2014-06-01

Convolutions and Convex Combinations of Harmonic Mappings of the Disk

Zachary M. Boyd
Brigham Young University - Provo

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Zachary M Boyd

A thesis submitted to the faculty of
Brigham Young University
in partial fulfillment of the requirements for the degree of

Master of Science

Michael Dorff, Chair
David Cardon
Gary Lawlor

Department of Mathematics

Brigham Young University

June 2014

ABSTRACT

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Zachary M Boyd
Department of Mathematics, BYU
Master of Science

Let f_1, f_2 be univalent harmonic mappings of some planar domain D into the complex plane $\mathbb{C}$. This thesis contains results concerning conditions under which the convolution $f_1 * f_2$ or the convex combination $tf_1 + (1-t)f_2$ is univalent. This is a long-standing problem, and I provide several partial solutions. I also include applications to minimal surfaces.

Keywords: Geometric function theory, harmonic maps, minimal surfaces, differential geometry

Acknowledgments

I would like to sincerely thank the research partners who patiently taught, mentored, and worked with me. This includes Robert Buss, Susanna Fulmer, Ife George, Rachel Messick, Matthew Romney, Ryan Viertel, and Daniel Walton. Dr. Dorff especially has taken hundreds of hours to work with me and has given me room and time to develop and grow. Above all, thanks are due to my wife, Katie, who has supported me in countless late nights, meetings, and working vacations. Without her, this thesis would have been impossible.

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Chapter 1. Introduction

1.1 Overview

Harmonic mappings in geometric function theory has received a lot of attention in the last three decades and has become a large and diverse subject, including topics as wide-ranging as p-valent functions, boundary characterizations, and coefficient bounds. This thesis focuses primarily on convolution of harmonic mappings and its relationship to univalence and other geometric conditions. Several original theorems on this subject comprise the bulk of this work, although I include substantial excursions into the study of convex combinations and minimal surface theory.

In this first chapter we will discuss some background material about planar harmonic mappings. These functions can be thought of as a generalization of analytic maps, and so we will first present a brief overview of analytic univalent mappings. Then we will discuss harmonic mappings with an emphasis on three topics: the shearing technique, inner mapping radius, and convolutions. Finally, we will discuss the connection between planar harmonic mappings and minimal surfaces.

1.2 Analytic univalent maps

Definition 1.1. Let $F\colon D \subset \mathbb{C} \to \mathbb{C}$. The function $F(x+iy) = u(x+iy) + iv(x+iy)$ is analytic if:

- u and v are real harmonic in D; and
- u and v are harmonic conjugates
  (that is, $u_x = v_y$ and $u_y = -v_x$).

In this context, a function $u(x,y)\colon \mathbb{R}^2 \to \mathbb{R}$ is called *real harmonic* if $u_{xx} + u_{yy} = 0$.

While analytic functions may map from any open, connected set in general, the following theorem allows us to restrict attention to the unit disk $\mathbb{D}$ in many cases.

Theorem 1.2 (Riemann Mapping Theorem)**.** *Let $G \neq \mathbb{C}$ be a simply-connected domain with $a \in G$. Then there exists a unique univalent, surjective, analytic function $F\colon G \to \mathbb{D}$ such that $F(a) = 0$ and $F'(a) > 0$.*

Thus if D is a simply-connected, proper subset of the complex plane, we may replace the function $f\colon D \to \mathbb{C}$ by the function $f \circ \phi\colon \mathbb{D} \to \mathbb{C}$, where the existence of $\phi\colon \mathbb{D} \to D$ is guaranteed. Therefore, in the study of univalent (one-to-one) analytic functions, we may restrict our attention to the following class of functions.

Definition 1.3. The family of analytic, normalized, univalent functions denoted by S is

$$S = \{F\colon \mathbb{D} \to \mathbb{C} \,|\, F \text{ is analytic and univalent}, F(0) = 0, F'(0) = 1\}.$$

The functions in this family are also known as *schlicht* functions. Note that $F \in S$ implies $F(z) = z + a_2z^2 + a_3z^3 + \cdots$. The following are two essential examples that will be used throughout the paper.

Example 1.4 (The analytic right half-plane mapping)**.**

$$F_h(z) = \frac{z}{1-z} = \sum_{n=1}^{\infty} z^n = z + z^2 + z^3 + \cdots \in S.$$

Example 1.5 (The Koebe function)**.**

$$F_k(z) = \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} nz^n = z + 2z^2 + 3z^3 + \cdots \in S.$$

Note that F_k maps to the entire complex plane minus a slit from $-1/4$ to ∞ (see figure 1.1).

Some important properties of the family S include

- The uniqueness condition in the Riemann Mapping Theorem.

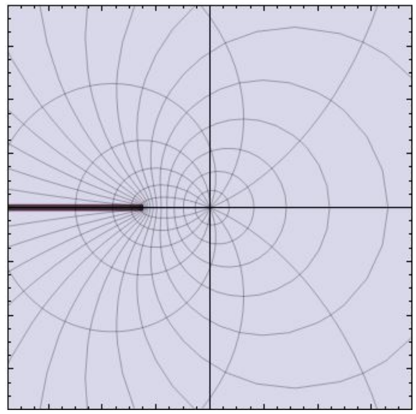


Figure 1.1: The image of $\mathbb{D}$ under $F_k(z) = \frac{z}{(1-z)^2} \in S$.

- (de Branges' Theorem) For $F \in S$, $|a_n| \leq n$, for all n.
- (Koebe $\frac{1}{4}$-Theorem) If $F \in S$, then $F(\mathbb{D})$ contains the disk $G = \{w : |w| < \frac{1}{4}\}$.

See [17] for more background in univalent analytic functions.

1.3 Harmonic univalent maps

Complex-valued harmonic functions are a generalization of the analytic functions in which the one of the requirements is relaxed.

Definition 1.6. Let $f\colon D \subset \mathbb{C} \to \mathbb{C}$. The function $f(x,y) = u(x,y) + iv(x,y)$ is a (complex-valued) harmonic function if:

- f is continuous; and
- u and v are real harmonic in D.

This definition views harmonic functions as being composed of real and imaginary parts. If D is simply-connected, we have a useful characterization (see [4]).

Theorem 1.7. *If $f = u + iv$ is harmonic in a simply-connected domain G, then $f = h + \overline{g}$, for some analytic functions h and g.*

Note that $f = h + \overline{g}$ is equivalent to $f = \mathrm{Re}\{h+g\} + i\,\mathrm{Im}\{h-g\}$. Also, one consequence of this theorem is that a harmonic function f is represented by a power series of the form

$$f(z) = h(z) + \overline{g(z)} = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n \overline{z}^n.$$

In particular, every harmonic function with domain $\mathbb{D}$ is just the sum of analytic and coanalytic parts, represented by h and $\overline{g}$, respectively. To see the geometric effect of including $\overline{g}$, we recall that an analytic map is called *conformal* if the angle between intersecting curves in the domain is the same as the angle between their image curves. A harmonic map is the sum of two maps, one which preserves angles, and another which reverses them. After some reflection, it should be clear that if $|h'(z_0)| > |g'(z_0)|$, then the map is *sense-preserving* at z_0, meaning that positive angles remain positive, and negative angles remain negative under the map f. Equivalently, we say that a function is sense-preserving if the left-hand side of a directed curve is mapped to the left-hand side of its image. The following theorem formalizes this intuition.

Theorem 1.8 (Lewy [30]). *$f(z) = h(z) + \overline{g(z)}$ is locally univalent and sense-preserving if and only if $|\omega(z)| = |g'(z)/h'(z)| < 1$ for all $z \in \mathbb{D}$.*

The function $\omega = g'/h'$ is known as the dilatation of $f = h + \overline{g}$.

In the pages that follow, graphs of functions are usually the image of the unit disk under the function in question. Also, many of these images have been created by the online applet *ComplexTool* [12].

Example 1.9.

- **Analytic polynomial map:** $F_p(z) = z - \frac{1}{2}z^2$

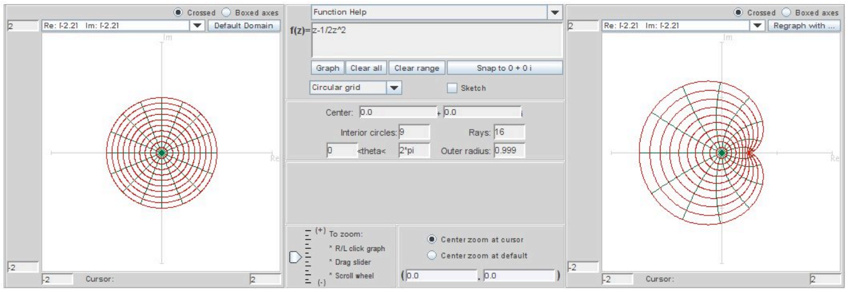


Figure 1.2: The image of $\mathbb{D}$ under F_p.

- **Harmonic polynomial map:** $f_p(z) = z + \frac{1}{2}\overline{z}^2$

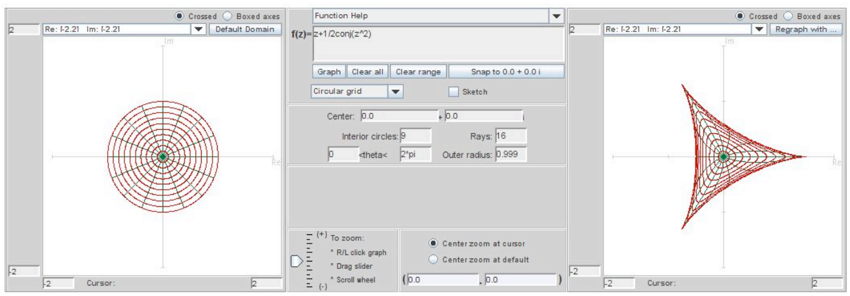


Figure 1.3: The image of $\mathbb{D}$ under f_p.

Example 1.10.

- **Analytic right half-plane map:** $F_h(z) = \frac{z}{1-z}$

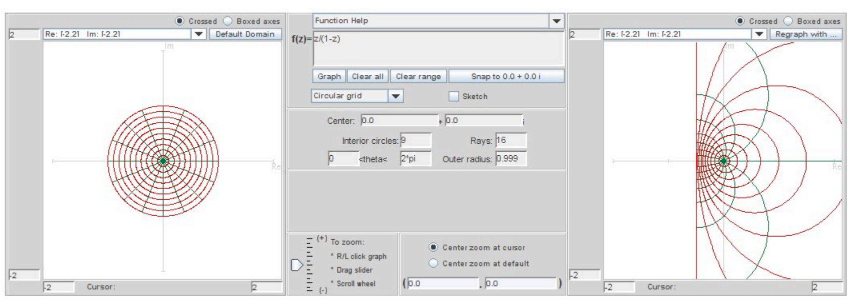


Figure 1.4: The image of $\mathbb{D}$ under F_h.

• **Harmonic right half-plane map:** $f_h(z) = \mathrm{Re}\left(\frac{z}{1-z}\right) + i\,\mathrm{Im}\left(\frac{z}{(1-z)^2}\right)$

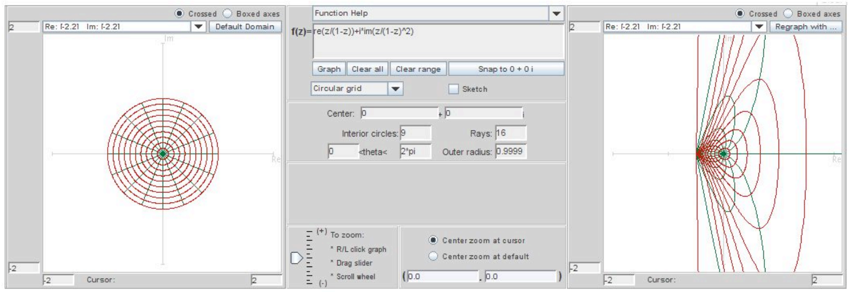


Figure 1.5: The image of $\mathbb{D}$ under f_h.

Observe that in the harmonic case, terms involving $\overline{z}$ are permissible, but terms involving $z\overline{z}$ are not. Also, the graphics highlight the fact that the images of radial and circular lines intersect at right angles in the conformal case, but not in the harmonic case.

The boundary of $f_p(\mathbb{D})$ in Example 1.9 consists of concave arcs, and the boundary of $f_h(\mathbb{D})$ in Example 1.10 gets mapped to just two points, $w = -\frac{1}{2}$ and $w = \infty$. These examples illustrate an important fact about the boundary behavior of certain harmonic functions:

Theorem 1.11 (Weitsman [42]). *Let f be a univalent harmonic mapping with smooth boundary extension. Suppose for any sequence $f(z_n) = \zeta_n \to \zeta \in \partial f(\mathbb{D})$, we have $|\omega(z_n)| \to 1$. Then the boundary of $f(\mathbb{D})$ is a concave arc.*

Example 1.10 also shows that the uniqueness part of the Riemann mapping theorem fails in the harmonic case, since both maps, F_h and f_h, send the disk to the same right half-plane.

As a final point in this section, we note that, in analogy to S, we define the classes S_H and S_H^O as follows.

Definition 1.12. Let S_H be the family of complex-valued, harmonic, univalent mappings that are normalized on the unit disk; that is,

$$S_H = \{f\colon \mathbb{D} \to \mathbb{C} \mid f \text{ is harmonic and univalent}$$
$$f(0) = a_0 = 0, f_z(0) = a_1 = 1\}.$$
$$S_H^O = \{f \in S_H \mid f_{\overline{z}}(0) = b_1 = 0\}.$$

Thus, $S \subset S_H^O \subset S_H$. Other important classes include K, K_H, and K_H^O, which are the subclasses of S, S_H, and S_H^O containing only the *convex* functions, which are exactly those whose image is a convex domain in $\mathbb{C}$.

We now introduce some major unsolved problems in the field that have obvious analogues in the theory of analytic functions. For years, the biggest problem in the theory of univalent analytic functions was the *Bieberbach Conjecture*, which was solved by DeBrange in 1984. Solving this problem allows us to know the sharp bounds on growth and distortion of harmonic maps, among other things. In the non-analytic case, we have the following. (See [16] for background on these fundamental conjectures.)

Conjecture 1 (Harmonic Bieberbach Conjecture)**.** *Let*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n \overline{z}^n \in S_H^O.$$

Then

- $|a_n| \leq \frac{1}{6}(n+1)(2n+1)$,
- $|b_n| \leq \frac{1}{6}(n-1)(2n-1)$,
- $\big||a_n| - |b_n|\big| \leq n$.

Currently, the best proven bound is that for all functions $f \in S_H^O$, $|a_2| < 49$. The conjecture is that $|a_2| \leq \frac{5}{2}$.

Recall that for analytic functions we have the Koebe $\frac{1}{4}$-Theorem, which gives the minimum distance between the origin and the boundary image of f for analytic functions. In the harmonic case, we have

Conjecture 2. *If $f \in S_H^O$, then $f(\mathbb{D})$ contains the disk $G = \left\{w : |w| < \frac{1}{6}\right\}$.*

Currently, the best result is that the range of $f \in S_H^O$ contains the disk $\left\{w : |w| < \frac{1}{16}\right\}$.

1.4 Shearing

In their paper, Clunie and Sheil-Small introduced the shearing technique that provides a procedure for constructing harmonic maps $f = h + \overline{g}$ that are univalent. Before describing the shearing technique, we need the following definition.

Definition 1.13. A domain Ω is convex in the horizontal direction (CHD) if every line parallel to the real axis has a connected intersection with Ω.

In Figure 1.6 we see examples of this concept. We can now state the Shearing Theorem.

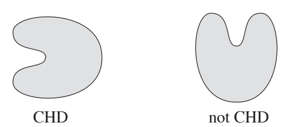


Figure 1.6: Convex in the horizontal direction (CHD).

Theorem 1.14 (Shearing Theorem, [4]). *Let $f = h + \overline{g}$ be a harmonic function that is locally univalent in $\mathbb{D}$ (i.e., $|\omega(z)| < 1$ for all $z \in \mathbb{D}$). The function $F = h - g$ is an analytic univalent mapping of $\mathbb{D}$ onto a CHD domain if and only if $f = h + \overline{g}$ is a univalent mapping of $\mathbb{D}$ onto a CHD domain.*

Summary of the Shearing Technique: To use the shearing technique we start with

- an analytic function F that is CHD, and
- an analytic function ω such that $|\omega(z)| < 1$ for all $z \in \mathbb{D}$.

Then we solve the system of equations $F = h - g$ and $\omega = g'/h'$ for g and h. The resulting harmonic function $f = h + \overline{g}$ is guaranteed to be univalent.

Notice that it is easy to reformulate Clunie and Sheil-Small's Shearing Theorem for functions which are convex in other directions. In particular, consider the case of convex in the vertical direction which we will use in this paper.

Definition 1.15. A domain Ω is convex in the vertical direction (CVD) if every line parallel to the imaginary axis has a connected intersection with Ω.

Theorem 1.16. *Let $f = h + \overline{g}$ be a harmonic function that is locally univalent in $\mathbb{D}$ (i.e., $|\omega(z)| < 1$ for all $z \in \mathbb{D}$). The function $F = h + g$ is an analytic univalent mapping of $\mathbb{D}$ onto a CVD domain if and only if $f = h + \overline{g}$ is a univalent mapping of $\mathbb{D}$ onto a CVD domain.*

More generally, shearing in any direction θ can be achieved by considering the function $h - e^{2i\theta}g$. In [4] it was shown that a harmonic function maps to a convex domain if and only if its shears in every direction are convex in that direction.

Example 1.17. Consider the analytic function

$$F_p(z) = z - \tfrac{1}{2}z^2.$$

This is the analytic polynomial map F_p given in Example 1.9. It is CHD. Now choose a dilatation. We will choose

$$\omega(z) = g'(z)/h'(z) = z.$$

Note that $|\omega(z)| < 1 \; \forall z \in \mathbb{D}$. Next, set $h(z) - g(z) = F_p(z) = z - \frac{1}{2}z^2$. Taking the derivative of both sides yields $h'(z) - g'(z) = 1 - z$. Since $g'(z) = zh'(z)$, we substitute $g'(z)$ into the previous equation to get $h'(z) = 1$. Integrating this and normalizing it so that $h(0) = 0$, yields $h(z) = z$. Because $g'(z) = zh'(z)$, we can solve for g to get $g(z) = \frac{1}{2}z^2$. Hence, by the Shearing Theorem

$$f_p(z) = h(z) + \overline{g(z)} = z + \tfrac{1}{2}\,\overline{z}^2 \in S_H^O.$$

Thus, we have constructed a harmonic function f_p that is univalent and CHD. Note that this is the harmonic polynomial function f_p in Example 1.9.

Example 1.18. Consider

$$F_k(z) = h(z) - g(z) = \frac{z}{(1-z)^2} \quad \text{with} \quad \omega(z) = z.$$

Using the same approach as above, we get

$$f_k(z) = h(z) + \overline{g(z)} = \operatorname{Re}\left(\frac{z + \frac{1}{3}z^3}{(1-z)^3}\right) + i\operatorname{Im}\left(\frac{z}{(1-z)^2}\right) \in S_H^O.$$

The harmonic function f_k is a slit mapping which maps $\mathbb{D}$ onto $\mathbb{C}$ minus a slit on the negative real axis with the tip of the slit at $-\frac{1}{6}$. There is considerable evidence that f_k can fill a role in harmonic function theory similar to that of the Koebe function from Example 1.5 in analytic function theory , and for this reason, f_k is called the *harmonic Koebe function.*

To help explore how shearing affects the geometry between analytic and harmonic mappings, one can use the online applet *ShearTool* [12]. The image below demonstrates the functionality of this applet, which simultaneously plots both $h - g$ and $h + \overline{g}$.

Almost all examples of shearing have used dilatations that are finite Blaschke products. One important type of mappings that are not finite Blaschke products is a singular inner function. We give a brief description of this topic. For more details, see [29].

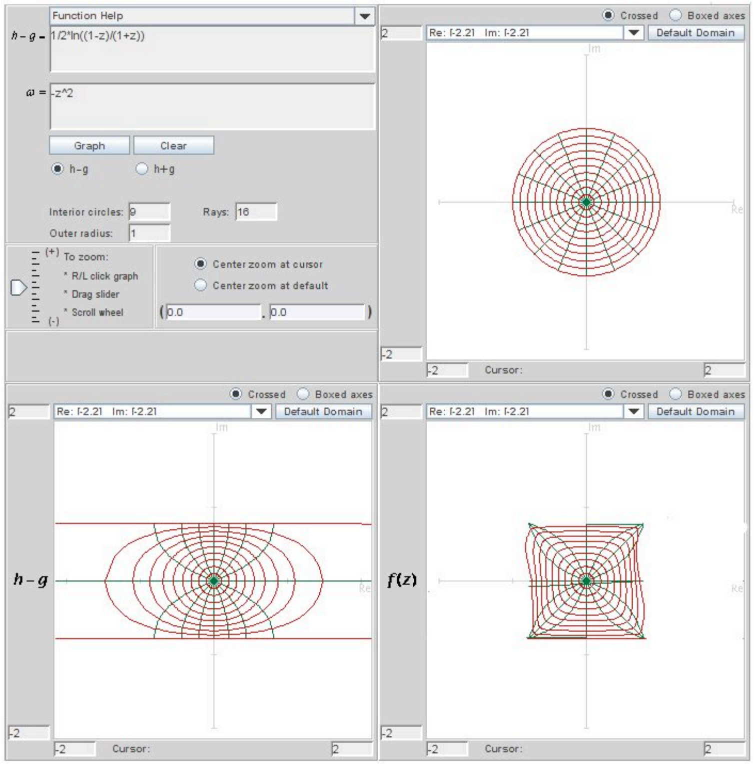


Figure 1.7: The image of $\mathbb{D}$ under the $f = h + \overline{g}$ is shown in the bottom right, where f is constructed from shearing $h(z) - g(z) = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$ with $\omega(z) = -z^2$.

Definition 1.19. A bounded analytic function f is called an *inner function* if $|\lim_{r\to 1^-} f(re^{i\theta})| = 1$ almost everywhere with respect to Lebesgue measure on $\partial\mathbb{D}$. If f has no zeros on $\mathbb{D}$, then f is called a *singular inner function.*

Every inner function can be expressed in the form

$$f(z) = e^{i\alpha}B(z)\exp\left(-\int \frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\mu\left(e^{i\theta}\right)\right),$$

where $\alpha, \theta \in R$, μ is a positive measure on $\partial\mathbb{D}$, and $B(z)$ is a Blaschke product, i.e. $B(z) = e^{i\theta}\prod_{j=1}^{\infty}\left(\frac{z-a_j}{1-\overline{a_j}z}\right)^{m_j}$, for some series of constants $|a_j| < 1$ satisfying $\sum_{n=1}^{\infty}(1-|a_n|) < \infty$.

The function $f(z) = e^{\frac{z+1}{z-1}}$ is an example of a singular inner function. Weitsman [41] provided the following example.

Example 1.20. Shear

$$h(z) - g(z) = \frac{z}{1-z} + \frac{1}{2}e^{\frac{z+1}{z-1}} \qquad \text{with} \qquad \omega(z) = e^{\frac{z+1}{z-1}}.$$

By a result by Pommenke [35], it can be shown that $h - g$ is convex in the direction of the real axis. Shearing $h - g$ with $\omega(z) = e^{\frac{z+1}{z-1}}$ and normalizing yields

$$h(z) = \int \frac{1}{(1-z)^2}\, dz = \frac{z}{1-z}.$$

Solving for g we get

$$g(z) = -\tfrac{1}{2}e^{\frac{z+1}{z-1}}.$$

The image given by the map is similar to the image given by the right half-plane map $\frac{z}{1-z}$ except that there are an infinite number of cusps (see Figure 1.8).

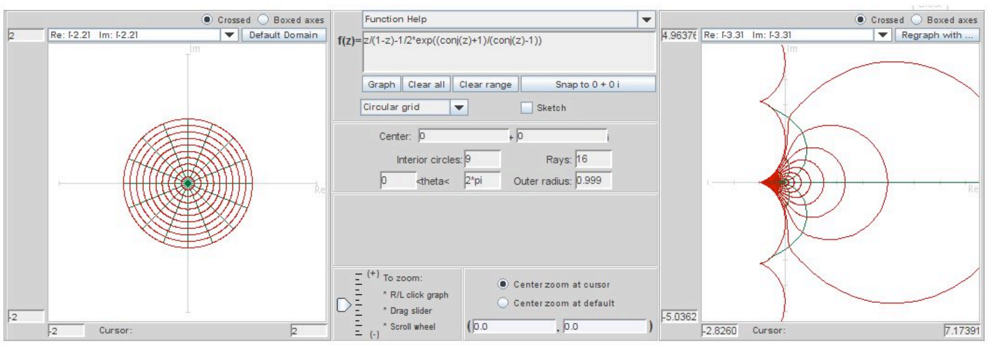


Figure 1.8: Image of $\mathbb{D}$ under $f(z) = \frac{z}{1-z} - \frac{1}{2}e^{\frac{\overline{z}+1}{\overline{z}-1}}$.

A technique to find harmonic mappings whose dilatations are singular inner functions involves using a theorem by Clunie and Sheil-Small [4].

Theorem 1.21. *Let $f = h + \overline{g}$ be locally univalent in $\mathbb{D}$ and suppose that $h + cg$ is convex for some $|c| \leq 1$. Then f is univalent.*

To develop the technique, we let $c = 0$ in Theorem 1.21. This means that if h is analytic convex and if ω is analytic with $|\omega(z)| < 1$, then $f = h + \overline{g}$ is a harmonic univalent mapping. To establish that a function f is convex, we will use the following theorem (see [17]).

Theorem 1.22. *Let f be analytic in $\mathbb{D}$ with $f(0) = 0$ and $f'(0) = 1$. Then f is univalent and maps onto a convex domain if and only if*

$$\operatorname{Re}\left[1 + \frac{zf''(z)}{f'(z)}\right] \geq 0 \text{ for all } z \in \mathbb{D}.$$

Example 1.23. Let

$$h(z) = z + 2\log(z+1) \quad \text{with} \quad \omega(z) = g'(z)/h'(z) = e^{\frac{z-1}{z+1}}.$$

Using Theorem 1.22, we can show that h is convex. Then solving for g we get $g(z) = (z+1)e^{(z-1)/(z+1)}$.

Hence,

$$f(z) = h(z) + \overline{g(z)} = z + 2\log(z+1) + (\overline{z}+1)e^{\frac{\overline{z}-1}{\overline{z}+1}}.$$

By Theorem 1.21, $f = h + \overline{g}$ is univalent. The image of $\mathbb{D}$ under f is shown in Figure 1.9.

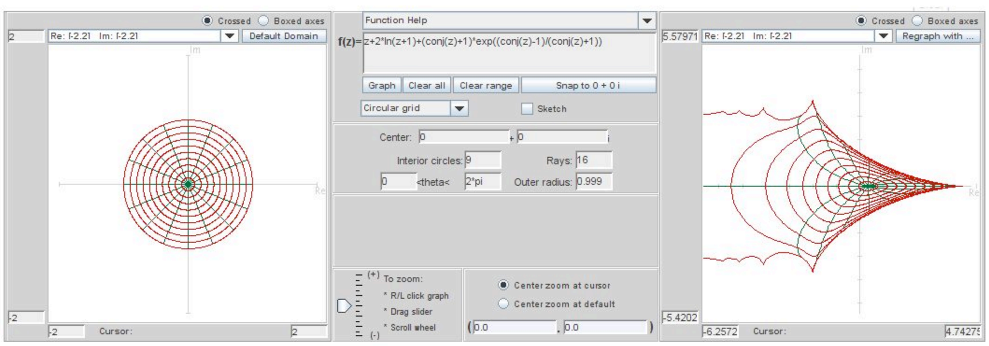


Figure 1.9: Image of $\mathbb{D}$ under $f(z) = h + \overline{g}$ in Example 1.23.

1.5 Convolutions

The shearing technique given in Theorem 1.14 provides a way to construct harmonic functions that are univalent. This approach requires certain conditions in order to apply the technique. Convolutions is another approach to construct harmonic univalent functions. It also requires certain conditions in order to guarantee that the resulting functions are univalent. In addition, the study of convolutions is an interesting topic on its own.

The convolution of harmonic functions is a generalization of the convolution of analytic functions which is an important area in the study of schlicht functions (see [37] for more information about the convolution of analytic functions). However, many of the nice theorems in the analytic case do not carry over to the harmonic case. For example, the Pólya-Schoenberg conjecture which was proved by Ruscheweyh and Sheil-Small states that convexity is preserved under analytic convolution. This convexity preserving property does not hold for harmonic convolutions. But there are several open areas related to harmonic convolutions to investigate. In this section we will explore some of these. For more details about harmonic convolutions, see [5].

Let's begin with with the definition of the convolution for analytic functions.

Definition 1.24 (Analytic Convolution)**.** Given $F_1, F_2 \in S$ represented by

$$F_1(z) = \sum_{n=0}^{\infty} A_n z^n \quad \text{and} \quad F_2(z) = \sum_{n=0}^{\infty} B_n z^n,$$

their convolution is defined as

$$F_1(z) * F_2(z) = \sum_{n=0}^{\infty} A_n B_n z^n.$$

As mentioned above, the analytic convolution preserves convexity since $F_1, F_2 \in K \Rightarrow F_1 * F_2 \in K$. The algebra of convolutions is also simplified by viewing certain functions as operators. For instance, $F(z) = \frac{z}{1-z}$ is the convolution identity because its power series is

$z + z^2 + z^3 + \cdots$.

We define an analogous operation for harmonic functions as follows:

Definition 1.25. Given

$$f_1 = h_1 + \overline{g_1} = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b}_n \, \overline{z}^n \quad \text{and}$$
$$f_2 = h_2 + \overline{g_2} = z + \sum_{n=2}^{\infty} c_n z^n + \sum_{n=1}^{\infty} \overline{d}_n \, \overline{z}^n,$$

define harmonic convolution as

$$f_1 * f_2 = h_1 * h_2 + \overline{g_1 * g_2} = z + \sum_{n=2}^{\infty} a_n c_n z^n + \sum_{n=1}^{\infty} \overline{b_n d_n} \, \overline{z}^n.$$

Harmonic convolutions involve difficulties not present in the analytic case. For instance, it is not difficult to find $f_1, f_2 \in K_H^O$ such that $f_1 * f_2 \notin K_H^O$. In fact, $f_1 * f_2$ may even fail to be univalent. The example below illustrates this.

Example 1.26. Let $f_h = h_h + \overline{g}_h \in K_H^O$ be the harmonic right half-plane map in Example 1.10, where

$$h_h(z) = \frac{z - \frac{1}{2}z^2}{(1-z)^2}, \quad g_h(z) = \frac{-\frac{1}{2}z^2}{(1-z)^2},$$

and let $f_2 = h_2 + \overline{g}_2 \in K_H^O$ be the canonical regular 6-gon map, where

$$h_2(z) = z + \sum_{n=1}^{\infty} \tfrac{1}{6n+1} z^{6n+1}, \quad g_2(z) = \sum_{n=1}^{\infty} \tfrac{-1}{6n-1} z^{6n-1}.$$

Then $f_h * f_2$ is not univalent, because

$$|(g_h(z) * g_2(z))'/(h_h(z) * h_2(z))'| = |z^4(2+z^6)/(1+2z^6)| \not< 1, \forall z \in \mathbb{D}.$$

Open Problem 1. *Let $f_1, f_2 \in K_H^O$. Since $f_1 * f_2$ is not necessarily univalent, what additional* *conditions can we impose upon f_1, f_2 so that $f_1 * f_2 \in S_H^O$?*

Several researchers have recently published results related to this question (see [3], [6], [11], [20], [31], [32]). Let's look at some of these results. Theorem 1.27 ([6]) gives conditions under which local univalence of the convolution is enough to establish global univalence.

Theorem 1.27. *Let $f_1 = h_1 + \overline{g}_1, f_2 = h_2 + \overline{g}_2 \in S_H^O$ such that $h_i(z) + g_i(z) = \frac{z}{1-z}$. Let $\widetilde{\omega}$ be the dilatation of $f_1 * f_2$. If $|\widetilde{\omega}(z)| < 1$ for all $z \in \mathbb{D}$, then $f_1 * f_2 \in S_H^O$ and is CHD.*

Theorem 1.27 has been used to determine specific cases in which harmonic convolutions preserve univalence. In [11], the following result is proved.

Theorem 1.28. *Consider the right half-plane map*

$$f_h(z) = h_1(z) + \overline{g_1(z)} = \frac{z - \frac{1}{2}z^2}{(1-z)^2} - \overline{\frac{\frac{1}{2}z^2}{(1-z)^2}},$$

and let $f = h + \overline{g} \in K_H^O$ with $h(z) + g(z) = \frac{z}{1-z}$ and $\omega = g'/h' = e^{i\theta}z^n$ $(n \in \mathbb{Z}^+, \theta \in \mathbb{R})$. If $n = 1, 2$, then $f_h * f \in S_H^O$ and is CHD.

The proof of this theorem relies on properties on analytic convolutions and results about the location of zeros of symmetric polynomials. If $n > 2$ in the above theorem, then $f_h * f$ fails to be univalent. In [3], we get the next theorem.

Theorem 1.29. *Let $f_\theta = h_\theta + \overline{g_\theta}, f_\rho = h_\rho + \overline{g_\rho} \in S_H^O$ such that $h_\theta(z) + g_\theta(z) = h_\rho(z) + g_\rho(z) = \frac{z}{1-z}$, $g_\theta'/h_\theta' = e^{i\theta}z$, and $g_\rho'/h_\rho' = e^{i\rho}z$ $(\theta, \rho \in \mathbb{R})$. Then $f_\theta * f_\rho \in S_H^O$ is CHD.*

The following theorem was proved in [32]

Theorem 1.30. *Let $f = h + \overline{g} \in S_H^O$ with $h(z) + g(z) = \frac{z}{1-z}$ and $\omega(z) = \frac{z+a}{1+\overline{a}z}$ with $|a| < 1$. Then $f_h * f \in S_H^O$ and is CHD if and only if*

$$(\text{Re } a)^2 + 9\,(\text{Im } a)^2 \leq 1.$$

There are other convolution problems that remain to be investigated. In many theorems, the canonical harmonic right half-plane function f_h is convoluted with other harmonic functions. Can similar theorems be proven if f_h is replaced with a different function? For example, consider the harmonic mapping f_1 formed by shearing $h_1(z) + g_1(z) = \frac{z}{1-z}$ with other dilatations such as $\omega(z) = e^{i\theta}\frac{1+a}{1+az}$ or $\omega(z) = z$.

Open Problem 2. *Let $f = h + \overline{g} \in S_H^O$ with $h(z) + g(z) = \frac{z}{1-z}$ and $\omega = g'/h' = e^{i\theta}z^n$ ($n \in \mathbb{Z}^+, \theta \in \mathbb{R}$). Determine the values of n for which $f_1 * f$ is univalent, where f_1 is one of the specific functions mentioned above.*

Many of the harmonic convolution results given above require that one of the functions be a sheared half-plane. In [11] and [20], results are proven about the harmonic convolutions of strip mappings and polygons.

Open Problem 3. *Determine more results about the convolutions of harmonic functions that are shears of vertical strips or polygons.*

1.6 Harmonic maps and minimal surfaces

Planar harmonic mappings with certain properties are related to minimal surfaces in $\mathbb{R}^3$, and it is possible to use results from one area to prove new results in the other area. Before discussing this further, we need to present some background material about minimal surfaces.

Minimal surfaces are one solution to the problem of finding the minimal surface area required to span a given curve. Minimal surfaces are guaranteed to minimize area only locally but often they provide the globally-minimal solution as well. One consequence of the area-minimizing property is that all minimal surfaces look like saddle surfaces at each point, and the bending upward in one direction is matched by the downward bending in the orthogonal direction. (This equal-but-opposite bending property will be defined later as "zero mean curvature.") The images below are some well-known minimal surfaces.

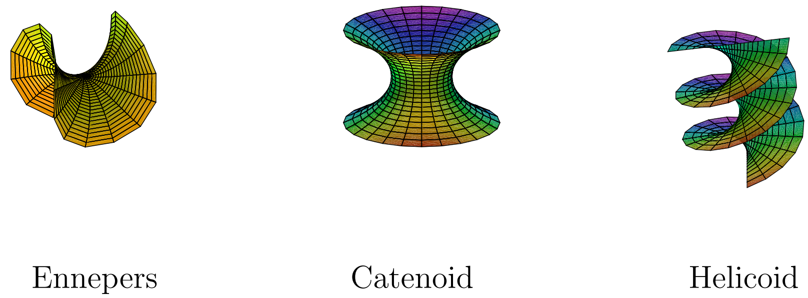


Figure 1.10: Examples of minimal surfaces

1.6.1 Background. In order to explore minimal surfaces more fully, we introduce three important concepts from differential geometry. For more details on the material from this section, see [8].

A surface, $M \in \mathbb{R}^3$, can be *parametrized* by a smooth function $\mathbf{x} \colon D \to \mathbb{R}^3$ if $\mathbf{x}(D) = M$ and $\mathbf{x}$ is one-to-one. Parameterizing a surface with smooth functions allows us to do calculus with the surface and gives us a way to translated geometric concepts into rigorous analytic language.

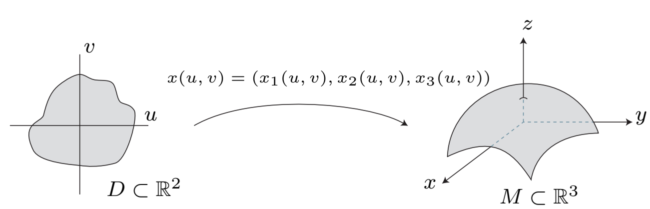


Figure 1.11: The parametrization of a surface.

Isothermal parameterizations are essential for the study of minimal surfaces. Basically, such parametrizations map small squares to small squares. Every minimal surface in $\mathbb{R}^3$ has an isothermal parametrization.

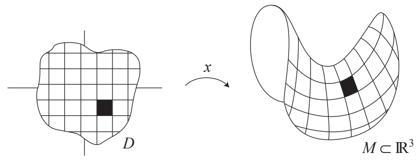


Figure 1.12: An isothermal parametrization maps small squares to small squares.

Next, we need to discuss the idea of *normal curvature*. At each point p on the surface M, there is a unit normal $\mathbf{n}$. The normal curvature measures how much the surface bends toward $\mathbf{n}$ as you travel in the direction of the tangent vector $\mathbf{w}$ at p. Specifically, given the normal vector $\mathbf{n}$ at each point $p \in M$, we can find a plane $\mathcal{P}$ containing $\mathbf{n}$ that intersects M in some curve $\mathbf{c}$, which has a curvature value k. As the plane $\mathcal{P}$ revolves around the unit normal $\mathbf{n}$ at p, we get a continuous function of curvature values $k(\theta)$. Let k_1 and k_2 be the maximum and minimum curvature values at p. The *mean curvature* of a surface M at p is $H = \frac{1}{2}(k_1 + k_2)$.

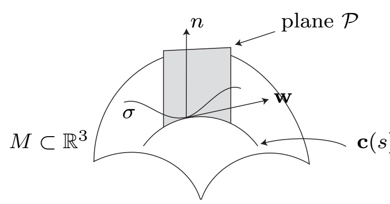


Figure 1.13: Normal curvature.

Definition 1.31. A minimal surface is a surface M with $H = 0$ at all $p \in M$.

Recall that the intuition behind vanishing mean curvature is that M is a saddle surface with positive curvature in one direction being matched by negative curvature in the orthogonal direction.

Just as the Shearing Theorem links analytic function theory to harmonic function theory, the Weierstrass Representation links harmonic function theory to minimal surface theory.

Theorem 1.32 (General Weierstrass Representation)**.** *If we have analytic functions φ_k ($k = 1, 2, 3$) such that*

- $\phi^2 = (\varphi_1)^2 + (\varphi_2)^2 + (\varphi_3)^2 = 0$
- $|\phi|^2 = |\varphi_1|^2 + |\varphi_2|^2 + |\varphi_3|^2 \neq 0$ *and is finite,*

then the parametrization

$$\mathbf{x} = \left(\operatorname{Re}\int \varphi_1(z)dz, \operatorname{Re}\int \varphi_2(z)dz, \operatorname{Re}\int \varphi_3(z)dz\right)$$

defines a minimal surface.

We also have the following converse.

Theorem 1.33. *Let M be a surface with parametrization $\mathbf{x} = (x_1, x_2, x_3)$ and let $\phi = (\varphi_1, \varphi_2, \varphi_3)$, where $\varphi_k = \frac{\partial x_k}{\partial z}$.*

$$\mathbf{x} \text{ is isothermal } \iff \phi^2 = (\varphi_1)^2 + (\varphi_2)^2 + (\varphi_3)^2 = 0.$$

If $\mathbf{x}$ is isothermal, then

$$M \text{ is minimal if and only if each } \varphi_k \text{ is analytic.}$$

We can apply the above theorems to planar harmonic mappings. First, recall $f = h + \overline{g} = \operatorname{Re}(h+g) + i\operatorname{Im}(h-g)$. In Theorem 1.32, choose $\varphi_1 = h' + g'$ and $\varphi_2 = -i(h' - g')$. Then we find φ_3 that will satisfy the requirements of the Weierstrass representation. That is,

$$\begin{aligned} 0 =& (\varphi_1)^2 + (\varphi_2)^2 + (\varphi_3)^2 \\ =& (h' + g')^2 + [-i(h' - g')]^2 + (\phi_3)^2. \end{aligned}$$

Solving for φ_3 yields $(\varphi_3)^2 = -4h'g'$, so $\varphi_3 = -2i\sqrt{h'g'}$.

Notice that $\sqrt{h'g'}$ may not always exist as an analytic function, but whenever it does, the Weierstrass representation applies. Since $\sqrt{h'g'} = h'\sqrt{\omega}$, it is enough for the dilatation to have an analytic square root. Thus, we have the following result.

Theorem 1.34 (Weierstrass Representation - (h, g))**.** *Let the harmonic mapping $f = h + \overline{g}$ be univalent with g'/h' being the square of an analytic function. Then the parametrization*

$$X = \left(\operatorname{Re}(h+g), \operatorname{Im}(h-g), 2\ \operatorname{Im}\int\sqrt{h'g'}\right)$$

defines a minimal graph whose projection is $f(\mathbb{D})$.

MinSurfTool [12] is another applet available online that allows for quick and easy visualization of minimal surfaces.

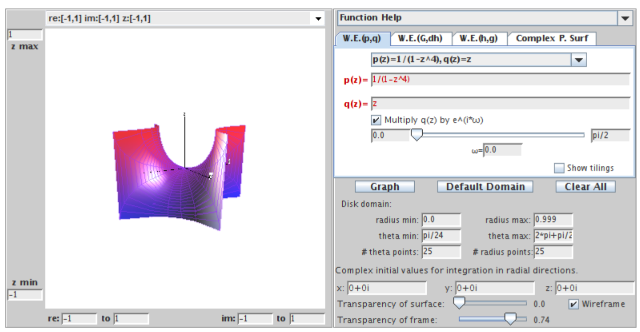


Figure 1.14: The *MinSurfTool* applet.

Example 1.35. Consider the harmonic map

$$f(z) = h(z) + \overline{g(z)} = \operatorname{Re}\left[\frac{i}{2}\log\left(\frac{i+z}{i-z}\right)\right] + i\operatorname{Im}\left[\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)\right].$$

It can be constructed by shearing $h(z) - g(z) = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$ with $g'(z)/h'(z) = -z^2$ and is therefore univalent. Note that $f(\mathbb{D})$ is a square region (see Figure 1.15).

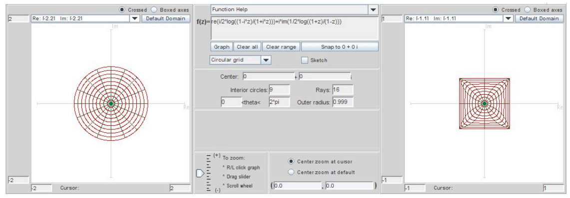


Figure 1.15: The image of $\mathbb{D}$ under f, the harmonic square map.

Since the dilatation is the square of an analytic function, we can apply Theorem 1.34. Then $x_3(z) = 2\ \operatorname{Im} \displaystyle\int \sqrt{h'g'} = \frac{1}{2} \operatorname{Im}\left[i \log\left(\frac{1+z^2}{1-z^2}\right)\right]$.

By the Weierstrass representation, we have the parametrization of a minimal graph given by

$$\begin{aligned}
\mathbf{x} &= \left(\operatorname{Re}(h+g), \operatorname{Im}(h-g), 2\ \operatorname{Im}\int \sqrt{h'g'} \right) \\
&= \left(\operatorname{Re}\left[\frac{i}{2}\log\left(\frac{i+z}{i-z}\right)\right], \operatorname{Im}\left[\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)\right], \operatorname{Im}\left[\frac{i}{2}\log\left(\frac{1+z^2}{1-z^2}\right)\right] \right).
\end{aligned}$$

This minimal surface is Scherk's doubly periodic surface. In Figure 1.16 the curved object is the minimal surface and the flat object is the image of the the unit disk under the corresponding harmonic map (it can also be viewed as the projection of the minimal surface into the complex plane).

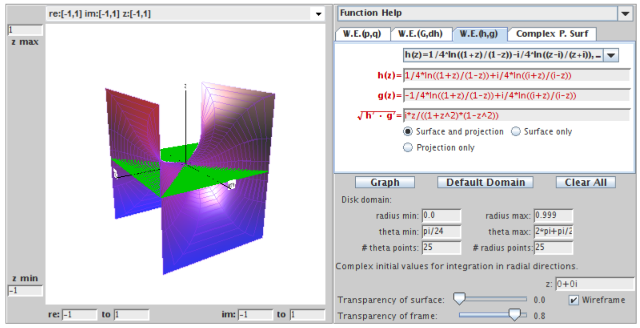


Figure 1.16: Scherk's doubly periodic minimal surface.

We might wonder if the integrals found in the Weierstrass representations are well-defined. In certain cases, they may indeed be multi-valued. But in such cases, the ill-definedness reflects the fact that surface is periodic in one or more of the coordinates, as is the case with the Scherk surfaces.

With the background we just discussed, we are ready to explore applications of harmonic maps to minimal surface theory.

1.6.2 Connecting harmonic maps to specific minimal graphs.

The Weierstrass Representation allows us to take an harmonic univalent function with an appropriate dilatation and lift it to a minimal graph. Several recent papers have used this technique ([14], [15], [19], [22], [33], [34]). However, it is often difficult to identify the resulting minimal graphs. One approach to recognizing the minimal surface is to use a change of variable (see [10]).

Example 1.36. Shearing $h(z) - g(z) = \frac{z}{(1-z)^2}$ with $\omega(z) = z^2$ yields the univalent harmonic slit-map

$$f(z) = \frac{z - z^2 + \frac{1}{3}z^3}{(1-z)^3} + \overline{\frac{\frac{1}{3}z^3}{(1-z)^3}}.$$

The parametrization of the corresponding minimal graph is

$$\mathbf{x} = \left(\mathrm{Re}\left\{\frac{z - z^2 + \frac{2}{3}z^3}{(1-z)^3}\right\}, \mathrm{Im}\left\{\frac{z}{(1-z)^2}\right\}, \mathrm{Im}\left\{\frac{2z^2 - \frac{2}{3}z^3}{(1-z)^3}\right\}\right).$$

This is not a standard form for a known minimal surface. However, using the substitution $z \to \frac{\widetilde{z}+1}{\widetilde{z}-1}$ and interchanging the second and third coordinate functions, we derive the parametrization

$$\widetilde{\mathbf{x}} = \left(-\tfrac{1}{4}\,\mathrm{Re}\left\{\widetilde{z} + \tfrac{1}{3}\widetilde{z}^3\right\}, \tfrac{1}{4}\,\mathrm{Im}\left\{\widetilde{z} - \tfrac{1}{3}\widetilde{z}^3\right\}, \tfrac{1}{4}\,\mathrm{Im}\left\{\widetilde{z}^2\right\}\right).$$

This is Ennepers surface. Thus, the original surface $\mathbf{x}$ is the part of Ennepers surface formed by using a right half-plane as the domain instead of the standard unit disk.

Open Problem 4. *Determine the minimal graphs formed by lifting harmonic univalent mappings in any of the following papers [14], [15], [19], [33], [34].*

Open Problem 5. *Use the shearing technique to generate a univalent harmonic map with a dilatation that is a perfect square and use the Weierstrass representation to construct the minimal graph. Then determine what surface it is.*

1.6.3 Connecting results about harmonic maps with results about minimal surfaces. Since certain types of harmonic univalent functions are related to minimal graphs, it should be true that theorems and concepts from one field should relate to theorems and concepts from the other field.

One example of this concerns a harmonic convolution theorem and Krust Theorem about conjugate minimal surfaces.

Definition 1.37. Let $\mathbf{x}$ and $\mathbf{y}$ be isothermal parametrizations of two minimal surfaces such that their component functions are pairwise harmonic conjugates. Then, $\mathbf{x}$ and $\mathbf{y}$ are called conjugate minimal surfaces.

The helicoid and the catenoid are conjugate surfaces. Any two conjugate minimal surfaces can be joined through a one-parameter family of *associated minimal surfaces* by the equation

$$\mathbf{z} = (\cos t)\mathbf{x} + (\sin t)\mathbf{y},$$

where $t \in \mathbb{R}$. Figure 1.17 displays several associated surfaces that are formed as the helicoid is transformed to the catenoid, its conjugate surface.

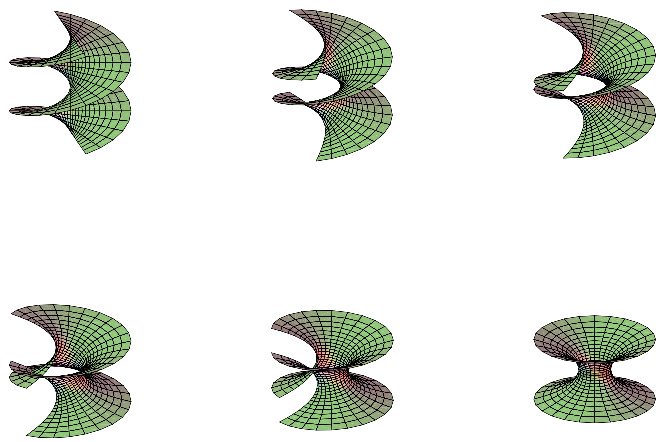


Figure 1.17: The helicoid, the catenoid, and some of their associated surfaces.

An important theorem in minimal surface theory is Krust Theorem.

Theorem 1.38 (Krust)**.** *If an embedded minimal surface $X\colon \mathbb{D} \to \mathbb{R}^3$ can be written as a graph over a convex domain in $\mathbb{C}$, then all associated minimal surfaces $Z\colon \mathbb{D} \to \mathbb{R}^3$ are graphs.*

Now consider the following less well known theorem about harmonic convolutions [4].

Theorem 1.39 (Clunie and Sheil-Small)**.** *If $f = h + \overline{g} \in K_H$ and $\varphi \in K$, then the functions*

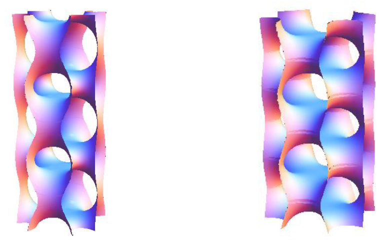
$$h * \varphi + \alpha \overline{g * \varphi}$$

are univalent and close-to-convex, where $(|\alpha| \leq 1)$ and $*$ denotes harmonic convolution.

Open Problem 6. *Determine theorems and properties of harmonic maps that relate to theorems and properties of minimal surfaces.*

As a second example, we will prove a result about minimal surfaces using results from harmonic univalent mappings. In particular, we will consider a family of minimal surfaces known as Scherk's dihedral surfaces and determine the parameter values for which these surfaces are embedded. First, some background information.

While minimal surfaces can be parametrized by the Weierstrass representation, there is no guarantee the surface will not have self-intersections. Minimal surfaces that have no self-intersections are known as embedded minimal surfaces, and they are a major interest in minimal surface theory. The family $\mathcal{F}_n(\varphi)$ of singly periodic Scherk surfaces with higher dihedral symmetry have n number of vertical planes that extend to infinity. The smallest angle, φ, between these symmetric planes varies (see Figure 1.18).

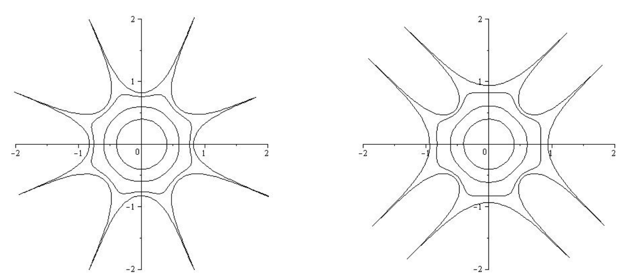


$n = 4$ $\quad$ $n = 4$

$\varphi = \frac{\pi}{2}$ $\quad$ $\varphi = \frac{\pi}{3}$

Figure 1.18: Two examples from the family of Scherk's dihedral surfaces.

We can look at the projection of one piece of these surfaces onto $\mathbb{C}$ which is also the image of the unit disk under the corresponding harmonic univalent mappings (sees Figure 1.19).



$n = 4$ $\quad$ $n = 4$

$\varphi = \frac{\pi}{2}$ $\quad$ $\varphi = \frac{\pi}{3}$

Figure 1.19: The projection onto $\mathbb{C}$ of one piece from each example in Figure 1.18.

These minimal surfaces are embedded provided that

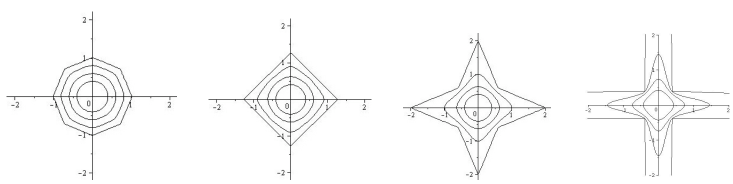
$$\frac{\pi}{2} - \frac{\pi}{n} < \frac{n-1}{n}\varphi < \frac{\pi}{2}.$$

We can prove this inequality using results planar harmonic mappings. We summarize the proof below.

Proof. Consider the following family of harmonic maps: $f_n(z) = h_n(z) + \overline{g_n(z)}$, $n \geq 2$, $\varphi \in [0, \frac{\pi}{2}]$, where

$$h_n'(z) = \frac{1}{(z^n - e^{i\varphi})(z^n - e^{-i\varphi})}, \quad g_n'(z) = \frac{z^{2n-2}}{(z^n - e^{i\varphi})(z^n - e^{-i\varphi})}$$

(see Figure 1.20).



$n = 4$	$n = 4$	$n = 4$	$n = 4$
$\varphi = \frac{\pi}{2}$	$\varphi = \frac{\pi}{3}$	$\varphi = \frac{\pi}{6}$	$\varphi = 0$

Figure 1.20: Images of the unit disk under $f = h_n + \overline{g}_n$.

It is known that $f_n = h_n + \overline{g_n}$ maps $\mathbb{D}$ onto a $2n$-gon, and in [33] it was shown that f_n is univalent and convex for every $\varphi \in \left(\frac{n}{n-1}\left(\frac{\pi}{2} - \frac{\pi}{n}\right), \frac{\pi}{2}\right]$. Using the Weierstrass representation, we can lift f_n to an embedded minimal surface X. Since X is over a convex domain, Krust's theorem guarantees that the conjugate surfaces Y are embedded. These conjugate surfaces Y are Scherk surfaces with higher dihedral symmetry and this establishes the inequality. $\square$

Open Problem 7. *Use theorems and properties about harmonic univalent mappings to prove results about minimal surfaces.*

1.6.4 Using harmonic maps to construct new minimal surfaces.

Finally, we show an example in which a harmonic univalent function is lifted to form a minimal graph that appears to be new. The construction is outlined below. Complete details are found in [2] and in chapter **??** of this thesis.

Let $f = h + \overline{g}$, where

$$h(z) = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right),$$

and let $\omega = \left(e^{\frac{z+1}{z-1}}\right)^2$. Since $g' = h'\omega = \frac{1}{1-z^2}e^{2\frac{z+1}{z-1}}$, we know that

$$g(z) = -\frac{1}{2}E_1\left(\frac{z+1}{-z+1}\right) + \frac{1}{2}E_1(1),$$

where $E_1(z)$ is the exponential integral function. By a result by Clunie and Sheil-Small, $f = h + \overline{g}$ is univalent. The image of $f(\mathbb{D})$ is shown in Figure 3.5.

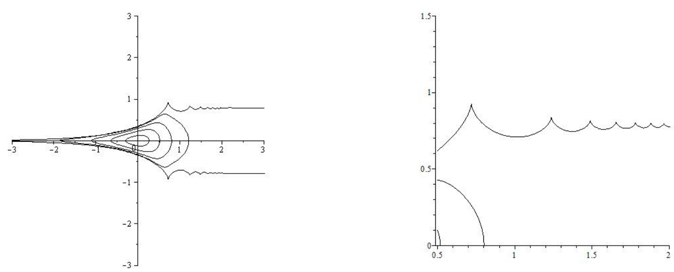


Figure 1.21: The image of $f(\mathbb{D})$ and a close-up of that image.

By the Weierstrass representation $f = h + \overline{g}$ lifts to an embedded minimal surface (see Figure 3.6).

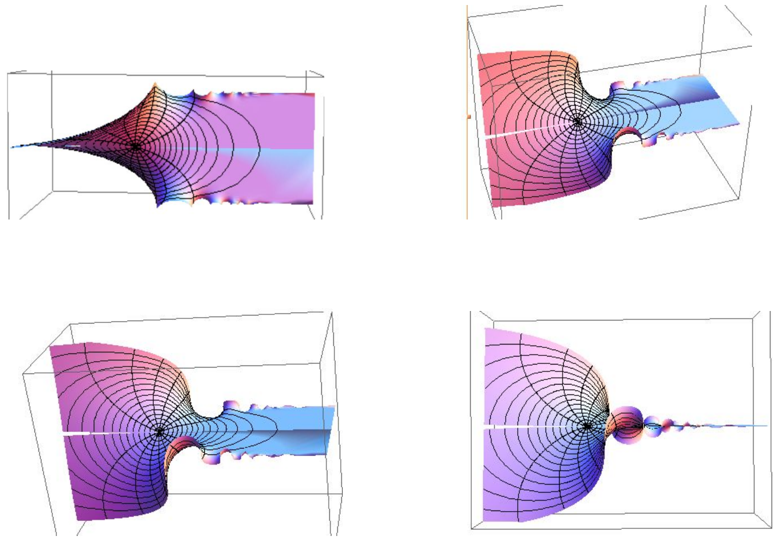


Figure 1.22: Images of the minimal surface constructed from f

This surface is constructed from a harmonic univalent map that has a dilatation being a *singular inner function* (i.e., a function which never equals zero and which has modulus equal to one on the unit disk). One consequence of having such a dilatation is that there is no (finite) point where the function is approximately analytic. This corresponds to the idea that the minimal surface never has zero Gauss curvature. The surface also has an infinite number of cusps and a singularity with unusual behavior.

Open Problem 8. *Construct other minimal surfaces from harmonic univalent maps with dilatations that are singular inner functions.*

Open Problem 9. *Determine the necessary and sufficient conditions for a harmonic function to have a singular inner function as its dilatation. Specifically, determine the kind of growth and boundary behavior exhibited by such harmonic functions.*

Chapter 2. Convolutions of Half-plane Maps

2.1 Background

It is shown in [6] that if $f = h + \overline{g} \in S_H^O$ maps the unit disk onto the right half-plane $R = \{w : \operatorname{Re} w > -1/2\}$, then it must satisfy the following condition

$$h(z) + g(z) = \frac{z}{1-z}. \tag{2.1}$$

Let $S_H^O(R)$ denote the class of harmonic mappings $f = h + \overline{g} \in S_H^O$ that satisfy (2.1). They are the so called vertical shears of the conformal half-plane mapping $\varphi(z) = z/(1-z)$. It was proved in [6] that if $f_j = h_j + \overline{g_j} \in S_H^O(R)$, $j = 1, 2$, and $f_1 * f_2$ is locally univalent and sense-preserving, then $f_1 * f_2 \in S_H^O$ and is convex in the direction of the real axis. As observed in [11], the assumption of the local univalency of the convolution function in this statement cannot be omitted. Our main theorem in this chapter is the following:

Theorem 2.1. *If $f_k = h_k + \overline{g}_k \in S_H^O(R)$, $k = 1, 2$, and $\omega_1(z) = g_1'(z)/h_1'(z) = -xz$, $\omega_2(z) = g_2'(z)/h_2'(z) = -yz$ with $|x| = |y| = 1$, then the function $\widetilde{f} = f_1 * f_2$ is convex in the direction of the real axis. In particular, if $x = y = -1$, then $\widetilde{f}$ is convex.*

In the proof of the main theorem, we will use the following characterization of the class of analytic functions mapping $\mathbb{D}$ conformally onto a domain convex in one direction due to Royster and Ziegler [36].

Theorem 2.2. *A nonconstant and analytic function F maps $\mathbb{D}$ univalently onto a domain Ω convex in the direction of the imaginary axis if and only if there are numbers $\mu \in [0, 2\pi)$ and $\nu \in [0, \pi]$, such that*

$$\operatorname{Re}\left\{-ie^{i\mu}\left(1 - 2\cos\nu e^{-i\mu} z + e^{-2i\mu} z^2\right) F'(z)\right\} \geq 0, \quad z \in \mathbb{D}.$$

In the proof of our theorem we will also take advantage of the theory of harmonic Hardy spaces. A function $f(z)$ harmonic in the unit disk is said to be of class h^p $(0 < p < \infty)$ if the integral means

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|f(re^{i\theta})\right|^p d\theta$$

are bounded for $0 \le r < 1$; h^∞ is simply the collection of bounded harmonic functions on $\mathbb{D}$. The usual Hardy spaces H^p consist of the functions in h^p that are analytic in $\mathbb{D}$. It is clear that an analytic function belongs to H^p if and only if its real and imaginary parts are both in h^p. It is well known that every function $f \in H^p$ has a non-tangential limit $f(e^{i\theta})$ for almost every $\theta \in (-\pi, \pi]$. We will apply the following theorem concerning harmonic Hardy spaces, (see [27, p. 15], [24, p. 38]).

Theorem 2.3. *Let $1 < p \le \infty$, and assume that $f \in h^p$. Then for almost all θ, $f(z)$ tends nontangentially to a finite limit $f(e^{i\theta})$, as $z \to e^{i\theta}$, $f(e^{i\theta}) \in L^p(-\pi,\pi)$, and, for $0 \le r < 1$,*

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1+r^2-2r\cos(\theta - t)} f(e^{i\theta})dt.$$

Let us emphasize that Theorem 2.3 does not hold for $p = 1$. In the remark we give an example of a function $f \in h^1$ which cannot be recovered from its boundary behavior.

We also note that this theorem actually implies an extended version of the maximum (minimum) principle for real harmonic functions in $\mathbb{D}$.

2.2 Proof of Theorem 2.1

The following lemma is a modified version of Lemma 2.5 in [23].

Lemma 2.4. *If $f = h + \overline{g} \in S_H^O(R)$, then $f(\mathbb{D})$ is convex.*

Proof. By Theorem 5.7 in [4], it suffices to show that the function $h - e^{2i\theta}g$ is convex in the direction θ for every $\theta \in [0, \pi)$ (see Theorem 1.14 for an intuition). The function $h - e^{2i\theta}g$ is convex in the direction θ if and only if $F_\theta = ie^{-i\theta}\left(h - e^{2i\theta}g\right)$ is convex in the vertical direction.

Let us first assume that $\theta \in [0, \pi/2]$. We apply Theorem 2.2 with $\mu = \nu = 0$. We have

$$\begin{aligned}
\operatorname{Re}\left\{-iF_\theta'(z)(1-z)^2\right\} &= \operatorname{Re}\left\{e^{-i\theta}[h'(z) - e^{2i\theta}g'(z)](1-z)^2\right\} \\
&= \operatorname{Re}\left\{[e^{-i\theta}h'(z) - e^{i\theta}g'(z)](1-z)^2\right\} \\
&= \operatorname{Re}\left\{\big[\,(h'(z) - g'(z))\cos\theta - i\,(h'(z) + g'(z))\sin\theta\big](1-z)^2\right\}.
\end{aligned}$$

Since $h'(z) + g'(z) = 1/(1-z)^2$, and because the function $\frac{h'(z)-g'(z)}{h'(z)+g'(z)} = \frac{1-g'(z)/h'(z)}{1+g'(z)/h'(z)} = \frac{1-\omega(z)}{1+\omega(z)} = p(z)$ has a positive real part, it follows that

$$\operatorname{Re}\left\{-iF_\theta'(z)(1-z)^2\right\} = \operatorname{Re} p(z)\cos\theta \geq 0$$

for $\theta \in [0, \pi/2]$. By Theorem 2.2, the function F_θ is convex in the direction of the imaginary axis for $\theta \in [0, \pi/2]$. The same conclusion can be drawn for the function F_θ with $\theta \in (\pi/2, \pi)$ if we apply Theorem 2.2 with $\mu = \nu = \pi$.

$\square$

<u>Proof of Theorem 2.1.</u> Let us set the notation $\widetilde{f} = f_1 * f_2 = h_1 * h_2 + \overline{g_1 * g_2} = H + \overline{G}$. By the result of Dorff [6], it suffices to show that $\widetilde{f}$ is locally univalent and sense-preserving. Additionally, if $x = 1$ or $y = 1$, then the assertion follows from Theorem 3 in [11]. Assume now that $x, y \neq 1$. Then we solve the system of equations $h_1 + g_1 = \frac{z}{1-z}$ and $g_1'/h_1' = -xz$, easily obtaining

$$h_1(z) = \frac{-\bar{x}}{1-\bar{x}}\frac{z}{1-z} - \frac{1}{(1-\bar{x})(1-x)}\log\frac{1-z}{1-xz}, \tag{2.2}$$

$$g_1(z) = \frac{1}{1-\bar{x}}\frac{z}{1-z} + \frac{1}{(1-\bar{x})(1-x)}\log\frac{1-z}{1-xz}. \tag{2.3}$$

Next, we use the following three identities:

- $z(F_1 * F_2)' = F_1 * (zF')$
- $\frac{z}{1-z} * F = F$

- $\log \frac{1-z}{1-xz} * F = \int \frac{-F(x)+F(zx)}{z} dz$

all three of which can easily be verified by computing the power series involved. Consequently, we get

$$zH'(z) = h_1(z) * zh_2'(z) = \frac{1}{1-\bar{x}} \left(-\bar{x} z h_2'(z) - \frac{1}{1-x}(-h_2(z) + h_2(zx)) \right)$$

and

$$zG'(z) = g_1(z) * zg_2'(z) = \frac{1}{1-\bar{x}} \left(zg_2'(z) + \frac{1}{1-x}(-g_2(z) + g_2(zx)) \right).$$

In order to prove the local univalence of $\widetilde{f}$, it suffices to show that for $z \in \mathbb{D}$ we have

$$|H'(z)|^2 > |G'(z)|^2,$$

that is,

$$\left| \bar{x} h_2'(z) + \frac{1}{z(1-x)}(-h_2(z) + h_2(zx)) \right|^2 > \left| g_2'(z) + \frac{1}{z(1-x)}(-g_2(z) + g_2(zx)) \right|^2.$$

Since

$$g_2'(z) = -yzh_2'(z),$$

the last inequality can be written in an equivalent way

$$\left| \bar{x} + \frac{1}{z(1-x)} \frac{h_2(zx) - h_2(z)}{h_2'(z)} \right|^2 > \left| -yz + \frac{1}{z(1-x)} \frac{g_2(zx) - g_2(z)}{h_2'(z)} \right|^2. \tag{2.4}$$

First note that for $z = 0$ the last inequality becomes $|1-\bar{x}| > 0$. Moreover, we can rewrite (2.4) as

$$\left| \bar{x} + \frac{1}{z(1-x)} \frac{h_2(zx) - h_2(z)}{h_2'(z)} \right|^2 > \left| -y + \frac{1}{z^2(1-x)} \frac{g_2(zx) - g_2(z)}{h_2'(z)} \right|^2 |z|^2.$$

It is therefore sufficient to show that

$$\left|\bar{x}+\frac{1}{z(1-x)}\frac{h_2(zx)-h_2(z)}{h_2'(z)}\right|^2 > \left|-y+\frac{1}{z^2(1-x)}\frac{g_2(zx)-g_2(z)}{h_2'(z)}\right|^2,$$

that is,

$$\begin{aligned}1+2\operatorname{Re}\left(\frac{x}{z(1-x)}\frac{h_2(zx)-h_2(z)}{h_2'(z)}\right)+\left|\frac{1}{z(1-x)}\frac{h_2(zx)-h_2(z)}{h_2'(z)}\right|^2\\ >1-2\operatorname{Re}\left(\frac{\bar{y}}{z^2(1-x)}\frac{g_2(zx)-g_2(z)}{h_2'(z)}\right)+\left|\frac{1}{z^2(1-x)}\frac{g_2(zx)-g_2(z)}{h_2'(z)}\right|^2.\end{aligned} \tag{2.5}$$

It follows from Lemma 2.4 that the function $f_2 = h_2 + \overline{g_2}$ is convex, hence a result of Clunie and Sheil-Small (Corollary 5.8 in [4]) applies, which gives the following

$$|K(z)| = \left|\frac{g_2(zx)-g_2(z)}{h_2(zx)-h_2(z)}\right| < 1, \qquad z \in \mathbb{D}.$$

Moreover, since $K(0) = 0$, the Schwarz lemma gives

$$\left|\frac{g_2(zx)-g_2(z)}{h_2(zx)-h_2(z)}\right| < |z|, \qquad z \in \mathbb{D}.$$

Consequently,

$$\left|\frac{1}{z(1-x)}\frac{h_2(zx)-h_2(z)}{h_2'(z)}\right|^2 > \left|\frac{1}{z^2(1-x)}\frac{g_2(zx)-g_2(z)}{h_2'(z)}\right|^2.$$

Thus inequality (2.5) will be proved once we establish that

$$\operatorname{Re} J(z) = \operatorname{Re}\left(\frac{x}{z(1-x)}\frac{h_2(zx)-h_2(z)}{h_2'(z)}+\frac{\bar{y}}{z^2(1-x)}\frac{g_2(zx)-g_2(z)}{h_2'(z)}\right) > 0.$$

Since h_2 and g_2 can be expressed by formulas (2.2) and (2.3) with x replaced by y, one can find that

$$J(z)=\frac{(1-z)(1-yz)}{z(1-y)}+\frac{\bar{y}(1-xyz)(1-z)^2(1-yz)}{|1-y|^2z^2(1-x)}\left(\log\frac{1-xz}{1-xyz}-\log\frac{1-z}{1-yz}\right). \quad (2.6)$$

Note that the function $J(z)$ is analytic in the unit disk and it takes value $(1-x)/2$ for $z=0$.

We first consider the case when $x=y$. Then the last formula is reduced to

$$J(z)=\frac{(1-z)(1-xz)}{z(1-x)}+\frac{\bar{x}(1-x^2z)(1-z)^2(1-xz)}{|1-x|^2z^2(1-x)}\left(\log\frac{1-xz}{1-x^2z}-\log\frac{1-z}{1-xz}\right). \quad (2.7)$$

It is easy to see that in this case the function $J(z)$ has a continuous extension to $\overline{\mathbb{D}}$. Therefore it suffices to prove that

$$\min\{\operatorname{Re} J(z) : |z|=1\}\geq 0. \quad (2.8)$$

Assume first that $x=e^{i\alpha}$, where $\alpha\in(0,\pi]$. Then for J given by (2.7),

$$\operatorname{Re} J(e^{it})=\operatorname{Re}\left(8i\frac{\sin\frac{\alpha+t}{2}\sin\left(\alpha+\frac{t}{2}\right)\sin^2\frac{t}{2}}{(2-2\cos\alpha)\sin\frac{\alpha}{2}}\left(\log\frac{1-e^{i(\alpha+t)}}{1-e^{i(2\alpha+t)}}-\log\frac{1-e^{it}}{1-e^{i(\alpha+t)}}\right)\right).$$

Since

$$\operatorname{Im}\log\frac{1-e^{i\alpha}e^{it}}{1-e^{i2\alpha}e^{it}}=\begin{cases}-\frac{\alpha}{2} & \text{if}\quad t\in(0,2\pi-2\alpha)\cup(2\pi-\alpha,2\pi)\\ \pi-\frac{\alpha}{2} & \text{if}\quad t\in(2\pi-2\alpha,2\pi-\alpha)\end{cases}$$

and

$$\operatorname{Im}\log\frac{1-e^{it}}{1-e^{i\alpha}e^{it}}=\begin{cases}-\frac{\alpha}{2} & \text{if}\quad t\in(0,2\pi-\alpha)\\ \pi-\frac{\alpha}{2} & \text{if}\quad t\in(2\pi-\alpha,2\pi),\end{cases}$$

we have

$$\operatorname{Im}\left(\log\frac{1-e^{i(\alpha+t)}}{1-e^{i(2\alpha+t)}}-\log\frac{1-e^{it}}{1-e^{i(\alpha+t)}}\right)=\begin{cases}0 & \text{if} \quad t\in(0,2\pi-2\alpha)\\ \pi & \text{if} \quad t\in(2\pi-2\alpha,2\pi-\alpha)\\ -\pi & \text{if} \quad t\in(2\pi-\alpha,2\pi).\end{cases}$$

From this and from the formula for $\operatorname{Re} J(e^{it})$ inequality (2.8) follows in the case of $x=e^{i\alpha}$, $\alpha\in(0,\pi]$. Similar arguments apply to the case $\alpha\in(-\pi,0)$.

We remark that in the case when $x\neq y$ the above reasoning cannot be applied because the function J does not have a continuous extension for $z=\bar{x}$. To prove the general case we will use Theorem 2.3.

Assume that $x=e^{i\alpha}$ and $y=e^{i\beta}$. Then for $e^{it}\neq 1, e^{-i\alpha}, e^{-i\beta}, e^{-i(\alpha+\beta)}$,

$$\operatorname{Re} J(e^{it})=\operatorname{Re}\left(8i\frac{\sin\left(\frac{\alpha+\beta+t}{2}\right)\sin\left(\frac{\beta+t}{2}\right)\sin^2\frac{t}{2}}{(2-2\cos\beta)\sin\frac{\alpha}{2}}\left(\log\frac{1-e^{i(\alpha+t)}}{1-e^{i(\alpha+\beta+t)}}-\log\frac{1-e^{it}}{1-e^{i(\beta+t)}}\right)\right).$$

A calculation shows that if $0<\alpha\le\beta<2\pi$ and $\alpha+\beta<2\pi$, then

$$\operatorname{Im}\log\frac{1-e^{i\alpha}e^{it}}{1-e^{i\alpha+\beta}e^{it}}=\begin{cases}-\frac{\beta}{2} & \text{if} \quad t\in(2\pi-\alpha,2\pi)\cup(0,2\pi-(\alpha+\beta))\\ \pi-\frac{\beta}{2} & \text{if} \quad t\in(2\pi-(\alpha+\beta),2\pi-\alpha)\end{cases}$$

and

$$\operatorname{Im}\log\frac{1-e^{it}}{1-e^{i\beta}e^{it}}=\begin{cases}-\frac{\beta}{2} & \text{if} \quad t\in(0,2\pi-\beta)\\ \pi-\frac{\beta}{2} & \text{if} \quad t\in(2\pi-\beta,2\pi).\end{cases}$$

Hence

$$\operatorname{Im}\left(\log\frac{1-e^{i\alpha}e^{it}}{1-e^{i\alpha+\beta}e^{it}}-\log\frac{1-e^{it}}{1-e^{i(\beta+t)}}\right)=\begin{cases}\pi & \text{if} \quad t\in(0,2\pi-\beta)\\ -\pi & \text{if} \quad t\in(2\pi-\alpha,4\pi-(\alpha+\beta))\\ 0 & \text{otherwise.}\end{cases}$$

From this and formula (2.6) for $\operatorname{Re} J(e^{it})$ we see that in this case $\operatorname{Re} J(e^{it}) \geq 0$. Now, suppose that $0 < \alpha \leq \beta < 2\pi$ and $\alpha + \beta > 2\pi$.

Then, analogously, we have

$$\operatorname{Im} \log \frac{1 - e^{i\alpha} e^{it}}{1 - e^{i\alpha+\beta} e^{it}} = \begin{cases} -\frac{\beta}{2} & \text{if} \quad t \in (2\pi - \alpha, 4\pi - (\alpha + \beta)) \\ \pi - \frac{\beta}{2} & \text{if} \quad t \in (4\pi - (\alpha + \beta), 2\pi) \cup (0, 2\pi - \alpha) \end{cases}$$

and

$$\operatorname{Im} \log \frac{1 - e^{it}}{1 - e^{i\beta} e^{it}} = \begin{cases} -\frac{\beta}{2} & \text{if} \quad t \in (0, 2\pi - \beta) \\ \pi - \frac{\beta}{2} & \text{if} \quad t \in (2\pi - \beta, 2\pi). \end{cases}$$

Hence

$$\operatorname{Im} \left(\log \frac{1 - e^{i\alpha} e^{it}}{1 - e^{i\alpha+\beta} e^{it}} - \log \frac{1 - e^{it}}{1 - e^{i(\beta+t)}} \right) = \begin{cases} \pi & \text{if} \quad t \in (0, 2\pi - \beta) \\ -\pi & \text{if} \quad t \in (2\pi - \alpha, 4\pi - (\alpha + \beta)) \\ 0 & \text{otherwise,} \end{cases}$$

and $\operatorname{Re} J(e^{it}) \geq 0$ also in this case. One can easily check that when $0 < \alpha < \beta < 2\pi$ and $\alpha + \beta = 2\pi$, then again $\operatorname{Re} J(e^{it}) \geq 0$. It then follows that $\operatorname{Re} J(e^{it}) \geq 0$ for $e^{it} \neq 1, e^{-i\alpha}, e^{-i\beta}, e^{-i(\alpha+\beta)}$ and $\alpha \leq \beta$. The symmetry of a convolution implies that $\operatorname{Re} J(e^{it}) \geq 0$ also if $\alpha > \beta$.

Now it will be useful to note that $J \in H^p$, $1 < p < \infty$. Indeed, analyticity guarantees boundedness of the integral means for $0 \leq |z| \leq r_0$, where $0 < r_0 < 1$. The first term in formula (2.6) is also bounded for $r_0 < |z| < 1$. This shows that the first term is in H^∞. Since the H^p spaces are nested with $H^\infty \subset H^p$ for all $p \geq 1$, it is now enough to show that the second term lies in H^p for some $p > 1$, and that the integral means of the second term behave as those of $\log(1 - z)$. So consider the case $p = 2$.

$$
\begin{aligned}
\int_0^{2\pi} |\log(1-re^{it})|^2 dt &= \int_0^{2\pi} \log(1-re^{it})\overline{\log(1-re^{it})}dt \\
&= \int_0^{2\pi} \left(\log|1-re^{it}|\right)^2 + \left(\arg(1-re^{it})\right)^2 dt \\
&= \int_{-\pi/2}^{\pi/2} \left(\log|1-re^{it}|\right)^2 dt + const \\
&= \frac{1}{4}\int_{-\pi/2}^{\pi/2} \left(\log|1-re^{it}|^2\right)^2 dt + const \\
&= \frac{1}{4}\int_{-\pi/2}^{\pi/2} \left(\log(1-re^{it})(1-re^{-it})\right)^2 dt + const \\
&= \frac{1}{4}\int_{-\pi/2}^{\pi/2} \left(\log(1-2r\cos t + r^2)\right)^2 dt + const \\
&\leq \frac{1}{4}\int_{-\pi/2}^{\pi/2} \left(\log(2(1-\cos t))\right)^2 dt + const \\
&= \frac{1}{4}\int_{-\pi/2}^{\pi/2} \left(\log\left(4\sin^2\left(\frac{t}{2}\right)\right)\right)^2 dt + const \\
&= \frac{1}{2}\int_{-\pi/4}^{\pi/4} \left(\log\left(4\sin^2(t)\right)\right)^2 dt + const \\
&= 4\int_0^{\pi/4} \left(\log\left(2\sin(t)\right)\right)^2 dt + const \\
&\leq 4\int_0^{\pi/4} \left(\log\left(\frac{4}{\pi}t\right)\right)^2 dt + const
\end{aligned}
$$

The latter integral is easily seen to be finite using calculus. So the second term is in H^2. Consequently, $\operatorname{Re} J \in h^p,\ 1 < p < \infty$. Next we appeal to Theorem 2.3 to deduce that $\operatorname{Re} J$ is the Poisson integral of its boundary value, so it is positive in $\mathbb{D}$.

Finally, we study the function $\tilde{f} = f * f$, where $f = h + \overline{g} \in S_H^O(R)$ and $\omega(z) = g(z)/h(z) = z$. Then

$$
h(z) = \frac{1}{4}\log\left(\frac{1+z}{1-z}\right) + \frac{1}{2}\frac{z}{1-z}, \tag{2.9}
$$

$$g(z) = -\frac{1}{4}\log\left(\frac{1+z}{1-z}\right) + \frac{1}{2}\frac{z}{1-z}. \tag{2.10}$$

Thus

$$\begin{aligned} h'(z) &= \frac{1}{2}\left(\frac{1}{(1-z)^2} + \frac{1}{1-z^2}\right), \\ g'(z) &= \frac{1}{2}\left(\frac{1}{(1-z)^2} - \frac{1}{1-z^2}\right). \end{aligned}$$

If now, as in the above, $\widetilde{f} = f * f = h * h + \overline{g * g} = H + \overline{G}$, then by examining the power series of the functions, we have

$$\begin{aligned} zH'(z) = h(z) * zh'(z) &= \frac{1}{2}\left(zh'(z) + \frac{1}{2}\left(h(z) - h(-z)\right)\right), \\ zG'(z) = g(z) * zg'(z) &= \frac{1}{2}\left(zg'(z) - \frac{1}{2}\left(g(z) - g(-z)\right)\right). \end{aligned}$$

Hence

$$\begin{aligned} zH'(z) &= \frac{1}{2}\left(\frac{z}{2(1-z)^2} + \frac{z}{1-z^2} + \frac{1}{4}\log\frac{1+z}{1-z}\right), \\ zG'(z) &= \frac{1}{2}\left(\frac{z}{2(1-z)^2} - \frac{z}{1-z^2} + \frac{1}{4}\log\frac{1+z}{1-z}\right). \end{aligned}$$

We will show that $\widetilde{f} = H + \overline{G}$ is convex. To this end, as in the proof of Lemma 2.4, it is enough to show that the function $H - e^{2i\theta}G$ is convex in the direction θ for all $\theta \in [0, \pi)$, or equivalently, that $F_\theta = ie^{-i\theta}\left(H - e^{2i\theta}G\right)$ is convex in the direction of the imaginary axis for all $\theta \in [0, \pi)$. We now apply Theorem 2.2 with constants $\mu = \nu = \pi/2$. Since

$H'(z) - G'(z) = 1/(1-z^2)$, we have

$$\begin{aligned}
\operatorname{Re}\left\{\left(1-z^2\right) F_\theta'(z)\right\} =& -\operatorname{Im}\left\{(1-z^2)\left(e^{-i\theta}H'(z) - e^{i\theta}G'(z)\right)\right\}\\
=& -\operatorname{Im}\left\{\frac{(1-z^2)}{z}\left(e^{-i\theta}zH'(z) - e^{i\theta}zG'(z)\right)\right\}\\
=& -\operatorname{Im}\left\{\frac{(1-z^2)}{z}\Big(-i\left(zH'(z)+zG'(z)\right)\sin\theta\right.\\
&\left.+\left(zH'(z)-zG'(z)\right)\cos\theta\Big)\right\}\\
=& -\operatorname{Im}\left\{-i\sin\theta\left(\frac{1}{2}\frac{1+z}{1-z}+\frac{1-z^2}{4z}\ln\frac{1+z}{1-z}\right)\right\}\\
=& \frac{1}{2}\sin\theta\operatorname{Re}\left\{\frac{1+z}{1-z}+\frac{1-z^2}{2z}\log\frac{1+z}{1-z}\right\} \geq 0,
\end{aligned}$$

where the last inequality is due to the fact that $z \mapsto \log(1+z)/(1-z)$ is convex. Indeed, if φ is a convex analytic function such that $\varphi(0) = 0$, then it is starlike, and so it satisfies the condition $\operatorname{Re}\frac{\varphi(z)}{z\varphi'(z)} > 0$.

$\square$

Remark. One can describe the image of the unit disk under the convex function $\widetilde{f} = f * f = H + \overline{G}$ considered in the last part of the proof. We first note that $f = h + \overline{g}$, where h and g are given by (2.9) and (2.10), maps univalently $\mathbb{D}$ onto the half-strip $\{w : \operatorname{Re}\{w\} > -1/2,\ |\operatorname{Im}\{w\}| < \pi/4\}$ (see e.g. [16, p. 42]). A calculation shows that

$$\begin{aligned}
H(z) &= \frac{1}{4}\left(\frac{z}{1-z} + \log\frac{1+z}{1-z} + \frac{1}{2}\operatorname{Li}_2(z) - \frac{1}{2}\operatorname{Li}_2(-z)\right),\\
G(z) &= \frac{1}{4}\left(\frac{z}{1-z} - \log\frac{1+z}{1-z} + \frac{1}{2}\operatorname{Li}_2(z) - \frac{1}{2}\operatorname{Li}_2(-z)\right),
\end{aligned}$$

where $\operatorname{Li}_2(z)$ denotes the dilogarithm function defined by the series

$$\operatorname{Li}_2(z) = \sum_{n=1}^{\infty}\frac{z^n}{n^2}, \quad \text{for } |z| \leq 1.$$

Hence

$$\widetilde{f}(z) = \frac{1}{4}\operatorname{Re}\frac{1+z}{1-z} + \frac{1}{2}\operatorname{Re}\sum_{n=0}^{\infty}\frac{z^{2n+1}}{(2n+1)^2} - \frac{1}{4} + \frac{i}{2}\arg\frac{1+z}{1-z}.$$

We observe first that the function $\widetilde{f}$ has unrestricted limit $\lim \widetilde{f}(z)$, as $z \to e^{i\theta}$, $z \in \mathbb{D}$, and is continuous for all points $e^{i\theta} \in \partial\mathbb{D} \setminus \{1, -1\}$. Moreover, $\widetilde{f}(e^{i\theta})$ is given by

$$\widetilde{f}(e^{i\theta}) = \begin{cases} -\frac{1}{4} + \frac{1}{2}\sum_{n=0}^{\infty}\frac{\cos(2n+1)\theta}{(2n+1)^2} + i\frac{\pi}{4}, & \theta \in (0,\pi) \\ -\frac{1}{4} + \frac{1}{2}\sum_{n=0}^{\infty}\frac{\cos(2n+1)\theta}{(2n+1)^2} - i\frac{\pi}{4}, & \theta \in (\pi, 2\pi). \end{cases}$$

It is worth noting here that the boundary function $\widetilde{f}(e^{i\theta})$ is bounded on $(0,\pi)\cup(\pi,2\pi)$ and has finite one-sided limits: $\lim_{\theta\to\pi^-}\widetilde{f}(e^{i\theta}) = -\frac{4+\pi^2}{16} + i\frac{\pi}{4}$, $\lim_{\theta\to\pi^+}\widetilde{f}(e^{i\theta}) = -\frac{4+\pi^2}{16} - i\frac{\pi}{4}$, and $\lim_{\theta\to 0^+}\widetilde{f}(e^{i\theta}) = \frac{-4+\pi^2}{16} + i\frac{\pi}{4}$, $\lim_{\theta\to 2\pi^-}\widetilde{f}(e^{i\theta}) = \frac{-4+\pi^2}{16} - i\frac{\pi}{4}$.

We remark that the function $\widetilde{f}$ is an example of the convex mappings described in Theorem 2.4 in [1]. This theorem says that if f is a univalent, harmonic orientation-preserving mapping from $\mathbb{D}$ onto an unbounded convex domain which is neither a strip nor a half-plane, then $f \in h^1$ and there is only one point $e^{i\lambda}$ that corresponds to ∞, and for some constant $A \in \mathbb{C}$,

$$f(z) = \int_0^{2\pi} P(z,\theta) f(e^{i\theta})d\theta + AP(z,\lambda)$$

where P denotes the Poisson kernel

$$P(z,\theta) = \frac{1}{2\pi}\operatorname{Re}\frac{e^{i\theta}+z}{e^{i\theta}-z}.$$

A calculation shows that our function $\widetilde{f}$ can be written as

$$\widetilde{f}(z) = \int_0^{2\pi} P(z,\theta)\widetilde{f}(e^{i\theta})d\theta + \frac{1}{4}P(z,0),$$

where $\widetilde{f}(e^{i\theta})$ is the function defined above. It then follows from the proof of Theorem 2.4

in [1] that the range of $\widetilde{f}$ is the half-strip

$$\left\{w: \ \mathrm{Re}\{w\} > -\frac{4+\pi^2}{16}, \ |\mathrm{Im}\{w\}| < \frac{\pi}{4}\right\}.$$

2.3 Conclusion

In summary, we have proved the following theorem

Theorem 2.5. *If* $f_k = h_k + \overline{g}_k, \ k = 1, 2,$ *map into the right half-plane* $\mathrm{Re}(z) > \frac{1}{2}$ *in* $\mathbb{C}$ *and satisfy* $\omega_1(z) = g_1'(z)/h_1'(z) = -xz$, $\omega_2(z) = g_2'(z)/h_2'(z) = -yz$ *with* $|x| = |y| = 1$, *then the function* $\widetilde{f} = f_1 * f_2$ *is convex in the direction of the real axis. In particular, if* $x = y = -1$, *then* $\widetilde{f}$ *is convex.*

This shows the broad applicability of an earlier theorem of Dorff in [6] by proving that a large family of functions satisfy its hypotheses. More broadly, it contributes to the search for understanding of when the convolution of two harmonic univalent functions is again univalent. Around the same time that this result was published in [3], other authors (see [28, 31, 32, 11]) also contributed to an expanded family of known result involving half-planes, and this is still an active area of discovery. It is expected that this will be an important class of convolution examples for years to come.

Chapter 3. Harmonic univalent mappings with singular inner function dilatation

3.1 Introduction

Most examples of univalent harmonic mappings have a dilatation that is a finite Blaschke product (see [14], [15], [18], [21], [40]). In [29], planar harmonic mappings with infinite Blaschke product dilatation are discussed. While finite Blaschke product dilatations have been the subject of much research, very little has been done with singular inner function dilatations (SIFD). Laugesen posed the problem of finding necessary and sufficient conditions on the boundary of a harmonic function f for it to be SIFD. However, he was unaware of any nontrivial examples of such functions. Since then, Weitsmann has given two examples [41]. In this chapter, we improve upon this by constructing an infinite family of SIFD functions and establishing a method to construct additional examples. We also consider questions about the boundary behavior of harmonic functions with singular inner dilatation.

3.2 Background

A *Blaschke product* is an expression of the form

$$B(z) = e^{i\theta} \prod_{j=1}^{\infty} \left(\frac{z - a_j}{1 - \overline{a_j} z} \right)^{m_j}.$$

Blaschke products are a part of the more general class of inner functions. By definition, an *inner function* is a bounded analytic function $f\colon \mathbb{D} \to \mathbb{C}$ such that $\lim_{|z|\to 1} |f(z)| = 1$. Any inner function can be written in the form

$$f(z) = e^{i\alpha} B(z) \exp\left(- \int \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu\left(e^{i\theta}\right) \right),$$

where $\alpha, \theta \in \mathbb{R}$, μ is a singular positive measure on $\partial\mathbb{D}$, and $B(z)$ is a Blaschke product [24]. In addition, we say that an inner function is *singular* if f has no zeroes on $\mathbb{D}$. A basic example of a singular inner function is $f(z) = e^{\frac{z+1}{z-1}}$.

Previous examples of SIFD functions have relied on the shearing technique. We restate the shearing theorem here for reference.

Theorem 3.1. *Let h and g be analytic functions on the unit disk $\mathbb{D}$ such that $f = h + \overline{g}$ is locally univalent. Then f is a univalent CHD function of $\mathbb{D}$ if and only if the analytic function $\varphi = h - g$ has the same property.*

As previously noted, the condition that $|\omega| < 1$ guarantees the local univalence of f. The advantage of the shearing technique is that it guarantees the univalence of f for a given CHD analytic function and a given dilatation.

We illustrate both the shearing technique and how it can be used to construct SIFD harmonic mappings with the following example, originally given by Weitsman [5] (for another example, see Greiner [21]).

Example 3.2. The analytic function

$$\varphi = \frac{z}{1-z} + \frac{1}{2} e^{\frac{z+1}{z-1}}$$

is univalent and CHD (see [5]). Hence shearing φ with dilatation $\omega(z) = e^{\frac{z+1}{z-1}}$, yields

$$h = \frac{z}{1-z} \quad \text{and} \quad g = \frac{-1}{2} e^{\frac{z+1}{z-1}},$$

which gives the harmonic univalent function

$$f = h + \overline{g} = \frac{z}{1-z} - \overline{\frac{1}{2} e^{\frac{z+1}{z-1}}}$$

with singular inner function dilatation.

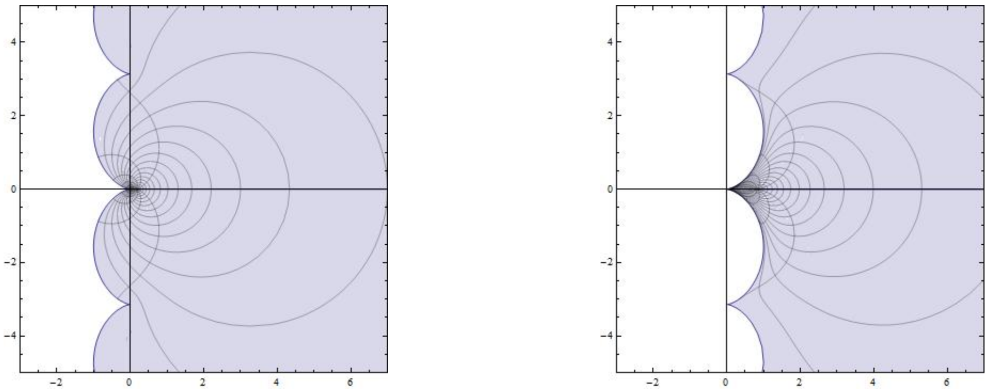


Figure 3.1: Images of $\varphi(\mathbb{D})$ and $f(\mathbb{D})$ in Example 3.2.

This approach requires a careful selection of the original analytic mapping to get an integral expression that can be handled algebraically. Currently, this approach has led to only a few examples of SIFD functions.

3.3 New Examples of SIFD Harmonic Functions

The following example extends Example 3.2. It utilizes the shearing theorem to construct an infinite family of harmonic functions that can be expressed in terms of exponential integrals. We will require a result by Royster and Zeigler [36].

Theorem 3.3. *Let φ be an analytic mapping of $\mathbb{D}$ that satisfies*

$$\operatorname{Re}\left\{-e^{i\mu}\left(1-2\cos(\nu)e^{-i\mu}z+e^{-2i\mu}z^2\right)\varphi'(z)\right\}\geq 0$$

for some $\mu,\nu\in[0,\pi]$. Then φ is a univalent mapping of $\mathbb{D}$ onto a CHD domain.

Example 3.4. Consider the analytic function

$$\varphi = h - g = \int_0^z \frac{1-e^{\frac{\zeta+1}{\zeta-1}}}{1-2\cos(\nu)\zeta+\zeta^2}d\zeta.$$

By setting $\mu=0$ in Theorem 3.3, we see that φ is univalent and CHD. Shearing φ with the dilatation $\omega = e^{\frac{\zeta+1}{\zeta-1}}$, yields

$$h(z) = \int_0^z \frac{1}{1 - 2\cos\nu\zeta + \zeta^2} d\zeta$$

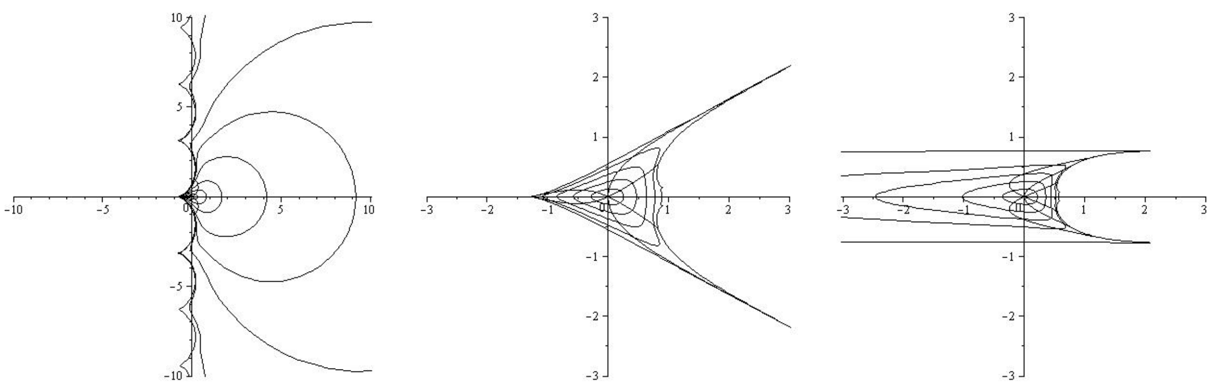
$$g(z) = \int_0^z \frac{e^{\frac{z+1}{z-1}}}{1 - 2\cos\nu\zeta + \zeta^2} d\zeta.$$

Let $\alpha = 1 + \cos\nu$, $\beta = 1 - \cos\nu$, and $u = \frac{z+1}{z-1}$. For $\nu \in (0, \pi)$, we can solve these integrals to get

$$h(z) = \frac{-1}{\sqrt{\alpha\beta}} \arctan\left(\frac{-z + \cos\nu}{\sqrt{\alpha\beta}}\right) - h_0(\nu)$$

$$g(z) = \frac{-1}{2\sqrt{-\alpha\beta}} \left(e^{-\sqrt{-\alpha/\beta}} E_1\left(-u - \sqrt{-\alpha/\beta}\right) - e^{\sqrt{-\alpha/\beta}} E_1\left(-u + \sqrt{-\alpha/\beta}\right)\right) - g_0(\nu),$$

where $h_0(\nu)$ and $g_0(\nu)$ are normalizing constants, and E_1 is the exponential integral. Theorem 3.1 guarantees that the harmonic mapping $f = h + \overline{g}$ will be CHD and univalent for every value of ν. Note that when $\nu = 0$ we get the normalized version of Weitsman's example.



$\nu = 0$ $\qquad$ $\nu = \frac{\pi}{2}$ $\qquad$ $\nu = \pi$

Figure 3.2: Images of $f(\mathbb{D})$ for various values of ν in Example 3.4.

We now present a new technique for creating SIFD harmonic functions, based on the following theorem by Clunie and Sheil-Small in [4].

Theorem 3.5. *Let $f = h + \overline{g}$ be locally univalent in $\mathbb{D}$ and suppose that $h + cg$ is convex for some $|c| \leq 1$. Then f is univalent.*

Example 3.6. Let

$$h(z) = \frac{1}{2} \log \left(\frac{1+z}{1-z} \right)$$

and let $\omega = e^{\frac{z+1}{z-1}}$. Then

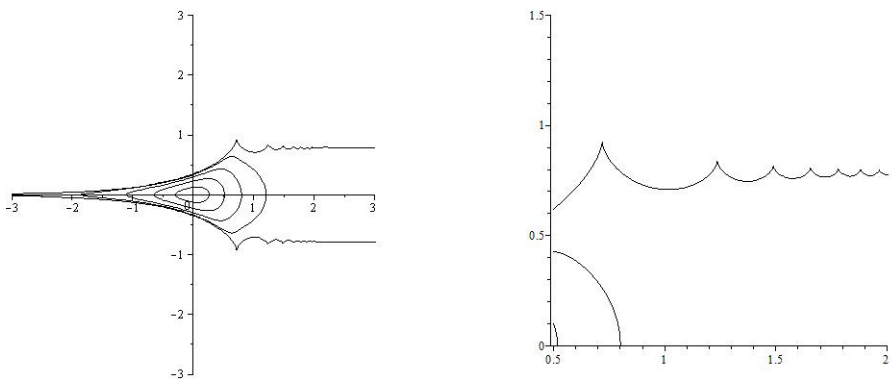
$$g' = h'\omega = \frac{1}{1-z^2} e^{\frac{z+1}{z-1}}.$$

It follows from Theorem 3.5 with $c = 0$ that $f = h + \overline{g}$ is univalent. Integrating g gives

$$g(z) = -\frac{1}{2} E_1 \left(\frac{z+1}{-z+1} \right) + \frac{1}{2} E_1(1),$$

where $E_n(z)$ is the exponential integral function

$$E_n(z) = \int_1^\infty \frac{e^{-zt}}{t^n} dt, \quad \operatorname{Re}(z) > 0.$$



(a) Image of $f(\mathbb{D})$ (b) Close up image

Figure 3.3: Images in Example 3.6.

Using Theorem 3.5 and the approach in Example 3.6, we can construct many examples of harmonic univalent SIFD functions. Figure 3.4 has a collection of more examples.

3.4 Boundary Behavior

In each of these examples, it appears that the image of the harmonic map has infinitely many cusps. We will now give a heuristic analysis that confirms this for Examples 3.2 and 3.4. We require a theorem by Greiner [21].

Theorem 3.7. *Let φ be a conformal mapping of the unit disk $\mathbb{D}$ onto a domain convex in the direction of the real axis and let ω be an analytic function with $|\omega(z)| < 1$ in $\mathbb{D}$. Let f be the horizontal shear of φ with dilatation ω. If $I = \{e^{i\theta} | \theta \in (a, b)\}$ is an arc up to which ω is continuous and if*

$$\omega\left(e^{i\theta}\right) \in \partial\mathbb{D} \setminus \{1\}, \quad \lim_{r\to 1} \frac{\partial \varphi}{\partial \theta}\left(re^{i\theta}\right) \in \mathbb{C} \setminus \mathbb{R}, \quad \textit{and} \quad \lim_{r\to 1} \frac{\partial^2 \varphi}{\partial \theta^2}\left(re^{i\theta}\right) \in \mathbb{C}$$

for all $\theta \in (a, b)$, then

$$\hat{f}(I) = \left\{\lim_{r\to 1} f\left(re^{i\theta}\right) | \theta \in (a, b)\right\}$$

is a strictly concave boundary arc.

We will approach this analysis by finding all points in the domain of φ that do not meet the above criteria. That is, we are looking for the set A of points on $\partial\mathbb{D}$ up to which

1. ω does not extend continuously, or

2. ω extends continuously but

   a) $\omega\left(e^{i\theta}\right) = 1$,

   b) $\lim_{r\to 1} \frac{\partial \varphi}{\partial \theta}\left(re^{i\theta}\right) \in \mathbb{R} \cup \{\infty\}$, or

   c) $\lim_{r\to 1} \frac{\partial^2 \varphi}{\partial \theta^2}\left(re^{i\theta}\right) \notin \mathbb{C}$.

First we note that ω is continuous up to $\partial\mathbb{D}$ except at $z = 1$. By taking the second partial derivative of φ with respect to θ it is easy to see that condition 2c) is satisfied only when $z = 1$ or $z = e^{\pm i\nu}$. Next, consider condition 2a). Notice

$$e^{\frac{e^{i\theta}+1}{e^{i\theta}-1}} = e^{-i \cot \frac{\theta}{2}}.$$

Thus $\omega\left(e^{i\theta}\right)=1$ exactly when $\theta=\pi$. Finally, consider condition $2b$). A simple computation yields

$$\frac{\partial\phi}{\partial\theta}\left(re^{i\theta}\right)=\frac{ire^{i\theta}\left(1-e^{\frac{re^{i\theta}+1}{re^{i\theta}+1}}\right)}{1-2\cos(\nu)re^{i\theta}+r^2e^{2i\theta}},$$

which extends continuously to $\partial\mathbb{D}$ except at $z=1$ and $z=e^{\pm i\nu}$. Thus for all other points on $\partial\mathbb{D}$,

$$\lim_{r\to1}\frac{\partial\phi}{\partial\theta}\left(re^{i\theta}\right)=\frac{\partial\phi}{\partial\theta}\left(e^{i\theta}\right).$$

Notice that for these points,

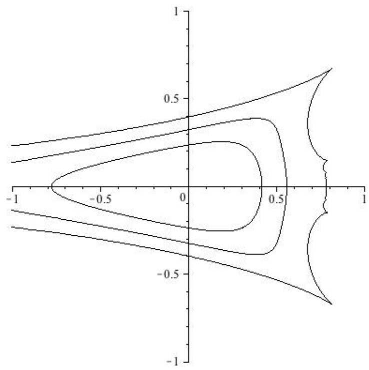
$$\begin{aligned}
&\lim_{r\to1}\frac{\partial\phi}{\partial\theta}\left(re^{i\theta}\right)\in\mathbb{R}\cup\{\infty\}\\
\iff&\operatorname{Im}\left\{\frac{\partial\phi}{\partial\theta}\left(e^{i\theta}\right)\right\}=\operatorname{Im}\left\{\frac{ie^{i\theta}\left(1-e^{-i\cot\frac{\theta}{2}}\right)}{1-2\cos(\nu)e^{i\theta}+e^{2i\theta}}\right\}=0\\
\iff&\operatorname{Im}\left\{ie^{i\theta}\left(1-e^{-i\cot\frac{\theta}{2}}\right)\left(1-2\cos(\nu)e^{-i\theta}+e^{-2i\theta}\right)\right\}\\
&=(2\cos\theta-2\cos\nu)\left(1-\cos\left(\cot\frac{\theta}{2}\right)\right)=0.
\end{aligned}$$

Thus for $z\neq1, z\neq e^{\pm i\nu}$,

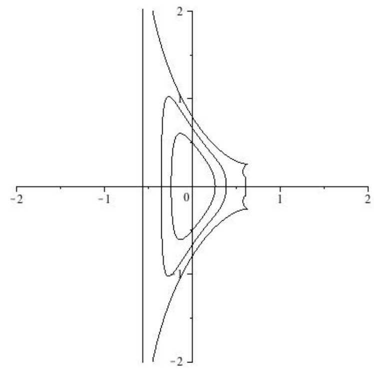
$$\operatorname{Im}\left\{\frac{\partial\phi}{\partial\theta}(e^{i\theta})\right\}=0\iff\cot\frac{\theta}{2}=2k\pi, k\in\mathbb{Z}.$$

In summary, the set A consists of the points $\left\{z=e^{i\theta}|\theta=\pm\nu,\text{or}\cot\frac{\theta}{2}=2k\pi,k\in\mathbb{Z}\right\}$. Theorem 3.7 says that at every point of $\partial\mathbb{D}$ not in A, the boundary of the image of φ is a concave arc. Thus the image of φ has infinitely many cusps which occur at the points of A.

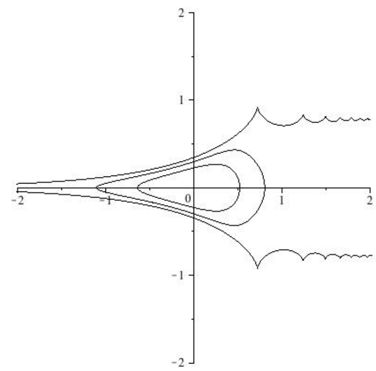
3.5 More Examples



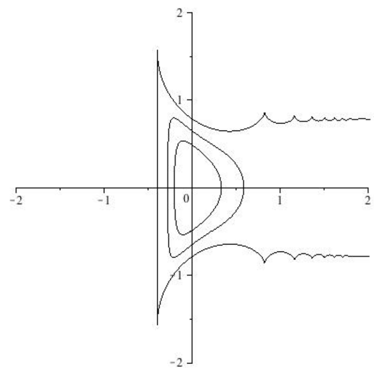
(a) $h = \log(1+z); \omega = e^{\frac{z+1}{z-1}}$



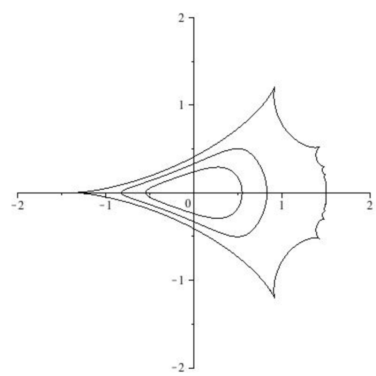
(b) $h = \log(1+z); \omega = -e^{\frac{z+1}{z-1}}$



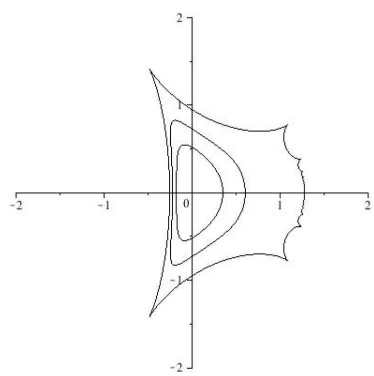
(c) $h = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right); \omega = e^{\frac{z+1}{z-1}}$



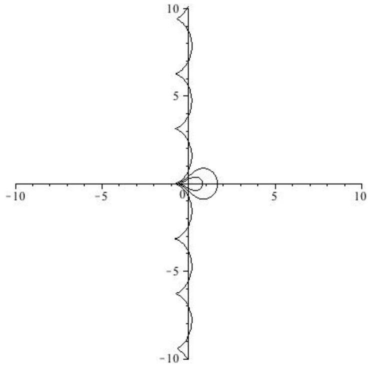
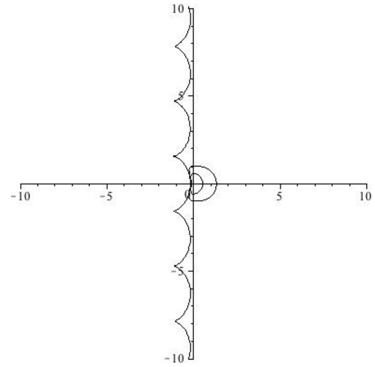
(d) $h = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right); \omega = -e^{\frac{z+1}{z-1}}$



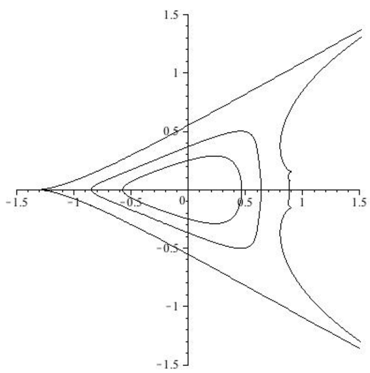
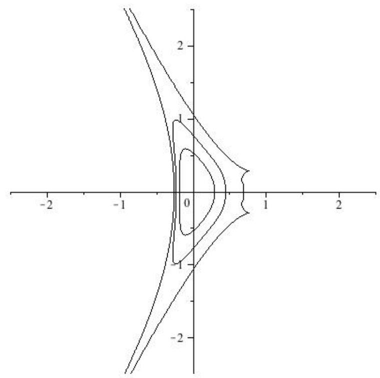
(e) $h = 2\log\left(\frac{z}{2-z}\right); \omega = e^{\frac{z+1}{z-1}}$



(f) $h = 2\log\left(\frac{z}{2-z}\right); \omega = -e^{\frac{z+1}{z-1}}$

(g) $h = \frac{z}{1-z}; \omega = e^{\frac{z+1}{z-1}}$ (h) $h = \frac{z}{1-z}; \omega = -e^{\frac{z+1}{z-1}}$

(i) $h = \frac{1}{2i} \log \left(\frac{1+iz}{1-iz} \right); \omega = e^{\frac{z+1}{z-1}}$ (j) $h = \frac{1}{2i} \log \left(\frac{1+iz}{1-iz} \right); \omega = -e^{\frac{z+1}{z-1}}$

Figure 3.4: Summary of Examples

3.6 Using harmonic maps to construct new minimal surfaces

In this section we show an example in which a harmonic univalent function is lifted to form a minimal graph that appears to be new. The construction is along the same lines as Example 3.6 and is only outlined below.

Let $f = h + \overline{g}$, where

$$h(z) = \frac{1}{2} \log \left(\frac{1+z}{1-z} \right)$$

and let $\omega = \left(e^{\frac{z+1}{z-1}} \right)^2$. Since $g' = h'\omega = \frac{1}{1-z^2} e^{2\frac{z+1}{z-1}}$, we know that

$$g(z) = -\frac{1}{2} E_1 \left(\frac{z+1}{-z+1} \right) + \frac{1}{2} E_1(1),$$

where $E_1(z)$ is the exponential integral function. Theorem 3.5 shows that, $f = h + \overline{g}$ is univalent. The image of $f(\mathbb{D})$ is shown in Figure 3.5.

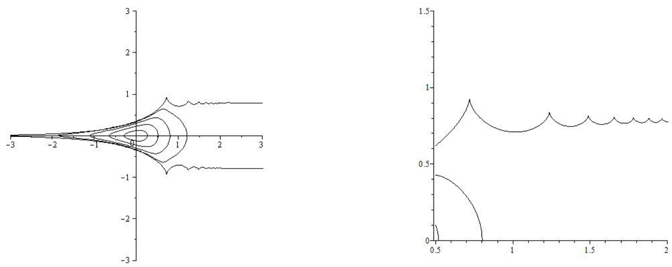


Figure 3.5: The image of $f(\mathbb{D})$ and a close-up of that image.

By the Weierstrass representation $f = h + \overline{g}$ lifts to an embedded minimal surface (see Figure 3.6).

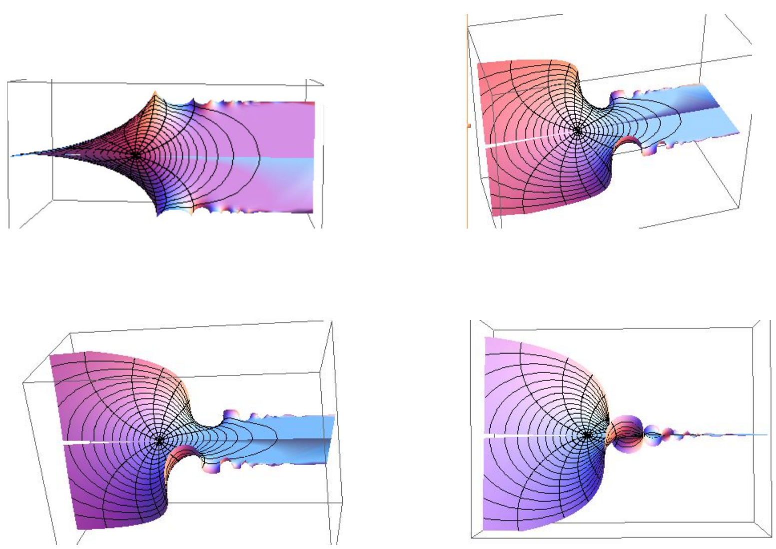


Figure 3.6: Images of the minimal surface constructed from f

The fact that the harmonic function has nonvanishing dilatation means that there is no (finite) point where the function is approximately analytic. In terms of minimal surfaces, this means that the minimal surface never has zero Gauss curvature. The many cusps of the harmonic map lift to wrinkles in the edge of the surface. (It can be seen that this the same is true of the corners in the map that induces Scherk's doubly-periodic surface.) It also has a singularity with unusual behavior when $z \to 1$.

Chapter 4. Convolutions of Non-vertical Shears

Up to this point, we have considered only shears in the vertical direction. While there are unique features of vertical shearing, we show that it is possible to generalize results about vertical shears to arbitrary directions. Thus working only in the vertical direction is often sufficient. This section provides one approach to the generalization and gives examples. We emphasize that the most important application is the transition between vertical and horizontal shearing.

To state the identity, we must first establish some more covenenient notation. We will use $f_{\alpha,\omega}$ to denote the shear of $\varphi(z) = \dfrac{z}{1-z}$ in the α direction with dilatation $e^{-2i\alpha}\omega$. That is, $f_{\alpha,\omega} = h_{\alpha,\omega} + \overline{g_{\alpha,\omega}}$, where $h_{\alpha,\omega} - e^{2i\alpha} g_{\alpha,\omega} = \dfrac{z}{1-z}$ and $g'/h' = e^{-2i\alpha}\omega$.

Lemma 4.1. *Let $\alpha, \beta \in \mathbb{R}$, and let ω be analytic on $\mathbb{D}$ with $|\omega| < 1$. Then $h_{\alpha,\omega} = h_{\beta,\omega}$ and $g_{\alpha,\omega} = e^{-2i(\alpha-\beta)} g_{\beta,\omega}$.*

Proof. We examine the steps of the shearing technique, keeping track of the relationship between the two shears. Differentiating the equation $\varphi = h_{\alpha,\omega} - e^{2i\alpha} g_{\alpha,\omega}$ gives

$$\begin{aligned}
\varphi' &= h'_{\alpha,\omega} - e^{2i\alpha} g'_{\alpha,\omega} \\
&= h'_{\alpha,\omega}(1 - e^{2i\alpha}(e^{-2i\alpha}\omega)) \\
&= h'_{\alpha,\omega}(1 - \omega).
\end{aligned}$$

Hence $h'_{\alpha,\omega} = \varphi'/(1-\omega)$. Likewise, we have $h'_{\beta,\omega} = \varphi'/(1-\omega)$. This shows that $h_{\alpha,\omega} = h_{\beta,\omega}$. This gives immediately $g_{\alpha,\omega} = e^{-2i\alpha}(h_{\alpha,\omega} - \varphi)$ and likewise $g_{\beta,\omega} = e^{-2i\beta}(h_{\beta,\omega} - \varphi)$. We conclude that $g_{\alpha,\omega} = e^{-2i(\alpha-\beta)} g_{\beta,\omega}$. $\square$

Theorem 4.2. *Let $\alpha, \beta \in \mathbb{R}$, and let ω_1, ω_2 be analytic on $\mathbb{D}$ with $|\omega_1| < 1$ and $|\omega_2| < 1$. Then $f_{\alpha,\omega_1} * f_{-\alpha,\omega_2} = f_{\beta,\omega_1} * f_{-\beta,\omega_2}$.*

Proof. This is a direct computation. We can express the various functions as follows.

$$
\begin{aligned}
f_{\alpha,\omega_1}(z) &= h_{\alpha,\omega_1}(z) + \overline{g_{\alpha,\omega_1}(z)} = \sum_{n=1}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b_n}\overline{z}^n \\
f_{-\alpha,\omega_2}(z) &= h_{-\alpha,\omega_2}(z) + \overline{g_{-\alpha,\omega_2}(z)} = \sum_{n=1}^{\infty} A_n z^n + \sum_{n=1}^{\infty} \overline{B_n}\overline{z}^n \\
f_{\beta,\omega_1}(z) &= h_{\beta,\omega_1}(z) + \overline{g_{\beta,\omega_1}(z)} = \sum_{n=1}^{\infty} c_n z^n + \sum_{n=1}^{\infty} \overline{d_n}\overline{z}^n \\
f_{-\beta,\omega_2}(z) &= h_{-\beta,\omega_2}(z) + \overline{g_{-\beta,\omega_2}(z)} = \sum_{n=1}^{\infty} C_n z^n + \sum_{n=1}^{\infty} \overline{D_n}\overline{z}^n
\end{aligned}
$$

We then have

$$
\begin{aligned}
(f_{\alpha,\omega_1} * f_{-\alpha,\omega_2})(z) &= \sum_{n=1}^{\infty} a_n A_n z^n + \sum_{n=1}^{\infty} \overline{b_n B_n}\overline{z}^n \\
(f_{\beta,\omega_1} * f_{-\beta,\omega_2})(z) &= \sum_{n=1}^{\infty} c_n C_n z^n + \sum_{n=1}^{\infty} \overline{d_n D_n}\overline{z}^n
\end{aligned}
$$

From the Lemma we have $a_n = c_n$, $b_n = e^{-2i(\alpha-\beta)} d_n$, $A_n = C_n$, and $B_n = e^{-2i(-\alpha+\beta)} D_n$ for all n. Hence

$$
\begin{aligned}
(f_{\beta,\omega_1} * f_{-\beta,\omega_2})(z) &= \sum_{n=1}^{\infty} a_n A_n z^n + \sum_{n=1}^{\infty} \overline{e^{-2i(\alpha-\beta)} d_n e^{-2i(-\alpha+\beta)} D_n}\overline{z}^n \\
&= \sum_{n=1}^{\infty} a_n A_n z^n + \sum_{n=1}^{\infty} \overline{d_n D_n}\overline{z}^n \\
&= (f_{\alpha,\omega_1} * f_{-\alpha,\omega_2})(z).
\end{aligned}
$$

$\square$

This theorem has the aesthetically displeasing aspect that the two functions being convolved are shears in different directions. The exceptions to this are the vertical and horizontal directions, which is where this theorem seems most useful. We demonstrate this with the following alternate versions of the same theorems.

4.1 First Example

The following is a theorem from [11], and the investigation of whether an analogue held in the horizontal direction was the impetus for discovering the relationship exposed in this chapter.

Theorem 4.3. *Consider the right half-plane map* $f_1 = h_1 + \overline{g_1} \in S_H^0$ *with* $h_1 + g_1 = z/(1-z)$ *and* $\omega(z) = -z$*. Let* $f \in h + \overline{g} \in S_H^0$ *with* $h + g = z/(1-z)$ *and* $\omega(z) = e^{i\theta} z^n$ *(*$n \in \mathbb{N}$ *and* $\theta \in \mathbb{R}$*). If* $n = 1, 2$*, then* $f_\alpha * f \in S_H^0$ *and is convex in the horizontal direction.*

The analogue is

Corollary 4.4. *Consider the right half-plane map* $f_1 = h_1 + \overline{g_1} \in S_H^0$ *with* $h_1 - g_1 = z/(1-z)$ *and* $\omega(z) = z$*. Let* $f \in h + \overline{g} \in S_H^0$ *with* $h - g = z/(1-z)$ *and* $\omega(z) = e^{i\theta} z^n$ *(*$n \in \mathbb{N}$ *and* $\theta \in \mathbb{R}$*). If* $n = 1, 2$*, then* $f_1 * f \in S_H^0$ *and is convex in the horizontal direction.*

4.2 Second Example

We can also generalize the main theorem of a previous chapter with almost no effort.

Theorem 4.5. *Let* f_1 *be the shear of* $\varphi(z) = z/(1-z)$ *in the vertical direction with dilatation* $\omega_1 = e^{i\theta} z$*,* $\theta \in \mathbb{R}$*. Let* f_2 *be the shear of* $\varphi(z) = z/(1-z)$ *in the vertical direction with dilatation* $\omega_2 = e^{i\rho} z$*,* $\rho \in \mathbb{R}$*. Then* $f_1 * f_2 \in S_H$ *and is convex in the horizontal direction.*

Here is the generalization:

Corollary 4.6. *Let* f_1 *be the shear of* $\varphi(z) = z/(1-z)$ *in the* α *direction with dilatation* $\omega_1 = e^{i\theta} z$*,* $\theta \in \mathbb{R}$*. Let* f_2 *be the shear of* $\varphi(z) = z/(1-z)$ *in the* $-\alpha$ *direction with dilatation* $\omega_2 = e^{i\rho} z$*,* $\rho \in \mathbb{R}$*. Then* $f_1 * f_2 \in S_H$ *and is convex in the horizontal direction.*

5.1 Introduction

It is known (see [6]) that any harmonic mapping $h+\overline{g}$ onto the strip

$$\Omega_\alpha = \left\{ w : \frac{\alpha-\pi}{2\sin\alpha} < \mathrm{Re}(w) < \frac{\alpha}{2\sin\alpha} \right\}$$

has the form $h+g = \frac{1}{2i\sin\alpha}\log\left(\frac{1+ze^{i\alpha}}{1+ze^{-i\alpha}}\right)$. Also in [6], there is the following theorem

Theorem 5.1. *Let $f = h+\overline{g} \in K_H^O$ be a vertical strip mapping. If $f*f$ is locally univalent and satisfies* $\mathrm{Re}\left((1-z)^2(h'-g')\right) > 0$, *then $f*f$ is CHD.*

The proof relies on a convexity criterion, a lemma of Ruscheweyh and Sheil-Small, and clever computations. Here are the criterion and lemma, which we will also use in our analysis.

Theorem 5.2 (Royster and Ziegler, [36])**.** *Let f be analytic in $\mathbb{D}$ with $f(0)=0$ and $f'(0)\neq 0$, and let*

$$\phi(z) = \frac{z}{(1+ze^{i\theta})(1+ze^{-i\theta})},$$

where $\theta \in \mathbb{R}$. If

$$\mathrm{Re}\left(\frac{zf'(z)}{\phi(z)}\right)$$

then f is CHD.

Notice that ϕ is the derivative of the functional form for a half-plane map. The following lemma was used to prove a famous conjecture in convolution theory (see [39]).

Lemma 5.3. *Let ϕ and G be analytic in $\mathbb{D}$ with $\phi(0)=G(0)=0$. If ϕ is convex and G is starlike, then for each function F analytic in $\mathbb{D}$ and satisfying* $\mathrm{Re}(F(z)) > 0$, *we have*

$$\mathrm{Re}\left(\frac{(\phi*FG)(z)}{(\phi*G)(z)}\right) > 0.$$

5.2 Main Theorem

We find that Theorem 5.1 can be generalized by allowing that the two convolution factors be different functions. Here is the theorem and proof.

Theorem 5.4. *Fix* $\alpha \in [\frac{\pi}{2}, \pi)$*, and let* $f_k = h_k + \overline{g_k}$ *be vertical strip mappings, i.e.* $h_k + g_k = \frac{1}{2i\sin(\alpha)} \log\left(\frac{1+e^{i\alpha}z}{1+e^{-i\alpha}z}\right)$ *for* $k = 1, 2$*. If* $f_1 * f_2$ *is locally-univalent and if* $\operatorname{Re}\left((1-z)^2(h_k' - g_k')\right) > 0$ *then* $f_1 * f_2$ *is CHD.*

Proof. Consider

$$F_1 = (h_1 + g_1)(h_2 - g_2) = h_1 * h_2 - h_1 * g_2 + h_2 * g_1 + g_1 * g_2$$

$$F_2 = (h_1 - g_1)(h_2 + g_2) = h_1 * h_2 + h_1 * g_2 - h_2 * g_1 + g_1 * g_2.$$

Then we have

$$\frac{1}{2}(F_1 + F_2) = h_1 * h_2 - g_1 * g_2.$$

So it is enough to show that $\operatorname{Re}\left(\frac{zF_k'(z)}{\phi(z)}\right) > 0$ for $\phi(z) = \frac{z}{(1+e^{i\alpha}z)(1+e^{-i\alpha}z)}$. Now $zF_1'(z) = z\left((h_1+g_1)*(h_2-g_2)\right)' = (h_1+g_1) * z(h_2' - g_2') = (h_1+g_1) * \left[\frac{z}{(1-z)^2}(1-z)^2(h_2' - g_2')\right]$ and $\phi(z) = \frac{z}{(1+e^{i\alpha}z)(1+e^{-i\alpha}z)} = (h_1+g_1) * \frac{z}{(1-z)^2}$, so by Theorem 5.3 and the hypothesis of Theoem 5.4, we have

$$\operatorname{Re}\left(\frac{zF_1'(z)}{\phi(z)}\right) = \operatorname{Re}\left(\frac{(h_1+g_1) * \left[\frac{z}{(1-z)^2}(1-z)^2(h_2' - g_2')\right]}{(h_1+g_1) * \frac{z}{(1-z)^2}}\right) > 0.$$

This completes the proof. $\square$

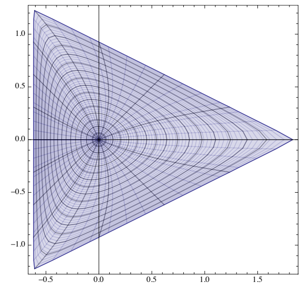


Figure 5.1: Image of $f_1 * f_1$.

5.3 Examples

5.3.1 Example 1. Consider $\alpha = \frac{\pi}{2}$ and $\omega = -z$. Then $h_1 + g_1 = \frac{1}{2i}\log\left(\frac{1+iz}{1-iz}\right)$. By shearing, we find h_1 and g_1.

$$h_1 = -\frac{1}{2}\log(1-z) + \frac{\sqrt{2}}{4}e^{i\frac{\pi}{4}}\log(1-iz) - \frac{\sqrt{2}}{4}e^{-i\frac{\pi}{4}}\log(1+iz)$$

$$g_1 = \frac{1}{2}\log(1-z) + \frac{\sqrt{2}}{4}e^{3i\frac{\pi}{4}}\log(1-iz) + \frac{\sqrt{2}}{4}e^{-3i\frac{\pi}{4}}\log(1+iz)$$

Now we need to verify $\mathrm{Re}\{(1-z)^2(h_1' - g_1')\}>0$. Computing, we find $(1-z)^2(h_1' - g_1') = \dfrac{1-z^2}{1+z^2}$. By subordination, it suffices to consider the map $\dfrac{1-z}{1+z}$, which clearly has positive real part.

5.3.2 Example 2. Now for f_2, consider $\alpha = \frac{\pi}{2}$ and $\omega = -z^2$. Again by shearing, we compute

$$h_2 = -\frac{1}{4}\log(1-z) + \frac{i}{4}\log(1-iz) - \frac{i}{4}\log(1+iz) + \frac{1}{4}\log(1+z)$$
$$g_2 = \frac{1}{4}\log(1-z) + \frac{i}{4}\log(1-iz) - \frac{i}{4}\log(1+iz) + \frac{1}{4}\log(1+z).$$

Again, we must check $\mathrm{Re}\{(1-z)^2(h_2' - g_2')\}>0$. Computing, we get $(1-z)^2(h_1' - g_1') = \dfrac{1-z}{1+z}$, which certainly has positive real part.

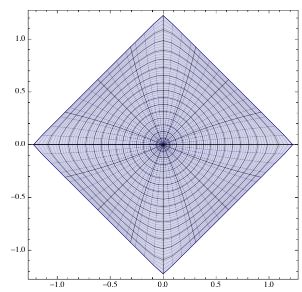


Figure 5.2: Image of $f_1 * f_2$.

The above calculations all assumed an α value of $\frac{\pi}{2}$. Just for interest, we include images of the corressponding calculations with different α values.

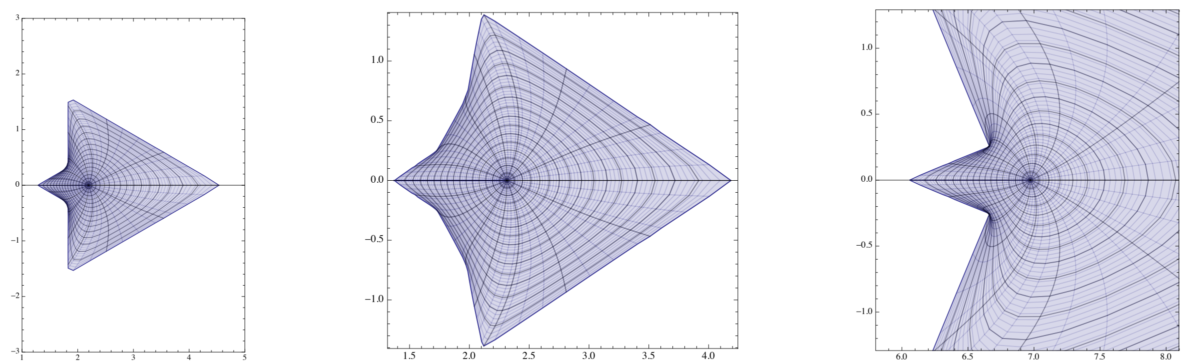


Figure 5.3: Images with $\omega = -z$ for f_1 and $\omega = -z^2$ for f_2 for $\alpha = \frac{2\pi}{3}$, $\frac{5\pi}{8}$, and $\frac{3\pi}{4}$, respectively.

Chapter 6. Convex Combinations

In this section, we present two main theorems, Theorem 6.3 and Theorem 6.5. The first concerns the convex combination of polygon maps, and the second generalizes a theorem by Dorff and Viertel, giving several applications, including an application to minimal surfaces.

6.1 Background

To proceed, we need just a little background. Consider the following condition, which is relevant when considering linear combinations of harmonic univalent functions.

Condition A. *For a function f, complex-valued harmonic and non-constant in $\mathbb{D}$, there exist sequences $\{z'_n\}, \{z''_n\}$ convergent to $z = 1$ and $z = -1$, respectively, such that*

$$\lim_{n\to\infty} \mathrm{Re}\{f(z'_n)\} = \sup_{|z|<1} \mathrm{Re}\{f(z)\}$$

$$\lim_{n\to\infty} \mathrm{Re}\{f(z''_n)\} = \inf_{|z|<1} \mathrm{Re}\{f(z)\}$$

This condition is important because of the following characterization, due to Hengartner and Schober.

Theorem 6.1. *Suppose f is analytic and non-constant in $\mathbb{D}$. Then*

$$\mathrm{Re}\{(1 - z^2)f'(z)\} \geq 0, \forall z \in \mathbb{D}$$

if and only if

- *f is convex in the vertical direction, and*
- *Condition A holds.*

There is no known general condition under which the linear combination of univalent harmonic functions is again univalent, and the search for partial results in special cases is an active area of research.

The following proposition justifies the proof strategy we will use. Indeed it generalizes the proof of Theorem 6.4.

Proposition 6.2. *Let f_1, f_2 be harmonic maps convex in the imaginary direction and satisfying Condition A, and let $t \in [0,1]$. Then their convex combination, $tf_1+(1-t)f_2$, is convex in the imaginary direction and satisfies Condition A if and only if it is locally univalent.*

Proof. Assume f_3 is locally-univalent. Then by the shearing theorem, $f_3 = h_3 + \overline{g_3}$ is convex in the vertical direction if and only if $h_3 + g_3$ is too. Using Hengartner and Schober's condition, we get

$$\begin{aligned} \mathrm{Re}\left((1-z^2)(h_3' + g_3')\right) &= \mathrm{Re}\left((1-z^2)\left[t(h_1'+g_1') + (1-t)(h_2'+g_2')\right]\right) \\ &= t\mathrm{Re}\left((1-z^2)(h_1'+g_1')\right) + (1-t)\mathrm{Re}\left((1-z^2)(h_2'+g_2')\right). \end{aligned}$$

Observe that by the shearing theorem, h_j+g_j is convex in the vertical direction. Also, h_j+g_j satisfies Condition A since $\mathrm{Re}\{f_j\} = \mathrm{Re}\{h_j+g_j\} = \mathrm{Re}\{h_j+\overline{g_j}\}$. Therefore, Hengartner and Schober's theorem guarantees that $\mathrm{Re}\left((1-z^2)(h_1'+g_1')\right) > 0$ and $\mathrm{Re}\left((1-z^2)(h_2'+g_2')\right) > 0$, so f_3 is convex in the vertical direction. □

6.2 Convex Combinations of Polygons

Mary Goodloe (see [20]) studied the canonical n-gon maps, which are given by

$$h_n' = \frac{1}{1-z^n}$$

and

$$g_n' = \frac{-z^{n-2}}{1-z^n}.$$

We seek to understand their behavior under convex combination. To do so, we will use Proposition 6.2.

Theorem 6.3. *Let f_n and f_m be the canonical n-gon and m-gon maps, respectively. Then their convex combination, $tf_n+(1-t)f_m$ is univalent and convex in the imaginary direction.*

Proof. Observe first that f_n and f_m are certainly convex in the imaginary direction. Furthermore, since these functions are constructed via Poisson integration of a piecewise-constant boundary function, the value as $z \to \pm 1$ along the real axis approach a real number which is either a vertex or the midpoint between two vertices. Therefore Condition A certainly applies. So Proposition 6.2 implies that the convex combination is univalent, if we can bound the dilatation by 1. If $m = n$, the claim is trivial, so assume without loss of generality that $m > n$. Then the dilatation is given by

$$\begin{aligned}
\frac{tg'_m + (1-t)g'_n}{th'_m + (1-t)h'_n} & \\
&= \frac{t\frac{-z^{m-2}}{1-z^m} + (1-t)\frac{-z^{n-2}}{1-z^n}}{t\frac{1}{1-z^m} + (1-t)\frac{1}{1-z^n}} \\
&= \frac{-tz^{m-2}(1-z^n) - (1-t)z^{n-2}(1-z^m)}{t(1-z^n) + (1-t)(1-z^m)} \\
&= z^{n-2}\frac{z^m - tz^{m-n} - (1-t)}{1 - tz^n - (1-t)z^m}.
\end{aligned}$$

Observe that, by symmetry of the coefficents, if the numerator above has a root z_0, then the denominator has root $1/\overline{z_0}$, so the dilatation factors as

$$z^{n-2}\prod_{k=1}^{m}\frac{z-a_i}{1-\overline{a_i}z},$$

where the a_i are the roots of $\phi(z) = z^m - tz^{m-n} - (1-t)$. Since the factors in the above expression map into the disk if $|a_i| < 1$, it would be enough to show that $|a_i| < 1$ for each i. For this, we may apply Rouche's Theorem to ϕ. So we consider a circle of radius $1+\epsilon$, where $\epsilon > 0$. Then we need to show that

$$|tz^{m-n} + (1-t)| < |z^m|$$

whenever $|z| = 1+\epsilon$. The LHS above is bounded by $t(1+\epsilon)^{m-n} + (1-t)$. observe that this expression is linear as a function of t and has positive slope as t increases, so a maximum is attained when $t = 1$. So the LHS is bounded by $(1+\epsilon)^{m-n}$, which is certainly strictly less than $(1+\epsilon)^m$, the value of the RHS. Thus, since ϵ was arbitrary, Rouche's Theorem guarantees that ϕ has all its zeros on the closed unit disk.

Suppose there is some root on the boundary of the unit disk. Then the presence of the leading z^{n-2} ensures that the dilatation is still strictly bounded by one.

So we see that the linear combination $tf_m + (1-t)f_n$ is locally univalent. □

6.3 Statement and Proof of Theorem 6.5

The main theorem of this section is a generalization of the following theorem.

Theorem 6.4. *Let $f_1 = h_1 + \overline{g_1}$ and $f_2 = h_2 + \overline{g_2}$ be mappings convex in the vertical direction and $\omega_2 = \omega_1$. If f_1 and f_2 satisfy Condition A, then $f_3 = tf_1 + (1-t)f_2$ where $0 \le t \le 1$ is convex in the vertical direction.*

This theorem is innovative in that it uses a dilatation condition to guarantee that the convex combination is univalent. Our aim is to improve on Theorem 6.4 by loosening the dilatation condition to include more cases. It will be seen that this loosening allows for applications beyond those of Theorem 6.5. Here is the extended theorem.

Theorem 6.5. *Let $f_1 = h_1 + \overline{g_1}$ and $f_2 = h_2 + \overline{g_2}$ be mappings convex in the vertical direction satisfying $\omega_2 = e^{i\theta}\omega_1$ for some $-\pi < \theta \le \pi$. If f_1 and f_2 satisfy Condition A, then for each $0 \le t \le 1$, the linear combination $f_3 = tf_1 + (1-t)f_2$ is convex in the vertical direction if $\theta = 0$ or $\operatorname{Re}\left(-\theta i e^{i\frac{\theta}{2}} \frac{h_2'}{h_1'}\right) > 0$.*

Proof. The case $\theta = 0$ is 6.4. So assume $\theta \neq 0$ and that $\operatorname{Re}\left(-\theta i e^{i\frac{\theta}{2}} \frac{h_2'}{h_1'}\right) > 0$.
The Shearing Theorem only applies to locally-univalent functions. To see that f_3 is locally

univalent, use $g_1' = \omega_1 h_1'$ and $g_2' = \omega_2 h_2' = e^{i\theta}\omega_1 h_2'$. Then

$$\omega_3 = \frac{tg_1' + (1-t)g_2'}{th_1' + (1-t)h_2'} = \frac{t\omega_1 h_1' + (1-t)e^{i\theta}\omega_1 h_2'}{th_1' + (1-t)h_2'} = \omega_1 \frac{1 + e^{i\theta}\frac{1-t}{t}\frac{h_2'}{h_1'}}{1 + \frac{1-t}{t}\frac{h_2'}{h_1'}}.$$

Observe that this is just a Mobius transformation of $\frac{1-t}{t}\frac{h_2'}{h_1'}$ that maps a half-plane to the unit disk. A quick calculation shows that the half-plane $\operatorname{Re}\left(-\theta i e^{i\frac{\theta}{2}} w\right) > 0$ is mapped to the unit disk. Therefore, we have local univalence if and only if $\operatorname{Re}\left(-\theta i e^{i\frac{\theta}{2}}\frac{h_2'}{h_1'}\right) > 0$. □

6.4 Applications

This theorem can be applied in several directions, some of which we illustrate now.

6.4.1 Minimal Surfaces. Theorem 6.5 applies to the theory of associated minimal surfaces.

Definition 6.6. Let M be a minimal surface parameterized by $\phi\colon \mathbb{C} \to \mathbb{R}^3$. Then the *conjugate minimal surface* is given by $\psi\colon \mathbb{C} \to \mathbb{R}^3$, where ψ is the harmonic conjugate of ϕ. There is then a family of *associated minimal surfaces* given by $(\cos\theta)\,\phi + (\sin\theta)\,\psi$.

The properties of associated and conjugate minimal surfaces are fascinating and have received well-deserved attention.

Here is how 6.5 applies to minimal surface theory: Let both $f = h+\overline{g}$ and $f_\theta = h+\overline{e^{i\theta}g}$, be convex in the vertical direction and satisfy Condition A. Then using the special case $h_1 = h_2$ of 6.5, we know that $tf+(1-t)f_\theta = h+\overline{[t + (1-t)e^{i\theta}]\,g}$ with $t \in [0,1]$ is univalent and convex in the vertical direction. If the dilatation of f is the square of an analytic function, then $h+\overline{g}$ induces a minimal graph. The associated minimal surfaces of f are easily calculated to be the same as the surfaces induced by f_θ. Therefore,the convex combination of associated minimal graphs is again a minimal graph even though the combined graph will not typically be an associated minimal surface in the sense of 6.6. Thus, this represents a new class of minimal surfaces that we can calculate explicitly in terms of the classical minimal surfaces.

Example 6.7. We can use this theorem to build a bridge between two classical minimal surfaces, namely Scherk's first and second surfaces. They are the first and last surfaces in Figure 6.1, respectively. The first Scherk surface is the surface that would be formed by a bubble spanning wire rods in a box shape, and the second Scherk surface is its harmonic conjugate surface. We've already taken the time to construct Scherk's first surface in an earlier example from the introduction, which is worth refering to again before reading this example. Here is the formula for the first surface's projection onto the complex plane. (Can you see the easy way to get the formula for the second surface from this one?)

$$f(z) = h(z) + \overline{g(z)} = \operatorname{Re}\left[\frac{i}{2}\log\left(\frac{i+z}{i-z}\right)\right] + i\operatorname{Im}\left[\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)\right].$$

The function above maps onto a square in the complex plane, and its conjugate function maps to a hypocycloid with four cusps. It is quite easy to show that these maps are convex in the vertical direction and satisfy Condition A, so we may apply Theorem 6.5 to see that the family of functions

$$h + \overline{(t + (1-t)(-1))\, g(z)},$$

together with their corresponding minimal surfaces, are univalent. Notice that the introduction of the parameter t simply has the effect of shrinking the dilatation and then reversing it. Indeed, when the dilatation is zero, the corresponding harmonic map is planar. Thus this family forms a minimal surface bridge between the two Scherk surfaces and the plane.

It is also interesting to note that the first three images correspond to four of the sides acting as rigid boundary conditions and the last three images correspond to the other four sides acting as rigid boundaries. Imagine this in terms of transforming wire frames spanned by bubbles.

6.4.2 Special Cases in the Plane.

Several well-known classes of functions h_1, h_2 satisfy the hypothesis of this theorem. To facilitate the discussion, we introduce the following special classes of functions, together with characterizations of them.

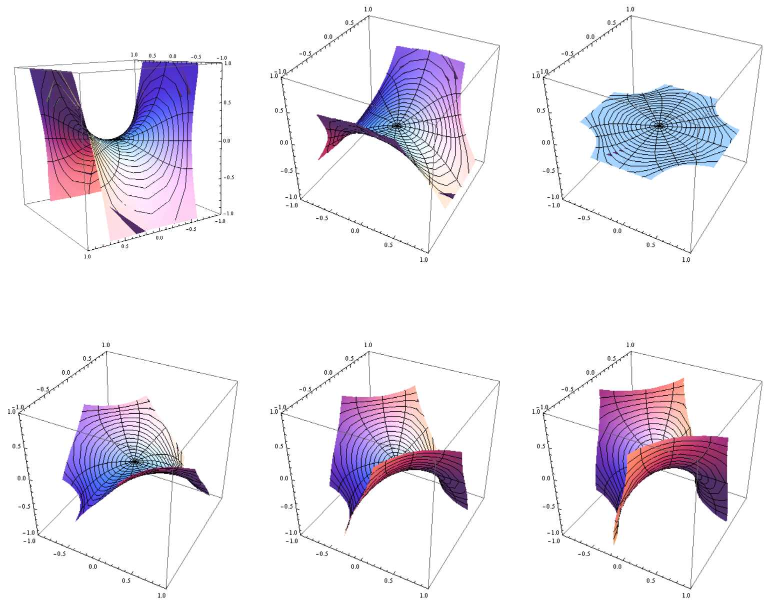


Figure 6.1: A transition between the two Scherk surfaces and the plane

Definition 6.8. A schlicht function f is called *convex* if $f(\mathbb{D})$ is a convex subset of the plane. These functions are characterized by fact that $zf'(z)$ is starlike.

Definition 6.9. A schlicht function f is called *close-to-convex* if there exist a real constant β and a convex function ϕ such that

$$\mathrm{Re}\left(e^{i\beta}\frac{f'(z)}{\phi'(z)}\right) > 0$$

for all $z \in \mathbb{D}$.

Definition 6.10. A schlicht function f is called *starlike* if for each z, the segment $tf(z), t \in [0, 1)$ lies entirely in $f(\mathbb{D})$. These functions are characterized by the relation

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)} > 0\right).$$

Definition 6.11. A schlicht function f is called *convex in one direction* if there is a real number α such that the intersection of any line that has direction $e^{i\alpha}$ with $f(\mathbb{D})$ is either empty or connected. These functions are characterized by the existence of constants b_1, θ_1, and θ_2 such that $\operatorname{Re}\left(\frac{zf'(z)}{\phi(z)}\right) > 0$, where $\phi(z) = \frac{b_1 z}{(1+e^{i\theta_1}z)(1+e^{i\theta_2}z)}$.

Each of these classes has been well-studied and is known to possess special properties. Here are some examples of applications to special subclasses. Some are quite general, and others are very restrictive. In each of these, we take $f_1 = h_1 + \overline{g_1}$, $f_2 = h_2 + \overline{g_2}$, each of of which is CVD and satisfies Condition A.

- If h_1 is close-to-convex and h_2 is an associated convex function from Definition 6.9 with constant $\beta = \frac{\theta - \pi}{2}$ (for $\theta \in [0, \pi]$), then Theorem 6.5 applies.

- If h_1 is starlike, and $h_2 = czh_1'$, where $\arg c = -\arg -\theta i e^{i\theta/2}$ then the theorem applies by the standard characterization of starlikeness.

- If h_2 is convex in one direction and $h_2 = \frac{b_1}{(1-e^{i\theta_1}z)(1-e^{i\theta-2}z)}$, for appropriate constants b_1, θ_1, and θ_2, then the theorem applies, which again follows from a standard characterization of functions convex in one direction.

- If $f_\theta = h + \overline{e^{i\theta} g}$ is convex in the vertical direction and satisfies Condition A for all θ, then $h + \epsilon \overline{g}$ is convex in the vertical direction for all $|\epsilon| < 1$. The theorem applies since every $\epsilon \in \mathbb{D}$ can be written in the form $t + (1-t)e^{i\theta}$ for some real θ.

- Under the assumptions of the previous bullet, h is convex in the vertical direction, which is seen by letting $\epsilon = 0$.

Chapter 7. Conclusion

7.1 Summary

In this thesis, I have presented several original theorems on the convolution and linear combination of harmonic univalent maps, including applications to minimal surface theory. In Chapter 2, we chose an infinite family of harmonic half-plane map and proved it is univalent by extensive computations and some h^p theory. In Chapter 3, we constructed a multi-parameter infinite family of harmonic functions, each of which has singular inner function dilatation. The boundary behavior of these functions is unusual, and we explained its causes. We lifted some of this family as minimal surfaces and investigated the unusual properties of these surfaces. In Chapter 4, we showed how many theorems about shearing in one direction can be automatically reformulated as theorems about shearing in other directions, including some prominent examples that influenced recent research attempts. In Chapter 5, we developed a criterion for the univalence of the convolution of vertical strip mappings. This parallels the very successful theory surrounding harmonic half-plane mappings. We included some geometrically-interesting examples in detail. In Chapter 6, we proved two main theorems concerning convex combinations. The first is that the convex combination of canonical polygon maps is univalent, and the second was a generalization of a condition on the dilatation of two maps such that their convex combination is univalent. We found that the extended dilatation condition allows for new applications to minimal surface theory, including the discovery of a nex class of minimal surfaces. As a special case, this gives a continuous deformation between the two Scherk surfaces that is different from the classical deformation through associate surfaces and which passes through the plane. This deformation has an interesting physical interpretation in terms of wire frames and boundary conditions.

7.2 Future Work

These theorems leave open several possibilities that are worth investigating.

- Our investigation of harmonic half-plane maps required extensive computation. Is there a simpler way? Can we make a broader determination about which dilatations in f_1 and f_2 yield local univalence in $f_1 * f_2$?

- For instance, our theorem on the convex combination of polygons relied heavily on the functional form of the standard polygon maps. Is there some other approach that generalizes to a larger class of polygon mappings, which do not have this convenient functional form? In light of the Rado-Knesser-Choquet theorem, we know that any piecewise-continuous curve which outlines a polygon induces a univalent harmonic map, so the class of polygon maps left to be investigated is still quite large.

- In our investigation of SIFD harmonic maps, we were able to construct an unusual minimal surface. What is the proper way to describe its ends? Also, can more such surfaces be constructed? What general properties is and SIFD function guaranteed to have?

- In our work on the relationship between linear combinations and dilatations, we relied on the functions being CVD and satisfying Condition A. Is this necessary? Are there less restrictive conditions we might impose? The class of CVD functions satisfying Condition A includes many familiar examples, but it would be nice to lose these restrictions.

- We have also given a condition on the dilatation of two harmonic strip maps that guarantees the univalence of their convex combination. The condition is still quite restrictive, so we might ask if there is a better condition that will include more pairs of functions? Indeed, it seems as though local univalence should be enough to guarantee global univalence in these situations.

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