Journal of Mathematical Psychology 81 (2017) 28–39

Contents lists available at ScienceDirect

Journal of Mathematical Psychology

journal homepage: www.elsevier.com/locate/jmp

A generalized extensive structure that is equipped with a right action and its representation

Yutaka Matsushita

Department of Media Informatics, Kanazawa Institute of Technology, 3-1 Yatsukaho, Hakusan, Ishikawa, 924-0838, Japan

HIGHLIGHTS

- The set of commodities (the base set) is a generalized extensive structure.
- Decomposability is supposed for the product of the base set and a set of durations.
- Both ordering and algebraic axioms are proposed for the decomposable structure.
- Properties derived under the axioms yield right action of duration on the base set.
- We get a weighted additive model so as to reflect nonconstant impatience.

ARTICLE INFO

Article history:
Received 31 March 2016
Received in revised form 29 May 2017
Available online 22 September 2017

Keywords:
Extensive structure
Right action
Weighted additive model
Time preference
Impatience
Stationarity

ABSTRACT

In intertemporal choice, it has been found that if the receipt time is closer to the present, then people tend to grow increasingly or decreasingly impatient. This paper develops an axiom system to construct a weighted additive model reflecting nonconstant impatience. By presupposing that an increment in duration is subjectively assessed according to the periods at which advancement occurs, we denote the one-period advanced receipt of outcomes by multiplying the outcomes by the increment on the right. By this right multiplication, we can regard the effect of advance as the decomposition into two factors, i.e., the factor of step-by-step advance accompanied by subdivided durations and the factor of advance based on the total duration. First, the conditions for enabling right multiplication are proposed for the Cartesian product of the underlying set of a generalized extensive structure and a set of durations. Second, the properties derived under these conditions yield a right action on the generalized extensive structure. Finally, the weighted additive model is obtained as a representation of the generalized extensive structure equipped with the right action.

1. Introduction

Preference for the advanced timing of satisfaction, called *impatience* (Koopmans, 1960), is a well-known concept in the field of intertemporal choice. The concept is illustrated by the following example: receiving \$1000 now is probably preferred to receiving \$1100 after one year. A major reason for this preference is that the value of outcomes decreases with the passage of time.

Koopmans (1972) axiomatized a utility model for infinite outcome sequences to enable it to deal with impatience, which is caused by advancing the receipt of outcomes by "any finite number of periods". By incorporating five postulates consisting of weaker independence, stationarity, [1] and monotonicity (which is different from monotonicity related to a binary operation) into the topological framework of Debreu (1960), the utility model was first constructed on the space of actually finite outcome sequences; then by the use of continuity it was extended to a utility model on the space of infinite outcome sequences. Denoting an infinite-period temporal sequence of outcomes by $(a_1, a_2, \ldots)$, his utility model is expressed as the power series $u(a_1, a_2, \ldots) = \sum_{i=1}^{\infty} \alpha^{i-1} u(a_i)$, where $0 < \alpha < 1$ is a constant discount factor. Furthermore, Bleichrodt, Rohde, and Wakker (2008) refined Koopmans' formulation theoretically. Aiming to make it possible to deal with unbounded outcome sequences, they weakened the continuity condition from an infinite-dimensional version to a finite-dimensional version by introducing two conditions (constant-equivalence, tail-robustness). This work might make a test of axioms feasible. However, although these theories might be suitable for outcomes expressed by real numbers, e.g., amounts of money, it is too restrictive to treat preferences among qualitative outcomes because the validity of topological conditions (connectedness, separability) is nearly impossible

E-mail address: yutaka@neptune.kanazawa-it.ac.jp.

[1] Stationarity means if the receipt times of two commodities are advanced or deferred by the same amount, the preference between two commodities is invariant.

http://dx.doi.org/10.1016/j.jmp.2017.09.001

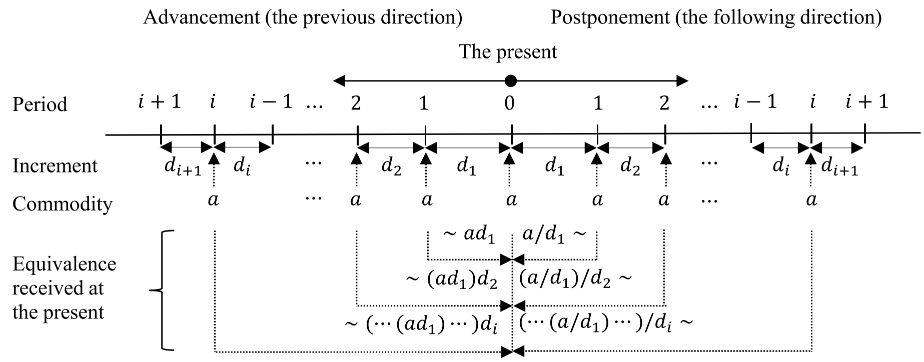

Fig. 1. Advanced receipt and postponed receipt of commodities by right multiplication and right division, respectively: in the case of advanced receipt, $(\cdots(ad_1)\cdots)d_i$ is defined as a solution x to $(x,\ 0) \sim (\cdots((a,\ 0)d_1)\cdots)d_i$ (Proposition 5); in the case of postponed receipt, $(\cdots(a/d_1)\cdots)/d_i$ is defined as a solution x to $(x,\ 0) \sim (\cdots((a,\ 0)/d_1)\cdots)/d_i$ (Proposition 7).

to test directly. To address the problem, Hübner and Suck (1993) adapted Koopmans' result to a general algebraic framework. That is, they extended the n-component, additive conjoint structure (Krantz, Luce, Suppes, & Tversky, 1971) to an infinite-dimensional version and added stationarity and monotonicity to derive the same utility model as above. Their utility model was similarly constructed using these two steps, apart from the substitution of restricted solvability for continuity in the step of extending to an infinite-dimensional model. They also could deal with unbounded outcome sequences. Meanwhile, Fishburn and Rubinstein (1982) constructed a utility model reflecting impatience in such a different way that it is determined as a function of a single outcome at a particular time. A major advantage of their work is to generalize stationarity to the Thomsen condition. Let X be a set of outcomes and let T be a set of times. Both sets are assumed to be nonnegative real intervals. With the help of the topological framework of Debreu (1960) (Fishburn, 1970), they also derived the multiplicative utility function $u(x,\ t) = \varphi(t)v(x)$, where v is increasing on X and φ is positive and decreasing on T (due to impatience). However, these two works possess drawbacks. First, as was pointed out by Hübner and Suck (1993), the validity of their conditions is difficult to test because they are formulated in the infinite-dimensional structure. Second, as in Fishburn and Rubinstein's (1982) work, the Thomsen condition is itself artificial, and unfortunately, the problem of topological conditions still remains to be solved.

Recently, Matsushita (2014) constructed a *weighted additive model*, which is an order-preserving function on a generalized extensive structure A that is of the form

$$u(ab) = \alpha u(a) + u(b),\ \alpha \geqslant 1, \tag{1}$$

where ab is the concatenation of a and b, and it implies receiving a one period earlier from now and b now. This is a representation of the generalized extensive structure, called a central left nonnegative concatenation structure with left identity. The left identity element e plays an important role in this construction. The right multiplication (resp. right division) of a by e indicates advancing (resp. postponing) its receipt by one period. Hence $(a/e)b$ implies receiving a and b in the same time period. By defining a new operation as $a \circ b = (a/e)b$, the central left nonnegative concatenation structure reduces to an extensive structure with respect to the operation $\circ$. Using the fact that ab is equivalent to $(ae/e)b$ and letting u be an additive representation of the extensive structure, one obtains $u(ab) = u(ae) + u(b)$; in view of u being a ratio scale, it is possible to derive $u(ae) = \alpha u(a)$. The inequality $\alpha \geqslant 1$ (which is interpreted as a markup factor) shows that the model reflects impatience. Indeed, this construction uses the axioms of r-nonnegativity ($ae \succsim a$) and monotonicity ($a \succsim b \Leftrightarrow ae \succsim be$), which correspond to impatience (in the wider sense) and stationarity, respectively. Since every sequence is expressed as a concatenation, the above model can evaluate preferences between outcome sequences with "any distinct finite number of periods". Hence the model solves the problem involved in Hübner and Suck's (1993) formulation.

However, Matsushita's (2014) formulation has a problem in that the condition of stationarity is used. It is well known (Loewenstein & Prelec, 1992) that stationarity (constant impatience) is often violated. The preference in the first paragraph can be reversed if the delay time is increased with the time lag held constant: receiving \$1100 after three years may be preferred to receiving \$1000 after two years, which is an example of decreasing impatience. Attema, Bleichrodt, Rohde, and Wakker (2010) showed the other type of violation of stationarity – increasing impatience[2] – by analyzing the behavior of subjects faced with intertemporal (delayed) choice problems through time-tradeoff sequences: subjects are increasingly impatient for periods close to the present and constantly impatient for later periods. Moreover, Takahashi, Han, and Nakamura (2012) showed that the exponential discount function with logarithmic time perception, a psychological time duration with the logarithmic unit, is transformed into the generalized hyperbolic discount function (Loewenstein & Prelec, 1992); in other words, perceiving time according to a logarithmic scale and constantly discounting in terms of this perceived time yields decreasing impatience (Attema et al., 2010), because the exponential discount function reflects constant impatience and the hyperbolic one captures decreasing impatience.

These works bring us the following concept: if a one-period advance occurs in a period closer to the present, then a person is sensitive to the advance, and if the person is increasingly (or decreasingly) impatient, then he/she may feel as if its time increment is smaller (or greater) than the actual increment. To allow for the effect of the time duration varying according to a period in which an advance occurs, we express a one-period advance by multiplying outcomes by an increment in a "subjective" duration (not e) corresponding to the period on the right. We then study a utility model reflecting nonconstant impatience under measurement theory. To be more precise, let d_{s_i} be an increment in a subjective duration when advancing the receipt of a from period $i-1$ to period i in the previous direction (see Fig. 1). We express the advanced receipt of a by n periods as $(\cdots(ad_{s_1})\cdots)d_{s_n}$, and construct a utility model of the form

$$u((\cdots(ad_{s_1})\cdots)d_{s_n}) = (\varphi(d_{s_1})\cdots\varphi(d_{s_n}))u(a),\ \varphi(d_{s_i}) > 1,$$

for which u is the weighted additive model of (1), and φ is a weight function of increments in the duration. Since right division

[2] Although theoretical studies commonly assumed decreasing impatience, several empirical studies (Attema, Bleichrodt, Gao, Huang, and Wakker 2016) have found increasing impatience.

is defined as the inverse of right multiplication (i.e., $(ad_{s_i})/d_{s_i} \sim a$), this utility model can represent also a preference for the postponed receipt of a by n periods:

$$u((\cdots(a/d_{s_1})/\cdots)/d_{sn}) = \frac{1}{\varphi(d_{s_1})\cdots\varphi(d_{s_n})}u(a).$$

The motivation for constructing the utility model is to solve the problem of nonconstant impatience raised as above. Indeed, the previous model of (1) cannot explain the decreasing impatience, for since a/e denotes the one-period postponed receipt of a and $u(a/e) = (1/\alpha)u(a)$, it follows that if $u(\$1000) > u(\$1100/e) = (1/\alpha)u(\$1100)$, then $(1/\alpha^2)u(\$1000) = u((\$1000/e)/e) > u((\$1100/e)/e)/e = (1/\alpha^3)u(\$1100)$. On the other hand, the proposed model can explain this nonconstant impatience: the allocation of suitable values to $\varphi(d_{s_1})$, $\varphi(d_{s_2})$, and $\varphi(d_{s_3})$ (see Example 2) yields $u(\$1000) > u(\$1100/d_{s_1})$ and $u((\$1000/d_{s_1})/d_{s_2}) < u(((\$1100/d_{s_1})/d_{s_2})/d_{s_3})$.

The substantial work of this study is to extend the central left nonnegative concatenation structure A in such a way that right multiplication by d_{s_i} has the same properties as right multiplication by e; that is, right multiplication by d_{s_i} is requested at least to inherit the properties of r-nonnegativity, monotonicity, and consistent advance from right multiplication by e. Moreover, right action to devise on A is an action based on right multiplication by durations having these properties. From the request, we will formulate the axioms in the situation of advanced receipt. The formulation is carried out in two steps. First, we consider the conditions for enabling right multiplication for the Cartesian product of the underlying sets $A \times T$. Second, by deriving the essential properties under the conditions, we define a central left nonnegative concatenation structure that is equipped with the right action algebraically. A major reason for considering right multiplication on $A \times T$ is to give a clear meaning of ad_{s_1} and to make experimental tests on the conditions simple. Indeed, ad_{s_1} is to be defined as a commodity received at the present that is equivalent to the commodity a at the advanced time d_{s_1}. Herein this equivalence is written in the form $(ad_{s_1}, 0) \sim (a, d_{s_1})$ because we assume an element $(a, d_{s_1}) \in A \times T$ to mean that one receives a at the advanced time point d_{s_1}. Hence all that we have to do for experimental checks is to evaluate the preferences between elements of $A \times T$. As for another reason, to maintain compatibility with the fact that A itself has a partial binary operation, it is also desirable that we define the right action as a partial operation; herein the domain of the right action can be specifically written as a subset of $A \times T$. Note that the proposed axioms are to be derived from a direct definition of right division (as will be shown in Section 3.4), so that we can test the axioms experimentally in the situation of postponed receipt. In addition, we will discuss the relation of the utility model with the exponential discount function.

The proofs of the lemmas, propositions, and theorem are given in the final section. The mathematical basis for regarding right multiplication by increments in the duration as a right action is shown in the Appendix.

2. Basic concepts

2.1. The base structure

Throughout this paper, $\mathbb{R}$ and $\mathbb{R}_0^+$ denote the sets of all real numbers and of all nonnegative real numbers, respectively. Let $\succsim_A$ be a binary relation on a nonempty set A that is interpreted as a preference relation. As usual, $\succ_A$ denotes the asymmetric part, $\sim_A$ the symmetric part, and $\precsim_A$, $\prec_A$ denote reversed relations. The binary relation $\succsim_A$ on A is a *weak order* if and only if it is connected and transitive. Let $\cdot$ be a "partial" binary operation on A, meaning a function from a subset B of $A \times A$ into A. The expression $a \cdot b$ is said to be *defined* (in A) if and only if $(a, b) \in B$. An element $e \in A$ denotes no change in the status quo with outcome sequences. That is, it is assumed that receiving e prior to a is no different from receiving a at the present; however, ae implies advancing the receipt of a by one period, so that $a \cdot e$ is not always $\sim a$.

In preparation, we will review the definition and properties of the base structure of this paper. In the following conditions, all the products are assumed to be defined.

A1. Weak order: $\succsim_A$ is a weak order on A.
A2. Local definability: if $a \cdot b$ is defined, $a \succsim_A c$, and $b \succsim_A d$, then $c \cdot d$ is defined.
A3. Monotonicity: $a \succsim_A b \Leftrightarrow a \cdot x \succsim_A b \cdot x \Leftrightarrow x \cdot a \succsim_A x \cdot b$ for all $a, b, x \in A$.
A4. Left identity: e is a *left identity element*; that is, $e \cdot a \sim_A a$ for all $a \in A$.
A5. R-nonnegativity: whenever $x \cdot a$ is defined, $x \cdot a \succsim_A x$.
A6. Left solvability: whenever $a \succ_A b$, there exists $x \in A$ such that $x \cdot b$ is defined and $a \sim x \cdot b$.

We will inductively define the nth "left" multiple of an element a by $a^0 = e$, $a^1 = a$ and

$a^n = a \cdot a^{n-1}$ if the right-hand side is defined

a^n is undefined otherwise.

A7. Left Archimedean: every bounded sequence $\{a^n\}$ consisting of the left multiples of $a \succ_A e$ is finite.

The system $\langle A, \succsim_A, \cdot, e\rangle$ is a *concatenation structure with left identity* if and only if A1–A4 are satisfied. A *left nonnegative concatenation structure with left identity* is a concatenation structure with left identity for which A5–A7 are satisfied. Regardless of the fact that A5 is written as a right sided concept, the concatenation structure is prefixed by the term "left nonnegative". The reason is that if a concatenation structure with left identity is r-nonnegative, then it is also l-nonnegative (i.e., $a \cdot x \succsim_A x$). Hence every left nonnegative concatenation structure with left identity is a nonnegative concatenation structure with left identity for which the solvability and Archimedean properties are satisfied only in relation to left-concatenation. It is worthwhile to state that A consists – at most – of elements such that $a \succsim e$ and that the subset $\{a \in A \,|\, a \succ e\}$ is a generalization of the PCS[3] (Luce, Krantz, Suppes, & Tversky, 1990). Throughout the paper, the trivial case where A has just a single element e is always excluded.

Since, in A6, x is uniquely determined up to $\sim_A$ by A3, we write $x \sim_A a/b$, and $a/a \sim_A e$ because $a \sim_A e \cdot a$. Thus a partial binary operation $/$, called *right division*, is defined on A.

Remark 1. Let $\langle A, \succsim_A, \cdot, e\rangle$ be a concatenation structure with left identity. If A6 holds, then for all $a, b, x \in A$, the following properties hold.

(i) $(a \cdot b)/b \sim_A a \sim_A (a/b) \cdot b$ whenever $a \cdot b$ is defined and $a \succsim_A b$.
(ii) Monotonicity of right division (Demko, 2001, Lemma 3.1):

$a \succsim_A b \Leftrightarrow a/x \succsim_A b/x$ whenever $a, b \succsim_A x$,
$a \succsim_A b \Leftrightarrow x/a \precsim_A x/b$ whenever $x \succsim_A a, b$.

A8. Weak associative-commutativity: whenever either of $a \cdot (b \cdot c)$ or $b \cdot (a \cdot c)$ is defined, the other expression is also defined and $a \cdot (b \cdot c) \sim_A b \cdot (a \cdot c)$.

[3] The PCS is a positive extensive structure satisfying a solvability property and the Archimedean property. The extensive structure is a weakly associative PCS.

A9. Consistent advance: whenever either of $(a \cdot b) \cdot e$ or $(a \cdot e) \cdot (b \cdot e)$ is defined, the other expression is also defined and $(a \cdot b) \cdot e \sim_A (a \cdot e) \cdot (b \cdot e)$.

Definition 1. A left nonnegative concatenation structure with left identity is said to be *central* if it satisfies A8 and A9.[4]

Define a partial binary operation $\circ$ on A by

$$a \circ b = (a/e) \cdot b. \tag{2}$$

Since $(a \cdot e)/e \sim_A a$ by Remark 1(i), it is rational to interpret a/e as meaning that the receipt of a is postponed by one period. Hence $a \circ b$ is regarded as the (concurrent) receipt of a and b in the same period. Therefore it is natural that $\circ$ has the properties of weak associativity and commutativity: $(a \circ b) \circ c \sim_A a \circ (b \circ c)$ and $a \circ b \sim_A b \circ a$. The next lemma guarantees these properties of $\circ$.

Lemma 1 (*Matsushita, 2014*, *Lemma 2*).[5] *Let A be a left nonnegative concatenation structure with left identity. If A8 is satisfied, then* $E(A) = \langle A, \succsim_A, \circ, e\rangle$ *is an extensive structure with identity.*

2.2. Expressing outcome sequences

Henceforth concatenations expressed implicitly by juxtaposition are meant to bind more strongly than right divisions to reduce the number of brackets in equivalences. For example, $(a \cdot b)/b$ reduces to ab/b.

We will write outcome sequences in a left-branching fashion, and enforce the rule that each period number in an outcome sequence is counted going back. Hence $(\cdots((a_1a_2)a_3)\cdots a_{n-1})a_n$ denotes that a person receives the last component a_n of the sequence in the latest period, the last component a_{n-1} of the first outside parenthesis in the period immediately before the latest period, $\cdots$, and so on; finally, the first component a_1 is received $n-1$ periods earlier. According to the rule, repeated use of right multiplication (or right division) by e implies a step-by-step advancement (or postponement) from the reference period, i.e., the present. An outcome is written in the following respective forms according to whether its receipt is advanced or postponed by n periods:

$$\text{Advancing case.} (\cdots(a\underbrace{e)\cdots e)e}_{n\text{ times}}.$$

$$\text{Postponing case.} (\cdots(a/\underbrace{e)\cdots/e)/e}_{n\text{ times}}.$$

3. Axioms on the decomposable structure

3.1. Basic properties

Henceforth assume that a central left nonnegative concatenation structure $\langle A, \succsim_A, \cdot, e\rangle$ with left identity has no minimal positive element. Let $[e]$ be the equivalence class of the left identity e, i.e., $[e] = \{x \in A \mid x \sim_A e\}$. We write $A \setminus [e]$ to mean $\{a \in A \mid a \notin [e]\}$. Throughout this section, let $T = \mathbb{R}$, which is equipped with the usual order $\geqslant$. Let $\langle T, \geqslant, \cdot_T, 0\rangle$ be an Archimedean simply[6] ordered group (Krantz et al., 1971 Definition 2.3). The structure consists of the sets of positive elements, negative elements, and an identity element. Let $t \in T$ be arbitrary. We write $t^{-1} = x$ for $x \in T$ such that $x \cdot_T t = 0$. By A3, $t^{-1} < 0 \Leftrightarrow t > 0$. The symbols t and t^{-1} are used to denote positive and negative elements, respectively. We provide a binary relation $\succsim$ on the Cartesian product of the underlying sets A and T. An element (a, t) or (a, t^{-1}) in $A \times T$ indicates the receipt of a commodity a "advanced" or "postponed" toward a duration t, respectively. Note that t or t^{-1} means an "absolute" time. In particular, $(a, 0)$ means receiving a at a time point of reference (e.g., the present). The following conditions are assumed for $\succsim$ on $A \times T$.

B1. Weak order: $\succsim$ is a weak order on $A \times T$.
B2. Independence: for all $a, b \in A$ and all $s, t \in T$ with $s, t \geqslant 0$,

(i) $a \succsim_A b \Leftrightarrow (a, t) \succsim (b, t) \Leftrightarrow (a, t^{-1}) \succsim (b, t^{-1})$.
(ii) $s \geqslant t \Leftrightarrow (a, s) \succsim (a, t)$.
(iii) $s \geqslant t \Leftrightarrow (a, s^{-1}) \precsim (a, t^{-1})$.

B3. Restricted solvability: given $(b, s) \in A \setminus [e] \times T$ and $t \in T$, whenever there exist $\bar{a}, \underline{a} \in A \setminus [e]$ for which $(\bar{a}, t) \succsim (b, s) \succsim (\underline{a}, t)$, there exists an $x \in A \setminus [e]$ such that $(x, t) \sim (b, s)$.
B4. Smallness at any duration level: for any $a \in A \setminus [e]$ and any $t \in T$ with $t \geqslant 0$, there exists an $x \in A \setminus [e]$ such that $(a, 0) \succsim (x, t)$.

The system $\langle A \times T, \succsim\rangle$ is a *decomposable structure* if and only if B1 and B2 are satisfied. In what follows, we will exclude elements such as (e, t) or (e, t^{-1}) from consideration because it makes little sense to evaluate the effect of the advanced or the postponed receipt of a null outcome. Axiom B2(ii) and B2(iii) imply impatience. Axiom B3 along with B4 guarantees that there conditionally exists a commodity received at the present that is equivalent to any commodity for which its receipt is advanced toward any time level (Proposition 1). The equivalence at the present is a similar concept to the "present value" of Bleichrodt, Keskin, Rohde, Spinu, and Wakker (2015), although they defined the value with respect to the postponed receipt of commodities. Axiom B4 prevents the dominance of the component T over A in ordering, i.e., whenever $t > 0$, then $(x, t) \succ (a, 0)$ for all $x, a \in A \setminus [e]$.

The next proposition is fundamental to the development of our theory in the next subsection.

Proposition 1. *Assume that* $\langle A \times T, \succsim\rangle$ *is a decomposable structure for which* B3 *and* B4 *are satisfied.*

(i) *For any* $a \in A \setminus [e]$ *and any* $t \in T$, *there exists* $x \in A \setminus [e]$ *such that* $(x, t) \sim (a, 0)$. *Moreover, x is uniquely determined up to* $\sim_A$.
(ii) *For any* $(a, t) \in A \setminus [e] \times T$, *there exists* $x \in A$ *such that* $(x, 0) \sim (a, t)$ *if and only if* $(\bar{a}, 0) \succsim (a, t)$ *for some* $\bar{a} \in A$. *Moreover, x is uniquely determined up to* $\sim_A$.

3.2. Right multiplication as an advancement operator

Henceforth assume that $\langle A \times T, \succsim\rangle$ is a decomposable structure for which B3 and B4 are satisfied. Let $d_o > 0$ denote an increment (e.g., one month, one year) in the objective (physical) duration of one period; on the other hand, $d_s > 0$ denotes an increment in a subjective (psychological) duration assessed by individuals, which is dependent on when a one-period advancement occurs. We write d_ε with $\varepsilon = o$ or s when it is unnecessary to make a distinction between d_o and d_s; further, it is always assumed that $d_\varepsilon > 0$. Henceforth we use $s, t \geqslant 0$ to denote an arbitrary time at which one receives commodities but use $d_\varepsilon > 0$ for the above increment in the duration for one period. It will be shown later that t can be expressed as the sum of these d_ε. This subsection devotes itself to giving a clear meaning to the right multiplication of a by d_ε.

We will express the advanced receipt of a commodity by multiplying it by an increment in the duration on the right. Right

---

[4] The term "central" is used as the concept to mean that it makes $\circ$ commutative and associative (Matsushita, 2011).

[5] Matsushita (2014) included the condition of A having no minimal positive element in the lemma. However, it is actually not necessary.

[6] A simple order is an antisymmetric weak order (Krantz et al., 1971).

multiplication is defined by the following rule: for any $a \in A$ and any $d_\varepsilon \in T$,

$$(a,\ 0)d_\varepsilon = \begin{cases}(a,\ d_\varepsilon) & \text{if } a \succ_A e\\ (e,\ 0) & \text{if } a \sim_A e.\end{cases} \tag{3}$$

Definition 2. Assume that $\langle A \times T,\ \succsim\rangle$ is a decomposable structure for which B3 and B4 are satisfied. Each product $(a,\ 0)d_\varepsilon$ is said to be *defined in* $A \times \{0\}$ if and only if $(\bar{a},\ 0) \succsim (a,\ 0)d_\varepsilon$ for some $\bar{a} \in A$. In this case, given $a \succ_A e$, $(x,\ 0) \sim (a,\ d_\varepsilon)$ has the (unique up to $\sim_A$) solution $x \in A$ (Proposition 1(ii)). Hence we introduce the *right multiplication* of $a \in A \setminus [e]$ by d_ε by setting $x = ad_\varepsilon$, so that

$$(a,\ 0)d_\varepsilon \sim (ad_\varepsilon,\ 0). \tag{4}$$

In particular, we define $ad_\varepsilon \sim_A e$ whenever $a \sim_A e$, and $(a, 0)0 = (a, 0)$, so that $a0 = a$. When $(a,\ 0)d_\varepsilon$ is defined in $A \times \{0\}$, ad_ε is also said to be *defined in* A.

The following equivalence that is obtained by (3) and (4) will be useful for an intuitive understanding of the meaning of ad_ε.

$$(ad_\varepsilon,\ 0) \sim (a,\ d_\varepsilon) \text{ whenever } a \succ_A e. \tag{5}$$

This allows us to interpret ad_ε as a commodity received at the present that is equivalent to a for which its receipt is advanced by one period at which the increment in the duration is assessed at d_ε.

Let

$$C = \{(a,\ t) \in A \times T \,|\, (\bar{a},\ 0) \succsim (a,\ t) \text{ for some } \bar{a} \in A\}.$$

Whenever $(a,\ d_\varepsilon) \in C$, $(a,\ 0)d_\varepsilon$ is defined in $A \times \{0\}$, so ad_ε is also defined in A. Hence the right multiplication of a by d_ε is a "partial" operation on $A \times T$. The next lemma guarantees that the product is uniquely determined.

Lemma 2. *Assume that $\langle A \times T,\ \succsim\rangle$ is a decomposable structure for which B3 and B4 are satisfied. Assume that ad_ε, bd_ε, and ad'_ε are defined in A. Then the following are true.*

(i) $a \succsim_A b \Leftrightarrow ad_\varepsilon \succsim_A bd_\varepsilon$.
(ii) $d_\varepsilon \geqslant d'_\varepsilon \Leftrightarrow ad_\varepsilon \succsim_A ad'_\varepsilon$ *whenever* $a \succ_A e$.

It is proper to define right division by increments in the duration for elements of $A \times \{0\}$, denoted /, as the inverse of right multiplication by increments in the duration; that is, right division is defined to mean that

$$((a,\ 0)d_\varepsilon)/d_\varepsilon = (a,\ 0) \text{ and } ((a,\ 0)/d_\varepsilon)d_\varepsilon = (a,\ 0). \tag{6}$$

Substituting (3) into the former part of (6) yields such a clear form that we can understand the operational role of right division by d_ε:

$$(a,\ d_\varepsilon)/d_\varepsilon = (a,\ 0) \text{ whenever } a \succ_A e. \tag{7}$$

We extend the application range of the former part of (6) to an equivalent element received at the present to define right division for elements of $A \times \{0\}$.

B5. $(a,\ 0) \sim (x,\ 0)d_\varepsilon \Leftrightarrow (a,\ 0)/d_\varepsilon \sim (x,\ 0)$.

Using the latter part of (6), this can be rewritten as

B5′. $(a,\ 0)/d_\varepsilon \sim (x,\ 0) \Leftrightarrow ((a,\ 0)/d_\varepsilon)d_\varepsilon \sim (x,\ 0)d_\varepsilon$,

which is an extension of the application range of the latter part of (6). Since in the case where $a \succ_A e$, $(x,\ d_\varepsilon) \sim (a,\ 0)$ has the (unique up to $\sim_A$) solution $x \in A \setminus [e]$ (Proposition 1(i)), it is seen from B5 that $(a,\ 0)/d_\varepsilon \sim (x,\ 0)$ always holds. Moreover, $(e,\ 0)/d_\varepsilon = (e,\ 0)$. Thus the following proposition is obtained.

Proposition 2. *Assume that B5 is satisfied. Then right division by elements of T is a binary operation on $A \times \{0\}$, where each quotient is uniquely determined up to $\sim$.*

Definition 3. Assume that B5 is satisfied. Then every quotient $(a,\ 0)/d_\varepsilon$ is said to be *defined in* $A \times \{0\}$ (Proposition 2). Since $((x,\ 0)d_\varepsilon =)(x,\ d_\varepsilon) \sim (a,\ 0)$ has the (unique up to $\sim_A$) solution x for any $a \succ_A e$ (Proposition 1(i)), we introduce the right division of $a \in A \setminus [e]$ by d_ε by setting $x = a/d_\varepsilon$, so that

$$(a,\ 0)/d_\varepsilon \sim (a/d_\varepsilon,\ 0). \tag{8}$$

In particular, $a/d_\varepsilon \sim_A e$ whenever $a \sim_A e$, and $a/0 \sim_A a$. We also say that a/d_ε *is defined in* A.

From (7) and (8) we can interpret a/d_ε as a commodity equivalent to the commodity a, the receipt of which is postponed by one period whose increment in the duration is assessed at d_ε. The next lemma gives the meaning of the notation ad_ε and a/d_ε.

Lemma 3. *Assume that ad_ε is defined in A and that B5 is satisfied. Then the following are true.*

(i) $ad_\varepsilon/d_\varepsilon \sim_A a$ *and* $(a/d_\varepsilon)d_\varepsilon \sim_A a$.
(ii) $b \sim_A ad_\varepsilon \Leftrightarrow b/d_\varepsilon \sim_A a$.

Proposition 3. *Assume that B5 is satisfied. Then the following are true.*

(i) $a \succsim_A b \Leftrightarrow a/d_\varepsilon \succsim_A b/d_\varepsilon$.
(ii) $d_\varepsilon \geqslant d'_\varepsilon \Leftrightarrow a/d_\varepsilon \precsim_A a/d'_\varepsilon$ *whenever* $a \succ_A e$.

Hereafter, we will clarify the meaning of elements in $A \times \{d_\varepsilon\}$ of the form $(ab,\ d_\varepsilon)$; more precisely, we examine how an increment d_ε in the duration acts on two commodities a and b. For this purpose, a concatenation operation on $A \times \{d_\varepsilon\}$ is devised such that it is an extension of the operation · on A. Since each $(a,\ 0)$ is identified with a, it should be the case that $(a,\ 0) \bigcirc (b,\ 0) = (ab,\ 0)$. Hence we obtain the following definition.

Definition 4. Let $a,\ b \succ_A e$ be arbitrary, and assume that $(a, b) \in B$. Let $t \in T$. Let $\bigcirc$ be a partial binary operation on $A \times \{t\}$ that is defined by

$$(a,\ t) \bigcirc (b,\ t) = (ab,\ t). \tag{9}$$

Each product $(a,\ t) \bigcirc (b,\ t)$ $(a,\ b \succ_A e)$ is said to be *defined on* $A \times \{t\}$ if and only if $(a,\ b) \in B$.

Remark 2. Apply (3) to both sides of (9) to obtain

$$(a,\ 0)d_\varepsilon \bigcirc (b,\ 0)d_\varepsilon \sim (ab,\ 0)d_\varepsilon.$$

This provides a regulation on the right multiplication of $(ab,\ 0)$ by increments in the duration; that is, ab, the receipt of which is advanced by one period with the increment in the duration assessed at d_ε, is decomposed into the concatenation of a and b, each receipt of which is advanced by one period with the assessed increment d_ε in the duration.

Let $a,\ b \succ_A e$ and $d_\varepsilon > 0$. Let $x,\ y \in A$ be solutions to $(x,\ 0) \sim (a,\ d_\varepsilon)$ and $(y,\ 0) \sim (b,\ d_\varepsilon)$, respectively. Then $\bigcirc$ is *well-defined* if whenever either $(a,\ d_\varepsilon) \bigcirc (b,\ d_\varepsilon)$ is defined on $A \times \{d_\varepsilon\}$ with $(\bar{a},\ 0) \succsim (ab,\ d_\varepsilon)$ for some $\bar{a} \in A$ or $(x,\ 0) \bigcirc (y,\ 0)$ is defined on $A \times \{0\}$, the other expression is also defined and $(a,\ d_\varepsilon) \bigcirc (b,\ d_\varepsilon) \sim (x,\ 0) \bigcirc (y,\ 0)$.

B6. The partial binary operation $\bigcirc$ is *well-defined*.

Lemma 4. *Let $\bigcirc$ be a partial binary operation defined on $A \times \{t\}$ for each $t \in T$. Let $a,\ b \succ_A e$ and $d_\varepsilon > 0$. Let $ad_\varepsilon,\ bd_\varepsilon \in A$ be solutions to $(ad_\varepsilon,\ 0) \sim (a,\ d_\varepsilon)$ and $(bd_\varepsilon,\ 0) \sim (b,\ d_\varepsilon)$, respectively. Then the following two conditions are equivalent.*

(i) *Whenever either $(a,\ d_\varepsilon) \bigcirc (b,\ d_\varepsilon)$ is defined on $A \times \{d_\varepsilon\}$ with $(\bar{a},\ 0) \succsim (ab,\ d_\varepsilon)$ for some $\bar{a} \in A$ or $(ad_\varepsilon,\ 0) \bigcirc (bd_\varepsilon,\ 0)$ is defined on $A \times \{0\}$, the other expression is also defined and $(a,\ d_\varepsilon) \bigcirc (b,\ d_\varepsilon) \sim (ad_\varepsilon,\ 0) \bigcirc (bd_\varepsilon,\ 0)$.*

(ii) *Whenever either of $(ab)d_\varepsilon$ or $(ad_\varepsilon)(bd_\varepsilon)$ is defined in A, the other expression is also defined and $(ab)d_\varepsilon \sim_A (ad_\varepsilon)(bd_\varepsilon)$.*

Remark 3. Well-definedness guarantees consistency in the definability of the concatenation operation and right multiplication by increments in the duration: $(ad_\varepsilon,\ bd_\varepsilon) \in B \Leftrightarrow (ab,\ d_\varepsilon) \in C$. Assume that either of $(ab)d_\varepsilon$ or $(ad_\varepsilon)(bd_\varepsilon)$ is defined in A. Lemma 4 asserts that in the presence of (5), a necessary and sufficient condition for well-definedness is that $(ab)d_\varepsilon \sim_A (ad_\varepsilon)(bd_\varepsilon)$. Indeed, since $(a,\ d_\varepsilon)\bigcirc(b,\ d_\varepsilon) = (ab,\ d_\varepsilon)$, $(ad_\varepsilon,\ 0)\bigcirc(bd_\varepsilon,\ 0) = ((ad_\varepsilon)(bd_\varepsilon),\ 0)$, and $(ab,\ d_\varepsilon) \sim ((ab)d_\varepsilon,\ 0)$, it is easily seen that $(a,\ d_\varepsilon) \bigcirc (b,\ d_\varepsilon) \sim (ad_\varepsilon,\ 0) \bigcirc (bd_\varepsilon,\ 0)$ if and only if $(ab)d_\varepsilon \sim_A (ad_\varepsilon)(bd_\varepsilon)$.

3.3. Right multiplication for multiple times

Hereafter, let d_{s_i} denote an increment in a subjective duration when an advancement or postponement occurs from period $i-1$ to period i in the previous or following direction (Fig. 1). For example, according to Takahashi et al. (2012), a cumulative subjective duration χ_i up to period i is calculated by

$$\lambda \cdot \ln(1 + \mu \cdot i) \text{ for some real constants } \lambda,\ \mu > 0, \tag{10}$$

and hence we have $d_{s_i} = \chi_i - \chi_{i-1}$. We define $d_i = d_o$ or d_{s_i} for each $i = 1, \ldots, n$. This subsection considers carrying out right multiplication by increments in the duration multiple times. As a result, we can express a commodity for which its receipt is advanced or postponed by n periods, using right multiplication or division by increments d_i in the duration, respectively:

Advancing case.$((\cdots((ad_1)d_2)\cdots)d_{n-1})d_n$.

Postponing case.$((\cdots((a/d_1)/d_2)/\cdots)/d_{n-1})/d_n$.

Fig. 1 gives a concise illustration of these two kinds of expressions.

However, since $((a,\ 0)d_\varepsilon)d'_\varepsilon = (a,\ d_\varepsilon)d'_\varepsilon$ by (3), right multiplication must be defined for elements of the form $(a,\ t)$ to carry out it multiple times. Note here that $(a,\ t)d_\varepsilon$ is not necessarily $\sim (ad_\varepsilon,\ t)$, because right multiplication for a commodity that is received at some advanced time might depend on the time (Bleichrodt et al., 2015, Theorem 6). Hence according to Smith's (2006, Chap. 10) approach, it is rational to define as follows: if $(\bar{a},\ t \cdot_T d_\varepsilon) \succsim (a,\ t)d_\varepsilon$ for some $\bar{a} \in A$, then

$$(a,\ t)d_\varepsilon \sim (a\tau[t,\ d_\varepsilon],\ t \cdot_T d_\varepsilon), \tag{11}$$

where τ is a function of t and d_ε. Here $\tau[t,\ d_\varepsilon]$ is interpreted as an amplification or attenuation operator that reflects the effect caused by advancing the receipt of a from the time point t to a still more preceding time point $t \cdot_T d_\varepsilon$. For example, if a person feels t to be (or not to be so) a sufficient duration, then $\tau[t,\ d_\varepsilon]$ may become an attenuation (or amplification) operator. Equivalence (11) implies that the two-period advanced receipt of a is equivalent to the receipt of $a\tau[t,\ d_\varepsilon]$ at the advanced time $t \cdot_T d_\varepsilon$. Henceforth unless otherwise specified, the premise for (11) is assumed to always hold. The operator has the monotonicity property.

Proposition 4. *If τ is defined by (11), then*

$$a \succsim_A b \Leftrightarrow a\tau[t,\ d_\varepsilon] \succsim_A b\tau[t,\ d_\varepsilon].$$

A monotonicity axiom is needed to carry out right multiplication by increments d_i multiple times: for any $(a,\ s),\ (b,\ t) \in A \times T$,

B7. $(a,\ s)d_1 \sim (b,\ t)$
$\Leftrightarrow (\cdots(((a,\ s)d_1)d_2)\cdots)d_n \sim (\cdots((b,\ t)d_2)\cdots)d_n$ for $n \geqslant 2$.

Axiom B7 implies that the equivalence between two commodities received at different time points is invariant under any common n-period advancement as long as the advancement is expressed by multiple use of right multiplication by increments in the duration. This axiom is a generalization of stationarity in the sense that the concept of a subjective duration is introduced. Substituting (4) for the antecedent of B7 gives the restricted version:

B7$_0$. $(a,\ 0)d_1 \sim (ad_1,\ 0)$
$\Leftrightarrow (\cdots(((a,\ 0)d_1)d_2)\cdots)d_n \sim (\cdots((ad_1,\ 0)d_2)\cdots)d_n$.

We first consider the property derived from B7$_0$, restricting the multiplicands to elements of $A \times \{0\}$.

Proposition 5. *Assume that $((\cdots(ad_1)\cdots)d_{n-1},\ 0)d_n$ is defined in $A \times \{0\}$ for $n \geqslant 2$. If B7$_0$ is satisfied, then*

$$(\cdots((a,\ 0)d_1)\cdots)d_n \sim ((\cdots(ad_1)\cdots)d_n,\ 0). \tag{12}$$

Proposition 5 tells us that the product $(\cdots(ad_1)\cdots)d_n$ is defined as a solution x to $(x,\ 0) \sim (\cdots((a,\ 0)d_1)\cdots)d_n$ and that the multiple right multiplication of a by increments in the duration is derived from the multiple right multiplication of $(a,\ 0)$ by increments in the duration. See the next subsection for a similar result (Proposition 7) regarding right division.

Example 1. Assume that the hypotheses of Proposition 5 hold. Using B7$_0$, (3) and (5) in turn, we have

$$((a,\ 0)d_1)d_2 \sim (ad_1,\ 0)d_2 = (ad_1,\ d_2) \sim ((ad_1)d_2,\ 0).$$

We next extend the multiplicands to elements of $A \times T$. Herein it is required of the extension that (12) is derived from (11). In what follows, we consider the construction of τ so as to meet this requirement.

Assume now that $(\cdots(ad_1)\cdots)d_n$ is defined in A. Let $d(n) = (\cdots(d_1 \cdot_T d_2) \cdot_T \cdots) \cdot_T d_n$. For example, (11) and B7 give

$$((a,\ d_1)d_2)d_3 \sim (a\tau[d_1,\ d_2],\ d(2))d_3 \sim ((a\tau[d_1,\ d_2])\tau[d(2),\ d_3],\ d(3)).$$

By repeated use of (11) and B7,

$$(\cdots((a,\ d_1)d_2)\cdots)d_n \sim ((\cdots(a\tau[d_1,\ d_2])\cdots)\tau[d(n-1), d_n],\ \ d(n)).$$

Hence (12) holds if and only if the following holds:

$$((\cdots(ad_1)\cdots)d_n,\ 0) \sim ((\cdots(a\tau[d_1,\ d_2])\cdots)\tau[d(n-1), d_n],\ \ d(n)). \tag{13}$$

Since the right-hand side of (13) is defined in $A \times \{0\}$, it follows from (5) that

$$((\cdots(a\tau[d_1,\ d_2])\cdots)\tau[d(n-1), d_n],\ \ d(n)) \sim (((\cdots(a\tau[d_1,\ d_2])\cdots)\tau[d(n-1), d_n])d(n),\ \ 0).$$

Hence we have by B1 and B2(i)

$$((\cdots(a\tau[d_1,\ d_2])\cdots)\tau[d(n-1),\ d_n])d(n) \sim_A (\cdots(ad_1)\cdots)d_n.$$

By Lemma 3(ii),

$$(\cdots(a\tau[d_1,\ d_2])\cdots)\tau[d(n-1),\ d_n] \sim_A (\cdots(ad_1)\cdots)d_n/d(n). \tag{14}$$

Consequently, it is seen that (12) is tantamount to (14).

Useful notation is provided to express the form of τ. For any $d_\varepsilon \in T$, define the mapping of a subset of A into A by the rule $R_{d_\varepsilon}(a) = ad_\varepsilon$ and $R_{d_\varepsilon^{-1}}(a) = a/d_\varepsilon$; further, set $R_{d_\varepsilon}^{-1} = R_{d_\varepsilon^{-1}}$. The composition of these mappings is denoted by juxtaposition, e.g., $R_{d_\varepsilon}R_{d'_\varepsilon}(a) = (ad_\varepsilon)d'_\varepsilon$. In order to make the expression for τ definite, we will express τ as an operator on the set of equivalence classes in A under $\sim_A$, denoted $A/\sim_A$. Precisely, letting $[a] = \{x \in A\,|x \sim_A a\}$, we define $[a]\tau[d_\varepsilon, d'_\varepsilon] = [a\tau[d_\varepsilon, d'_\varepsilon]]$, where $\tau[d_\varepsilon, d'_\varepsilon]$ in the left-hand side is an operator on $A/\sim_A$. This definition is possible by Proposition 4. Since $(a, d_\varepsilon) \sim (a\tau[0, d_\varepsilon], d_\varepsilon)$ by B1, (3), and (11), it follows that $\tau[0, d_\varepsilon] = id$ (an identity mapping on $A/\sim_A$).

Proposition 6. *Assume that* $(\cdots(ad_1)\cdots)d_n$ *is defined in A for* $n \geqslant 2$. *Assume that* (11) *and B7 are satisfied. Then* (14) *holds if and only if* τ *is expressed as* $\tau[d(k-1), d_k] = (R_{d(k-1)}R_{d_k})R_{d(k)}^{-1}$ *on* $A/\sim_A$ *for each* $k = 2, \ldots, n$.

Substituting (14) into the right-hand side of (13) yields

$$((\cdots(ad_1)\cdots)d_n, 0) \sim ((\cdots(ad_1)\cdots)d_n/d(n), d(n)). \tag{15}$$

By means of (15), it is possible to regard the "total" effect of the advance expressed by $(\cdots(ad_1)\cdots)d_n$ as consisting of two factors, $(\cdots(ad_1)\cdots)d_n/d(n)$ and $R_{d(n)}$, the former of which is attributed to the way an advance occurs: the n-step advance accompanied by subdivided durations d_i, and the latter of which is attributed to the advance by the total duration $d(n)$. As for the former factor, since $((\cdots(ad_1)\cdots)d_n/d(n), 0)$ has no advancement, it turns out that the first component expresses not a pure effect of advance but the effect of the procedure for generating the advance: the n-times iteration of a small advance. To explain more precisely, three cases are provided.

Case 1. $(\cdots(ad_1)\cdots)d_n \succ_A ad(n)$, or $(\cdots(ad_1)\cdots)d_n/d(n) \succ_A a$. This case implies that the n-step advance with a set of increments $(d_1, \ldots, d_n)$ in the duration enhances the value of a more than the one-step advance whose duration is subjectively assessed at $d(n)$.

Case 2. $(\cdots(ad_1)\cdots)d_n \prec_A ad(n)$, or $(\cdots(ad_1)\cdots)d_n/d(n) \prec_A a$. This case implies that the n-step advance reduces the value of a more than the one-step advance whose duration is subjectively assessed at $d(n)$.

Case 3. $(\cdots(ad_1)\cdots)d_n \sim_A ad(n)$, or $(\cdots(ad_1)\cdots)d_n/d(n) \sim_A a$. This case implies that there is no effect of the step-by-step advance, and the total effect of advance depends only on the advance of the duration $d(n)$.

We consider right multiplication by an increment d_o in the objective duration. Recall that ae indicates advancing the receipt of a by one period for which the increment in duration is a physical (objective) one. Hence it is rational to consider

$$(a, 0)d_o \sim (ae, 0),$$

which gives

$$ad_o \sim_A ae. \tag{16}$$

Our concept is that stationarity holds if a person can regard each one-period advancement as merely an advancement in which the increment in duration is objectively assessed no matter when the advancement occurs. In other words, if the increment in duration for each advancement is objectively assessed, then the effect of advance is invariable regardless of receipt time. We require that right multiplication by d_o be independent of the receipt time $t \geqslant 0$ of commodities; hence we define, for all $a \succ_A e$,

$$(a, t)d_o \sim (ae, t) \text{ whenever } (a, e) \in B. \tag{17}$$

In the preceding and following equivalences, note that by (16), $ad_\varepsilon \sim_A ae$ if $\varepsilon = o$. It must be verified here that (17) is consistent with (11).

B8. Right multiplication by an increment in the objective duration: $(a, d_\varepsilon)d_o$ is defined by means of (17).

Lemma 5. *Assume that* $B7_0$ *and B8 are satisfied. Then* $(ad_\varepsilon, e) \in B$ *if and only if* $(ae, d_\varepsilon) \in C$ *for all* $a \in A\backslash[e]$. *Moreover,* $(ad_\varepsilon)e \sim_A (ae)d_\varepsilon$ *for all* $a \in A$.

This lemma guarantees the consistency between (17) and (11). Indeed, by (11), $(a, d_\varepsilon)d_o \sim (a\tau[d_\varepsilon, d_o], d_\varepsilon\cdot_T d_o)$. Since $(a, d_\varepsilon)d_o$ is defined in $A \times \{0\}$ by the hypothesis of Lemma 5, and since $a\tau[d_\varepsilon, d_o] = (ad_\varepsilon)e/(d_\varepsilon\cdot_T d_o)$ by Proposition 6, it follows that $(a\tau[d_\varepsilon, d_o], d_\varepsilon\cdot_T d_o) \sim ((ad_\varepsilon)e, 0)$. On the other hand, $(ae, d_\varepsilon) \sim ((ae)d_\varepsilon, 0)$. Hence by the latter result of Lemma 5, $(a\tau[d_\varepsilon, d_o], d_\varepsilon\cdot_T d_o) \sim (ae, d_\varepsilon)$, as required.

3.4. Right division as a postponement operator

So far we have defined right division as the inverse of right multiplication. However, by contrast with (3) or (11), it might be natural to define right division directly as a postponement operator. In this subsection, we devise a uniform frame to define right multiplication and right division simultaneously and show that right division defined in this frame can be an inverse of right multiplication in the sense of (6). Hence the properties of right division in the preceding two subsections can be reproduced from the direct definition. This implies that we can experimentally test these properties in a postponement situation.

First, right division is defined as a postponement operator by the rule:

$$(a, 0)/d_\varepsilon = (a, d_\varepsilon^{-1}) \text{ if } a \succ_A e;\ (e, 0)/d_\varepsilon = (e, 0).$$

Second, a dual axiom to B4 is provided as follows:

B4′. Smallness at the present: for any $a \in A \setminus [e]$ and any $t^{-1} \in T$ with $t^{-1} \leqslant 0$, there exists an $\underline{a} \in A \setminus [e]$ such that $(a, t^{-1}) \succsim (\underline{a}, 0)$.

Since $(a, 0) \succsim (a, d_\varepsilon^{-1})$ by B2(iii), it follows from B3 and B4′ that there exists an $x \in A \setminus [e]$ such that $(x, 0) \sim (a, d_\varepsilon^{-1})$. Hence the right division of a by d_ε is defined by setting $x = a/d_\varepsilon$, so that we reproduce (8):

$$(a/d_\varepsilon, 0) \sim (a, 0)/d_\varepsilon.$$

A similar method to the proof of Lemma 2 guarantees monotonicity regarding right division, i.e., the properties of Proposition 3.

We show that this right division can be the inverse of right multiplication. For this purpose, the symbol t^δ, $\delta = \pm 1$, is used to mean positive or negative elements; that is, $t^\delta = t$ if $\delta = 1$, and $t^\delta = t^{-1}$ if $\delta = -1$. For convenience, we write $(a, t)d_\varepsilon^{-1} = (a, t)/d_\varepsilon$. A uniform expression is provided such that one can define right multiplication and right division as an advancement and a postponement operator, respectively:

$$(a, t^\delta)d_\varepsilon^\delta \sim (a\tau[t^\delta, d_\varepsilon^\delta], t^\delta\cdot_T d_\varepsilon^\delta). \tag{18}$$

Since $(a, d_\varepsilon^{-1})d_\varepsilon \sim (a\tau[d_\varepsilon^{-1}, d_\varepsilon], 0)$ and $(a, d_\varepsilon)/d_\varepsilon \sim (a\tau[d_\varepsilon, d_\varepsilon^{-1}], 0)$ by (18), it is seen that (6) holds up to indifference $\sim$:

$$((a, 0)d_\varepsilon)/d_\varepsilon \sim (a, 0) \text{ and } ((a, 0)/d_\varepsilon)d_\varepsilon \sim (a, 0)$$

if and only if $\tau[d_\varepsilon, d_\varepsilon^{-1}] = \tau[d_\varepsilon^{-1}, d_\varepsilon] = id$ (as an operator on $A/\sim_A$). (This equality suggests that τ be defined by $\tau[t^\delta, d_\varepsilon^\delta] = (R_t^\delta R_{d_\varepsilon}^\delta)(R_{t^\delta\cdot_T d_\varepsilon^\delta})^{-1}$.) Consequently, it turns out that right division based on (18) can be identical to right division that was defined in the previous subsection. Further assuming that B5 holds, we can obtain the properties of Lemma 3.

A monotonicity axiom for right division based on (18) is needed to derive the multiple right division of a by increments in the duration from the multiple right division of $(a, 0)$ by increments in the duration.

B7′. $(a, s)/d_1 \sim (b, t)$
$\Leftrightarrow (\cdots(((a, s)/d_1)/d_2)\cdots)/d_n \sim (\cdots((b, t)/d_2)\cdots)/d_n$.

The next proposition is an adapted version of Proposition 5 for right division.

Proposition 7. *If B7′ is satisfied, then*

$(\cdots((a, 0)/d_1)\cdots)/d_n \sim ((\cdots(a/d_1)/\cdots)/d_n, 0)$.

4. Axioms on the base structure

4.1. Generalized extensive structure with a right action

In this subsection, we will transform the axioms on $A \times T$ raised in the previous section into the ones on A. Such an axiomatization enables us to utilize the same method as the proof of the previous theorem (Matsushita, 2014). Recall here that in either case of right multiplication or right division, elements of T operating on A are always nonnegative, e.g., $ad_\varepsilon,\ a/d_\varepsilon,\ d_\varepsilon \geqslant 0$. Henceforth in this section, we assume that $T = \mathbb{R}_0^+$, and let $\langle T, \geqslant, \cdot_T, 0\rangle$ be a "positive" closed extensive structure (Krantz et al., 1971, Definition 3.1) with an identity element 0 (in which $\geqslant$ is a simple order).

We list the conditions on A that have been derived from the lemmas, definitions, and propositions in the previous section (see Table 1 for a concise correspondence between them). In the following conditions, all products $ad_\varepsilon,\ bd_\varepsilon,\ ad'_\varepsilon$ are assumed to be defined in A.

A10. $a \succsim_A b \Leftrightarrow ad_\varepsilon \succsim_A bd_\varepsilon$.
A11. $d_\varepsilon \geqslant d'_\varepsilon \Leftrightarrow ad_\varepsilon \succsim_A ad'_\varepsilon$ whenever $a \succ_A e$.
A12. $(a/d_\varepsilon)d_\varepsilon \sim_A a$ and $ad_\varepsilon/d_\varepsilon \sim_A a$.
A13. $ed_\varepsilon \sim_A e$ and $e/d_\varepsilon \sim_A e$.
A14. For any $a \in A$ and any $d_\varepsilon \in T$, there exists $x \in A$ such that $xd_\varepsilon \sim_A a$.
A15. Whenever either of $(ab)d_\varepsilon$ or $(ad_\varepsilon)(bd_\varepsilon)$ is defined in A, the other expression is also defined and $(ab)d_\varepsilon \sim_A (ad_\varepsilon)(bd_\varepsilon)$.
A16. $(ad_\varepsilon, e) \in B \Leftrightarrow (ae, d_\varepsilon) \in C$ for all $a \in A \setminus [e]$. Moreover, $(ad_\varepsilon)e \sim_A (ae)d_\varepsilon$ for all $a \in A$.

Attention should be paid to the fact that

$$ad_\varepsilon \succ_A a \text{ whenever } a \succ_A e, \tag{19}$$

because $ad_\varepsilon \succ_A a0 = a$ if $d_\varepsilon > 0$ by A11 and Definition 2.

On the other hand, the dual properties hold for right division. Proposition 3 gives the dual properties to A10 and A11. Moreover, we obtain

$a \succsim_A a/d_\varepsilon \succ_A e$ whenever $a \succ_A e$.

Indeed, in view of (19), it follows from A12 and Proposition 3(i) that $a \succ_A a/d_\varepsilon$. Meanwhile since $a \succ_A e$, by Proposition 3(i) and A13, $a/d_\varepsilon \succ_A e$.

Axiom A15 implies that the mapping R_{d_ε} is a homomorphism with respect to the concatenation operation on A, i.e., $R_{d_\varepsilon}(ab) \sim_A R_{d_\varepsilon}(a)R_{d_\varepsilon}(b)$. By A10, R_{d_ε} is order-preserving. We call an order-preserving homomorphism such as R_{d_ε} an *order-homomorphism*. In particular, an onto order-homomorphism is called an *order-isomorphism*. Axiom A16 implies that the composition of an advance operator with an increment d_s in a subjective duration and an advance operator with an increment d_o in the objective duration is commutative, i.e., $R_{d_o}R_{d_s} = R_{d_s}R_{d_o}$.

Table 1
Correspondence between conditions and original statements.

Conditions on A	Original statements
A10	Lemma 2(i), Proposition 5
A11	Lemma 2(ii), Proposition 5
A12	Lemma 3(i)
A13	Definitions 2 and 3
A14	Proposition 1(i)
A15	Lemma 4
A16	Lemma 5

Lemma 6. *Let $\langle T, \geqslant, \cdot_T, 0\rangle$ be a positive closed extensive structure with identity. Let $\langle A, \succsim_A, \cdot, e\rangle$ be a central left nonnegative concatenation structure with left identity. Assume that a partial binary operation $\circ$ on A is defined by (2). If* A15 *and* A16 *are satisfied, then for any $a, b \in A$,*

$$ad_\varepsilon \circ bd_\varepsilon \sim_A (a \circ b)d_\varepsilon. \tag{20}$$

Although the mathematical basis is omitted here (it is given in the Appendix), a correspondence $d_\varepsilon \mapsto R_{d_\varepsilon}$ is a representation of the simply ordered set T on A. Therefore it seems appropriate to regard right multiplication of A by T as a "right action"[7] of T on A in the measurement-theoretic sense.

Definition 5. Let $\langle T, \geqslant, \cdot_T, 0\rangle$ be a positive closed extensive structure with identity. A *central left nonnegative concatenation structure with left identity that is equipped with a right action of* T is a central left nonnegative concatenation structure with left identity $\langle A, \succsim_A, \cdot, e\rangle$ for which A10–A16 are satisfied.

We present the representation theorem.

Theorem 1. *Let $\langle T, \geqslant, \cdot_T, 0\rangle$ be a positive closed extensive structure with identity. Let $\langle A, \succsim_A, \cdot, e\rangle$ be a central left nonnegative concatenation structure with left identity that has a right action of T and no minimal positive element. Then there exist functions $\varphi : T \to [1, \infty)$ with $\varphi(0) = 1,\ \varphi(d_o) = \alpha$ and $u : A \to \mathbb{R}_0^+$ that is the weighted additive model of (1) with $u(e) = 0$ such that*

(i) $u((\cdots(ad_1)\cdots)d_n) = (\varphi(d_1)\cdots\varphi(d_n))u(a)$ *whenever* $(\cdots(ad_1)\cdots)d_n$ *is defined in A,*
(ii) $u((\cdots(a/d_1)/\cdots)/d_n) = \frac{1}{\varphi(d_1)\cdots\varphi(d_n)}u(a)$,
(iii) $s \geqslant t \Leftrightarrow \varphi(s) \geqslant \varphi(t)$.

Moreover, other functions φ' and u' satisfy the above properties if and only if $\varphi' = \varphi$ and $u' = \gamma u$ for some real number $\gamma > 0$.

Remark 4. We mention the reason why φ has been restricted to an absolute scale. According to Fishburn and Rubinstein (1982, Theorem 3) since u is of the multiplicative form (by (i)), two factors φ and u are to be unique up to power transformations with common exponents, i.e., $\varphi' = \gamma_1\varphi^\beta,\ u' = \gamma_2 u^\beta$ with $\gamma_1, \gamma_2, \beta > 0$. Meanwhile since u is an additive representation, it must be a ratio scale. This forces β into being $=$ 1. Since $u(ad_\varepsilon) = \varphi(d_\varepsilon)u(a)$ and $u'(ad_\varepsilon) = \varphi'(d_\varepsilon)u'(a)$, from the permissible transformations we obtain $u'(ad_\varepsilon) = \gamma_2 u(ad_\varepsilon) = \gamma_2\varphi(d_\varepsilon)u(a)$ and $u'(ad_\varepsilon) = \varphi'(d_\varepsilon)u'(a) = \gamma_1\gamma_2\varphi(d_\varepsilon)u(a)$. Hence it holds that $\gamma_2\varphi(d_\varepsilon)u(a)(1 - \gamma_1) = 0$. In view of the fact that a and d_ε are arbitrary, it follows that $1 - \gamma_1 = 0$, or $\gamma_1 = 1$, so that $\varphi' = \varphi$.

The utility model of Theorem 1 implies that right multiplication by increments in the duration is commutative, i.e., $(ad_1)d_2 \sim (ad_2)d_1$. However, the order of operating two one-period advancements having different increments in a subjective duration is likely

[7] For example, see Lang (1993) for the mathematical meanings of terminology, representation, action, and group action (which appears later).

Table 2
Increments in a subjective duration and weight values.

Period	1	2	3
d_s	1.4	1.0	0.77
$\varphi(d_s)$	1.151	1.105	1.080

to have an effect on preferences. It is a future research topic to develop a utility model that can reflect this order effect. Further, note that φ does not represent the algebraic structure of T. We will later consider the case where φ is related to an additive representation of T.

The utility model of Theorem 1 can deal with intertemporal choice problems in which decreasing or increasing impatience is involved. This is exemplified in a postponement situation as follows.

Example 2. Assume that a person is faced with the following choice problems.

A: receiving \$1000 now vs. B: receiving \$1100 after one year

A′: receiving \$1000 after two years vs. B′: receiving \$1100 after three years

Decreasing impatience can give rise to preferences: A is preferred to B, while B′ is preferred to A′. To calculate a utility value of each option, let $u(\$1000) = 1.0$ and $u(\$1100) = 1.1$, and $\alpha = \varphi(d_o) = 1.105$. Assume that an increment in a subjective duration for each year and the corresponding weight value are determined as in Table 2. Here they are calculated on the basis of (10) with $\lambda = 3.46$, $\mu = 0.5$ and the equation in the corollary below with $\nu = 0.1$. Then by Theorem 1, we have $u(\$1000) > u(\$1100/d_1)$ and $u((\$1000/d_1)/d_2) < u(((\$1100/d_1)/d_2)/d_3)$, implying the above preferences. On the other hand, the previous model of (1) gives $u(\$1000) > u(\$1100/d_o)$ and $u((\$1000/d_o)/d_o) > u(((\$1100/d_o)/d_o)/d_o)$, as mentioned in the introduction.

4.2. Relation to the exponential discount function

We will consider the case where the total effect of advance depends only on the advance of the sum of increments $d(n)$ (the third case raised after (15)). For this, the following axiom is provided.

A17. $a(d_\varepsilon \cdot_T d'_\varepsilon) \sim_A (ad_\varepsilon)d'_\varepsilon$.

Indeed, A17 and A10 yield $(\cdots(ad_1)\cdots)d_n \sim_A ad(n)$. Axiom A17 is expressed as $R_{d_\varepsilon \cdot_T d'_\varepsilon} = R_{d_\varepsilon} R_{d'_\varepsilon}$; in view of the fact that $R_0 = id$, R is a homomorphism of T into the set of those maps R_{d_ε} (roughly speaking, this allows us to call R a group action). It should be emphasized that A17 drastically alters our interpretation of the right action of increments in the duration. Although the right action has been regarded as a step-by-step advance operator, the alternation to a group action invalidates the step-by-step property of the advance operator. Indeed, A17 means that the receipt of a is advanced toward the amount of duration $d_\varepsilon \cdot_T d'_\varepsilon$, regardless of whether the advancement is carried out by the procedure repeated once or twice.

Proposition 8. *Let $\langle T, \geqslant, \cdot_T, 0\rangle$ be a positive closed extensive structure with identity, and let w be an additive representation of T. Assume that all of the hypotheses of Theorem 1 are satisfied. If* A17 *is satisfied, then the function $\varphi : T \to [1, \infty)$ of Theorem 1 (of course, having property* (iii)*) is of the multiplicative form $\varphi(s \cdot_T t) = \varphi(s)\varphi(t)$, where*

$\varphi(t) = e^{\nu w(t)}$ *for some real constant $\nu > 0$.*

Since a discount rate is evaluated by the reciprocal of the weight, $1/\varphi(t) = e^{-\nu w(t)}$, Proposition 8 asserts that our weight function is related to the exponential discount function only if we acknowledge A17. Since any w is strictly increasing, when $\cdot_T$ reduces to the usual addition +, it is seen from Theorem 3.2 Falmagne (1985) that w is of the linear form, i.e., $w(t) = \gamma t$ for $\gamma > 0$. Thus we obtain the corollary to Proposition 8.

Corollary 1. *If $\cdot_T$ reduces to +, then $\varphi(t) = e^{\nu t}$.*

The corollary implies that

$$\varphi(s + t) = e^{\nu(s+t)}.$$

According to Takahashi et al. (2012), if a cumulative duration is calculated by (10), then the exponential discount function transforms into the general hyperbolic one. A tractable method for measuring the discount function without requiring any knowledge of utility functions was proposed by Attema et al. (2016), which enables us to elicit a cumulative discount weight only from preferences among outcome streams.

5. Conclusion

This study extended the weighted additive model of (1) such that it can reflect nonconstant impatience. Under the presupposition that the set of commodities is a central left nonnegative concatenation structure with left identity and a set of durations is an Archimedean simply ordered group, a decomposable structure was assumed for the Cartesian product of the underlying sets of these two structures. Further, right multiplication and right division by increments in the duration and a concatenation operation were devised as operations on the decomposable structure. An element of the decomposable structure denotes a commodity received at a time point advanced by the specified duration. Ordering and algebraic axioms were proposed for the decomposable structure. In particular, restricted solvability played an important role. First, it gave a clear meaning of a right action: the image of a right action is regarded as a commodity received at the present that is equivalent to a commodity received one period earlier. Second, with the help of well-definedness, restricted solvability provided the compatibility of a right action with the concatenation operation for commodities. The proposed axioms seem to be easy to test experimentally because each axiom is described as a preference between commodities whose receiving time points are different. Finally, by transforming the axioms into the ones on the central left nonnegative concatenation structure, we developed a structure that is equipped with a right action. In this structure, the advanced receipt of commodities is expressed by multiple use of right multiplication of commodities by increments in the duration. By using this right multiplication and division, we can decompose the effect of advance into two factors, i.e., the factor of step-by-step advance accompanied by subdivided durations and the factor of advance based on the total duration. Note that the proposed utility model can address intertemporal postponed choice problems because right division is defined as the inverse of right multiplication. In a postponement situation, it was shown that our utility model reduces to the exponential discount function if the effect of advancement (dually, postponement) depends only on the total duration. A topic for future research is to develop a utility model that can allow for the order effect when operating the right actions of two different increments in a subjective duration.

6. Proofs

In the following proofs, we will utilize the fact (Matsushita, 2014, Proposition 1) that A consists – at most – of elements such that $a \succsim e$.

6.1. Proposition 1

Proof. (i) The proposition is obvious when $t = 0$. Since $(a, t) \succ (a, 0)$ for $t > 0$ by B2(ii), we obtain the conclusion from B3 and B4. The uniqueness of x follows from B2(i).

(ii) Since necessity is obvious, we prove only sufficiency. Assume that $(\bar{a}, 0) \succsim (a, t)$. Then since $(a, t) \succsim (a, 0)$ by B2(ii), B3 guarantees the existence of $x \in A$ such that $(x, 0) \sim (a, t)$. The uniqueness is similar to (i). □

6.2. Lemma 2

Proof. (i) The lemma is obvious in the cases where $a \sim_A b \sim_A e$ and $a \succ_A b \sim_A e$. Hence we prove the case where $a, b \succ_A e$. By (5), $(ad_\varepsilon, 0) \sim (a, d_\varepsilon)$ and $(bd_\varepsilon, 0) \sim (b, d_\varepsilon)$. Assume that $a \succsim_A b$. Since $(a, d_\varepsilon) \succsim (b, d_\varepsilon)$ by B2(i), it follows from B1 that $(ad_\varepsilon, 0) \succsim (bd_\varepsilon, 0)$, and hence by B2(i), $ad_\varepsilon \succsim_A bd_\varepsilon$. The converse is also proved.

(ii) In a similar way to (i), we prove (ii). □

6.3. Lemma 3

Proof. (i) If $a \sim_A e$, then the lemma is trivial. Let $a \succ_A e$. Using (8) and applying B5 to (4), we have $(ad_\varepsilon/d_\varepsilon, 0) \sim (ad_\varepsilon, 0)/d_\varepsilon \sim (a, 0)$. By B1 and B2(i), $ad_\varepsilon/d_\varepsilon \sim_A a$. Similarly, we use (4) and apply B5′ to (8) to obtain $((a/d_\varepsilon)d_\varepsilon, 0) \sim (a/d_\varepsilon, 0)d_\varepsilon \sim (a, 0)$. By B1 and B2(i), $(a/d_\varepsilon)d_\varepsilon \sim_A a$, as required.

(ii) We first consider the case of $a \sim_A e$. Assume that $b \sim_A ad_\varepsilon$. Then since $b \sim_A e$, b/d_ε is also $\sim_A e$ by Definition 3. Hence $b/d_\varepsilon \sim_A a$. Assume next that $b/d_\varepsilon \sim_A e$. By part (i) of the lemma, $(b/d_\varepsilon)d_\varepsilon \sim_A b$. By Lemma 2(i) and Definition 2, $(b/d_\varepsilon)d_\varepsilon \sim_A ed_\varepsilon \sim_A e$. Hence by A1, $b \sim_A e (\sim_A ad_\varepsilon)$. We next consider the case of $a \succ_A e$. By the hypothesis, let b solve $(b, 0) \sim (a, d_\varepsilon)$. Then since $(b, 0) \sim (a, 0)d_\varepsilon$ by (3), it follows from B5 that $(b, 0)/d_\varepsilon \sim (a, 0)$; meanwhile by (8), $(b, 0)/d_\varepsilon \sim (b/d_\varepsilon, 0)$. Hence by B1 and B2(i), $b/d_\varepsilon \sim_A a$. On the other hand, since $(a, d_\varepsilon) \sim (ad_\varepsilon, 0)$ by (5), B2(i) along with B1 implies that $b \sim_A ad_\varepsilon$. Thus we obtain the conclusion. □

6.4. Proposition 3

Proof. (i) The proposition is obvious in the cases of $a \sim_A b \sim_A e$ and $a \succ_A b \sim_A e$. Hence we consider the case of $a, b \succ_A e$. From (5), Lemma 3(i), B1, and B2(i), we obtain

$(a, 0) \sim (a/d_\varepsilon, d_\varepsilon)$ and $(b, 0) \sim (b/d_\varepsilon, d_\varepsilon)$.

Hence by B1 and B2(i), $a \succsim_A b \Leftrightarrow a/d_\varepsilon \succsim_A b/d_\varepsilon$.

(ii) Given $d_\varepsilon > d'_\varepsilon$, assume, contrarily to Proposition 3(ii), that $a/d_\varepsilon \succsim_A a/d'_\varepsilon$. Recall from the proof of (i) that $(a, 0) \sim (a/d'_\varepsilon, d'_\varepsilon)$. Hence by B2(i), $(a, 0) \sim (a/d_\varepsilon, d_\varepsilon) \succ (a/d'_\varepsilon, d'_\varepsilon) \sim (a, 0)$, or $(a, 0) \succ (a, 0)$, in contradiction to B1. Hence a/d_ε must be $\prec_A a/d'_\varepsilon$. Next, when $a/d_\varepsilon \prec_A a/d'_\varepsilon$, assume that $d_\varepsilon \leqslant d'_\varepsilon$. Also, by B2(i), $(a, 0) \sim (a/d_\varepsilon, d_\varepsilon) \prec (a/d'_\varepsilon, d'_\varepsilon) \sim (a, 0)$, which contradicts B1. Hence d_ε must be $> d'_\varepsilon$. Obviously, $d_\varepsilon = d'_\varepsilon \Leftrightarrow a/d_\varepsilon \sim_A a/d'_\varepsilon$. □

6.5. Lemma 4

Proof. We first show that (i) implies (ii). By (i), either $(ad_\varepsilon, 0) \bigcirc (bd_\varepsilon, 0) = ((ad_\varepsilon)(bd_\varepsilon), 0)$ or $(a, d_\varepsilon) \bigcirc (b, d_\varepsilon) = (ab, d_\varepsilon)$ with $(\bar{a}, 0) \succsim (ab, d_\varepsilon)$ for some $\bar{a} \in A$. It is obvious from the former equality that $(ad_\varepsilon)(bd_\varepsilon)$ is defined in A. We consider the latter case. Proposition 1(ii) guarantees the existence of $x \in A$ such that $(x, 0) \sim (ab, d_\varepsilon)$; according to (5), this is written as $((ab)d_\varepsilon, 0) \sim (ab, d_\varepsilon)$, so that $(ab)d_\varepsilon$ is defined in A. It is clear from (i) that if either $(ad_\varepsilon)(bd_\varepsilon)$ or $(ab)d_\varepsilon$ is defined, then so is the other. Since $(a, d_\varepsilon) \bigcirc (b, d_\varepsilon) \sim (ad_\varepsilon, 0) \bigcirc (bd_\varepsilon, 0)$, it follows from B1 that $((ab)d_\varepsilon, 0) \sim ((ad_\varepsilon)(bd_\varepsilon), 0)$, and hence by B2(i), $(ab)d_\varepsilon \sim_A (ad_\varepsilon)(bd_\varepsilon)$.

We next show that (ii) implies (i). Assume that $(a, d_\varepsilon) \bigcirc (b, d_\varepsilon)$ is defined on $A \times \{d_\varepsilon\}$ and $(\bar{a}, 0) \succsim (ab, d_\varepsilon)$ for some $\bar{a} \in A$. Then the method used in the previous proof gives $(ab, d_\varepsilon) \sim ((ab)d_\varepsilon, 0)$. Meanwhile since $(ad_\varepsilon, bd_\varepsilon) \in B$ by (ii), $(ad_\varepsilon, 0) \bigcirc (bd_\varepsilon, 0)$ is defined on $A \times \{d_\varepsilon\}$; since $(ad_\varepsilon)(bd_\varepsilon) \sim_A (ab)d_\varepsilon$ by (ii), $(ad_\varepsilon, 0) \bigcirc (bd_\varepsilon, 0) \sim ((ab)d_\varepsilon, 0)$. Hence by B1, $(ad_\varepsilon, 0) \bigcirc (bd_\varepsilon, 0) \sim (a, d_\varepsilon) \bigcirc (b, d_\varepsilon)$. Assume next that $(ad_\varepsilon, 0) \bigcirc (bd_\varepsilon, 0)$ is defined on $A \times \{0\}$. Then $(ad_\varepsilon, bd_\varepsilon) \in B$. Since $(ad_\varepsilon)(bd_\varepsilon) \sim_A (ab)d_\varepsilon$ by (ii), we have $(ad_\varepsilon, 0) \bigcirc (bd_\varepsilon, 0) \sim ((ab)d_\varepsilon, 0)$. By Proposition 1(i), $(x, d_\varepsilon) \sim ((ab)d_\varepsilon, 0)$ for some $x \in A$. Here by (5), $(xd_\varepsilon, 0) \sim (x, d_\varepsilon)$. By B1 and B2(i), $xd_\varepsilon \sim_A (ab)d_\varepsilon$. It follows from Lemma 2(i) that $x \sim_A ab$. Hence $(a, d_\varepsilon) \bigcirc (b, d_\varepsilon)$ is defined on $A \times \{d_\varepsilon\}$, and $((ab)d_\varepsilon, 0) \sim (ab, d_\varepsilon)$. By B1, $(a, d_\varepsilon) \bigcirc (b, d_\varepsilon) \sim (ad_\varepsilon, 0) \bigcirc (bd_\varepsilon, 0)$. □

6.6. Proposition 4

Proof. Assume that $a \succsim_A b$, so that $at \succsim_A bt$ by Lemma 2(i). Since $(at, d_\varepsilon) \succsim (bt, d_\varepsilon)$ by B2(i), it follows from (3) and B7$_0$ that $((a, 0)t)d_\varepsilon \succsim ((b, 0)t)d_\varepsilon$. On the other hand, by (3), B7, and (11), we have $((a, 0)t)d_\varepsilon \sim (a\tau[t, d_\varepsilon], t \cdot_T d_\varepsilon)$ and $((b, 0)t)d_\varepsilon \sim (b\tau[t, d_\varepsilon], t \cdot_T d_\varepsilon)$. Hence by B1, $(a\tau[t, d_\varepsilon], t \cdot_T d_\varepsilon) \succsim (b\tau[t, d_\varepsilon], t \cdot_T d_\varepsilon)$, and by B2(i), we obtain the demanded equivalence. The converse is also proved. □

6.7. Proposition 5

Proof. By induction on n. Example 1 proves the case of $n = 2$. Assume that the proposition holds if $n = k$. Set $x = ad_1$. By B7$_0$, $(\cdots(((a, 0)d_1)d_2)\cdots)d_{k+1} \sim (\cdots((x, 0)d_2)\cdots)d_{k+1}$. Since $((\cdots(xd_2)\cdots)d_k, 0)d_{k+1}$ is defined in $A \times \{0\}$, the element $(\cdots(xd_2)\cdots)d_{k+1}$ exists in A. From the induction hypothesis we obtain $(\cdots((x, 0)d_2)\cdots)d_{k+1} \sim ((\cdots(xd_2)\cdots)d_{k+1}, 0)$. Hence by B1, $(\cdots((a, 0)d_1)\cdots)d_{k+1} \sim ((\cdots(ad_1)\cdots)d_{k+1}, 0)$. □

6.8. Proposition 6

Proof. Assume that (14) holds. Substituting the equivalence of (14) for $n = k - 1$ into the left-hand side of the equivalence of (14) for $n = k$ under Proposition 4, we obtain

$$(((\cdots(ad_1)\cdots)d_{k-1})/d(k-1))\tau[d(k-1), d_k] \sim_A ((\cdots(ad_1)\cdots)d_k)/d(k).$$

Set $D(a) = ((\cdots(ad_1)\cdots)d_{k-1})/d(k-1)$. Then since

$$((\cdots(ad_1)\cdots)d_k)/d(k) \sim_A ((D(a)d(k-1))d_k)/d(k)$$

by Lemmas 2(i), 3(i), and Proposition 3(i), it follows again from the lemmas and proposition that

$$(((D(a)\tau[d(k-1), d_k])d(k))/d_k)/d(k-1) \sim_A D(a).$$

This implies that $((\tau[d(k-1), d_k])R_{d(k)})R_{d_k}^{-1}R_{d(k-1)}^{-1} = id$ holds as an operator on $A/\sim_A$. Hence $\tau[d(k-1), d_k] = (R_{d(k-1)}R_{d_k})R_{d(k)}^{-1}$, as required. Conversely, if this equation holds, then it is easily seen that (14) holds. □

6.9. Lemma 5

Proof. Let $a \succ_A e$. Assume that $(ad_\varepsilon, e) \in B$. By (17), $(ad_\varepsilon, 0)d_o \sim ((ad_\varepsilon)e, 0)$. Hence $(ad_\varepsilon, 0)d_o$ is defined in $A \times \{0\}$. Since $(a, d_\varepsilon)d_o = ((a, 0)d_\varepsilon)d_o \sim (ad_\varepsilon, 0)d_o$ by (3) and B7$_0$ and since $(a, d_\varepsilon)d_o \sim (ae, d_\varepsilon)$ by (17), it follows that $(ad_\varepsilon, 0)d_o \sim (ae, d_\varepsilon)$. We can use (5) to obtain $(ae, d_\varepsilon) \sim ((ae)d_\varepsilon, 0)$. This implies that $(ae, d_\varepsilon) \in C$. The converse is also proved. Finally, from the above equivalences and B2(i), we obtain $(ad_\varepsilon)e \sim_A (ae)d_\varepsilon$. Obviously, the equivalence holds if $a \sim_A e$. □

6.10. Proposition 7

Proof. The proof is similar to that of Proposition 5. Substituting (8) for the antecedent of B7′ gives $(\cdots(((a,\,0)/d_1)/d_2)\cdots)/d_n \sim (\cdots((a/d_1,\,0)/d_2)\cdots)/d_n$. From the induction hypothesis we obtain

$$(\cdots((a/d_1,\,0)/d_2)\cdots)/d_n \sim (\cdots(((a/d_1)/d_2)\cdots)/d_n,\,0).$$

Hence B1 guarantees the conclusion. □

6.11. Lemma 6

Proof. If either $a \sim_A e$ or $b \sim_A e$, then the lemma is trivial (recall that e is an identity element with respect to $\circ$). Let $a,\,b \succ_A e$. We claim that A16 implies that $(a/e)d_\varepsilon \sim_A ad_\varepsilon/e$. Set $b = a/e$. Since $(be,\,d_\varepsilon) \in C$, A16 implies that $(bd_\varepsilon)e \sim_A (be)d_\varepsilon$. From Remark 1(i) and (ii) we obtain $bd_\varepsilon \sim_A (be)d_\varepsilon/e$, which implies that $(a/e)d_\varepsilon \sim_A ad_\varepsilon/e$. Assume that $(ad_\varepsilon,\,bd_\varepsilon) \in B$ and $(a \circ b,\,d_\varepsilon) \in C$ with $(a,\,b) \in B$. Then

$$\begin{aligned}(a \circ b)d_\varepsilon &= ((a/e)b)d_\varepsilon && (2)\\ &\sim_A ((a/e)d_\varepsilon)(bd_\varepsilon) && (\text{A15})\\ &\sim_A (ad_\varepsilon/e)(bd_\varepsilon) && (\text{The above claim, A3})\\ &= ad_\varepsilon \circ bd_\varepsilon, && (2)\end{aligned}$$

as required. □

6.12. Theorem 1

We provide two concepts for the proof.

- Let $\langle A,\,\succsim_A,\,\circ,\,e\rangle$ be an extensive structure with identity, and let S be a nonempty subset of A. A relation is defined on S by the restriction of $\succsim_A$ to S. A partial binary operation is defined by the restriction of $\circ$ to S such that $a \circ b$ $(a,\,b \in S)$ is defined in A and belongs to S. Then S is called an *extensive substructure with identity of A* if it contains the identity e and is an extensive structure with respect to the relation and operation defined above. (See Theorem 3.5 in Krantz et al., 1971.) Hereafter, we will denote this relation and operation on S by the same symbols $\succsim_A$ and $\circ$, respectively.
- Extensive structures A and A' are *order-isomorphic* if there exists an order-isomorphism $\iota : A \to A'$.

Proof. Theorem 1 (Matsushita, 2014) guarantees the existence of the weighted additive representation $u : A \to \mathbb{R}_0^+$ with $u(e) = 0$. See Remark 4 for the uniqueness assertion. Hence it remains to prove that u satisfies (i) and (ii) and φ satisfies (iii). Since it is clear from A13, the order-preserving property, and the equality $u(e) = 0$ that (i) and (ii) hold whenever $a \sim_A e$, we prove (i) and (ii) for $a \succ_A e$. For this it suffices to show that $u(ad_\varepsilon) = \varphi(d_\varepsilon)u(a)$ for some $\varphi(d_\varepsilon) > 1$. Note then that $\varphi(d_o) = \alpha$. Indeed, it follows from (16) and the order-preserving property that $u(ad_o) = \varphi(d_o)u(a) = \alpha u(a) = u(ae)$. The following lemma is provided for the proof.

Lemma 7. *For any given d_ε, let $A_{d_\varepsilon} = [e] \cup \{ad_\varepsilon \,|\, (a,\,d_\varepsilon) \in C\}$ and $A' = [e] \cup \{a \in A \,|\, (a,\,d_\varepsilon) \in C\}$. Let $E(A_{d_\varepsilon}) = \langle A_{d_\varepsilon},\,\succsim_A,\,\circ,\,e\rangle$ and $E(A') = \langle A',\,\succsim_A,\,\circ,\,e\rangle$. Then both $E(A_{d_\varepsilon})$ and $E(A')$ are extensive substructures with identity of $E(A)$, and are order-isomorphic.*

Proof. The proof is similar to that of Lemma 3 (Matsushita, 2014); that is, its steps are as follows.

1. It is verified that A1–A7 hold for $E(A_{d_\varepsilon})$.
2. Using the presupposition of (20): $(ad_\varepsilon,\,bd_\varepsilon) \in B$ if and only if $(a,\,b) \in B$ and $(a \circ b,\,d_\varepsilon) \in C$, it is shown that $E(A_{d_\varepsilon})$ is an extensive structure with respect to the above-defined $\succsim_A$ and $\circ$ if and only if $E(A')$ is.
3. It is shown that a mapping ι of A' to A_{d_ε} defined by $\iota(a) = ad_\varepsilon$ is an order-isomorphism, so $E(A_{d_\varepsilon})$ and $E(A')$ are order-isomorphic.

In what follows, we will prove A6 relating to $\circ$ to specify a difference between the present and previous proofs. Assume that $ad_\varepsilon \succ bd_\varepsilon$. By A6 relating to $\cdot$, there exists $x \in A$ such that $ad_\varepsilon \sim_A x(bd_\varepsilon)$. Since $(x,\,bd_\varepsilon) \in B$ here, A2 relating to $\cdot$ implies that $xe \in A$. Hence A14 guarantees the existence of $y \in A$ such that $xe \sim_A yd_\varepsilon$, and by Remark 1(i) and (ii), $x \sim_A (yd_\varepsilon)/e$. By using A3 relating to $\cdot$, $x(bd_\varepsilon) \sim_A (yd_\varepsilon/e)(bd_\varepsilon) = (yd_\varepsilon) \circ (bd_\varepsilon)$. This along with A1 implies that $ad_\varepsilon \sim_A (yd_\varepsilon) \circ (bd_\varepsilon)$, as required. □

This lemma implies that the restriction of u is an additive representation on $E(A_{d_\varepsilon})$ and on $E(A')$. Hence the equation $u(ad_\varepsilon) = \varphi(d_\varepsilon)u(a)$ where $\varphi(d_\varepsilon) \geqslant 1$ is deduced in the same way as in the proof of Theorem 1 (Matsushita, 2014). Inductive use of this result yields (i), e.g., $u((ad_1)d_2) = \varphi(d_2)u(ad_1) = \varphi(d_2)(\varphi(d_1)u(a))$. We next prove (ii). It must be valid from A12 and the order-preserving property that $u((a/d_\varepsilon)d_\varepsilon) = u(a)$. By (i), $\varphi(d_\varepsilon)u(a/d_\varepsilon) = u(a)$, or $u(a/d_\varepsilon) = (1/\varphi(d_\varepsilon))u(a)$. Again by inductive use of this equation, we obtain (ii). Before proving (iii), we must show that $\varphi(d_\varepsilon) > 1$ (not $\varphi(d_\varepsilon) \geqslant 1$). Recall that $a \prec_A ad_\varepsilon$ for all $a \succ_A e$ with $(a,\,d_\varepsilon) \in C$ ((19)). Then by (i) and the order-preserving property, $u(a) < u(ad_\varepsilon) = \varphi(d_\varepsilon)u(a)$. Thus $\varphi(d_\varepsilon) > 1$. To determine the value of $\varphi(0)$, we apply the expression $u(at) = \varphi(t)u(a)$, $t = d_\varepsilon$ to the case of $t = 0$. It then follows from the equality $a0 = a$ (Definition 2) that $\varphi(0) = 1$. For (iii), let $d_\varepsilon \geqslant d'_\varepsilon$. Then by A11 and (i), $\varphi(d_\varepsilon)u(a) \geqslant \varphi(d'_\varepsilon)u(a) \Leftrightarrow ad_\varepsilon \succsim_A ad'_\varepsilon$, and hence we obtain $\varphi(d_\varepsilon) \geqslant \varphi(d'_\varepsilon) \Leftrightarrow d_\varepsilon \geqslant d'_\varepsilon$. □

6.13. Proposition 8

Proof. In view of A17, it follows from (i) and the order-preserving property of Theorem 1 that $\varphi(s{\cdot_T}t)u(a) = (\varphi(s)\varphi(t))u(a)$, so that $\varphi(s{\cdot_T}t) = \varphi(s)\varphi(t)$ because a is arbitrary. Now set $\phi = \ln \varphi$ to obtain $\phi(s{\cdot_T}t) = \phi(s) + \phi(t)$ and $\phi(0) = 0$. It is clear from property (iii) that $s \geqslant t \Leftrightarrow \phi(s) \geqslant \phi(t)$. These results show that ϕ is an additive representation of T. Hence by the admissible transformation (Krantz et al., 1971 Theorem 3.1), we have $\phi(t) = \nu w(t)$ for some $\nu > 0$, or $\varphi(t) = e^{\nu w(t)}$. □

Acknowledgment

This work was supported by JSPS KAKENHI Grant Number 17K00392. The author is extremely grateful to the anonymous reviewers for carefully reading the original manuscript and for the many invaluable suggestions that have considerably improved the quality of the paper.

Appendix

Given $d_\varepsilon \geqslant 0$, let $C_{d_\varepsilon} = [e] \cup \{a \in A \,|\, (a,\,d_\varepsilon) \in C\}$, which is the domain of the mapping R_{d_ε}. Note that by B4, $C_{d_\varepsilon} \setminus [e] \neq \varnothing$. We claim that

$$d_\varepsilon \geqslant d'_\varepsilon \Rightarrow C_{d_\varepsilon} \setminus [e] \subseteq C_{d'_\varepsilon} \setminus [e]. \tag{A.1}$$

Proof. Let $d_\varepsilon \geqslant d'_\varepsilon$. Since $C_{d_\varepsilon} \setminus [e] \neq \varnothing$, take an element $x \in C_{d_\varepsilon} \setminus [e]$. Then $(\bar{a},\,0) \succsim (x,\,d_\varepsilon)$ for some $\bar{a} \in A$. Since $(x,\,d_\varepsilon) \succsim (x,\,d'_\varepsilon)$ by B2(ii), we have $(\bar{a},\,0) \succsim (x,\,d'_\varepsilon)$ by B1, which implies that $x \in C_{d'_\varepsilon} \setminus [e]$. □

Let $M = \{R_{d_\varepsilon} | d_\varepsilon \in T\}$ (recall from A10 that R_{d_ε} is order-preserving). In view of (A.1), we introduce a binary relation $\geqslant_M$ on M by setting

$$\begin{aligned} &R_{d_\varepsilon} \geqslant_M R_{d'_\varepsilon} \\ &\quad\Leftrightarrow R_{d_\varepsilon}(a) \succsim_A R_{d'_\varepsilon}(a) \text{ for some (and hence for all)} \\ &\quad\; a \in C_{d_\varepsilon} \setminus [e] \cap C_{d'_\varepsilon} \setminus [e]. \end{aligned} \tag{A.2}$$

Proposition A. *Let $\langle T, \geqslant, \cdot_T, 0\rangle$ be a positive closed extensive structure with identity. Let $\langle A, \succsim_A, \cdot, e\rangle$ be a central left nonnegative concatenation structure with left identity. Assume that $\geqslant_M$ is defined by (A.2). If A11 is satisfied, then $d_\varepsilon \geqslant d'_\varepsilon \Leftrightarrow R_{d_\varepsilon} \geqslant_M R_{d'_\varepsilon}$, and hence $\geqslant_M$ is a simple order on M.*

Proof. If $d_\varepsilon > d'_\varepsilon$, then since $C_{d_\varepsilon} \setminus [e] \subseteq C_{d'_\varepsilon} \setminus [e]$ by (A.1), we have by A11 $ad_\varepsilon \succ_A ad'_\varepsilon$ for some $a \in C_{d_\varepsilon} \setminus [e]$, implying that $R_{d_\varepsilon} >_M R_{d'_\varepsilon}$. Obviously, if $d_\varepsilon = d'_\varepsilon$, then $R_{d_\varepsilon} = R_{d'_\varepsilon}$. Thus $d_\varepsilon \geqslant d'_\varepsilon \Leftrightarrow R_{d_\varepsilon} \geqslant_M R_{d'_\varepsilon}$. Since $\geqslant$ is a simple order, this implies that $\geqslant_M$ is also. □

References

Attema, A. E., Bleichrodt, H., Gao, Y., Huang, Z., & Wakker, P. P. (2016). Measuring discounting without measuring utility. *American Economic Review*, *106*, 1476–1494.

Attema, A. E., Bleichrodt, H., Rohde, K. I. M., & Wakker, P. P. (2010). Time-tradeoff sequences for analyzing discounting and time inconsistency. *Management Science*, *56*, 2015–2030.

Bleichrodt, H., Keskin, U., Rohde, K. I. M., Spinu, V., & Wakker, P. P. (2015). Discounted utility and present value–A close relation. *Operations Research*, *63*, 1420–1430.

Bleichrodt, H., Rohde, K. I. M., & Wakker, P. P. (2008). Koopmans' constant discounting for intertemporal choice: A simplification and a generalization. *Journal of Mathematical Psychology*, *52*, 341–347.

Debreu, G. (1960). Topological methods in cardinal utility theory. In K. J. Arrow, S. Karlin, & P. Suppes (Eds.), *Mathematical methods in the social sciences*. Stanford, CA: Stanford Univ. Press.

Demko, M. (2001). Lexicographic product decompositions of partially ordered quasigroups. *Mathematica Slovaca*, *51*, 13–24.

Falmagne, J. C. (1985). *Elements of psychophysical theory*. Oxford University Press.

Fishburn, P. C. (1970). *Utility theory for decision making*. New York: Wiley.

Fishburn, P. C., & Rubinstein, A. (1982). Time preference. *International Economic Review*, *23*, 677–694.

Hübner, R., & Suck, R. (1993). Algebraic representation of additive structures with an infinite number of components. *Journal of Mathematical Psychology*, *37*, 629–639.

Koopmans, T. C. (1960). Stationary ordinal utility and impatience. *Econometrica*, *28*, 287–309.

Koopmans, T. C. (1972). Representation of preference orderings over time. In C. B. McGuire, & R. Radner (Eds.), *Decision and organization*. Amsterdam/London: North-Holland.

Krantz, D. H., Luce, R. D., Suppes, P., & Tversky, A. (1971). *Foundations of measurement, Vol. 1*. New York: Academic Press..

Lang, S. (1993). *Algebra*. (3rd ed.). Addison-Wesley.

Loewenstein, G., & Prelec, D. (1992). Anomalies in intertemporal choice: evidence and an interpretation. *Quarterly Journal of Economics*, *107*, 573–597.

Luce, R. D., Krantz, D. H., Suppes, P., & Tversky, A. (1990). *Foundations of measurement, Vol. 3*.. New York: Academic Press.

Matsushita, Y. (2011). Central r-naturally fully ordered groupoids with left identity. *Quasigroups and Related Systems*, *19*, 287–300.

Matsushita, Y. (2014). Generalization of extensive structures and its representation. *Journal of Mathematical Psychology*, *62–63*, 16–21.

Smith, J. D. H. (2006). *An introduction to quasigroups and their representations*. Boca Raton: Chapman and Hall/CRC.

Takahashi, T., Han, R., & Nakamura, F. (2012). Time discounting: Psychophysics of intertemporal and probabilistic choices. *Journal of Behavioral Economics and Finance*, *5*, 10–14.