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Periodic Points and Surfaces Given by Trace Maps

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ABSTRACT

Periodic Points and Surfaces Given by Trace Maps

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In this thesis, we consider the properties of diffeomorphisms of $\mathbb{R}^3$ called trace maps. We begin by introducing the definition of the trace map. The group B_3 acts by trace maps on $\mathbb{R}^3$. The first two chapters deal with the action of a specific element of B_3, called α_n. In particular, we study the fixed points of α_n lying on a topological subspace contained in $\mathbb{R}^3$, called $\mathcal{T}$. We investigate the duality of the fixed points of the action of α_n, which will be defined later in the thesis.

Chapter 3 involves the study of the fixed points of an element called γ_{nm}, and it generalizes the results of chapter 2. Chapter 4 involves a study of the period two points of γ_{nm}.

Chapters 5-8 deal with surfaces and curves induced by trace maps, in a manner described in chapter 5. Trace maps define surfaces, and we study the intersection of those surfaces. In particular, we classify each such possible intersection.

Keywords: trace maps, diffeomorphism, fixed points, automorphisms

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Contents

List of Figures

Chapter 1. Preliminary Results

In chapters $1-3$ of this thesis, we examine curves of fixed points of diffeomorphisms of $\mathbb{R}^3$ induced by automorphisms of a free group.

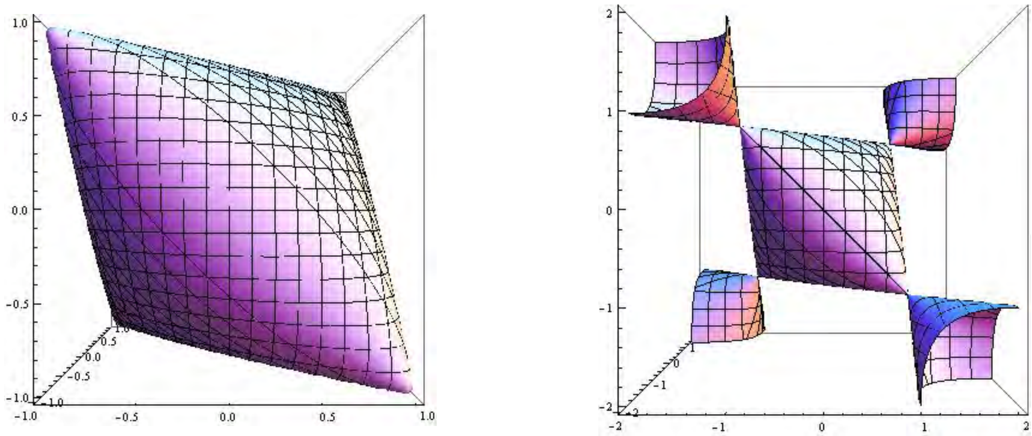
We start by noting that an element $A \in SL(2,\mathbb{Z})$ with $|\text{trace}(A)| > 2$ determines a hyperbolic automorphism of the torus, $\mathbb{T}^2$, which we view as the quotient $\mathbb{R}^2/\mathbb{Z}^2$. Identifying a point $(\theta_1, \theta_2) \in \mathbb{R}^2/\mathbb{Z}^2$ with $(-\theta_1, -\theta_2) \in \mathbb{R}^2/\mathbb{Z}^2$, gives a surface $\mathbb{T}^2/((x,y) \sim (-x,-y))$ which is homeomorphic to a 2-sphere. The matrix A will induce a map on the quotient space that is a diffeomorphism except at the singular points of the space.

Define the three strand braid group $B_3 = \langle \tau_1, \tau_2 \mid \tau_1\tau_2\tau_1 = \tau_2\tau_1\tau_2 \rangle$. The group B_3 acts on $\mathbb{R}^3$ by diffeomorphisms called trace maps, which will be defined presently.

The *Fricke Character* is defined to be

$$E(x,y,z) := x^2 + y^2 + z^2 - 2xyz. \tag{1.1}$$

(This was introduced in [1]). It determines level surfaces that are smooth except in the case of $E^{-1}(1)$ (see Proposition 1.3 and [2]).



(a) $E^{-1}(1)$ viewed in the unit cube.

(b) expanded view of $E^{-1}(1)$.

Figure 1.1: Views of $E^{-1}(1)$.

The surface $E^{-1}(1) \cap [-1,1]^3$, shown in figure 1.1(b), is a "curvilinear tetrahedron" that we call $\mathcal{T}$, which we will identify with the quotient space $\mathbb{T}^2/\sim$. We will show that

$\mathcal{T}$ is invariant under the action of each element of the braid group B_3. Thus the action of an element $\alpha \in B_3$ restricted to $\mathcal{T}$, yields a diffeomorphism of the surface $\mathcal{T}$. We will show that for each $\alpha \in B_3$, there is a corresponding element of $SL(2, \mathbb{Z})$, whose action on the torus $\mathbb{T}^2$ induces the same diffeomorphism of $\mathcal{T}$, as does the action of α restricted to $\mathcal{T}$. We will analyze the fixed points of the action of the braid group on $\mathbb{R}^3$, and in so doing, we analyze the fixed points of the action of the corresponding toral automorphism on $\mathcal{T}$. In particular, we study two families of elements $\alpha_n, \gamma_{nm} \in B_3$, with parameters n and m in the integers, which correspond respectively to the family of matrices

$$\begin{pmatrix} 1 & n \\ -n & 1-n^2 \end{pmatrix}, \qquad \begin{pmatrix} 1 & n \\ -m & 1-mn \end{pmatrix} \in SL(2, \mathbb{Z}). \tag{1.2}$$

The diffeomorphisms induced by the actions of α_n and γ_{nm} are of particular interest to us. In [2], Humphries studied the fixed points of the action of α_n on $\mathbb{R}^3$. In chapter 2, we will do the same, expanding on the results in [2]. The third chapter will focus on the fixed points of the action of γ_{nm} on $\mathbb{R}^3$.

For the first family of maps, i.e., the ones corresponding to the action of α_n, we will require that $n \in 2\mathbb{N}$. For the second family of maps, we allow $m, n \in 2\mathbb{Z}$, so that this case is a generalization of the first. The reason for doing so, is that restricting the parameters to these values yields symmetry in the dynamics of the system that is not otherwise present.

The elements α_n and γ_{nm} act on $\mathbb{R}^3$ by diffeomorphisms, and they will both yield curves of fixed points that will intersect $\mathcal{T}$. This will follow by the implicit function theorem, and will be proven later in the chapter (see Section 1.3). In chapters 1, 2, and 3 of this thesis, we will investigate characteristics of the curves of fixed points induced by the actions of α_n and γ_{nm}.

1.1 Automorphisms of the Free Group on 2 generators

Let F_2 denote the free group on two generators x_1 and x_2, and let $\sigma_i \in \mathrm{Aut}(F_2)$, $i = 1, 2$, be defined by

$$\begin{aligned} \sigma_1(x_1) &= x_1x_2, & \sigma_1(x_2) &= x_2, \\ \sigma_2(x_1) &= x_1, & \sigma_2(x_2) &= x_1^{-1}x_2. \end{aligned}$$

Proposition 1.1. *The elements σ_1 and σ_2 satisfy the braid relation, $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$.*

Proof. We show the Proposition holds when we act on x_1:

$$\sigma_1\sigma_2\sigma_1(x_1) = \sigma_1\sigma_2(x_1x_2) = \sigma_1(x_2) = x_2,$$

while

$$\sigma_2\sigma_1\sigma_2(x_1) = \sigma_2\sigma_1(x_1) = \sigma_2(x_1x_2) = x_2.$$

The case where we act on x_2 is similar. □

Now let the x_i be represented by elements of $SL(2, \mathbb{C})$. In general, 2 elements of $SL(2, \mathbb{C})$ will generate a free group of rank 2 [3]. Let

$$t_1 = \mathrm{trace}(x_1), \quad t_2 = \mathrm{trace}(x_2), \quad t_{12} = \mathrm{trace}(x_1x_2).$$

Let $x = t_1/2$, let $y = t_2/2$, and let $z = t_{12}/2$. We use the standard trace identities for 2×2 matrices:

$$\mathrm{tr}(A^{-1}) = \mathrm{tr}(A), \;\; \mathrm{tr}(I_2) = 2, \;\; \mathrm{tr}(AB) = \mathrm{tr}(A)\,\mathrm{tr}(B) - \mathrm{tr}(AB^{-1}),$$

to determine an action of σ_1, σ_2 on $\mathbb{Z}[x, y, z]$. For example, $\sigma_1(x_1) = x_1x_2$ which gives the action $\sigma_1(t_1) = (t_{12})$ so that $\sigma_1(x) = z$. Using our identities, we can determine where σ_1 maps x, y and z:

$$\sigma_1(x) = z, \quad \sigma_1(y) = y, \qquad\qquad \sigma_1(z) = 2yz - x;$$

$$\sigma_2(x) = x, \quad \sigma_2(y) = 2xy - z, \quad \sigma_2(z) = y.$$

This action extends to an action on the ring $\mathbb{Q}[x, y, z]^3$, by acting on each polynomial in a triple individually. This can be viewed as a map on $\mathbb{R}^3$, if we act on the triple (x, y, z), and view x, y, and z as the usual coordinate functions for $\mathbb{R}^3$. Diffeomorphisms of $\mathbb{R}^3$ created in this way are called trace maps. The action of σ_1 and σ_2 will be discussed later in more depth, and we will also provide a matrix representation for this action (see Lemma 2.1). It will follow from Proposition 1.1 that the group $\langle \sigma_1, \sigma_2 \rangle$ satisfies the braid relation. Thus there is a surjective homomorphism from B_3 to $\langle \sigma_1, \sigma_2 \rangle$ given by mapping the generators τ_1 and τ_2 of B_3 to σ_1 and σ_2 respectively. In fact, this homomorphism is a representation of B_3 in $\text{Aut}(\mathbb{Q}[x, y, z])$. We define the action of B_3 on $\mathbb{R}^3$ by defining the action of an element $\alpha \in B_3$, as the action of its image in $\langle \sigma_1, \sigma_2 \rangle$. We will later show that the kernel of the action of B_3 is $\langle (\tau_1 \tau_2)^3 \rangle$. This subgroup is central in B_3. We will also show that the group $\langle \sigma_1, \sigma_2 \rangle$ is antiisomorphic to $PSL(2, \mathbb{Z})$ (see Theorem 5.2).

From the previous, each element of $\langle \sigma_1, \sigma_2 \rangle$ acts on the triple (x, y, z), viewed either in $\mathbb{Q}[x, y, z]^3$ or in $\mathbb{R}^3$, through diffeomorphisms induced by automorphisms of a free group, as previously described. Additionally, each element in $\langle \sigma_1, \sigma_2 \rangle$ acts on a triple (x, y, z) through Nielsen transformations (see [4]). This will be discussed more fully in Lemma 5.1, where we will show that automorphisms and Nielsen transformations induce the same action on the triple $(x, y, z) \in \mathbb{Q}[x, y, z]^3$. For the remainder of the paper, a left action of α will refer to the action induced by automorphisms, and a right action of α will refer to an action induced by Nielsen transformations.

Trace maps have been studied by various authors([6],[7],[2]). For example, in [10], the Fibonnaci trace map $(x, y, z) \to (y, z, 2yz - x)$ is studied. While we will not investigate such applications here, the main purpose for studying trace maps involves applications to the study of quasicrystals. This is a natural application when one realizes that the substitutions involved in Nielsen transformations are closely analogous to the structure of quasicrystals. Studying trace maps gives information on the quasicrystal analogous to the Nielsen transformation that induces that trace map (see [10]).

Natural topics to analyze when working with trace maps are fixed points, period two points([10],[2],[11]). The first four chapters of this thesis focus on fixed points and period two points of trace maps.

Lemma 1.2. *Every element of* $\langle \sigma_1, \sigma_2 \rangle$ *fixes* $\mathcal{T}$.

Proof. We show that the statement of the lemma holds for σ_1. The fact that it holds for σ_2 will follow similarly. We have:

$$\begin{aligned}\sigma_1(x^2+y^2+z^2-2xyz) &= \sigma_1(x)^2+\sigma_1(y)^2+\sigma_1(z)^2-2\sigma_1(x)\sigma_1(y)\sigma_1(z)\\ &= z^2+y^2+(2yz-x)^2-2(z)(y)(2yz-x)\\ &= z^2+y^2+4y^2z^2-4xyz+x^2-4z^2y^2+2xyz\\ &= x^2+y^2+z^2-2xyz.\end{aligned}$$

The σ_2 case follows similarly. As the generators for the group fix the *Fricke character*, so will each element of $\langle \sigma_1, \sigma_2 \rangle$.

One of the characteristics of the surface $\mathcal{T}$, is that any point on the surface can be written as a triple $(\cos(2\pi\theta_1), \cos(2\pi\theta_2)), \cos(2\pi(\theta_1+\theta_2)))$, for real numbers θ_1, θ_2, with $0 \le \theta_1, \theta_2 < 1$. The action of σ_1 on such a triple is as follows:

$$\begin{aligned}&(\cos(2\pi\theta_1), \cos(2\pi\theta_2)), \cos(2\pi(\theta_1+\theta_2)))\sigma_1\\ &\qquad = (\cos(2\pi(\theta_1+\theta_2)), \cos(2\pi\theta_2), 2\cos(2\pi\theta_1)\cos(2\pi(\theta_1+\theta_2)) - \cos(2\pi\theta_1)).\end{aligned}$$

The first two components of the resulting triple clearly lie in the interval $[-1,1]$. To show the last component lies in the interval $[-1,1]$, consider the following:

$$\begin{aligned}&2\cos(2\pi\theta_1)\cos(2\pi(\theta_1+\theta_2)) - \cos(2\pi\theta_1)\\ &\qquad = \cos(2\pi(-\theta_2) + \cos(2\pi(\theta_1+2\theta_2)) - \cos(2\pi\theta_1) = \cos(2\pi(\theta_1+2\theta_2)).\end{aligned}$$

This implies that the image of a point in the unit cube $[-1,1]^3$ lying on $\mathcal{T}$, remains inside the unit cube. The result follows similarly for σ_2. This implies that $\mathcal{T}$ is invariant under σ_1 and σ_2. □

The previous lemma implies that $\mathcal{T}$ is invariant under the action of each element of B_3. We now show that $\mathcal{T}$ has singular points only at the corners of the curvilinear tetrahedron, which are the points of the set

$$V = \{(1,1,1),\ \ (-1,-1,1),\ \ (-1,1,-1),\ \ (1,-1,-1)\}.$$

Proposition 1.3. *The only singular points of $\mathcal{T}$ lie at the points of V.*

Proof. Consider the gradient of the Fricke character, $(2x-2yz, 2y-2xz, 2z-2xy)$. The singular points are the points where the gradient is 0 [5]. Thus finding the singular points is equivalent to finding simultaneous solutions to the equations

$$\begin{aligned}
&2x - 2yz = 0,\\
&2y - 2xz = 0,\\
&2z - 2yx = 0,\\
&x^2 + y^2 + z^2 - 2xyz = 1.
\end{aligned}$$

The points in V are the only such solutions on $\mathcal{T}$. □

Let $\theta = (\theta_1, \theta_2)^T$, with $0 \leq \theta_1, \theta_2 < 1$. This defines coordinates on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$.

Define

$$\Pi\colon \mathbb{T}^2 \to \mathcal{T}, \quad (\theta_1, \theta_2)^T \to (\cos(2\pi\theta_1), \cos(2\pi\theta_2), \cos(2\pi(\theta_1+\theta_2))). \tag{1.3}$$

To show that the image of the map lies on $\mathcal{T}$, note that for all real numbers θ_1, θ_2, with

$0 \leq \theta_1, \theta_2 < 1$, we have:

$$
\begin{aligned}
&\cos(2\pi\theta_1)^2 + \cos(2\pi\theta_2)^2 + \cos(2\pi(\theta_1+\theta_2)^2 - 2\cos(2\pi\theta_1)\cos(2\pi\theta_2)\cos(2\pi(\theta_1+\theta_2)) \\
&\quad = \cos(2\pi\theta_1)^2 + \cos(2\pi\theta_2)^2 + (\cos(2\pi\theta_1)\cos(2\pi\theta_2) - \sin(2\pi\theta_1)\sin(2\pi\theta_2))^2 \\
&\qquad - 2\cos(2\pi\theta_1)\cos(2\pi\theta_2)(\cos(2\pi\theta_1\cos(2\pi(\theta_2) - \sin(2\pi\theta_1)\sin(2\pi\theta_2)) \\
&\quad = \cos(2\pi\theta_1)^2 + \cos(2\pi\theta_2)^2 - \cos^2(2\pi\theta_1)\cos^2(2\pi\theta_2) + \sin^2(2\pi\theta_1)\sin^2(2\pi\theta_2)) \\
&\quad = \cos^2(2\pi\theta_1)\sin^2(2\pi\theta_2) + \cos(2\pi\theta_2) + \sin^2(2\pi\theta_1)\sin^2(2\pi\theta_2)) \\
&\quad = \sin^2(2\pi\theta_2) + \cos^2(2\pi\theta_2) \\
&\quad = 1.
\end{aligned}
$$

We see that $\Pi(\theta_1, \theta_2)^T = \Pi(-(\theta_1, \theta_2)^T)$, and in fact, $(\theta_1, \theta_2)^T$ and $-(\theta_1, \theta_2)^T$ are the only points identified under this map. This identifies $\mathcal{T}$ and $\mathbb{T}^2/\sim$. In θ coordinates, the preimages of the points of V take the form

$$V' = \{(0,0)^T, (0,1/2)^T, (1/2,0)^T, (1/2,1/2)^T\}.$$

It is easy to see that for each element $p \in V'$, $p = -p \mod \mathbb{Z}^2$. In fact, these are the only such points on the torus where this relation holds. Taken together, this implies that the map Π is a branched double cover, with V' as its branch set.

We now show the relation between a diffeomorphism on $\mathcal{T}$ induced by a hyperbolic automorphism of the torus, and the restriction to $\mathcal{T}$ of the action of an element of B_3.

Consider the following antihomomorphism:

$$\Phi \colon B_3 \to SL(2,\mathbb{Z}),\ \tau_1 \to \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \tau_2 \to \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}. \tag{1.4}$$

We note that τ_1 and τ_2 generate B_3, while $\Phi(\sigma_1)$, and $\Phi(\sigma_2)$ generate $SL(2,\mathbb{Z})$. Also, $\Phi(\sigma_1)$, and $\Phi(\sigma_2)$ satisfy the braid relation, which implies that Φ does define an antihomomorphism. Since Φ is surjective, to every toral automorphism in $SL(2,\mathbb{Z})$, there is a corresponding element of B_3. We now show how the action of elements of B_3 and its image under Φ are related.

Proposition 1.4. *The functions* Π *and* Φ *are related in the following manner. If* $\alpha \in B_3$*, and* $\theta = (\theta_1, \theta_2)^T \in \mathbb{T}^2$*,*

$$(\Pi\theta)\alpha = \Pi(\Phi(\alpha)\theta). \tag{1.5}$$

Proof. Now, Φ is an antihomomorphism, and σ_1 and σ_2 generate B_3. Initially, we show that the result holds with $\alpha = \tau_1^{\pm 1}$, and $\alpha = \tau_2^{\pm 1}$. Consider τ_1. Recall that $\Phi(\tau_1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Thus,

$$(\Pi(\theta))\tau_1 = (\cos(2\pi(\theta_1 + \theta_2), \cos(2\pi(\theta_2)), 2\cos(2\pi\theta_2)\cos(2\pi(\theta_1 + \theta_2)) - \cos(2\pi\theta_1)),$$

and

$$\Pi(\Phi(\tau_1))(\theta)) = \cos(2\pi(\theta_1 + \theta_2)), \cos(2\pi\theta_2), \cos(2\pi(\theta_1 + 2\theta_2)).$$

The fact that the two triples are equal follows from a simple application of trigonometric identities. The rest of the cases follow similarly.

Now, let $\alpha \in B_3$. Assume the result holds for α. We now induct on the length of α as a word in $\tau_i^{\pm 1}$. Let $\alpha' = \alpha\tau_1$. By hypothesis,

$$(\Pi\theta)\alpha = \Pi(\Phi(\alpha)(\theta)).$$

Next, we have

$$(\Pi\theta)\alpha' = \Pi(\Phi(\alpha)(\theta))\tau_1 = \Pi(\Phi(\tau_1)(\Phi(\alpha)\theta)) = \Pi(\Phi(\alpha')\theta).$$

This proves the result when $\alpha' = \alpha\sigma_1$. Showing the result is true for the remaining cases, $\alpha' = \alpha\sigma_i^{\pm 1}$, $i = 1, 2$, follows similarly. This completes the induction. □

Now the point of the last result is this: Φ is an antihomomorphism from B_3 to $SL(2,\mathbb{Z})$, so the following holds:

$$\Phi(\tau_1^n \tau_2^n) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}^n \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n \\ -n & 1-n^2 \end{pmatrix}. \tag{1.6}$$

Similarly,

$$\Phi(\tau_1^n \tau_2^m) = \begin{pmatrix} 1 & m \\ -n & 1-mn \end{pmatrix}. \tag{1.7}$$

Thus studying the diffeomorphisms of $\mathcal{T}$ induced by the matrices given in equation (1.2), reduces to understanding the action of an element in B_3 that corresponds to them. By the previous two equations, such elements can be chosen to be $\tau_1^n \tau_2^n$, and $\tau_1^n \tau_2^m$ respectively. Thus we will let $\alpha_n = \tau_1^n \tau_2^n$, and $\gamma_{nm} = \tau_1^n \tau_2^m$. The action of α_n on $\mathbb{R}^3$ is given by the diffeomorphism $\sigma_1^n \sigma_2^n$, and the action of γ_{nm} is given by the diffeomorphism $\sigma_1^n \sigma_2^m$.

In addition to the previous restrictions given on m and n, namely that for the diffeomorphisms corresponding to α_n, n must be an even integer greater than 0, and for the second case n, m are even integers, we will add the following: for the first case, let $n > 2$, and for the second case, let $|m|, |n| > 2$. This guarantees that the matrices given by $\Phi(\alpha_n)$ and $\Phi(\gamma_{nm})$ are hyperbolic, meaning that the absolute value of their traces is greater than 2.

We conclude the section with a classification of fixed points. We will prove later, (see Proposition 1.8) that the action of any $\alpha \in B_3$ permutes the elements of V. Thus we will focus on the fixed points of $\mathcal{T} \backslash V$. If $\alpha \in B_3$ acts on $\mathcal{T} \backslash V$, there are two possible types of fixed points, for if $\Pi(\theta) \in \mathcal{T} \backslash V$ is fixed by α_n, then by (1.5) we have $\Phi(\alpha)(\theta) = \pm\theta$. We call the fixed point a *preserving fixed point* if $\Phi(\alpha)(\theta) = \theta$. Otherwise, we call it a *reversing fixed point*. Additionally, we will be interested in a concept called *duality*. We will shortly prove that each fixed point of the action of an element $\alpha \in B_3$ restricted to $\mathcal{T}$ lies on a curve of points fixed by the action of α on $\mathbb{R}^3$. We define two fixed points on $\mathcal{T}$ to be *dual*, if they are joined by a smooth curve of fixed points.

1.2 Chebyshev Polynomials

The analysis of the actions of α_n and γ_{nm}, which are given by the diffeomorphisms $\sigma_1^n\sigma_2^n$, and $\sigma_1^n\sigma_2^m$, depends heavily on the use of Chebyshev polynomials. We will present their definition and some of their basic properties.

The following is the recursive definition for the Chebyshev polynomial of the second kind, which we designate as U-Type polynomials, or as $U_n(x)$:

$$U_{-1}(x) = 0, \;\; U_0(x) = 1, \;\; U_1(x) = 2x, \;\; U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x).$$

The U-Type Chebyshev polynomials have the following properties:

$$U_n(\cos(\theta)) = \frac{\sin((n+1)\theta)}{\sin(\theta)}, \tag{1.8}$$

$$U_m(-x) = (-1)^m U_m(x), \tag{1.9}$$

$$U_m(x)^2 - 2xU_{m-1}(x)U_m(x) + U_{m-1}(x)^2 = 1, \tag{1.10}$$

$$U_{2m}(x) = U_m(x)^2 - U_{m-1}(x)^2, \tag{1.11}$$

$$U_{2m-1}(x) = 2U_m(x)U_{m-1}(x) - 2xU_{m-1}(x)^2, \tag{1.12}$$

$$U_m(-1) = (-1)^m(m+1). \tag{1.13}$$

We will prove equation (1.9) in this thesis (see Lemma 2.15), the rest of the proofs may be found in [2].

Additionally, we can extend the U-Type Chebyshev polynomials to the negative integers by:

$$U_{-n}(x) = -U_{n-2}(x), \tag{1.14}$$

see [8].

While we will deal mainly with the polynomials U_n, we will also have occasion to use Chebyshev polynomials of the first kind, designated as T-Type polynomials, or as $T_n(x)$. These are defined recursively in the following way:

$$T_0(x) = 1,\ T_1(x) = x,\ T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x). \tag{1.15}$$

The Chebyshev T-Type polynomials have the property that

$$T_n(\cos(x)) = \cos(nx). \tag{1.16}$$

Again, these can be extended to the negative integers by:

$$T_{-n}(x) = T_n(x). \tag{1.17}$$

The U-Type Chebyshev polynomials and the T-Type Chebyshev polynomials are related by the following equation:

$$T_n(x) = \frac{1}{2}(U_n(x) - U_{n-2}(x)). \tag{1.18}$$

1.3 Curves of Fixed Points

Several of the following proofs use ideas from [9].

Lemma 1.5. *At any fixed point $p \in \mathcal{T}$ of the action of some $\alpha \in B_3$ restricted to $\mathcal{T}$, the eigenvalues of the Jacobian of α at p are $\{1, \pm\lambda, \pm\frac{1}{\lambda}\}$, where λ and $\frac{1}{\lambda}$ are the same eigenvalues as the matrix $\Phi(\alpha)$.*

Proof. It is easy to note that $E \circ \sigma_i(x, y, z) = E(x, y, z)$, and since this holds for the generators, it must hold for each element in $\langle \sigma_1, \sigma_2 \rangle$. This yields the following by the chain rule.

$$DE(\alpha(x, y, z))D\alpha(x, y, z) = DE(x, y, z).$$

If p is a fixed point of α this reduces to saying

$$\nabla E(p)D\alpha(p) = \nabla E(p).$$

Thus, 1 is an eigenvalue of $D\alpha(x, y, z)^T$ with eigenvector $\nabla E(x, y, z)$. Hence 1 is an eigenvalue of $D\alpha(x, y, z)$. Now $\mathcal{T}$ is invariant under α, so that the corresponding eigenvector of $D\alpha(p)$ must point out of the plane tangent to $\mathcal{T}$ at (p). The other two eigenvalues are determined by the action of α restricted to $\mathcal{T}$, because $\mathcal{T}$ is invariant under α. But this action is given by the induced action of the matrix $\Phi(\alpha)$. Thus the eigenvalues of $D\alpha(p)$ are the eigenvalues of $\Phi(\alpha)$ up to sign, as the Jacobian of a linear transformation is itself. The signs come from noting that if p is a reversing fixed point, the directions of the eigenvectors are reversed, so the eigenvalues at a reversing fixed point are the negative eigenvalues of $\Phi(\alpha)$. □

This leads to the following theorem (see [10]).

Theorem 1.6. *Let $\alpha \in \langle \sigma_1, \sigma_2 \rangle$, such that $\Phi(\alpha)$ is a hyperbolic matrix. Let p be a fixed point on $\mathcal{T}$ such that $p \notin V$. Then there is a smooth curve of fixed points of the action of α on $\mathbb{R}^3$ such that p is on this curve.*

Proof. We will apply the implicit function theorem (see [12]). Let p a fixed point of the action of α restricted to $\mathcal{T}$, with p not in V. Choose α so that $\Phi(\alpha)$ is a hyperbolic matrix. In this case, we note that the eigenvalues of α at the point p are $\{1, \pm\lambda, \pm\frac{1}{\lambda}\}$, where the last two eigenvalues are plus or minus the eigenvalues of $\Phi(\alpha)$, and λ does not lie on the unit circle.

Locally, $\mathcal{T}$ is an embedded submanifold in a neighborhood of p. Additionally, the level sets of the Fricke character, $E(x, y, z)$, locally foliate the space [2]. Thus we can change the coordinates in a neighborhood of p so that the first coordinate is given by $E(x, y, z)$, and the last two correspond to a location on a level set of E. Call this new set of coordinates (E, u_1, u_2). Now, there is a diffeomorphism corresponding to α in the new coordinate system, which we will call $A\colon \mathbb{R}^3 \to \mathbb{R}^3$, where $A(E, u_1, u_2) = (A_1(E, u_1, u_2), A_2(E, u_1, u_2), A_3(E, u_1, u_2))$, locally. Now, the eigenvalues of the Jacobian are invariant under a change of basis, thus A has eigenvalues $\{1, \pm\lambda, \pm\frac{1}{\lambda}\}$. Also, note that the eigenvalue 1 corresponds to an eigenvector that is not in the $u_1 - u_2$ plane. This implies that the matrix

$$\begin{pmatrix} \frac{\partial A_2}{\partial u_1} & \frac{\partial A_2}{\partial u_2} \\ \frac{\partial A_3}{\partial u_2} & \frac{\partial A_3}{\partial u_2} \end{pmatrix} - I_2,$$

is invertible at p, because the projection of the action of A onto the $u_1 - u_2$ plane yields a map whose Jacobian has eigenvalues $\{\pm\lambda, \pm\frac{1}{\lambda}\}$. Define $f\colon \mathbb{R}^3 \to \mathbb{R}^2$ to be $f = \pi \circ (A - Id)$, where π is projection onto the second and third coordinates. Then, as p is a fixed point of A, $f(p) = 0$. In the new set of coordinates, let $p = (p_1, p_2, p_3)$. By the implicit function theorem, there exists a neighborhood U in $\mathbb{R}$ containing p_1, and a neighborhood V in $\mathbb{R}^2$, containing (p_2, p_3), and a smooth function $g\colon U \to V$, such that $\{(x, \mathbf{g(x)}) | x \in U\} = \{(x, \mathbf{y}) \in U \times V \mid f(x, \mathbf{y}) = 0\}$. As this is precisely the set of fixed points in $U \times V$, we see that p lies on a curve of fixed points. The implicit function theorem implies that this curve is smooth. Finally, as the results of the implicit function thoerem are not affected by a change of coordinates, we have proven the theorem. □

We now show that the diffeomorphisms $\langle\sigma_1, \sigma_2\rangle$, are volume preserving. For $\alpha \in \langle\sigma_1, \sigma_2\rangle$, denote the Jacobian of α by J_α.

Proposition 1.7. *If $\alpha \in \langle\sigma_1, \sigma_2\rangle$, then α is volume preserving.*

Proof. By the chain rule, it suffices to show that $\det(J_{\sigma_1}) = \det(J_{\sigma_2}) = 1$. Note that

$$J_{\sigma_1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 2z & 2y \end{pmatrix}.$$

This clearly has determinant 1. Similarly, this holds for σ_2. □

This implies that B_3 acts on $\mathbb{R}^3$ through volume preserving diffeomorphisms.

Now, we want to determine the properties of the action of $\alpha \in \langle\sigma_1, \sigma_2\rangle$ on the elements of V. This will follow a proof in [9].

Proposition 1.8. *Let $\alpha \in \langle\sigma_1, \sigma_2\rangle$. Then the action of α permutes the elements of V.*

Proof. Previously, we had $E \circ \sigma_i(x, y, z) = E(x, y, z)$, and since this holds for the generators, it must hold for each element in $\langle \sigma_1, \sigma_2 \rangle$. Thus by the chain rule, we have the following:

$$DE(\alpha(x, y, z))D\alpha(x, y, z) = DE(x, y, z).$$

Since $E \colon \mathbb{R}^3 \to \mathbb{R}$, we have the following:

$$D\alpha^T(x, y, z)\nabla E(\alpha(x, y, z)) = \nabla(x, y, z).$$

$|D\alpha| = 1$, because α is volume preserving. Thus, if (x, y, z) is a critical point of $E(x, y, z)$, so is it's image. $\square$

Chapter 2. The Fixed Points of the action of α_n and their Duality

We begin with the analysis of the diffeomorphisms induced by the element, $\alpha_n = \tau_1^n \tau_2^n \in B_3$. We give a simple form for the action of α_n on $\mathbb{R}^3$ using Chebyshev polynomials. Recall that the action of α_n is given by the diffeomorphism $\sigma_1^n \sigma_2^n$. The following action will be of primary importance for the remainder of this chapter. Several of the following theorems come from [2].

Lemma 2.1. *If $k \in \mathbb{Z}$, then*

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \sigma_1^k = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 2y \end{pmatrix}^k \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \sigma_2^k = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2x & -1 \\ 0 & 1 & 0 \end{pmatrix}^k \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Denote the above 3×3 matrices by $M_1 = M_1(y)$ and $M_2 = M_2(x)$. Then we obtain

$$M_1^k = \begin{pmatrix} -U_{k-2}(y) & 0 & U_{k-1}(y) \\ 0 & 1 & 0 \\ -U_{k-1}(y) & 0 & U_k(y) \end{pmatrix},$$

$$M_2^k = \begin{pmatrix} 1 & 0 & 0 \\ 0 & U_k(x) & -U_{k-1}(x) \\ 0 & U_{k-1}(x) & -U_{k-2}(x) \end{pmatrix}.$$

Proof. The first follows from the action of σ_i. We prove the second using induction. We begin with M_1. For $k = 1$, we have

$$M_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 2y \end{pmatrix} = \begin{pmatrix} -U_{1-2}(y) & 0 & U_{1-1}(y) \\ 0 & 1 & 0 \\ -U_{1-1}(y) & 0 & U_1(y) \end{pmatrix}.$$

Thus the initial case holds. Assume now, that

$$M_1^k = \begin{pmatrix} -U_{k-2}(y) & 0 & U_{k-1}(y) \\ 0 & 1 & 0 \\ -U_{k-1}(y) & 0 & U_k(y) \end{pmatrix}.$$

Then

$$\begin{aligned} M_1^k M_1 &= \begin{pmatrix} -U_{k-2}(y) & 0 & U_{k-1}(y) \\ 0 & 1 & 0 \\ -U_{k-1}(y) & 0 & U_k(y) \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 2y \end{pmatrix} \\ &= \begin{pmatrix} -U_{k-1}(y) & 0 & -U_{k-2}(y) + 2yU_{k-1}(y) \\ 0 & 1 & 0 \\ -U_k & 0 & -U_{k-1} + 2yU_k \end{pmatrix} \\ &= \begin{pmatrix} -U_{k-1}(y) & 0 & U_k(y) \\ 0 & 1 & 0 \\ -U_k(y) & 0 & U_{k+1}(y) \end{pmatrix} = M_1^{k+1}, \end{aligned}$$

where the penultimate equality follows from (1.2). This completes the induction. □

From the previous lemma, it is relatively simple to derive the following.

Lemma 2.2. *For all $n \in \mathbb{N}$ and $(x, y, z)^T \in \mathbb{R}^3$ we have*

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \alpha_n = \begin{pmatrix} -xU_{n-2}(y) + zU_{n-1}(y) \\ U_n(x^*)y - U_{n-1}(x^*)[-xU_{n-1}(y) + zU_n(y)] \\ U_{n-1}(x^*)y - U_{n-2}(x^*)[-xU_{n-1}(y) + zU_n(y)] \end{pmatrix}. \tag{2.1}$$

Where $x^* = -xU_{n-2}(y) + zU_{n-1}(y)$.

In particular, if $(x, y, z)^T \in \mathbb{R}^3$ is a fixed point of α_m and $U_{n-1}(y) \neq 0$ then

$$z = \frac{x(1 + U_{n-2}(y))}{U_{n-1}(y)}. \tag{2.2}$$

Proof. To prove the first, let $k = n$ in Lemma 2.1. Multiplying $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ by $M_1^n(y)$ gives the matrix

$$\begin{pmatrix} -xU_{n-2}(y) + zU_{n-1}(y) \\ y \\ -xU_{n-1} + zU_k(y) \end{pmatrix}.$$

Multiplying the previous matrix by $M_2^n(x^*)$ gives the matrix

$$\begin{pmatrix} -xU_{n-2}(y) + zU_{n-1}(y) \\ U_n(x^*)y - U_{n-1}(x^*)[-xU_{n-1}(y) + zU_n(y)] \\ U_{n-1}(x^*)y - U_{n-2}(x^*)[-xU_{n-1}(y) + zU_n(y)] \end{pmatrix}.$$

The second statement follows by noting that if (x, y, z) is a fixed point, $x^* = x$. Solving $x^* = x$ for z yields the formula for z. □

From this presentation of the diffeomorphism induced by α_n, we can derive certain facts about the fixed points of α_n. In [2], the authors determined that there were three possibilities for the curves of fixed points in $\mathbb{R}^3$. These possibilities are that the curves are either straight lines, the curves lie in the planes $x = \pm y$, or the curves do not intersect the planes $x = \pm y$. We say a few words about the first two cases, but focus mainly on the third and most complicated case for the remainder of this chapter.

2.1 The Straight Line Cases.

In this section, we consider curves of fixed points that are vertical lines. The main result of this section is that there is a vertical line of fixed points running through most fixed points lying on $\mathcal{T}$ whose y coordinate satisfy $U_{n-1}(y) = 0$.

For any $N \in \mathbb{N}$, let $K_N \subset SL(2,\mathbb{Z})$ denote the kernel of the homomorphism $SL(2,\mathbb{Z}) \to SL(2,\mathbb{Z}/N\mathbb{Z})$. Note, this kernel is the set of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with the property that a and b are equivalent to $1 \mod n$, and that b and c are equivalent to $0 \mod n$. So we have $\Phi(\alpha_n) = \begin{pmatrix} 1 & n \\ -n & 1-n^2 \end{pmatrix} \in K_n$.

Proposition 2.3. *For any $k, m \in \mathbb{Z}$, and any $\beta \in B_3$ such that $\Phi(\beta) \in K_n$, $\Pi(k/n, m/n)^T$ is a preserving fixed point of β. In Particular, this is the case for α_n.*

Proof. We use (1.5). Note that $\Phi(\beta)(\theta)$ can be written in the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} k/n \\ m/n \end{pmatrix},$$

where $a, d \equiv 1 \mod (n)$, and $b, c \equiv 0 \mod (n)$. Thus we can write $a = a_1 n + 1$, $b = b_1 n$, etc. Substituting into the previous equation yields

$$\begin{pmatrix} (a_1 n + 1)\frac{k}{n} + b_1 n \frac{m}{n} \\ (c_1 n)\frac{k}{n} + (d_1 n + 1)\frac{m}{n} \end{pmatrix} = \begin{pmatrix} a_1 k = \frac{k}{n} + b_1 n \\ c_1 k + d_1 m + \frac{m}{n} \end{pmatrix} = \begin{pmatrix} \frac{k}{n} \\ \frac{m}{n} \end{pmatrix} \mod \mathbb{Z}^2.$$

Thus for $\theta = \begin{pmatrix} k/n \\ m/n \end{pmatrix}$, and for $\beta \in K_n$, we have that $(\Pi\theta)\beta = (\Pi\theta)$, by equation (1.5). Also, $\Phi(\beta)(\theta) = \theta$, which implies that this θ is a preserving fixed point for any $\beta \in K_n$, and thus for α_n. □

We will now show that for most integer values of k, m, and n, the vertical line

$$p(z) = \left(\cos\left(\frac{2\pi k}{n}\right), \cos\left(\frac{2\pi m}{n}\right), z\right) \tag{2.3}$$

is a line of fixed points of the action of α_n, and is not tangential to $\mathcal{T}$.

Proposition 2.4. *Let n, j, and $k \in \mathbb{N}$, with $j,k \not\equiv 0 \mod \frac{n}{2}$, and n even. Let $a = \cos(\frac{2\pi j}{n})$.*

(i) We have $M_1^n(a) = M_2^n(a) = I_3$.

(ii) If $\alpha \in \langle \sigma_1^n, \sigma_2^n \rangle$ and $v = \left(\cos\left(\frac{2\pi j}{n}\right), \cos\left(\frac{2\pi k}{n}\right), z\right)^T$, then $(v)\alpha = v$. In particular, we have $(v)\alpha_n = v$ and the vertical line $p(z) = \left(\cos\left(\frac{2\pi j}{n}\right), \cos\left(\frac{2\pi k}{n}\right), z\right)^T$ is a curve of fixed points of α_n.

(iii) If $j \equiv 0 \mod \frac{n}{2}$ or $k \equiv 0 \mod \frac{n}{2}$, then $\left(\cos\left(\frac{2\pi j}{n}\right), \cos\left(\frac{2\pi k}{n}\right), z\right)^T$ is not a curve of fixed points of α_n.

Proof. (i) From Lemma 2.2, we have

$$M_1^n(a) = \begin{pmatrix} -U_{n-2}(a) & 0 & U_{n-1}(a) \\ 0 & 1 & 0 \\ -U_{n-1}(a) & 0 & U_n(a) \end{pmatrix}.$$

Using (1.8), we have, as $j \not\equiv 0 \mod \frac{n}{2}$ implies that $\sin(\frac{2\pi j}{n}) \neq 0$, that

$$
\begin{aligned}
U_n\left(\cos\left(\frac{2\pi j}{n}\right)\right) &= \sin\left(\frac{(n+1)2\pi j}{n}\right)/\sin\left(\frac{2\pi j}{n}\right) = 1,\\
U_{n-1}\left(\cos\left(\frac{2\pi j}{n}\right)\right) &= \sin\left(\frac{2n\pi j}{n}\right)/\sin\left(\frac{2\pi j}{n}\right) = 0,\\
U_{n-2}\left(\cos\left(\frac{2\pi j}{n}\right)\right) &= \sin\left(\frac{(n-1)2\pi j}{n}\right)/\sin\left(\frac{2\pi j}{n}\right) = -1.
\end{aligned}
$$

This gives $M_1^n(a) = I_3$. The proof that $M_2^n(a) = I_3$ is similar.

(ii) From Lemma 2.2 and (i), we have $v\sigma_1^n = M_1^n\left(\cos\left(\frac{2\pi k}{n}\right)\right)v = I_3 v = v$, and $v\sigma_2^n = M_2^n(a) = I_3 v = v$. This allows us to conclude that

$$
v\alpha_n = (v\sigma_1^n)\sigma_2^n = v\sigma_2^n = v.
$$

(iii) First we do the case where $k = \frac{n}{2}$. Here we have $v = \left(\cos\left(\frac{2\pi j}{n}\right), -1, z\right)^T$, which from Lemma 2.2 implies that the first coordinate of $v\alpha_n$ is

$$
-\cos\left(\frac{2\pi j}{n}\right)U_{n-2}(-1) + U_{n-1}(-1)z = -(n-1)\cos\left(\frac{2\pi j}{n}\right) - nz \neq \cos\left(\frac{2\pi j}{n}\right).
$$

Note that we used equation (1.9) for the second equality.

If $k = 0$, then $v = \left(\cos\left(\frac{2\pi j}{n}\right), 1, z\right)^T$, and the first coordinate of $v\alpha_n$ is

$$
nz - (n-1)\cos\left(\frac{2\pi j}{n}\right) \neq \cos\left(\frac{2\pi j}{n}\right).
$$

The cases where $j \equiv 0, \frac{n}{2} \mod n$ are similar. □

The line $p(z)$ meets $\mathcal{T}$ at another point which must be $\Pi(k/n, -m/n)^T$. So these two points are dual. Now, the solutions of $U_{n-1}(y) = 0$ are $\cos\left(\frac{2\pi j}{2n}\right)$. However, it is easy to see that a fixed point $(x, y, z) \in \mathcal{T}$ with $U_{n-1}(y) = 0$ can only be fixed by the action of α_n if j is even. Thus, the fixed points with y coordinate given by $\cos\left(\frac{2\pi j}{n}\right)$, where $j \neq 0 \mod \frac{n}{2}$ are precisely the fixed points where $U_{n-1}(y)$ is 0.

Lemma 2.5. *If (x, y, z) is a fixed point of the action of α_n lying on $\mathcal{T}$, and $y = \cos\left(\frac{2\pi j}{n}\right)$ for some integer j, then $x = \cos\left(\frac{2\pi k}{n}\right)$ for some integer k.*

Proof. The θ coordinates of such a fixed point are given by $(\theta_1, \frac{j}{n})^T$. If (x, y, z) is such a fixed point, then $(\theta_1, \frac{j}{n})^T$ is fixed by $\Phi(\alpha_n)$. We have

$$\begin{pmatrix} 1 & n \\ -n & 1-n^2 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \frac{j}{n} \end{pmatrix} = \begin{pmatrix} \theta_1 + j \\ -n\theta_1 + \frac{j}{n} - nj \end{pmatrix}.$$

But the only way $(\theta_1 + j, -n\theta_1 + \frac{j}{n} - nj)^T = \pm(\theta_1, \frac{j}{n})^T \mod \mathbb{Z}^2$, is if $\theta_1 = \frac{k}{n}$ for some integer k. □

This implies that the points discussed in this section are the only fixed points of the action of α_n not covered by Lemma 2.2.

Proposition 2.6. *If n is a multiple of 4, then the x, y, and z axes are fixed by α_n.*

Proof. We note that

$$(0, 0, z)\sigma_i^{\epsilon 4} = (0, 0, z),$$

for $i = 1, 2$ and $\epsilon = \pm 1$. This holds for each such triple representing an element of an axis. □

Thus when n is a multiple of 4, the x, y, and z axes are a line of fixed points that run through the axial points

$$\{\{(1,0,0), (-1,0,0)\}, \{(0,1,0), (0,-1,0)\}, \{(0,0,1), (0,0,-1)\}\},$$

making these pairs of points dual in this case (see [2], Lemma 2.4).

2.2 The $x = \pm y$ case

We will describe situation where the curves of fixed points are in the planes $x = y$. The situation where they lie in the $x = -y$ plane follows similarly. We note that some of the straight line curves lie in the plane $x = y$, thus some of these curves have already been discussed. It can be shown that any curve intersecting the $x = y$ plane is completely contained in this plane ([2], see Lemma 3.2). We also study the points of V in this section, as they lie in the planes $x = \pm y$.

Lemma 2.7. *Any fixed point not on one of the vertical lines discussed in the previous section, must lie on a curve with equation*

$$(x, x, x(1 + U_{n-2}(x))/U_{n-1}(x)). \tag{2.4}$$

Proof. By equation (2.2), such a point must take the form

$$(x, y, x(1 + U_{n-2}(y))/U_{n-1}(y)),$$

and since it lies in the plane $y = x$, it must take the above form. □

Theorem 2.8. *Let $\gamma_+(x) = (x, x, x(1 + U_{n-2}(x))/U_{n-1}(x))$, a curve in $\mathbb{R}^3$. The curve $\gamma_+(x)$ is fixed by α_n.*

Proof. We have the following:

$$M_1^n \begin{pmatrix} x \\ x \\ x(1 + U_{n-2}(x))/U_{n-1}(x) \end{pmatrix} = \begin{pmatrix} x \\ x \\ -xU_{n-1}(x) + xU_n x(1 + U_{n-2}(x))/U_{n-1}(x) \end{pmatrix}.$$

Substituting for $U_n(x)$ by the definition of the U-Type Chebyshev polynomial yields the following:

$$\begin{pmatrix} x \\ x \\ 2x^2 + (-xU_{n-1}^2(x) + 2x^2U_{n-1}(x)U_{n-2}(x) - xU_{n-2}(x) - xU_{n-2}^2(x))/U_{n-1}(x) \end{pmatrix}.$$

Note that

$$2x^2 + (-xU_{n-1}^2(x) + 2x^2U_{n-1}(x)U_{n-2}(x) - xU_{n-2}(x) - xU_{n-2}^2(x))/Un - 1(x)$$

$$= 2x^2 + [x(-U_{n-1}^2(x) + 2xU_{n-1}(x)U_{n-2}(x) - U_{n-2}(x)^2) - xU_{n-2}(x)]/U_{n-1}(x)$$

$$= 2x^2 + (-x - xU_{n-2}(x))/U_{n-1}(x)$$

$$= 2x^2 - x(1 + U_{n-2}(x))/U_{n-1}(x),$$

where the last equality follows by equation (1.10). Next,

$$M_2^n \begin{pmatrix} x \\ x \\ 2x^2 - x(1 + U_{n-2}(x))/U_{n-1}(x) \end{pmatrix} =$$

$$\begin{pmatrix} x \\ x(1 + U_{n-2}(x) - 2xU_{n-1}(x) + U_n(x) \\ xU_{n-1}(x) + [xU_{n-2}(x)(1 + U_{n-2}(x) - 2xU_{n-1}(x)]/U_{n-1}(x) \end{pmatrix}.$$

simplifying the above using equation (1.10) yields $\gamma_+(x)^T$. There is a similar result in the $x = -y$ plane. □

Figure 2.1 shows the fixed curves of α_{20} in the $x = y$ plane. The blue lines are the fixed curves, the red curve is the curve $z = 2x^2 - 1$, and the yellow curve is the line $z = 1$. Solving $E(x, x, z) = 1$ for z yields the solutions 1 and $2x^2 - 1$.

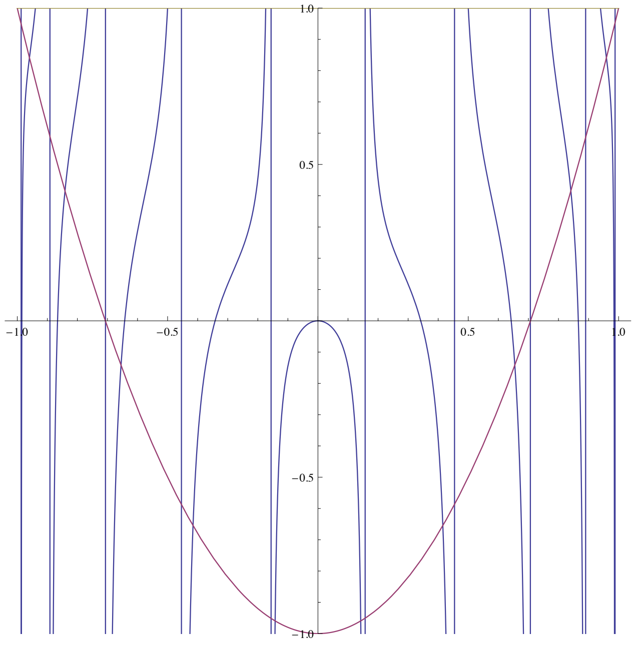


Figure 2.1: In blue: fixed curves of α_{20} in the $x = y$ plane. In red and yellow: the intersection of $\mathcal{T}$ and the $x = y$ plane.

We now discuss the points in V. Recall that these were the corners of the tetrahedron which in Cartesian coordinates take the form

$$V = \{(1,1,1),\ \ (-1,-1,1),\ \ (-1,1,-1),\ \ (1,-1,-1)\}.$$

Proposition 2.9. *Each element of V is fixed by any $\alpha \in \langle \sigma_1^2, \sigma_2^2 \rangle$. The fixed curve that intersects each element of V does not enter the convex hull of $\mathcal{T}$.*

Proof. We have that $(x,y,z)\sigma_1^2 = (-x+2yz, y, -2xy+(-1+4y^2)z)$. We see by evaluating at each point of V, that σ_1^2 fixes V. The σ_2 case follows similarly. The second part of this proof is given in [2] in Lemma 2.11. □

A consequence of the previous proposition is that no point in V is dual to a fixed point of the action of α_n restricted to $\mathcal{T}$.

2.3 Fixed Curves not Intersecting the Planes $x = \pm y$

We now look at the fixed points of the action of α_n with $U_{n-1}(y) \neq 0$. We will give explicit coordinates for these points dependent only on the parameter n.

In this section, we study the fixed points of the action of α_n using equations (2.1) and (2.2). Let

$$Y = U_n(x^*)y - U_{n-1}(x^*)[-xU_{n-1}y + zU_n(y)];$$
$$Z = U_{n-1}(x^*)y - U_{n-2}(x^*)[-xU_{n-1}(y) + zU_n(y)].$$

We consider the simultaneous solutions to the two equations

$$Y - y = 0,$$
$$Z - z = 0,$$

where $z = x(1 + U_{n-2}(y))/U_{n-1}(y)$. These will yield the x and y coordinates of the fixed points of the action of α_n, with $U_{n-1}(y) \neq 0$. Now, to find the simultaneous solutions to the above equations, it suffices to find the greatest common denominator of $Y - y$ and $Z - z$. We will call this $G_n(x, y)$. For the rest of this chapter, n will be a multiple of 2, and $m = n/2$. We require this restriction because if n is odd, there will be no symmetry in the set of fixed curves lying on $\mathcal{T}$. This will be explored further in the next section.

Theorem 2.10. *Let n a multiple of 2, and let $m = n/2$, then*

$$G_n(x, y) = U_{m-2}(y)U_{m-1}(x)(1 - x^2)y + U_{m-1}(x)U_{m-1}(y)(x^2 - y^2) \qquad (2.5)$$
$$+ U_{m-1}(y)U_{m-2}(x)(y^2 - 1)x.$$

Proof. See [2] Proposition 4.1. Alternatively, apply the formula for the γ_{nm} case, and let $n = m$. (see equation (3.2)). □

The zeroes of $G_n(x, y)$ give the set of fixed curves of the action of α_n, projected to the $x - y$ plane. We can explicitly find the points of intersection in the $x - y$ plane of $G_n(x, y)$ and $\mathcal{T}$ (where we have substituted $z = x(1 + U_{n-2}(y))/U_{n-1}(y))$). This is shown in the following few theorems.

Theorem 2.11. *For $(x, y, z) \in \mathcal{T}$, with (x, y, z) fixed under the action of α_n, write the element (x, y, z) in terms of θ_1 and θ_2. That is,*

$$(x, y, z) = (\cos(2\pi\theta_1), \cos(2\pi\theta_2), \cos(2\pi(\theta_1 + \theta_2))).$$

Then θ_1 and θ_2 must satisfy one of the following relations. Given $k \in \mathbb{Z}$

$$\theta_2 = \frac{k}{n}, \qquad (2.6)$$
$$\theta_1 = -\frac{n}{2}\theta_2 + \frac{k}{2}. \qquad (2.7)$$

Proof. If (x, y, z) is a fixed point on $\mathcal{T}$, then we know from Lemma 2.2 that $z = x(1 + U_{n-2}(y))/U_{n-1}(y)$ and by equation (1.3) we have that $z = cos(2\pi(\theta_1 + \theta_2))$. We rewrite the first equation in terms of θ_1 and θ_2, and then use equation (1.8) to rewrite the first equation as

$$z = \frac{\cos(2\pi\theta_1)\left(1 + \frac{\sin(2\pi(n-1)\theta_2)}{\sin(2\pi\theta_2)}\right)}{\frac{\sin(2\pi n\theta_2)}{\sin(2\pi\theta_2)}}.$$

This gives us the relation, after simplification,

$$\frac{\cos(2\pi\theta_1)(\sin(2\pi\theta_2) + \sin(2\pi(n-1)\theta_2))}{\sin(2\pi n\theta_2)} = \cos(2\pi(\theta_1 + \theta_2)).$$

Multiplying by the denominator yields the following:

$$\cos(2\pi\theta_1)\sin(2\pi\theta_2) + (\sin(2\pi n\theta_2)(\cos(2\pi\theta_2) - \cos(2\pi n\theta_2)\sin(2\pi\theta_2))\cos(2\pi\theta_1)$$
$$-(\cos(2\pi\theta_1)\cos(2\pi\theta_2) - \sin(2\pi\theta_1)\sin(2\pi\theta_2))\sin(2\pi n\theta_2) = 0.$$

Simplifying gives

$$\sin(2\pi\theta_2)(\cos(2\pi\theta_1) - \cos(2\pi n\theta_2)\cos(2\pi\theta_1) + \sin(2\pi\theta_1)\sin(2\pi n\theta_2)) = 0.$$

A final cancellation yields

$$\sin(2\pi\theta_2)(\cos(2\pi(n\theta_2 + \theta_1)) - \cos(2\pi\theta_1)) = 0.$$

This yields the solutions

$$\theta_2 = \frac{k}{n},$$
$$\theta_1 = -\frac{n}{2}\theta_2 + \frac{k}{2},$$

for $k \in \mathbb{Z}$. Note that, if $k = 0 \mod \frac{n}{2}$, $\sin(2\pi n\theta_2)$ is 0, which makes the initial equation in the previous proof undefined. However, we have previously shown, in Section 2.1, that such k values yield preserving fixed points, although they do not lie on a straight line of fixed points. Thus there need be no restriction on the θ_2 values. □

We note that letting $\theta_2 = \frac{k}{n}$, yields only the previously shown solutions for the y value of the fixed point. That is, we obtain only the points previously discussed in Section 2.1.

We will prove later, (see Theorem (3.8)) that in the case where $\theta_1 = -\frac{n}{2}\theta_2 + \frac{k}{2}$, substituting this relation into $G_n(x, y)$ gives the equation

$$\frac{(-1)^{km-k+1}}{2}\cos^2(2\pi\theta_2)\sin(2\pi n\theta_2)\sin(2\pi(m^2-1)\theta_2) = 0. \tag{2.8}$$

It is easy to see the solutions of this equation take one of the forms:

$$\theta_2 = \frac{j}{2n},$$
$$\theta_2 = \frac{j}{2(m^2-1)}, \quad j \in \mathbb{Z}.$$

These correspond with the y values:

$$y = \cos\left(\frac{\pi j}{n}\right), \tag{2.9}$$
$$y = \cos\left(\frac{\pi j}{m^2-1}\right), \quad j \in \mathbb{Z}. \tag{2.10}$$

This gives a set containing all of the x and y coordinates of solutions of the intersection of $\mathcal{T}$ and the $G_n(x, y)$. There are several of these solutions that are not actually intersection points. This occurs because several of them lie on lines where the z value diverges. To determine which of these should be eliminated from our set of solutions, we need the following.

Theorem 2.12. *Let $k \in \mathbb{Z}$:*

(i) $\lim_{y\to\cos(\frac{k\pi}{n})} \frac{x(1+U_{n-2}(y))}{U_{n-1}(y)} = \infty$ *for k odd, and $x \neq 0$.*

(ii) $\lim_{y\to\cos(\frac{k\pi}{n})} \frac{x(1+U_{n-2}(y))}{U_{n-1}(y)}$ *is finite for k even.*

Proof. In the unit square, we can write $\frac{x(1+U_{n-2}(y))}{U_{n-1}(y)}$ as

$$\frac{x\left(1+\frac{\sin(2\pi(n-1)\theta_2)}{\sin(2\pi\theta_2)}\right)}{\frac{\sin(2\pi n\theta_2)}{\sin(2\pi\theta_2)}}.$$

This simplifies to the following:

$$\frac{x(\sin(2\pi\theta_2)+\sin(2\pi(n-1)\theta_2))}{\sin(2\pi n\theta_2)}.$$

We can expand this into the following:

$$\frac{x(\sin(2\pi\theta_2)+\sin(2\pi n\theta_2)\cos(2\pi\theta_2)-\sin(2\pi\theta_2)\cos(2\pi n\theta_2))}{\sin(2\pi n\theta_2)}.$$

So our problem reduces to finding

$$\lim_{\theta_2\to\frac{k}{2n}}\frac{x(\sin(2\pi\theta_2)+\sin(2\pi n\theta_2)\cos(2\pi\theta_2)-\sin(2\pi\theta_2)\cos(2\pi n\theta_2))}{\sin(2\pi n\theta_2)}.$$

Now, if k is odd, the top part of the fraction is $2x$ and the bottom goes to zero, so this diverges unless $x=0$. If k is even, we have a $0/0$ limit, and apply l'Hopital's rule to find the limit. After taking derivatives, we take the limit of the function

$$\frac{x(\cos(2\pi\theta_2)+n\cos(2\pi n\theta_2)\cos(2\pi\theta_2)-(\cos(2\pi\theta_2)\cos(2\pi n\theta_2))}{n\cos(2\pi n\theta_2)} \\ -\frac{\sin(2\pi n\theta_2)\sin(2\pi\theta_2)+n\sin(2\pi\theta_2)\sin(2\pi n\theta_2)}{n\cos(2\pi n\theta_2)}.$$

Substituting gives

$$\frac{x\left(\cos\left(\frac{k\pi}{n}\right)+n\cos\left(\frac{k\pi}{n}\right)-\cos\left(\frac{k\pi}{n}\right)\right)}{n\cos(\frac{k\pi}{n})}=\frac{x\left(n\cos\left(\frac{k\pi}{n}\right)\right)}{n}=\cos\left(\frac{k\pi}{n}\right),$$

which is finite. □

This implies that the solutions that we obtain from equation (2.8) that have $y=\cos(\frac{k\pi}{n})$ for k odd are not fixed points of the action of α_n restricted to $\mathcal{T}$. If k is even, we again obtain the straight line fixed curves discussed in Section 2.1.

We now determine the type (preserving or reversing) of the remaining fixed points.

Lemma 2.13. *The points given by*

$$\left(\cos\left(\frac{-2\pi nj}{4(m^2-1)} - \pi k\right), \cos\left(\frac{2\pi j}{2(m^2-1)}\right), \cos\left(\frac{-2\pi nj}{4(m^2-1)} - \pi k + \frac{2\pi j}{2(m^2-1)}\right)\right)$$

are reversing fixed points of the action of α_n.

Proof. Since the points given by the coordinates above lie on $\mathcal{T}$, we know that $(\theta_1, \theta_2)^T = \pm\left(\frac{-nj}{4(m^2-1)} - \frac{k}{2}, \frac{j}{2(m^2-1)}\right)^T$ are the points on the torus that are mapped to the fixed point $(\cos(2\pi\theta_1), \cos(2\pi\theta_2), \cos(2\pi(\theta_1+\theta_2))$. We now show that the positive choice for $(\theta_1, \theta_2)^T$ is a reversing fixed point for any choices of j and k in the integers, the negative choice follows identically. Therefore, we have that:

$$\begin{pmatrix} 1 & n \\ -n & 1-n^2 \end{pmatrix} \begin{pmatrix} \frac{-nj}{4(m^2-1)} - \frac{k}{2} \\ \frac{j}{2(m^2-1)} \end{pmatrix} = \begin{pmatrix} \frac{-nj}{4(m^2-1)} - \frac{k}{2} + \frac{nj}{2(m^2-1)} \\ \frac{n^2 j}{4(m^2-1)} + \frac{nk}{2} + \frac{j-jn^2}{(2(m^2-1)} \end{pmatrix} = \begin{pmatrix} \frac{nj}{4(m^2-1)} + \frac{k}{2} \\ \frac{-j}{2(m^2-1)} \end{pmatrix} \mod \mathbb{Z}^2.$$

□

We have one more case to consider. On the boundary of the unit square, we have exceptions to Proposition 2.4. These are precisely the cases where $j = 0 \mod \frac{n}{2}$. Hence, they are not dual as straight line curves. We require the following lemma to identify these points. (see equation 1.9)

Lemma 2.14. *For $m \in \mathbb{N}$, $U_m(1) = m+1$.*

Proof. Now, $U_0(1) = 1$ by equation (1.2). Assume the hypothesis holds for the natural numbers up to m. Thus, $U_m = m+1$, and $U_{m-1}(1) = m$. $U_{m+1}(1) = 2(m+1)-m = m+2$. This completes the proof. □

We also require the next lemma.

Lemma 2.15. *For $m \in \mathbb{N}$, $U_m(-x) = (-1)^m U_m(x)$.*

Proof. This holds for U_0 by equation (1.2). Assume it holds for the natural numbers up to m. Then $U_m(-x) = (-1)^m U_m(x)$, and $U_{m-1}(-x) = (-1)^{m-1}U_{m-1}(x)$.. Therefore $U_{m+1}(x) = -2x(-1)^m U_m(x) - (-1)^{m-1}U_{m-1}(x) = (-1)^{m+1}(2xU_m(x) - U_{m-1}(x)) = (-1)^{m+1}U_{m+1}(x)$. This completes the proof. □

We note that the cases where $j = 0 \mod \frac{n}{2}$ are precisely the cases when $\theta_2 = 0$ or $\theta_2 = 1/2$. We now find the coordinates of the fixed points in this case.

Theorem 2.16. *If $\theta_2 = \frac{n}{2}$, and $y = \cos(2\pi\theta_2)$, then Equation 2.5 reduces to $\pm(x^2 - 1)U_{m-1}(x)$. The roots of this equation are ± 1, and $\cos(2\pi\theta_1)$, where $\theta_1 = \frac{k}{n}$, for $0 < k < m$. If $\theta_2 = 0$, then reducing the equation, we again have $\pm(x^2 - 1)U_{m-1}(x)$.*

Proof. This follows immediately from Lemmas 2.14, 2.15, and Equation (1.8). □

Theorem 2.16 completes our characterization of the set of intersections of fixed curves of α_n, and $\mathcal{T}$. We summarize the results of the previous section with the following theorem.

Theorem 2.17. *A point (x, y, z) lies at the intersection of a fixed curve of α_n and $\mathcal{T}$ if and only if its x and y coordinates take one of the following forms:*

$$
\begin{aligned}
&x = \pm 1, y = \cos\left(\frac{\pi k}{m}\right), \quad 0 \le k \le m; \\
&x = \cos\left(\frac{\pi k}{m}\right), y = \pm 1, \quad 0 \le k \le m; \\
&x = \cos\left(\frac{2\pi j}{n}\right), y = \cos\left(\frac{2\pi k}{n}\right), \quad k, j \in \mathbb{N}, \quad k, j \ne 0 \mod \frac{n}{2}; \\
&x = \cos\left(\frac{\pi m j}{m^2 - 1} - \pi k\right), \quad y = \cos\left(\frac{\pi j}{m^2 - 1}\right), k, j \in \mathbb{Z}.
\end{aligned}
$$

For these fixed points, the first, second, and third set are preserving, and the fourth set is reversing. The fourth set has the property that the z coordinate is given by $z = x(1 + U_{n-2}(y))/U_{n-1}(y)$, and the third set lies on a straight line of fixed points.

When discussing duality, we deal only with the fixed points of the action of α_n that do not lie in V. The points in V are precisely the points with x and y coordinates on the corners of the unit square, with z coordinate $(x(1+U_{n-2}(y))/U_{n-1}(y)$. They have been dealt with previously.

2.4 Dual Points

We are now in a position to determine which fixed points of the action of α_n, restricted to $\mathcal{T}\backslash V$, are dual. We have already given the duality of almost all of the preserving fixed points. We now determine the duality of the reversing fixed points, and the remaining preserving fixed points. The following lemma simplifies the process.

Lemma 2.18. *If $\alpha_n = \tau_1^n\tau_2^n$, then there is an 8 way symmetry of the resulting fixed curves. Precisely, they are symmetric across the x and y axes, and across $y = \pm x$. If n is a multiple of 2, then $\mathcal{T}$, given by the equation $E(x, y, z) = x^2 + y^2 + z^2 - 2xyz - 1$, and projected to the $x - y$ plane using the substitution $z = \frac{x(1+U_{n-2}(y))}{U_{n-1}(y)}$, is symmetric over the x and y axes.*

Proof. From (2.5), we have an equation for the greatest common denominator,

$$G(x, y) = U_{m-2}(y)U_{m-1}(x)(1 - x^2)y + U_{m-1}(x)U_{m-1}(y)(x^2 - y^2) \\ + U_{m-1}(y)U_{m-2}(x)(y^2 - 1)x.$$

(a) Substituting $-x$ for x yields the equation

$$(-1)^{m-1}U_{m-2}(y)U_{m-1}(x)(1 - x^2)y + (-1)^{m-1}U_{m-1}(x)U_{m-1}(y)(x^2 - y^2) \\ (-1)^{m-1}U_{m-1}(y)U_{m-2}(x)(y^2 - 1)x.$$

Setting this equal to 0, we see it has the same solutions as equation (2.5).

(b) Switching x and y gives the following equation:

$$U_{m-2}(x)U_{m-1}(y)(1-y^2)x + U_{m-1}(y)U_{m-1}(x)(y^2-x^2) \\ + U_{m-1}(x)U_{m-2}(y)(x^2-1)y,$$

which has the same solutions as (2.5).

The rest of the cases follow easily.

For the Fricke character, substituting $-x$ for x yields

$$x^2 + y^2 + \left(\frac{-x(1+U_{n-2}(y))}{U_{n-1}(y)}\right)^2 - 2xy\left(\frac{x(1+U_{n-2}(y))}{U_{n-1}(y)}\right).$$

Simplifying this yields the original equation. The other case follows easily, using Lemma 2.15, and the fact that n is even. □

We can use this symmetry to simplify our analysis, by analyzing the first quadrant, and applying this analysis to the rest of the plane using symmetry.

Using Theorem 2.5, we note that a curve of fixed points cannot cross the line $y = \cos\left(\frac{j\pi}{n}\right)$ for j an odd integer, except when x is 0. Similarly a curve of fixed points cannot cross the line $x = \cos\left(\frac{j\pi}{n}\right)$, for j an odd integer, except when y is 0. It turns out that this is sufficient to characterize the duality of these curves.

Lemma 2.19. *For $k \in \mathbb{N}$, $k < 2n$, we have $\lceil \frac{k(m^2-1)}{2m} \rceil = \lceil \frac{km}{2} \rceil$.*

Proof. We divide the proof into cases.

If m is even, then $\lceil \frac{km}{2} \rceil = \frac{km}{2}$. But $\frac{k(m^2-1)}{2m} < \frac{km}{2}$ for all $m > 0$, because if not we have the equations

$$\frac{k(m^2-1)}{2m} > \frac{km}{2}$$
$$km^2 - k > km^2$$
$$-k > 0,$$

which results in a contradiction.

We now show that $\frac{k(m^2-1)}{2m} > \frac{km}{2}$. If not, then we have the equations

$$\begin{aligned} \frac{k(m^2-1)}{2m} &< \frac{km}{2} - 1, \\ km^2 - k &< km^2 - 2m, \\ 2m - k &< 0, \end{aligned}$$

which is again a contradiction. The case where m is odd follows similarly. Additionally, if m is odd, $\lceil \frac{km}{2} \rceil = \frac{km+1}{2}$. □

Let n be even. The analysis of the duality of the fixed points of the action of α_n, restricted to $\mathcal{T}$ separates into two cases. The case where m is odd, and the case where $m = n/2$ is even. The case where m is even is dealt with in [2], and we will not repeat the analysis here. For the rest of this section, assume that m is odd. We know that the y coordinate of the reversing fixed points is given by $y = \cos(\frac{\pi j}{m^2-1})$. By Lemma 2.19, we know that given an integer $0 < k < 2m-2$, if an integer $j \in \left(\frac{k(m^2-1)}{2m}, \frac{(k+2)(m^2-1)}{2m}\right)$, then it lies in the interval $[\frac{km+1}{2}, \frac{(k+2)m-1}{2}]$. We first note that there are always m integers in the interval. This follows as there are $\frac{(k+2)m-1}{2} - \frac{km+1}{2} = m - 1 + 1 = m$ integers in the interval.

We have the following:

Lemma 2.20. *If $j \in [\frac{km+1}{2}, \frac{(k+2)m-1}{2}]$ with $0 < k \leq m-2$, with k odd, then*

$$\cos\left(\frac{(k+2)\pi}{2m}\right) < \cos\left(\frac{j\pi}{m^2-1}\right) < \cos\left(\frac{k\pi}{2m}\right).$$

Proof. This follows as if $j \in [\frac{km+1}{2}, \frac{(k+2)m-1}{2}]$, then $j \in \left(\frac{k(m^2-1)}{2m}, \frac{(k+2)(m^2-1)}{2m}\right)$. Dividing by m^2-1 preserves the direction of the inequalities. For $0 < k < m-2$, cosine is a monotonically decreasing function. Thus applying cosine simply switches the direction of the inequality. □

This essentially allows us to determine precisely which j values give us y values between two given planes $y = \cos\left(\frac{k\pi}{n}\right)$, $y = \cos\left(\frac{(k+2)\pi}{n}\right)$. We now determine which of these are dual.

Lemma 2.21. *For a reversing fixed point (x, y, z), let $y = \cos\left(\frac{m(k+1)\pi}{2(m^2-1)}\right)$. Then the corresponding x value is larger than $\cos\left(\frac{\pi}{2m}\right)$.*

Proof. The x coordinate is given by $x = \cos\left(\frac{-\pi m^2(k+1)}{2(m^2-1)} + \pi l\right)$, for some integer l. We have

$$\begin{aligned}\cos\left(\frac{-\pi m^2(k+1)}{m^2-1} + \pi l\right) &= \cos\left(-\frac{\pi(m^2-1)(k+1)}{2(m^2-1)} - \frac{\pi(k+1)}{2(m^2-1)} + \pi l\right)\\ &= \cos\left(-\frac{\pi(k+1)}{2} - \frac{\pi(k+1)}{2(m^2-1)} + \pi l\right).\end{aligned}$$

Since $k+1$ is even, we can choose l so that the equation simplifies to $\cos\left(\frac{\pi(k+1)}{2(m^2-1)}\right)$. Now, from here, it is sufficient to show that $\frac{(k+1)}{2(m^2-1)} < \frac{1}{2m}$, because applying cosine reverses the sign, and we will have $\cos\left(\frac{\pi(k+1)}{2(m^2-1)}\right) > \cos\left(\frac{\pi}{2m}\right)$, which is what we want to show. We have

$$\frac{(k+1)}{2(m^2-1)} < \frac{m-1}{2(m^2-1)} < \frac{1}{2(m+1)} < \frac{1}{2m},$$

which completes the result. □

Our next step is to examine the duality of the rest of the points with $j \in [\frac{km+1}{2}, \frac{(k+2)m-1}{2}]$. We prove the following.

Theorem 2.22. *Let i an integer such that $0 \leq i < \lfloor \frac{m}{2} \rfloor$. Then in the first quadrant, a point with y coordinate $\cos\left(\frac{\pi(km+1+2i)}{2(m^2-1)}\right)$ is dual to a point with y coordinate $\cos\left(\frac{\pi((k+2)m-1-2i)}{2(m^2-1)}\right)$. Each y value corresponds to a unique point in the first quadrant, and thus these two points must be dual.*

Proof. The fact that there is one specific x coordinate corresponding to each y coordinate follows from Theorem 2.17, as this shows there is a unique x coordinate up to an integer multiple of π. This implies that there is a unique x coordinate in the first quadrant.

Consider first the coordinate $y = \cos\left(\frac{\pi(km+1+2i)}{2(m^2-1)}\right)$. This has corresponding x coordinate $\cos\left(\frac{\pi((km+1+2i)m)}{2(m^2-1)}\right)$. We start by simplifying the term $\frac{(km+1+2i)m}{2(m^2-1)}$.

$$\frac{(km+1+2i)m}{2(m^2-1)} = \frac{km^2-k+k+m+2im}{2(m^2-1)},$$
$$= \frac{k}{2} + \frac{k+m(2i+1)}{2(m^2-1)}.$$

We consider the number $\frac{k+(1+2i)m}{2(m^2-1)} + \frac{k}{2}$. In particular, assume that

$$\frac{j}{2m} < \frac{((km+1+2i)m)}{2(m^2-1)} + \frac{k}{2} < \frac{j+2}{2m},$$

for j an odd integer less than $m-2$. Solving for i, we get the equation

$$\frac{(j-km)(m^2-1)-km-m^2}{2m^2} < i < \frac{(j+2-km)(m^2-1)-km-m^2}{2m^2}.$$

Simplifying yields

$$\frac{j-1-km}{2} - \frac{j}{2m^2} < i < \frac{j+1-km}{2} - \frac{j+2}{2m^2}.$$

Since $\frac{j}{2m^2} < \frac{1}{2}$, and since $\frac{j+2}{2m^2} < \frac{1}{2}$, we have that, since i is an integer,

$$\lceil\frac{j-1-km}{2}\rceil \leq i \leq \lfloor\frac{j+1-km}{2}\rfloor.$$

Thus $i = \frac{j-mk}{2}$, (recall that j, k and m are odd), so that $j = 2i + km$. This j value may correspond to a line in the left half of the plane. To produce one in the correct half of the plane, adjust the value of l in (2.17) by 1.

We next prove that the fixed point with y coordinate $\cos\left(\frac{\pi((k+2)m-1-2i)}{2(m^2-1)}\right)$ has x coordinate lying within the same range. We examine the term $\frac{(k+2)m-1-2i)}{2(m^2-1)}$. Assume that

$$\frac{j}{2m} < \frac{(k+2)m-1-2i)}{2(m^2-1)} < \frac{j+2}{2m}.$$

Then

$$\frac{-j+km+2m-1}{2}+\frac{j}{2m^2}>i>\frac{-j-3+km+2m}{2}+\frac{j+2}{2m^2}.$$

Similarly to the first case, this gives $j = km + 2m - 2i - 2$. It remains to show that this j value gives the partition as the j value for the fixed point with y coordinate $\cos\left(\frac{\pi(km+1+2i)}{2(m^2-1)}\right)$, which had corresponding j value $(2i + mk)$. let $j = (2i + mk)$ and $j' = km + 2m - 2i - 2$. It is trivial to note that $\cos\left(\frac{\pi j}{2m}\right) = \cos\left(\frac{\pi(j'+2)}{2m}\right)$ and that $\cos\left(\frac{\pi(j+2)}{2m}\right) = \cos\left(\frac{\pi j'}{2m}\right)$. Since the specified j value is unique for each of the pairs of fixed points indicated in the statement of the theorem, this concludes the theorem. □

We now examine the fixed points with y coordinate $\cos\left(\frac{\pi i}{m^2-1}\right)$, with $i \leq \frac{m-1}{2}$.

Theorem 2.23. *Assume that $i \leq \frac{m-1}{2}$. Then the reversing fixed point with x and y coordinates $\left(\cos\left(\frac{\pi m i}{m^2-1}\right), \cos\left(\frac{\pi i}{m^2-1}\right)\right)$ is dual to the preserving fixed point with x and y coordinates $\left(\cos\left(\frac{\pi i}{m}\right), 1\right)$.*

Proof. First, we note that if

$$\frac{j}{2m}<\frac{im}{m^2-1}<\frac{j+2}{2m},$$

where j is an odd integer, then $j = 2i - 1$, using the same method as in the previous proof. Since $i < \frac{m-1}{2}$, this implies that no two fixed points with y coordinates $\cos\left(\frac{\pi i}{m^2-1}\right)$, for different i values, are dual. By necessity, they are each dual to a fixed point with y coordinate 1, and x coordinate $\cos\left(\frac{\pi k}{m}\right)$ for some k. Choosing $k = i$ yields the only possible such value that will lie between $\cos\left(\frac{\pi(2i-1)}{2m}\right)$ and $\cos\left(\frac{\pi(2i+1)}{2m}\right)$. Thus these two points must be dual. □

The last thing we have to deal with are the points given in Lemma 2.21.

Lemma 2.24. *Fix k an odd integer less than $m - 2$. The reversing fixed point with x and y coordinates $\left(\cos\left(\frac{m^2(k+1)\pi}{2(m^2-1)} - \pi l\right), \cos\left(\frac{m(k+1)\pi}{2(m^2-1)}\right)\right)$, where l is chosen so that the x coordinate lies in the first quadrant, is dual to the preserving fixed point with x and y coordinates $\left(1, \cos\left(\frac{\pi(k+1)}{2m}\right)\right)$.*

Proof. Again, let k an odd integer less than $m-2$. Consider the fixed point with x and y coordinates $\left(\cos\left(\frac{m^2(k+1)\pi}{2(m^2-1)}-\pi l\right), \cos\left(\frac{m(k+1)\pi}{2(m^2-1)}\right)\right)$, where l is chosen so that the x coordinate lies in the first quadrant. By Lemma 2.21, this must be dual to a point with x and y coordinates given by $\left(1, \cos\left(\frac{2\pi i}{2m}\right)\right)$, with i an even integer less than n. Assume that

$$\frac{j}{2m} < \frac{m(k+1)}{2(m^2-1)} < \frac{j+2}{2m},$$

for j an odd integer. Then

$$j-1-\frac{j}{m^2} < k < j+1-\frac{j+2}{m^2}.$$

From this we deduce that $j=k$, or $j=k-1$. Since k is odd, $j=k$. Now, we require

$$\frac{j}{2m} < \frac{i}{2m} < \frac{j+2}{2m},$$

by Lemma 2.12. Since $j=k$ and i is even we deduce that the reversing fixed point with x and y coordinates $\left(\cos\left(\frac{m(k+1)\pi}{2(m^2-1)}\right), \cos\left(\frac{m^2(k+1)\pi}{2(m^2-1)}-\pi l\right)\right)$ is dual to the preserving fixed point with x and y coordinates $\left(1, \cos\left(\frac{\pi(k+1)}{2m}\right)\right)$. $\square$

By Lemma 2.18, we have completely determined the duality of this family of diffeomorphisms, as we have done so for the first quadrant. The analysis for the case where m is even can be found in [2], which uses a similar methodology for the proof.
We end this section by recapitulating the results in a more convenient format.

Theorem 2.25. *Let n an even integer, and let $n=2m$. Further assume that m is an odd integer. The fixed points of the action $\sigma_1^n\sigma_2^n$ on $\mathbb{R}^3$ lying on the surface $\mathcal{T}$, were given in Theorem 2.17. For those fixed points not lying on a straight line of fixed points, their z coordinates are determined by their x and y coordinates, and their duality is given as follows*

(1) The reversing fixed point with x and y coordinates $(\cos(\frac{\pi m(km+1+2i)}{2(m^2-1)}-\pi r), \cos(\frac{\pi(km+1+2i)}{2(m^2-1)}))$ is dual to the reversing fixed point with x and y coordinates $(\cos(\frac{\pi m(km+1+2i)}{2(m^2-1)}-\pi r'), \cos(\frac{\pi(km+1+2i)}{2(m^2-1)}))$, where i, k, r, r' are integers,

$0 \leq i < \lfloor \frac{m}{2} \rfloor$, *$k$ is an odd integer such that $0 < k < m-2$, and r and r' are chosen so that the x coordinates are positive, so that each point lies in the 1st quadrant.*

(2) *The reversing fixed point with x and y coordinates $(\cos(\frac{m^2(k+1)\pi}{2(m^2-1)} - \pi r), \cos(\frac{m(k+1)\pi}{2(m^2-1)}))$, is dual to the preserving fixed point with x and y coordinates $(1, \cos(\frac{\pi(k+1)}{2m}))$, where k is an odd integer greater than 0 and less than $m-2$, and r is an integer chosen so that the x coordinate is positive.*

(3) *The reversing fixed point with x and y coordinates $(\cos(\frac{\pi m i}{m^2-1} - r\pi), \cos(\frac{\pi i}{m^2-1}))$ is dual to the preserving fixed point with x and y coordinates $(\cos(\frac{\pi i}{m}), 1)$, where i is an integer such that $0 \leq i \leq \lfloor \frac{m}{2} \rfloor$, and r is an integer chosen so that the x coordinate is positive.*

By Lemma 2.18, this is sufficient to completely determine the duality of every fixed point of the action of α_n on $\mathcal{T}$.

2.5 Example

We now demonstrate results of the previous two sections using an example. For this section let $n = 22$, so that $m = 11$. We now plot the zeroes of equation (2.5), for $m = 11$.

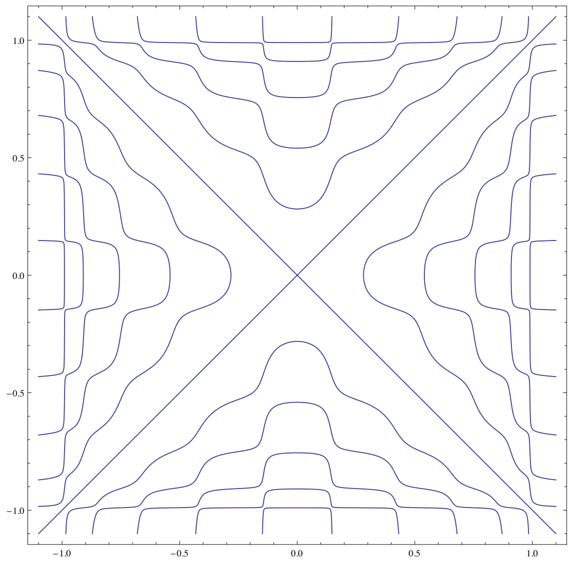


Figure 2.2: Contour Plot of the curves given $G_{22}(x, y) = 0$.

We now focus on the first quadrant, as we did in the last section. The following picture contains the solutions of equation (2.5), and the projection of $\mathcal{T}$, which we obtained by substituting equation (2.2) for z in the equation $x^2+y^2+z^2-2xyz=1$. The intersections of these two sets of curves will show the fixed points discussed in the previous section, excluding the straight line curves.

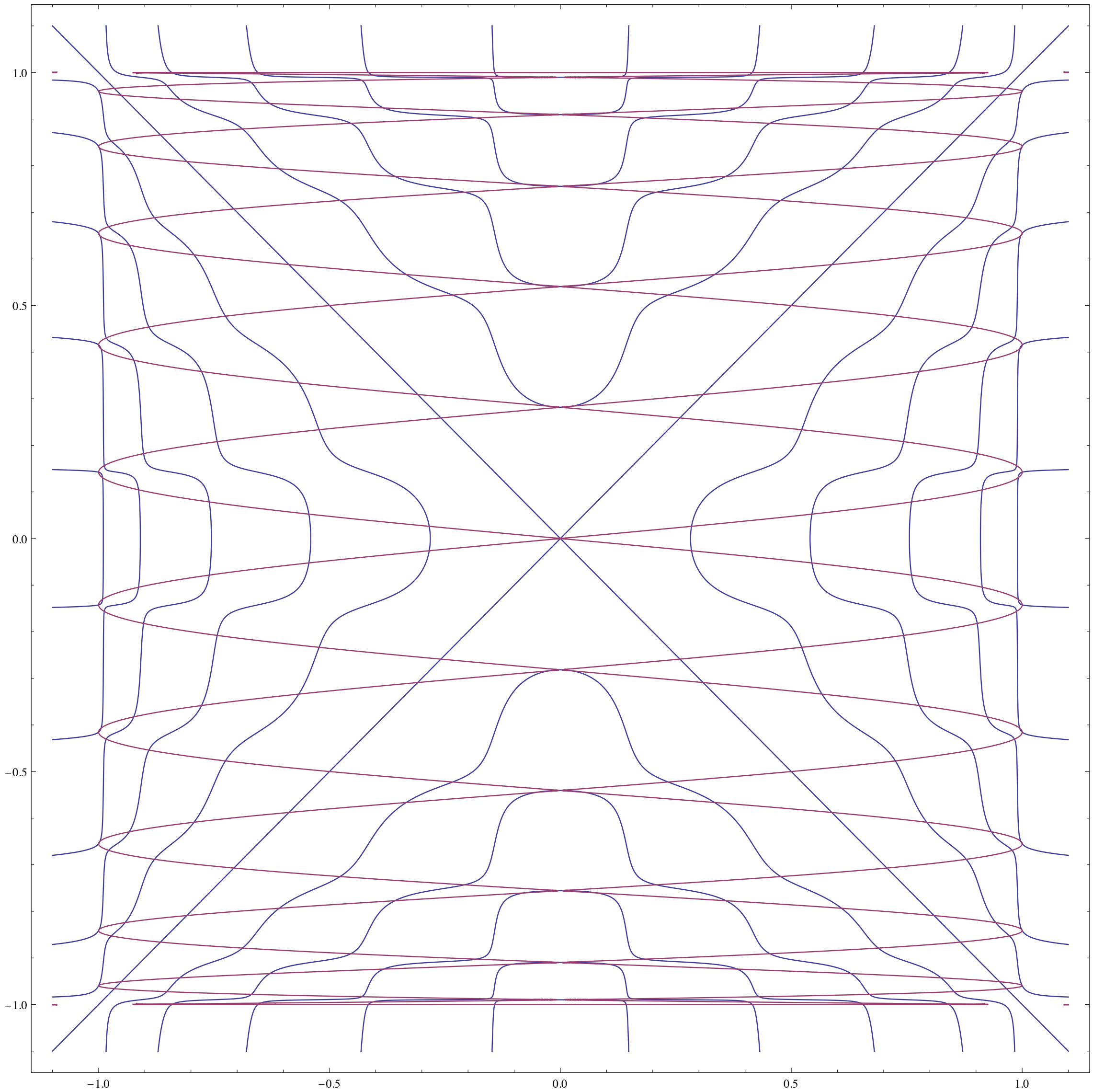


Figure 2.3: Projection of $\mathcal{T}$ and curves given by G_{22}.

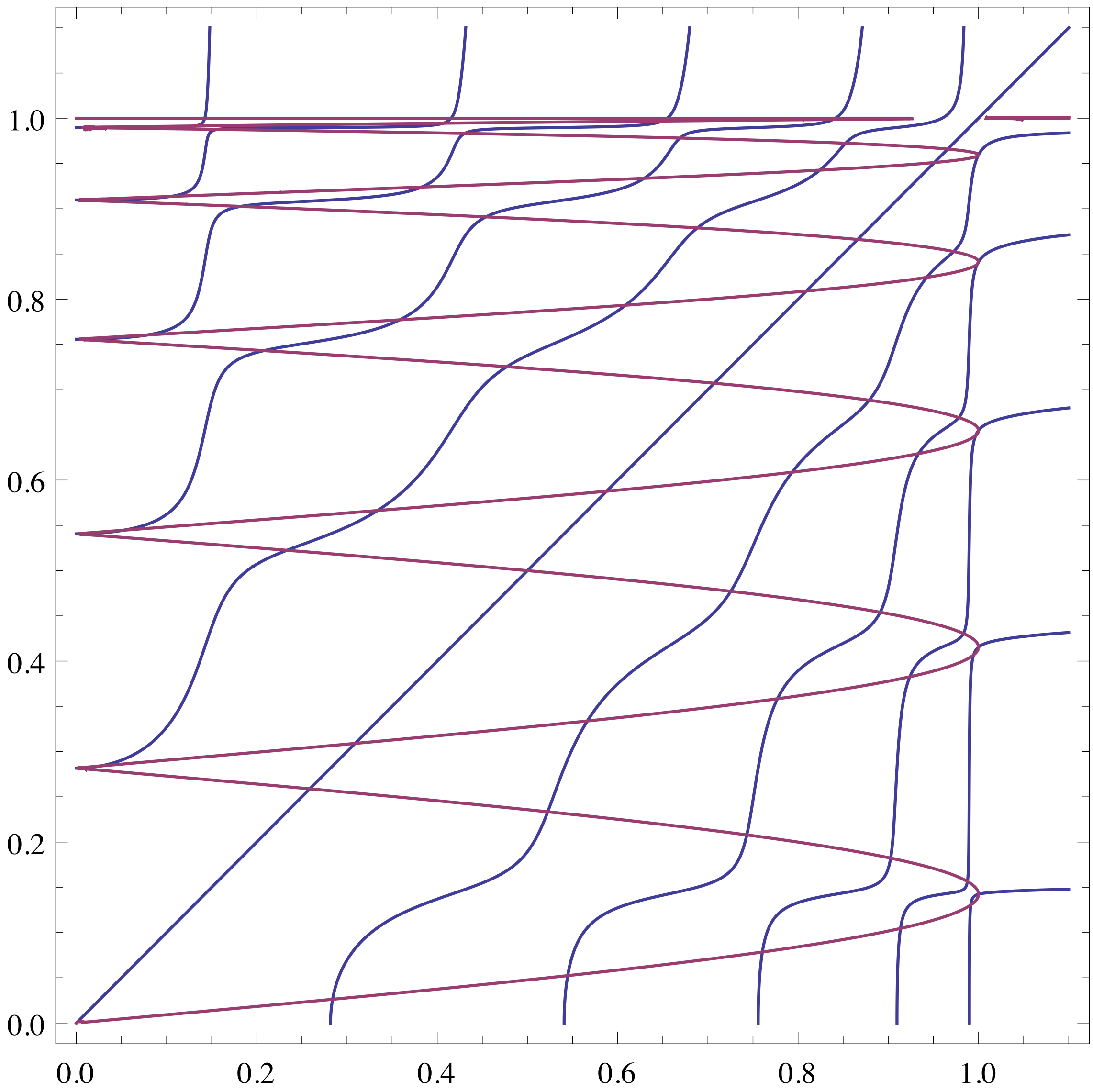


Figure 2.4: Projection of $\mathcal{T}$ and curves given by G_{22} in the first quadrant.

We now show a figure that contains one of each type of dual points as demonstrated in the previous section.

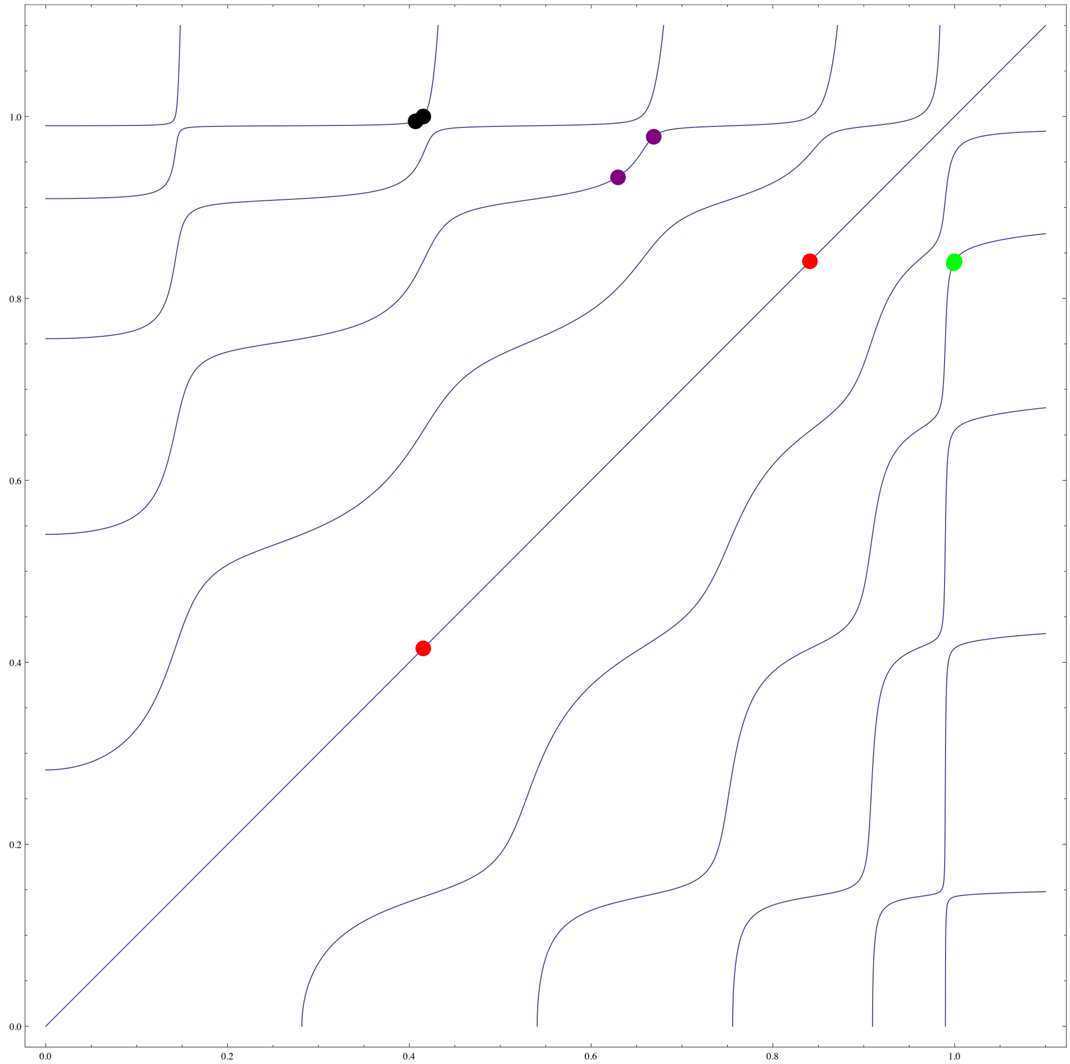


Figure 2.5: Examples of dual points on $G_{22} = 0$.

In the above, the points of the same color, except for the red points, are dual. The red points are examples of those discussed in Section 2.1. There are two green points, although they are close enough that it is hard to differentiate between them. These are given by the equation in Lemma 2.24, with $k = 3$. The purple points came from Theorem 2.22, with $k = 1$, and $i = 2$. The black points came from Theorem 2.23, with $i = 4$. This figure shows the curves of fixed points projected to the $x - y$ plane, the dual points, and the projection of $\mathcal{T}$.

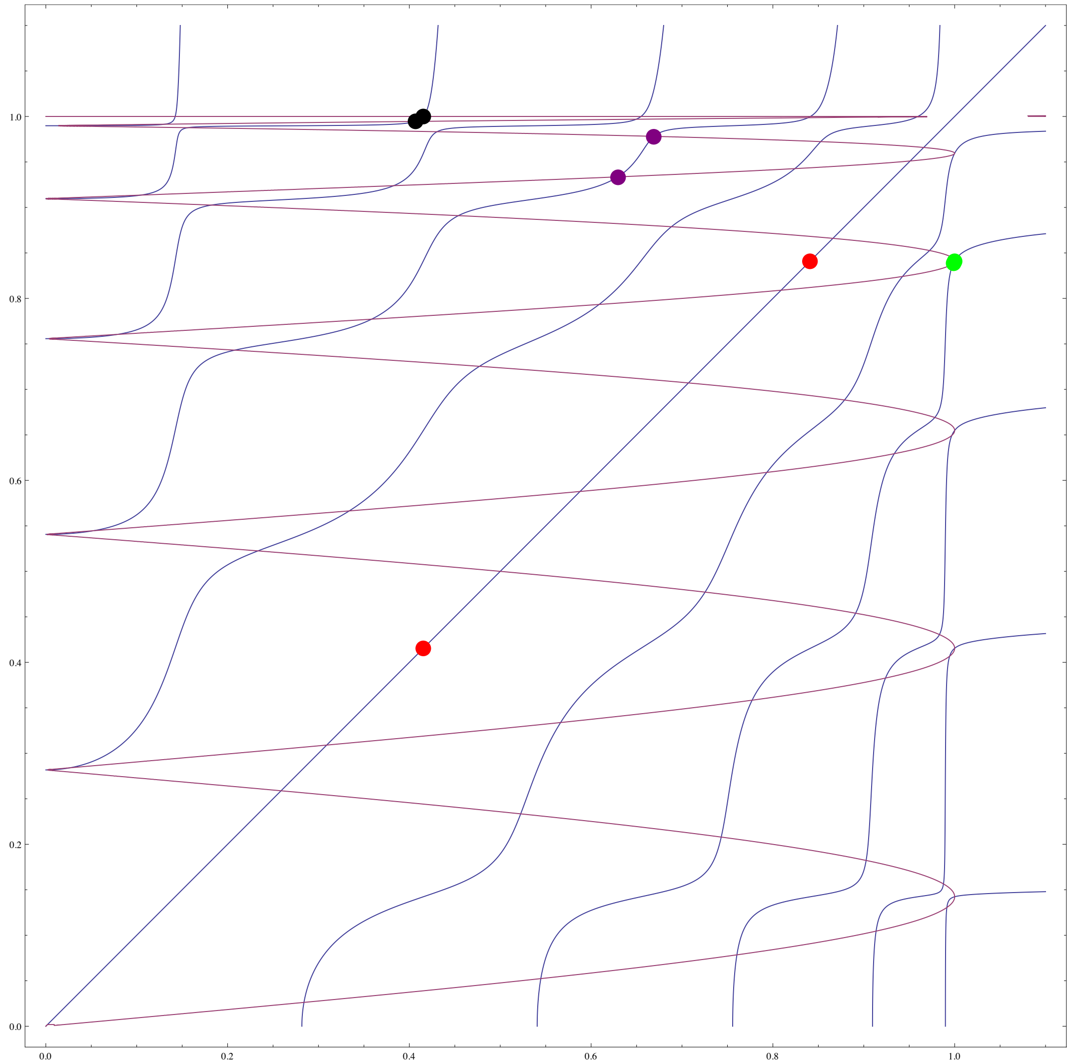


Figure 2.6: Projection of $\mathcal{T}$, curves given by G_{22} and dual points.

The next figure shows the curves of fixed points projected to the $x - y$ plane, the projection of $\mathcal{T}$, and every reversing fixed point, as well as preserving fixed points lying on the edge of the unit square.

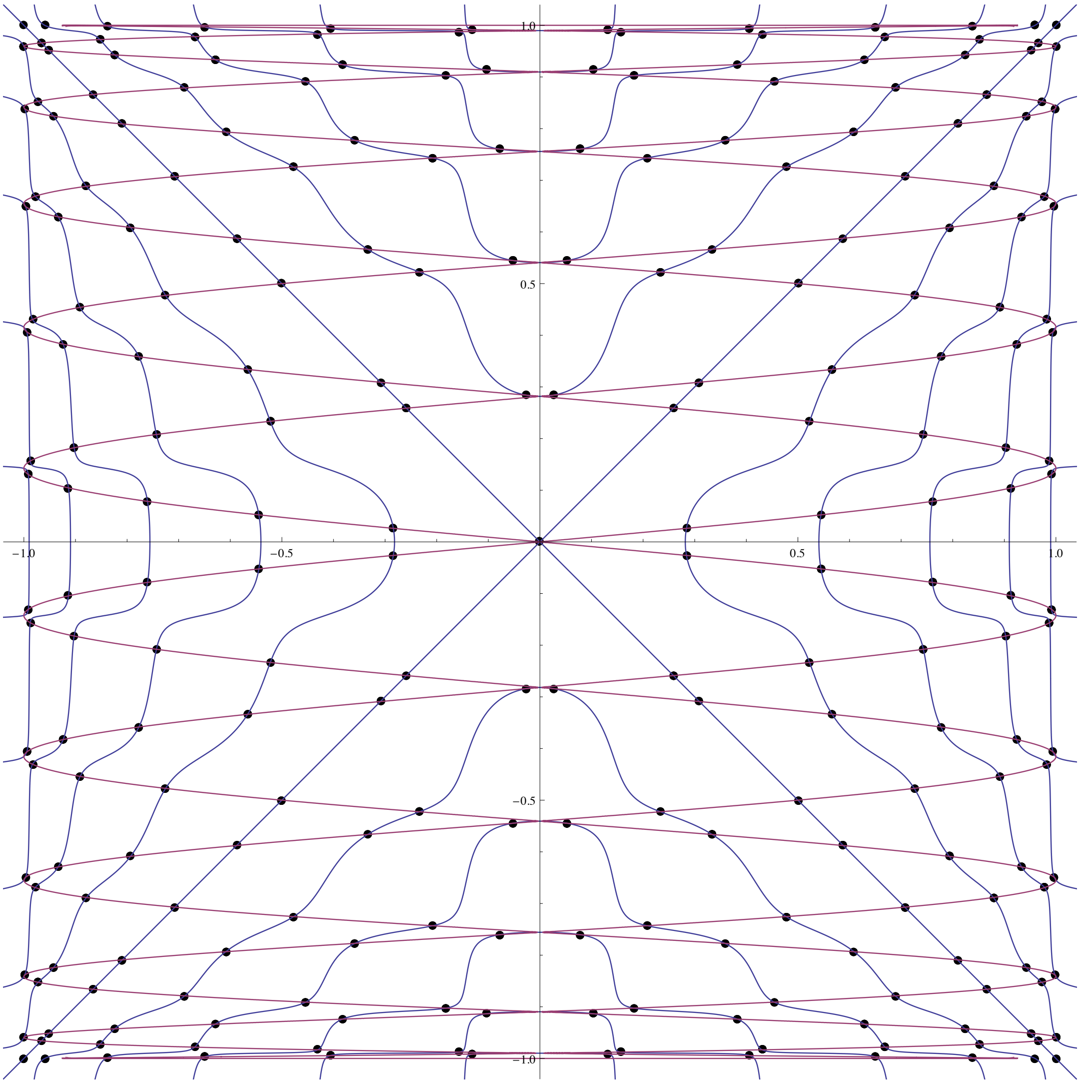


Figure 2.7: Projection of $\mathcal{T}$, curves given by G_{22}, and fixed points.

Finally, we have a copy of the previous image, with the lines $y = \cos(\frac{\pi j}{22})$ and $x = \cos(\frac{\pi k}{22})$, for j and k odd integers shown. These are the lines that determine the duality of the fixed points.

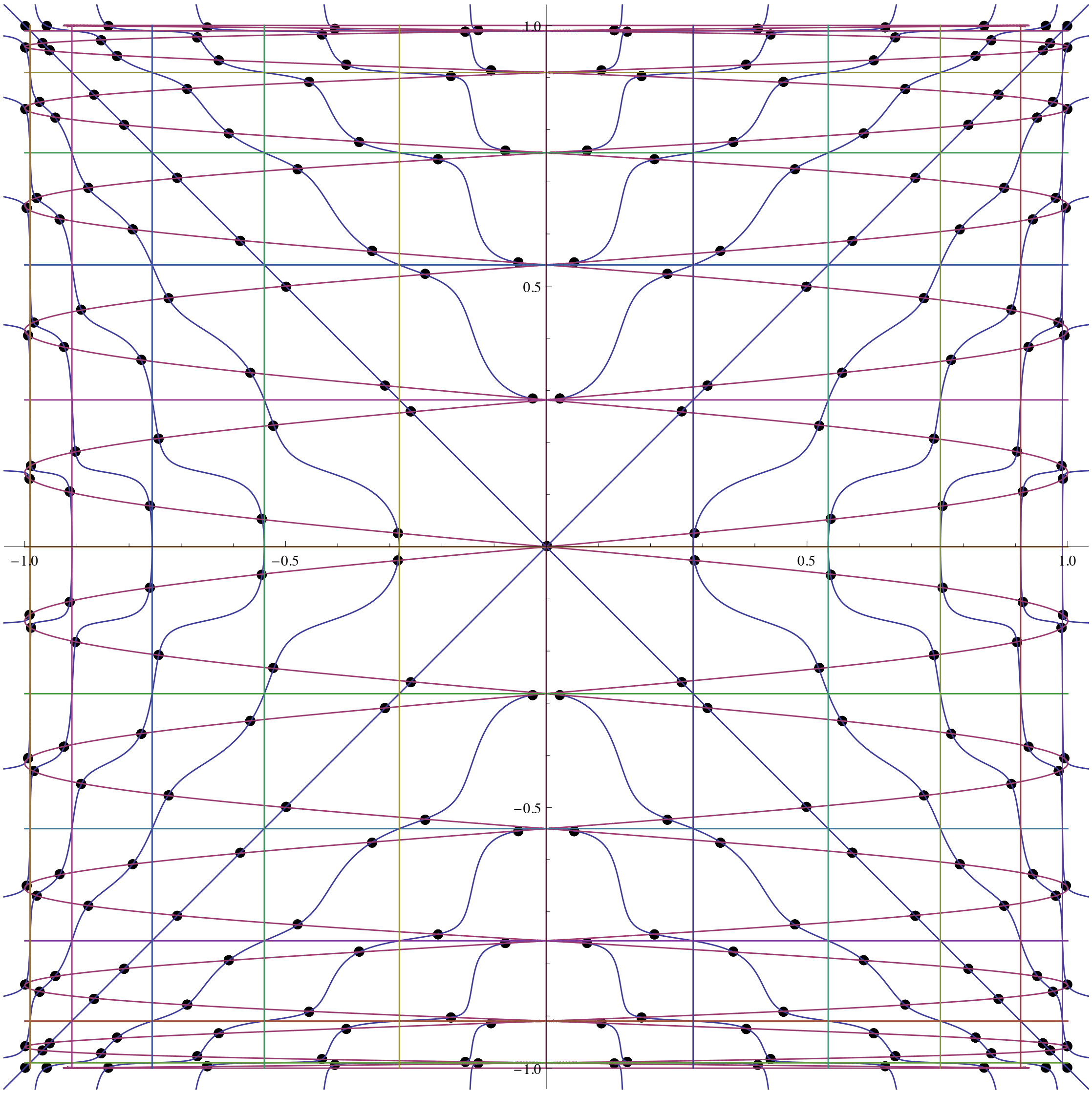


Figure 2.8: This image demonstrates the duality of each of the points in the previous image. the plotted lines are the lines $y = \cos(\frac{\pi j}{22})$ and $x = \cos(\frac{\pi k}{22})$, for j and k odd integers.

Chapter 3. The Fixed Points of the action of γ_{nm}

We now move to the more general case of investigating fixed points and duality for the maps $\gamma_{nm} = \tau_1^n \tau_2^m$. The action of γ_{nm} on $\mathbb{R}^3$ is given by the diffeomorphism $\sigma_1^n \sigma_2^m$. In this section, we will apply a very similar methodology as we did in the α_n case. The convenient thing about this is that many of the previous theorems hold true. For example, our solution for z in terms of Chebyshev polynomials will remain the same. Recall the following.

Theorem 3.1. *In the case where n and m are two positive integers, we obtain the following:*

$$M_1^n = \begin{pmatrix} -U_{n-2}(y) & 0 & U_{n-1}(y) \\ 0 & 1 & 0 \\ -U_{n-1}(y) & 0 & U_n(y) \end{pmatrix};$$
$$M_2^m = \begin{pmatrix} 1 & 0 & 0 \\ 0 & U_m(x) & -U_{m-1}(x) \\ 0 & U_{m-1}(x) & -U_{m-2}(x) \end{pmatrix}.$$

Proof. This follows exactly as in the α_n case. □

Lemma 3.2. *The action of σ_1^{-n} on the triple (x, y, z), is equivalent to the action of M_1^{-n} on (x, y, z), that is, $(x, y, z)\sigma_1^{-n} = M_1^{-n}(x, y, z)^T$. Similarly, $(x, y, z)\sigma_2^{-n} = M_2^{-n}(x, y, z)^T$.*

Proof. First, note that $(x, y, z)\sigma_1^{-1} = (2xy - z, y, x) = M_1^{-1}(x, y, z)^T$. From this, the result follows by induction, using the same method as in the proof of Lemma 2.1. □

It is easy to show that M_1^{-n} is given by substituting $-n$ in for n in M_1^n, with a similar result for M_2, so that we can proceed as we did in chapter 2. The following is an analogue of Lemma 2.2

Lemma 3.3. *For all $n, m \in \mathbb{Z}$ and $(x, y, z)^T \in \mathbb{R}^3$ we have*

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \sigma_1^n \sigma_2^m = \begin{pmatrix} -xU_{n-2}(y) + zU_{n-1}(y) \\ U_m(x^*)y - U_{m-1}(x^*)[-xU_{n-1}(y) + zU_n(y)] \\ U_{m-1}(x^*)y - U_{m-2}(x^*)[-xU_{n-1}(y) + zU_n(y)] \end{pmatrix}. \tag{3.1}$$

Where $x^* = -xU_{n-2}(y) + zU_{n-1}(y)$.

In particular, if $(x, y, z)^T \in \mathbb{R}^3$ *is a fixed point of the action of* γ_{nm} *and* $U_{n-1}(y) \neq 0$ *then*

$$z = x(1 + U_{n-2}(y))/U_{n-1}(y).$$

Proof. This follows as with the α_n case. □

3.1 The Straight Line Case for γ_{nm}

The solutions to the equation $U_{n-1}(y) = 0$ are given by $y = \cos(\frac{\pi j}{n})$. Assume (x, y, z) is a fixed point on $\mathcal{T}$ with $y = \cos(\frac{\pi j}{n})$ and $x = \cos(2\pi\theta_1)$. Then the preimage of this point in θ coordinates is $\pm(\theta_1, \frac{j}{2n})$. Consider the following:

$$\begin{pmatrix} 1 & n \\ -m & 1-nm \end{pmatrix} \begin{pmatrix} \theta_1 \\ \frac{k}{2n} \end{pmatrix} = \begin{pmatrix} \theta_1 + \frac{k}{2} \\ -m\theta_1 + \frac{k}{2n} - m\frac{k}{2} \end{pmatrix}.$$

From this, we can see that $\Phi(\gamma_{nm})(\theta_1, \frac{k}{2n})^T = \pm(\theta_1, \frac{k}{2n})^T \mod \mathbb{Z}^2$ if and only if k is an even integer, and $\theta_1 = \frac{j}{m}$, for some integer j. This yields a set of preserving fixed points of the action of γ_{nm} restricted to $\mathcal{T}$. That is, the set $\left\{\Pi\begin{pmatrix} \frac{j}{m} \\ \frac{k}{n} \end{pmatrix}\right\}$, for $j, k \in \mathbb{Z}$, is a set of preserving fixed points of the action of γ_{nm} restricted to $\mathcal{T}$. This set includes the only fixed points of γ_{nm} on $\mathcal{T}$ where $U_{n-1}(y) = 0$.

Proposition 3.4. *Let* n*,* j*,* k*, and* $m \in \mathbb{N}$*, with* $j \not\equiv 0 \mod \frac{m}{2}$*,* $k \not\equiv 0 \mod \frac{n}{2}$ *and* n, m *even. Let* $a = \cos\left(\frac{2\pi j}{m}\right)$*, and let* $b = \cos\left(\frac{2\pi k}{n}\right)$*.*

(i) We have $M_1^n(b) = M_2^m(a) = I_3$*.*

(ii) If $v = \left(\cos\left(\frac{2\pi j}{m}\right), \cos\left(\frac{2\pi k}{n}\right), z\right)^T$*, then* $v\gamma_{nm} = v$*. Thus the vertical line* $p(z) = \left(\cos\left(\frac{2\pi j}{m}\right), \cos\left(\frac{2\pi k}{n}\right), z\right)^T$ *is a curve of fixed points of* γ_{nm}*.*

(iii) If $j \equiv 0 \mod \frac{m}{2}$ *or if* $k \equiv 0 \mod \frac{n}{2}$*, then* $\left(\cos\left(\frac{2\pi j}{m}\right), \cos\left(\frac{2\pi k}{n}\right), z\right)^T$ *is not a curve of fixed points of* γ_{nm}*.*

Proof. The proof is almost identical to the proof of Proposition 2.4. □

Thus, again we have that under the conditions assumed at the beginning of Proposition 3.4, $p(z) = \left(\cos\left(\frac{2\pi j}{m}\right), \cos\left(\frac{2\pi k}{n}\right), z\right)^T$ is a vertical straight line of fixed points, linking the preserving fixed points $\Pi\left(\frac{j}{m}, \frac{k}{m}\right)^T$ and $\Pi\left(\frac{j}{m}, -\frac{k}{m}\right)^T$, making these two points dual in this case.

3.2 Fixed Points with $U_{n-1}(y) \neq 0$.

We begin our analysis of the fixed points $(x, y, z) \in \mathcal{T}$ with $U_{n-1}(y) \neq 0$. By Lemma 3.3, we end up with the following equations for these fixed points:

$$x = -xU_{n-2}(y) + zU_{n-1}(y);$$
$$y = U_m(x^*)y - U_{m-1}(x^*)[-xU_{n-1}y + zU_n(y)];$$
$$z = U_{m-1}(x^*)y - U_{m-2}(x^*)[-xU_{n-1}(y) + zU_n(y)].$$

For the rest of this section, let n, m be even integers, and let $k = n/2$, and let $l = m/2$.

Theorem 3.5. *A fixed point (x, y, z) of the action of γ_{nm} on $\mathbb{R}^3$, with $U_{n-1}(y) \neq 0$, is determined by the location of the x and y coordinates. The x and y coordinates satisfy the following equation:*

$$G(x, y) = U_{k-1}(y)U_{l-2}(x)(y^2 - 1)x + U_{k-1}(y)U_{l-1}(x)(x^2 - y^2) + U_{l-1}(x)U_{k-2}(y)(1 - x^2)y = 0, \tag{3.2}$$

where $2k = n$, and $2l = m$. The z coordinate satisfies $z = x(1 + U_{n-2}(y))/U_{n-1}(y)$.

Proof. We shall go through this derivation in detail. We start with Lemma 3.3. This gives us two equations in (x, y, z). Subtract y from the second equation, and z from the third. This will allow us to find the fixed points of these curves simply by finding the greatest common denominator of the two new equations, after substituting for z and taking the numerators. We start by substituting in for z as a function of x and y, and then factor the resulting equations. For the first equation, we obtain the following:

$$
\begin{aligned}
Y(x,y)-y=\frac{1}{U_{n-1}(y)}&(yU_m(x)U_{n-1}(y)+xUm-1(x)U_{n-1}(y)^2\\
&-xU_{m-1}(x)U_n(y)-xU_{m-1}(x)U_n(y)U_{n-2}(y)-yU_{n-1}(y).
\end{aligned}
$$

The second equation becomes:

$$
\begin{aligned}
Z(x,y)-z(x,y)=\frac{1}{U_{n-1}(y)}&(yU_{m-1}(x)U_{n-1}(y)+xU_{m-2}(x)U_{n-1}(y)^2\\
&-xU_{m-2}(x)U_n(y)-xU_{m-2}(x)U_n(y)U_{n-2}(y)-x-xU_{n-2}(y).
\end{aligned}
$$

We then focus on the numerator of the equations. We proceed by substituting for each $U_{n-2}(y)$ and $U_{m-2}(x)$ term, terms of the form U_{m-1}, U_m, U_{n-1}, and U_n, using the definition of the Chebyshev polynomial, (1.2). This gives the following:

$$
\begin{aligned}
n_Y=&\,yU_m(x)U_{n-1}(y)+xUm-1(x)U_{n-1}(y)^2-xU_{m-1}(x)U_n(y)\\
&-xU_{m-1}(x)U_n(y)(2yU_{n-1}(y)-U_n(y))-yU_{n-1}(y),\\
n_Z=&\,yU_{m-1}(x)U_{n-1}(y)+2x^2U_{m-1}(x)U_{n-1}(y)^2-xU_{n-1}(y)^2U_m(x)-2x^2U_{m-1}(x)\\
&U_n(y)+xU_n(y)U_m(x)-4x^2yU_n(y)U_{m-1}(x)U_{n-1}(y)\\
&+2x^2U_n(y)^2U_{m-1}(x)+2xyU_n(y)U_m(x)U_{n-1}(y)\\
&-xU_n(y)^2U_m(x)-x-2xyU_{m-1}(y)+xU_n(y).
\end{aligned}
$$

Recall that m and n are multiples of 2, where $n=2k$ and $m=2l$. We then use the Chebyshev identity formulas in Section 2 (see equations (1.11),(1.12)). This gives the following two formulas:

$$
\begin{aligned}
n_Y =& y(U_l(x)^2 - U_{l-1}(x)^2)(2U_k(y)U_{k-1}(y) - 2yU_{k-1}(y)^2) + (2U_l(x)U_l - 1(x) \\
& - 2xU_{l-1}(x)^2)x(2U_k(y)U_{k-1}(y) - 2yU_{k-1}(y)^2)^2 - (2U_l(x)U_{l-1}(x) - 2xU_{l-1}(x)^2) \\
& x(U_k(y)^2 - U_{k-1}(y)^2) - (2(2U_l(x)U_{l-1}(x) - 2xU_{l-1}(x)^2))x(U_k(y)^2 - U_{k-1}(y)^2)y \\
& (2U_k(y)U_{k-1}(y) - 2yU_{k-1}(y)^2) + (2U_l(x)U_{l-1}(x) - 2xU_{l-1}(x)^2)x(U_k(y)^2 \\
& - Uk - 1(y)^2)^2 - y(2U_k(y)U_{k-1}(y) - 2yU_{k-1}(y)^2), \\
n_Z =& y(2U_l(x)U_{l-1}(x) - 2xU_{l-1}(x)^2)(2U_k(y)U_{k-1}(y) - 2yU_{k-1}(y)^2) \\
& + 2x^2(2U_k(y)U_{k-1}(y) - 2yUk - 1(y)^2)^2(2U_l(x)U_{l-1}(x) - 2xU_{l-1}(x)^2) \\
& - x(2U_k(y)U_{k-1}(y) - 2yU_{k-1}(y)^2)^2(U_l(x)^2 - U_{l-1}(x)^2) - 2x^2(U_k(y)^2 \\
& - U_{k-1}(y)^2)(2U_l(x)U_{l-1}(x) - 2xU_{l-1}(x)^2) + x(U_k(y)^2 - U_{k-1}(y)^2)(U_l(x)^2 \\
& - U_{l-1}(x)^2) - 4x^2(U_k(y)^2 - U_{k-1}(y)^2)(2U_l(x)U_{l-1}(x) - 2xU_{l-1}(x)^2)y \\
& (2U_k(y)U_{k-1}(y) - 2yU_{k-1}(y)^2) + 2x^2(U_k(y)^2 - U_{k-1}(y)^2)^2(2U_l(x)U_{l-1}(x) \\
& - 2xU_{l-1}(x)^2) + 2x(U_k(y)^2 - U_{k-1}(y)^2)y(U_l(x)^2 - U_{l-1}(x)^2)(2U_k(y)U_{k-1}(y) \\
& - 2yU_{k-1}(y)^2) - x(U_k(y)^2 - U_{k-1}(y)^2)^2(U_l(x)^2 - U_{l-1}(x)^2) - x \\
& - 2xy(2U_k(y)U_{k-1}(y) - 2yU_{k-1}(y)^2) + x(U_k(y)^2 - U_{k-1}(y)^2).
\end{aligned}
$$

We now use equation (1.10) to simplify the previous equations. This yields the following two formulas:

$$
\begin{aligned}
n_Y =& - U_{l-1}(x)(-U_{k-1}(y)y^2U_{l-1}(x) + U_{k-1}(y)U_{l-1}(x)x^2 - U_{k-1}(y)U_l(x)x \\
& + U_{k-1}(y)xy^2U_l(x) + U_{l-1}(x)yU_k(y) - U_{l-1}(x)yU_k(y)x^2), \\
n_Z =& - U_{k-1}(y)U_l(x)U_{l-1}(x)y^2 + U_{k-1}(y)xU_{l-1}(x)^2y^2 + U_{k-1}(y)x^2U_l(x)U_{l-1}(x) \\
& + U_{k-1}(y)xy^2 - U_{k-1}(y)x + U_{k-1}(y)xU_{l-1}(x)^2 - 2U_{k-1}(y)x^3U_{l-1}(x)^2 \\
& + yU_l(x)U_{l-1}(x)U_k(y) - 2xyU_{l-1}(x)^2U_k(y) + 2x^3U_{l-1}(x)^2yU_k(y) \\
& - x^2U_l(x)U_{l-1}(x)yU_k(y).
\end{aligned}
$$

We now claim that the greatest common denominator of the two functions is the following:

$$G(x,y) = U_{k-1}(y)U_{l-2}(x)(y^2-1)x + U_{k-1}(y)U_{l-1}(x)(x^2-y^2) + U_{l-1}(x)U_{k-2}(y)(1-x^2)y.$$

If this is true, then there is some $f_1(x,y)$ and $f_2(x,y)$ such that $f_1(x,y)G(x,y) - n_Y = 0$, and $f_2(x,y)G(x,y) - n_Z = 0$. From examples we have calculated, we claim that $f_1(x,y) = U_{l-1}(x)$ and $f_2(x,y) = U_{l-2}(x)$.

We now show this is the case. Now $U_{l-1}(x)G(x,y) - n_Y = 0$ and $U_{l-2}(x)G(x,y) - n_Z = 0$ if and only if,

$$U_{l-1}(x)U_{l-2}G(x,y) - U_{l-2}n_Y = 0 \text{ and } U_{l-2}(x)U_{l-1}(x)G(x,y) - U_{l-1}(x)n_Z = 0.$$

But the last statement holds if and only if $U_{l-2}(x)n_Y - U_{l-1}(x)n_Z = 0$. We now prove this is the case. In this part, we will skip one step, where we rewrite $U_{l-2}(x)$ using the definition of the U-Type Chebyshev polynomials. We begin with the following.

$$U_{l-2}(x)n_Y - U_{l-1}(x)n_Z = U_{l-1}(x)U_{k-1}(y)x(y-1)(y+1)(U_{l-1}(x)^2 + U_l(x)^2 - 1 - 2xU_l(x)U_{l-1}(x)).$$

We now examine the last part of the equation $U_{l-1}(x)^2 + U_l(x)^2 - 1 - 2xU_l(x)U_{l-1}(x)$. By one of the Chebyshev identities (see (1.10)), we have

$$U_{l-1}(x)^2 - 1 - 2xU_l(x)U_{l-1}(x) = -U_l(x)^2.$$

This allows us to conclude that $U_{l-2}(x)n_Y - U_{l-1}(x)n_Z = 0$.

Thus we find that

$$G(x,y) = U_{k-1}(y)U_{l-2}(x)(y^2-1)x + U_{k-1}(y)U_{l-1}(x)(x^2-y^2) + U_{l-1}(x)U_{k-2}(y)(1-x^2)y,$$

is the greatest common denominator of n_Y and n_Z. □

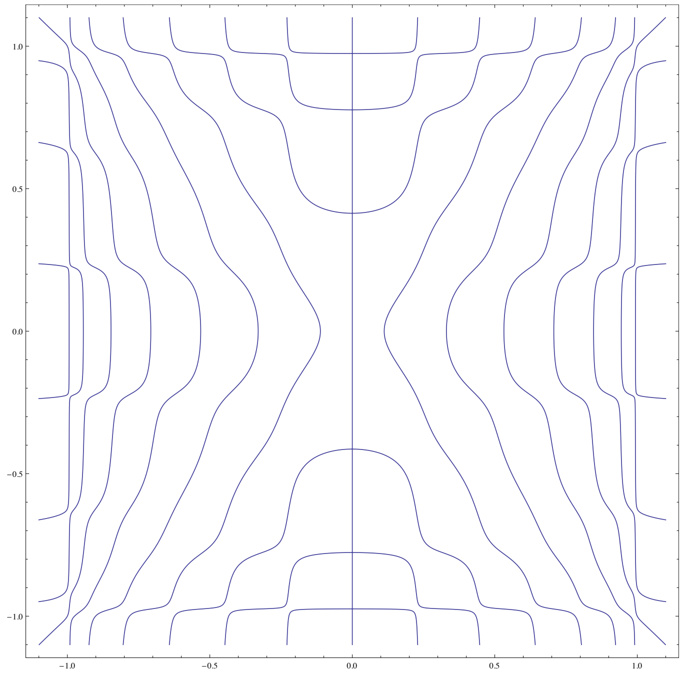


Figure 3.1: The curves given by $G(x, y) = 0$ for $n = 14$ and $m = 28$.

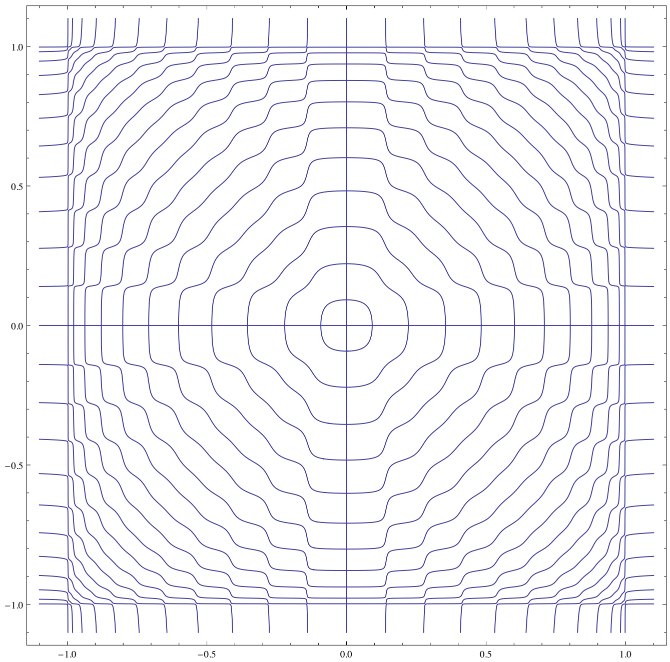


Figure 3.2: The curves given by $G(x, y) = 0$ for n=44 and m=-44.

The fixed points of γ_{nm} on $\mathcal{T}$, with x and y coordinates given by the intersection of the curves $G(x, y) = 0$ and the curves $E(x, y, z) = 1$ with the substitution $z = x(1 + U_{n-2}(y))/U_{n-1}(y)$, which are the fixed points that were not discussed in Section 3.1, are precisely the reversing fixed points of the action of γ_{nm}. This follows by a proof similar to that of Lemma 2.13.

3.3 Analysis of the curves given by $G(x, y)$.

Now that we have $G(x, y)$, we can perform some of the same analyses that we performed in the α_n case. In particular, we will investigate geometric properties of the x and y coordinates of the fixed points of γ_{nm}. We begin by discussing the symmetry of the system.

Theorem 3.6. *The curves given by the zeroes of $G(x, y)$ are symmetric about the x-axis and the y-axis. They are symmetric about the lines $y = \pm x$, if $k = \pm l$.*

Proof. We note that

$$G(-x, y) = U_{k-1}(y)U_{l-2}(-x)(y^2-1)(-x) + U_{k-1}(y)U_{l-1}(-x)(x^2-y^2) \\ + U_{l-1}(-x)U_{k-2}(y)(1-x^2)y.$$

By Lemma 2.15, we have

$$G(-x, y) = (-1)^{l-1}U_{k-1}(y)U_{l-2}(x)(y^2-1)(x) + (-1)^{l-1}U_{k-1}(y)U_{l-1}(x)(x^2-y^2) \\ + (-1)^{l-1}U_{l-1}(x)U_{k-2}(y)(1-x^2)y.$$

This clearly gives the same set of curves as $G(x, y)$. The symmetry about the y-axis follows similarly.

Now, the case $k = l$ has already been done (see Lemma 2.18). Now let $k = -l$. Substituting into $G(x, y)$, gives

$$\begin{aligned} G(y, x) &= U_{-l-1}(x)U_{l-2}(y)(x^2-1)y + U_{-l-1}(x)U_{l-1}(y)(y^2-x^2) \\ &\qquad + U_{l-1}(y)U_{-l-2}(x)(1-y^2)x \\ &= -U_{l-1}(x)U_{l-2}(y)(x^2-1)y - U_{l-1}(x)U_{l-1}(y)(y^2-x^2) \\ &\qquad - U_{l-1}(y)U_{l}(x)(1-y^2)x \\ &= U_{l-1}(x)U_{l-2}(y)(1-x^2)y + U_{l-1}(x)U_{l-1}(y)(x^2-y^2) \\ &\qquad + U_{l-1}(y)U_{l}(x)(y^2-1)x \end{aligned}$$

$$= -G(x, y),$$

which concludes the proof. □

In the following, we use the notation $G(x, y, k, l)$, to refer to $G(x, y)$ with parameters k and l.

Theorem 3.7. *We have the following:*

$$G(x, y, k, l) = -G(x, y, -k, -l),$$

for any integers k and l.

Proof. We have that

$$G(x, y, -k, -l) = U_{-(k+1)}(y)U_{-l}(x)(y^2 - 1)x + U_{-(k+1)}(y)U_{-(l+1)}(x)(x^2 - y^2) \\ + U_{-(l+1)}(x)U_{-k}(y)(1 - x^2)y,$$

which simplifies to

$$x(y^2 - 1)U_{k-1}(y)U_l(x) + (1 - x^2)yU_k(y)U_{l-1}(X) + (x^2 - y^2)U_{k-1}(y)U_{l-1}(x),$$

by equation (1.14), after canceling the negatives. Substituting for $U_k(y)$ and $U_l(x)$ using equation (1.2), and simplifying yields the following equation:

$$-x(y^2 - 1)U_{k-1}(y)U_{l-2}(x) - y(1 - x^2)U_{k-2}(y)U_{l-1}(x) \\ + U_{k-1}(y)U_{l-1}(x)(x(y^2 - 1)2 + y(1 - x^2)^2y + (x^2 - y^2),$$

which clearly simplifies to $-G(x, y, k, l)$. □

We can explicitly solve for the points of intersection of $\mathcal{T}$, under the substituting $z = x(1 + U_{n-2}(y))/U_{n-1}(y)$, and the roots of $G(x, y)$. We note that when we write

(x, y, z) in terms of θ_1 and θ_2, we will have the same relations between θ_1 and θ_2 we had in the previous two examples. That is, one of the following holds.

$$\begin{aligned} \theta_2 &= \frac{r}{2}, \\ \theta_2 &= \frac{r}{n}, \\ \theta_1 &= -\frac{n}{2}\theta_2 + \frac{r}{2}, \end{aligned}$$

where $r \in \mathbb{Z}$. Note that we use the integer r instead of k as in the previous formulation of the theorem, because we reserve k as the integer equal to $\frac{n}{2}$.

Theorem 3.8. *In the case where the relation between θ_1 and θ_2 takes the form*

$$\theta_1 = -\tfrac{n}{2}\theta_2 + \tfrac{r}{2},$$

the intersection points of $\mathcal{T}$ substituting $z = x(1 + U_{n-2}(y))/U_{n-1}(y)$, and the roots of the function $G(x, y)$, are the solutions of the equation:

$$\frac{(-1)^{lr-r+1}}{2} \cos^2(2\pi\theta_2) \sin(4\pi k\theta_2) \sin(2\pi(lk-1)\theta_2) = 0. \tag{3.3}$$

Proof. We start by noting the following:

$$\begin{aligned} U_{k-1}(y) &= \frac{\sin(2\pi k\theta_2)}{\sin(2\pi\theta_2)}; \\ U_{l-1}(x) &= \frac{(-1)^{rl-r}\sin(2\pi lk\theta_2)}{\sin(2\pi k\theta_2)}; \\ U_{k-2}(y) &= \frac{\sin(2\pi(k-1)\theta_2)}{\sin(2\pi\theta_2)}; \\ U_{l-2}(x) &= \frac{(-1)^{r(l-1)-r}\sin(2\pi(l-1)k\theta_2))}{\sin(2\pi k\theta_2)}. \end{aligned}$$

Substituting into $G(x,y)$ yields

$$\begin{aligned}
&\frac{(-1)^{rl-r}}{\sin(2\pi\theta_2)}((\sin(2\pi k\theta_2)\cos(2\pi lk\theta_2) - \sin(2\pi lk\theta_2)\cos(2\pi k\theta_2))\sin^2(2\pi\theta_2)\\
&\qquad \cos(2\pi k\theta_2) + \sin(2\pi lk\theta_2)\cos^2(2\pi k\theta_2) - \sin(2\pi lk\theta_2)\cos^2(2\pi\theta_2)\\
&\qquad + \sin(2\pi lk\theta_2)(\sin(2\pi k\theta_2)\cos(2\pi\theta_2) - \sin(2\pi\theta_2)\cos(2\pi k\theta_2))\\
&\qquad \sin(2\pi k\theta_2)\cos(2\pi\theta_2))\\
&= \frac{(-1)^{rl-r}}{\sin(2\pi\theta_2)}(\sin(2\pi k\theta_2)\cos(2\pi lk\theta_2)\sin^2(2\pi\theta_2)\cos(2\pi k\theta_2) - \sin(2\pi lk\theta_2)\\
&\qquad \cos^2(2\pi k\theta_2)\sin^2(2\pi\theta_2) + \sin(2\pi lk\theta_2)\cos^2(2\pi k\theta_2) - \sin(2\pi lk\theta_2)\\
&\qquad \cos^2(2\pi\theta_2) + \sin(2\pi lk\theta_2)\sin^2(2\pi k\theta_2)\cos^2(2\pi\theta_2)\\
&\qquad - \sin(2\pi lk\theta_2)\sin(2\pi\theta_2)\cos(2\pi k\theta_2))\sin(2\pi k\theta_2)\cos(2\pi\theta_2))\\
&= \frac{(-1)^{rl-r}}{\sin(2\pi\theta_2)}(\cos^2(2\pi\theta_2)\sin(2\pi lk\theta_2)(\sin^2(2\pi k\theta_2) - 1) + \sin(2\pi k\theta_2)\sin(2\pi\theta_2)\\
&\qquad \cos(2\pi k\theta_2)(\cos(2\pi lk\theta_2)\sin(2\pi\theta_2) - \sin(2\pi lk\theta_2)\cos(2\pi\theta_2))\\
&\qquad + \sin(2\pi lk\theta_2)\cos^2(2\pi\theta_2)(1 - \sin^2(2\pi\theta_2)))\\
&= \frac{(-1)^{rl-r}}{\sin(2\pi\theta_2)}(-\cos^2(2\pi\theta_2)\sin(2\pi lk\theta_2)\cos^2(2\pi k\theta_2) + \sin(2\pi lk\theta_2)\cos^2(2\pi k\theta_2)\\
&\qquad \cos^2(2\pi\theta_2) + \sin(2\pi k\theta_2)\sin(2\pi\theta_2)\cos(2\pi k\theta_2)\sin(2\pi(lk-1)\theta_2)\\
&= \frac{(-1)^{rl-r}}{\sin(2\pi\theta_2)}(-\cos^2(2\pi\theta_2)\sin(2\pi k\theta_2)\sin(2\pi\theta_2)\cos(2\pi k\theta_2)\sin(2\pi(lk-1)\theta_2)\\
&= \frac{(-1)^{rl-r+1}}{2}\cos^2(2\pi\theta_2)\sin(4\pi k\theta_2)\sin(2\pi(lk-1)\theta_2).
\end{aligned}$$

This gives $G(x,y)$ in terms of θ_2, based on a relation between θ_1 and θ_2 given by the Fricke character. Hence, solving the

$$\tfrac{(-1)^{rl-r+1}}{2}\sin(4\pi k\theta_2)\sin(2\pi(lk-1)\theta_2) = 0,$$

and using the relations between θ_1 and θ_2 will give us the x and y coordinates of the intersection points of $\mathcal{T}$ and curves given by the zeroes of $G(x,y)$. $\square$

The solutions to the previous equation come from letting $\theta_2 = \frac{j}{2(lk-1)}$, and letting $\theta_2 = \frac{j}{4k} = \frac{j}{2n}$, for any $j \in \mathbb{Z}$. From this, we can easily determine θ_1 using the fact that $\theta_1 = -\frac{n}{2}\theta_2 + \frac{r}{2}$, for some $r \in \mathbb{Z}$. We now seek to find a simpler expression for

$G(x,y)$, inside of the unit square. Since we are inside the unit square, we can again write $x = \cos(2\pi\theta_1)$ and $y = \cos(2\pi\theta_2)$.

Theorem 3.9. *Inside of the unit square, we can rewrite $G(x,y)$ as*

$$\tan(2\pi\theta_2)\tan(2\pi k\theta_2) - \tan(2\pi\theta_1)\tan(2\pi l\theta_1). \tag{3.4}$$

Proof. Substituting for $x = \cos(2\pi\theta_1)$ and $y = \cos(2\pi\theta_2)$ into $G(x,y)$, and using standard trigonometric identities, we obtain the following.

$$\begin{aligned}\frac{-1}{\sin(2\pi\theta_2)\sin(2\pi\theta_1)}&(\sin(2\pi k\theta_2)\cos(2\pi\theta_1)\cos(2\pi l\theta_1)\cos^2(2\pi\theta_2)\sin(2\pi\theta_1)\\ &- \sin(2\pi k\theta_2)\cos(2\pi\theta_1)\cos(2\pi l\theta_1)\sin(2\pi\theta_1) + \sin(2\pi l\theta_1)\\ &\cos(2\pi\theta_2)\cos(2\pi k\theta_2)\sin(2\pi\theta_2) - \sin(2\pi l\theta_1)\cos(2\pi\theta_2)\cos(2\pi k\theta_2)\\ &\cos^2(2\pi\theta_1)\sin(2\pi\theta_2).\end{aligned}$$

But we can combine the first two terms inside the parentheses to reduce to the following:

$$\begin{aligned}&\sin(2\pi k\theta_2)\cos(2\pi\theta_1)\cos(2\pi l\theta_1)\sin(2\pi\theta_1)(\cos^2(2\pi\theta_2) - 1)\\ &\qquad = \sin(2\pi k\theta_2)\cos(2\pi\theta_1)\cos(2\pi l\theta_1)\sin(2\pi\theta_1)(-\sin^2(2\pi\theta_2)).\end{aligned}$$

and the second two similarly combine:

$$\begin{aligned}&\sin(2\pi l\theta_1)\cos(2\pi\theta_2)\cos(2\pi k\theta_2)\sin(2\pi\theta_2)(1 - \cos^2(2\pi\theta_1))\\ &\qquad = \sin(2\pi l\theta_1)\cos(2\pi\theta_2)\cos(2\pi k\theta_2)\sin(2\pi\theta_2)(-\sin^2(2\pi\theta_1)).\end{aligned}$$

Canceling then gives the following equation for our curves.

$$\begin{aligned}&\cos(2\pi\theta_1)\cos(2\pi l\theta_1)\sin(2\pi\theta_2)\sin(2\pi k\theta_2) - \cos(2\pi\theta_2\cos(2\pi k\theta_2)\\ &\qquad\qquad\qquad \sin(2\pi\theta_1)\sin(2\pi l\theta_1) = 0.\end{aligned}$$

This simplifies easily by dividing both sides by

$$\cos(2\pi\theta_1)\cos(2\pi\theta_2)\cos(2\pi k\theta_2)\cos(2\pi l\theta_1),$$

yields the following

$$\tan(2\pi\theta_2)\tan(2\pi k\theta_2) - \tan(2\pi\theta_1)\tan(2\pi l\theta_1),$$

which concludes the theorem. □

While this is a much simpler equation, one must be careful as it is not defined where the roots of the sine portions don't exist, i.e., multiples of π/k and π/l.

Our goal now is to characterize what happens to the set of curves given by $G(x, y) = 0$, outside of the unit square. We will show that if k and l are positive even integers, there are $k + 1$ points of intersection of the curves given by $G(x, y)$ and the line $x = 1$ with $-1 \leq y \leq 1$, and $l + 1$ points of intersection of these curves with the line $y = 1$ with $-1 \leq x \leq 1$. Then we will show that the curves do not intersect outside of the unit square. There is an analogous result for different signs for k and l.

Applying Lemma 2.14, we see that on the line $x = 1$,

$$G(1, y) = U_{k-1}(y)(l + 1)(y^2 - 1) + U_{k-1}(y)l(1 - y^2) + lU_{k-2}(y)(1 - 1^2)y.$$

The last term is 0, and we can easily combine the first two terms into the equation

$$(y^2 - 1)U_{k-1}(y).$$

Using our regular trigonometric substitution, which is equation (1.8), we obtain the equation

$$G(1, \cos(2\pi\theta_2)) = -\sin(2\pi\theta_2)\sin(2\pi k\theta_2). \tag{3.5}$$

Letting $y = \cos(2\pi\theta_2)$, we see that equation (3.5) yields $k + 1$ distinct points where the level curves intersect the line $x = 1$.

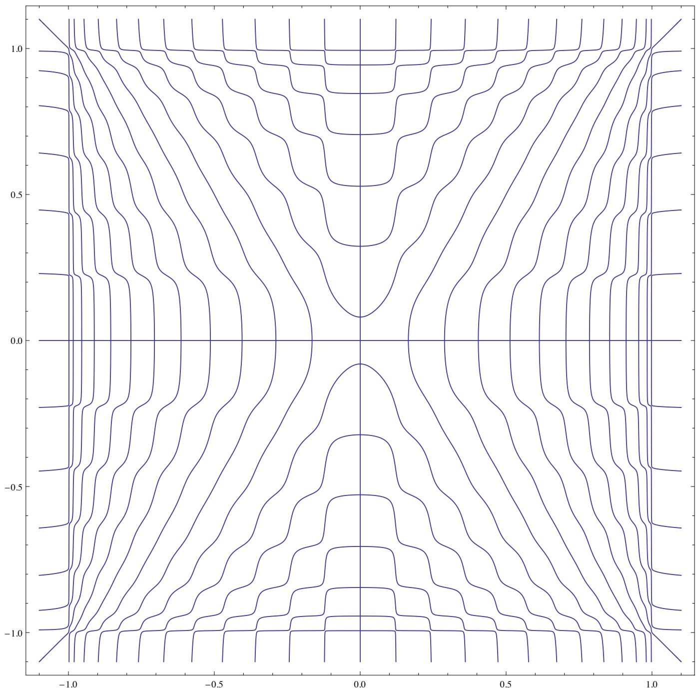


Figure 3.3: Solutions to $G(x, y) = 0$, where $n = 28$ and $m = 52$.

The preceding picture shows the curves given by $G(x, y) = 0$, where $n = 28$ and $m = 52$. One can easily count the curves exiting the top and right edges of the unit square, and see that there are 15 curves exiting the right side, and 27 exiting the top. We now must show that the curves given by $G(x, y) = 0$ do not intersect outside of the unit square. Again we will consider the $k + 1$ curves entering from the right. The result for the $l + 1$ curves follow identically.

Consider the solutions for the y values we obtained at $x = 1$. They all took the form $y = \cos(\pi j/k), j \in \{0, 1, ..., k\}$. We now show that the only times that a curve intersects these lines is when that curve lies in the unit square. This will imply that outside of the unit square, our curves cannot intersect the lines $y = \cos(\frac{\pi j}{k})$, and hence cannot intersect each other.

Substituting $y = \cos(\frac{\pi j}{k})$ in for y, and letting $x = \cos(2\pi\theta_1)$, we obtain the following:

$$G\left(\cos\left(\frac{\pi j}{k}\right), \cos(2\pi\theta_1)\right) = \frac{\sin(2\pi l\theta_1)\sin\left(\frac{(k-1)\pi j}{k}\right)(1-\cos^2(2\pi\theta_1))\cos\left(\frac{\pi j}{k}\right)}{\sin(2\pi\theta_1)\sin\left(\frac{\pi j}{k}\right)}.$$

This simplifies to

$$\pm \sin(2\pi l\theta_1)\sin(2\pi\theta_1)\cos(\pi j/k).$$

This has roots $\theta_1 = \frac{i}{2l}, i \in (0, 1, ...l)$. This yields gives us $l + 1$ distinct solutions. As these all lie in the unit square, and because $G(x, y)$ has degree $l + 1$ in the x variable, we conclude that no curve intersects the lines $y = \cos\left(\frac{\pi j}{k}\right)$ outside of the unit square. So that the curves outside of the unit square cannot intersect.

We now note that the proof of the curves entering from the top follows identically to this one, and that the proof that the curves coming from the bottom and the left then follow by the symmetry of our curves. Thus, we have characterized the behavior of our curves outside of the unit square.

Conjecture 3.10. *Inside the unit square, the curves given by the solutions of the equation* $G(x, y) = 0$ *are nonintersecting for any choice of* n *and* m*.*

As yet, we have been unable to prove the preceding conjecture.

We would like to study the duality of the general case at this point, however, in general, the curves given by $G(x, y)$ are not symmetric about the lines $y = \pm x$. Therefore, while the curves cannot pass the lines $y = \cos\left(\frac{\pi q}{2l}\right)$, for q odd, unless $x = 0$, we do not have the symmetry that says they cannot pass the lines $x = \cos\left(\frac{\pi q'}{2k}\right)$. However, plugging in the element $\cos\left(\frac{\pi q'}{2k}\right)$, with q' odd, into $G(x, y)$, requires $y = \cos\left(\frac{\pi q}{2l}\right)$ for q some odd integer, in order for $G(x, y)$ to be 0. Therefore, we know that the fixed curves of the action of γ_{nm} cannot cross the lines $x = \cos\left(\frac{\pi q'}{2k}\right)$, where q' is some odd integer, unless $y = 0$. This gives sufficient restrictions to study duality, however, due to time and space constraints, the duality of the general case will not be discussed here. An analysis of the duality of the general case would follow the method of Section 2.4, although there may need to be multiple cases depending on the parities and signs of k and l.

Chapter 4. Points of Period Two

We now begin a discussion of period two points of the maps induced by γ_{nm}. We will not classify all of the period two points of these maps, as the equations become too unwieldy. We shall identify period two points having one of the following properties.

$$(x, y, z)\alpha \to (-x, y, -z)\alpha \to (x, y, z), \tag{4.1}$$

or

$$(x, y, z)\alpha \to (x, -y, -z)\alpha \to (x, y, z), \tag{4.2}$$

for $\alpha \in \langle \sigma_1, \sigma_2 \rangle$.

We begin with a couple of theorems from [2].

Lemma 4.1. *The automorphism $S \colon \mathbb{Q}[x, y, z]^3 \to \mathbb{Q}[x, y, z]^3$ given by $(x, y, z)S = (-x, y, -z)$ centralizes any $\alpha \in \langle \sigma_1, \sigma_2^2 \rangle$, so that $\alpha S = S\alpha$. Particularly, let m and n be even integers. We note that S centralizes both the diffeomorphism representing α_n and the diffeomorphisms representing γ_{nm}.*

Proof. We have

$$\begin{aligned}
(x, y, z)\sigma_1 S &= (z, y, 2yz - x)S = (-z, y, -2yz + x), \\
(x, y, z)S\sigma_1 &= (-x, y, -z)\sigma_1 = (-z, y, -2yz + x).
\end{aligned}$$

Showing this holds for σ_2^2 follows a similar pattern. □

Theorem 4.2. *Let $\alpha \in \langle \sigma_1, \sigma_2^2 \rangle$. suppose that $(x_0, y_0, z_0)\alpha = (-x_0, y_0, -z_0)$. Then (x_0, y_0, z_0) is a period two point for α.*

Proof. Let $S : (x, y, z)S = (-x, y, -z)$. Then by the previous result we know that S centralizes α. Thus, we have that

$$(x_0, y_0, z_0)\alpha = (-x_0, y_0, -z_0) = (x_0, y_0, z_0)S.$$

So

$$\begin{aligned}(-x_0, y_0, -z_0)\alpha &= (x_0, y_0, z_0)S\alpha \\ &= (x_0, y_0, z_0)\alpha S \\ &= (-x_0, y_0, -z_0)S \\ &= (x_0, y_0, z_0).\end{aligned}$$

The $(x, -y, -z)$ case follows similarly. $\square$

4.1 Period Two Points of the Action of γ_{nm}

We analyze the period two points of γ_{nm} which acts by the diffeomorphisms $\sigma_1^n\sigma_2^m$. We assume that n and m are integer multiples of 2, and thus Theorem 4.2 applies. Let $n = 2k$ and $m = 2l$. We find the period two points under the map induced by $\sigma_1^n\sigma_2^m$, that have the property that

$$(x, y, z)\sigma_1^n\sigma_2^m = (x, -y, -z) \tag{4.3}$$

or

$$(x, y, z)\sigma_1^n\sigma_2^m = (-x, y, -z). \tag{4.4}$$

We will start with the $(x, -y, -z)$ case. We have the following from Lemma 3.3

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \sigma_1^n\sigma_2^m = \begin{pmatrix} -xU_{n-2}(y) + zU_{n-1}(y) \\ U_m(x^*)y - U_{m-1}(x^*)[-xU_{n-1}(y) + zU_n(y)] \\ U_{m-1}(x^*)y - U_{m-2}(x^*)[-xU_{n-1}(y) + zU_n(y)] \end{pmatrix}, \tag{4.5}$$

where $x^* = xU_{n-2}(y) + zU_{n-1}(y)$.

We note that again, we obtain the solution $z = x(1+U_{n-2}(y))/U_{n-1}(y)$, for $U_{n-1}(y) \neq 0$, after solving the first equation for z. We then substitute z into the other two equations, and since we want $(x, y, z)\sigma_1^n\sigma_2^m = (x, -y, -z)$, we have the following two equations.

$$n_Y = y + U_m(x)y - U_{m-1}(x)[-xU_{n-1}(y) + zU_n(y)] = 0;$$

$$n_Z = z + U_{m-1}(x)y - U_{m-2}(x)[-xU_{n-1}(y) + zU_n(y)] = 0,$$

where we have noted that $x = x^*$, and z was left in the equation for simplicity.

Theorem 4.3. *Consider the equation:*

$$H(x, y) = -yT_l(x)U_k(y) + U_{k-1}(y)(-xU_{l-1}(x) + y^2U_l(x)) = 0. \tag{4.6}$$

If (x, y) is a solution to the previous equation, and $U_{n-1}(y) \neq 0$, there is a corresponding point $(x, y, z) \in \mathbb{R}^3$, which is a period two point of the action of γ_{nm}. The point (x, y, z) has the property that $z = x(-1 + U_{n-2}(y))/U_{n-1}(y)$ and $(x, y, z)\sigma_1^n\sigma_2^m = (x, -y, -z)$.

Proof. We proceed as usual by finding the greatest common denominators of the previous two equations. We label this common denominator $H(x, y)$. We start by looking at the numerators of n_Y and n_Z. We start by substituting for $U_{l-2}(x)$ and $U_{k-2}(y)$ according to the definition of the Chebyshev polynomial, and then we use the fact that m and n are multiples of two, and use equations (1.11) and (1.12) to write our equations in the following forms.

$$\begin{aligned}
n_Y =& 2(-(U_{k-1}(y)^2(U_{l-1}(x)^2(2x^2(2y^2+1)U_k(y)^2 + x^2 - y^2) - x((4y^2+2)U_k(y)^2 \\
&+ 1)U_{l-1}(x)U_l(x) + y^2(U_l(x)^2+1))) + yU_k(y)U_{k-1}(y)(U_{l-1}(x)^2(4x^2U_k(y)^2 - 1) \\
&- 4xU_k(y)^2U_{l-1}(x)U_l(x) + U_l(x)^2 + 1) + xU_{k-1}(y)^4U_{l-1}(x)(U_l(x) - xU_{l-1}(x)) \\
&+ 4xyU_k(y)U_{k-1}(y)^3U_{l-1}(x)(xU_{l-1}(x) - U_l(x)) - xU_k(y)^2(U_k(y)^2 - 1)U_{l-1}(x) \\
&(xU_{l-1}(x) - U_l(x)));
\end{aligned}$$

$$
\begin{aligned}
n_Z =& U_{k-1}(y)^2(xU_{l-1}(x)^2(-2(4x^2-1)(2y^2+1)U_k(y)^2-4x^2+4y^2+1)+4U_l(x)\\
& U_{l-1}(x)(2x^2(2y^2+1)U_k(y)^2+x^2-y^2)-x(((4y^2+2)U_k(y)^2+1)U_l(x)^2\\
& +4y^2-1))+xU_{k-1}(y)^4((1-4x^2)U_{l-1}(x)^2+4xU_l(x)U_{l-1}(x)-U_l(x)^2)\\
& +4xyU_k(y)U_{k-1}(y)^3((4x^2-1)U_{l-1}(x)^2-4xU_l(x)U_{l-1}(x)+U_l(x)^2)+4yU_k(y)\\
& U_{k-1}(y)(xU_{l-1}(x)^2((4x^2-1)U_k(y)^2-1)+U_l(x)U_{l-1}(x)(1-4x^2U_k(y)^2)\\
& +xU_k(y)^2U_l(x)^2+x)-x(U_k(y)^4((4x^2-1)U_{l-1}(x)^2-4xU_l(x)U_{l-1}(x)\\
& +U_l(x)^2)+U_k(y)^2((1-4x^2)U_{l-1}(x)^2+4xU_l(x)U_{l-1}(x)-U_l(x)^2+1)-1).
\end{aligned}
$$

We then simplify using equation (1.10). We note that when we factor n_Y over $H(x,y)$ in examples, that $n_Y/H(x,y) = T_l(x)$, and $n_Z/H(x,y) = T_{l-1}(x)$. we now show that in general,

$$n_Y/T_l(x) = n_Z/T_{l-1}(x),$$

which follows if

$$n_Y T_{l-1}(x) - n_Z T_l(x) = 0.$$

We substitute for the T-Type Chebyshev polynomials in terms of U-Type Chebyshev polynomials, and reducing the powers of U-Type Chebyshev polynomials using equation 1.10, gives us the fact that $n_Y T_{l-1}(x) - n_Z T_l(x) = 0$.

It remains to show that

$$n_Y/T_l(x) - (-yT_l(x)U_k(y) + U_{k-1}(y)(-xU_{l-1}(x) + y^2U_l(x))) = 0,$$

but this follows by the exact same method. We show that

$$n_Y - T_l(x)(-yT_l(x)U_k(y) + U_{k-1}(y)(-xU_{l-1}(x) + y^2U_l(x))) = 0.$$

This follows by substituting in U-Type Chebyshev polynomials for the T-Type Chebyshev polynomials, after which reducing the powers of the U-Type Chebyshev polynomials yields the results.

We conclude that the greatest common denominator of n_Y and n_Z is

$$(-yT_l(x)U_k(y) + U_{k-1}(y)(-xU_{l-1}(x) + y^2U_l(x))),$$

and that all points satisfying

$$(-yT_l(x)U_k(y) + U_{k-1}(y)(-xU_{l-1}(x) + y^2U_l(x))) = 0,$$

are period two points, and satisfy the condition that $(x, y, z)\sigma_1^n\sigma_2^m = (x, -y, -z)$. $\square$

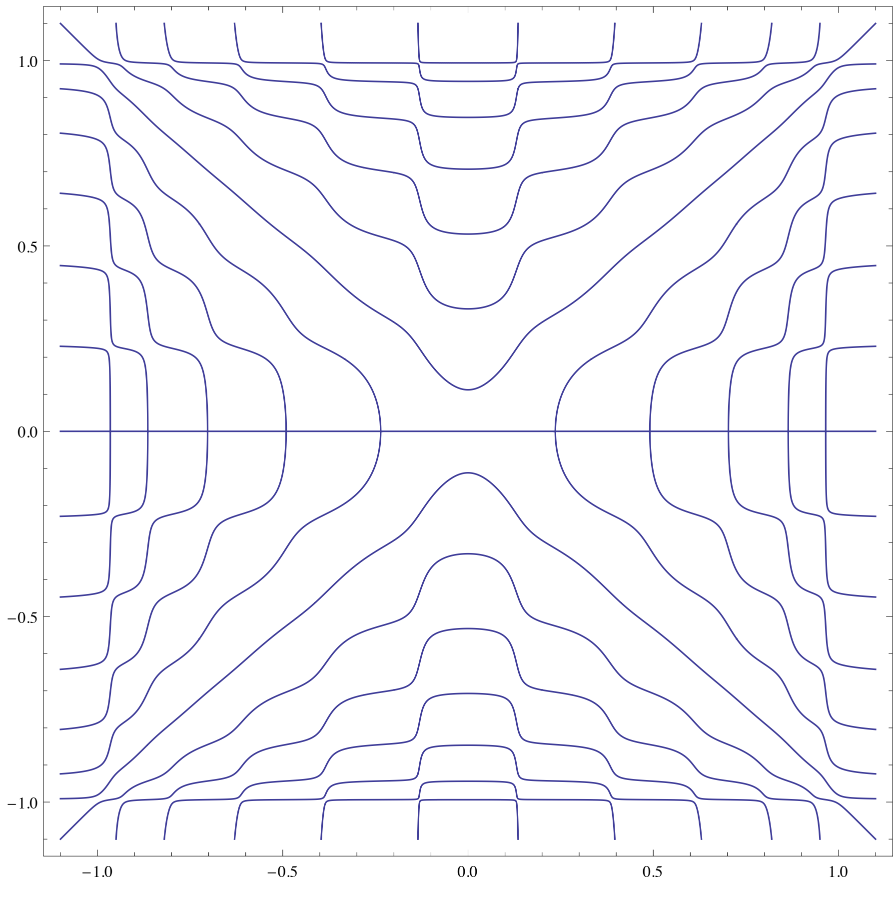


Figure 4.1: The x and y coordinates for period two points of the action of γ_{nm} obtained from Theorem 4.3, with $n = 28, m = 24$.

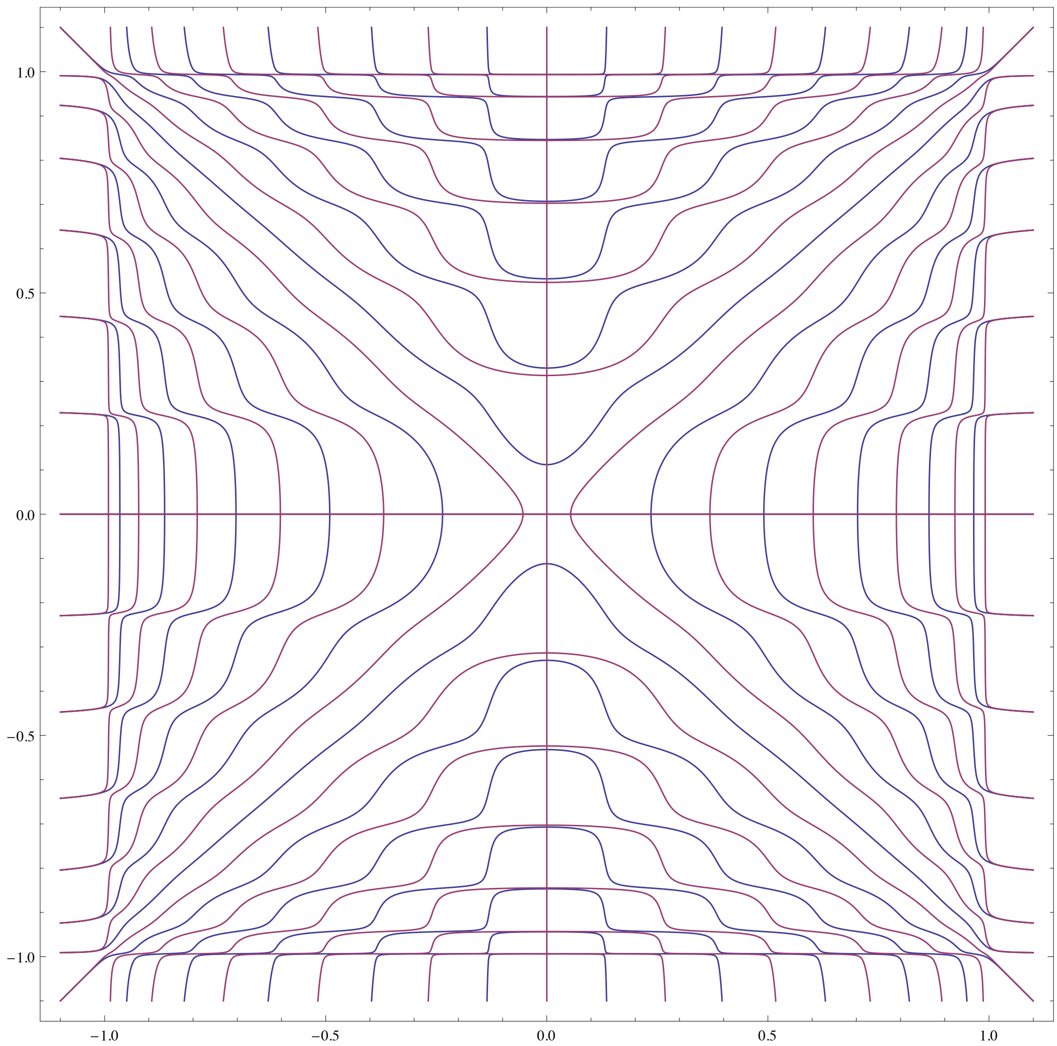


Figure 4.2: In blue, the points from the previous image. In red, the x and y coordinates of the fixed points of γ_{nm} given by equation (3.2) with $n = 28$ and $m = 24$.

We now assume that $(x, y, z)\gamma_{nm} = (-x, y, -z)$. This yields another set of period two points given by the following equations. The proof is omitted.

Theorem 4.4. *Consider the equation:*

$$H'(x, y) = -xT_k(y)U_l(x) + x^2 U_k(y)U_{l-1}(x) - yU_{k-1}(y)U_{l-1}(x) = 0. \tag{4.7}$$

If (x, y) is a solution to the previous equation, and $U_{n-1}(y) \neq 0$, there is a corresponding point $(x, y, z) \in \mathbb{R}^3$, which is a period two point of the action of γ_{nm}, with the property that $z = x(1 + U_{n-2}(y))/U_{n-1}(y)$ and $(x, y, z)\sigma_1^n \sigma_2^m = (-x, y, -z)$.

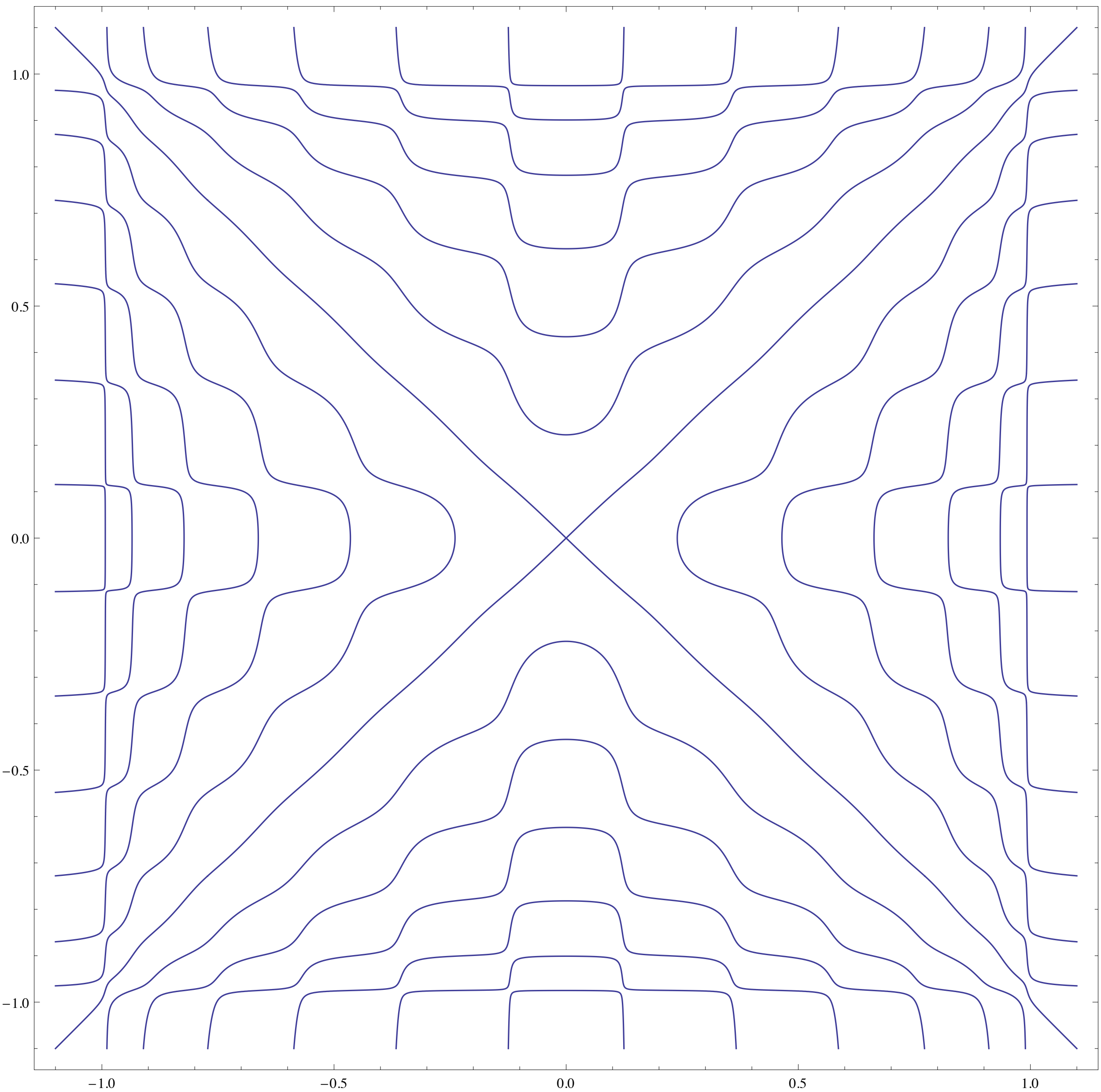


Figure 4.3: The x and y coordinates of period 2 points given by Theorem 4.4, with $n = 28$, and $m = 26$.

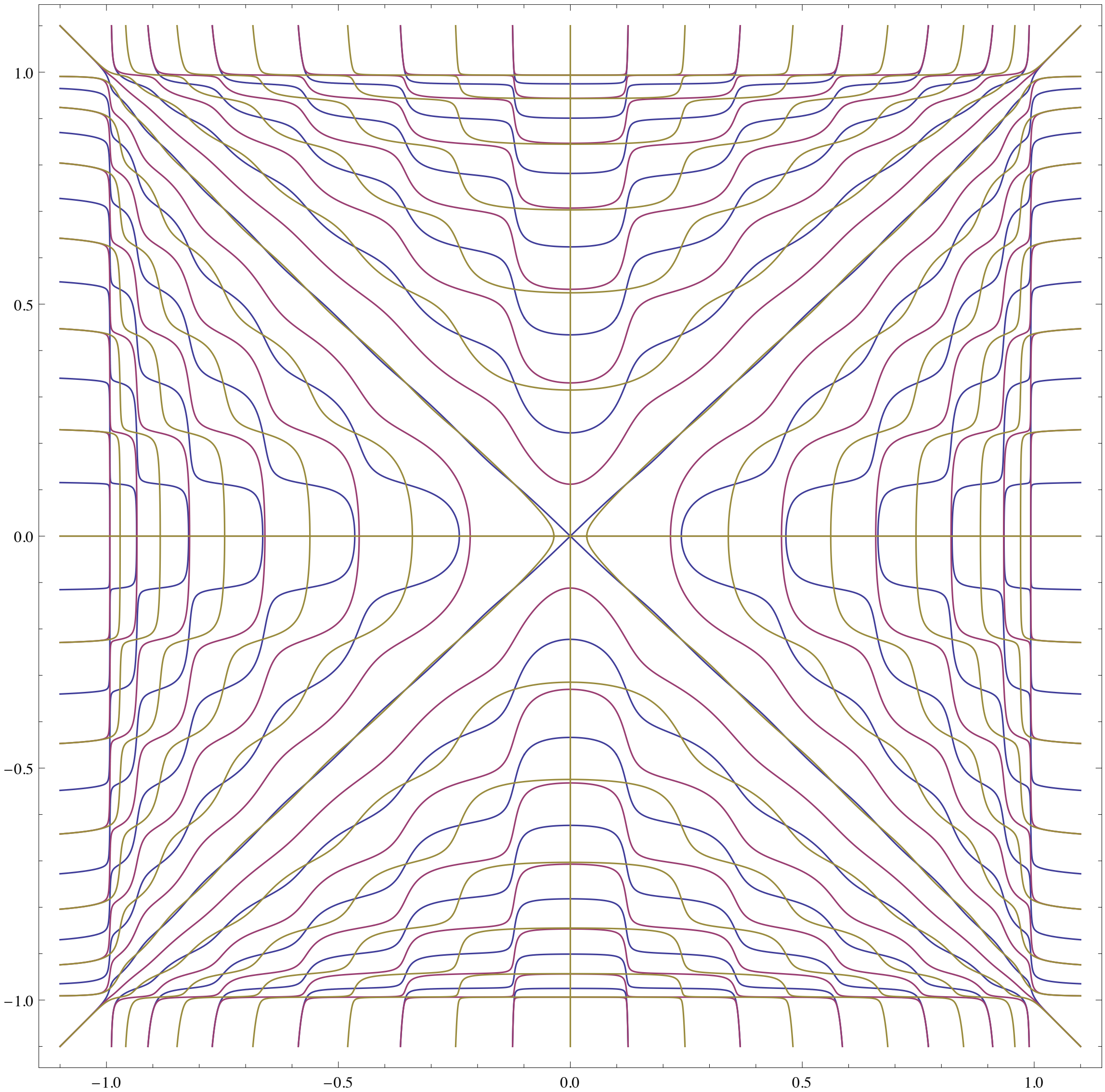


Figure 4.4: Here $n = 28$ and $m = 26$. The yellow curves are given by Theorem 4.3, the blue curves are given by Theorem 4.4, and the red curves are given by equation (3.2).

There are several geometric features to be noted in the above figures. Primarily, note that the x and y coordinate of the period 2 points, given by Theorems 4.3 and 4.4 lie between the curves given by equation (3.2). This happens for every example we have checked, although we have not proven this in general as of yet. Also note that there are points of intersection in the above figure, between the curves given by Theorems 4.3 and 4.4. Given the conditions on such points, this can only happen if $U_{n-1}(y) = 0$. In fact, if we plot those solutions as lines on the previous figure, we can verify that each intersection point does have y coordinate satisfying $U_{n-1}(y) = 0$.

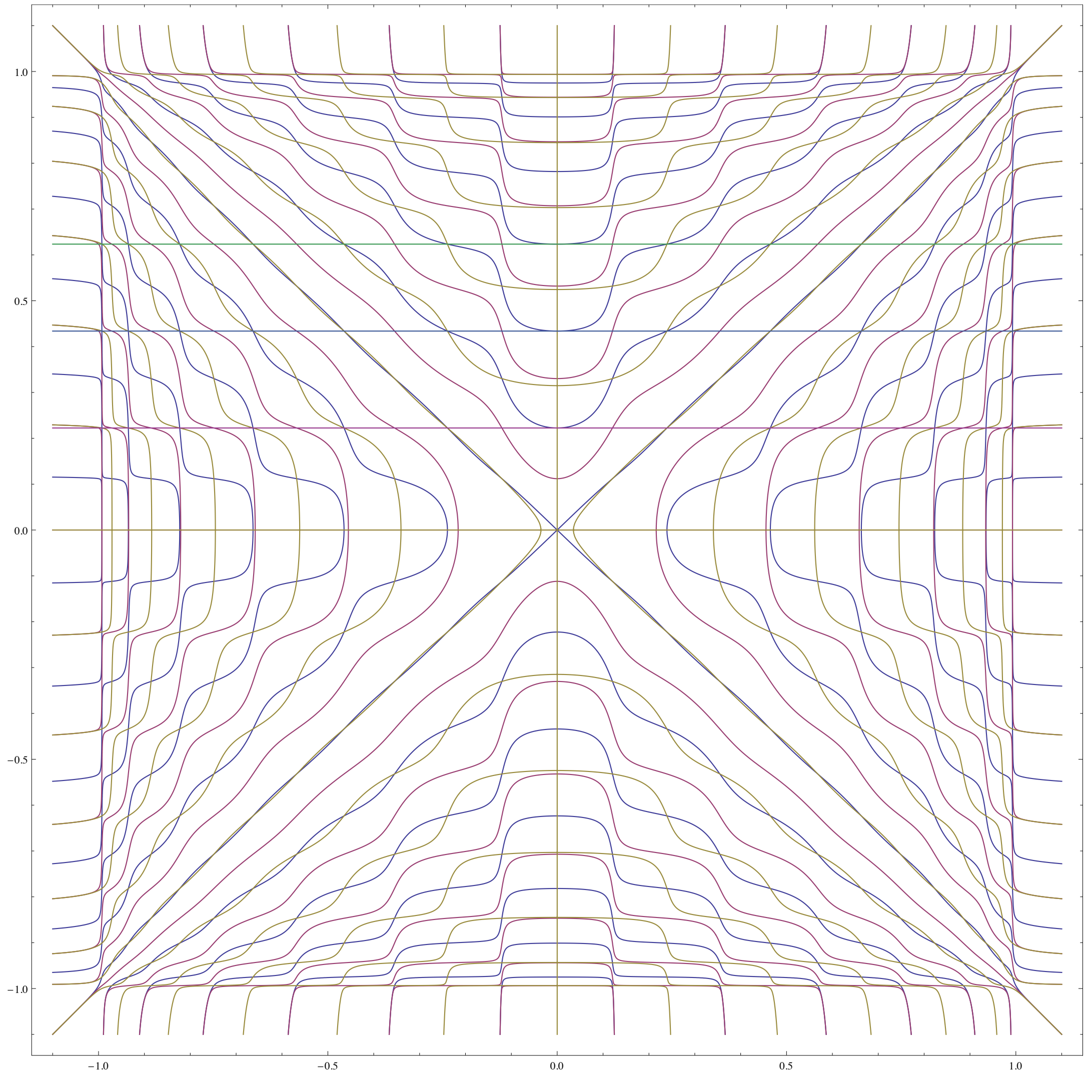


Figure 4.5: This shows the previous image plotted with the lines $y = \cos\left(\frac{\pi j}{28}\right)$, with $j = 8$, 10, and 12.

4.2 Period Two Points of the Element w_n

It is of some interest to consider the element $w_n = \sigma_1^{n-1}\sigma_2^{-1}\sigma_1^{-1} \in \langle \sigma_1, \sigma_2 \rangle$, because of its relation to the element $\sigma_1^n \sigma_2^n$ discussed in chapter 2.

Theorem 4.5.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} w_n^2 = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \alpha_n. \tag{4.8}$$

Proof. The actions of σ_1 and σ_2 are given by the matrices M_1 and M_2 given in Lemma 2.1. We note that the inverse of these matrices give the actions of σ_1^{-1} and σ_2^{-1} respectively, as per Lemma 3.2.

Applying the action of w_n twice gives the following:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} w_n^2 = \begin{pmatrix} x^* \\ -yU_{n-2}(x^*) + U_{n-1}(x^*)(-xU_{n-3}(y) + zU_{n-2}(y) \\ -yU_{n-3}(x^*) + U_{n-2}(x^*)(-xU_{n-3}(y) + zU_{n-2}(y)) \end{pmatrix}, \tag{4.9}$$

where $x^* = -xU_{n-2}(y) + zU_{n-1}(y)$.

We now show that the second two terms reduce to the terms in equation (2.1). For the first equation, we note that using the definition of the Chebyshev polynomials, it can be rewritten in the form

$$yU_n(x^*) - (U_{n-1}(x^*)(-xU_{k-1}(y) + zU_k(y)) + (2x^*y + 2xyU_{n-2}(y) - 2yzU_{n-1}(y))).$$

Thus, it remains to show that $2x^*y + 2xyU_{n-2}(y) - 2yzU_{n-1}(y) = 0$ Substituting back in for x^*, and expanding, we find that $(2x^*y + 2xyU_{n-2}(y) - 2yzU_{n-1}(y) = 0$. The second equation follows almost identically. □

This shows that the dual points of w_n are the fixed points of α_n, which we have already discussed.

In general, finding the period two points of the action of an element of B_3 is a difficult problem, which we have not been able to solve completely, except in the various special cases previously discussed. In particular, we can find all of the period two points of w_n, and a subset of the period two points of the action of γ_{nm}.

Chapter 5. Polynomials and Surfaces Induced by Trace Maps

We continue by expanding upon our knowledge of the group $\langle \sigma_1, \sigma_2 \rangle$. A significant portion of the following chapter, as well as the first theorem in chapter 2, which forms the supporting framework for the rest of the thesis, comes from [13].

We know that $\alpha \in \langle \sigma_1, \sigma_2 \rangle$ acts on $\mathbb{R}^3$ on the right through Nielsen transformations. We also know that $\alpha \in \langle \sigma_1, \sigma_2 \rangle$ acts on elements of $\mathbb{Q}[x, y, z]$ on the left by automorphisms. This can be expanded to an action on $\mathbb{Q}[x, y, z]^3$ by acting on each element of a triple in $\mathbb{Q}[x, y, z]^3$ individually. We show that these actions correspond in a sense described below.

Let σ_1 and σ_2 act on the left by automorphisms. Their action on $(x, y, z) \in \mathbb{Q}[x, y, z]^3$ is given by

$$\sigma_1(x, y, z) = (z, y, 2yz - x);$$

$$\sigma_2(x, y, z) = (x, 2xy - z, y).$$

The right action of σ_1 and σ_2 on $\mathbb{R}^3$ is given by

$$(X, Y, Z)\sigma_1 = (Z, Y, 2YZ - X);$$

$$(X, Y, Z)\sigma_2 = (X, 2XY - Z, Y),$$

where $(X, Y, Z) \in \mathbb{R}^3$. If we view (X, Y, Z) as elements of $\mathbb{Q}[x, y, z]^3$, we obtain an induced right action of α on $\mathbb{Q}[x, y, z]^3$. We will now show that the right action of α and the left action of α yield the same action on the triple $(x, y, z) \in \mathbb{Q}[x, y, z]^3$.

Lemma 5.1. *For $\alpha \in \langle \sigma_1, \sigma_2 \rangle$ we have*

$$\alpha(x, y, z) = (x, y, z)\alpha.$$

Proof. We induct on $|\alpha|$, which we define to be the length of α as a reduced word in the

σ_i generators. If $|\alpha| = 0, 1$ the statement holds by the definitions of the actions.

Assume the result holds for each α such that $|\alpha| < n$. Let $\alpha = \alpha_1\sigma_1$, with $|\alpha_1| = n-1$ so that $|\alpha| = n$. By the inductive hypothesis let

$$\alpha_1(x, y, z) = (\alpha_1(x), \alpha_1(y), \alpha_1(z)) = (X(x, y, z), Y(x, y, z), Z(x, y, z)) = (x, y, z)\alpha_1.$$

Then we have the following:

$$\begin{aligned}
(x, y, z)\alpha &= (x, y, z)\alpha_1\sigma_1 \\
&= (X, Y, Z)\sigma_1 \\
&= (Z, Y, 2YZ - X) \\
&= (\alpha_1(z), \alpha_1(y), 2\alpha_1(y)\alpha_1(z) - \alpha_1(x)) \\
&= \alpha_1(z, y, 2yz - x) \\
&= \alpha_1\sigma_1(x, y, z) \\
&= \alpha(x, y, z).
\end{aligned}$$

The remaining three cases, where $\alpha = \alpha_1\sigma_1^{-1}$, $\alpha = \alpha_1\sigma_2$, and $\alpha = \alpha_1\sigma_2^{-1}$ follow similarly.

□

Theorem 5.2. *There is an antiisomorphism from the group $\langle\sigma_1, \sigma_2\rangle$, where the elements of $\langle\sigma_1, \sigma_2\rangle$ are viewed as acting on the right, to the group $PSL(2, \mathbb{Z})$.*

Proof. Recall that for each element α of B_3, we have assigned a matrix using the antihomomorphism Φ, where

$$m_1 := \Phi(\tau_1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$
$$m_2 := \Phi(\tau_2) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix},$$

where τ_1 and τ_2 are the generators of B_3.

Let $\Phi' : \langle\sigma_1, \sigma_2\rangle \to PSL(2, \mathbb{Z})$, defined by $\Phi'(\sigma_1) = [m_1]$, $\Phi'(\sigma_2) = [m_2]$, where $[m_1]$

and $[m_2]$ denote the cosets containing m_1 and m_2. Now as m_1 and m_2 generate $SL(2,\mathbb{Z})$, $[m_1]$ and $[m_2]$ must generate $PSL(2,\mathbb{Z})$. Thus we can write every element of $PSL(2,\mathbb{Z})$ as a word in $[m_1]$ and $[m_2]$. Therefore, Φ' is a surjective map. We denote elements of $PSL(2,\mathbb{Z})$ as square bracketed matrices.

For $\alpha \in \langle \sigma_1, \sigma_2 \rangle$, let $(x,y,z)\alpha = (X,Y,Z)$ where $X,Y,Z \in \mathbb{Q}[x,y,z]$. Let $\Phi'(\alpha) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. From (1.5) we have that $\Phi'(\alpha) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ if and only if the elements in B_3 it represents, call them υ_i, have the property that $\Phi(\upsilon_i) = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, if and only if

$$X(\cos(2\pi\theta_1), \cos(2\pi\theta_2), \cos(2\pi(\theta_1+\theta_2))) = \cos(2\pi(a\theta_1 + b\theta_2)), \tag{5.1}$$

$$Y(\cos(2\pi\theta_1), \cos(2\pi\theta_2), \cos(2\pi(\theta_1+\theta_2))) = \cos(2\pi(c\theta_1 + d\theta_2)), \tag{5.2}$$

$$Z(\cos(2\pi\theta_1), \cos(2\pi\theta_2), \cos(2\pi(\theta_1+\theta_2))) = \cos(2\pi((a+c)\theta_1 + (b+d)\theta_2)). \tag{5.3}$$

Note that for both matrices $\pm\Phi(\upsilon_i)$, the equations above are the same.

We show that for $\alpha \in \langle \sigma_1, \sigma_2 \rangle$, $\Phi'(\alpha\sigma_i^j)$, for $i = 1,2$ $j = \pm 1$, is equal to $\Phi'(\sigma_i^j)\Phi'(\alpha)$. We proceed by induction on the length of α. Let $|\alpha|$ be as defined in the proof of Lemma 5.1. The $|\alpha| = 0, 1$ cases are clear.

Assume that equations (5.1)-(5.3) hold for α, where $(x,y,z)\alpha = (X,Y,Z)$ and $\Phi'(\alpha) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and consider $\beta = \alpha\sigma_1$. The other cases are similar.

Then $(x,y,z)\alpha = (X,Y,Z)$ gives $(x,y,z)\alpha\sigma_1 = (Z, Y, 2YZ - X) = (X', Y'Z')$. We have the following:

$$\Phi'(\sigma_1)\Phi'(\alpha) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ c & d \end{bmatrix} = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}. \tag{5.4}$$

We now show that (5.4) satisfies equations (5.1)-(5.3) for $\alpha\sigma_1$. By our inductive assumption, equations (5.1)-(5.3) hold true for α. Thus we have the following:

$$
\begin{aligned}
\cos(2\pi(a'\theta_1 + b'\theta_2) &= \cos(2\pi((a+c)\theta_1 + (b+d)\theta_2))) \\
&= Z(\cos(2\pi\theta_1), \cos(2\pi\theta_2), \cos(2\pi(\theta_1+\theta_2))) \\
&= X'(2\pi\cos(\theta_1), \cos(2\pi\theta_2), \cos(2\pi(\theta_1+\theta_2))).
\end{aligned}
$$

This satisfies equation (5.1). The proof that equation (5.2) is satisfied follows similarly. Finally:

$$
\begin{aligned}
&Z'(\cos(2\pi\theta_1), \cos(2\pi\theta_2), \cos(2\pi(\theta_1+\theta_2))) \\
&\quad =2Y(\cos(2\pi\theta_1, \cos(2\pi\theta_2), \cos(2\pi(\theta_1+\theta_2)))Z(\cos(2\pi\theta_1), \cos(2\pi\theta_2), \\
&\qquad \cos(2\pi(\theta_1+\theta_2))) - X(\cos(2\pi\theta_1), \cos(2\pi\theta_2), \cos(2\pi(\theta_1+\theta_2))) \\
&\quad = 2\cos(2\pi(c\theta_1 + d\theta_2))\cos(2\pi((a+c)\theta_1 + (b+d)\theta_2)) - \cos(2\pi(a\theta_1 + b\theta_2)) \\
&\quad = 2\cos(2\pi(c\theta_1 + d\theta_2))\cos(2\pi((a+c)\theta_1 + (b+d)\theta_2)) \\
&\qquad - \cos(2\pi([a+c]\theta_1 + [b+d]\theta_2] - [c\theta_1 + d\theta_2])). \qquad (5.5)
\end{aligned}
$$

We also have

$$
\cos((a'+c')\theta_1 + (b'+d')\theta_2) = \cos((a+2c)\theta_1 + (b+2d)\theta_2). \qquad (5.6)
$$

Let $u = a\theta_1 + b\theta_2, v = c\theta_1 + d\theta_2, w = u + v$. Then the right side of (5.6) equals $\cos(u+v)$, and (5.5) gives

$$
\begin{aligned}
2\cos(v)\cos(w) - \cos(w-v) &= 2\cos(v)\cos(w) - (\cos(w)\cos(v) + \sin(w)\sin(v)) \\
&= \cos(v)\cos(w) - \sin(w)\sin(v) = \cos(v+w).
\end{aligned}
$$

This is sufficient to show that $\Phi'(\alpha\sigma_1) = \Phi'(\sigma_1)\Phi'(\alpha)$. The remaining cases follow similarly.

Continuing with our proof that $\Phi' : \langle\sigma_1, \sigma_2\rangle \to PSL(2, \mathbb{Z})$ is an antiisomomorphism. We want to show that $\Phi'(\alpha\beta) = \Phi'(\beta)\Phi'(\alpha)$ for all $\alpha, \beta \in \langle\sigma_1, \sigma_2\rangle$. We induct on the length of β. The theorem clearly holds if $|\beta|= 0$, and if $|\beta|= 1$, the theorem holds by our previous result.

We know that $\beta=\beta_1\sigma_i^j$, where $|\beta_1| < |\beta|$. This yields the following:

$$\Phi'(\alpha\beta) = \Phi'(\alpha\beta_1\sigma_i^j) = \Phi'(\sigma_i^j)\Phi'(\alpha\beta_1) = \Phi'(\sigma_i^j)\Phi'(\beta_1)\Phi'(\alpha) = \Phi'(\beta)\Phi'(\alpha)$$

This shows that our map Φ' is an antihomomorphism onto $PSL(2, \mathbb{Z})$.

Now, it is easily seen that $(x, y, z)(\sigma_1\sigma_2)^3 = (x, y, z)$, so that the normal closure of $\langle(\sigma_1\sigma_2)^3\rangle$ lies in the kernel of $\langle\sigma_1, \sigma_2\rangle$. It turns out that this subgroup is central. We note that, by [14], $PSL(2, \mathbb{Z})$ has the presentation

$$\langle[m_1], [m_2] \mid [m_1][m_2][m_1] = [m_2][m_1][m_2], ([m_1][m_2])^3\rangle.$$

Now, as Φ' is a surjective antihomomorphism, and since $\langle\sigma_1, \sigma_2\rangle$ satisfies the braid relations $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$, and has kernel containing $\langle(\sigma_1\sigma_2)^3\rangle$, which is a central subgroup, Φ' must be an antiisomorphism. □

A consequence of the previous theorem is that to each coset $m \in PSL(2, \mathbb{Z})$ we can associate a triple $(X(m), Y(m), Z(m))$ of polynomials in $\mathbb{Q}[x, y, z]$ by choosing some $\alpha \in \langle\sigma_1, \sigma_2\rangle$ and letting $(X(m), Y(m), Z(m)) = (x, y, z)\alpha$, where $\Phi'(\alpha) = m$. We write $\Psi(m) = (X(m), Y(m), Z(m))$. So α corresponds to m. From this point forward, to reduce notational inconvenience, we will write the generators of $PSL(2, \mathbb{Z})$ as m_1 and m_2, so that $PSL(2, \mathbb{Z})$ has presentation $\langle m_1, m_2 \mid m_1m_2m_1 = m_2m_1m_2, (m_1m_2)^3\rangle$.

We now prove the following lemma:

Lemma 5.3. *Let $n_1, n_2 \in PSL(2, \mathbb{Z})$.*

(i) The following are equivalent:

(a) n_1 and n_2 have coset representatives having the same first row:

(b) $\Psi(n_1)$ and $\Psi(n_2)$ have the same first entry:

(c) $n_1 = \Phi'(\sigma_2^k)n_2$ for some $k \in \mathbb{Z}$.

(ii) n_1 and n_2 have coset representatives with the same second row if and only if $\Psi(n_1)$ and $\Psi(n_2)$ have the same second entry if and only if $n_1 = \Phi'(\sigma_1^k)n_2$ for some $k \in \mathbb{Z}$.

Proof. (i) suppose that n_1 and n_2 have coset representatives with the same first row: $n_1' = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $ad - bc = 1$, $n_2' = \begin{pmatrix} a & b \\ c' & d' \end{pmatrix}$, with $ad' - bc' = 1$. Then

$$n_1 n_2^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d' & -b \\ -c' & a \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -k & 1 \end{bmatrix}.$$

for some $k \in \mathbb{Z}$, which shows that $n_1 = \Phi'(\sigma_2^k)n_2$ for some $k \in \mathbb{Z}$. Thus (a) implies (c).

Let α_1, α_2 correspond to n_1, n_2 so that $\Phi'(\alpha_1) = n_1, \Phi'(\alpha_2) = n_2$. Now $n_1 = \Phi'(\sigma_2^k)n_2$ gives

$$\Phi'(\alpha_1) = \Phi'(\sigma_2^k)\Phi'(\alpha_2) = \Phi'(\alpha_2\sigma_2^k).$$

Thus, by Theorem 5.2, $\alpha_1 = \alpha_2\sigma_2^k$. Then $\Psi(n_1) = (X, Y, Z) = (x, y, z)\alpha_1$ and $\Psi(n_2) = (X', Y', Z')$. This gives

$$(X', Y', Z') = (x, y, z)\alpha_2 = (x, y, z)\alpha_1\sigma_2^{-k} = (X, Y, Z)\sigma_2^{-k}.$$

However, the action of σ_2 fixes the first entry so that the first rows of $\Psi(m_1)$ and $\Psi(m_2)$ are the same. So (c) implies (b).

Now assume that the first entries of $\Psi(m_1)$, and $\Psi(m_2)$ are the same. Let $n_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $n_2 = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$. Let $\Psi(m_1) = (X, Y, Z)$ and $\Psi(m_2) = (X', Y', Z')$. Let $\alpha_i, i = 1, 2$ correspond to the n_i. Then

$$\alpha_1(x, y, z) = (x, y, z)\alpha_1 = (X, Y, Z), \;\; \alpha_2(x, y, z) = (x, y, z)\alpha_2 = (X, Y', Z').$$

We have that $\alpha_1(x) = X = \alpha_2(x)$ and thus $\alpha_2^{-1}\alpha_1(x) = x$. Thus

$$(x, y, z)\alpha_2^{-1}\alpha_1 = \alpha_2^{-1}\alpha_1(x, y, z) = (x, Y_1, Z_1),$$

for some $Y_1, Z_1 \in \mathbb{Q}[x,y,z]$. By Theorem (5.2), we have

$$\Phi'(\alpha_2^{-1}\alpha_1) = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix},$$

which implies that $n_1 n_2^{-1} = \Phi'(\alpha_1)\Phi'(\alpha_2^{-1}) = \Phi'(\alpha_2^{-1}\alpha_1) = \Phi'(\sigma_2^k)$. So (b) implies (c). The proof of (ii) is similar. $\square$

Definition 5.4. Let

$$\chi = \{\alpha(x) \colon \alpha \in \langle \sigma_1, \sigma_2 \rangle\}.$$

Since $(x,y,z)\sigma_1\sigma_2 = (z,x,y)$ and as $(x,y,z)(\sigma_1\sigma_2)^2 = (y,z,x)$, we see that $\chi = \{\alpha(y) \colon \alpha \in \langle \sigma_1, \sigma_2 \rangle\} = \{\alpha(z) \colon \alpha \in \langle \sigma_1, \sigma_2 \rangle\}$.

Let $\hat{\mathbb{Q}} = \mathbb{Q} \bigcup \{\infty\}$. We can define $\bar{\Psi} : \hat{\mathbb{Q}} \to \chi$ by $\bar{\Psi}(a/b) = X$, where $m = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\alpha(x) = X$, $\Phi'(\alpha) = m$. Here we assume that $\gcd(a,b) = 1$. The map $\bar{\Psi}$ is well defined by Lemma 5.3.

Definition 5.5. For $U \in \mathbb{Q}[x,y,z]$, let

$$\nu(U) = \{(x,y,z) \in \mathbb{R}^3 \colon U(x,y,z) = 0\}.$$

We also define $P_{a/b} = X$ and $S_{a/b} = \nu(P_{a/b})$. Because $\langle \sigma_1, \sigma_2 \rangle \cong PSL(2,\mathbb{Z})$, and by Lemma 5.3, $P_{(-a)/(-b)} = P_{a/b}$, and thus $S_{a/b} = S_{(-a)/(-b)}$. Also, since σ_2 fixes x, $P_{1/0} = x$. For future reference, the case where $a = 0$ only defines a polynomial if $b = \pm 1$. This case is only important in that it implies that there exists polynomials corresponding to a denominator b, for any $b \in \mathbb{Z}$.

Lemma 5.6. *Let $X = P_{a/b}$, $a/b \in \hat{\mathbb{Q}}$, and $\alpha \in \langle \sigma_1, \sigma_2 \rangle$ where $\Phi'(\alpha) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $X = \alpha(x)$ and*

$$\nu(X) = \nu(x)\alpha^{-1}.$$

Proof. The first statement has already been shown. To prove the second, let $p = (u, v, w) \in \nu(X)$ so that $X(u, v, w) = 0$. Let $(x, y, z)\alpha = (X, Y, Z)$. Then

$$\begin{aligned} p = (u, v, w) &= (x, y, z)|_p \\ &= (x, y, z)\alpha\alpha^{-1}|_p \\ &= (\alpha(x, y, z))\alpha^{-1}|_p \\ &= (X, Y, Z)\alpha^{-1}|_p \\ &= (X(p), Y(p), Z(p))\alpha^{-1} \\ &= ((0, Y(p), Z(p))\alpha^{-1} \in \nu(x)\alpha^{-1}. \end{aligned}$$

This shows that $\nu(X) \subset \nu(x)\alpha^{-1}$.

If we let $q = (r, s, t) \in \nu(x)\alpha^{-1}$, then $(r, s, t)\alpha = (0, u, v)$ for some $u, v \in \mathbb{R}$. This gives the following:

$$\begin{aligned} (0, u, v) &= (r, s, t)\alpha \\ &= (x, y, z)\alpha|_q \\ &= (\alpha(x, y, z))|_q \\ &= (X, Y, Z)|_q \\ &= (X(q), Y(q), Z(q)). \end{aligned}$$

This shows that $X(q) = 0$, so that $q \in \nu(X)$. This gives that $\nu(X) = \nu(x)\alpha^{-1}$. $\square$

For $a/b \in \hat{\mathbb{Q}}$, and $m\colon = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, define $(u/v)m$ to mean r/s where $(u, v)m' = \pm(r, s)$, and m' is a coset representative of m.

Corollary 5.7. *Let* $\alpha \in \langle \sigma_1, \sigma_2 \rangle$:

(i) Let $X = P_{a/b}$, *for some* $a/b \in \hat{\mathbb{Q}}$, *and let* $m\colon = \Phi'(\alpha) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. *Then* $S_{a/b} = (S_{1/0})\alpha^{-1}$.

(ii) Let $u/v \in \hat{\mathbb{Q}}$. *Then* $S_{(u,v)m} = S_{u/v}\alpha^{-1}$.

(iii) Let $(u/v)m^{-1} = (r/s)$*. If*

$$(S_{a/b} \cap S_{u/v})\alpha = S_{1/0} \cap S_{(u/v)m^{-1}},$$

Then $s = av - cu$*.*

Proof. (i) This follows immediately from Lemma 5.6.

(ii) Let $\beta \in \langle \sigma_1, \sigma_2 \rangle$ where $n = \Phi'(\beta) = \begin{bmatrix} u & v \\ u' & v' \end{bmatrix}$. Then by (5.2) we have $\Phi'(\alpha\beta) = nm$, and by (i) we have

$$\begin{aligned} S_{(u/v)m} &= S_{(1/0)nm} \\ &= S_{(1/0)}(\alpha\beta)^{-1} \\ &= (S_{1/0}(\beta)^{-1})\alpha^{-1} \\ &= S_{(1/0)n}\alpha^{-1} \\ &= S_{u/v}\alpha^{-1}. \end{aligned}$$

(iii) $(S_{a/b} \cap S_{u/v})\alpha = (S_{a/b})\alpha \cap (S_{u/v})\alpha$, and $(S_{a/b})\alpha = S_{(a/b)m^{-1}} = S_{1/0}$ and $(S_{u/v})\alpha = S_{(u/v)m^{-1}}$, which gives the first part of the result. The rest follows easily. □

Theorem 5.8. *Let* $a, b, c, d \in \mathbb{Z}$ *with* $gcd(a, b) = gcd(c, d) = 1$*, and let* $f = ad - bc$*. Then* $\mathcal{T}$ *hits* $S_{a/b} \cap S_{c/d}$ *in* $2|f|$ *points. The set* $S_{a/b} \cap S_{c/d}$ *is a union of* $|f|$ *copies of* $\mathbb{R}$*, each an unbounded 1-manifold.*

Proof. Since $\gcd(a, b) = 1$, the Euclidean algorithm guarantees a coset representative m for a coset $m' \in PSL(2, \mathbb{Z})$ such that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} m^{-1} = \begin{pmatrix} 1 & 0 \\ e & f \end{pmatrix}$, with $e \in \mathbb{Z}$, and where $f = ad - bc$, as the determinant of m^{-1} is 1. Now $m' = \Phi'(\alpha)$ for some element $\alpha \in \langle \sigma_1, \sigma_2 \rangle$, and by Corollary (5.7), we have that $S_{a/b} \cap S_{c/d}$ and $S_{1/0} \cap S_{(c/d)m^{-1}}$ are images of each other under α. Thus we can assume that $a/b = 1/0$, $c/d = e/f$, where $\gcd(e, f) = 1$.

Thus we now solve

$$x = P_{1/0} = 0, \quad P_{e/f} = 0.$$

In this situation, we see that $S_{1/0}$ meets $\mathcal{T}$ in the circle $y^2 + z^2 = 1$, as evaluating $x^2 + y^2 + z^2 - 2xyz = 1$ at $x = 0$, leaves one with $y^2 + z^2 = 1$. So $S_{1/0} \cap \mathcal{T} \cap S_{e/f}$ consists of a finite number of points on this circle. We show that there are $2|f|$ solutions to the corresponding equations

$$\cos(2\pi\theta_1) = 0, \quad \cos(2\pi(e\theta_1 + f\theta_2)) = 0.$$

Without loss of generality, assume that f is greater than 0. On the torus, $S_{1/0} \cap S_{e/f}$ corresponds to the lines

$$1\theta_1 + 0\theta_2 = \pm\tfrac{1}{4} \text{ and } e\theta_1 + f\theta_2 = \pm\tfrac{1}{4} + k, k \in \mathbb{Z},$$

respectively. This gives the solutions

$$\theta_1 = \frac{\epsilon_1}{4}, \quad \theta_2 = \frac{1}{f}\Big(\frac{\epsilon_2}{4} + k - e\epsilon_1\frac{1}{4}\Big),$$

where $\epsilon_i = \pm 1, i = 1, 2$. This gives $4f$ possibilities. However, these come in pairs that are negatives of each other. Thus there are really only $2f$ such points of $S_{1/0} \cap S_{e/f}$ on $\mathcal{T}$. □

Call these $2f$ points $p_1, \ldots, p_{2f}$. Note that each surface $S_{1/0}$, $S_{e/f}$ intersects transversely near each p_k, as the corresponding lines in θ space are not parallel. This gives a curve of intersection $\gamma_k(s)$ near p_k.

Theorem 5.9. *The points $p_1, \ldots, p_{2f}$ are connected in pairs by smooth curves of $S_{1/0} \cap S_{e/f}$. Each such smooth arc is given by a single Puiseux-Newton series solution to one of the irreducible factors of a polynomial $P(y, z) = P_{a,b,c,d}(y, z)$.*

Proof. The surface $S_{e/f}$ is determined by $P_{e/f}(x, y, z)$; however, for the case $S_{1/0} \cap S_{e/f}$, our polynomial has $x = 0$, so $P_{e/f} = P_{e/f}(0, y, z)$. Without loss of generality, we can assume that $P_{e/f}(x, y, z)$ is the first entry of $(x, y, z)\alpha$, for some $\alpha \in \langle\sigma_1, \sigma_2\rangle$.

Lemma 5.10. *For all $\alpha \in \langle\sigma_1, \sigma_2\rangle$ we have*

$$(\alpha(x, y, z))|_{x=0} = ((x, y, z)\alpha)|_{x=0} = (0, y, z)\alpha.$$

Proof. The first equality has already been shown in Lemma 5.1. We now proceed by induction on the length $n = |\alpha|$ as word in the generators $\sigma_i^{\pm 1}$. This is clear for the case $n \leq 1$. Assume the statement holds for $|\alpha| = n$, and consider $\alpha' = \sigma_1\alpha$, with the rest of the cases following similarly.

$$(x, y, z)\alpha = (X, Y, Z), \text{ for } X = X(x, y, z), Y = Y(x, y, z), Z = Z(x, y, z).$$

This gives the following:

$$\begin{aligned}
(x, y, z)\sigma_1\alpha|_{x=0} &= (z, y, 2yz - x)\alpha|_{x=0} \\
&= (X(z, y, 2yz - x), Y(z, y, 2yz - x), Z(z, y, 2yz - x))|_{x=0} \\
&= (X(z, y, 2yz), Y(z, y, 2yz), Z(z, y, 2yz)) \\
&= (z, y, 2yz)\alpha \\
&= (0, y, z)\sigma_1\alpha.
\end{aligned}$$

Thus we have $(\alpha(x, y, z))|_{x=0} = ((x, y, z)\alpha)|_{x=0} = (0, y, z)\alpha$. $\square$

We consider again $\gamma_i(s)$, the smooth arc created by the intersection of $S_{1/0}$ and $S_{e/f}$. We can choose $\gamma_i(0) = p_i$. This arc is determined by some irreducible factor of $P_{e/f}$. Call this factor $P_{e/f,i}(y, z)$. By the Newton-Puiseux theorem (see [15]), the factor $P_{e/f,i}(y, z)$ and the point p_i determine a Puiseux series that we denote by $\gamma_i(t)$, that parameterizes the curve $S_{1/0} \cap S_{e/f}$ near p_i. Thinking of points in the plane $S_{1/0}$ (i.e. $x = 0$), that are near $\gamma_i(t)$, we see that for $|t|$ small, on one side of $\gamma(t)$, the function $P_{e/f,i}(y, z)$ will be positive, while on the other side, the function will be negative. This shows that each arc of $S_{1/0} \cap S_{e/f}$ continues until it meets a non-manifold point p. Further, there is an interval $[r, s)$ such that $\gamma(r) = p_i$, and the image $\gamma([r, s))$ is a part of $S_{1/0} \cap S_{e/f}$. It follows by continuity, that $\gamma(s) = p$. Now, there is a Puiseux-Newton series $g(t)$ for $S_{1/0} \cap S_{e/f}$ centered at the point p. The series $g(t)$ must agree, perhaps after reparameterization, with the Puiseux series $\gamma(t)$ for points of $S_{1/0} \cap S_{e/f}$ just before p, but $g(t)$ is defined for points of $S_{1/0} \cap S_{e/f}$ not in $f([r, s))$. Thus we have points of $S_{1/0} \cap S_{e/f}$ that determine a smooth curve that contains p in its interior. We continue along this curve until we either hit one of the p_j, or we hit another non-manifold point. In the second situation we repeat the previous argument. This process terminates as there are only finitely many non-manifold points. As a consequence, we obtain an infinitely long smooth curve connecting p_i to some p_j, $j \neq i$ as required. $\square$

From the above, the following theorem follows easily.

Proposition 5.11. *The set of curves* $S_{a/b}(0, y, z) = \nu(P_{a/b}(0, y, z))$ *hits* $\mathcal{T}$ $2|b|$ *times.*

Proof. The determinant of $\begin{pmatrix} 1 & 0 \\ a & b \end{pmatrix}$ is b. $\square$

For the remainder of this thesis, we study the polynomials $P_{a/b}(0, y, z)$, with their corresponding varieties $S_{a/b}(0, y, z) = \nu(P_{a/b}(0, y, z))$. This is equivalent to studying the curves given by $S_{a/b} \cap S_{c/d}$, by Lemma 5.10.

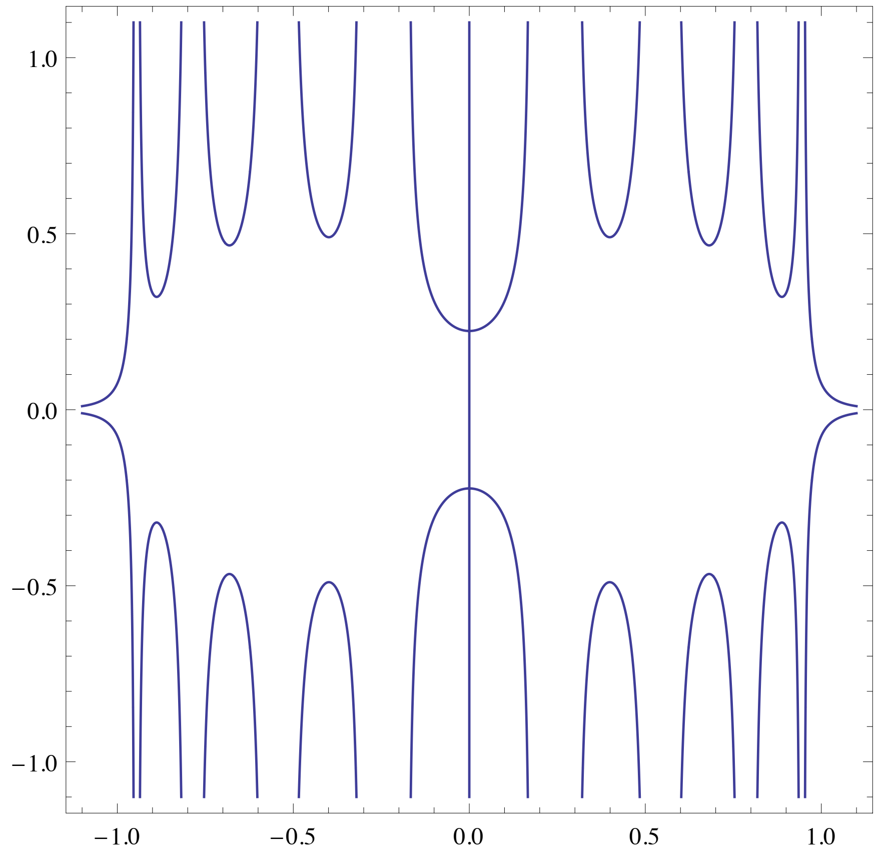


Figure 5.1: Contour Plot of $S_{2/19} \cap S_{1/0}$.

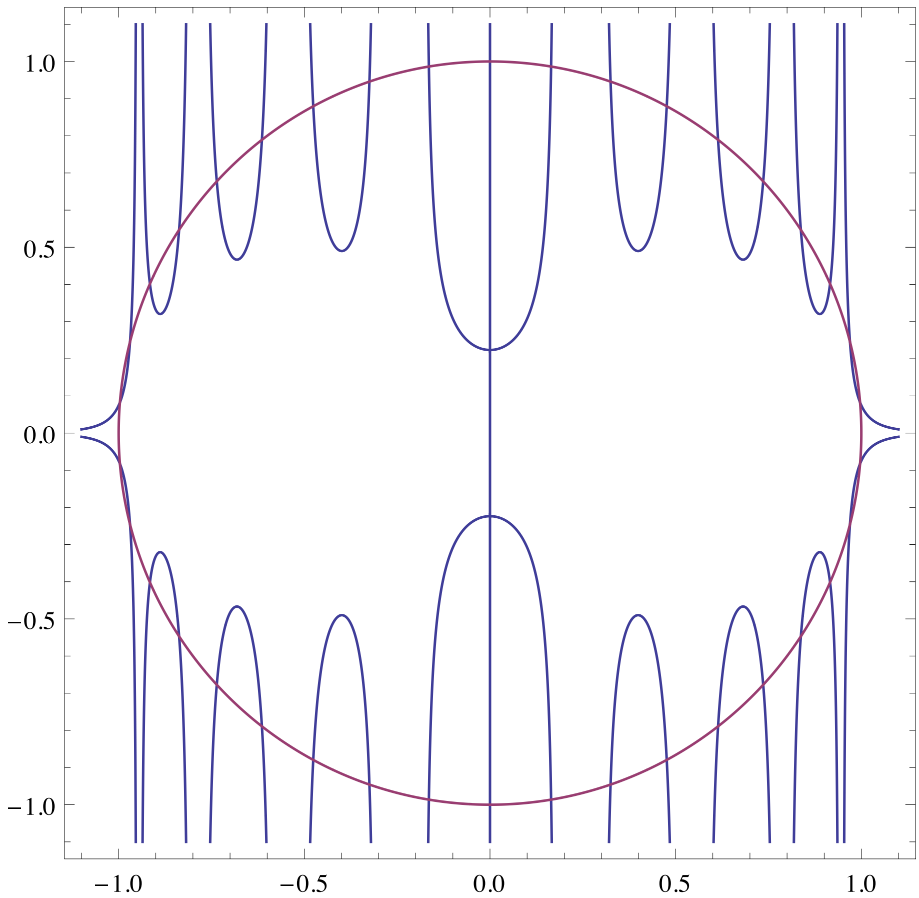


Figure 5.2: Contour Plot of $S_{2/19} \cap S_{1/0}$, with the unit circle $y^2 + z^2 = 1$ which comes from the equation $E(0, y, z) = 1$.

Chapter 6. Relations Between the Polynomials $P_{a/b}(0, y, z)$

We now begin our study of the polynomials $P_{a/b}(0, y, z)$. In particular, we desire to know how many distinct polynomials there are for a given b value. This will take up most of the following two chapters. In this chapter, we will study relations between the polynomials $P_{a/b}(0, y, z)$ and $P_{-a/b}(0, y, z)$, and we will study relations between the polynomials $P_{a/b}(0, y, z)$ and $P_{(b-a)/b}(0, y, z)$. In the first case, we will show that $P_{a/b}(0, y, z) = \pm P_{-a/b}(0, y, z)$ and in the second case, we will show that $P_{(b-a)/b}(0, z, y) = P_{a/b}(0, y, z)$.

Theorem 6.1. *Let $b \in \mathbb{Z}$. Then*

$$\{\{\pm P_{a/b}(0, y, z)\} \colon a \in \mathbb{Z}, \gcd(a, b) = 1\}$$

is a finite set of cardinality at most $2\varphi(b)$, where φ is the euler phi function.

Proof. We show that $P_{(a+4b)/b}(0, y, z) = P_{a/b}(0, y, z)$. Let
$\alpha = \alpha_{a/b}$, so that $\Phi'(\alpha) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We note that $(0, y, z)\sigma_2^{-4} = (0, y, z)$, thus

$$(0, y, z)\sigma_2^{-4}\alpha = (0, y, z)\alpha.$$

Thus

$$P_{(1/0)\Phi'(\sigma_2^{-4}\alpha)} = P_{(1/0)\Phi'(\alpha)}$$

But

$$\Phi'(\sigma_2^{-4}\alpha) = \Phi'(\alpha)\Phi'(\sigma_2^{-4}) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} = \begin{pmatrix} a + 4b & b \\ c + 4d & d \end{pmatrix}.$$

This implies that we have at most $4\varphi(b)$ polynomials $P_{a/b}(0, y, z)$.

Let $G = \langle \sigma_1, \sigma_2 \rangle$, and let $H = \langle \sigma_1^2, \sigma_2 \rangle$. The Reidemeister-Schreier process (see [16]) shows that $[G \colon H] = 3$. Here, a right transversal is $1, \sigma_1, \sigma_1\sigma_2$.

Lemma 6.2. *For all $\alpha \in H$, if $(x, y, z)\alpha = (X, Y, Z)$, then $(x, -y, -z)\alpha = (X, -Y, -Z)$.*

Proof. We induct on the length $n = |\alpha|$, of α as a product of the generators $\sigma_i^{\pm 1}$. The case $n = 0$ holds. Assume that $\alpha \in H$ satisfies the inductive hypothesis. First, let $\beta = \alpha\sigma_2$, so $|\beta| > |\alpha|$. Then

$$(x, y, z)\beta = (x, y, z)\alpha\sigma_2 = (X, Y, Z) = (X, 2XY - Z, Y).$$

Then we have

$$\begin{aligned}(x, -y, -z)\beta &= (x, -y, -z)\alpha\sigma_2 = (X, -Y, -Z)\sigma_2 \\ &= (X, -(2XY - Z), -Y).\end{aligned}$$

The case $\beta = \alpha\sigma_2^{-1}$ is similar.

If $\beta = \alpha\sigma_1^2$, then we have

$$(x, y, z)\beta = (x, y, z)\alpha\sigma_1^2 = (X, Y, Z)\sigma_1^2 = (2YZ - X, Y, 4Y^2Z - 2XY - Z),$$

and

$$\begin{aligned}(x, -y, -z)\beta &= (x, -y, -z)\alpha\sigma_1^2 = (X, -Y, -Z)\sigma_1^2 \\ &= (2(-Y)(-Z) - X, -Y, 4Y^2(-Z) - 2X(-Y) - (-Z)) \\ &= (2YZ - X, -Y, -(4Y^2Z - 2XY - Z)),\end{aligned}$$

as required. The case $\beta = \alpha\sigma_1^{-2}$ is similar. □

Now suppose that $(0, y, z)\alpha = (X, Y, Z)$, where $\alpha \in H$. Then we have

$$(0, y, z)\sigma_2^{-2}\alpha = (0, -y, -z)\alpha = (X, -Y, -Z).$$

Thus for $\Phi'(\alpha) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ as above, we see that

$$P_{a/b}(0, y, z) = X = P_{e/f}(0, y, z),$$

where (e, f) is the first row of a coset representative of

$$\Phi'(\sigma_2^{-2}\alpha) = \Phi'(\alpha)\Phi'(\sigma_2^{-2}) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} a + 2b & b \\ c + 2d & d \end{bmatrix},$$

so that $(e, f) = \pm(a + 2b, b)$.

We note that H is the subgroup of cosets that have even $(1, 2)$ entry. Thus if b is even, then there are at most $2\varphi(b)$ values. If b is odd, the α is either in the coset $H\sigma_1$ or in the coset $H\sigma_1\sigma_2$.

First suppose that $\alpha \in H\sigma_1$, then $\alpha\sigma_1 \in H$. Now let

$$(0, y, z)\alpha = (X, Y, Z),$$

so that

$$(0, y, z)\alpha\sigma_1 = (X, Y, Z)\sigma_1 = (Z, Y, 2YZ - X).$$

Since $\alpha\sigma_1 \in H$, we see that

$$(0, -y, -z)\alpha\sigma_1 = (X, Y, Z)\sigma_1 = (Z, -Y, -(2YZ - X)),$$

which gives

$$(0, -y, -z)\alpha = (0, -y, -z)\alpha\sigma_1\sigma_1^{-1} = (Z, -Y, -(2YZ - X))\sigma_1^{-1} = (-X, -Y, -Z).$$

This shows that

$$(0, y, z)\sigma_2^{-2}\alpha = (0, -y, -z)\alpha = (-X, -Y, Z),$$

so we have that $P_{(a+2b)/b}(0, y, z) = -P_{a/b}(0, y, z)$, which gives the result in this case.

Lastly, suppose that $\alpha \in H\sigma_1\sigma_2$. As in the above, let $(0, y, z)\alpha = (X, Y, Z)$, so that

$$(0, y, z)\alpha\sigma_2^{-1}\sigma_1^{-1} = (X, Y, Z)\sigma_1^{-1}\sigma_2^{-1} = (Y, Z, X).$$

Since $\alpha\sigma_2^{-1}\sigma_1^{-1} \in H$, we see that

$$(0, -y, -z)\alpha\sigma_2^{-1}\sigma_1^{-1} = (Y, -Z, -X).$$

So

$$\begin{aligned}(0, -y, -z)\alpha &= (0, -y, -z)\alpha(\sigma_2^{-1}\sigma_1^{-1}(\sigma_2^{-1}\sigma_1^{-1})^{-1} \\ &= (Y, -Z, -X)(\sigma_2^{-1}\sigma_1^{-1})^{-1}\end{aligned}$$

$$= (Y, -Z, -X)\sigma_1\sigma_2$$

$$= (-X, Y, -Z).$$

This shows that $P_{(a+2b)/b}(0, y, z) = -P_{a/b}(0, y, z)$. The lemma follows. $\square$

We now continue with a technical lemma which will lead us to further classify the polynomials $P_{a/b}(0, y, z)$. The following shows that given a coset $\omega = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the parity of c and d, depend only on the parity of a and b, and the length of ω as a word in m_1, and m_2. We note that it makes sense at this point to think of the word length of ω as having a parity, as based on the presentation $\langle m_1, m_2 \mid m_1m_2m_1 = m_2m_1m_2, (m_1m_2)^3 \rangle$ for $PSL(2, \mathbb{Z})$, we can easily see that if two words are equivalent, then they have the same parity as words in m_1, and m_2. This follows easily from the fact that the only relations possible to change words, are to switch an odd number of elements for an odd number of elements, as in applying $m_1m_2m_1 = m_2m_1m_2$, or by canceling an even number of elements, as in $(m_1m_2)^3$, or as in canceling adjacent elements that are inverses of each other.

Lemma 6.3. *For* $\omega = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in PSL(2, \mathbb{Z})$*:*

1. *Assume a is odd and b is odd:*
   - *(i) If the length of ω is even, then c is odd and d is even.*
   - *(ii) If the length of ω is odd, then c is even and d is odd.*
2. *Assume a is odd and b is even:*
   - *(i) If the length of ω is even, then c is even and d is odd.*
   - *(ii) If the length of ω is odd, then c is odd and d is odd.*
3. *Assume a is even and b is odd:*
   - *(i) If the length of ω is even, then c is odd and d is odd.*
   - *(ii) If the length of ω is odd, then c is odd and d is even.*

Proof. We start by noting that this includes all of the cases, as if ω is in $PSL(2,\mathbb{Z})$, a and b must be coprime. We induct on the length of ω as a word in m_1 and m_2.

We note that in the initial case $\omega = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, the length of ω is even, a is odd, b is even, c is even and d is odd. This matches $2(i)$. Next, assume that a is odd and b is odd, and the length of ω is even. Let $\omega = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ by the inductive hypothesis, c is odd and d is even. Consider

$$\beta = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} = \begin{bmatrix} a & a+b \\ c & c+d \end{bmatrix}.$$

The end result has the length of β being odd, a' odd and b' even. Additionally, c' is odd and d' is odd, which matches 2(ii). Next,

$$\beta = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 10 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} = \begin{bmatrix} a-b & b \\ c-d & d \end{bmatrix}.$$

The end result has the length of β being odd, a' even, b' odd. Additionally, c' is odd and d' is even. This matches $3(ii)$. The other five cases follow identically. □

For notational convenience, let $P_{a/b}(y,z) = P_{a/b}(0,y,z)$.

Theorem 6.4. *For $a, b \in \mathbb{Z}$, with $\gcd(a,b) = 1$, we have $P_{a/b}(y,z) = \pm P_{a/-b}(y,z)$.*

Proof. We start by proving the following lemma. Let $\omega = \omega(m_1, m_2)$ a word in the generators m_1 and m_2.

Lemma 6.5. *If $\omega(m_1,m_2) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $\omega(m_1^{-1}, m_2^{-1}) = \begin{bmatrix} a & -b \\ -c & d \end{bmatrix}$.*

Proof. We proceed by induction on $n = |\omega(m_1, m_2)|$. There is a case for each possibility given in Lemma 6.3, but we will only demonstrate one of these cases. The initial case, the empty word corresponding to the identity, is clear. Assume that

$$\omega(m_1,m_2) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \omega(m_1^{-1}, m_2^{-1}) = \begin{bmatrix} a & -b \\ -c & d \end{bmatrix}.$$

Then

$$\omega(m_1, m_2)m_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & a+b \\ c & c+d \end{bmatrix},$$

$$\omega(m_1^{-1} m_2^{-1})m_1^{-1} = \begin{bmatrix} a & -b \\ -c & d \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & -a-b \\ -c & c+d \end{bmatrix},$$

and

$$\omega(m_1, m_2)m_2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} a-b & b \\ c-d & d \end{bmatrix},$$

$$\omega(m_1^{-1} m_2^{-1})m_2^{-1} = \begin{bmatrix} a & -b \\ -c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} a-b & -b \\ d-c & d \end{bmatrix}.$$

Thus the induction holds in these cases. □

We now classify the actions of $\alpha(\sigma_1, \sigma_2)$ and $\alpha(\sigma_1^{-1}, \sigma_2^{-1})$ by the length of α in regards to its generators, σ_1 and σ_2, and in regards to the top row of a matrix representing the coset corresponding to α.

Lemma 6.6. *Assume the following:*

(a) $(0, y, z)\alpha(\sigma_1, \sigma_2) = (X(y, z), Y(y, z), Z(y, z))$,

(b) $(0, y, z)\alpha(\sigma_1^{-1}, \sigma_2^{-1}) = (X'(y, z), Y'(y, z), Z'(y, z))$,

(c) $\Phi'(\alpha(\sigma_1, \sigma_2)) = \omega(m_1, m_2) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

1. *Assume a is odd and b is odd:*

   (i) If the length of alpha is even, then $X' = -X$, $Y' = -Y$, $Z' = 2YX - Z$.

   (ii) If the length of α is odd, then $X' = -X$, $Y' = Y$, $Z' = Z - 2XY$.

2. *Assume a is odd and b is even:*

   (i) If the length of α is even, then $X' = -X$, $Y' = Y$, $Z' = Z - 2XY$.

   (ii) If the length of α is odd, then $X' = -X$, $Y' = -Y$, $Z' = 2YX - Z$.

3. *Assume a is even and b is odd:*

   (i) *If the length of α is even, then $X' = X$, $Y' = -Y$, $Z' = Z - 2XY$.*

   (ii) *If the length of α is odd, then $X' = X$, $Y' = -Y$, $Z' = Z - 2XY$.*

Proof. The result will follow by induction on the length of α. There are six cases, we will prove one of them, the rest follow identically. Assume that a is odd, b is odd, and $|\alpha|$ is even. Then by the previous lemma, we know that c is odd and d is even. Also, by assumption, $X' = -X$, $Y' = -Y$, and $Z' = 2XY - Z$. Then

$$(0, y, z)\alpha(\sigma_1, \sigma_2)\sigma_1 = (Z, Y, 2YZ - X),$$

and

$$\begin{aligned}(0, y, z)\alpha(\sigma_1^{-1}, \sigma_2^{-1})\sigma_1^{-1} &= (2X'Y' - Z', Y', X') \\ &= (Z, -Y, -X).\end{aligned}$$

We note that in the resulting case, the length of $\beta = \alpha\sigma_1$ is odd, and also that if a' and b' are in the top row of the matrix representing the coset corresponding to β:

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ c & d \end{bmatrix} = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$$

implies that a' is even and b' is odd. The results we obtained from $(0, y, z)\beta(\sigma_1, \sigma_2)$ and $(0, y, z)\beta(\sigma_1^{-1}, \sigma_2^{-1})$ correspond with this result. That is, applying $3(ii)$, we see that given

$$(0, y, z)\beta(\sigma_1, \sigma_2) = (Z, Y, 2YZ - X),$$

we should obtain the corresponding term to be

$$(Z, -Y, 2YZ - X - 2YZ) = (Z, -Y, -X),$$

which indeed was our result. Doing the same thing with σ_2, we obtain

$$(0, y, z)\alpha(\sigma_1, \sigma_2)\sigma_2 = (X, 2XY - Z, Y),$$

and

$$\begin{aligned}(0, y, z)\alpha(\sigma_1^{-1}, \sigma_2^{-1})\sigma_2^{-1} &= (X', Z', 2X'Z' - Y') \\ &= (-X, 2XY - Z, Y - 4X^2Y + 2XZ).\end{aligned}$$

Multiplication of any coset in $PSL(2, \mathbb{Z})$ on the left by m_2 does not change the parity of a' and b'. This results in the order of $\beta = \alpha\sigma_2$ being odd, with a' and b' odd. The relations between $(0, y, z)\beta(\sigma_1, \sigma_2)$ and $(0, y, z)\beta(\sigma_1^{-1}, \sigma_2^{-1})$ correspond with this result. The remaining five cases follow easily. □

Theorem 6.4 now follows, for in every case, the first term X, in $(0, y, z)\alpha(\sigma_1, \sigma_2)$ always equals the first term of $(0, y, z)\alpha(\sigma_1^{-1}, \sigma_2^{-1})$ up to sign. □

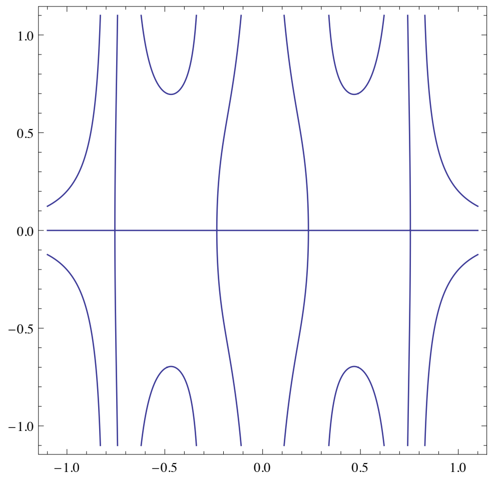


Figure 6.1: Contour Plot of $S_{3/13} \cap S_{1/0}$ and $S_{-3/13} \cap S_{1/0}$.

Lemma 6.7. *Assume that* $(0, y, z)\alpha(\sigma_1, \sigma_2) = (X(y, z), Y(y, z), Z(y, z))$, *then* $(0, y, z)\sigma_2^{-1}\alpha(\sigma_1^{-1}, \sigma_2^{-1}) = (X(z, y), Y(z, y), (2XY - Z)(z, y))$.

Proof. We induct on the length of α. If the length of α is zero, then we have for the first term $(0, y, z)$ and for the second, $(0, z, -y)$, which satisfies the conclusions of the theorem.

Assume the length of α is not zero. Let $(0, y, z)\alpha(\sigma_1, \sigma_2) = (X(y, z), Y(y, z), Z(y, z))$. By the inductive hypothesis, $(0, y, z)\sigma_2^{-1}\alpha(\sigma_1^{-1}, \sigma_2^{-1}) = (X(z, y), Y(z, y), (2XY - Z)(z, y))$. Proceeding first with the term $\alpha(\sigma_2, \sigma_1)\sigma_1$, we obtain the following.

$$\begin{aligned}(0, y, z)\alpha(\sigma_1, \sigma_1) &= \sigma_1(X(y, z), Y(y, z), Z(y, z))\sigma_1 \\ &= (X(y, z), Y(y, z), (2YZ - X)(y, z)), \\ (0, y, z)\alpha(\sigma_1^{-1}, \sigma_2^{-1}) &= \sigma(X(z, y), Y(z, y), (2XY - Z)(z, y))\sigma_1^{-1} \\ &= (Z(z, y), Y(z, y), X(z, y)).\end{aligned}$$

As $2Z(y, z)Y(y, z) - (2Z(y, z)Y(y, z) - X(y, z)) = X(y, z)$, the new element satisfies the conclusions of the theorem. We now show the same is true for $\alpha(\sigma_1, \sigma_2)\sigma_2$.

$$\begin{aligned}(0, y, z)\alpha(\sigma_1, \sigma_2)\sigma_2 &= (X(y, z), Y(y, z), Z(y, z))\sigma_2 \\ &= (X(y, z), (2XY - Z)(y, z), Y(y, z)),\end{aligned}$$

while

$$\begin{aligned}(0, y, z)\alpha(\sigma_1^{-1}, \sigma_2^{-1}) =& (X(z, y), Y(z, y), (2XY - Z)(z, y))\sigma_2^{-1} \\ =& (X(z, y), (2XY - Z)(z, y), \\ & \quad 2X(z, y)(2XY - Z)(z, y) - Y(z, y)).\end{aligned}$$

As $2X(y, z)(2XY - Z)(y, z) - Y(y, z)$ matches the last term, this new element satisfies the conclusion of the theorem. □

Theorem 6.8. *For all $a, b \in \mathbb{Z}$ with $\gcd(a, b) = 1$, $P_{a/b}(y, z) = P_{(b-a)/b}(z, y)$.*

Proof. Assume that $\Phi'(\alpha(\sigma_1, \sigma_2)) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $\Phi'(\alpha(\sigma_1^{-1}, \sigma_2^{-1})) = \begin{bmatrix} a & -b \\ -c & d \end{bmatrix}$.

We know that $\sigma_2^{-1}\alpha(\sigma_1^{-1}, \sigma_2^{-1})$ corresponds to

$$\begin{bmatrix} a & -b \\ -c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} a-b & -b \\ d-c & d \end{bmatrix}.$$

Which implies that, by Lemma 6.7, $P_{a/b}(y,z) = P_{(a-b)/-b}(z,y) = P_{(b-a)/b}(z,y)$. □

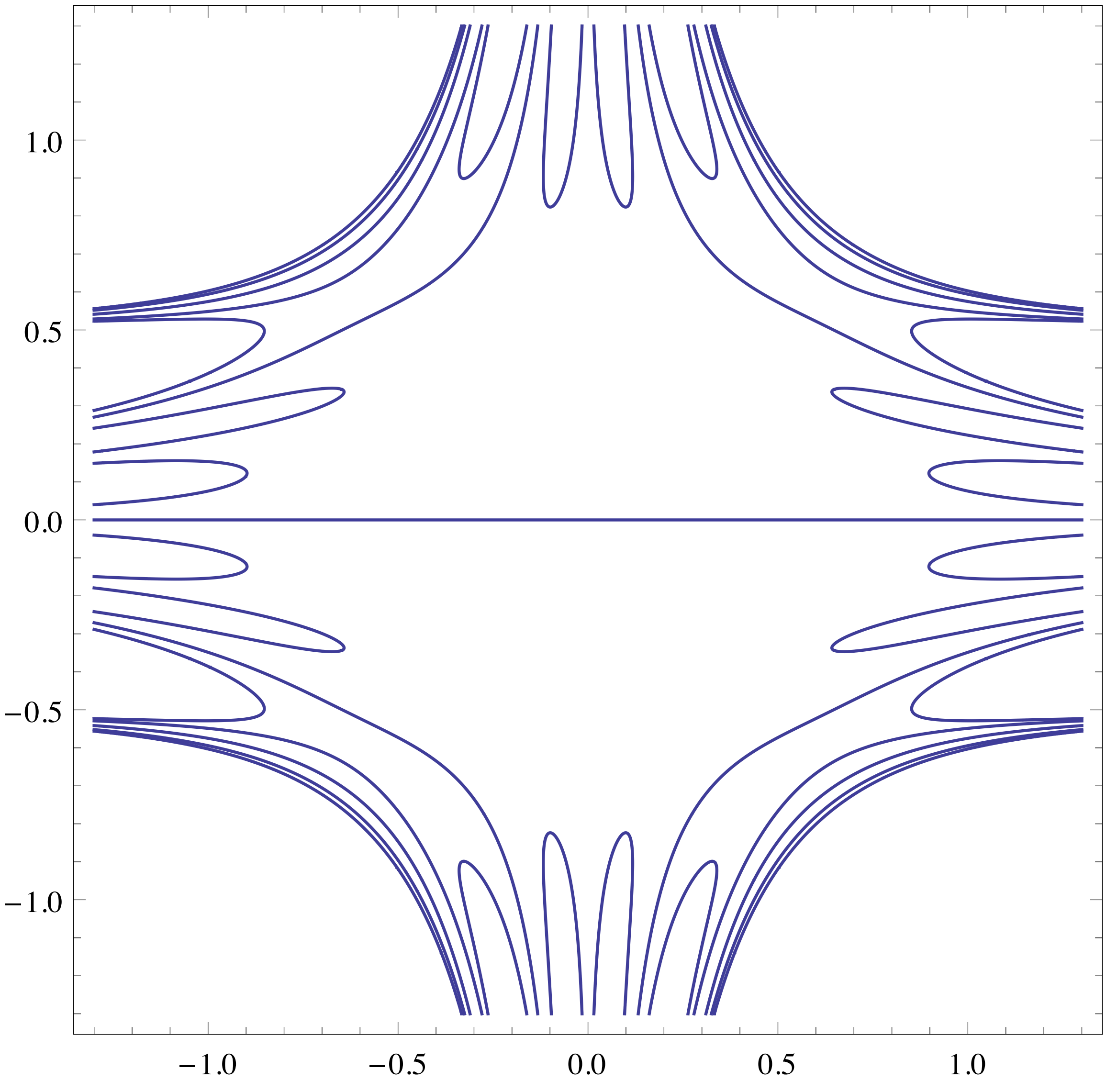


Figure 6.2: Contour Plot of $S_{23/41} \cap S_{1/0}$.

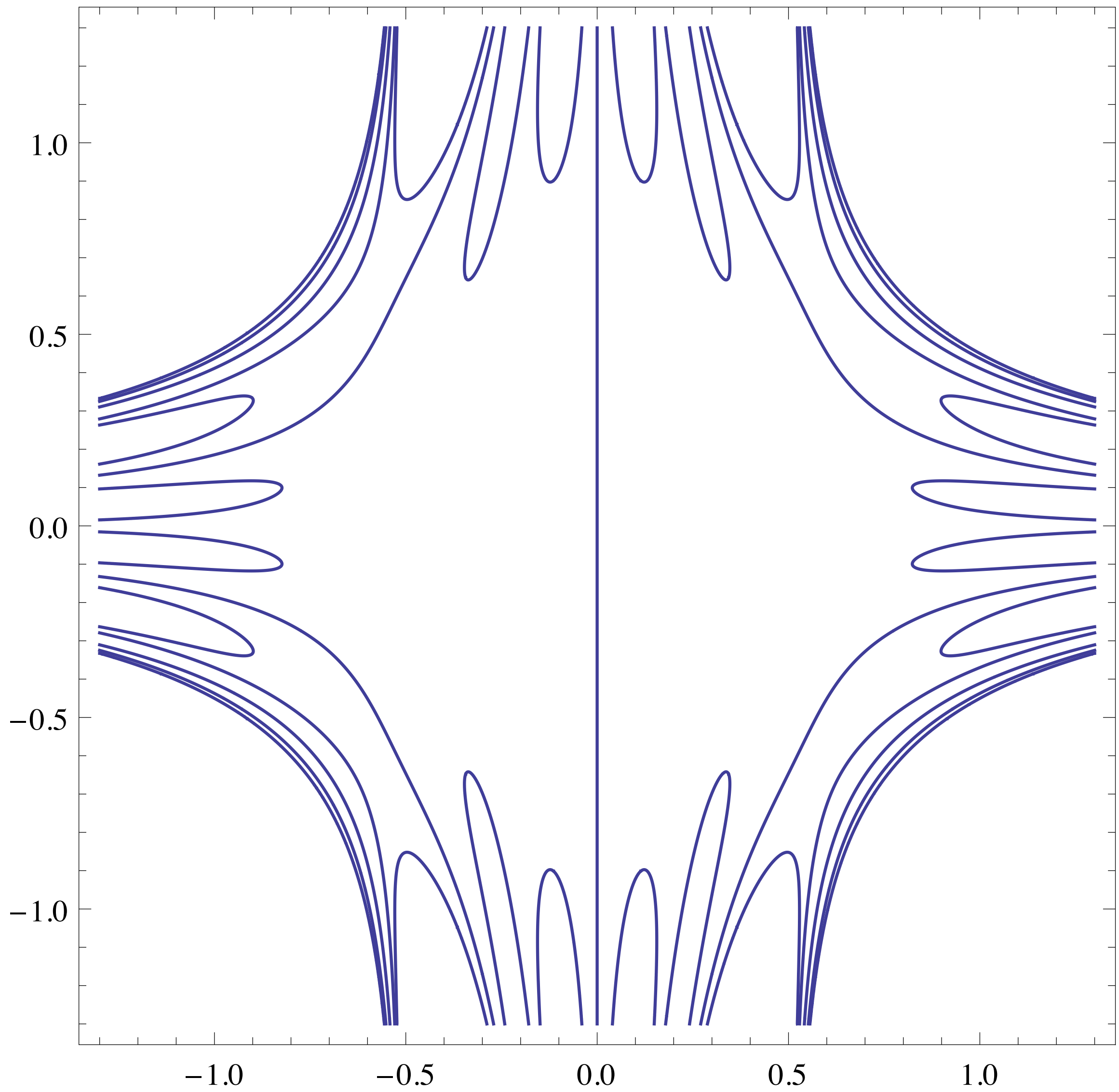


Figure 6.3: Contour Plot of $S_{18/41} \cap S_{1/0}$.

Our eventual goal, which will be achieved in the next chapter, is to prove that there are precisely $\phi(b)$ sets of curves $S_{a/b}(y, z)$ associated to the polynomials $P_{a/b}(y, z)$.

Chapter 7. Uniqueness of the Curves $S_{a/b} \cap S_{0/1}$

The ultimate goal of this chapter will be to show that there are precisely $\phi(b)$ polynomials $P_{a/b}(y,z)$, up to sign, for each $b \in \mathbb{Z}$. We will obtain many interesting results along the way. Of particular note, we will determine the degree of a polynomial $P_{a/b}(y,z)$ in terms of a and b.

We start by classifying the polynomials $P_{a/b}(y,z)$ mod the ideal $I = (y^2 + z^2 - 1)$. Let $U_n = U_n(z), T_n = T_n(z)$. Acting on $(0,y,z) \mod I$ actually only yields six possible sets of polynomials for (X,Y,Z) These are

(i) (T_a, yU_{a+b}, yU_b);

(ii) (yU_a, T_b, yU_{a+b});

(iii) (yU_a, yU_{a+b}, T_b);

(iv) $(-T_a, yU_{a+b}, -yU_b)$;

(v) $(-yU_a, -T_b, yU_{a+b})$;

(vi) $(yU_a, -yU_{a+b}, -T_b)$.

Here the a and b do not necessarily correspond to the matrices corresponding to the images of $(0,y,z)$. Now, recall that $U_{-n} = -U_{n-2}$ and $T_{-n} = T_n$. We will show that this set of types is preserved $\mod I$, under the action of $\langle\sigma_1, \sigma_2\rangle$.

Lemma 7.1. *The previously discussed types of polynomials are the only possible polynomials generated by* $\langle\sigma_1, \sigma_2\rangle \mod I$.

Proof. It is clear that the actions of σ_1 and σ_2 take one of these forms mod I

$$(1)(T_a, yU_{a+b}, uU_b)\sigma_1 = (yU_b, yU_{a+b}, -T_{a+2b+2}) = (yU_b, -yU_{-a-b-2}, -T_{-a-2b-2})$$

$$(2)(T_a, yU_{a+b}, yU_b)\sigma_2 = (T_a, yU_{2a+b}, yU_{a+b})$$

(3)$(yU_a, T_b, yU_{a+b})\sigma_1 = (yU_{a+B}, T_b, yU_{a+2b})$

(4)$(yU_a, T_b, yU_{a+b})\sigma_2 = (yU_a, yU_{a-b}, T_{-b})$

(5)$(yU_a, yU_{a+b}, T_b)\sigma_1 = (T_b, yU_{a+b}, yU_{a+2b})$

(6)$(yU_a, yU_{a+b}, T_b)\sigma_2 = (yU_a, -T_{2a+b+2}, yU_{a+b}) = (-yU_{-a-2}, -T_{2a+b+2}, yU_{a+b})$

(7)$(-T_a, yU_{a+b}, -yU_b)\sigma_1 = (-yU_b, yU_{a+b}, T_{a+2b+2}) = (yU_{-b-2}, yU_{a+b}, T_{a+2b+2})$

(8)$(-T_a, yU_{a+b}, -yU_b)\sigma_2 = (-T_a, -yU_{2a+b}, yU_{a+b}) = (-T_{-a}, yU_{-2a-b-2}, yU_{-a-b-2})$

(9)$(-yU_a, -T_b, yU_{a+b})\sigma_1 = (yU_{a+b}, -T_{-b}, yU_{-a-b-2})$

(10)$(-yU_a, -T_b, yU_{a+b})\sigma_2 = (-yU_a, yU_{b-a-2}, -T_b) = (yU_{-a-2}, yU_{b-a-2}, -T_b)$

(11)$(yU_a, -yU_{a+b}, -T_b)\sigma_1 = (-T_b, -yU_{a+b}, yU_{a+b}) = (-T_b, yU_{-a-b-2}, -yU_{-a-2b-2})$

(12)$(yU_a, -yU_{a+b}, -T_b)\sigma_2 = (yU_a, T_{-2a-b-2}, yU_{-a-b-2})$.

Thus we see that that the only possible polynomials generated by $\langle\sigma_1, \sigma_2\rangle \mod I$ are the ones previously described. □

Viewing the actions of σ_1 and σ_2 as permutations, we induce a homomorphism $\Sigma\colon PSL(2,\mathbb{Z}) \to S_6$:

$$\sigma_1 \to (1, 6, 4, 3), \quad \sigma_1 \to (2, 3, 5, 6).$$

This image is isomorphic to S_4. We now have the following lemma.

Lemma 7.2. *Let* $\alpha \in \langle\sigma_1, \sigma_2, \rangle$, *where* $\Phi'(\alpha) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. *Let* $(0, y, z)\alpha = (X, Y, Z)$. *Then*

(i) $deg(X) = |b|$;

(ii) $deg(Y) = |d|$;

(iii) $deg(Z) = |b - d|$.

Proof. We can assume that $b > 0$, using the fact that $P_{a/b} = P_{(-a)/(-b)}$. Let $(x, y, z)\alpha = (X', Y', Z')$, and let $(0, y, z)\alpha = (X, Y, Z)$. Then by Lemma 5.10 we have that $X'(0, y, z) = X, Y'(0, y, z) = Y, Z'(0, y, z) = Z$. Each of these polynomials has a unique term of highest degree. Further, considering X, then if we reduce $X \mod (y^2 + z^2 - 1)$, which eliminates all even powers of y from X, the leading term reduces to dz^ϵ or dyz^ϵ, for some constant d, and $\epsilon \in \mathbb{N}$. We note that this does not change the total degree of X. Now, let $\theta_1 = \frac{1}{4}$. We have that

$$\begin{aligned} X'(\cos(2\pi\theta_1), \cos(2\pi\theta_2), \cos(2\pi(\theta_1+\theta_2))) =& X'(0, \cos(2\pi\theta_2), -\sin(2\pi\theta_2)) \\ =& X(0, \cos(2\pi\theta_2), -\sin(2\pi\theta_2)). \end{aligned}$$

There are similar equations for Y' and Z'. However, using equation (5.1), we know that

$$\begin{aligned} X(0, \cos(2\pi\theta_2), -\sin(2\pi\theta_2)) =& X'(\cos(2\pi\theta_1), \cos(2\pi\theta_2), \cos(2\pi(\theta_1+\theta_2))) \\ =& \cos\left(\frac{a\pi}{2} + 2\pi b\theta_2\right). \end{aligned} \tag{7.1}$$

Now, letting $z = -\sin(2\pi\theta_2), y = \cos(2\pi\theta_2)$, we see that we already have that $y^2+z^2 = 1$, which implies that $X(0, \cos(2\pi\theta_2), -\sin(2\pi\theta_2)$ is of the form $T_n(-\sin(2\pi\theta_2))$, or is of the form $\cos(2\pi\theta_2)U_{n-1}(-\sin(2\pi\theta_2))$. Assume that we have an equation of the first form. Then

$$T_n(-\sin(2\pi\theta_2)) = T_n\left(\cos\left(\frac{\pi}{2} + 2\pi\theta_2\right)\right) = \cos\left(\frac{n\pi}{2} + 2n\pi\theta_2\right).$$

The equality (7.1) holds only if $n = b$. Which implies that $X(0, y, z) \mod (x^2+y^2-1)$ is of degree b. As $X(0, y, z) \mod (x^2+y^2-1)$ has the same degree as $X(0, y, z)$, $X(0, y, z)$ must have degree b. The other case follows similarly. □

This is important in proving the next theorem.

Theorem 7.3. *Let $\alpha \in \langle\sigma_1, \sigma_2\rangle$. Assume $(0, y, z)\alpha = (X(y, z), Y(y, z), Z(y, z))$. Also assume that ν as a word in $\sigma_i^{\pm 1}$ corresponds to some word ω in $m_1^{\pm 1}, m_2^{\pm 1}$, where ω is the coset $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and has the property that with the property that $|a| < |b|$, and $|c| < |d|$. Then the degrees of the leading terms of the polynomials X, Y, and Z are given by the following:*

(i) $X(y, z)$ has leading term $c_1 y^{||b|-|a||} z^{|b|-||b|-|a||}$

(ii) $Y(y, z)$ has leading term $c_2 y^{|d|-|c||} z^{|d|-||d|-|c||}$

(iii) $Z(y,z)$ *has leading term* $c_3 y^{||b+d|-|a+c||} z^{|b+d|-||b+d|-|a+c||}$

where c_i*,* $i = 1,2,3$ *are constants.*

Proof. We prove this for the Z case, the rest follow similarly. We induct on $|a|+|b|+|c|+|d|$. If $|a|+|b|+|c|+|d| = 2$, the statement holds, as the possibilities are only $\pm I$. This is the smallest possible case in $PSL(2,\mathbb{Z})$. It suffices to show that if the result holds for $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, it holds for $m_i^{\epsilon} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$, where $|a'| < |b'|$, $|c'| < |d'|$, $i = 1,2$ and $\epsilon = \pm 1$. Also, it is easy to note that since $a'd' - b'c' = 1$, if $|a|$ increases, $|b|$ increases (assuming c,d fixed as in the σ_1 case). The same holds true for $|c|$ and $|d|$ in the σ_2 case. Finally, the fact that $a'd' - b'c' = 1$ allows us to determine the sign of the third element given the other three. We can assume that $a > 0$, by representing our coset with the matrix which has a positive a value. This significantly reduces the number of cases we have to deal with. We proceed with one nontrivial case, the rest follow similarly. Consider the element $m_1 \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Assume $a > 0$, $d > 0$, $c < 0$, and $b < 0$. Then $|c| > |2a|$ and $|d| > |2b|$, or else the magnitude of the values in the $(1,1)$ and $(1,2)$ slots decreases. We can assume that each polynomial can be built up without ever reducing its degree at any step, which allows us to say that we simply require $deg(YZ) = deg(Z')$, where Y,Z are determined by a,b,c,d, and Z' is determined by a', b', c', d'. Thus we desire

$$||d| - |c|| + ||b+d| - |a+c|| = ||b+2d| - |a+2c||.$$

If this holds, then the exponents of y in the third term of $(0,y,z)\alpha$, and the expected values given by the action corresponding to $\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$, are the same. By Lemma 7.2, this determines the exponent of z in the highest degree term. Hence, the theorem is proved for this case. The rest of the cases follow similarly. □

For the last theorem, we assumed that $|a| < |b|$. We now show that every polynomial is equivalent up to sign to a polynomial satisfying the above condition.

Lemma 7.4. *Every polynomial* $P_{a/b}$ *is equivalent up to sign, with a polynomial* $P_{a'/b}$ *where* $|a'| < |b|$*.*

Proof. Again assume that $b > 0$. Note that $(0, y, z)\sigma_2^{\pm 2} = (0, -y, -z)$, and $(0, y, z)\sigma_4^{\pm 4} = (0, y, z)$. Now, by Lemma 6.2, if $(0, y, z)\alpha = (X(y, z), Y(y, z), Z(y, z))$, then $(0, -y, -z)\alpha = (-X(y, z), Y(y, z), -Z(y, z))$. Assume that $\Phi'(\alpha) = A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in PSL(2, \mathbb{Z})$. Now,

$$Am_2^{2k} = \begin{bmatrix} a - 2bk & b \\ c - 2dk & d \end{bmatrix}.$$

The fact that there exists a $k \in \mathbb{Z}$ that forces $|a - 2bk| < |b|$ is a trivial algebraic result. Choosing such k, we note that the coset Am_2^{2k} is the image of $\sigma_2^{2k}\alpha$ under Φ'. Now, $(0, y, z)\sigma_2^{2k}\alpha = (0, \pm y, \pm z)\alpha = (\pm X, \pm Y, \pm Z)$, and this completes the lemma. □

This gives us the main result of the section.

Theorem 7.5. *Each polynomial $P_{a/b}(y, z)$, $0 \leq a < b$ is unique, and each defines a unique set of curves $S_{a/b}(y, z)$. Every other polynomial $P_{a/b}(y, z)$. with a any element of $\mathbb{Z}$ such that* $\gcd(a, b) = 1$, *is equivalent, up to sign, to a polynomial $P_{a'/b}$, $0 \leq a' < b$.*

Chapter 8. Characteristics of the Polynomials $P_b(y,z)$

Definition 8.1. For $b \in \mathbb{N}$, define

$$P_b(y,z) = \sum_{\substack{i=1 \\ \gcd(i,b)=1}}^{b-1} P_{a/b}(y,z). \tag{8.1}$$

The polynomial $P_b(y,z)$ is the object of study for this chapter of the thesis. The polynomial P_b is of interest, because it is the sum of all polynomials corresponding to a particular $b \in \mathbb{Z}$. It has some interesting properties, one of which we can state immediately.

Theorem 8.2. *For any $b \in \mathbb{N}$, $(y+z)$ is always a factor of $P_b(y,z)$*

To prove this, we use the following lemma:

Lemma 8.3. *Assume that $(0,y,z)\alpha = (X,Y,Z)$ is associated to some coset of the form $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where we assume $0 < |a| < |b|$, and $0 < |c| < |d|$. Then, each of $X(y,z)$, $Y(y,z)$, and $Z(y,z)$ is either odd or even in both y and z.*

Proof. We prove this by induction on the length of α as a word in σ_1 and σ_2. It clearly holds in the case where $|\alpha| = 0, 1$. Note that since we don't care that much about the eventual sign, we can apply Theorem 6.4, which will allow us to assume that a, b, c, d are positive, this reduces the number of cases that have to be done. Now assume the lemma is true for $X(y,z)$, $Y(y,z)$ and $Z(y,z)$. The triple (X,Y,Z) is associated to some coset $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. We note that by Theorem 7.3, we can determine the parities of X, Y and Z in the variables y and z in which variable based on the parity of a, b, c, and d. We will do one case. Assume that b is odd, a is even, d is even and c is odd. Then we have that, by Theorem 7.3, that $X(y,z)$ is odd in y, and even in z, $Y(y,z)$ is odd in y and odd in z, and $Z(y,z)$ is even in y, and odd in z. Consider $(X,Y,Z)\sigma_1 = (Z,Y,2YZ-X)$. In this case, the first two elements of the triple, Z and Y, clearly satisfy the lemma, and it remain to show that the last element of the triple does. For the last element, note that

YZ is odd in y and even in z, which matches X, hence $2YZ - X$ is odd in y and even in z, and thus satisfies the lemma. The rest of the cases follow similarly. □

We now prove Theorem 8.2.

Proof. We note that since $\gcd(a,b) = 1$, $X(y,z)$ is odd in exactly of the variables y and z, by Theorem 7.3. Thus plugging in $y = -z$ into $P_{a/b}(t,z)$ and into $P_{b-a/b}(y,z)$ we see that since $P_{a/b}(y,z) = P_{b-a/b}(z,y)$, we have $P_{a/b}(-z,z) + P_{b-a/b}(-z,z) = 0$. Thus, since for each a there is a corresponding $b-a$ that is also relatively prime to b, all terms must cancel out, and we are left with $P_b(-z,z) = 0$ This implies that $(y+z)$ is a factor of $P_b(y,z)$ □

Returning to the polynomial types of $P_{a/b}(y,z) \mod I$ given in Lemma 7.1, we derive further information on the polynomials $P_b(y,z)$. Recall that we have a homomorphism $\Sigma\colon PSL(2,\mathbb{Z}) \to S_6$ defined by letting

$$\sigma_1 \to (1,6,4,3), \sigma_1 \to (2,3,5,6).$$

This image is isomorphic to S_4. A magma calculation tells us that the kernel of Σ is the congruence mod 4 subgroup. This implies that the index of the kernel of Σ is order 24. Magma gives a transversal for $ker(\Sigma)$ that we order as follows.

$$(1)\ \ id = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad (2)\ \ m_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \ (3)\ \ m_2 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix},$$

$$(4)\ \ m_1^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \ (5)\ \ m_2^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ (6\ \ )m_1^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix},$$

$$(7)\ \ m_1 m_2 = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}, \ (8)\ \ m_1 m_2^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \ (9)\ \ m_2 m_1 = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix},$$

$$(10)\ \ m_2^2 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}, \ (11)\ \ m_2 m_1^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}, \ (12)\ \ m_1^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix},$$

$$(13)\ \ m_1^{-1} m_2^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}, \ (14)\ \ m_2^{-1} m_1 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \ (15)\ \ m_2^{-1} m_1^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix},$$

$$(16)\quad m_1^2m_2 = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix},\quad (17)\quad m_1^2m_2^{-1} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix},\quad (18)\quad m_1, m_2m_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

$$(19) m_1m_2^2 = \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix},\quad (20)\quad m_1m_2m_1^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix},\quad (21)\quad m_1m_2^{-1}m_1 = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix},$$

$$(22)\quad m_1m_2^{-1}m_1 = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix},\quad (23)\quad m_2^2m_1 = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix},\quad (24)\quad m_1^2m_2^2 = \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix}.$$

The matrices above correspond to the images $(0, y, z)\alpha$ as follows:

(1) $(0, y, z) = (yU_{-1}, yU_0, T_1)$;

(2) $(0, y, z)\sigma_1 = (z, y, 2yz) = (T_1, yU_0, yU_1)$;

(3) $(0, y, z)\sigma_2 = (0, -z, y) = (yU_{-1}, T_1, yU_0)$;

etc.

Proposition 8.4. *(a) If $a, b \in \mathbb{Z}$ are odd with $a \equiv b \mod 4$, and $\gcd(a, b) = 1$, then*

$$P_{a/b}(y, z) \equiv T_b(z) \mod I.$$

(b) If $a, b \in \mathbb{Z}$ are odd with $a \not\equiv b \mod 4$ and $\gcd(a, b) = 1$, then

$$P_{a/b}(y, z) \equiv -T_b(z) \mod I.$$

(c) If $a, b \in \mathbb{Z}$, $\gcd(a, b) = 1$, with $(a, b) \mod 4$ in $\{(3, 0), (1, 2), (2, 3), (0, 1)\}$, then

$$P_{a/b}(y, z) \equiv yU_{b-1}(z) \mod I.$$

(d) If $a, b \in \mathbb{Z}$, with $\gcd(a, b) = 1$, with $(a, b) \mod 4$ in $\{(0, 3), (2, 1), (3, 2), (1, 0)\}$, then

$$P_{a/b}(y, z) \equiv -yU_{b-1}(z) \mod I.$$

Proof. For $\alpha \in \langle \sigma_1, \sigma_2 \rangle$, we can write $\alpha = \alpha_i \beta$ where $\beta \in \ker(\Sigma)$, and α_i is one of the coset representatives. Let $(X, Y, Z) = (0, y, z)\alpha_i$.
Here, β is in the kernel of Σ so that β does not change the type of the first entry of $(0, y, z)\alpha_i \mod I$. Hence if $(0, y, z)\alpha_i = (T_c, yU_{c+d}, yU_d)$, then

$$(0, y, z)\alpha = (0, y, z)\alpha_i\beta = (T_{c'}, yU_{c'+d'}, yU_{d'}).$$

Given this, we simply check the 24 α_i cases. We have already checked the first few above. □

This finally allows us to prove the following.

Corollary 8.5. *Let p be an odd prime.*

(i) If $p \equiv 1 \mod 4$, then $\sum_{i=1}^{p-1} P_{i/p}(y, z) \in I$.

(ii) If $p \equiv 3 \mod 4$, then $\sum_{i=1}^{p-1} P_{i/p}(y, z) \notin I$.

Proof. (i) if $i \equiv 1 \mod p$, then Proposition 8.4 (a) gives $P_{i/p}(y, z) = T_p(z) \mod I$. If $i \equiv 2 \mod p$, then Proposition 8.4 (d) gives $P_{i/p}(y, z) = -T_p(z) \mod I$. If $i \equiv 3 \mod p$, then Proposition 8.4 (b) gives $P_{i/p}(y, z) = -T_p(z) \mod I$. If $i \equiv 0 \mod p$, then Proposition 8.4 (c) gives $P_{i/p}(y, z) = yU_{p-1}(z) \mod I$.
Thus, considering the sum $\sum_{i=1}^{p-1} P_{i/p}$ we see that writing $i = 1 + 4k, 2 + 4k, 3 + 4k, 4 + 4k$, we obtain the terms $T_p(z) - yU_{p-1}(z) - T_p(z) + yU_{p-1}(z) = 0$. Given that $p \equiv 1 \mod 4$, the result follows, as $p - 1 \equiv 0 \mod 4$.

(ii) Using the same idea, Proposition 8.4 tells us that the contribution to the sum $\sum_{i=1}^{p-1} P_{i/p}$ from $i = 1 + 4k, 2 + 4k, 3 + 4k, 4 + 4k$ is $-T_p(z) + yU_{p-1}(z) + T_p(z) - yU_{p-1}(z) = 0$. However, since $p \equiv 3 \mod 4$, and the cycle that leads to cancellation is of length four, the sum cannot be $0 \mod I$. In particular, the sum is equal to $-T_p(z) + yU_{p-1}(z) \not\equiv 0 \mod I$.

This concludes the proof. □

Corollary 8.5 implies several ideas. The first is that $(x^2 + y^2 - 1)$ is a factor of $P_b(y, z)$ for all b given that b is prime and $b \equiv 1 \mod 4$. A consequence of this is that the intersection of $P_b(y, z)$ and the level surface $\mathcal{T}$ in the plane $x = 0$ is the entirety of $\mathcal{T}$ intersected with the $x = 0$ plane.

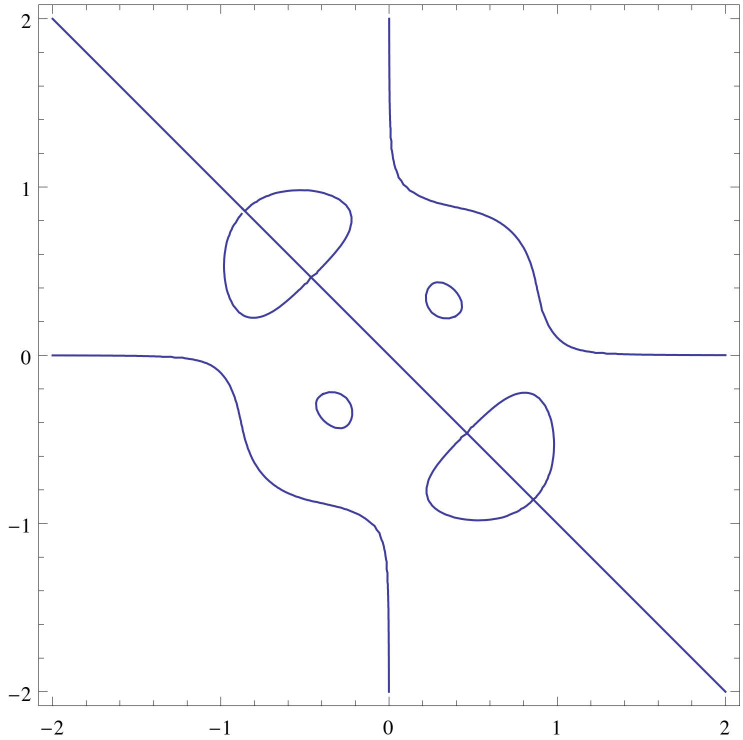


Figure 8.1: Contour Plot of $P_7(y, z) = 0$

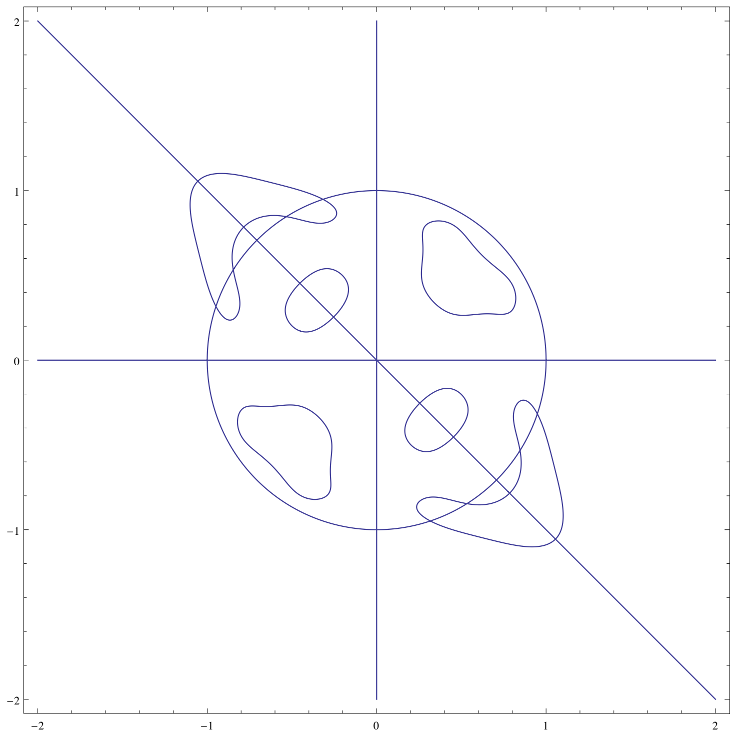


Figure 8.2: Contour Plot of $P_{13}(y, z) = 0$

We conclude this thesis with a review of the results. We have studied the duality of the fixed points on $\mathcal{T}$ of the action induced by α_n. We have not studied this in general for γ_{nm}, but we know how to do so. We understand the fixed points of the action of α_n and γ_{nm} on $\mathcal{T}$. We have identified and studied a class of period two points of the action of γ_{nm} on $\mathbb{R}^3$. We have not been able to study the fixed points on $\mathcal{T}$ for most elements of B_3. We do not yet have a complete understanding of the period two points of either α_n or γ_{nm}, nor have we studied the duality of the period two points we have found. In the second part of the thesis, we have a fairly good understanding of the properties of the polynomials $P_{a/b}(y,z)$. We have shown that understanding these polynomials reduces to understanding the polynomials $P_{a'/b}(y,z)$, where $0 < a' < b$, and we understand several of the properties that the sum $P_b(y,z)$ has, where $P_b(y,z)$ is the sum of the distinct polynomials $P_{a/b}(y,z)$, for a specific b value.

9.1 Further Research

Further research questions might include: explicitly calculating the dual points of the action of γ_{nm}; investigating the duality of period two points; investigating other trace maps, along with their invariant sets; and determining what families of diffeomorphisms of $\mathbb{R}^3$ are induced by trace maps.

In addition to the previous, there are several other things that we do not understand very well, that have not received any attention in the thesis so far, but are directly related to the results therein. Consider the following image, given in Section 2.5.

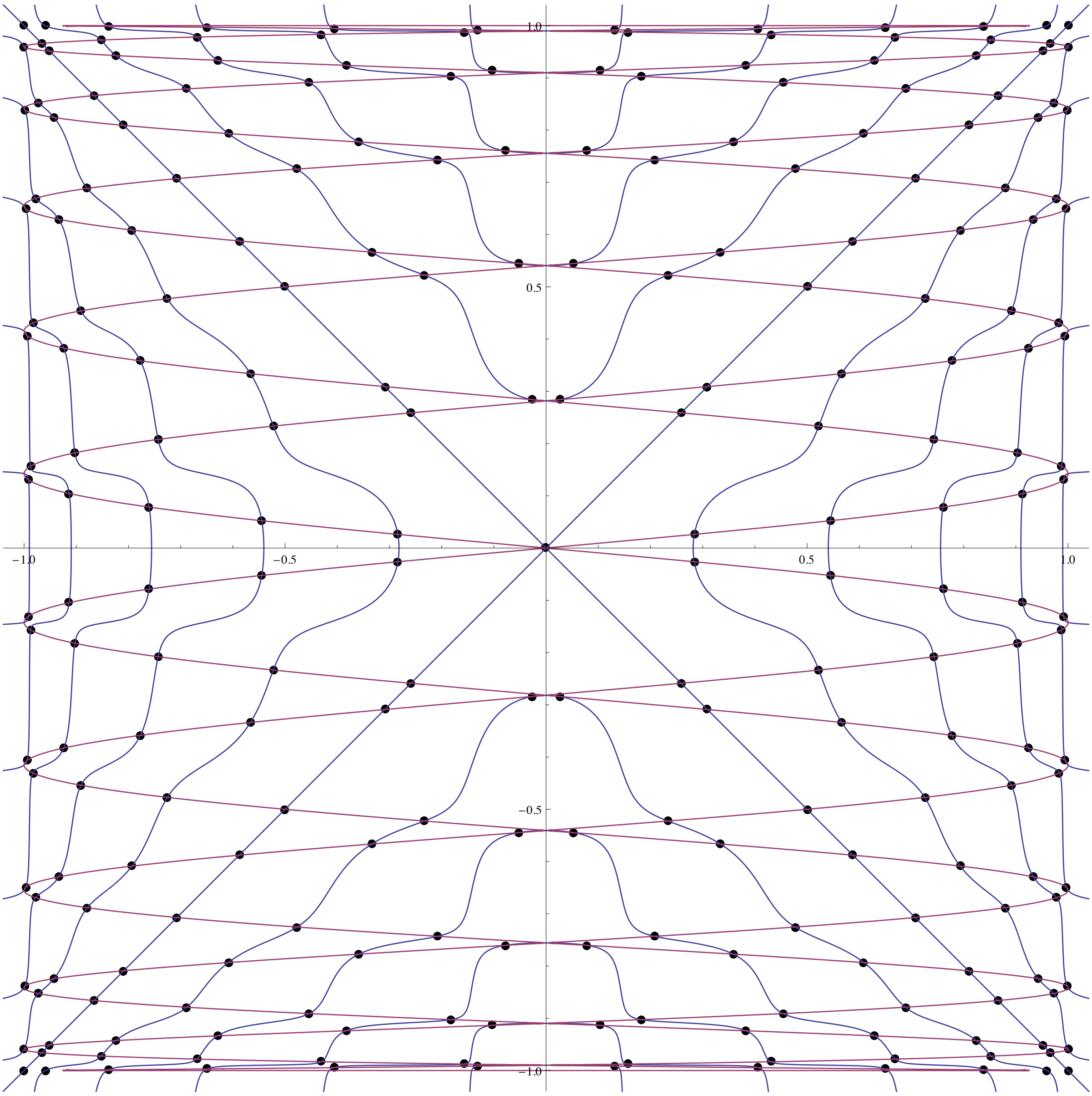


Figure 9.1: Projection of $\mathcal{T}$, curves given by G_{22}, and fixed points.

Consider the set of intersection points marked on the above image. Recall that these are the x and y coordinates of the reversing fixed points of the action of α_{22}, unioned with the preserving fixed points of α_{22} that lie on the boundary of the unit square.

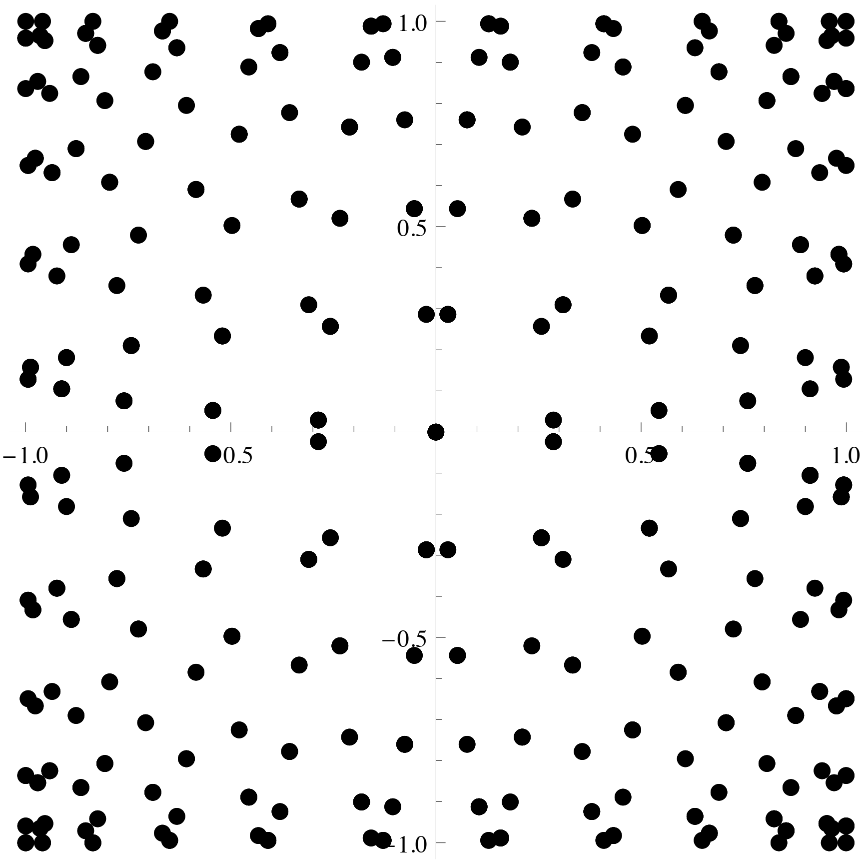


Figure 9.2: The x and y coordinates of reversing and boundary points of the action of α_{22} on $\mathcal{T}$

If we take the curves given by projecting $\mathcal{T}$ to the $x-y$ plane using the substitution $z = x(1 + U_{n-2}(y))/U_{n-1}(y)$, and rotate them 90°, and superimpose the points given in Figure 9.2 onto these curves, we obtain the following.

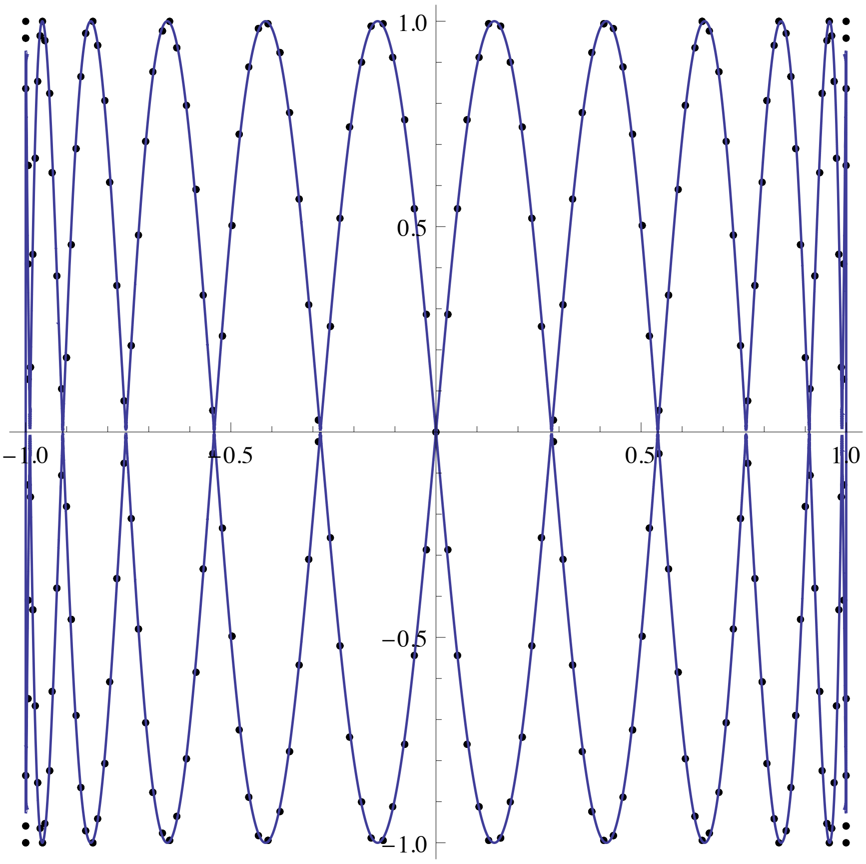


Figure 9.3: The x and y coordinates of the reversing and boundary points, with a rotation of the projection of $\mathcal{T}$.

We see that each reversing fixed point, and each boundary point lies not only on the projection of $\mathcal{T}$, but also on the projection of $\mathcal{T}$ rotated 90°. This isn't that surprising, as the system is very symmetrical. However, if we look at the projection of $\mathcal{T}$ and its rotation, as well as the reversing fixed points and the boundary points of the action of α_{22}, we obtain the following image.

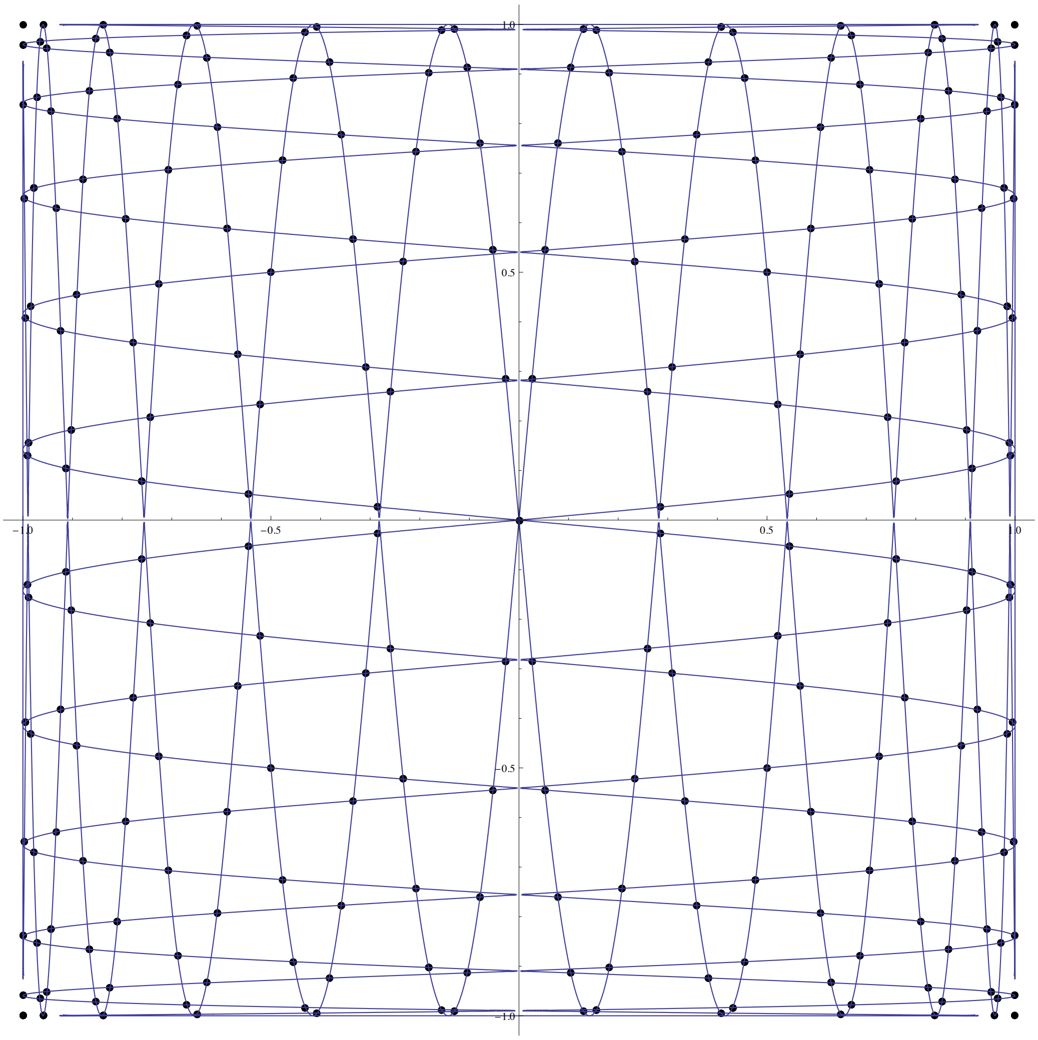


Figure 9.4: The x and y coordinates of the reversing and boundary points, with the projection of $\mathcal{T}$ and its rotation.

Figure 9.4 shows that each of the previous points lie on the intersection of the projection of $\mathcal{T}$, and its 90° rotation. However, these points only account for approximately half of the intersection points. Consider the following.

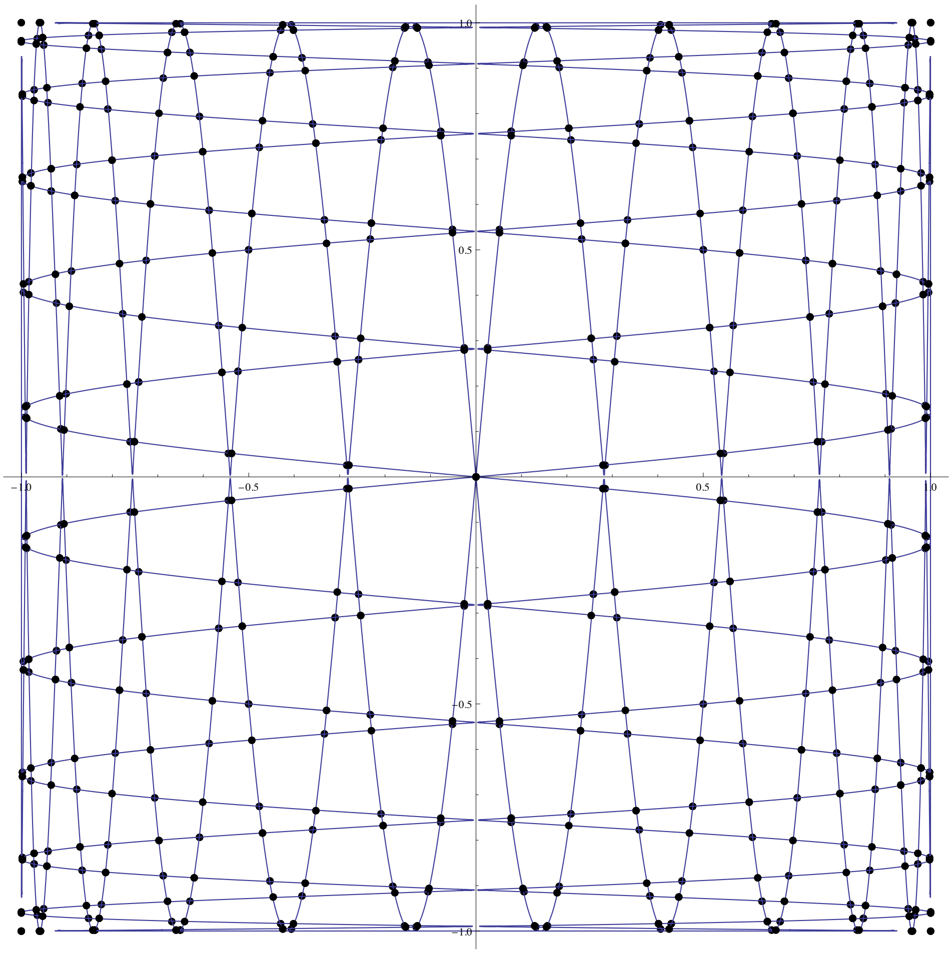


Figure 9.5: The x and y coordinates of the intersection of the projection of $\mathcal{T}$ and its rotation.

Recall that the element $\alpha_{22} \in B_3$ acts on $\mathbb{R}$ via the diffeomorphism $\sigma_1^{22}\sigma_2^{22}$. In Figure 9.5, we have plotted the x and y coordinates of the reversing and boundary points of the diffeomorphism $\sigma_1^{22}\sigma_2^{-22}$ restricted to $\mathcal{T}$, in addition to the x and y coordinates of the reversing and boundary points of the diffeomorphism $\sigma_1^{22}\sigma_2^{22}$.

The fact that the intersections of the projection of $\mathcal{T}$ and its rotation yield the reversing and boundary points of the diffeomorphisms $\sigma_1^n\sigma_2^n$ and $\sigma_1^n\sigma_2^{-n}$ holds for each example we have tried, and in fact would not be particularly difficult to prove in general, but we do not know why this occurs. It is worth investigating, as it could shed light on the relationship between trace maps and invariant sets.

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