy^2 + \frac{1}{\alpha e^{a_0 y} + \beta e^{-2a0y}}dx^2. \tag{75}$$

Identifying x by θ and comparing the both sides of Eq. (75), we obtain

Table 2
Conditions lead to a Riemannian configuration space for the mechanical system (53).

Case	Regions	Conditions	Type
1.	Whole the xy-plane	$\alpha > 0, \beta > 0$	Riemannian
2.	In the region $\left\{(x,y) \in \mathbb{R}\times]\frac{1}{3}\ln\left(-\frac{\beta}{\alpha}\right), \infty[\right\}$	$\alpha > 0, \beta < 0$	Riemannian
3.	In the region $\left\{(x,y) \in \mathbb{R}\times]-\infty, \frac{1}{3}\ln\left(-\frac{\beta}{\alpha}\right)[\right\}$	$\alpha < 0, \beta > 0$	Riemannian
4.	Whole the xy-plane	$\alpha < 0, \beta < 0$	Pseudo-Riemannian
5.	In the region $\left\{(x,y) \in \mathbb{R}\times]-\infty, \frac{1}{3}\ln\left(-\frac{\beta}{\alpha}\right)[\right\}$	$\alpha > 0, \beta < 0$	Pseudo-Riemannian
6.	In the region $\left\{(x,y) \in \mathbb{R}\times]\frac{1}{3}\ln\left(-\frac{\beta}{\alpha}\right), \infty[\right\}$	$\alpha < 0, \beta > 0$	Pseudo-Riemannian

$$r = \sqrt{\frac{1}{\alpha e^{a_0 y} + \beta e^{-2a0y}}}, \tag{76}$$

and thus, we can evaluate z from the following first order differential equation

$$\left(\frac{dz}{dy}\right)^2 = 1 - \left(\frac{dr}{dy}\right)^2.$$

thus, we get

$$z = \frac{1}{2}\int_0^y \sqrt{4 - \frac{a_0^2(\alpha e^{a_0 y} - 2\beta e^{-2a_0 y})^2}{(\alpha e^{a_0 y} + \beta e^{-2a_0 y})^3}}dy. \tag{77}$$

The surface of revolution is described by the two Eqs. (76) and (77). This surface is dependent on the parameters a_0, α, β and it is presented in Fig. 2 for certain values of these parameters. Thus the integrable mechanical system describing by (53) and (54) can be physically interpreted as the motion of a particle with a unit mass on a surface of revolution with the parametric Eqs. (76) and (77).

For the case 1.b, the metric of the configuration space of the mechanical system describing by the Lagrangian (57) admits the form

$$dS^2 = dy^2 + \frac{y^6}{g(y)}dx^2, \tag{78}$$

where

$$g(y) = \alpha y^6 + \beta. \tag{79}$$

Thus the configuration space associated the Lagrangian (57) is Riemannian or Pseudo-Riemannian depending on the sign of the function $g(y)$. These conditions are collected in Table 3. The cases for which the metric of a configuration space is Riemannian, we can find the equation of the surface on which the motion occurs. In an analogy way to the above procedures, we can find the family of the surface of the revolution on which the motion of the particle takes place. Now we write down the equations describing it as

$$r = \frac{y^3}{\sqrt{\alpha y^6 + \beta}},$$
$$z = \int_0^y \sqrt{1 - \frac{9y^4}{\alpha y^6 + \beta}\left[1 - \frac{\alpha y^6}{\sqrt{\alpha y^6 + \beta}}\right]^2}dy. \tag{80}$$

It relies on the two parameters α and β. It is appeared in Fig. 1 for certain values of these values. Thus, the integrable system (57) and (58) characterize physically the motion of a particle with a unit

Table 3
Conditions lead to Riemannian configuration space for the mechanical system (57).

Case	Regions	Conditions	Type
1.	Whole the *xy*-plane	$\alpha > 0, \beta > 0$	Riemannian
2.	In the region $\left\{(x, y) \in \mathbb{R}\times]\sqrt[6]{-\frac{\beta}{\alpha}}, \sqrt[6]{\frac{\beta}{\alpha}}[\right\}$	$\alpha < 0, \beta > 0$	Riemannian
3.	In the region $\left\{(x, y) \in \mathbb{R}\times]-\infty, -\sqrt[6]{-\frac{\beta}{\alpha}}[\cup]\sqrt[6]{\frac{\beta}{\alpha}}, \infty[\right\}$	$\alpha > 0, \beta < 0$	Riemannian
4.	Whole the *xy*-plane	$\alpha \neq 0, \beta = 0$	Plane
5.	Whole the *xy*-plane	$\alpha < 0, \beta < 0$	Pseudo-Riemannian
6.	In the region $\left\{(x, y) \in \mathbb{R} \times \mathbb{R}\backslash]-\sqrt[6]{-\frac{\beta}{\alpha}}, \sqrt[6]{\frac{\beta}{\alpha}}[\right\}$	$\alpha < 0, \beta > 0$	Pseudo-Riemannian
7.	In the region $\left\{(x, y) \in \mathbb{R}\times]-\sqrt[6]{-\frac{\beta}{\alpha}}, \sqrt[6]{\frac{\beta}{\alpha}}[\right\}$	$\alpha > 0, \beta < 0$	Pseudo-Riemannian

mass on a surface with parametric Eq. (80) under the action of potential and gyroscopic forces. Notice the other cases can be investigated in analogy method.

Finally, we obtain a new interesting rare integrable problem in a rigid body dynamic which generalizes all previous cases in this field. This case is interpreted physically as the motion of a heavy magnetized gyrostat carrying electric charges in an axially symmetric combination of the three classical fields (for more details about such interpretation see, e.g., [13]).

Acknowledgments

The author is grateful to Deanship of Scientific Research, King Faisal University, for supporting this work (Project No. 160276).

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