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Compression Bodies and Their Boundary Hyperbolic Structures

Vinh Xuan Dang
Brigham Young University - Provo

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Vinh Xuan Dang

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Jessica S. Purcell, Chair
Gregory R. Conner
Darrin Doud
Stephen Humphries
Eric Swenson

Department of Mathematics

Brigham Young University

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ABSTRACT

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Vinh Xuan Dang
Department of Mathematics, BYU
Doctor of Philosophy

We study hyperbolic structures on the compression body C with genus 2 positive boundary and genus 1 negative boundary. We consider individual hyperbolic structures as well as special regions in the space of all such hyperbolic structures. We use some properties of the boundary hyperbolic structures on C to establish an interesting property of cusp shapes of tunnel number one manifolds. This extends a result of Nimershiem in [26] to the class of tunnel number one manifolds. We also establish convergence results on the geometry of compression bodies. This extends the work of Ito in [13] from the punctured-torus case to the compression body case.

Keywords: Hyperbolic Manifolds, Kleinian Groups, Compression Bodies

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Contents

Chapter 1. Introduction

An *unknotting tunnel* for a 3-manifold M with torus boundary components is defined to be an arc τ whose endpoints are on the boundary of M such that the complement of a neighborhood of τ in M is a handlebody. *Tunnel number one manifolds* are manifolds (other than a solid torus) that admit as single unknotting tunnel. In [2], Adams initiated the study of the relationship between unknotting tunnels and hyperbolic structures on tunnel number one manifolds. This relationship is part of a major task in the study of 3-manifolds to relate topological invariants to geometric ones. Recently, there has been further process in the study of the geometry of unknotting tunnels. In [9], Cooper, Lackenby and Purcell investigated lengths of unknotting tunnels. In [10], this was extended by Cooper, Futer and Purcell. Throughout these studies they answered multiple long-standing questions posed by Adams in [2] and Sakuma and Weeks in [28] and also posed more open questions for further investigations. A natural generalization of the study of tunnel number one manifolds is the study of compression bodies. A compression body is either a handlebody or the result of attaching 1-handles to a boundary of a surface crossed with an interval as in Figure 2.3. The reason compression bodies come into the study of tunnel number one manifolds and unknotting tunnels is because any tunnel number one manifold is built by attaching a compression body to a handlebody, and the unknotting tunnel corresponds to an arc in the compression body. Furthermore, compression bodies are interesting in themselves because every compact, orientable 3-manifold admits a *Heegaard splitting* into two compression bodies. Thus, compression bodies are buiding blocks for many manifolds. In [15], Lackenby and Purcell initiated the study of hyperbolic structures on *compression bodies* and their relationship with unknotting tunnels in tunnel number one manifolds. Some of this work was extended to more complicated compression bodies by Burton and Purcell in [6]. However, there is still much to discover concerning hyperbolic structures even for simple compression bodies.

This thesis is an attempt to understand more about the hyperbolic structures of the simplest possible compression body, the $(1;2)$-compression body. We consider individual hyperbolic structures as well as special regions in the space of all such hyperbolic structures. This can be used to establish interesting properties of tunnel number one manifolds, and to establish convergence results on the geometry of compression bodies.

In the first part of the thesis, which comprises Chapter 1 to 5, we apply techniques introduced in [9], [10], [6], and [15], especially an understanding of boundary of the space of hyperbolic structures on the $(1;2)$-compression body, to prove two theorems which reveal more attractive features of tunnel-number-one manifolds, their unknotting tunnels, and their *cusp shapes*. In particular, we prove

Theorem 1.1 (Cusp Shapes Theorem). *For any $\epsilon > 0$ and any similarity class of flat metrics $[T]$ on the torus, there exists a complete, finite volume, tunnel number one hyperbolic manifold M with a single rank-2 cusp $T_\epsilon \times [0, \infty)$ whose shape $[T_\epsilon]$ is ϵ-close to $[T]$.*

Cusps of hyperbolic manifolds and cusp shapes will be defined in more details in Section 2.1 of Chapter 2. Basically, the theorem says that the set of possible cusp shapes corresponding to cusps of tunnel number one hyperbolic manifolds is dense in the set of possible Euclidean metrics on a torus. This extends a result by Nimershiem [26] to the class of tunnel number one manifolds.

Moreover, in trying to generalize Theorem 1.1 to manifolds with large tunnel numbers, we have constructed manifolds which have a cusp whose shape is arbitrarily close to a prescribed shape and a tunnel system of arbitrarily many tunnels, each of which satisfies certain geometric property. Specifically, we prove

Theorem 1.2. *Let $[T]$ be any similarity class of flat metrics on the torus. For any natural number n, any real number $R > 0$ and any $\epsilon > 0$, there exists a complete, finite volume hyperbolic manifold with a single rank-2 cusp whose shape is ϵ-close to $[T]$ and which admits a system of tunnels $\{\tau_1, \ldots, \tau_n\}$ such that the geodesic representative of the homotopy class of each tunnel τ_i has length at least R.*

Some interesting open questions about about cusp shapes and cusp areas of manifolds with unknotting tunnels still remain and these are described at the end of of Chapter 5.

The key in proving Theorem 1.1 and Theorem 1.2 lies in the fact that *maximally cusped structures*, which are special structures on the boundary of the space of hyperbolic structures on the $(1;2)$-compression body, are dense on this boundary (see [7]). Yet, there is much that is unknown about this boundary.

In the second part of this thesis, which comprises Chapter 6 and 7, we prove some interesting properties about convergence of "slices" of this boundary. The foundational work of Culler and Shalen in [11] has provided the framework to parameterize the space of hyperbolic structures on 3-manifolds using trace coordinates. Furthermore, work of Bromberg in [5] and Magid in [18] has provided a *local model* for the subspace consisting of *minimally parabolic* and *geometrically finite* hyperbolic structures on many manifolds. In [13], Ito applied these theories to the special case of the manifold $M = S \times I$ where S is a punctured torus and I is an interval. He proved some convergence results on the slices of the space of hyperbolic structures on M. These results definitely give a better quantitative understanding of this space. We are the first to consider the analogue of Ito's result to the case of the $(1;2)$-compression body. In particular, our main theorem in the second part of the thesis is the following:

Theorem 1.3 (Slice Convergence Theorem)**.** *If $\{c_n\}$ is a sequence of complex numbers in $\mathbb{C} \setminus [-2, 2]$ such that $\{c_n\}$ converges to 2 horocyclically, then the slices $\mathcal{L}(c_n)$ converge to the slice $\mathcal{L}(2)$ in the sense of Hausdorff.*

The various types of convergence and "slices" of the boundary of the space of hyperbolic structures on the $(1;2)$-compression body will be discussed carefully in Chapter 6 and 7. Roughly speaking, Theorem 1.3 says that these special slices are determined by the trace coordinates. When the traces converge in a certain sense (*horocyclically*), the slices will converge to a slice which contains special structures on the boundary.

Here is a more detailed summary of the content of each chapter in this thesis:

- **Chapter 2** introduces the central players in the Cusp Shape Theorem 1.1. In section 2.1, we define cusp shapes, which are similarity classes of flat metrics on cusp cross-sections of a hyperbolic manifold and we survey some known results about cusp shapes of hyperbolic manifolds. Section 2.2 introduces compression bodies, especially the $(1;2)$-compression bodies, which are building blocks for tunnel-number-one manifolds. Section 2.3 gives some concrete examples of tunnel-number-one manifolds.

- **Chapter 3** discusses hyperbolic structures on compression bodies. We start by briefly describing the classification of elements of $\mathrm{PSL}(2;\mathbb{C})$ based on their actions on the upper half space model $\mathbb{H}^3$. In section 3.2, we define precisely what it means to give the $(1;2)$-compression body a hyperbolic structure and define two special types of hyperbolic structures, geometrically finite structures and minimally parabolic structures. We give an explicit example of a hyperbolic structure on the $(1;2)$-compression body that is both geometrically finite and minimally parabolic. In section 3.3, we carefully explain the Poincare Polyhedron Theorem 3.10, the main tool for proving that a given subgroup of $\mathrm{PSL}(2;\mathbb{C})$ is discrete and geometrically finite, hence, gives rise to a hyperbolic manifold. We repeatedly apply the Poincare Polyhedron Theorem throughout this thesis to construct hyperbolic structures on $(1;2)$-compression bodies.

- **Chapter 4** focuses on very special hyperbolic structures on the boundary of the space $\mathrm{MP}(C,T)$ of all geometrically finite and minimally parabolic structures on the $(1;2)$-compression body C, called cusped structures. In section 4.2, we give an example of a class of cusped structures which will be very useful in the proof of the Cusp Shape Theorem 5.1. In section 4.3, we present an explicit construction of maximally cusped structures, which are special cusped structures for which every component of the boundary of the convex core of the corresponding hyperbolic manifold is a 3-punctured sphere. These structures are dense on the boundary of $\mathrm{MP}(C,T)$ and are used in a crucial way in the proof of Theorem 1.1.

- **Chapter 5** is the heart of part one. We prove the Cusp Shape Theorem in section 5.1. We also present a more general result on cusp shapes of manifolds with tunnel number at most n, namely Theorem 1.2. This chapter concludes with section 5.3 where we discuss some open questions about cusp shapes and cusp areas of manifolds with unknotting tunnels.

- **Chapter 6** is the beginning of part two of the thesis where we shift our attention to the space $\mathrm{MP}(C, T)$. Section 6.1 describes a local model for this space due to Bromberg. In section 6.2 and 6.3, we prove some important properties of this local model which will then be applied in the proof of the Slice Convergence Theorem 1.3.

- **Chapter 7** contains the proof of two important results. Theorem 7.2 is about neighborhoods of points in $\mathcal{M}$, here $\mathcal{M}$ is roughly a model for the class of cusped structures described in Chapter 4. This theorem is a key step in the proof of the Slice Convergence Theorem, which is proved in section 7.3.

Chapter 2. Cusp Shapes, Compression Bodies and Tunnel-Number-One Manifolds

This chapter introduces the central players in the Cusp Shape Theorem 5.1, which is the main theorem in the first part of the thesis.

2.1 Cusp Shapes

For ease of exposition, we assume in this section that M is a complete, finite-volume, non-compact, orientable hyperbolic manifold. Such a manifold can be decomposed into a compact manifold M_{core} with toroidal boundary components and finitely many ends (see Figure 2.1).

Each end of M is isometric to a manifold of the form $T_j \times [0, \infty)$ where T_j is homeomorphic to a torus. Such an end is called a cusp of M and T_j is called a cusp cross-section. The

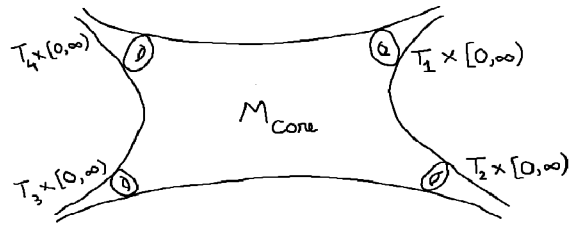

Figure 2.1: Thick-Thin Decomposition of a Finite Volume, Non-compact Hyperbolic Manifold

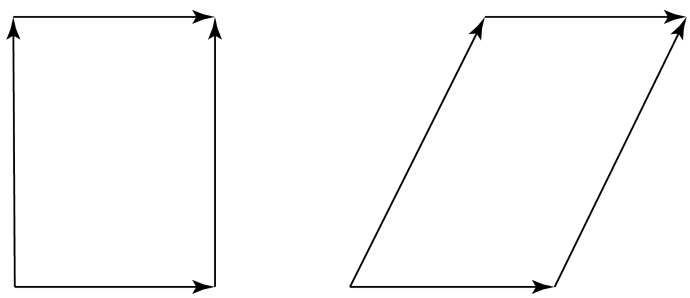

Figure 2.2: Different Similarity Classes on a Torus

hyperbolic metric on M induces a flat metric on the cusp cross-section T_j which is well-defined up to similarity. These similarity classes of flat metrics on the cusp cross-sections are called the cusp shapes of M. We shall denote the cusp shape corresponding to a cusp cross-section T_j by $[T_j]$ (see Figure 2.2).

Since there are uncountably many similarity classes of flat metrics on the torus $\mathbb{T}^2$ and there are only countably many finite volume manifolds (see [30]), not all similarity classes of flat metrics on $\mathbb{T}^2$ can appear as cusp shapes of finite-volume, hyperbolic manifolds. Nevertheless, in [26], Nimershiem proved that the set of similarity classes of $\mathbb{T}^2$ occurring as cusp shapes of finite-volume hyperbolic manifolds is dense in the set $\mathcal{S}(\mathbb{T}^2)$ of all similarity classes of flat metrics on $\mathbb{T}^2$. In the same paper she extended the above result and proved that the set of similarity classes of $\mathbb{T}^2$ that bound 1-cusped hyperbolic manifolds is also dense in $\mathcal{S}(\mathbb{T}^2)$. These results were later generalized to higher dimensions by McReynolds who proved in [24] that given a flat n-manifold Q, the set of similarity classes of flat metrics on Q occurring as cusp shapes of hyperbolic $(n+1)$-orbifolds is dense in the set of all similarity classes of flat metrics on Q. He also proved that if Q is a flat n-torus then "orbifolds" can

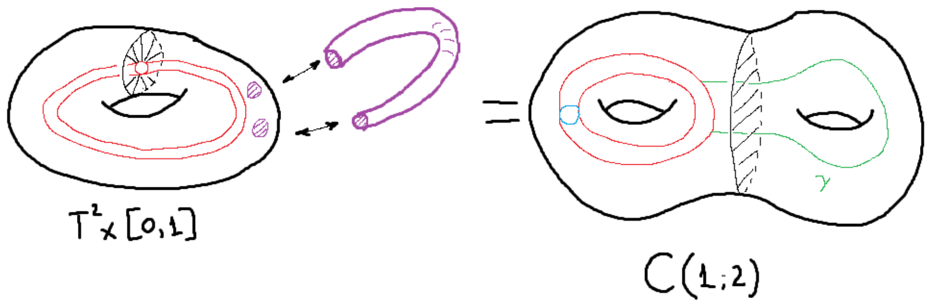

Figure 2.3: The $(1;2)$-compression body

be replaced by "manifolds."

The main result of the first part of this thesis generalizes Nimershiem's result to a class of manifolds called *tunnel-number-one manifolds*. The building blocks for constructing these tunnel-number-one manifolds are simple manifolds called *compression bodies*.

2.2 Compression Bodies

To define a compression body, we start with S, which is a closed, orientable (possibly disconnected) surface whose genus is at least 1 and the interval $I = [0, 1]$. A *compression body* C is either a handlebody or the result of attaching 1-handles to $S \times I$ along $S \times \{1\}$ such that the result is connected. The *negative boundary* is $S \times \{0\}$ and is denoted $\partial_- C$. The *positive boundary* is $\partial C \setminus \partial_- C$ and is denoted $\partial_+ C$.

The compression body that we primarily study here is one for which $\partial_- C$ is a torus and $\partial_+ C$ is a genus 2 surface. We will call this the $(1;2)$-compression body and denote it by $C(1;2)$ or just C when there is no confusion. Note that $C(1;2)$ is obtained by attaching a single 1-handle to $\mathbb{T}^2 \times I$ (See Figure 2.3).

The fundamental group of $C = C(1;2)$ is $\pi_1(C) = (\mathbb{Z}\times\mathbb{Z})*\mathbb{Z}$ because $C(1;2)$ deformation retracts to a torus union the arc γ in Figure 2.3. This arc is called the *core tunnel* of C. From now on, we will denote the generators of the $\mathbb{Z} \times \mathbb{Z}$ factor by α and β and by abuse of notation we will denote the generator of the last $\mathbb{Z}$ factor by γ.

A more general compression body, which is the object of study of [8] or [16] for example,

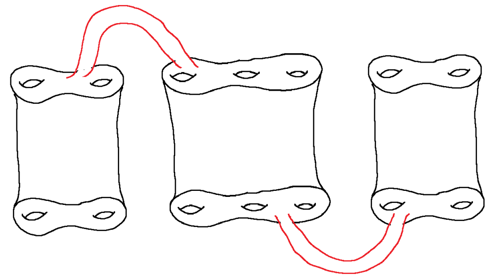

Figure 2.4: A more general compression body

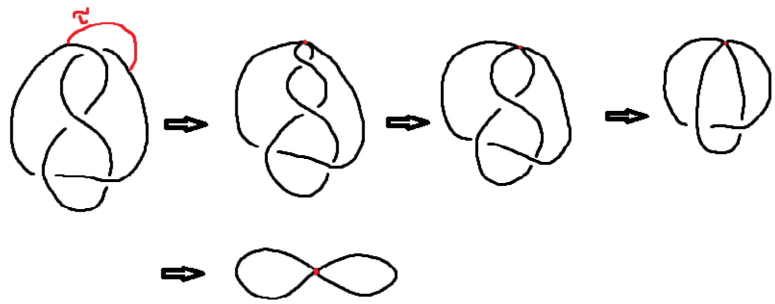

Figure 2.5: A tunnel-number-one manifold

is shown in Figure 2.4.

Compression bodies can be used to build manifolds with *unknotting tunnels*. In particular, the $(1;2)$-compression body can be used to build *tunnel-number-one manifolds* which we will define in the following section.

2.3 Tunnel-Number-One Manifolds

A *tunnel-number-one manifold* is a manifold which admits a properly embedded arc τ such that the complement in M of a neighborhood of τ is a handlebody. The complement of the knot in Figure 2.5 is an example of a tunnel-number-one manifold.

The geometry of tunnel-number-one manifolds is an interesting subject of study. See [2], [3], [9], [6], and [15] for some results and open questions.

Note that a tunnel-number-one manifold can be constructed by gluing a $(1;2)$-compression body and a genus 2 handlebody along their genus 2 boundaries. As a result, a complete understanding of the geometric structures on compression bodies could lead to many interesting results about the geometry of tunnel-number-one manifolds. For example in [9], Cooper, Lackenby and Purcell built tunnel-number-one manifolds with arbitrarily long unknotting tunnels by studying hyperbolic structures on the $(1;2)$-compression body. One of the main goals of this thesis is to understand some aspects of the space of geometric structures on the $(1;2)$-compression body. In the next Chapter, we will provide some background material on geometric structures, specifically, hyperbolic structures on the $(1;2)$-compression body.

Chapter 3. Hyperbolic Structures on Compression Bodies

To give a compression body a hyperbolic structure amounts to constructing a *discrete and faithful* representation from the fundamental group of the compression body to $\mathrm{PSL}(2,\mathbb{C})$. In the first section, we briefly review some important properties of elements and subgroups of $\mathrm{PSL}(2,\mathbb{C})$.

3.1 Geometry of $\mathrm{PSL}(2,\mathbb{C})$

Recall that $\mathrm{PSL}(2,\mathbb{C}) = \mathrm{SL}(2,\mathbb{C})/\{\pm I\}$ where $\mathrm{SL}(2,\mathbb{C})$ is the group of 2-by-2 matrices of determinant 1 and I is the identity matrix. An element A of $\mathrm{PSL}(2,\mathbb{C})$ acts on the sphere at infinity $\mathbb{S}_\infty = \mathbb{C} \cup \{\infty\}$ of the upper half space $\mathbb{H}^3$ as a Möbius transformation. Specifically, the action of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ on $\mathbb{C} \cup \{\infty\}$ is given by

$$z \longmapsto A(z) = \frac{az+b}{cz+d}, \forall z \in \mathbb{C} \cup \{\infty\}.$$

Since a Möbius transformation is a composition of reflections in circles in $\mathbb{C} \cup \{\infty\}$ and a reflection in a circle in $\mathbb{C} \cup \{\infty\}$ can be extended to a reflection in the hemisphere in $\mathbb{H}^3$ bounded by that circle, this action can be extended to an action on $\mathbb{H}^3$ (see [19] for more details). Elements of $\mathrm{PSL}(2,\mathbb{C})$ can be classified based on their action on $\mathbb{H}^3$ (see Chapter 1 of [22] for a very nice exposition). More algebraically, one can classify these elements based on their traces. In particular, an element $A \in \mathrm{PSL}(2,\mathbb{C})$ is

- *parabolic* if $\mathrm{tr}(A) = \pm 2$,
- *loxodromic* if $\mathrm{tr}(A) \in \mathbb{C} \setminus [-2,2]$,
- *elliptic* if $\mathrm{tr}(A) \in (-2,2)$.

Note that a parabolic element has exactly one fixed point, a loxodromic element has exactly two fixed points, and an elliptic element has infinitely many fixed points in $\mathbb{H}^3 \cup \mathbb{S}_\infty$. The fixed points of parabolic and loxodromic elements belong to the sphere at infinity (see [19], Section 1.1).

We will be concerned with subgroups of $\mathrm{PSL}(2,\mathbb{C})$ which are images of representations of the fundamental group $\pi_1(C)$ of the $(1;2)$-compression body C into $\mathrm{PSL}(2,\mathbb{C})$. For such a representation to correspond to a hyperbolic structure on C, it is necessary that its image is a *discrete* subgroup of $\mathrm{PSL}(2,\mathbb{C})$. A subgroup Γ of $\mathrm{PSL}(2,\mathbb{C})$ is *discrete* if the identity is *isolated* in Γ, i.e., there is no infinite sequence of distinct elements in Γ which converges to the identity (see [19], Section 2.2 for equivalent conditions of "discreteness"). Since a discrete subgroup Γ of $\mathrm{PSL}(2,\mathbb{C})$ acts on $\mathbb{H}^3$, we can study its quotient space $\mathbb{H}^3/\Gamma$. If Γ is torsion-free, i.e., it contains no elliptic elements, then $\mathbb{H}^3/\Gamma$ is a complete, hyperbolic manifold (see [19], Chapter 2 for more details).

We are now ready to discuss some background material on hyperbolic structures on the $(1;2)$-compression body.

3.2 Hyperbolic Structures on the $(1;2)$-compression body

Definition 3.1. Let $C = C(1;2)$ be the $(1;2)$-compression body. To give C a *hyperbolic structure* is to construct a discrete and faithful representation $\rho : \pi_1(C) \longrightarrow \mathrm{PSL}(2,\mathbb{C})$ such that the manifold $\mathbb{H}^3/\Gamma$ is homeomorphic to the interior of C. Here $\Gamma = \rho(\pi_1(C))$ is the image of $\pi_1(C)$ under ρ. The representation ρ is *discrete* if Γ is discrete and it is *faithful* if it is a monomorphism.

We are mostly interested in special hyperbolic structures on C called *geometrically finite* and *minimally parabolic* structures.

Definition 3.2. A hyperbolic structure on C is *geometrically finite* if Γ admits a finite-sided, convex fundamental domain.

Definition 3.3. A hyperbolic structure on C is *minimally parabolic* if Γ has no rank one parabolic subgroups. Here a rank one parabolic subgroup of Γ is a cyclic group generated by a parabolic element of Γ. We will also call such a subgroup a *rank-1 cusp.*

Note that $\Gamma = \rho(\pi_1(C))$ is minimally parabolic if for all $g \in \pi_1(C)$, the element $\rho(g)$ of Γ is parabolic if and only if g is conjugate to an element of the fundamental group of the torus boundary component of C.

Here is an example of a discrete and faithful representation which gives C a geometrically finite and minimally parabolic hyperbolic structure:

Example 3.4. Recall that $\pi_1(C) = (\underbrace{\mathbb{Z}}_{\langle\alpha\rangle} \times \underbrace{\mathbb{Z}}_{\langle\beta\rangle}) * \underbrace{\mathbb{Z}}_{\langle\gamma\rangle}$. Construct $\rho : \pi_1(C) \longrightarrow \mathrm{PSL}(2, \mathbb{C})$ by specifying ρ on the generators α, β and γ as

$$\rho(\alpha) = \begin{pmatrix} 1 & 100 \\ 0 & 1 \end{pmatrix}, \rho(\beta) = \begin{pmatrix} 1 & 100i \\ 0 & 1 \end{pmatrix}, \rho(\gamma) = \begin{pmatrix} 6i & -1 \\ 1 & 0 \end{pmatrix}$$

and extend ρ to the whole group.

To prove that ρ is discrete and gives C a geometrically finite structure, we will construct a special fundamental domain for the action of $\Gamma = \rho(\pi_1(C))$ on $\mathbb{H}^3$ called the *Ford domain* (see Figure 3.1) and apply the Poincaré Polyhedron Theorem. To prove that the structure is minimally parabolic, we will need to analyze the *visibility* of the *isometric spheres* corresponding to elements of Γ. Since we will use these tools repeatedly later on, we recall them in a more general setting in the next section.

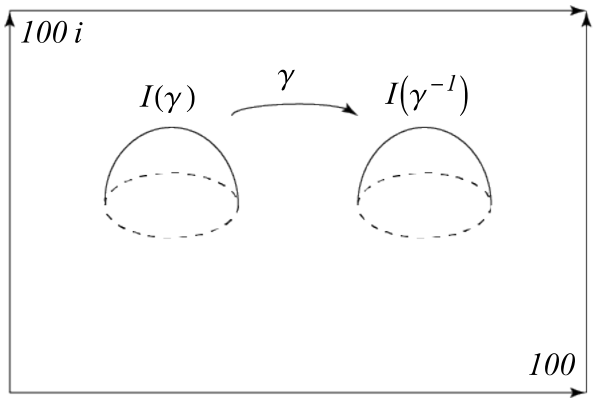

Figure 3.1: A Ford Domain for the action of Γ

3.3 Isometric Spheres, Ford Domains and the Poincaré Polyhedron Theorem

This section closely follows [15]; the reader can consult that paper for proofs of various results cited here.

Let Γ be a discrete subgroup of $\mathrm{PSL}(2,\mathbb{C})$ and let $M = \mathbb{H}^3/\Gamma$ be the quotient manifold corresponding to Γ. We assume that Γ has a single *rank-2 cusp*, i.e., a free Abelian subgroup Γ_∞ of rank 2 generated by 2 parabolic elements. By conjugating Γ, we can assume that Γ_∞ admits $\{\infty\}$ as its unique fixed point. Note that if an element g of Γ fixes $\{\infty\}$, then $g \in \Gamma_\infty$. Also note that topologically, a rank-2 cusp corresponds to the cross product of a torus and an interval as in Section 2.1.

Definition 3.5. Let $g \in \Gamma \setminus \Gamma_\infty$ and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The *isometric sphere* of g, denoted by $I(g)$, is the Euclidean hemisphere centered at $g^{-1}(\infty) = -d/c$ with radius $1/|c|$.

Remark. (i) Note that if H is any horosphere centered at ∞ in the upper half space $\mathbb{H}^3$, then $I(g)$ is the set of points in $\mathbb{H}^3$ equidistant from H and $g^{-1}(H)$ (see Section 2 of [15]).

(ii) The element g maps $I(g)$ isometrically to $I(g^{-1})$ and it maps the half ball $B(g)$ bounded by $I(g)$ to the exterior of the half ball $B(g^{-1})$ bounded by $I(g^{-1})$ (Lemma 2.15 of [15]).

Definition 3.6. The *equivariant Ford domain* for the action of Γ, denoted by $\mathcal{F}$, is the set

$$\mathcal{F} = \mathbb{H}^3 \setminus \bigcup_{g \in \Gamma \setminus \Gamma_\infty} B(g).$$

Definition 3.7. A *vertical fundamental domain* for Γ_∞ is a fundamental domain for the action of Γ_∞ cut out by finitely many vertical geodesics planes in $\mathbb{H}^3$.

Definition 3.8. A *Ford domain* for the action of Γ is the intersection of $\mathcal{F}$ with a vertical fundamental domain for the action of Γ_∞.

Example 3.9. Let $\Gamma = \rho(\pi_1(C))$ where ρ is as in Example 3.4. Then the isometric sphere $I(\rho(\gamma))$ has center 0 and radius 1 and the isometric sphere $I(\rho(\gamma^{-1}))$ has center $6i$ and radius 1. In this case, the group Γ_∞ is generated by $\rho(\alpha)$ and $\rho(\beta)$ and it acts by Euclidean translations on $\mathbb{H}^3$. By choice of $\rho(\alpha)$ and $\rho(\beta)$, all translates of $I(\rho(\gamma))$ and $I(\rho(\gamma^{-1}))$ are disjoint. It turns out that the equivariant Ford domain $\mathcal{F}$ consists of the exterior of the half balls $B(\rho(\gamma))$ and $B(\rho(\gamma^{-1}))$ and their translates under Γ_∞. We will also prove later that the Ford domain for ρ is as shown in Figure 3.1.

Remark. Note that the manifold $M = \mathbb{H}^3/\Gamma$ is geometrically finite if and only if Γ admits a Ford domain with a finite number of faces (Corollary 2.10. of [15]).

We are now in the position to state the Poincaré Polyhedron Theorem and an important corollary.

Theorem 3.10 (Poincaré Polyhedron Theorem)**.** *Let $g_1, \ldots, g_n$ be elements of $PSL(2,\mathbb{C})$ and let $\Gamma_\infty \cong \mathbb{Z} \times \mathbb{Z}$ be a parabolic subgroup of $PSL(2,\mathbb{C})$ fixing $\{\infty\}$. Let P be the polyhedron cut out by isometric spheres corresponding to $\{g_1, \ldots, g_n\}$ and $\{g_1^{-1}, \ldots, g_n^{-1}\}$ and a vertical fundamental domain for the action of Γ_∞. Let M be the object obtained from P by gluing isometric spheres corresponding to g_j and g_j^{-1} via the isometry g_j and gluing faces of the vertical fundamental domain by parabolic isometries in Γ_∞. Suppose that each face pairing maps a face of P isometrically to another face of P and let Γ be the group generated by* *all the face pairings. If the sum of the dihedral angles about each edge of M is 2π and the monodromy around the edge is the identity. Then*

- *M is a smooth hyperbolic manifold with $\pi_1(M) \cong \Gamma$, and*

- *Γ is discrete.*

Proof. See Theorem 2.25 of [15]. ∎

The Poincaré Polyhedron Theorem has the following useful Corollary

Corollary 3.11. *Let Γ be a subgroup of $PSL(2,\mathbb{C})$ with a rank 2 parabolic subgroup Γ_∞ fixing ∞. Suppose the isometric spheres corresponding to a finite set of elements of Γ as well as a vertical fundamental domain for Γ_∞ cut out a polyhedron P so that face pairings given by the isometries corresponding to isometric spheres and to elements of Γ_∞ yield a smooth hyperbolic manifold with fundamental group Γ. Then Γ is discrete, the manifold $\mathbb{H}^3/\Gamma$ is geometrically finite and P is a Ford domain of $\mathbb{H}^3/\Gamma$.*

Proof. See Lemma 2.26. of [15]. ∎

We give an example where the Poincaré Polyhedron Theorem and Corollary 3.11 is applied to show that a representation of $\pi_1(C)$ is discrete and gives C a geometrically finite structure.

Example 3.12. The representation $\rho : \pi_1(C) \longrightarrow \mathrm{PSL}(2,\mathbb{C})$ defined on the generators α, β and γ of $\pi_1(C)$ as

$$\rho(\alpha) = \begin{pmatrix} 1 & 100 \\ 0 & 1 \end{pmatrix}, \rho(\beta) = \begin{pmatrix} 1 & 100i \\ 0 & 1 \end{pmatrix}, \rho(\gamma) = \begin{pmatrix} 6i & -1 \\ 1 & 0 \end{pmatrix}$$

is a discrete representation of $\pi_1(C)$ into $\mathrm{PSL}(2,\mathbb{C})$ and ρ gives C a geometrically finite hyperbolic structure.

In fact, as discussed above, the isometric spheres $I(\rho(\gamma))$ and $I(\rho(\gamma^{-1}))$ and their translates under Γ_∞ are all disjoint. By choice of $\rho(\alpha)$ and $\rho(\beta)$, we can choose a vertical fundamental domain for Γ_∞ which contains $I(\rho(\gamma))$ and $I(\rho(\gamma^{-1}))$. Let P be the region in the interior of the vertical fundamental domain and exterior to the half balls $B(\rho(\gamma))$ and $B(\rho(\gamma^{-1}))$. Let M be the object obtained by gluing the vertical sides of P via elements of Γ_∞ and gluing $I(\rho(\gamma))$ to $I(\rho(\gamma^{-1}))$ via $\rho(\gamma)$. Since the only edges of M come from the vertical fundamental domain, the Poincaré Polyhedron Theorem applies to show that M is a smooth hyperbolic manifold with $\pi_1(M) \cong \Gamma$ and Γ is discrete. Moreover, Corollary 3.11 implies that P is a Ford domain for the action of Γ and M is geometrically finite. Finally, when we glue the sides of the vertical fundamental domain of Γ_∞, the result is homeomorphic to $\mathbb{T}^2 \times (0,1)$. And gluing the face $I(\rho(\gamma))$ and $I(\rho(\gamma^{-1}))$ is topologically equivalent to attaching a 1-handle to $\mathbb{T}^2 \times (0,1)$, resulting in a manifold homeomorphic to the interior of the $(1;2)$-compression body C. So, the geometrically finite hyperbolic manifold $M = \mathbb{H}^3/\Gamma$ is homeomorphic to the interior of C. Therefore, the representation ρ gives C a geometrically finite hyperbolic structure by definition.

Finally, to prove that ρ gives C a minimally parabolic structure, we apply the following result of [15].

Proposition 3.13. *Suppose that $\rho : \pi_1(C) \longrightarrow PSL(2,\mathbb{C})$ gives C a geometrically finite hyperbolic structure. If none of the visible isometric spheres of the Ford domain for the action of $\Gamma = \rho(\pi_1(C))$ are visibly tangent on their boundaries, then ρ gives C a minimally parabolic structure.*

Proof. See Lemma 2.18. of [15] ∎

Now, by the discussion above, we have seen that the Ford domain for $\Gamma = \rho(\pi_1(C))$ in Example 3.4 is as shown in Figure 3.1. By inspection, the only visible isometric spheres of Γ are $I(\rho(\gamma))$ and $I(\rho(\gamma^{-1}))$ and they are not visibly tangent on their boundaries. Therefore, Proposition 3.13 implies that this structure is minimally parabolic.

This concludes our discussion of putting a geometrically finite, minimally parabolic structure on the $(1;2)$-compression body. Proving our Cusp Shape Theorem will also require an understanding of the space of all hyperbolic structures on C, and special structures on the boundary of this space. This is the subject of the next chapter.

Chapter 4. Cusped Structures and Maximally Cusped Structures

Throughout this chapter, $C = C(1;2)$ is the $(1;2)$-compression body and T is the torus boundary component of C.

4.1 Space of Hyperbolic Structures - $\mathrm{AH}(C,T)$ and $\mathrm{MP}(C,T)$

We first define the *representation space* of C with *parabolic locus* T to be the space

$$\mathrm{R}(C,T) = \{\text{representation } \rho : \pi_1(C) \longrightarrow \mathrm{PSL}(2,\mathbb{C}) : g \in \pi_1(T) \Longrightarrow \rho(g) \text{ is } \textit{parabolic}\}.$$

The space $\mathrm{AH}(C,T)$ of hyperbolic structures on C is the set (defined below) equipped with the topology of *algebraic convergence*

$$\mathrm{AH}(C,T) = \{\rho \in \mathrm{R}(C,T) : \rho \text{ is discrete and faithful}\}/\sim,$$

where we quotient out by the equivalence relation $\sim$ given by $\rho \sim \rho'$ if they are conjugate representations.

The topology on $\mathrm{AH}(C,T)$ is given by *algebraic convergence*. We say that a sequence of representations $\{\rho_n\}$ in $\mathrm{AH}(C,T)$ converges *algebraically* to a representation ρ if the sequence $\{\rho_n(g)\}$ converges to $\rho(g)$ for every $g \in \pi_1(C)$. Here we regard $\rho_n(g)$ as a point in the manifold $\mathrm{PSL}(2,\mathbb{C})$ and so $\{\rho_n(g)\}$ converges to $\rho(g)$ in the sense of convergence of points in $\mathrm{PSL}(2,\mathbb{C})$.

Suppose $\{\rho_n\}$ converges to ρ algebraically. Algebraic convergence does not reveal much geometric information, i.e., we do not know much about the relationship between the sequence of quotient manifolds $\mathbb{H}^3/\rho_n(\pi_1(C))$ and the limiting manifold $\mathbb{H}^3/\rho(\pi_1(C))$. Later on, we will need a stronger notion of convergence, called *geometric convergence*. We will define *geometric convergence* of the images $\Gamma_n = \rho_n(\pi_1(C))$ and $\Gamma = \rho(\pi_1(C))$ of the representations.

Definition 4.1. We say that a sequence of subgroups Γ_n of $\mathrm{PSL}(2,\mathbb{C})$ converges *geometrically* to a subgroup Γ if

(i) For each $\zeta \in \Gamma$, there exists $\zeta_n \in \Gamma_n$ such that $\{\zeta_n\}$ converges to ζ.

(ii) If $\zeta_{n_j} \in \Gamma_{n_j}$, and $\{\zeta_{n_j}\}$ converges to ζ, then $\zeta \in \Gamma$.

The following Theorem (see [14]) relates the two notions of convergence

Theorem 4.2 (Jorgensen and Marden)**.** *Suppose the sequence of discrete representations $\{\rho_n\}$ converges to a discrete representation ρ algebraically. Let $\Gamma_n = \rho_n(\pi_1(C))$ and $\Gamma = \rho(\pi_1(C))$. Then there exists a subsequence $\{\Gamma_{n_j}\}$ of $\{\Gamma_n\}$ such that $\{\Gamma_{n_j}\}$ converges geometrically to a discrete group Γ'. Moreover, Γ is a subgroup of Γ'. Hence, the manifold $M = \mathbb{H}^3/\Gamma$ is a covering manifold of the manifold $M' = \mathbb{H}^3/\Gamma'$.*

Finally, the space $\mathrm{MP}(C,T)$ of geometrically finite, minimally parabolic hyperbolic structures on C is a subspace of $\mathrm{AH}(C,T)$ equipped with the subspace topology. It is defined as

$$\mathrm{MP}(C,T) = \{[\rho] \in \mathrm{AH}(C,T) : \rho \text{ gives } C \text{ a geometrically finite, minimally parabolic structure}\}.$$

The relationship between $\mathrm{MP}(C,T)$ and $\mathrm{AH}(C,T)$ is completely described in the following two difficult theorems.

Theorem 4.3 (Marden [20] and Sullivan [29]).

$$MP(C,T) = Int(AH(C,T)).$$

And

Theorem 4.4 (Density Theorem, see [25] and [27]).

$$AH(C,T) = \overline{MP(C,T)}$$

We are now in the position to discuss the structures on the boundary of $\mathrm{MP}(C,T)$.

4.2 Cusped Structures

From Theorem 4.4, the boundary $\partial\mathrm{MP}(C,T) = \mathrm{AH}(C,T)\setminus\mathrm{MP}(C,T)$ of the space $\mathrm{MP}(C,T)$ consists of equivalence classes of representations which give C structures that are geometrically finite but not minimally parabolic or structures that are *geometrically infinite* (not geometrically finite). Also from Theorem 4.4, every structure on the boundary of $\mathrm{MP}(C,T)$ is the algebraic limit of a convergent sequence in $MP(C,T)$. We call structures that are geometrically finite but not minimally parabolic *cusped structures*. The following Lemma provides the construction of a family of examples of cusped structures. Such structures will be very useful in our proof of the Cusp Shape Theorem.

Lemma 4.5. *Let $\rho : \pi_1(C) \longrightarrow PSL(2,\mathbb{C})$ be defined on the generators α, β and γ of $\pi_1(C)$ as*

$$\rho(\alpha) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \rho(\beta) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \rho(\gamma) = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$$

where a and b are complex numbers which are linearly independent over $\mathbb{R}$ with $|a| > 4$ and $|b| > 4$. Then ρ gives C a cusped structure.

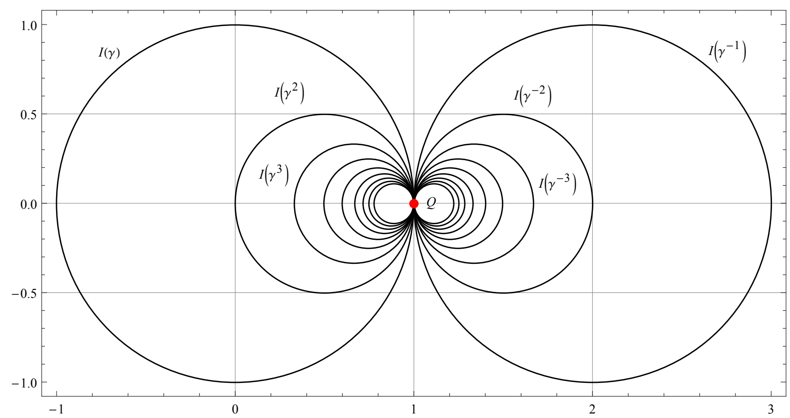

Figure 4.1: A Ford Domain for the action of Γ in Lemma 4.5

Proof. We first prove that $[\rho]$ is in $\mathrm{AH}(C,T)$. It suffices to prove that ρ is discrete. We will see that the Ford domain for the action of $\Gamma = \rho(\pi_1(C))$ is as shown in Figure 4.1.

By inspection, the only visible isometric spheres of Γ are $I(\rho(\gamma))$, $I(\rho(\gamma^{-1}))$ and their translates under $\Gamma_\infty = \langle \rho(\alpha), \rho(\beta) \rangle$. The isometric spheres corresponding to $\rho(\gamma^k)$ for $k \in \mathbb{Z} \setminus \{\pm 1\}$ are all invisible, so are their translates under Γ_∞. By choice of a and b, we can choose a vertical fundamental domain for Γ_∞ which contains $I(\rho(\gamma))$ and $I(\rho(\gamma^{-1}))$. Let P be the polyhedron cut out by the isometric spheres $I(\rho(\gamma))$ and $I(\rho(\gamma^{-1}))$ and a vertical fundamental domain for the action of Γ_∞ as in Figure 4.1. The polyhedron P has six faces and the face pairings are $\rho(\alpha)$, $\rho(\beta)$ and $\rho(\gamma)$. Each of them maps a face of P isometrically to another face of P. Moreover, P has one edge corresponding to the vertical fundamental domain, which satisfies the requirements of the Poincaré Polyhedron Theorem 3.10. The theorem applies to show that $\mathbb{H}^3/\Gamma$ is a smooth hyperbolic manifold and Γ is discrete, i.e., ρ is discrete. Furthermore, the manifold obtained by gluing the faces of P is homeomorphic to C. Also, Corollary 3.11 implies that P, which is finite-sided, is a Ford domain for the action of Γ. Thus, ρ gives C a geometrically finite hyperbolic structure.

However, $\rho(\gamma)$ is parabolic but γ is not an element of the free Abelian rank 2 subgroup $\pi_1(T)$ of $\pi_1(C)$. Hence, ρ is not minimally parabolic by definition. Thus, ρ gives C a structure which is geometrically finite but not minimally parabolic, i.e., a cusped structure. ■

4.3 Maximally Cusped Structures

Maximally cusped structures are special cusped structures on the boundary of $\mathrm{MP}(C,T)$ which will play an important role in our proof of the Cusp Shape Theorem. We define them and explicitly compute some examples in this section. By abuse of language, from now on we will call a representation ρ that gives C a hyperbolic structure a structure.

Definition 4.6. A *maximally cusped structure* is a cusped structure ρ such that every component of the boundary of the convex core of $\mathbb{H}^3/\rho(\pi_1(C))$ is a 3-punctured sphere.

Maximally cusped structures are useful because of the following result (see Theorem 16.2. of [7]):

Theorem 4.7 (Canary - Culler - Hersonsky - Shalen)**.** *Maximally cusped structures are dense on the boundary $\partial MP(C,T)$ of the space $MP(C,T)$.*

Although it is not necessary to explicitly construct maximally cusped structures to prove our Cusp Shape Theorem, the computation of maximally cusped structures on the $(1,2)$-compression body is both interesting and nontrivial. Therefore, we will make a detour here to present our construction.

4.3.1 Construction of Maximally Cusped Structures. To produce a maximally cusped structure on the $(1;2)$-compression body C, we must pinch 3 curves that form a pants decomposition of the genus 2 boundary ∂_+C of C. Suppose we start with a geometrically finite, minimally parabolic structure in $\mathrm{MP}(C,T)$; such a structure is given by, for example, the representation ρ with

$$\rho(\alpha) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \rho(\beta) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \rho(\gamma) = \begin{pmatrix} c & -1 \\ 1 & 0 \end{pmatrix}$$

where a and b are complex numbers which are linearly independent over $\mathbb{R}$ with $|a| > 2|c|$ and $|b| > 2|c|$, and $c \in \mathbb{R}$.

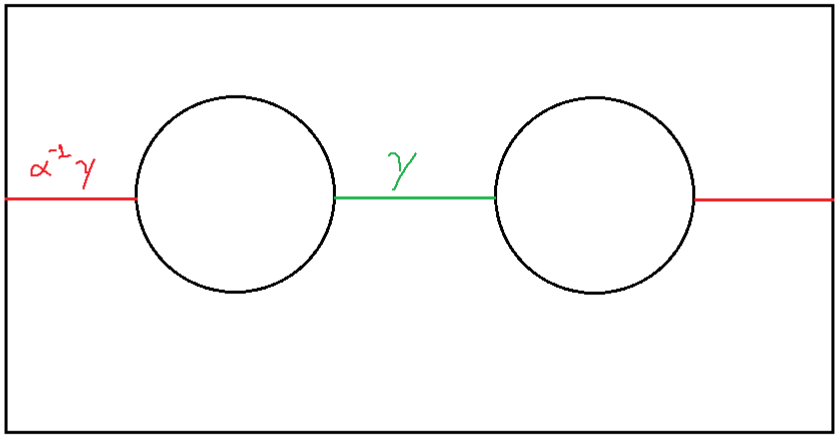

Figure 4.2: Two Curves To Be Pinched on $\partial_+ C$

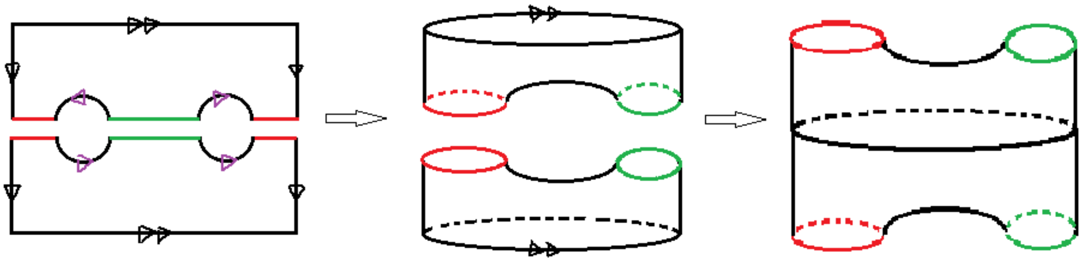

Figure 4.3: 4-punctured sphere obtain from cutting along 2 curves on $\partial_+ C$

We start by choosing 2 convenient curves to pinch, namely, the red and the green curve in Figure 4.2, where the endpoints of these curves are identified under $\rho(\pi_1(C))$.

Note that if we fix an appropriate base point, then the green curve corresponds to the word γ and the red curve corresponds to the word $\alpha^{-1}\gamma$ in $\pi_1(C)$. Now, we need to figure out the last curve which, together with the red and the green curve, will form a pants decomposition of $\partial_+ C$. Cut along the red and the green curve; we obtain a 4-punctured sphere $S_{0,4}$. The process is depicted in Figure 4.3.

It is well-known that there is a one-to-one correspondence between simple closed curves

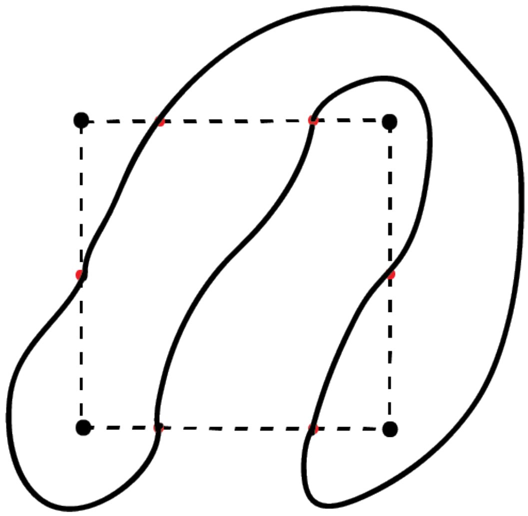

Figure 4.4: The 2/1 curve on the 4-punctured sphere

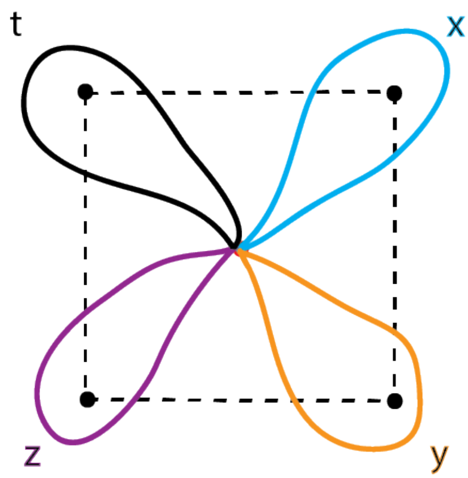

Figure 4.5: Generators of $\pi_1(S_{0,4})$

on $S_{0,4}$ and the set $\mathbb{Q} \cup \{1/0\}$ (see Section 2 of [9]). For example, Figure 4.4 shows the curve 2/1 where the 4-punctured sphere is drawn in a projection plane with 4 points removed.

Any p/q curve on $S_{0,4}$, together with the curve γ and $\alpha^{-1}\gamma$, will form a pants decomposition of $\partial_+ C$. It remains to express the p/q curve as a word in the generators α, β and γ of $\pi_1(C)$. We know $\pi_1(S_{0,4}) = \langle x, y, z, t | xyzt = 1\rangle$ where x, y, z, t are the 4 curves in Figure 4.5.

Therefore, it suffices to express x, y and z as words in the generators α, β and γ. We do so by choosing a base point $\mathcal{O}$ for $\partial_+ C$ and performing curve tracing as in Figure 4.6.

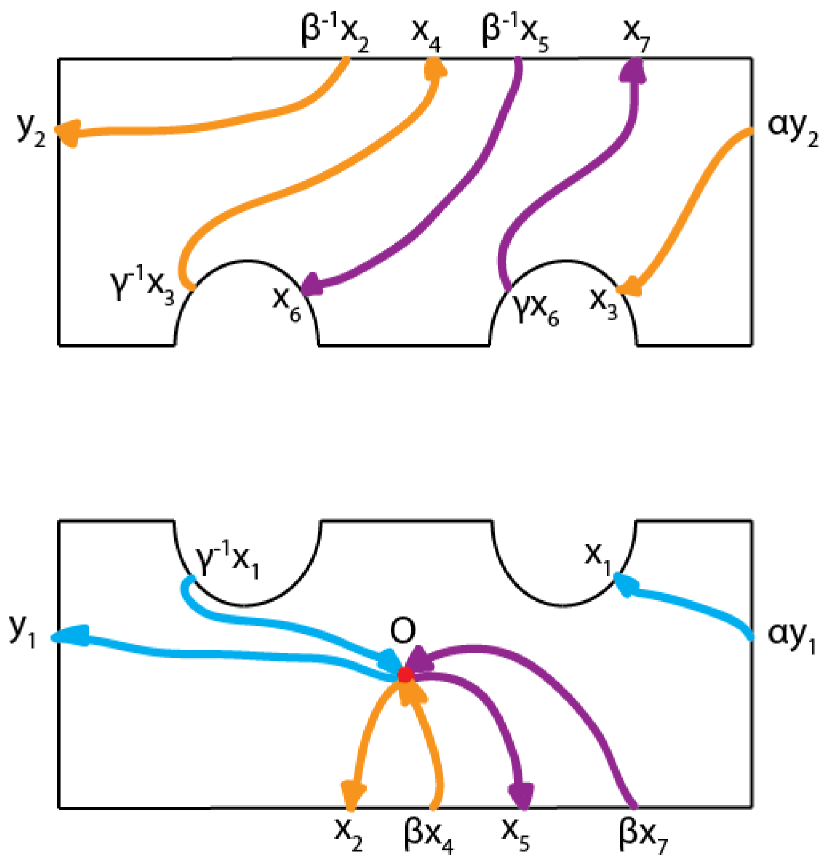

Figure 4.6: Generators of $\pi_1(S_{0,4})$ on $\partial_+ C$

On $\partial_+ C$, we see that the simple closed curve x goes from the base point $\mathcal{O}$ to the point y_1. The point y_1 is then identified to the point αy_1 in the opposite side. The curve then goes from αy_1 to x_1 which is then identified to the point $\gamma^{-1}x_1$. Finally, the curve goes from $\gamma^{-1}x_1$ back to the base point $\mathcal{O}$. We use the following arrow diagram to trace this curve

$$\mathcal{O} \longrightarrow y_1 \sim \alpha y_1 \longrightarrow x_1 \sim \gamma^{-1}x_1 \longrightarrow \mathcal{O}.$$

It follows that the curve x corresponds to the word $\gamma^{-1}\alpha$ in $\pi_1(C)$. Similarly, the diagram we obtain when tracing the curve y is

$$\mathcal{O} \longrightarrow x_2 \sim \beta^{-1}x_2 \longrightarrow y_2 \sim \alpha y_2 \longrightarrow x_3 \sim \gamma^{-1}x_3 \longrightarrow x_4 \sim \beta x_4 \longrightarrow \mathcal{O}.$$

Thus, the curve y corresponds to the word $\beta\gamma^{-1}\alpha\beta^{-1}$ in $\pi_1(C)$. Finally, the diagram for the curve z is

$$\mathcal{O} \longrightarrow x_5 \sim \beta^{-1}x_5 \longrightarrow x_6 \sim \gamma x_6 \longrightarrow x_7 \sim \beta x_7 \longrightarrow \mathcal{O}.$$

Hence, the curve z corresponds to the word $\beta\gamma\beta^{-1}$ in $\pi_1(C)$. Since $xyzt = 1$, we can also express t as a word in α, β and γ if we need to.

Now, suppose we want to express a p/q curve on the 4-punctured sphere, say the $2/1$ curve as a word in α, β and γ. We first note that in terms of the generators x, y, z, t for $\pi_1(S_{0,4})$, the $2/1$ curve is $zxyx^{-1}$. Therefore, by the above result, the $2/1$ curve corresponds to the word $\beta\gamma\beta^{-1}\gamma^{-1}\alpha\beta\gamma^{-1}\alpha\beta^{-1}\alpha^{-1}\gamma$.

Now, the green and the red curve in Figure 4.2 together with the $2/1$ curve on $S_{0,4}$ form a pant decomposition of $\partial_+ C$. To obtain a maximally cusped structure on C, all we need to do is to pinch these 3 curves. This amounts to setting the trace of images under the representation ρ of the words in the generators α, β and γ corresponding to these 3 curves

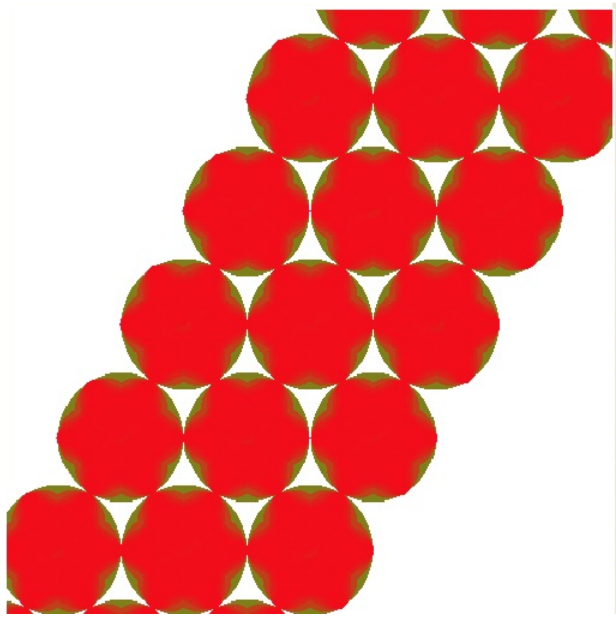

Figure 4.7: Isometric Spheres Pattern for a Maximally Cusped Structure

equal to 2 or -2. For example, we set

$$\mathrm{tr}(\rho(\gamma)) = 2 \text{ to pinch the green curve in Figure 4.2,}$$

$$\mathrm{tr}(\rho(\alpha^{-1}\gamma)) = 2 \text{ to pinch the red curve in Figure 4.2}$$

$$\mathrm{tr}(\rho(\beta\gamma\beta^{-1}\gamma^{-1}\alpha\beta\gamma^{-1}\alpha\beta^{-1}\alpha^{-1}\gamma)) = 2 \text{ to pinch the 2/1 curve on } S_{0,4}.$$

A solution for these 3 trace equations is $c = 2$, $a = 4$ and $b = -1 + i\sqrt{3}$. Therefore, a representation ρ which gives C a maximally cusped structure is

$$\rho(\alpha) = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}, \rho(\beta) = \begin{pmatrix} 1 & -1 + i\sqrt{3} \\ 0 & 1 \end{pmatrix}, \rho(\gamma) = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$$

A portion of the isometric sphere pattern for the group $\Gamma = \rho(\pi_1(C))$, drawn by a program written by Dr. Jessica Purcell, is shown in Figure 4.7. The full pattern would include spheres tangent in a hexagon pattern covering the entire plane.

We can apply the above procedure to pinch the red and green curve in Figure 4.2 and any p/q curve on $S_{0,4}$. However, the computations become more complex rather quickly. A more general procedure for constructing maximally cusped structures would be interesting.

Chapter 5. Cusp Shapes of Tunnel-Number-One Manifolds

5.1 Cusp Shapes of Tunnel-Number-One Manifolds

Throughout this section, $C = C(1;2)$ is the $(1;2)$-compression body. We state and prove the main theorem in this first part of the thesis here.

Theorem 5.1 (Cusp Shape Theorem)**.** *For any $\epsilon > 0$ and any similarity class of flat metrics $[T]$ on the torus, there exists a complete, finite volume, tunnel number one hyperbolic manifold M with a single rank-2 cusp $T_\epsilon \times [0, \infty)$ whose shape $[T_\epsilon]$ is ϵ-close to $[T]$.*

Before we go into more details, here is an outline of the proof:

(1) Construct a cusped structure ρ on the $(1;2)$-compression body C such that ρ gives C cusp shape $[T]$.

(2) Find a maximally cusped structure ρ' on C that is very close to ρ, hence, ρ' gives C a cusp shape close to $[T]$.

(3) Glue the convex core of C equipped with ρ' to the convex core of a genus two handlebody H equipped with a maximally cusped structure along the genus two boundary components to obtain a finite volume, tunnel number one manifold $\widehat{M}$ with 4 rank-2 cusps, one of which has shape very close to $[T]$.

(4) Apply Thurston's Hyperbolic Dehn Surgery Theorem to fill in 3 of the rank-2 cusps of $\hat{M}$ to obtain M.

Step (1) is accomplished by the following Lemma

Lemma 5.2. *Let $[T]$ be any similarity class of flat metrics on the torus. There exists a cusped structure ρ on C such that ρ gives C cusp shape $[T]$.*

Proof. Let the representation $\rho : \pi_1(C) \longrightarrow \mathrm{PSL}(2, \mathbb{C})$ be defined on the generators α, β and γ of $\pi_1(C)$ as

$$\rho(\alpha) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \rho(\beta) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \rho(\gamma) = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix},$$

where a and b are complex numbers such that

(i) a and b are linearly independent over $\mathbb{R}$ with $|a| > 4$ and $|b| > 4$, and

(ii) $[\mathbb{R}^2/\langle \rho(\alpha), \rho(\beta) \rangle] = [T]$.

Condition (i) ensures that a and b satisfy the hypothesis of Lemma 4.5. Hence, ρ gives C a cusped structure. Condition (ii) ensures that the shape on the rank-2 cusp of C equipped with this structure is $[T]$. ■

Step (2) essentially follows from Theorem 4.7. We show this more precisely in another Lemma.

Lemma 5.3. *Let $[T]$ be any similarity class of flat metrics on the torus. For any $\epsilon > 0$, there exists a maximally cusped structure ρ' on C such that ρ' gives C cusp shape $[T_{\epsilon/2}]$ which is $\epsilon/2$-close to $[T]$.*

Proof. Construct a cusped structure ρ such that ρ gives C cusp shape $[T]$ as in Lemma 5.2. By Theorem 4.7, maximally cusped structures are dense on the boundary $\partial\mathrm{MP}(C, T)$ of the space $\mathrm{MP}(C, T)$ of geometrically finite, minimally parabolic structures on C. Therefore, the fact that $\rho \in \partial MP(C)$ implies that there exists a sequence $\{\rho_n\}$ of maximally cusped structures on C which converges algebraically to ρ. Thus, we can choose a maximally cusped structure ρ' which is $\epsilon/2$-close to ρ. In particular, $\rho'(\alpha)$ and $\rho'(\beta)$ are $\epsilon/2$-close to $\rho(\alpha)$ and $\rho(\beta)$ because the convergence is algebraic. As a result, ρ' gives C cusp shape $[T_{\epsilon/2}]$ which is $\epsilon/2$-close to $[T]$. ■

We now construct a finite volume, tunnel number one manifold $\widehat{M}$ with 4 rank-2 cusps, one of which has shape $[T_{\epsilon/2}]$.

Lemma 5.4. *Let $[T]$ be any similarity class of flat metrics on the torus. For any $\epsilon > 0$, there exists a complete, finite volume, tunnel-number-one hyperbolic manifold $\widehat{M}$ with 4 rank-2 cusps, one of which has shape $[T_{\epsilon/2}]$ which is $\epsilon/2$-close to $[T]$.*

Proof. The main idea of this proof is summarized in Figure 5.1.

Let C be the compression body with the maximally cusped structure ρ' as in Lemma 5.3. Let H be a genus two handlebody. There exists a maximally cusped structure on H. Indeed, Canary, Culler, Hersonsky and Shalen (See Corrollary 15.1 of [7]) proved that such structures are dense in the boundary of geometrically finite structures on H. Therefore, there exists a hyperbolic manifold $\mathbb{H}^3/\Gamma_1$ homeomorphic to the interior of H such that the boundary of the convex core of $\mathbb{H}^3/\Gamma_1$ consists of two 3-punctured spheres.

Glue the convex cores of C and H (equipped with these maximally cusped structures) along the 3-punctured spheres via an isometry on each sphere. Since both C and H are geometrically finite, their convex cores have finite volumes and by a result of Adams (see [1]) the resulting manifold $\widehat{M}$ is a complete, finite volume hyperbolic manifold. The manifold $\widehat{M}$ is a tunnel-number-one manifold, being the result of gluing a $(1;2)$-compression body and the genus 2 handle body along their genus 2 boundaries. Furthermore, $\widehat{M}$ has 4 rank-2 cusps, three of which come from gluing the 3 rank-1 cusps on the boundaries of the convex cores of C and H and the remaining one is the rank-2 cusp of C whose shape is $[T_{\epsilon/2}]$.

■

Before we proceed with the proof of Theorem 5.1, we briefly recall Thurston's Hyperbolic Dehn Surgery Theorem specialized to our context (See Chapter 4 of [19] for the most general version of the theorem) .

Suppose the 3 rank-2 cusps of $\widehat{M}$ which come from gluing the 3 rank-1 cusps on the boundaries of the convex cores of C and H are $T_i \times [0, \infty)$, $i = 1, 2, 3$. Let $\widehat{M}'$ be the compact manifold bounded by the 3 tori $T_i, 1 \leq i \leq 3$, resulting from removing the interior

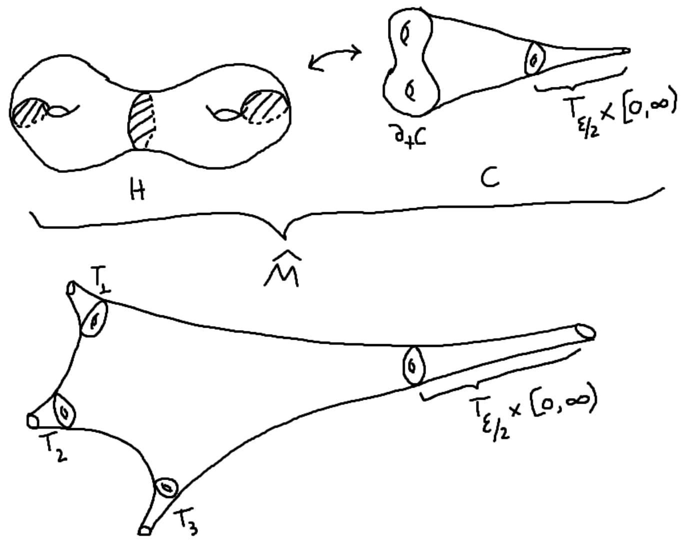

Figure 5.1: Constructing $\widehat{M}$

of the cusps of $\widehat{M}$. For each i choose a homology basis (μ_i, λ_i) for T_i. Let $\mathbb{S}^2$ denote the extended complex plane $\mathbb{C} \cup \{\infty\}$. Let $d_i = (p_i, q_i) \in \mathbb{Z}^2 \subset \mathbb{S}^2$ where p_i, q_i are relatively prime integers and let $d = (d_1, d_2, d_3) \in \mathbb{Z}^2 \times \mathbb{Z}^2 \times \mathbb{Z}^2 \subset \mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2$. Let $\widehat{M}_d$ be the manifold obtained by doing (p_i, q_i)-Dehn surgery on T_i for each i. This means that we glue to $\hat{M}'$ a solid torus along each T_i such that the curve $p_i\mu_i + q_i\lambda_i$ bounds a disk in the new solid torus. Then we have:

Theorem 5.5 (Hyperbolic Dehn Surgery Theorem)**.** *There exists a neighborhood U of $\infty = (\infty, \infty, \infty) \in \mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2$ such that for all $d \in U$, the manifold $\widehat{M}_d$ is hyperbolic. Moreover, as $d \to \infty$, the hyperbolic manifolds $\widehat{M}_d$ converge to $\widehat{M}$ geometrically.*

We are now ready to prove Theorem 5.1

Proof of Theorem 5.1. Let $\widehat{M}$ be the manifold constructed in Lemma 5.4. Note that $\widehat{M}$ has a Heegaard surface S resulting from gluing the boundary of H and the positive boundary of C. Let $\widehat{M}'$ be the manifold obtained by removing the interior of the cusps of $\widehat{M}$. Then $S \cap \widehat{M}$ intersects $\widehat{M}'$ in a surface S' with boundary on T_1, T_2, T_3. Choose a homology basis (μ_i, λ_i) for each T_i where λ_i is a component of $\partial S' \cap T_i$ and μ_i is any curve with intersection number

1 with λ_i. Let $d_i = (1, k_i), 1 \leq i \leq 3$ where k_i is a positive integer. Let $d = (d_1, d_2, d_3)$. As in the proof of Lemma 4.6 of [6], the manifold $\widehat{M}_d$ obtained by doing $(1, k_i)$-Dehn surgery on T_i for each i always has Heegaard surface S, one side of which is a genus 2 handlebody and the other side is a $(1; 2)$ compression body. Thus, $\widehat{M}_d$ is a tunnel-number-one manifold. Moreover, $\widehat{M}_d$ has only one rank-2 cusp by construction. Let $k_i \to \infty$. Then $d \to \infty$ and Theorem 5.5 gives us a sequence of complete, finite volume, tunnel-number-one hyperbolic manifolds $\widehat{M}_d$ which converges geometrically to $\widehat{M}$.

It follows that there exists a base point $\mathcal{O}$ in the universal cover $\mathbb{H}^3$ and a sequence of fundamental polyhedra $\mathcal{P}(\widehat{M}_d)$ for $\widehat{M}_d$ centered at $\mathcal{O}$ which converges to a fundamental polyhedron $\mathcal{P}(\widehat{M})$ for $\widehat{M}$, also centered at $\mathcal{O}$ (See Proposition 4.3.2 of [19]: geometric convergence is equivalent to polyhedral convergence). As a result, the sequence of links in $\mathcal{P}(\widehat{M}_d)$ for the cross-section of the unique rank-2 cusp of each of the manifolds $\widehat{M}_d$ converges to the link for the cross-section of the rank-2 cusp of $\widehat{M}$ whose shape is $[T_{\epsilon/2}]$. Hence, there exists a manifold M whose unique rank-2 cusp has shape $\epsilon/2$-close to $[T_{\epsilon/2}]$. Therefore, M is a tunnel-number-one manifold with a single rank-2 cusp whose shape is ϵ-close to $[T]$. This proves Theorem 5.1. ■

5.2 Towards a Generalization

We would like to extend Theorem 5.1 to the case of manifolds with an arbitrarily large tunnel number. However, we have not been able to achieve that goal yet. What we are able to do, nevertheless, is to construct manifolds which have a rank-2 cusp whose shape is arbitrarily close to a prescribed shape and a tunnel system of arbitrarily many tunnels, each of which satisfies certain geometric property. In particular, we make use of a construction by Cooper, Lackenby and Purcell in [9] to obtain the following theorem:

Theorem 5.6. *Let $[T]$ be any similarity class of flat metrics on the torus. For any natural number n, any real number $R > 0$ and any $\epsilon > 0$, there exists a complete, finite volume hyperbolic manifold with a single rank-2 cusp whose shape is ϵ-close to $[T]$ and which admits* *a system of tunnels $\{\tau_1, \ldots, \tau_n\}$ such that the geodesic representative of the homotopy class of each tunnel τ_i has length at least R.*

We start by constructing a hyperbolic structure ρ on a $(1; n+1)$-compression body C which gives C a prescribed cusp shape and a system of long tunnels. The construction of ρ which gives C long tunnels is a straightforward generalization of the construction in Example 3.2. of [9].

Proposition 5.7. *Let $[T]$ be any similarity class of flat metrics on the torus. For any natural number n and any real number $R > 0$, there exists a geometrically finite, minimally parabolic structure ρ on C such that ρ gives C cusp shape $[T]$ and a system of n tunnels $\{\tau_1, \ldots, \tau_n\}$ in which the length of the geodesic representative of the homotopy class of each tunnel τ_i is at least R.*

Proof. We first note that for the $(1; n+1)$ compression body C, the fundamental group of C is

$$\pi_1(C) = (\underbrace{\mathbb{Z}}_{\langle\alpha\rangle} \times \underbrace{\mathbb{Z}}_{\langle\beta\rangle}) * \underbrace{\mathbb{Z}}_{\langle\gamma_1\rangle} \cdots * \underbrace{\mathbb{Z}}_{\langle\gamma_n\rangle}$$

where $\gamma_1, \ldots, \gamma_n$ can be taken to be freely homotopic to the core tunnel of the 1-handles.

We define a representation $\rho : \pi_1(C) \longrightarrow PSL(2, \mathbb{C})$ by defining ρ on the generators of $\pi_1(C)$ as follows:

$$\rho(\gamma_1) = \begin{pmatrix} \frac{i(1+\lambda)}{\sqrt{\lambda}} & \frac{i}{\sqrt{\lambda}} \\ -\frac{i}{\sqrt{\lambda}} & -\frac{i}{\sqrt{\lambda}} \end{pmatrix}$$

where λ is a real number satisfying $0 < \lambda < exp(-R)$.

Note that the isometric spheres $I(\rho(\gamma_1))$ and $I(\rho(\gamma_1^{-1}))$ have the same radius $\sqrt{\lambda}$ and they have centers at -1 and $-1-\lambda$, respectively. The isometric spheres $I(\rho(\gamma_1^2))$ and $I(\rho(\gamma_1^{-2}))$ have the same radius 1 and they have centers at 0 and $-2-\lambda$, respectively. The spheres $I(\rho(\gamma_1^k))$ where $k = \pm 1, \pm 2$ are shown (side view) in Figure 5.2.

Direct computations show that all the isometric spheres $I(\rho(\gamma_1^k))$ where $k \notin \{\pm 1, \pm 2\}$ are invisible: they are contained in the above four isometric spheres. The picture (viewed

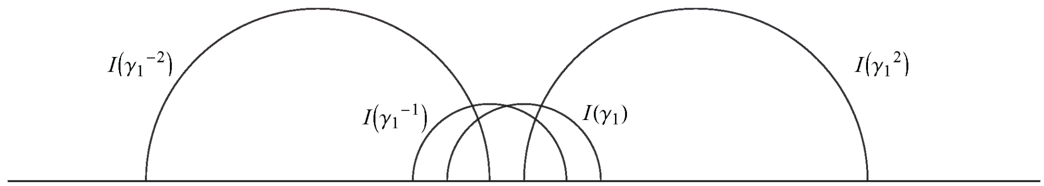

Figure 5.2: Visible Isometric Spheres $\rho(\gamma_1^{\pm1,\pm2})$ in Proposition 5.7

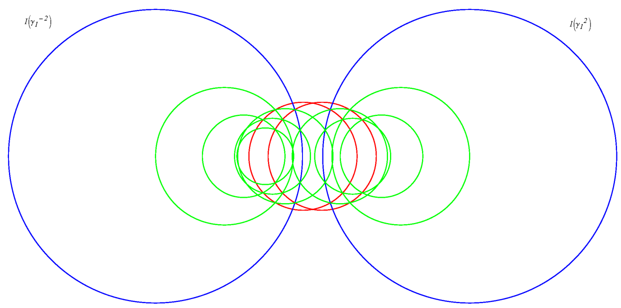

Figure 5.3: A lot of Isometric Spheres $\rho(\gamma_1^k)$ in Proposition 5.7

from the point at ∞ in $\mathbb{H}^3$) is shown in Figure 5.3 where the red and blue spheres are those of Figure 5.2 and the green spheres are some of the spheres $I(\rho(\gamma_1^k))$ with $k \notin \{\pm1, \pm2\}$.

Now we shall define $\rho(\gamma_2)$. Let $A = \begin{pmatrix} 1 & K \\ 0 & 1 \end{pmatrix}$ where $K > 0$ is to be specified shortly. Note that

$$\begin{aligned} I(A\rho(\gamma_1)A^{-1}) &= \{x \in \mathbb{H}^3 : d(x, H) = d(x, A\rho(\gamma_1^{-1})A^{-1}H)\} \\ &= \{x \in \mathbb{H}^3 : d(x, H) = d(x, A\rho(\gamma_1^{-1})H)\} \\ &= \{x \in \mathbb{H}^3 : d(A^{-1}x, H) = d(A^{-1}x, \rho(\gamma_1^{-1})H)\} \\ &= A\{y \in \mathbb{H}^3 : d(y, H) = d(y, \rho(\gamma_1^{-1})H)\} \\ &= AI(\rho(\gamma_1)). \end{aligned}$$

Here H is a horosphere centered at ∞ and we have used Remark (i) that follows Definition 3.5. The above calculation shows that the isometric sphere of the element $A\rho(\gamma_1)A^{-1}$ is obtained from that of $\rho(\gamma_1)$ by horizontal translation by K. Similarly, for any integer k, the isometric spheres of $A\rho(\gamma_1^k)A^{-1}$ are obtained from the corresponding spheres of $\rho(\gamma_1^k)$ by hori-

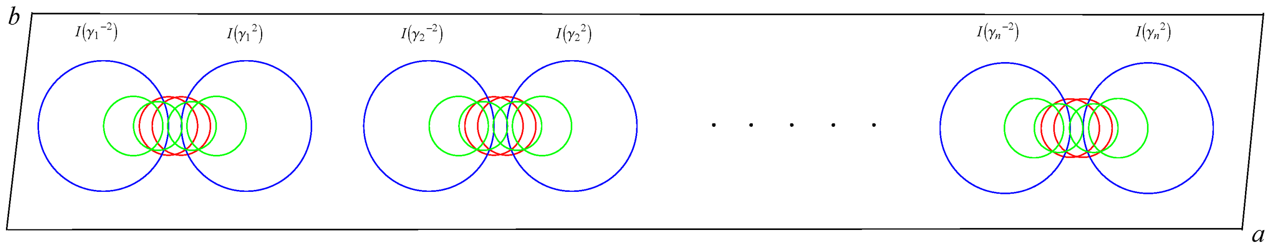

Figure 5.4: Isometric Sphere Pattern for $\rho(\pi_1(C))$ in Proposition 5.7

zontal translation by K. Thus, we can choose K sufficiently large such that the configuration of spheres $\rho(\gamma_1^k)$ in Figure 5.3 does not intersect with the K-translated configuration.

With such a choice for K, define $\rho(\gamma_2) = A\rho(\gamma_1)A^{-1}$. Similarly, define $\rho(\gamma_{i+1}) = A\rho(\gamma_i)A^{-1}$ for $2 \leq i \leq n-1$. The configuration of isometric spheres corresponding to $\rho(\gamma_i), 1 \leq i \leq n$ and their powers is shown in Figure 5.4.

Now we define

$$\rho(\alpha) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \rho(\beta) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

where a and b are complex number linearly independent over $\mathbb{R}$ chosen such that

- The magnitudes $|a|$ and $|b|$ are sufficiently large that none of $I(\rho(\gamma_i))$ and $I(\rho(\gamma_i^{-1}))$ meets any of their translates under $\rho(\alpha)$ and $\rho(\beta)$ and

- $[\mathbb{R}^2/\langle\rho(\alpha), \rho(\beta)\rangle] = [T]$.

Choose a vertical fundamental domain for the action of $\Gamma_\infty = \langle\rho(\alpha), \rho(\beta)\rangle$ that avoids the isometric spheres $I(\rho(\gamma_i)^k)$. Let P be the polyhedron cut out by the visible isometric spheres $I(\rho(\gamma_i^{\pm 1}))$, $I(\rho(\gamma_i^{\pm 2}))$ for $1 \leq i \leq n$ and the vertical fundamental domain for Γ_∞. Let N be the object obtained from P by gluing isometric spheres corresponding to g_j and g_j^{-1} via the isometry g_j (here g_j is one of the $\rho(\gamma_i^{\pm 1})$ or $\rho(\gamma_i^{\pm 2})$), and gluing the faces of the vertical fundamental domain by the isometries $\rho(\alpha)$ and $\rho(\beta)$. Let Γ be the group generated by the face pairing isometries of P. Then we claim that

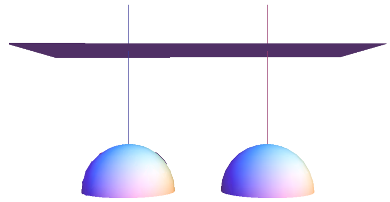

Figure 5.5: Dual Edges to Isometric Spheres

Lemma 5.8. *The group Γ is discrete and N is a smooth, geometrically finite hyperbolic manifold with $\pi_1(N) \cong \Gamma$. Moreover, N is homeomorphic to a $(1; n+1)$ compression body C. Thus, the representation ρ constructed above gives C a minimally parabolic and geometrically finite hyperbolic structure.*

Assume Lemma 5.8 is true. The fact that ρ gives C cusp shape $[T]$ immediately follows from the choice of a and b. Now it remains to prove that ρ gives C a system of n long tunnels. For each i, $1 \leq i \leq n$, let $\tilde{d}_i$ be the vertical geodesic in $\mathbb{H}^3$ dual to the isometric sphere $I(\rho(\gamma_i^{-1}))$ (see Figure 5.5). By Lemma 2.16 of [6], in the quotient manifold $\mathbb{H}^3/\rho(\pi_1(C))$, the images of the $\tilde{d}_i$ are homotopic to the core tunnels of C.

Now let $\{\tau_1, \ldots, \tau_n\}$ be a tunnel system of C where each τ_i is homotopic to a core tunnel of C. In $\mathbb{H}^3$, choose a maximal horoball neighborhood of ∞ and let H be the horosphere boundary of this neighborhood. For each i, let P_i and Q_i be the points of intersection between $\tilde{d}_i$ and $I(\rho(\gamma_i^{-1}))$ and H, respectively. We note that by definition, the length of the geodesic representative of a homotopy class of a tunnel τ_i of C is measured outside a maximal neighborhood of the cusp. Therefore, for each i this length is at least the length of the image of P_iQ_i in the quotient manifold which is the same as the hyperbolic length of P_iQ_i in the universal cover $\mathbb{H}^3$. Since $I(\rho(\gamma_i^2))$ has radius 1, the height of H is at least 1, so is the height of the point Q_i. Thus,

$$l_{hyp}(P_iQ_i) = \left| \ln \frac{z(Q_i)}{z(P_i)} \right| \geq \ln \left(\frac{1}{\sqrt{\lambda}} \right) \geq R.$$

The last inequality holds by choice of λ. Therefore, the geodesic representative of a homotopy class of each τ_i has length at least R.

So, we just proved that ρ gives C cusp shape $[T]$ and a system of n tunnels $\{\tau_1, \ldots, \tau_n\}$ each of whose length is at least R. ■

It remains to prove Lemma 5.8.

Proof of Lemma 5.8. We will again apply Poincaré Polyhedron Theorem 3.10 to show that Γ is discrete and N is a smooth hyperbolic manifold. To that end, it suffices to verify the monodromy condition and the dihedral angle condition about each edge class of P. Note that the polyhedron P has $3n$ edges which consists of 3 visible edges in the configuration of isometric spheres corresponding to $\rho(\gamma_i)$ and its powers for each i. Moreover, for each i, the 3 edges $e_{i,1} = I(\rho(\gamma_i)) \cap I(\rho(\gamma_i^2))$, $e_{i,2} = I(\rho(\gamma_i^{-1})) \cap I(\rho(\gamma_i))$ and $e_{i,3} = I(\rho(\gamma_i^{-1})) \cap I(\rho(\gamma_i^{-2}))$ are glued to a single edge class $[e_i]$. In particular, $e_{i,1}$ is glued to $e_{i,2}$ via γ_i, $e_{i,2}$ is glued to $e_{i,3}$ also via γ_i and $e_{i,3}$ is glued back to $e_{i,1}$ via γ_i^{-2} which completes the edge cycle. Thus the monodromy about $[e_i]$ is $\gamma_i \circ \gamma_i \circ \gamma_i^{-2} = Id$, which also implies that the dihedral angle about $[e_i]$ is 2π. Hence, the conditions for Theorem 3.10 are satisfied. As a result, Γ is discrete and N is a smooth hyperbolic manifold with $\pi_1(N) \cong \Gamma$. Moreover, by Corollary 3.11, P is a Ford domain for the action of Γ on $\mathbb{H}^3$ and N is a geometrically finite manifold.

To see that N is homeomorphic to a $(1; n+1)$-compression body C, we note that when the sides of the vertical fundamental domain are glued via $\rho(\alpha)$ and $\rho(\beta)$, the result is homeomorphic to $T^2 \times (0,1)$. Moreover, for each i, the gluing of the isometric spheres $I(\rho(\gamma_i^{-2}))$ and $I(\rho(\gamma_i^{-1}))$ to $I(\rho(\gamma_i^2))$ and $I(\rho(\gamma_i))$ is equivalent (after applying a homeomorphism) to attaching a 1-handle to $T^2 \times (0,1)$.

Finally, none of the visible isometric spheres in P are visibly tangent on their boundaries and so Proposition 3.13 applies to show that ρ gives C a minimally parabolic structure. ■

Now, unlike the situation in Lemma 5.2, the representation ρ in Proposition 5.7 belongs to the interior of $MP(C)$. The next step is to deform ρ until we obtain a structure on the boundary of $\mathrm{MP}(C, T)$.

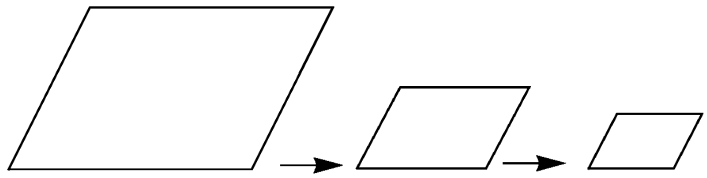

Figure 5.6: Deforming ρ while preserving the cusp shape

Proposition 5.9. *Let $[T]$ be any similarity class of flat metrics on the torus. For any natural number n and any real number $R > 0$, there exists a structure ρ' on C such that*

(i) The structure ρ' belongs to the boundary of $MP(C, T)$, and

(ii) ρ' gives C cusp shape $[T]$, and

(iii) ρ' gives C a system of n tunnels $\{\tau_1, \ldots, \tau_n\}$ in which the length of the geodesic representative of the homotopy class of each tunnel τ_i is at least R.

Proof. Let ρ be the representation constructed in Proposition 5.7. To obtain ρ', we deform ρ by keeping $\rho(\gamma_i)$ fixed for $1 \leq i \leq n$ and decreasing the translation distance of $\rho(\alpha)$ and $\rho(\beta)$ while preserving the cusp shape $[T]$, i.e., preserving the similarity class $[\mathbb{R}^2/\langle\rho(\alpha), \rho(\beta)\rangle]$ (see Figure 5.6).

As soon as the minimal translation distance of $\rho(\alpha)$ or $\rho(\beta)$ becomes smaller than the radius of an isometric sphere, we obtain an indiscrete representation of $\pi_1(C)$; this is an application of Shimizu's Lemma (see Lemma 2.28. of [15] for a statement directly applicable to our case and See Proposition II.C.5 of [21] for a nice proof). Thus, this deformation gives rise to a continuous path from ρ which is in the interior of $\mathrm{MP}(C)$ to an indiscrete representation ρ_I in the representation space $\mathrm{R}(C, T)$. Since $\mathrm{R}(C, T)$ contains $\overline{\mathrm{MP}(C, T)}$, this path must meet the boundary $\partial\mathrm{MP}(C, T)$ of $\mathrm{MP}(C, T)$ at some point ρ'. Note that ρ' gives C the same cusp shape as ρ does by the way we deform ρ. Moreover, for each i, the radii of the isometric spheres $I(\rho(\gamma_i^{\pm 1}))$ and $I(\rho(\gamma_i^{\pm 2}))$ remain unchanged during this deformation.

Therefore, by the proof of Proposition 5.7, the length of the geodesic representative of each core tunnel of C is still at least R. So, ρ' is a structure on C which satisfies all the requirements stated in the Proposition. ■

From here the proof of Theorem 5.6 follows in a way similar to that of Theorem 5.1.

Proof of Theorem 5.6. Let ρ' be the representation constructed in Proposition 5.9. Since $\rho' \in \partial \mathrm{MP}(C, T)$, there exists a maximally cusped structure ρ'' of C which is arbitrarily close to ρ' by Theorem 4.7. As a result, ρ'' gives C a cusp shape $[T_\epsilon]$ which is arbitrarily close to $[T]$ and also a tunnel system in which the geodesic representative of each tunnel has length at least R.

We then glue the convex cores of C equipped with ρ'' and of a genus n handlebody H also equipped with a maximally cusped structure along their boundaries. The manifold $\widehat{M}$ obtained is a finite volume manifold with $(3n + 1)$ rank-2 cusps, $3n$ of which come from gluing rank-1 cusps on the boundaries of the convex cores of C and H. The remaining cusp is the cusp of C with shape $[T_\epsilon]$. Moreover, $\widehat{M}$ also has a tunnel system with n long tunnels. Fill in $3n$ cusps of $\widehat{M}$, and apply Theorem 5.5, we obtain the manifold M in Theorem 5.6.

■

We conclude the first part of this thesis by posing some open questions about cusp shapes and cusp areas of manifolds with unknotting tunnels.

5.3 Some open questions about cusp shapes and cusp areas of manifolds with unknotting tunnels

We already mentioned the first question at the beginning of the previous section. More precisely, the question is

Question 5.10. For each natural number n, is the set of cusp shapes of complete, finite-volume, tunnel-number-n hyperbolic manifolds dense in the set $\mathcal{S}(\mathbb{T}^2)$ of all similarity classes of flat metrics on the torus?

Note that Theorem 5.6 constructs manifolds which admits a system of n tunnels, that is, the tunnel number of the manifold is at most n, but we do not know whether the tunnel number could be lower.

A second question, which actually motivates our work in the second part of this thesis is this

Question 5.11. Can we construct tunnel-number-one hyperbolic manifolds with cusp areas arbitrarily close to a prescribed cusp area?

Recall that a cusp of a hyperbolic manifold corresponds to a torus boundary component. To define the area of the cusp, we expand a horoball neighborhood of this torus boundary component until it meets itself. This *maximal horoball neighborhood* completely determines a flat metric on the torus and we can measure the area on the torus using the metric. So, the cusp area depends on a maximal horoball neighborhood of the torus boundary component corresponding to the cusp. On the other hand, the cusp shape only depends on the similarity class of the flat metric on the torus induced by the hyperbolic metric of the manifold.

In our case, the key in proving our cusp shape theorem is a sequence of maximally cusped structures ρ_n converging **algebraically** to a structure ρ with a prescribed cusp shape. Algebraic convergence is enough to say that $\rho_n(\alpha) \to \rho(\alpha)$ and $\rho_n(\beta) \to \rho(\beta)$. The cusp shape of ρ depends on the similarity class of $\langle \rho(\alpha), \rho(\beta) \rangle$ only. The cusp shape of ρ_n depends on the similarity class of $\langle \rho_n(\alpha), \rho_n(\beta) \rangle$ only. So, the cusp shape of ρ_n is close to that of ρ when n is large. If we want cusp area, we will need a tube around the thin part of $M_n = \mathbb{H}^3/\Gamma_n$ (this is the quotient of $\mathbb{H}^3$ by the maximal horoball neighborhood described above) to be very close to that of $M = \mathbb{H}^3/\Gamma$ when n is large. Algebraic convergence is **not** enough to guarantee this. Geometric convergence would guarantee it, but we do not have geometric convergence here.

Chapter 6. A Local Model for $\mathrm{MP}(C,T)$

This chapter is the beginning of the second part of the thesis. In this part, we shift our focus to the space $\mathrm{MP}(C,T)$ of geometrically finite, minimally parabolic hyperbolic structures on the $(1;2)$-compression body. From work of Bromberg in [5], this space can be parameterized locally by a subset of $\mathbb{C}^3$. We analyze the slices of this subset and prove a convergence theorem about certain slices (See Theorem 7.1).

In this chapter, we will discuss Bromberg's local model for $\mathrm{MP}(C,T)$ and prove Theorem 6.2 which relates a point in the model to the trace coordinates of the representation corresponding to that point. This result will then be used in the next chapter in proving our Slice Convergence Theorem 7.1.

We will start by defining various subsets of $\mathbb{C}^3$ and $\mathbb{C}^2$ which are the key ingredients in constructing a local model for $\mathrm{MP}(C,T)$.

6.1 Local Model for $\mathrm{MP}(C,T)$

6.1.1 The Space $\mathcal{T}$. Recall that $\mathrm{R}(C,T)$ is the $\mathrm{PSL}(2;C)$-representation space for $\pi_1(C)$ where each representation takes elements of $\pi_1(T)$ to parabolics. Let $\overline{\mathrm{R}}(C,T)$ be $\mathrm{R}(C,T)$ modulo conjugation. By abuse of notation, we will use ρ to denote the equivalence class of a representation ρ. Define the map $t : \overline{\mathrm{R}}(C,T) \longrightarrow \mathbb{C}^3$ as

$$t(\rho) = (t_1(\rho), t_2(\rho), t_3(\rho)),$$

where $t_1(\rho) = \mathrm{tr}(\rho(\alpha\gamma))$, $t_2(\rho) = \mathrm{tr}(\rho(\beta\gamma))$, and $t_3(\rho) = \mathrm{tr}(\rho(\gamma))$.

By work of Culler-Shalen (see Section 1 of [11]) and Gonzalez-Montesinos (see Section 3 of [12]), t is an embedding of $\overline{\mathrm{R}}(C,T)$ into $\mathbb{C}^3$.

Recall from Section 4.1. that $\mathrm{AH}(C,T)$ is the subspace of $\overline{\mathrm{R}}(C,T)$ consisting of conjugacy classes of discrete and faithful representations. We define

$$\mathcal{T} = t(\mathrm{AH}(C, T)).$$

More explicitly,

$$\mathcal{T} = \{(a, b, c) \in \mathbb{C}^3 : \exists \rho \in \mathrm{AH}(C, T) \text{ with } t(\rho) = (a, b, c)\}.$$

We can think about the space $\mathcal{T}$ as the "trace coordinates" for points in $\mathrm{AH}(C, T)$.

6.1.2 The Space $\mathcal{M}$. Let $\mathcal{P} = T \cup A_\gamma$ where A_γ is an annulus on the positive boundary of $C(1; 2)$ whose core curve is freely homotopic to γ. We define the spaces $R(C, \mathcal{P})$, $\mathrm{AH}(C, \mathcal{P})$ and $\mathrm{MP}(C, \mathcal{P})$ in analogy with $R(C, T)$, $\mathrm{AH}(C, T)$ and $\mathrm{MP}(C, T)$ by extending the parabolic locus to $\mathcal{P}$. Specifically,

$$R(C, \mathcal{P}) = \{\text{representations } \rho : \pi_1(C) \longrightarrow \mathrm{PSL}(2; C) | \rho(g) \text{ is parabolic for each } g \in \mathcal{P}\}.$$

$$\mathrm{AH}(C, \mathcal{P}) = \{\rho \in R(C, \mathcal{P}) | \rho \text{ is discrete and faithful}\}/\sim,$$

where we quotient out by the action of $\mathrm{PSL}(2, \mathbb{C})$ via conjugation.

$$\mathrm{MP}(C, \mathcal{P}) = \{[\rho] \in \mathrm{AH}(C, \mathcal{P}) | \rho \text{ is geometrically finite and minimally parabolic}\}.$$

Now for $(a, b) \in \mathbb{C}^2$, we construct a representation $\sigma_{a,b} \in R(C, \mathcal{P})$ as

$$\sigma_{a,b}(\alpha) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \sigma_{a,b}(\beta) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \sigma_{a,b}(\gamma) = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}.$$

Define

$$\mathcal{M} = \{(a, b) \in \mathbb{C}^2 | \sigma_{a,b} \in \mathrm{MP}(C, \mathcal{P})\}.$$

Remark. Note that all the "special" representations which give C the cusped structures that appear in the proof of our Cusp Shape Theorem 5.1 are contained in the space $\mathcal{M}$.

6.1.3 The Space $\mathcal{B}$.

Let W be a tubular neighborhood of the core tunnel γ in $C(1;2)$. Let T_γ be the boundary of W. Define $\widehat{C} = C \setminus W$ and $\widehat{\mathcal{P}} = \mathcal{P} \cup T_\gamma$. We can define $R(\widehat{C}, \widehat{\mathcal{P}})$, $\mathrm{AH}(\widehat{C}, \widehat{\mathcal{P}})$ and $\mathrm{MP}(\widehat{C}, \widehat{\mathcal{P}})$ in analogy with $R(C, \mathcal{P})$, $\mathrm{AH}(C, \mathcal{P})$ and $\mathrm{MP}(C, \mathcal{P})$ by considering the $\mathrm{PSL}(2, \mathbb{C})$-representation space of $\pi_1(\widehat{C})$ with parabolic locus $\widehat{\mathcal{P}}$. We note that

$$\pi_1(\widehat{C}) = (\underbrace{\mathbb{Z}}_{\langle\alpha\rangle} \times \underbrace{\mathbb{Z}}_{\langle\beta\rangle}) * (\underbrace{\mathbb{Z}}_{\langle\gamma\rangle} \times \underbrace{\mathbb{Z}}_{\langle\delta\rangle}).$$

Now for $(a, b, d) \in \mathbb{C}^3$, construct a representation $\sigma_{a,b,d} \in R(\widehat{C}, \widehat{\mathcal{P}})$ as

$$\sigma_{a,b,d}(\alpha) = \sigma_{a,b}(\alpha), \sigma_{a,b,d}(\beta) = \sigma_{a,b}(\beta), \sigma_{a,b,d}(\gamma) = \sigma_{a,b}(\gamma), \sigma_{a,b,d}(\delta) = \begin{pmatrix} d+1 & -d \\ d & 1-d \end{pmatrix}.$$

We define

$$\mathcal{B} = \{(a, b, d) \in \mathbb{C}^3 : \sigma_{a,b,d} \in \mathrm{MP}(\widehat{C}, \widehat{\mathcal{P}})\}.$$

We are now ready to describe a local model for the space $\mathrm{MP}(C, T)$.

6.1.4 Bromberg's Local Model for MP(C, T).

Let $\mathcal{A} = \mathcal{B} \cup (\mathcal{M} \times \{\infty\})$. We note that a point (a, b) is in $\mathcal{M}$ if and only if $(a, b, \infty) \in \mathcal{A}$. It follows from Bromberg's work (see section 3 of [5]) that we have the following:

Theorem 6.1 (Local Model for MP(C, T))**.** *Given (a, b) in $\mathcal{M}$, there exists a neighborhood U of the point (a, b, ∞) in $\mathcal{A}$, a neighborhood V of $\sigma_{a,b}$ in $MP(C, \mathcal{P}) \cup MP(C, T)$ and a homeomorphism Φ from $U \cap \mathcal{B}$ to $V \cap MP(C, T)$.*

Most of the work to establish this theorem has already been done by Bromberg in [5], the only difference is that in [5] the manifolds have incompressible boundary whereas our compression body has compressible boundary. This requires a slight modification in Proposition

3.7 of [5]. We give an outline of Bromberg's proof here.

Proof. We will construct two maps: Ψ from $AH(C,T)$ to $\mathcal{A}$ and Φ from $\mathcal{A}$ to $AH(C,T)$ and show that we can choose appropriate neighborhoods of these spaces such that these maps are continuous and are inverses of each other on these neighborhoods.

We start with Ψ. Fix a representation ρ in $MP(C,\mathcal{P})$. By Lemma 3.3 and Lemma 3.4 of [5], there exists a neighborhood V' of ρ in $AH(C,T)$ and a map $\omega : V' \longrightarrow AH(C,\mathcal{P})$ such that ω is continuous at all points in $V' \cap MP(C,\mathcal{P})$. The map ω is defined as follows:

For a representation σ in V'. Assume that $\sigma(\gamma)$ is not parabolic. Let $M_\sigma = \mathbb{H}^3/\sigma(\pi_1(C))$. Fix a smooth embedding $s_\sigma : C \longrightarrow M_\sigma$. Let γ_σ be the geodesic representative of γ in M_σ. Let $\widehat{M_\sigma}$ be the γ_σ-drilling of M_σ given by the Filling-Drilling Theorem of Hodgson, Kerckhoff, and Bromberg. Let

$$\psi_\sigma : M_\sigma - \{\text{thin parts}\} \longrightarrow \widehat{M_\sigma} - \{\text{thin parts}\}$$

be the drilling map. Let $\overline{M_\sigma}$ be the cover of $\widehat{M_\sigma}$ induced by the image of $\pi_1(C)$ under $(\psi_\sigma \circ s_\sigma)_*$. Let $\overline{s_\sigma} : N \longrightarrow \overline{M_\sigma}$ be the lift of $\psi_\sigma \circ s_\sigma$. Lemma 3.3 of [5] goes through without change to give us that the pair $(\overline{M_\sigma}, \overline{s_\sigma})$ is a marked hyperbolic 3-manifold. As a result, $(\overline{M_\sigma}, \overline{s_\sigma})$ determines a representation in $AH(C,\mathcal{P})$. The map ω is then given by

$$\omega(\sigma) = \begin{cases} (\overline{M_\sigma}, \overline{s_\sigma}) & \text{if } \sigma(\gamma) \text{ is not parabolic} \\ \sigma & \text{if } \sigma(\gamma) \text{ is parabolic} \end{cases}$$

Assume that $\sigma(\gamma)$ is not parabolic; we can conjugate so that $\omega(\sigma)(\gamma) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. Let δ be an essential loop in $\widehat{M_\sigma}$ such that $\psi_\sigma^{-1}(\delta)$ bounds a disk in the Margulis tube in M_σ, then there is a unique element of $\pi_1(\widehat{M_\sigma})$ freely homotopic to δ whose $\mathrm{PSL}(2,\mathbb{C})$-representative is of the form $\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$ with $\mathrm{Im}(z) > 0$. The map Ψ is defined as

$$\Psi(\sigma) = \begin{cases} (\omega(\sigma), z) & \text{if } \sigma(\gamma) \text{ is not parabolic} \\ (\omega(\sigma), \infty) & \text{if } \sigma(\gamma) \text{ is parabolic} \end{cases}$$

The proof of Proposition 3.8 in [5] goes through to show that the map Ψ is continuous on $V' \cap MP(C, \mathcal{P})$.

Now we define the map Φ from $\mathcal{A}$ to $AH(C, T)$. By abuse of notation, we identify a point (a, b) in $\mathcal{M}$ with the corresponding representation $\sigma_{a,b}$. Hence, a point in $\mathcal{A}$ can be identified with the ordered pair $(\sigma_{a,b}, d)$ where d is possibly ∞. Now we will need to apply the filling map in the Drilling-Filling Theorem and for it to work, we must restrict to the subspace $\mathcal{A}_K$ of $\mathcal{A}$ consisting of pairs $(\sigma_{a,b}, d)$ where $\frac{|d|}{\sqrt{\mathrm{Im(d)}}} > K$ (see the proof of Theorem 6.2 for a more detailed explanation of this quantity). Moreover, it is sufficient for our purpose to further restrict to the subspace $\mathring{\mathcal{A}}_K$ consisting of pairs $(\sigma_{a,b}, d)$ where $\sigma_{a,b,d} \in \mathrm{MP}(\widehat{C}, \widehat{\mathcal{P}})$ or $d = \infty$. For $(\sigma_{a,b}, d) \in \mathring{\mathcal{A}}_K$, let $\widehat{M_{\sigma_{a,b},d}}$ be the quotient hyperbolic manifold for the representation $\sigma_{a,b,d}$. Let $M_{\sigma_{a,b},d}$ be the filled manifold. Let

$$\phi_{\sigma_{a,b},d} : \widehat{M_{\sigma_{a,b},d}} - \{\text{thin parts}\} \longrightarrow M_{\sigma_{a,b},d} - \{\text{thin parts}\}$$

be the filling map. Let $f_{\sigma_{a,b}} : N \longrightarrow M_{\sigma_{a,b}}$ be a smooth marking map for the hyperbolic manifold $M_{\sigma_{a,b}}$ which is the quotient manifold for the representation $\sigma_{a,b}$. Since $\sigma_{a,b,d}$ is an extension of $\sigma_{a,b}$, $M_{\sigma_{a,b}}$ is a cover of $\widehat{M_{\sigma_{a,b},d}}$. Let $\pi_{\sigma_{a,b},d}$ be the covering map. Let $f_{\sigma_{a,b},d} = \phi_{\sigma_{a,b},d} \circ \pi_{\sigma_{a,b},d} \circ f_{\sigma_{a,b}}$. Lemma 3.6 of [5] goes through without change to give us that $(f_{\sigma_{a,b},d})_*$ is an isomorphism and $(M_{\sigma_{a,b},d}, f_{\sigma_{a,b},d})$ is a marked hyperbolic manifold in $AH(C, T)$. The map Φ is defined as

$$\Phi(\sigma_{a,b}, d) = \begin{cases} (M_{\sigma_{a,b},d}, f_{\sigma_{a,b},d}) & \text{if } d \neq \infty \\ \sigma_{a,b} & \text{if } d = \infty \end{cases}$$

Now we come to the key difference between Bromberg's case and ours. That is to show that the map Φ, so defined, is continuous on $\mathring{\mathcal{A}}_K$. To do so, we need to invoke the Ahlfors-Bers parameterization $\mathcal{AB}$ for $\mathrm{MP}(C, T)$ and the parametrization $\widehat{\mathcal{AB}}$ for $\mathrm{MP}(\widehat{C}, \widehat{\mathcal{P}})$. Since the compression body C has compressible boundary, the images of these Ahlfors-Bers parameterizations are quotients of Teichmuller spaces. More specifically, we have

$$\mathcal{AB} : \mathrm{MP}(C, T) \longrightarrow \mathrm{Teich}(\partial_+ C)/\mathrm{Mod}_0(C)$$

and

$$\widehat{\mathcal{AB}} : \mathrm{MP}(\widehat{C}, \widehat{\mathcal{P}}) \longrightarrow \mathrm{Teich}(\partial_+ C)/\mathrm{Mod}_0(\widehat{C}).$$

Here, recall that for a hyperbolic manifold M, $\mathrm{Mod}_0(M)$ is the group of isotopy classes of homeomorphisms of M which are homotopic to the identity. Since $\mathrm{Mod}_0(\widehat{C})$ is a subgroup of $\mathrm{Mod}_0(C)$, we have a covering map $\Pi : \mathrm{Teich}(\partial_+ C)/\mathrm{Mod}_0(\widehat{C}) \longrightarrow \mathrm{Teich}(\partial_+ C)/\mathrm{Mod}_0(C)$. Let τ be the map defined by $\tau((\sigma_{a,b}, d)) = \sigma_{a,b,d}$. Then by Lemma 3.2 of [5], τ is a local homeomorphism at all points $(\sigma_{a,b}, d)$ such that $\sigma_{a,b,d} \in AH(\widehat{C}, \widehat{\mathcal{P}})$. We also have

$$\Phi(\sigma_{a,b}, d) = \mathcal{AB}^{-1} \circ \Pi \circ \widehat{\mathcal{AB}} \circ \tau(\sigma_{a,b}, d).$$

From this, the rest of Bromberg's argument in Proposition 3.7 goes through to show that Φ is continuous on $\mathring{\mathcal{A}}_K$.

To sum up, we constructed two continuous maps $\Psi : \mathrm{AH}(C, T) \longrightarrow \mathcal{A}$ and $\Phi : \mathcal{A} \longrightarrow \mathrm{AH}(C, T)$. Proposition 3.9 in Bromberg's paper goes through to show that if we restrict to appropriate neighborhoods U of $\mathcal{A}$ and V of the subspace $\mathrm{MP}(C, \mathcal{P}) \cup \mathrm{MP}(C, T)$ of $\mathrm{AH}(C, T)$, then $\Psi \circ \Phi$ is the identity on $U \cap \mathcal{B}$. Moreover, the proof of Proposition 3.10 can be simplified significantly in our case to show that $\Phi \circ \Psi$ is the identity on $V \cap \mathrm{MP}(C, T)$. In fact, the key in his proof of Proposition 3.10 (translated to our case) is to show that the two manifolds $\widehat{M_{\sigma_{a,b}}}$, which is the manifold obtained by drilling the curve γ from $M_{\sigma_{a,b}}$, and $\widehat{M_{\sigma_{a,b},d}}$, which is the manifold corresponding to the extended representation $\sigma_{a,b,d}$, are the same. This requires a difficult topological lemma for the incompressible boundary case in Bromberg's paper. In our case, both manifolds correspond to the same representation, hence, they are the same. This concludes the proof. $\blacksquare$

Now given (a, b, ∞) in $\mathcal{A}$ let (a', b', d) be a point in the neighborhood $U \cap \mathcal{B}$ of (a, b, ∞) as in Theorem 6.1. Then $\Phi(a', b', d)$ is a representation in $V \cap \mathrm{MP}(C, T)$. The trace coordinates for $\Phi(a', b', d)$ are $(x, y, z) = t(\Phi(a', b', d))$. Our goal now is to relate these trace coordinates to the point (a', b', d). Before we do so, we briefly review the concept of the *complex length* of a loxodromic element in $\mathrm{PSL}(2, \mathbb{C})$.

6.2 Complex Lengths of Loxodromic Elements

Let $A \in \mathrm{PSL}(2, \mathbb{C})$ be loxodromic, then A has two distinct fixed points on the boundary at infinity $\mathbb{C} \cup \{\infty\}$ of the upper half space $\mathbb{H}^3$. The geodesic joining these two fixed points is called the axis of A. The axis of A is invariant under the action of A. Moreover, A acts as a translation on its axis and it rotates points in $\mathbb{H}^3$ around its axis. This information is encoded in the *complex length* of A which is denoted by $\ell(A)$. Specifically, the imaginary part of $\ell(A)$ encodes the angle incurred in translating along the axis of A a distance equal to the real part of $\ell(A)$.

Note that A is conjugate to one of the matrices $\pm \begin{pmatrix} \exp(\ell(A)/2) & 0 \\ 0 & \exp(-\ell(A)/2) \end{pmatrix}$ and one can also prove (see Lemma 12.1.2 of [17]) that the trace of a representative of A is related to the complex length of A by the formula

$$\mathrm{tr}^2(A) = 4\cosh^2\left(\frac{\ell(A)}{2}\right).$$

Note that $\ell(A)$ is uniquely determined if we take it in the strip

$$\Lambda = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0 \text{ and } -\pi \leq \mathrm{Im}(z) < \pi\}.$$

The map $z \mapsto 2\cosh(z/2)$ takes Λ to the right-half plane $\mathbb{C}_+ = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$. Following Ito's notation (see Section 5.1 of [13]), we will denote the inverse of this map by λ. So λ is the map $\lambda : \mathbb{C}_+ \setminus (0,2) \longrightarrow \Lambda$ with

$$\lambda(z) = 2\cosh^{-1}\left(\frac{z}{2}\right).$$

Note that for a loxodromic element A with $\mathrm{tr}\, A \in \mathbb{C}_+ \setminus (0,2)$, we have $\lambda(\mathrm{tr}\, A) = \ell(A)$.

6.3 The Neighborhood Refinement Theorem

We have the following theorem which relates a point in the model to the trace coordinates of the representation corresponding to that point.

Theorem 6.2. *Given (a,b) in $\mathcal{M}$ and $\epsilon > 0$, we can choose the neighborhood U in Theorem 6.1 such that the following is satisfied:*

For any (a',b',d) in $U \cap \mathcal{B}$, we have

(i) $\max\{|(a'+2)-x|, |(b'+2)-y|\} < \epsilon$ *and*

(ii) $\left| d - \dfrac{2\pi i}{\lambda(z)} \right| < \epsilon Im(d).$

where $(x,y,z) = t(\Phi(a',b',d))$.

Proof. First, choose neighborhoods U and V with a homeomorphism $\Phi : U \cap \mathcal{B} \longrightarrow V \cap \mathrm{MP}(C,T)$ as in Theorem 6.1. We shall prove that we can refine U so that the inequality in Theorem 6.2 holds.

Now take any (a',b',d) in $U \cap \mathcal{B}$; let ρ denote the representation $\Phi(a',b',d)$. Then $\rho \in \mathrm{MP}(C,T)$. Let

$$M = \mathbb{H}^3/\rho(\pi_1(C)) \text{ and } N = \mathbb{H}^3/\sigma_{a',b',d}(\pi_1(\widehat{C})).$$

Take ϵ_1 smaller than the 3-dimensional Margulis constant. Let $\mathbb{T}_{\epsilon_1}(T_\gamma)$ be the component of the thin part of N corresponding to the rank-2 cusp T_γ. Then

$$\pi_1\left(\partial \mathbb{T}_{\epsilon_1}(T_\gamma)\right) = \langle \sigma_{a',b',d}(\gamma), \sigma_{a',b',d}(\delta) \rangle = \left\langle \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} d+1 & -d \\ d & 1-d \end{pmatrix} \right\rangle.$$

We can conjugate this group by the matrix $A = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$ to obtain

$$A\langle \sigma_{a',b',d}(\gamma), \sigma_{a',b',d}(\delta) \rangle A^{-1} = \left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2d \\ 0 & 1 \end{pmatrix} \right\rangle.$$

It follows that the normalized length of the geodesic representative of δ in $\partial \mathbb{T}_{\epsilon_1}(T_\gamma)$ is

$$\frac{\text{length}(\delta)}{\sqrt{\text{Area}(\partial \mathbb{T}_{\epsilon_1}(T_\gamma)}} = \frac{|d|}{\sqrt{\text{Im}(d)}}$$

The inequality in Theorem 6.2 follows from the Filling theorem of Hodgson, Kerckhoff, and Bromberg (see [5]) and a theorem of Magid (Theorem 1.2 in [18]). The key requirement in both theorems is that we can choose the normalized length of δ to be sufficiently large. We can achieve this here by choosing the neighborhood U such that d is close to ∞ in a way that $\frac{|d|}{\sqrt{\text{Im}(d)}} \longrightarrow \infty$. The detail of the proof is as follows.

By the Filling theorem of Hodgson, Kerckhoff and Bromberg, for (a', b', d) in U, there exists an embedding $f_{a',b',d}$, called the filling map with $f_{a',b',d} : N \longrightarrow M$. The manifold M is called the filled manifold and the manifold N is called the drilled manifold.

Moreover, let $\mathbb{T}_{\epsilon_1}(\gamma)$ be the ϵ_1-Margulis tube around the geodesic representative of γ in M; the Filling theorem says that for every $\epsilon_2 > 0$, if the normalized length of δ is sufficiently large, then the filling map $f_{a',b',d}$ can be chosen so that it restricts to a $(1+\epsilon_2)$-biLipschitz diffeomorphism

$$f_{a',b',d} : N \setminus \mathbb{T}_{\epsilon_1}(T_\gamma) \longrightarrow M \setminus \mathbb{T}_{\epsilon_1}(\gamma).$$

Now the existence of such a diffeomorphism allows us to invoke a result of McMullen (see Lemma 3.20 of [23]) to compare the traces $\mathrm{tr}(\sigma_{a',b',d}(\alpha\gamma))$ and $\mathrm{tr}(\sigma_{a',b',d}(\beta\gamma))$ with a' and b'.

In fact, McMullen's result says that suppose we have a pair of hyperbolic solid tori, M_1 and M_2 where $M_j = \mathbb{H}^3/\langle\gamma_j\rangle$, $j = 1, 2$ whose core geodesics g_j have lengths bounded by some upper bound R, and suppose further that we have a $(1+\epsilon_2)$-quasi-isometric embedding ψ of a unit neighborhood of g_1 in M_1 to M_2 such that $\psi(g_1)$ is homotopic to g_2. Then

$$\left|\mathrm{tr}^2(\gamma_1) - \mathrm{tr}^2(\gamma_2)\right| < C(R)\epsilon_2$$

where $C(R)$ is a constant which depends on R only.

Now in our case, consider the two pairs of hyperbolic solid tori, namely, the pair

$$\mathbb{H}^3/\langle\sigma_{a',b',d}(\alpha\gamma)\rangle \text{ and } \mathbb{H}^3/\langle\rho(\alpha\gamma)\rangle,$$

and the pair

$$\mathbb{H}^3/\langle\sigma_{a',b',d}(\beta\gamma)\rangle \text{ and } \mathbb{H}^3/\langle\rho(\beta\gamma)\rangle.$$

The $(1+\epsilon_2)$-biLipschitz diffeomorphism $f_{a',b',d}$ restricts to a $(1+\epsilon_2)$-quasi-isometric embedding of a unit neighborhood of the core geodesic of one torus to the other torus in each pair, and the core geodesics of these solid tori have bounded length. Therefore, McMullen's result applies and we have

$$\left|\mathrm{tr}^2(\sigma_{a',b',d}(\alpha\gamma)) - \mathrm{tr}^2(\rho(\alpha\gamma))\right| < (\text{constant } K_1)\epsilon_2,$$

and

$$\left|\mathrm{tr}^2(\sigma_{a',b',d}(\beta\gamma)) - \mathrm{tr}^2(\rho(\beta\gamma))\right| < (\text{constant } K_2)\epsilon_2,$$

Note that $\mathrm{tr}(\sigma_{a',b',d}(\alpha\gamma)) = a' + 2$, $\mathrm{tr}(\sigma_{a',b',d}(\beta\gamma)) = b' + 2$, $\mathrm{tr}(\rho(\alpha\gamma)) = x$ and $\mathrm{tr}(\rho(\beta\gamma)) = y$.

Thus, for a given $\epsilon > 0$, we can choose ϵ_2 small enough so that $\max\{K_1, K_2\}\epsilon_2 < \epsilon$ and we can choose the normalized length of δ sufficiently large so that the above inequalities hold. We then obtain

$$|(a'+2)-x|<\epsilon \text{ and } |(b'+2)-y|<\epsilon,$$

Now to obtain $\left|d-\dfrac{2\pi i}{\lambda(z)}\right|<\epsilon$. We shall apply Theorem 1.2 of [18].

Let L^2 denote the square of the normalized length of δ, i.e.,

$$L^2=\frac{|d|^2}{\mathrm{Im}(d)}$$

and let

$$A^2=\frac{|d|^2}{\mathrm{Re}(d)}$$

Since $\mathrm{tr}(\rho(\gamma))=z$, the complex length of the core curve γ of the filling torus $\mathbb{T}_{\epsilon_1}(\gamma)$ is $\lambda(z)$. Suppose $\lambda(z)=l+i\theta$. Then Theorem 1.2 of Magid (see [18]) says that

$$\left|l-\frac{2\pi}{L^2}\right|\le\frac{C_1}{L^4},$$

and

$$\left|\theta-\frac{2\pi}{A^2}\right|\le\frac{C_2}{L^4},$$

where C_1 and C_2 are some constants. It follows from the triangle inequality that for some constant C, we have

$$\begin{aligned}\left|\lambda(z)-\left(\frac{2\pi}{L^2}+i\frac{2\pi}{A^2}\right)\right|&\le\frac{C}{L^4}\\ \left|\lambda(z)-\frac{2\pi i}{|d|^2}\left(-i\mathrm{Im}(d)+\mathrm{Re}(d)\right)\right|&\le\frac{C}{L^4}\\ \left|\lambda(z)-\frac{2\pi i}{d\overline{d}}\overline{d}\right|&\le\frac{C}{L^4}.\end{aligned}$$

Hence,

$$\left|\lambda(z)-\frac{2\pi i}{d}\right|\le\frac{C}{L^4}=C\frac{(\mathrm{Im}(d))^2}{|d|^4}$$

From $Ł^2 = \dfrac{|d|^2}{\mathrm{Im}(d)}$, we have $\mathrm{Re}\left(\dfrac{2\pi i}{d}\right) = \dfrac{2\pi}{L^2}$. This together with the above inequality implies that $|\lambda(z)| > \dfrac{1}{2}\left|\dfrac{2\pi i}{d}\right| = \dfrac{\pi}{|d|}$ when L is sufficiently large. Now, multiply both sides of the above inequality by $\left|\dfrac{d}{\lambda(z)}\right|$, we have

$$\left|d - \frac{2\pi i}{\lambda(z)}\right| \leq C\frac{(\mathrm{Im}(d))^2}{|\lambda(z)||d|^3} \leq C\frac{(\mathrm{Im}(d))^2}{\pi|d|^2} = \frac{C}{\pi L^2}\mathrm{Im}(d) \leq \epsilon \mathrm{Im}(d)$$

when L is sufficiently large.

Thus, when the normalized length L of δ is sufficiently large, we have

(i) $\max\{|(a'+2) - x|, |(b'+2) - y|\} < \epsilon$ and

(ii) $\left|d - \dfrac{2\pi i}{\lambda(z)}\right| < \epsilon \mathrm{Im}(d)$.

■

We are now in a position to study certain slices of the space $\mathcal{T}$, i.e., slices of hyperbolic structures on the $(1;2)$-compression body C; and to prove our Slice Convergence Theorem 7.1 in the next chapter.

Chapter 7. Convergence of Slices of Hyperbolic Structures on the (1; 2)-Compression Body

7.1 Slices of Hyperbolic Structures

Recall from the previous chapter that the space $\mathcal{T}$ is the subset of $\mathbb{C}^3$ consisting of points (a, b, c) which are trace coordinates for representations in $\mathrm{AH}(C, T)$. We now look at slices of this space. For each complex number c, define

$$\mathcal{L}(c) = \{(a, b) \in \mathbb{C}^2 : (a, b, c) \in \mathcal{T}\}.$$

We note that $\mathcal{L}(c) \neq \emptyset$ for any $c \in \mathbb{C}$ with $|c| > 2$. Indeed, given such a c, let a and b in $\mathbb{C}$ be linearly independent over $\mathbb{R}$ with $|a| > 2|c|$ and $|b| > 2|c|$.

Let $\rho : \pi_1(C) \longrightarrow \mathrm{PSL}(2, \mathbb{C})$ be the representation defined by

$$\rho(\alpha) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \rho(\beta) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \rho(\gamma) = \begin{pmatrix} c & -1 \\ 1 & 0 \end{pmatrix}.$$

By Lemma 2.27 in [15], $\rho \in \mathrm{MP}(C, T) \subset \mathrm{AH}(C, T)$. Thus, $(a, b) \in \mathcal{L}(c)$.

Note also that if $c = 2$, then $\mathcal{L}(2) \neq \emptyset$ because all the cusped structures in Lemma 4.5 belong to $\mathcal{L}(c)$.

It turns out that the convergence of the slices $\mathcal{L}(c)$ depends only on the behavior of the complex number c. In particular, in this chapter we prove the following theorem which generalizes a result of Kentaro Ito (see [13]) from the once-punctured torus case to the compression body case.

Theorem 7.1 (Slice Convergence Theorem)**.** *If $\{c_n\}$ is a sequence of complex numbers in $\mathbb{C} \setminus [-2, 2]$ such that $\{c_n\}$ converges to 2 horocyclically, then the slices $\mathcal{L}(c_n)$ converge to the slice $\mathcal{L}(2)$ in the sense of Hausdorff.*

Here, a sequence of complex numbers $\{z_n\}$ converges to 2 *horocyclically* if for every $\epsilon > 0$, there exists a natural number N such that $\lambda(z_n)$ lies in the ball $B_\epsilon(\epsilon)$ for each $n \geq N$. Equivalently, $\{z_n\}$ converges to 2 *horocyclically* if and only if the sequence $\{\mathrm{Im}\left(\frac{2\pi i}{\lambda(z_n)}\right)\}$ approaches ∞. See Figure 7.1.

Also, a sequence of subsets X_n of $\mathbb{C}^2$ *converges in the sense of Hausdorff* to a subset X of $\mathbb{C}^2$ if the following two conditions are satisfied

(1) For each $x \in X$, there exists $x_n \in X_n$ such that $\{x_n\}$ converges to x.

(2) If $x_{n_j} \in X_{n_j}$, and $\{x_{n_j}\}$ converges to x, then $x \in X$.

Note the similarity between this and geometric convergence of a sequence of discrete subgroups Γ_n of $\mathrm{PSL}(2, \mathbb{C})$ to a subgroup Γ.

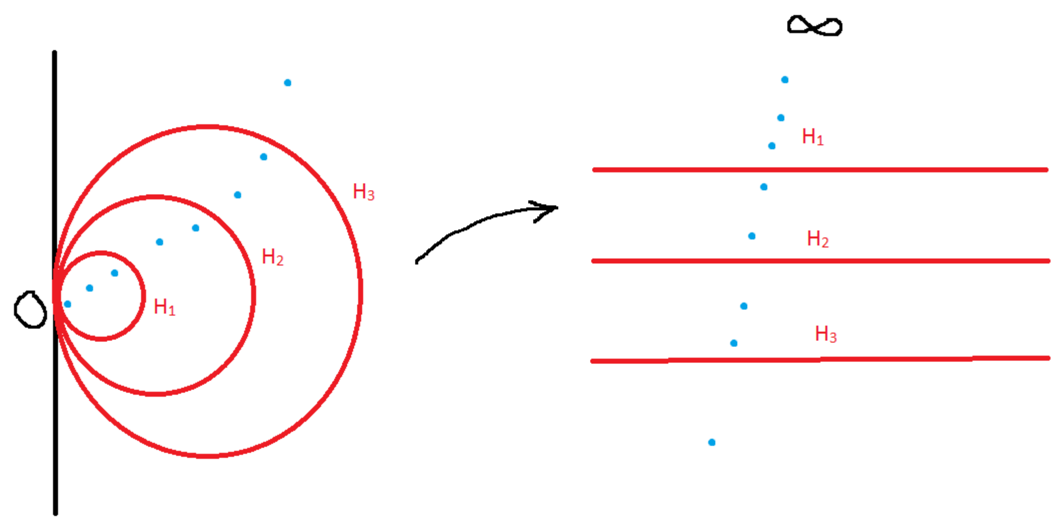

Figure 7.1: Horocyclic Convergence

The remaining of this chapter is devoted to the proof of Theorem 7.1. We follow the strategy that Ito used for the punctured-torus case in [13]. We will make use of the local model for MP(C, T) as well as the estimates for points in the model as provided by Theorem 6.1 and Theorem 6.2. The main innovation is Theorem 7.2 which is proved in the next section.

7.2 Neighborhoods of points in $\mathcal{M}$

Suppose that (a, b) is a point in $\mathcal{M}$, equivalently, (a, b, ∞) is in $\mathcal{A}$. Let U and V be the neighborhoods provided by Theorem 6.2.

Theorem 7.2 shows that there is a nice subset of $\mathbb{C}^3$ which is contained in $U \cap \mathcal{B}$. More specifically, we have the following

Theorem 7.2. *Given (a, b) in $\mathcal{M}$, there exists $\kappa > 0$ and $I > 0$ such that the set $N_{\kappa,I}(a, b)$ is a subset of $U \cap \mathcal{B}$ where $N_{\kappa,I}(a, b)$ is defined as*

$$N_{\kappa,I}(a, b) = B_\kappa(a, b) \times \{z \in \mathbb{C} : Im(z) > I\}.$$

Note that by restricting U, we can assume that for sufficiently small $\kappa > 0$ and sufficiently large $I > 0$, every point (a', b', d) in the neighborhood $N_{\kappa,I}(a, b)$ must also be in U. The difficult part is to prove that there exists a choice of κ and I such that every point (a', b', d) in $N_{\kappa,I}(a, b)$ is in $\mathcal{B}$ as well, that is, the representation $\sigma_{a',b',d}$ is in $\mathrm{MP}(\widehat{C}, \widehat{\mathcal{P}})$. (See Section 6.1.3 to recall the definition of $\sigma_{a',b',d}$)

This is done via a few lemmas. Lemma 7.3 and Lemma 7.4 establish some important properties of the isometric sphere pattern of the Ford domain corresponding to the action of $\sigma_{a',b',d}(\pi_1(\widehat{C}))$ where (a', b', d) is in $N_{\kappa,I}(a, b)$ for appropriate choices of small κ and large I. Then Lemma 7.5 and Lemma 7.6 show that for such choices of κ and I, the representation $\sigma_{a',b',d}$ is indeed a representation in $\mathrm{MP}(\widehat{C}, \widehat{\mathcal{P}})$.

Lemma 7.3. *There exists $\kappa > 0$ such that the number of visible isometric spheres for all representations $\sigma_{a',b'}$ as (a', b') runs through the entire ball $B_\kappa(a, b)$ is finite.*

Proof of Lemma 7.3. Since $(a, b) \in \mathcal{M}$, the representation $\sigma_{a,b}$ is geometrically finite and minimally parabolic, that is $\sigma_{a,b}$ is in $\mathrm{MP}(C, \mathcal{P})$. Since $\mathrm{MP}(C, \mathcal{P})$ is open, there exists an open ball about $\sigma_{a,b}$. Pulling this back to $\mathcal{M}$, we can find an $\kappa_1 > 0$ such that the open ball $B_{\kappa_1}(a, b)$ is contained in $\mathcal{M}$. Choose $\kappa > 0$ such that the closed ball $B_\kappa(a, b)$ is contained in $B_{\kappa_1}(a, b)$. For each point (a', b') in $B_\kappa(a, b)$, the number of visible isometric sphere corresponding to any Ford domain of $\sigma_{a',b'}$ is finite. We need to prove that the number of visible isometric spheres for *all representations* $\sigma_{a',b'}$ as (a', b') runs through the entire ball $B_\kappa(a, b)$ is finite.

Suppose to the contrary that there were infinitely many visible isometric spheres. It follows that there exists an infinite sequence of points $\{(a_n, b_n)\}$ in the ball $B_\kappa(a, b)$ such that all the representations σ_{a_n,b_n} corresponding to these points have Ford domains with distinct visible isometric spheres. The representations σ_{a_n,b_n} all belong to a closed ball in $\mathrm{MP}(C, \mathcal{P})$; therefore, there exists a subsequence $\sigma_{a_{n_j},b_{n_j}}$ which converges algebraically to a representation ρ. By Theorem 4.2, there exists a subsequence which, by abuse of notation is denoted also by $\sigma_{a_{n_j},b_{n_j}}$ which converges geometrically to a representation ρ'. Since these representations are minimally parabolic, they are *type-preserving* as in [4], it follows from Theorem 4.1 of [4] that the geometric limit and the algebraic limit agree, that is, $\rho = \rho'$. Now, by Proposition 4.3.2 of [19], there exists a sequence $\mathcal{F}(\sigma_{a_{n_j},b_{n_j}})$ of Ford domains for $\sigma_{a_{n_j},b_{n_j}}$ which converges to a Ford domain $\mathcal{F}(\rho)$ of ρ. Since ρ is geometrically finite, $\mathcal{F}(\rho)$ has only finitely many visible isometric spheres which contradicts the assumption that the number of isometric spheres in the sequence $\mathcal{F}(\sigma_{a_{n_j},b_{n_j}})$ is infinite. $\blacksquare$

Next, for any (a', b') in the ball $B_\kappa(a, b)$ conjugate the representation $\sigma_{a',b'}$ by the matrix $A = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$. Algebraically, we conjugate the group $\Gamma' = \langle \sigma_{a',b'}(\alpha), \sigma_{a',b'}(\beta), \sigma_{a',b'}(\gamma) \rangle$ by the matrix A. So the fixed point $\{\infty\}$ of $\Gamma'_\infty = \langle \sigma_{a',b'}(\alpha), \sigma_{a',b'}(\beta) \rangle$ is taken to the fixed point $\{-1\}$ of $A\Gamma'_\infty A^{-1}$ and the fixed point $\{1\}$ of $\sigma_{a',b'}(\gamma)$ is taken to $\{\infty\}$. An example of a Ford domain for the conjugated representation is shown in Figure 7.2.

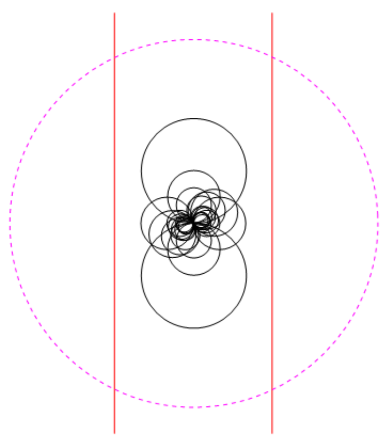

Figure 7.2: An example of the conjugated structure $A\sigma_{a',b'}A^{-1}$

Now we claim that

Lemma 7.4. *There exists $R > 0$ such that for an arbitrary (a', b') in $B_\kappa(a, b)$, the half ball with center -1 and radius R on $\widehat{\mathbb{C}}$ contains all the finitely many visible isometric spheres of any Ford domain for $A\sigma_{a',b'}A^{-1}$. (An example of such a ball is depicted by the dotted pink circle in Figure 7.2)*

Proof of Lemma 7.4. By Lemma 7.3, we have a list $\{g_1, g_2, \ldots, g_l\}$ of the finitely many group elements of $\pi_1(C)$ giving visible isometric spheres of the Ford domains for all representations $A\sigma_{a',b'}A^{-1}$ as (a', b') runs through the entire ball $B_\kappa(a, b)$. That is, there exist elements $g_1, g_2, \ldots, g_l$ of $\pi_1(C)$ such that for each j, there exists (a', b') in $B_\kappa(a, b)$ for which the isometric sphere corresponding to $\sigma_{a',b'}(g_j)$ is visible. For each j, consider the function $\mathrm{Rad}_j(I(\sigma_{a',b'}(g_j)))$ which gives the radius of the sphere $I(\sigma_{a',b'}(g_j))$ for an arbitrary point (a', b') in $B_\kappa(a, b)$. Note that Rad_j is continuous because Rad_j can be read off of the matrix $\sigma_{a',b'}(g_j)$ (see Definition 3.5).Therefore, as (a', b') runs through the closed ball $\overline{B_\kappa(a, b)}$, Rad_j attains a maximum value, called it R_j.

Choose R such that $R > \sum_{j=1}^{l} 2R_j$. We claim that such a choice satisfies the condition in Lemma 7.4. Indeed, let (a', b') be an arbitrary point in $B_\kappa(a, b)$. The isometric sphere pattern of any Ford domain for $\sigma_{a',b'}$ is connected.

In fact, we can first choose (a^*, b^*) with $|a^*| > 4$ and $|b^*| > 4$, then σ_{a^*,b^*} has a connected isometric sphere pattern which is explicitly described in Lemma 4.5. Since there exists an analytic path in $\mathcal{M}$ from (a^*, b^*) to (a, b), Lemma 5.4 of [15] applies to show that $\sigma_{a,b}$ has a connected isometric sphere patten. Since (a', b') is in a small neighborhood of (a, b), the same argument applies to show that the isometric sphere pattern of a Ford domain of $\sigma_{a',b'}$ is connected as well.

Now, in the worst case scenario, the visible isometric spheres of a Ford domain for $\sigma_{a',b'}$ are all the possible visible isometric spheres from the list $\{g_1, \ldots, g_l\}$ and they are arranged in a way that each sphere is visibly tangent to exactly one other sphere and intersects no other spheres. In this case, the half ball $B(-1, R)$ with center -1 and radius R on $\widehat{\mathbb{C}}$ will still contain all these spheres for the above choice of R. Thus, the condition in the Lemma is satisfied. This will hold for all the conjugated representations $A\sigma_{a',b'}A^{-1}$ as well. ■

Now, for the isometric sphere pattern of each $A\sigma_{a',b'}A^{-1}$, we can then add two vertical planes outside of the ball $B(-1, R)$ which contains all the existing isometric spheres. See Figure 7.3 for an example; here we added two green vertical planes to the existing isometric sphere pattern in Figure 7.2.

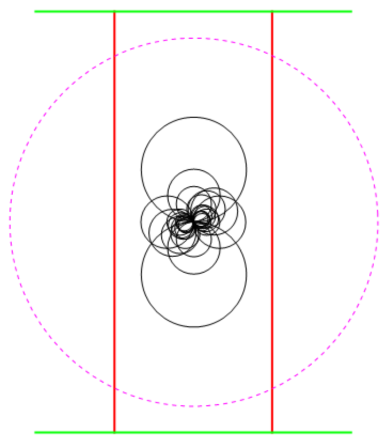

Figure 7.3: Adding two vertical planes to the existing sphere pattern

Identify these two added planes via translation by the matrix $D = \begin{pmatrix} 1 & 2d \\ 0 & 1 \end{pmatrix}$ for d with $\mathrm{Im}(d)$ sufficiently large. In particular, if we choose $I > R$, then the choice of $\mathrm{Im}(d) > I$ will determine a pair of planes outside $B(-1, R)$. Hence, the two added vertical planes together with the vertical planes corresponding to $\langle A\sigma_{a',b'}(\gamma)A^{-1}\rangle$ provide us with a vertical fundamental domain for the action of $\langle A\sigma_{a',b'}(\gamma)A^{-1}, D\rangle$. Now, this vertical fundamental domain together with the isometric spheres contained in the ball $B(-1, R)$ provides us with a candidate for the Ford domain for the action of the group $A\sigma_{a',b',d}(\pi_1(\widehat{C}))A^{-1}$. We will show that it is indeed a Ford domain for the action of $A\sigma_{a',b',d}(\pi_1(\widehat{C}))A^{-1}$ via Lemma 7.5 and 7.6.

Lemma 7.5. *There exists $\kappa > 0$ and $I > 0$ such that for all $(a', b', d) \in N_{\kappa,I}(a, b)$, the representation $\sigma_{a',b',d}$ is a discrete, faithful and geometrically finite representation of $\pi_1(\widehat{C})$.*

Proof of Lemma 7.5. We have established above that if κ is sufficiently small and I is sufficiently large, then a candidate for the Ford domain for the action of $A\sigma_{a',b',d}(\pi_1(\widehat{C}))A^{-1}$ where $(a', b') \in B_\kappa(a, b)$ and $\mathrm{Im}(d) > I$ consists of finitely many isometric spheres contained in a half ball $B(R)$ and a vertical fundamental domain for the action of $\langle A\sigma_{a',b'}(\gamma)A^{-1}, D\rangle$. Note that this vertical fundamental domain certainly satisfies the dihedral angle condition and the face pairing condition of Poincaré Polyhedron Theorem 3.10. The finitely many isometric spheres contained in $B(R)$ also satisfy these conditions because they are obtained by conjugating the isometric sphere pattern of the Ford domain for the action of $\sigma_{a',b'}(\pi_1(C))$ and such a pattern satisfies these conditions by assumption on κ. Therefore, when we glue up all the isometric spheres contained in $B(R)$ and the vertical fundamental domain for $\langle A\sigma_{a',b'}(\gamma)A^{-1}, D\rangle$, we obtain a smooth manifold Q with $\pi_1(Q) \cong A\sigma_{a',b',d}(\pi_1(\widehat{C}))A^{-1} \cong \sigma_{a',b',d}(\pi_1(\widehat{C}))$. Moreover, it also follows from Theorem 3.10 that the group $A\sigma_{a',b',d}(\pi_1(\widehat{C}))A^{-1}$ is discrete and geometrically finite. Hence, so is $\sigma_{a',b',d}(\pi_1(\widehat{C}))$. ■

It remains to prove that the manifold $\mathbb{H}^3/\sigma_{a',b',d}(\pi_1(\widehat{C}))$ is homeomorphic to $\widehat{C}$. This is done in the next lemma.

Lemma 7.6. *The manifold $\mathbb{H}^3/\sigma_{a',b',d}(\pi_1(\widehat{C}))$ is homeomorphic to the interior of $\widehat{C}$.*

Proof of Lemma 7.6. Consider the manifold $\mathbb{H}^3/\sigma_{a',b',d}(\pi_1(\widehat{C}))$ which is homeomorphic to the manifold Q obtained by gluing the isometric spheres contained in the half ball $B(R)$ and the faces of the vertical fundamental domain for the action of $\langle A\sigma_{a',b'}(\gamma)A^{-1}, D\rangle$. We will show that Q is homeomorphic to $\widehat{C}$. The boundary of $B(R)$ intersects the interior of the aforementioned vertical fundamental domain in a disk. The blue disk shaded in Figure 7.4 illustrates this disk for the example of Figure 7.3. Two curves on this disk are the intersection between the boundary of $B(R)$ and the two vertical walls corresponding to the translation by $A\sigma_{a',b'}(\gamma)A^{-1}$. In the example in Figure 7.4, these two curves are the pink curves with arrows. This forms an annulus $A_1(R)$ from the blue disk. This annulus separates Q into two parts. One part contains the point at infinity, the other one contains finitely many isometric spheres.

The interior of the part which contains the point at infinity is homeomorphic to $\mathbb{T}^2 \times (0,1)$. Indeed, a side view of this part looks like Figure 7.5 where each purple curve in the figure corresponds to a torus $\mathbb{T}^2$.

Now, the torus $\mathbb{T}^2 \times \{0\}$ is made up of two annuli $A_1(R)$ and $A_2(R)$ where $A_2(R)$ is the annulus shaded in yellow in Figure 7.4.

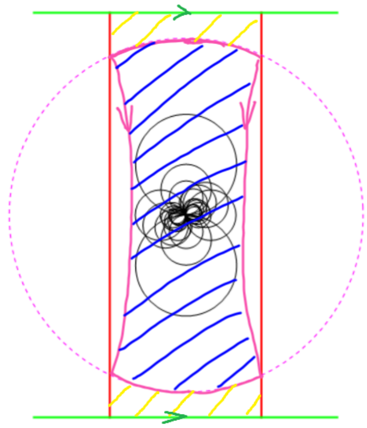

Figure 7.4: The annulus in Lemma 7.6

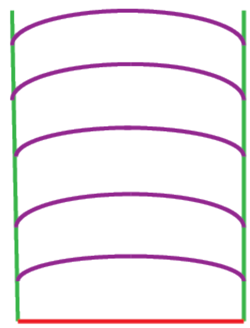

Figure 7.5: $\mathbb{T}^2 \times (0, 1)$ in Lemma 7.6

Since $A_1(R)$ only intersects the two vertical walls corresponding to the translation by $A\sigma_{a',b'}(\gamma)A^{-1}$, it will only intersect exactly two isometric spheres in the "pre-conjugate structure." The situation is illustrated in Figure 7.6.

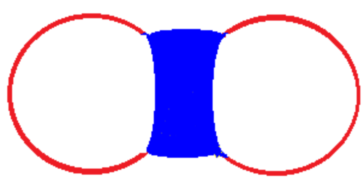

Figure 7.6: The annulus $A_1(R)$ is the pre-conjugate structure

The gluing of all isometric spheres is unchanged from $\sigma_{a',b'}$ aside from the isometric spheres corresponding to $\sigma_{a',b'}(\gamma)$ and $\sigma_{a',b'}(\gamma^{-1})$, which are still glued, only with $A_1(R)$ removed. Thus, the part of Q which contains the isometric spheres is homeomorphic to a

$(1;2)$-compression body together with $A_1(R)$ a marked annulus on $\partial_+ C$. So Q is obtained by gluing a $(1;2)$-compression body C and $\mathbb{T}^2 \times (0,1)$ along the annulus $A_1(R)$. We need to determine the curve of $A_1(R)$ on $\partial_+ C$ to finish the proof.

Now, let d_1 and d_2 be the geodesics dual to the 2 isometric spheres corresponding to $\sigma_{a',b'}(\gamma)$ and $\sigma_{a',b'}(\gamma^{-1})$ which intersect $A_1(R)$ (see Figure 5.5 for an example of dual geodesics to isometric spheres). The geodesics d_1 and d_2 intersect some horosphere about infinity at P_1 and P_2. Note that P_1P_2 is homotopic to a separating curve of the annulus $A_1(R)$. By Lemma 2.16 of [6], the geodesics d_1 and d_2 and P_1P_2 glue up to a curve homotopic to the core curve γ of the compression body. It then follows that the annulus $A_1(R)$ is parallel to the core curve γ of the compression body.

When we glue the annulus $A_1(R)$ in $\mathbb{T}^2 \times (0,1)$ to $A_1(R)$ in ∂_+C, the annulus $A_2(R)$ is attached to $\partial A_1(R)$ along its boundary curves. Hence, we obtain the torus $\mathbb{T}^2 \times \{0\}$ which is parallel to the core curve γ of the compression body. So Q is obtained by taking a $(1;2)$-compression body and stacking on $\mathbb{T}^2 \times (0,1)$ to a torus which is parallel to its core curve. It then follows that Q is homeomorphic to $\widehat{C}$. ■

Proof of Theorem 7.2. Choose κ as in Lemma 7.3 and $I > R$ where R is chosen as in Lemma 7.4. Then from Lemma 7.5 and Lemma 7.6, we conclude that the representation $\sigma_{a',b',d}$ corresponding to a point $(a',b',d) \in N_{\kappa,I}(a,b)$ is in $\mathrm{MP}(\widehat{C},\widehat{\mathcal{P}})$. This, together with the remark right after the statement of the theorem shows that with such a choice of κ and I, (a',b',d) is in $U \cap \mathcal{B}$. Hence, $N_{\kappa,I}(a,b) \subset U \cap \mathcal{B}$. ■

7.3 Proof of the Slice Convergence Theorem

We first define a map $\Psi : \mathbb{C}^2 \times (\mathbb{C}_+ \setminus (0,2)) \longrightarrow \mathbb{C}^2 \times \widehat{\mathbb{C}}$ as

$$\Psi(u,v,w) = (u,v,\frac{2\pi i}{\lambda(w)}),$$

For a given $(a,b) \in \mathcal{M}$ and given $\epsilon > 0$, choose the neighborhood U of (a,b,∞) as in the

Neighborhood Refinement Theorem and consider the sequence of maps

$$U \cap \mathcal{B} \xrightarrow{\Phi} V \cap \mathrm{MP}(C,T) \xrightarrow{t} \mathbb{C}^2 \times (\mathbb{C}_+ \setminus (0,2)) \xrightarrow{\Psi} \mathbb{C}^2 \times \widehat{\mathbb{C}},$$

Here t is the trace map defined in Section 6.1, Φ is the homeomorphism from Theorem 6.1 and U, V are the neighborhoods provided by Theorem 6.2.

Let $\chi = \Psi \circ t \circ \Phi$. Then χ is a homeomorphism from U to its image. We have the following lemma

Lemma 7.7. *Given $(a,b) \in \mathcal{M}$, there exists $\kappa_0 > 0$ such that for every κ with $0 < \kappa < \kappa_0$, there exists $I > 0$ such that $N_{\kappa/2,2I}(a+2,b+2) \subset \chi\left(N_{\kappa,I}(a,b)\right)$.*

Proof. Let U, V and χ be as above. By Theorem 7.2, there exist $\kappa_0 > 0$ and $I > 0$ such that

$$N_{\kappa_0,I}(a,b) \subset U \cap \mathcal{B}$$

Take κ with $0 < \kappa < \kappa_0$. For sufficiently large I, the following estimates are satisfied for each point (u,v,w) in $N_{\kappa,I}(a,b)$ by the Neighborhood Refinement Theorem 6.2:

$$|(u+2) - u'| < \frac{\kappa}{8} \tag{7.1}$$

$$|(v+2) - v'| < \frac{\kappa}{8} \tag{7.2}$$

$$|w - w'| < \frac{\kappa}{8}\mathrm{Im}(w) \tag{7.3}$$

where $(u',v',w') = \chi(u,v,w)$

We claim that for such a choice of κ and I, $N_{\frac{\kappa}{2},2I}(a+2,b+2) \subset \chi\left(N_{\kappa,I}(a,b)\right)$.

Suppose, by way of contradiction, that there exists a point

$$(a',b',d') \in N_{\frac{\kappa}{2},2I}(a+2,b+2) \setminus \chi\left(N_{\kappa,I}(a,b)\right).$$

Let $(x,y,z) = \chi(a'-2, b'-2, d')$. Since $(a',b') \in B_{\frac{\kappa}{2}}(a+2,b+2)$, we have

$$\sqrt{|a'-(a+2)|^2 + |b'-(b+2)|^2} < \frac{\kappa}{2}$$

It follows that $|a'-(a+2)| < \frac{\kappa}{2}$ and $|b'-(b+2)| < \frac{\kappa}{2}$. Equivalently, $|(a'-2)-a| < \frac{\kappa}{2}$ and $|(b'-2)-b| < \frac{\kappa}{2}$, that is, $(a'-2, b'-2) \in B_{\frac{\kappa}{2}}(a,b)$. Hence, $(a'-2, b'-2, d') \in N_{\frac{\kappa}{2},2I}(a,b) \subset N_{\kappa,I}(a,b)$. As a result, $(x,y,z) = \chi(a'-2,b'-2,d')$ satisfies the estimates (7.1), (7.2) and (7.3). Specifically, we have

$$|a'-x| < \frac{\kappa}{8} \tag{7.4}$$

$$|b'-y| < \frac{\kappa}{8} \tag{7.5}$$

$$|d'-z| < \frac{\kappa}{8}\mathrm{Im}(d') \tag{7.6}$$

Now consider the line segment connecting (x,y,z) and (a',b',d'), that is,

$$\gamma(t) = (1-t)(x,y,z) + t(a',b',d'), 0 \leq t \leq 1.$$

We have $\gamma(0) = (x,y,z) \in \chi(N_{\kappa,I}(a,b))$ and $\gamma(1) = (a',b',d') \notin \chi(N_{\kappa,I}(a,b))$. Since $|a'-x| < \frac{\kappa}{8}$ and $|a'-(a+2)| < \frac{\kappa}{2}$ (because $(a',b') \in B_{\frac{\kappa}{2}}(a+2,b+2)$), we have $|x-(a+2)| < \frac{5\kappa}{8}$. Similarly, $|y-(b+2)| < \frac{5\kappa}{8}$.

Now from $|d'-z| < \frac{\kappa}{8}\mathrm{Im}(d')$ and $\mathrm{Im}(d') > 2I$, we get

$$\mathrm{Im}(z) > 2I - \frac{\kappa}{8} \cdot 2I = \frac{8-\kappa}{4} \cdot I$$

We just established that both endpoints (x,y,z) and (a',b',d') are contained in the neighborhood $N_{\frac{5\kappa}{8},\frac{8-\kappa}{4}I}(a+2,b+2)$. Thus, $\gamma([0,1]) \subset N_{\frac{5\kappa}{8},\frac{8-\kappa}{4}I}(a+2,b+2)$.

Let $t_\infty = \{t : \gamma(t) \notin \chi\left(N_{\kappa,I}(a,b)\right)\}$. Then $0 < t_\infty \leq 1$. Choose a sequence t_n approaching

t_∞ from below and let $(a'_n, b'_n, d'_n) = \gamma(t_n)$. Then $(a'_n, b'_n, d'_n) = \gamma(t_n) \in \chi(N_{\kappa,I}(a,b))$. It follows that $(a_n, b_n, d_n) \in N_{\kappa,I}(a,b)$ where $(a_n, b_n, d_n) = \chi^{-1}(a'_n, b'_n, d'_n)$.

We will show that (a_n, b_n, d_n) belongs to a slightly smaller neighborhood, namely, $N_{\frac{3\kappa}{4}, I+8}(a,b)$.

First, since (a_n, b_n, d_n) is in $N_{\kappa,I}(a,b)$ and $(a'_n, b'_n, d'_n) = \chi(a_n, b_n, d_n)$, Theorem 6.2 gives the estimates $|a'_n - (a_n+2)| < \frac{\kappa}{8}$, $|b'_n - (b_n+2)| < \frac{\kappa}{8}$ and $|d'_n - d_n| < \frac{\kappa}{8}\mathrm{Im}(d_n)$.

Second, since $(a'_n, b'_n, d'_n) = \gamma(t_n) \in N_{\frac{5\kappa}{8}, \frac{8-\kappa}{4}I}(a+2, b+2)$, we have $|a'_n - (a+2)| < \frac{5\kappa}{8}$, $|b'_n - (b+2)| < \frac{5\kappa}{8}$ and $\mathrm{Im}(d'_n) > \frac{8-\kappa}{4}I$.

It follows that

$$|a_n - a| \leq |(a_n+2) - a'_n| + |a'_n - (a+2)| < \frac{\kappa}{8} + \frac{5\kappa}{8} = \frac{3\kappa}{4}$$

Similarly, $|b_n - b| < \frac{3\kappa}{4}$. And so $(a_n, b_n) \in B_{\frac{3\kappa}{4}}(a,b)$.

Now, we claim that $\mathrm{Im}(d_n) > I + 8$. Suppose not, that is, $\mathrm{Im}(d_n) \leq I + 8$. Then

$$|d_n - d'_n| < \frac{\kappa}{8}\mathrm{Im}(d_n) \leq \frac{\kappa}{8}(I+8) = \frac{\kappa}{8}I + \kappa.$$

Thus, $\mathrm{Im}(d'_n) < \mathrm{Im}(d_n) + \frac{\kappa}{8}I + \kappa \leq I + 8 + \frac{\kappa}{8}I + \kappa$.

But we already have that $\mathrm{Im}(d'_n) > \frac{8-\kappa}{4}I = 2I - \frac{\kappa}{4}I$. This yields a contradiction for sufficiently small κ.

Therefore, we have established that (a_n, b_n, d_n) belongs to the neighborhood $N_{\frac{3\kappa}{4}, I+8}(a,b)$. It follows that the accumulation point $(a_\infty, b_\infty, d_\infty)$ of the sequence (a_n, b_n, d_n) belongs to the neighborhood $N_{\kappa,I}(a,b)$. Consequently, $\gamma(t_\infty) = \chi(a_\infty, b_\infty, d_\infty)$ belongs to $\chi(N_{\kappa,I}(a,b))$. Since χ is a homeomorphism, this contradicts the definition of t_∞. This contradiction shows that $N_{\frac{\kappa}{2}, 2I}(a+2, b+2) \subset \chi(N_{\kappa,I}(a,b))$. ■

Lemma 7.8. *Suppose that $\{c_n\} \subset \mathbb{C}\backslash[-2,2]$ and $\{c_n\}$ converges to 2 horocyclically. Then for any $(a,b) \in \mathcal{M}$, there exists $\epsilon > 0$ and a natural number N such that the ball $B_\epsilon(a+2, b+2)$ is contained in $\mathcal{L}(c_n)$ for all $n \geq N$.*

Proof. Given $(a, b) \in \mathcal{M}$, choose U as in Theorem 6.2. By Theorem 7.2 and Lemma 7.7, we can choose $\kappa > 0$ and $I > 0$ such that $N_{\kappa,I}(a, b) \subset U \cap \mathcal{B}$ and $N_{\frac{\kappa}{2},2I}(a+2, b+2) \subset \chi(N_{\kappa,I}(a, b))$.

Choose ϵ such that $0 < \epsilon < \frac{\kappa}{2}$. Suppose that $(u, v) \in B_\epsilon(a + 2, b + 2)$. It follows that $(u, v) \in B_{\frac{\kappa}{2}}(a + 2, b + 2)$. Since $\{c_n\}$ converges to 2 horocyclically, $\mathrm{Im}\left(\dfrac{2\pi i}{\lambda(c_n)}\right) \longrightarrow \infty$, therefore, there exists $N > 0$ such that if $n \geq N$, $\mathrm{Im}\left(\dfrac{2\pi i}{\lambda(c_n)}\right) > 2I$. It follows that the point $\left(u, v, \dfrac{2\pi i}{\lambda(c_n)}\right)$ lies in the set $N_{\frac{\kappa}{2},2I}(a+2, b+2)$ by definition. Since $N_{\frac{\kappa}{2},2I}(a+2, b+2) \subset \chi(N_{\kappa,I}(a, b))$, there exists a point $(a', b', d) \in N_{\kappa,I}(a, b) \subset U \cap \mathcal{B}$ such that $\left(u, v, \dfrac{2\pi i}{\lambda(c_n)}\right) = \chi(a', b', d)$. It then follows that $(u, v, c_n) = t(\Phi(a', b', d))$. Since $\Phi(a', b', d)$ is a representation in $V \cap \mathrm{MP}(C, T) \subset \mathrm{MP}(C, T) \subset \mathrm{AH}(C, T)$, we must have $(u, v, c_n) \in \mathcal{T} = t(\mathrm{AH}(C, T))$ by definition. Therefore, $(u, v) \in \mathcal{L}(c_n)$.

We just proved that every point in the ball $B_\epsilon(a + 2, b + 2)$ also lies in $\mathcal{L}(c_n)$. This concludes the proof. ∎

We are now in a position to prove our main theorem, the Slice Convergence Theorem 7.1

Proof of the Slice Convergence Theorem 7.1. To establish that $\mathcal{L}(c_n)$ converges to $\mathcal{L}(2)$ in the sense of Hausdorff, there are two things to prove:

(i) Given $(a, b) \in \mathcal{L}(2)$, there exists a sequence $\{(a_n, b_n)\} \in \mathcal{L}(c_n)$ such that $\{(a_n, b_n)\}$ converges to (a, b) (as a sequence in $\mathbb{C}^2$).

(ii) If $(a_{n_j}, b_{n_j}) \in \mathcal{L}(c_{n_j})$ and the sequence $\{(a_{n_j}, b_{n_j})\}$ converges to (a, b) as a sequence in $\mathbb{C}^2$, then $(a, b) \in \mathcal{L}(2)$.

Proof of (i):

Let $(a, b) \in \mathcal{L}(2)$. There exists a sequence $\{(a_{m_j}, b_{m_j})\} \subset \mathcal{L}(2)$ such that $\{(a_{m_j}, b_{m_j})\}$ converges to (a, b) since $\mathcal{L}(2)$ is a closed subspace of $\mathbb{C}^2$. Apply Lemma 7.8 to the point $(a - 2, b - 2) \in \mathcal{M}$, we can choose $\epsilon > 0$ such that the ball $B_\epsilon(a, b)$ is contained in $\mathcal{L}(c_n)$ for sufficiently large n. Now, since $\{(a_{m_j}, b_{m_j})\} \longrightarrow (a, b)$, $(a_{m_j}, b_{m_j}) \in B_\epsilon(a, b)$ for m_j large enough. Thus, for every m_j sufficiently large, we can choose M_j such that $(a_{m_j}, b_{m_j}) \in \mathcal{L}(c_n)$

for all $n \geq M_j$. Choose the sequence $\{M_j\}$ to be increasing. Put $(a_n, b_n) = (a_{m_j}, b_{m_j})$ for every n with $M_j \leq n \leq M_{j+1}$. The sequence $\{(a_n, b_n)\}$ so constructed converges to (a, b). Moreover, $(a_n, b_n) \in \mathcal{L}(c_n)$ for sufficiently large n.

Proof of (ii):

Suppose that $(a_{n_j}, b_{n_j}) \in \mathcal{L}(c_{n_j})$ and the sequence $\{(a_{n_j}, b_{n_j})\}$ converges to (a, b). Since $(a_{n_j}, b_{n_j}) \in \mathcal{L}(c_{n_j})$, there exists a discrete and faithful representation $\rho_{a_{n_j}, b_{n_j}}$ such that $\mathrm{tr}(\rho_{a_{n_j}, b_{n_j}}(\alpha\gamma)) = a_{n_j}$, $\mathrm{tr}(\rho_{a_{n_j}, b_{n_j}}(\beta\gamma)) = b_{n_j}$ and $\mathrm{tr}(\rho_{a_{n_j}, b_{n_j}}(\gamma)) = c_{n_j}$.

Conjugate $\rho_{a_{n_j}, b_{n_j}}$ so that it has the form

$$\rho_{a_{n_j}, b_{n_j}}(\alpha) = \begin{pmatrix} 1 & a_{n_j} - c_{n_j} \\ 0 & 1 \end{pmatrix}, \rho_{a_{n_j}, b_{n_j}}(\beta) = \begin{pmatrix} 1 & b_{n_j} - c_{n_j} \\ 0 & 1 \end{pmatrix}, \rho_{a_{n_j}, b_{n_j}}(\gamma) = \begin{pmatrix} c_{n_j} & -1 \\ 1 & 0 \end{pmatrix}.$$

Let $\Gamma_0 = \rho_{a_{n_1}, b_{n_1}}(\pi_1(C))$ and for each $n_j \neq n_1$, let $\Gamma_j = \rho_{a_{n_j}, b_{n_j}}(\pi_1(C))$. Then Γ_0 and $\Gamma_j, j \geq 1$ are non-elementary Kleinian groups and $\Gamma_j \cong \Gamma_0 (\cong \pi_1(C))$ for each $j \geq 1$. Moreover, the maps

$$\rho_j = \rho_{a_{n_j}, b_{n_j}} \circ \rho^{-1}_{a_{n_1}, b_{n_1}} : \Gamma_0 \longrightarrow \Gamma_j$$

are isomorphisms.

Now, since $\{(a_{n_j}, b_{n_j})\} \longrightarrow (a, b)$ and $\{c_{n_j}\} \longrightarrow 2$, the representations $\rho_{a_{n_j}, b_{n_j}}$ converges to the representation σ of $\pi_1(C)$ with

$$\sigma(\alpha) = \begin{pmatrix} 1 & a-2 \\ 0 & 1 \end{pmatrix}, \sigma(\beta) = \begin{pmatrix} 1 & b-2 \\ 0 & 1 \end{pmatrix}, \sigma(\gamma) = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}.$$

Let $\rho = \sigma \circ \rho^{-1}_{a_{n_1}, b_{n_1}}$. Then for each $g \in \pi_1(C)$, $\rho_j(g) \longrightarrow \rho(g)$. Let

$$\Gamma = \{\rho(g) : g \in \pi_1(C)\},$$

then Jorgensen's Theorem (see Theorem 4.1.2 of [19]) says that Γ is a non-elementary Kleinian group and ρ is an isomorphism. This implies that the representation σ must be discrete and faithful. Since $\sigma(\alpha\gamma) = a$, $\sigma(\beta\gamma) = b$ and $\sigma(\gamma) = 2$, we must have $(a, b) \in \mathcal{L}(2)$ by definition. $\blacksquare$

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