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Busemann G-Spaces, CAT(k) Curvature, and the Disjoint $(0, n)$-Cells Property

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ABSTRACT

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A review of geodesics and Busemann G-spaces is given. Aleksandrov curvature and the disjoint $(0,n)$-cells property are defined. We show how these properties are applied to and strengthened in Busemann G-spaces. We examine the relationship between manifolds and Busemann G-spaces and prove that all Riemannian manifolds are Busemann G-spaces, though not all metric manifolds are Busemann G-spaces. We show how Busemann G-spaces that also have bounded Aleksandrov curvature admit local closest-point projections to geodesic segments. Finally, we expound local properties of Busemann G-spaces and define a new property which we call the symmetric property. We show that Busemann G-spaces which have the disjoint $(0,n)$-cells property for every value of n cannot have the symmetric property.

Keywords: Busemann, geodesic, disjoint $(0,n)$-cells, Aleksandrov curvature

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Chapter 1

Introduction

A manifold is defined in purely toplogical terms is a Hausdorff and second-countable topological space in which every point is contained in a neighborhood homeomorphic to $\mathbb{R}^n$ for some integer n. We may later impose a metric on the manifold. But that metric must respect the topology. Having a metric is favorable because it introduces a geometric structure on the manifold. With this geometry, we get geodesics, curvature, and other tools useful for providing a geometric description of our manifold.

Because the topology of the manifold is introduced first, we are limited in what metrics we can impose; the metric must respect the topology. What if we choose the metric first? We cannot, of course, always be sure that a metric space will have a manifold topology, but are there certain properties which if possessed by a metric space guarantee that it is a manifold? If so, we have another method for constructing a manifold which is perhaps more natural and intuitive. This is the question which Herbert Busemann attempted to answer, and the focus of this thesis.

Busemann laid out certain axioms which, if satisfied by a metric space, guarantee that the metric space has some of the same properties as manifolds. Such a space has become known as a *Busemann G-space*. In this thesis, we will study the relationship between Busemann G-spaces and manifolds.

As we shall see, it is not currently known if all Busemann G-spaces are manifolds. We shall prove a partial converse: every smooth manifold may be given a metric making it a Busemann G-space. This is an important fact to establish because it shows that Busemann G-spaces are a diverse and interesting class of metric spaces.

We will further support the relationship between manifolds and Busemann G-spaces by examining the role of curvature in Busemann G-spaces. We will recall a definition of curvature used in the absence of a smooth structure, and show how a Busemann G-space with bounded curvature must have certain manifold-like properties.

Finally, we will explore the issue of dimension. While there are examples of Busemann G-spaces of each finite dimension, there are no known examples of infinite dimensional Busemann G-spaces. We will examine a property of metric spaces known as the disjoint $(0,n)$-cells property which is closely related to the property of dimension. Infinite dimensional spaces analogous to manifolds have the disjoint $(0,n)$-cells property for every value of n. We will then define a new property possessed by some Busemann G-spaces which we will call the symmetric property. We will show that a Busemann G-space which has the disjoint $(0,n)$-cells property for every value of n cannot have the symmetric property. While a proof that there are no infinite dimensional Busemann G-spaces eludes us, this is a step toward that conclusion.

1.1 Paths and Geodesics in a Metric Space

In a topological space X, a continuous function $\gamma : [a,b] \to X$ is a *path* from $\gamma(a)$ to $\gamma(b)$. The following definitions demonstrate how we can calculate the length of a path whose codomain is a metric space:

Definition 1.1.1. A *partition of the interval* $[a,b]$ is any finite set $\{t_0, \cdots, t_n\} \subset [a,b]$ such that $a = t_0 < t_1 < t_2 < \cdots < t_n = b$. We will denote the set of all partitions of the interval by $\mathscr{P}([a,b])$.

Definition 1.1.2. Let (X,d) be a metric space, and let $\gamma : [a,b] \to X$. Let

$$\ell = \sup_{\{t_0,\cdots,t_n\}\in\mathscr{P}([a,b])} \sum_{k=1}^{n} d(\gamma(t_{k-1}),\gamma(t_k)).$$

If ℓ is finite, we say that γ is a *rectifiable* path, and ℓ is its length. If ℓ is infinite, we say that γ is *non-rectifiable*.

In this thesis, we will deal only with rectifiable paths. Therefore, unless otherwise stated, we will implicitly assume that every path we work with is rectifiable. The length of a rectifiable path γ is $\ell(\gamma)$.

A geodesic is a generalization of a straight line from Euclidean space in spaces where the notion of “straight” is not immediately and intuitively clear. Chief among the properties of straight lines is that a straight line segment is the shortest path between its endpoints. A geodesic path is also a shortest path between its endpoints. The definition we will use, which is equivalent to Busemann’s definition, is somewhat stronger.

Definition 1.1.3. Let (X,d) be a metric space. A geodesic path is an isometry of a closed interval of the real line into X.

Let (X,d) be a metric space. If $\gamma : [a,b] \to X$ is a geodesic path, and $\{t_0,\cdots,t_n\}$ is a partition of $[a,b]$, then

$$\sum_{k=1}^{n} d(\gamma(t_{k-1}),\gamma(t_k)) = \sum_{k=1}^{n} t_k - t_{k-1} = t_n - t_0 = b-a = d(\gamma(a),\gamma(b)).$$

Thus, $\ell(\gamma) = d(\gamma(a),\gamma(b))$. Since any path is at least as long as the distance between its endpoints, a geodesic path from $\gamma(a)$ to $\gamma(b)$ is a shortest path from $\gamma(a)$ to $\gamma(b)$.

Metric spaces do not always admit geodesic paths between each pair of points. A simple example is $\mathbb{R}^2 \setminus \{(0,0)\}$. There are a multitude of paths from $(-1,0)$ to $(1,0)$, but none of them have length exactly equal to 2. Even if there are shortest paths, there are not necessarily geodesic paths. Consider the unit circle in $\mathbb{R}^2$ with the induced metric. Since it inherits its metric from

$\mathbb{R}^2$, the shortest path within the circle between two antipodal points has length π, but the distance between antipodal points is 2, so the shortest path is not an isometry, and not a geodesic path. Is there a metric we can put on the circle so that the shortest paths, if they exist, are geodesic paths? The answer is yes. Such a metric is called the inner metric.

Definition 1.1.4. Let (X,d) be a metric space in which every pair of points can be joined by a rectifiable path. For $x,y \in X$, let $P_{x,y}$ be the set of all rectifiable paths from x to y. The *inner metric* $d'(\cdot,\cdot)$ on X is defined as

$$d'(x,y) = \inf\{\ell(\gamma) : \gamma \in P_{x,y}\}.$$

It is a simple exercise to show that the inner metric is in fact a metric. We will use this metric in several examples.

If $x,y \in X$ the domain of a geodesic path joining x and y, if it exists, must be an interval of length $d(x,y)$, such as $[0,d(x,y)]$. This is cumbersome to write, and in many cases throughout this thesis, we would prefer a function the domain $[0,1] = I$. If $\gamma : [0,d(x,y)] \to X$ is a geodesic path, then we define the *scaled geodesic path* $\tilde{\gamma} : [0,1] \to X$ by $\tilde{\gamma}(t) = \gamma(t \cdot d(x,y))$.

Just as a line is made up of line segments, a geodesic, which we are about to define, is made up of geodesic paths.

Definition 1.1.5. Let (X,d) be a metric space. A geodesic is a continuous map $\gamma : \mathbb{R} \to X$ such that for each $t \in \mathbb{R}$, t is contained in the interior of some closed interval $[a,b]$ such that $\gamma|_{[a,b]}$ is a geodesic path. In other words, γ is a local isometry.

It should be noted that there is some disagreement in the literature as to whether the term "geodesic" ought to refer to a function or the image of that function. As indicated by their definitions, we will use the terms "geodesic path" and "geodesic" to refer to functions, and use the terms "geodesic segment" and "geodesic line" to refer to their respective images.

1.2 Busemann G-Spaces

Busemann discovered a set of properties of metric spaces which, if held, result in many interesting properties which are generally attributed to manifolds. These properties became the axioms of the metric spaces which are now known as Busemann G-spaces [1].

Definition 1.2.1. A *Busemann G-space* is a metric space (X,d) with the following properties:

1. (Finite compactness) Every bounded infinite subset of X has an accumulation point.

2. (Menger convexity) For every pair of distinct points $x,z \in X$ there is a another point y distinct from x and z such that $d(x,z) = d(x,y) + d(y,z)$.

3. (Local extension) For each $p \in X$, there exists $\varepsilon_p > 0$ so that if $x,y \in B_{\varepsilon_p}(p)$ are distinct points, then there exists another point z distinct from x and y so that $d(x,z) = d(x,y) + d(y,z)$.

4. (Uniqueness of extension) If $x,y,z_1,z_2 \in X$ such that $d(x,z_i) = d(x,y) + d(y,z_i)$ for $i = 1,2$ and $d(y,z_1) = d(y,z_2)$, then $z_1 = z_2$.

Busemann presented these axioms in his 1955 book *The Geometry of Geodesics*. Of course, as with every newly defined classification of spaces, it behooves us to show that the class of Busemann G-spaces is non empty, and indeed contains some interesting examples. We will prove in Chapter 2 that all closed, connected smooth manifolds are Busemann G-spaces.

The next theorem lays out several key properties of Busemann G-spaces which we will find useful.

Theorem 1.2.2. *Let (X,d) be a Busemann G-space.*

1. *Every two points of X are joined by a geodesic segment.*

2. *If $p \in X$ and $y,z \in B_{\varepsilon_p}(p)$ (where ε_p is given in Definition 1.2.1 (3)), then there is a unique geodesic path joining y and z.*

3. *If $\gamma : [a,b] \to X$ is a geodesic path, then there exists a unique geodesic $\hat{\gamma} : \mathbb{R} \to X$ so that $\hat{\gamma}|_{[a,b]} = \gamma$.*

4. *For every $x, y \in X$, there is a homeomorphism $h : X \to X$ so that $h(x) = y$. If $d(x,y) < \delta \leq \varepsilon_x/4$, then h can be chosen so that it is the identity outside of $B_{3\delta}(x)$.*

5. *Let $p \in X$ and let $\gamma : \mathbb{R} \to X$ be a geodesic. If the image of $\gamma|_{[a,b]}$ is contained in $B_{\varepsilon_p}(p)$, then $\gamma|_{[a,b]}$ is a geodesic path.*

Proof. The proofs of (1) and (3) are contained in Busemann's book [1]. The proof of (4) is contained in Thurston's dissertation [2]. It remains to prove (2) and (5).

2 Existence of a geodesic path is guaranteed by (1), so we will focus on uniqueness. Suppose that $\gamma_1, \gamma_2 : [0, d(y,z)] \to X$ are geodesic paths from y to z. By the axiom of local extension, there is a point $x \in X$ such that

$$d(x,z) = d(x,y) + d(y,z).$$

Choose $t \in [0, d(y,z)]$. Let $z_i = \gamma_i(t)$. Since the geodesic paths are isometries, we have that $d(y, z_i) = t$ and

$$d(z_i, z) = d(y,z) - t.$$

Therefore,

$$d(x,z) = d(x,y) + d(y,z) = d(x,y) + d(y,z_i) + d(z_i,z) \geq d(x,z_i) + d(z_i,z) \geq d(x,z).$$

We conclude that the inequalities are equalities, and so

$$d(x,y) + d(y,z_i) + d(z_i,z) = d(x,z_i) + d(z_i,z).$$

Canceling the $d(z_i,z)$ term, we are left with $d(x,z_i) = d(x,y) + d(y,z_i)$. By the axiom of uniqueness of extension, we have that $\gamma_1(t) = z_1 = z_2 = \gamma_2(t)$. Since t is arbitrary, we conclude that $\gamma_1 = \gamma_2$.

5 Since γ is a geodesic, there is some $r > a$ so that $\gamma|_{[a,r]}$ is a geodesic path. Let

$$c = \sup\{t \in [a,b] : \hat{\gamma}|_{[a,t]} \text{ is a geodesic path}\}.$$

We know that $c \geq r > a$ since $\gamma_{[a,r]}$ is a geodesic path. Let $x = \gamma(a)$ and $y = \gamma(c)$. Suppose that $c < b$. Then y must be contained in $B_{\varepsilon_p}(p)$. By the property of local extension, there exists $z \in X$ so that $d(x,y) + d(y,z) = d(x,z)$. Let $\lambda : [c,e]$ be a geodesic path from y to z where $e = c + d(y,z)$. Define $\psi : [0,e] \to X$ by

$$\psi(t) = \begin{cases} \gamma(t) & a \leq t < c \\ \lambda(t) & c \leq t \leq e \end{cases}.$$

We will show that ψ is an isometry. Suppose $a \leq s \leq t \leq e$. If s and t both fall on the same side of c, then $\psi(s)$ and $\psi(t)$ are both in the image of $\gamma|_{[a,c]}$ or λ, both of which are isometries. Therefore, we will consider the case that $s < c \leq t$. By the triangle inequality,

$$d(\psi(s), \psi(t)) \leq d(\psi(s), y) + d(y, \psi(t)).$$

We claim that these are in fact equal. If not, then

$$\begin{aligned} d(x,z) &\leq d(x, \psi(s)) + d(\psi(s), \psi(t)) + d(\psi(t), z) \\ &< d(x, \psi(s)) + d(\psi(s), y) + d(y, \psi(t)) + d(\psi(t), z) \\ &= d(x,y) + d(y,z), \end{aligned}$$

which is a contradiction. Therefore, we must have that

$$d(\psi(s), \psi(t)) = d(\psi(s), y) + d(y, \psi(t)) = d(x,y) - s + t - d(x,y) = t - s.$$

Hence, ψ is an isometry and so it is a geodesic path. Suppose that z is not in the image of γ. Then ψ extends to a geodesic $\hat{\psi} : \mathbb{R} \to X$ which agrees with γ on $[a,c]$, but which is distinct from γ. This is a contradiction to the uniqueness of extension of geodesic paths. Therefore, z

is in the image of γ. This means that $\gamma_{[a,e]}$ is an isometry. Since $e > c$, this is a contradiction. We therefore conclude that $c \geq b$ and that therefore, $\gamma_{[a,b]}$ is a geodesic path.

□

We will rely on the fact that geodesics paths with endpoints contained in the neighborhood of the form $B_{\varepsilon_p}(p)$ depend continuously on their endpoints.

Proposition 1.2.3. *Let (X,d) be a Busemann G-space, and let $p \in X$. For $x,z \in B_{\varepsilon_p}(p)$, we know there is a unique scaled geodesic path from x to z, $\gamma_{x,z} : [0,1] \to X$. Then $\gamma_{x,z}$ depends continuously on all three entries.*

Proof. Let (x_n) and (z_n) be sequences of points in $B_{\varepsilon_p}(p)$ converging to x and z respectively. Let $(t_n) \subset [0,1]$ converge to t. Let $y_n = \gamma_{x_n,z_n}(t_n)$. We know that for each n,

$$d(x_n,y_n) + d(y_n,z_n) = d(x_n,z_n).$$

Since $(z_n) \subset B_{\varepsilon_p}(p)$, (y_n) is a bounded sequence and by the finite compactness property of Busemann G-spaces, it must have a convergent subsequence. This subsequence converges to a point y, which by continuity of the metric, has the property that

$$d(x,y) + d(y,z) = d(x,z).$$

There exists a unique geodesic path from x to y, and that geodesic path extends to a unique geodesic. By the uniqueness of extension property, that geodesic must pass through z. Therefore, y lies in the image of the scaled geodesic segment $\gamma_{x,y}$.

Next we need to establish that $\gamma_{x,z}(t) = y$. We know by the definition of the scaled geodesic path that

$$d(x,\gamma_{x,z}(t)) = td(x,z).$$

Furthermore,

$$d(x_n,y_n) = t_n d(x_n,z_n),$$

so as $n \to \infty$, we get that $d(x,y) = td(x,z)$. By the same argument, we get that $d(\gamma_{x,z}(t),z) = d(y,z) = (1-t)d(x,z)$. Since y and $\gamma_{x,z}(t)$ all lie on the geodesic segment joining x and z, one of the following is true:

$$d(x,y) + d(y,\gamma_{x,z}(t)) + d(\gamma_{x,z}(t),z) = d(x,z),$$

$$d(x,\gamma_{x,z}(t)) + d(\gamma_{x,z}(t),y) + d(y,z) = d(x,z).$$

In either case, this reduces to

$$d(x,z) = td(x,z) + d(y,\gamma_{x,z}(t)) + (1-t)d(x,z) = d(y,\gamma_{x,z}(t)) + d(x,z).$$

Therefore, $d(y,\gamma_{x,z}(t)) = 0$, and $y = \gamma_{x,z}(t)$. Thus, we have proved that every convergent subsequence of (y_n) converges to $\gamma_{x,z}(t)$.

The last thing to do is to show that every subsequence of (y_n) converges. Suppose that there is a subsequence (y_{n_k}) of (y_n) that does not converge. This means that there is some $\varepsilon > 0$ so that infinitely many of the terms of (y_{n_k}) lie outside of $B_\varepsilon(y)$. These infinitely many points constitute another subsequence, called $(y_{n_{k_m}})$ which is bounded, and is also bounded away from y. Therefore, $(y_{n_{k_m}})$ contains yet *another* subsequence which converges. But every convergent subsequence must converge to y. Since $(y_{n_{k_m}})$ is bounded away from y, none of its subsequences can possibly converge to y. This is a contradiction. So we conclude that every subsequence of $(y_n) = (\gamma_{x_n,z_n}(t_n))$ converges to $\gamma_{x,z}(t)$. So $\gamma_{x,z}$ is continuous in x and z. □

The question of principal importance since Busemann first introduced his geodesic spaces is the question which has since become known as the Busemann conjecture [1]:

Conjecture 1.2.4. *Every n dimensional Busemann G-space is an n dimensional topological manifold.*

This conjecture, if true, gives us a way to describe manifolds from a completely geometric point of view. Busemann himself proved the conjecture for Busemann G-spaces of dimensions $n = 1,2$

in his 1955 book [1]. In 1968, Krakus proved that the conjecture holds for dimension $n = 3$ [3]. Then in 1993, Thurston proved that the conjecture holds in dimension $n = 4$ [2]. Though the conjecture has been shown to hold in some other special cases (which we will discuss later), the question remains open for dimension greater than four.

1.3 Aleksandrov Curvature

Curvature is a central topic in differential geometry [4]. As reproducing results from differential geometry is an aim of the theory of Busemann G-spaces, we ought to have a notion of curvature which depends only on a metric. Such a notion for curvature was introduced in 1948 by Aleksandr Aleksandrov, and applies to length spaces [5]. Length spaces are metric spaces in which every pair of points can be joined by a geodesic path, as is the case in a Busemann G-space. Therefore, Aleksandrov's notion of curvature applies to Busemann G-spaces. The name for this type of curvature is an acronym created from the names of the three mathematicians who first used it–Cartan, Aleksandrov, and Toponogov. Hence, it is called CAT curvature.

Just as curvature in Riemannian manifolds can be described with a real scalar, we can do the same for CAT curvature. For a given real number k, CAT curvature is a property possessed by so-called CAT(k) spaces. In order to define CAT(k) spaces, we need a number of preliminary definitions.

Definition 1.3.1. Suppose (X,d) is a length space. Let $x,y,z \in X$ be distinct. The *geodesic triangle* with endpoints x, y, and z is the union of the images of the three geodesic paths joining respectively x and y, y and z, and z and x. This is denoted Δxyz, or simply Δ when the endpoints are not important.

Each of the three geodesic paths that make up a geodesic triangle is one of its edges. As with a triangle in $\mathbb{R}^2$, we may calculate the perimeter of a geodesic triangle by summing the lengths of each of its three geodesic edges. Aleksandrov's CAT(k) curvature relies on comparing a length space to a Riemannian manifold where curvature is already unambiguously defined. Specifically, we focus on 2-manifolds of constant curvature.

Definition 1.3.2. For all real numbers k, the *comparison space* of curvature k is the unique simply connected Riemannian 2-manifold of constant sectional curvature k. The comparison space inherits a length metric d_k, and is denoted (M_k, d_k).

For $k < 0$, M_k is a hyperbolic plane. If $k = 0$, then M_k is $\mathbb{R}^2$. When $k > 0$, M_k is a 2-sphere.

Definition 1.3.3. Given a real number k, the *diameter* of the comparison space M_k is

$$D_k = \begin{cases} \infty & k \leq 0 \\ \frac{\pi}{\sqrt{k}} & k > 0 \end{cases}.$$

Note that if $k > 0$, then M_k is a sphere. The metric placed on it is the standard Riemannian metric, which is the same as the inner metric induced by the standard embedding of S^2 into $\mathbb{R}^3$ so that the resulting sphere has radius $1/\sqrt{k}$. The distance between two points in the sphere under the inner metric is the length of the shortest path in the sphere where the path length is measured using the standard metric on $\mathbb{R}^3$. Thus, the diameter of the sphere under in Definition 1.3.3 is seen to be the farthest two points can be separated under the inner metric. The more common usage of the term *diameter* refers to the distance between antipodes of the sphere using the metric in $\mathbb{R}^3$–the induced metric–which is distinct from the Riemannian metric, or the inner metric, which we will be using. We compare a length space to a Riemannian manifold by mapping geodesic triangles into the manifold using an expanding map, which we define here.

Definition 1.3.4. Let (X,d) and (Y,ρ) be metric spaces. A continuous function $f : X \to Y$ is said to be an *expanding map* if for all $x, y \in X$,

$$d(x,y) \leq \rho(f(x), f(y)).$$

Definition 1.3.5. For a length space (X,d), a geodesic triangle $\Delta \subset X$ is said to satisfy the *CAT*(k) *inequality* if there exists an expanding map $f : \Delta \to M_k$ which restricts to an isometry on each of the edges of Δ.

Finally, we can characterize the curvature of a length space globally by examining whether the geodesic triangles satisfy the CAT(k) inequality.

Definition 1.3.6. A length space (X,d) is said to be a *CAT*(k) *space* if every geodesic triangle in (X,d) with perimeter less than $2D_k$ satisfies the CAT(k) inequality.

For some results, we need not have curvature globally bounded, but merely that we can bound the curvature locally.

Definition 1.3.7. A length space (X,d) has *locally bounded curvature* if every point in X is contained in a geodesically convex neighborhood which is CAT(k) space for some value of k.

We have a few results regarding CAT curvature. None of these facts is hard to prove, but see [5] for details.

Proposition 1.3.8. *The following are properties of CAT curvature.*

1. *A length space (X,d) is a CAT(k) space if and only if it is also a CAT(k') space for all $k' \geq k$.*

2. *Suppose (X,d) is a CAT(k) space. If $x,y \in X$ with $d(x,y) < D_k$, then there is a unique geodesic path joining x and y, and this geodesic path depends continuously on its endpoints.*

3. *The space M_k is a CAT(k) space.*

The following result, proved in 2001 by Berestovskii [6], shows the importance of Busemann spaces with bounded Aleksandrov curvature:

Theorem 1.3.9. *Busemann G-spaces of dimension $n \geq 5$ having Aleksandrov curvature bounded above (that is, which are CAT(k) spaces for some value of k) are topological n-manifolds.*

1.4 The Disjoint $(0,n)$-cells Property

When working with manifolds and similar structures, a useful property related to dimension is known as the *disjoint (m,n)-cells property*. We will not define the full disjoint (m,n)-cells property, as we will only need a weaker version of it [7]:

Definition 1.4.1. Let (X,d) be a metric space, and let B^n denote the n-disk for $n \in \mathbb{N} \cup \{0\}$. If for every point $x \in X$, continuous map $f : B^n \to X$, and $\varepsilon > 0$, there exists a point $x' \in X$ and a map $f' : B^n \to X$ such that

- $d(x,x') < \varepsilon$,
- $d(f(y), f'(y)) < \varepsilon$ for all $y \in B^n$, and
- $x' \notin f'(B_n)$

then we say that (X,d) has the *disjoint $(0,n)$-cells* property.

The following proposition is important in establishing the link between dimension and the disjoint $(0,n)$-cells property. It is proved in [8].

Proposition 1.4.2. *Manifolds of dimension k have the disjoint $(0,n)$-cells property for each $n = 0, 1, \cdots, k-1$.*

We will also make use of the following fact, whose proof is a simple exercise.

Proposition 1.4.3. *If X has the disjoint $(0,n)$-cells property, then every open subset of X also has the disjoint $(0,n)$-cells property.*

Proposition 1.4.4. *If (X,d) has the disjoint $(0,n)$-cells property, it also has the disjoint $(0,k)$ cells property for all $0 \leq k \leq n$.*

Proof. If $k < n$, we may view B^k as a subset of B^n. In fact, B^k is actually a retract of B^n, so any map $f : B^k \to X$ extends to a map $F : B^n \to X$. The result follows. $\square$

By the contrapositive, we can also say that if X does not have the disjoint $(0,n)$-cells property, then it does not have the disjoint $(0,k)$-cells property for each $k \geq n$. We therefore have three classes of spaces:

1. Spaces which do not have the disjoint $(0,n)$-cells property for any value of n. Such spaces include discrete metric spaces.

2. Spaces for which there exists an integer n so that the space has the disjoint $(0,k)$-cells property for $k = 0, 1, \cdots, n$, but not the disjoint $(0,k)$-cells property for $k > n$. Such spaces include $(n+1)$ dimensional manifolds.

3. Spaces which have the disjoint $(0,n)$-cells property for every value of n. Such spaces include Hilbert cube manifolds (Hilbert cube manifolds are infinite dimensional spaces analogous to manifolds in the sense that every point therein is contained in a neighborhood homeomorphic to the Hilbert cube).

While the disjoint $(0,n)$-cells property correlates nicely with dimension in sufficiently nice spaces, we should be careful to keep separate the two ideas. For example, consider the Cantor set C. It is completely disconnected, so any image of an n-cell in the Cantor set is a point. But every point of the Cantor set is a limit point of the Cantor set, so we conclude that if $f : B^n \to C$ and $\varepsilon > 0$ then there is a map $g : B^n \to C$ which maps B^n to a point ε-close to the image of f, which is necessarily a point by connectedness. Thus, the Cantor set has the disjoint $(0,n)$-cells property for all n. But the Cantor set is a zero-dimensional space. Furthermore, Halverson and Daverman and Walsh [9, 10] constructed spaces which are four dimensional, but which do not posses the disjoint $(0,2)$-cells property.

In the case of a Busemann G-space possessing the disjoint $(0,n)$-cells property, we can infer a slightly stronger property which we will utilize in the proofs contained in this thesis.

Proposition 1.4.5. *Suppose (X,d) is a Busemann G-space which has the disjoint $(0,n)$-cells property. Then for every $x \in X$, every map $f : B^n \to X$, and every $\varepsilon > 0$, there exists a map $g : B^n \to X$ such that*

- *$x \notin g(B^n)$,*
- *$d(f(y),g(y)) < \varepsilon$ for each $y \in B^n$, and*
- *if $d(f(y),x) > \varepsilon$, then $f(y) = g(y)$.*

Note the two differences between this result and the disjoint $(0,n)$-cells property. The first is subtle–the image of g excludes x itself, not just a point close to x. Next, the perturbation of f need only affect the function in a neighborhood of x.

Proof. Suppose that (X,d) is a Busemann G-space which has the disjoint $(0,n)$-cells property. Choose $x \in X$, let $f : B^n \to X$, and fix $\varepsilon > 0$. Choose $\delta > 0$ so that $3\delta < \varepsilon_x/2$ and $\delta < \varepsilon/7$. By the disjoint $(0,n)$-cells property, there exists a map $f' : B^n \to X$ and a point x' so that

- $d(x,x') < \delta$,
- $d(f(y),f'(y)) < \delta$ for all $y \in B^n$, and
- $x' \notin f'(B_n)$.

If $a,b \in B_{\varepsilon_x}(x)$, then there is a unique scaled geodesic path a to b by Theorem 1.2.2 (2). We know that $\delta < \varepsilon_x$. Therefore, there is a unique scaled geodesic path $\gamma_y : [0,1] \to X$ which joins $f(y)$ and $f'(y)$. By Proposition 1.2.3, $\gamma_y(t)$ is continuous in both y and t. Define the map $f'' : B^n \to X$ as follows:

$$f''(y) = \begin{cases} f(y) & d(x, f(y)) \geq 3\delta \\ \gamma_y\left(3 - \frac{d(f(y),x)}{\delta}\right) & 2\delta \leq d(x, f(y)) < 3\delta \\ f'(y) & d(x, f(y)) < 2\delta. \end{cases}$$

By the pasting lemma, f'' is continuous.

Let $y \in B^n$. We have three cases:

(1) $d(x, f(y)) \geq 3\delta$,

(2) $2\delta \leq d(x, f(y)) < 3\delta$, and

(3) $d(x, f(y)) < 2\delta$.

For each of these three cases, we will prove two things: $d(f(y), f''(y)) < \delta$ and $f''(y) \neq x'$.

Case 1: If $d(x, f(y)) \geq 3\delta$, then $f(y) = f''(y)$, so $d(f(y), f''(y)) = 0 < \delta$ and $d(x, f''(y)) \geq 3\delta$. Since $d(x,x') < \delta$, we conclude that $f''(y) \neq x'$.

Case 2: If $2\delta \leq d(x, f(y)) < 3\delta$, we know that $f''(y)$ lies on the geodesic segment from $f(y)$ to $f''(y)$. Therefore, $d(f(y), f''(y)) \leq d(f(y), f'(y)) < \delta$. Furthermore,

$$d(f''(y),x) \geq d(f(y),x) - d(f''(y), f(y)) > 2\delta - \delta = \delta.$$

Since $d(x,x') < \delta$, $f''(y) \neq x'$.

Case 3: Finally, if $d(x, f(y)) < 2\delta$, then $f''(y) = f'(y)$, so $d(f(y), f''(y)) = d(f(y), f'(y)) < \delta$. Since the image of f' does not contain x', $f''(y) \neq x'$.

By Theorem 1.2.2 (4), there exists a homeomorphism $h : X \to X$ so that $h(x') = x$ and h is the identity outside of $B_{3\delta}(x)$. Note then that for $z \in X$, $d(z, h(z)) < 6\delta$. Define $g = h \circ f'' : B^n \to X$. Since x' is the only point that maps to x under h, $x \notin f''(B^n)$. Let $y \in B^n$. Then

$$d(f(y), g(y)) \leq d(f(y), f''(y)) + d(f''(y), g(y)) < \delta + 6\delta = 7\delta < \varepsilon.$$

□

Chapter 2

Busemann G-spaces and Manifolds

In the first section of this chapter, we will show that the class of Busemann G-spaces contains all closed, connected, smooth manifolds. This is important because it shows that Busemann G-spaces form a nontrivial class of spaces, and gives a rich class of examples of Busemann G-spaces. In the second section, we will demonstrate an important point: not all metrics placed on topological manifolds yield Busemann G-spaces.

2.1 Closed, Connected, Smooth Manifolds

The proof that all closed, connected, smooth manifolds are Busemann G-spaces is rather long, and relies heavily on facts from differential geometry [4] [1]. To simplify the proof, we will first remind the reader of some of these basic facts.

A smooth manifold M of dimension n admits a Riemannian metric, which in turn both guarantees the existence of and provides a method to calculate equations for geodesics in M. This method involves solving the so-called *geodesic equation*, a second order ordinary differential equation. Thus, by the uniqueness of solutions to differential equations, if two geodesics $g, h : \mathbb{R} \to M$ pass

[1] See DoCarmo's book for more information on each of these facts

through the same point, and at this point have the same derivative, then there is some $T \in \mathbb{R}$ so that $g(t) = h(t+T)$.

Each point $p \in M$ has an associated *tangent plane* T_pM which is a copy of $\mathbb{R}^n$. There is a neighborhood U_p of the origin in T_pM so that for each vector $v \in U_p$, there is a unique scaled geodesic path $\gamma_v : [0,1] \to M$ such that $\gamma_v(0) = p$ and $\frac{d\gamma_v(t)}{dt}\Big|_{t=0} = v$. If U_p can be chosen to be all of T_pM for each $p \in M$, then M is called a *geodesically complete manifold*.

The *exponential map* $\exp_p : U_p \to M$ is defined as $\exp_p(v) = \gamma_v(1)$. For each point p, there is an $\varepsilon_p > 0$ such that the $\exp_p$ is injective on $B_{\varepsilon_p}(0)$. Furthermore, observe that the image of $B_{\varepsilon_p}(0)$ under $\exp_p$ is precisely $B_{\varepsilon_p}(p) \subset M$. Finally the *injectivity radius* of M is defined as

$$\varepsilon = \inf_{p \in M} \sup\{\delta : \exp_p \text{ is injective on } B_\delta(0) \subset T_pM\}.$$

If the injectivity radius is positive, this means points separated by no more than ε can be joined by a unique geodesic. In a compact manifold, it can be shown that the injectivity radius is positive.

The exponential map on compact manifolds gives us another way to represent geodesics and geodesic paths. If $\gamma : \mathbb{R} \to M$ is a geodesic on a Riemannian manifold with $\gamma(0) = x$ and $\gamma'(0) = v \in T_xM$, then we have the following equality:

$$\gamma(t) = \exp_x(tv).$$

Note that since a geodesic is a local isometry, v is necessarily a unit vector. Moreover, the length of the path traced out by $\exp_x(tv)$ for $a \leq t \leq b$ is exactly $b-a$.

An important theorem which we will cite is the Hopf-Rinow theorem, which is as follows:

Theorem 2.1.1. *(Hopf-Rinow) Let M be a connected Riemannian manifold and let $p \in M$. The following are equivalent:*

(a) $\exp_p$ *is defined on all of T_pM.*

(b) The closed and bounded sets of M are compact.

(c) M is complete as a metric space.

(d) M is geodesically complete.

(e) There is a sequence of compact subsets $K_n \subset M$, $K_n \subset K_{n+1}$, and $\bigcup_n K_n = M$, such that if $q_n \notin K_n$, then $d(p, q_n) \to \infty$.

Furthermore, each of the above imply that

(f) For any $q \in M$ there exists a geodesic path from p to q.

We now have the foundation we need to prove this section's theorem.

Theorem 2.1.2. *Every closed, connected Riemannian manifold is a Busemann G-space.*

Proof. Let M be a closed, connected manifold with a Riemannian metric which gives rise to a distance metric $d(\cdot,\cdot)$. A closed manifold is compact, so any metric imposed upon it must be complete. The Hopf-Rinow theorem states that any two points in a complete and connected Riemannian manifold are joined by at least one geodesic path. With this in mind, we may confirm each of the axioms for a Busemann G-space.

1. Since a closed Riemannian manifold is a compact metric space, it is also sequentially compact. Therefore, it has the Bolzano-Weierstrass property. Specifically, every sequence therein has a convergent subsequence. Every infinite set, therefore, contains a sequence of distinct points which in turn has a convergent subsequence which converges to a limit point of that set.

2. Choose $x, z \in M$ to be distinct points. They are joined by a geodesic path
$$\gamma : [0, d(x,z)] \to M.$$
Let $y = \gamma(d(x,z)/2)$. Since γ is an isometry, we have
$$d(x,y) + d(y,z) = \left(\frac{1}{2}d(x,z) - 0\right) + \left(d(x,z) - \frac{1}{2}d(x,z)\right) = d(x,z).$$

3. Let $p \in M$. Since M is compact, it has a positive injectivity radius ε. Let $x, y \in B_{\varepsilon/2}(p)$. We know there is a geodesic path from x to y, $\gamma : [0, d(x,y)] \to M$. Again, by the Hopf-Rinow theorem, a manifold which is a complete metric space is also geodesically complete. This means that if $v = \gamma'(0)$, then the geodesic $\exp_x(tv)$ is an extension of γ and is defined for all $t \in \mathbb{R}$. Choose $t_0 \in (d(x,y), \varepsilon)$, and let $z = \exp(t_0 v)$. Then $d(x,z) = t_0 < \varepsilon$, so there is a unique geodesic path $\lambda : [0, t_0] \to M$ joining x and z. Let $w = \lambda'(0)$. Then $z = \exp_x(t_0 w)$. We conclude that since $d(x,y)$ is less than the injectivity radius ε that $w = v$. This means that $\lambda|_{[0,d(x,y)]} = \gamma$, and so the geodesic segment joining x and z contains y. Thus,

$$d(x,y) + d(y,z) = d(x,z).$$

4. Again, since M is compact, it has a positive injectivity radius ε. Suppose that $x, y, z_1, z_2 \in M$ such that $d(x,y) + d(y,z_i) = d(x,z_i)$ for $i = 1,2$ and that $d(y,z_1) = d(y,z_2)$. First, we will show that if there is a unique geodesic path from x to y, then $z_1 = z_2$. A geodesic in a Riemannian manifold is given by the solution to a second-order differential equation (the so-called *geodesic equation*). The uniqueness of solutions to differential equations guarantees that γ extends to a unique geodesic $\hat{\gamma}$. Since $d(x,y) + d(y,z_i) = d(x,z_i)$, we know that y lies on the geodesic path γ_i from x to z_i. Since the geodesic path from x to y is unique, we conclude that γ_i is an extension of γ, and since $\hat{\gamma}$ is also the unique extension of γ to all of $\mathbb{R}$, we conclude that γ_i also extends to $\hat{\gamma}$. Thus, $z_i = \hat{\gamma}(d(x,z_i))$. Finally, since $d(x,z_1) = d(x,z_2)$, we conclude that $z_1 = z_2$.

Case (a) First, suppose that x, y, z_1, and z_2 are all contained in a neighborhood of diameter $\varepsilon/2$. Because $d(x,y) < \varepsilon$, there is a unique geodesic path γ from x to y, so we are done.

Case (b) Now suppose that y, z_1, and z_2 are all contained in a neighborhood U of diameter $\varepsilon/2$. If there is a unique geodesic path joining x and y, we can apply the same argument as above to show that $z_1 = z_2$. So we suppose, by way of contradiction, that there are two

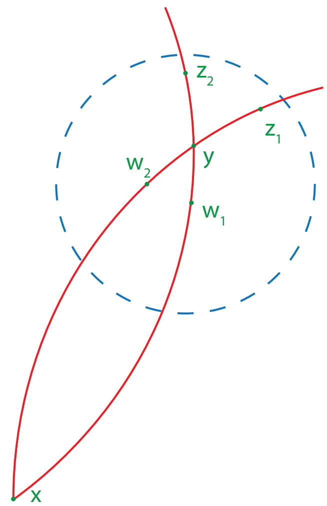

Figure 2.1 The points considered in case 4b in the proof of Theorem 2.1.2

distinct geodesic paths joining x and y. Let $\gamma_i : [0, d(x,y)]$ be a geodesic path from x to y which extends to a geodesic through z_i. Let $\delta > 0$ such that $B_\delta(y) \subset U$. If γ_1 and γ_2 agree on all of $(d(x,y) - \delta, d(x,y)]$, we would conclude that $\gamma_1 = \gamma_2$ by the uniqueness of solutions to differential equations. Otherwise, there is a $t \in (d(x,y) - \delta, d(x,y)]$ such that $\gamma_1(t) \neq \gamma_2(t)$. Let $w_i = \gamma_i(t)$ for $i = 1, 2$. See Figure 2.1 for a diagram. Observe that

$$d(x,z_1) = d(x,y) + d(y,z_1) = d(x,w_i) + d(w_i,y) + d(y,z_1) \geq d(x,w_i) + d(w_i,z_1) \geq d(x,z_1).$$

We conclude that all of the above inequalities are equalities and subtracting $d(x,w_i)$, we see that $d(w_i,y) + d(y,z_1) = d(w_i,z_1)$ for $i = 1, 2$. Finally, $d(w_i,y) = d(x,y) - t$ for $i = 1, 2$. By case (a), we conclude that $w_1 = w_2$, a contradiction. Thus, there is a unique geodesic from x to y, and so we are done.

Case (c) For the final case, we place no restrictions on the given points. Let

$$\gamma_i : [0, d(x,z_i)] \to M$$

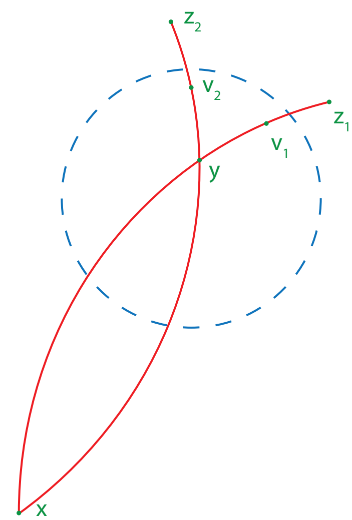

Figure 2.2 The points considered in case 4c in the proof of Theorem 2.1.2

be a geodesic path passing through from x to z_i passing through y for $i = 1,2$. Suppose, to the contrary, that $\gamma_1 \neq \gamma_2$. Then the two do not agree somewhere on the interval $[d(x,y), d(x,y) + \varepsilon/2)$ (again, so as to abide by the uniqueness of solutions to differential equations). We may therefore choose $t \in [d(x,y), d(x,y) + \varepsilon/2)$ such that $\gamma_1(t) \neq \gamma_2(t)$. Let $v_i = \gamma_i(t)$ for $i = 1,2$. See Figure 2.2 for a diagram. Observe that $d(x,y) + d(y,v_i) = d(x,v_i)$ and $d(y,v_i) = t - d(x,y) < \varepsilon/2$ for $i = 1,2$. We therefore use case (b) to conclude that $v_1 = v_2$, a contradiction, and so there is a unique geodesic from x to y.

Thus we see that the Riemannian metric placed on M makes M a Busemann G-space. $\square$

Corollary 2.1.3. *All connected, closed, smooth manifolds admit a metric consistent with a Busemann G-space.*

Proof. Since every smooth manifold admits a Riemannian metric, this fact is proved. $\square$

2.2 The Cone

At this point we must ask ourselves, what metric might we place on a manifold so that it is not a Busemann G-space? Consider the cone. Specifically, consider the set of points

$$C = \{(x,y,z) \in \mathbb{R}^3 : z = \sqrt{3(x^2+y^2)}\},$$

metric given for points $x, y \in C$,

$$d(x,y) = \inf\left\{\ell(\gamma) : \gamma \text{ a path in } C \text{ from } x \text{ to } y \text{ with length measured in } \mathbb{R}^3\right\}.$$

Recall that this is the inner metric on C. This metric cannot be induced by a Riemannian metric because such a metric could not be smooth at the cone point. And indeed, it is at the cone point that we will find problems with the properties of a Busemann G-space.

Let $\varepsilon > 0$, and $y = (0,0,0)$ and choose $x = (a_1, b_1, c_1) \in B_\varepsilon(y)$. Let $z = (a_2, b_2, c_2) \in C$. Let us estimate the distance from x to z. Figure 2.3 shows these points on the cone. Without loss of generality, suppose that z is closer to y than x. There is a path from x to z which follows a radial segment of the cone toward point until its last coordinate is c_2, and then follows the arc of a circle around the cone to z. This arc of a circle need not be any larger than a semicircle. Call the length of the radial segment L. The length of the circular arc is bounded by $\pi\sqrt{a_2^2+b_2^2}$. Therefore,

$$d(x,z) \leq L + \pi\sqrt{a_2^2+b_2^2}.$$

On the other hand, the shortest path between any point in C and y is a straight line, so we can use the Pythagorean theorem to calculate

$$d(x,y) = L + \sqrt{a_2^2+b_2^2+c_2^2} = L + \sqrt{a_2^2+b_2^2+3(a_2^2+b_2^2)} = L + 2\sqrt{a_2^2+b_2^2},$$

and

$$d(y,z) = \sqrt{a_2^2+b_2^2+c_2^2} = \sqrt{a_2^2+b_2^2+3(a_2^2+b_2^2)} = 2\sqrt{a_2^2+b_2^2}.$$

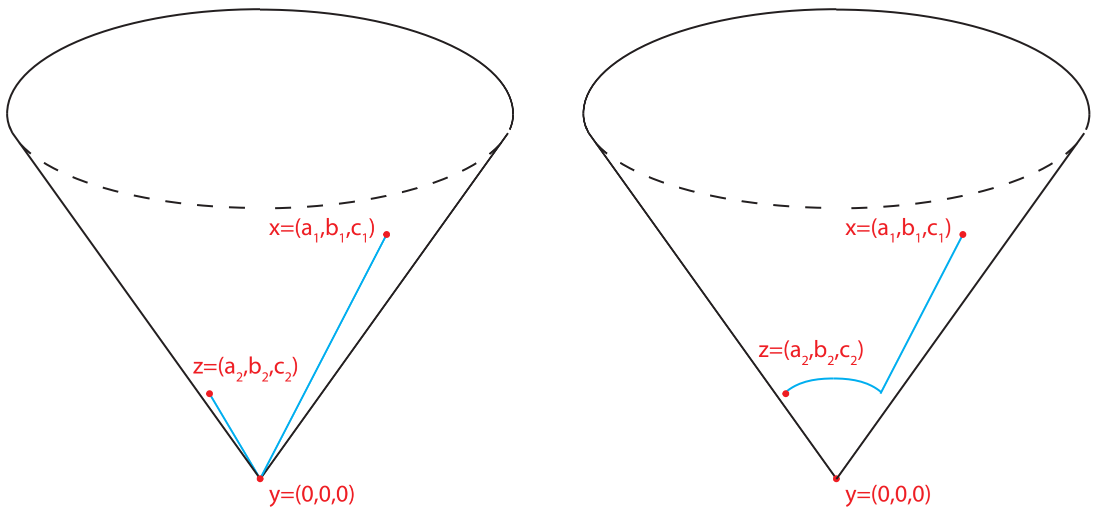

Figure 2.3 On the left, a path from x to y to z. On the right, a shorter path from x to z.

Thus we see that

$$d(x,z) \leq L + \pi\sqrt{a_2^2 + b_2^2} < L + 4\sqrt{a_2^2 + b_2^2} = d(x,y) + d(y,z).$$

This violates local extension property of geodesics about y (Definition 1.2.1 (3)), so the set C with the inner metric is not a Busemann G-space, and not all metric manifolds are Busemann G-spaces.

Chapter 3

Busemann G-Spaces with Bounded Aleksandrov Curvature

In this chapter we will explore some properties of Busemann G-spaces which have bounded CAT curvature. Note, for instance, that for a point x in a Busemann G-space and a geodesic path γ with an image in that space, there is not always a unique closest point to x in the image of γ. For example, the space may be a sphere, with γ realizing the equator, and x the north pole. Then every point in the image of γ is equidistant from x. We will show that if a Busemann G-space is a $\mathrm{CAT}(k)$ space, then there is a neighborhood U of γ in which every point does have a unique closest point in the image of γ.

3.1 A Lemma on Spheres

The focus of this section will be a useful, if rather technical lemma. The lemma is an analogous result in spherical geometry to a simple fact from Euclidean geometry. For a Euclidean isosceles triangle, with its unequal side as the base, its altitude is shorter than the length of the equal sides. Though the Euclidean result follows easily from Euclid's postulates, the lemma we will prove here requires some knowledge of spherical geometry, so we start by presenting a few facts.

First, we will formalize the notion of angles on a sphere. Consider the standard unit sphere S in $\mathbb{R}^3$ and distinct points x, y, and z in S, with each pair joined by a unique geodesic path. At the point x, there is a unique geodesic path γ_y from x to y, and a unique geodesic path γ_z from x to z. The derivatives $\gamma_y'(0)$ and $\gamma_z'(0)$ are elements of the tangent plane of the sphere based at x, so they are vectors. As vectors, it is possible to define an angle between them using the standard dot product and the inverse cosine. We call this angle $\angle yxz$, or equivalently $\angle zxy$.

Next, one will recall that the law of cosines from Euclidean geometry has an analog from spherical geometry [11].

Theorem 3.1.1. *(Law of Cosines for Spheres) Let S be the standard unit 2-sphere in $\mathbb{R}^3$. Consider a triangle in S whose edges are great circles with lengths A, B, and C. Let α be the angle of the triangle opposite the side of length A. Then we have that*

$$\cos(A) = \cos(B)\cos(C) + \sin(B)\sin(C)\cos(\alpha).$$

Note that we are taking sines and cosines of side lengths here. But these sides are actually arcs of circles of radius 1. Therefore, the side length is actually equal to the angle swept out by the arc. The idea also extends to non-unit spheres. If we project a sphere S' of radius $1/r$ radially onto the unit sphere S, then an arc of length A in S' projects to an arc of length $A \cdot r$ in S. All angles between arcs are unchanged by the projection. Thus, we have the following corollary:

Corollary 3.1.2. *Let S' be the standard 2-sphere of radius $1/r$ in $\mathbb{R}^3$. Consider triangle a in S' whose edges are great circles with lengths A, B, and C. Let α be the angle of the triangle opposite the side of length A. Then we have that*

$$\cos(A \cdot r) = \cos(B \cdot r)\cos(C \cdot r) + \sin(B \cdot r)\sin(C \cdot r)\cos(\alpha).$$

We are now ready to state and prove the lemma for this section.

Lemma 3.1.3. *Let $k > 0$ and let X be the sphere $X = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1/k\}$ endowed with the inner metric $d(\cdot,\cdot)$ inherited from $\mathbb{R}^3$. Let Δabc be a geodesic triangle in X such that $d(b,c) = d(a,c)$. Let m be the midpoint of the geodesic segment joining a and b. If $0 < d(b,c)\sqrt{k} < \pi/2$ and $0 < d(a,b)\sqrt{k} < \pi$, then $d(c,m) < d(b,c)$.*

Proof. If $d(c,m) = 0$, the result follows trivially, so we will consider $d(c,m) > 0$. To simplify the notation, we define the following:

$$d(a,b) = C, \quad d(a,c) = B, \quad d(b,c) = A, \quad d(c,m) = D.$$

note that $A = B$, and that the hypotheses give us

$$0 < A\sqrt{k} < \frac{\pi}{2},$$

and

$$0 < C\sqrt{k} < \pi.$$

Figure 3.1 shows a diagram of this geodesic triangle.

By the law of cosines for spheres,

$$\cos(A\sqrt{k}) = \cos(C\sqrt{k}/2)\cos(D\sqrt{k}) + \sin(C\sqrt{k}/2)\sin(D\sqrt{k})\cos(\angle bmc)$$

and

$$\cos(B\sqrt{k}) = \cos(C\sqrt{k}/2)\cos(D\sqrt{k}) + \sin(C\sqrt{k}/2)\sin(D\sqrt{k})\cos(\angle amc).$$

Of course, since $A = B$, $\cos(A\sqrt{k}) = \cos(B\sqrt{k})$. By construction, there is also the angle relation $\angle bmc = \pi - \angle amc$, which implies $\cos(\angle bmc) = -\cos(\angle amc)$. Thus, we can make these substitutions to get

$$\cos(A\sqrt{k}) = \cos(C\sqrt{k}/2)\cos(D\sqrt{k}) + \sin(C\sqrt{k}/2)\sin(D\sqrt{k})\cos(\angle bmc)$$
$$\cos(A\sqrt{k}) = \cos(C\sqrt{k}/2)\cos(D\sqrt{k}) - \sin(C\sqrt{k}/2)\sin(D\sqrt{k})\cos(\angle bmc).$$

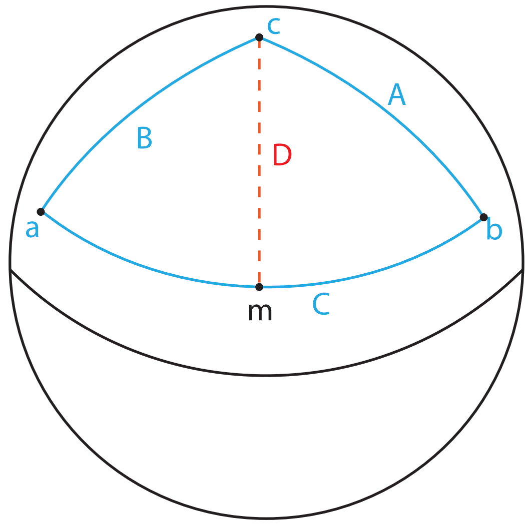

Figure 3.1 The endpoints geodesic triangle in the sphere considered in Lemma 3.1.3

We conclude that

$$\sin(C\sqrt{k}/2)\sin(D\sqrt{k})\cos(\angle bmc) = 0,$$

and thus

$$\cos(A\sqrt{k}) = \cos(C\sqrt{k}/2)\cos(D\sqrt{k}).$$

It follows from the hypothesis that $0 < \cos(C\sqrt{k}/2) < 1$, so $\cos(D\sqrt{k}) > \cos(A\sqrt{k})$. By the triangle inequality,

$$D\sqrt{k} \leq C\sqrt{k}/2 + A\sqrt{k} < \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

Therefore, we have that both $D\sqrt{k}$ and $A\sqrt{k}$ lie in the interval $(0,\pi)$. Since the cosine function is strictly decreasing on the interval $(0,\pi)$, it must be that $D\sqrt{k} < A\sqrt{k}$. It follows immediately that $D < A$, or rather,

$$d(c,m) < d(b,c)$$

as desired. $\square$

3.2 Projection to a Geodesic

The most important idea in this chapter is that in a CAT(k) Busemann G-space X, for every geodesic segment Γ and for every point x sufficiently close to Γ, there is a unique closest point in Γ to x. This paves the way for a natural closest-point projection of a small neighborhood of Γ onto Γ.

Proposition 3.2.1. *Let (X,d) be a Busemann G-space which is also a CAT(k) space. Let $\Gamma \subset X$ be a geodesic segment. Let $x \in X$ with $d(x,\Gamma) < D_k/2$. Then there is a unique point $y \in \Gamma$ such that $d(x,y) = d(x,\Gamma)$.*

Recall that D_k is the diameter of the unique Riemannian 2-manifold with constant curvature k. That is,

$$D_k = \begin{cases} \infty & k \leq 0 \\ \frac{\pi}{\sqrt{k}} & k > 0 \end{cases}.$$

Proof. Define a function $f : \Gamma \to \mathbb{R}$ by $f(z) = d(x,z)$. It is clear that f is continuous by the continuity of the metric. By the Extreme Value Theorem, there exists a point $y \in \Gamma$ which minimizes f. Thus,

$$d(x,y) = \min\{d(x,z) : z \in \Gamma\} = d(x,\Gamma) < D_k/2.$$

Suppose, by way of contradiction, that there exists a point $w \neq y$ such that $d(x,w) = d(x,y)$. Then we may construct an isosceles geodesic triangle Δxyw which has perimeter

$$P = d(x,y) + d(y,w) + d(w,x) \leq d(x,y) + d(y,x) + d(x,w) + d(w,x) < 4(D_k/2) = 2D_k.$$

Since (X,d) is a CAT(k) space, this triangle satisfies the CAT(k) inequality, so there exists an expanding map $h : \Delta xyw \to M_k$ which restricts to an isometry on each of the edges of Δxyw, and whose image is another geodesic triangle $\Delta x'y'w' \subset M_k$. We consider the midpoint z of the geodesic segment from y to w. Note that Γ is a geodesic segment containing both y and w, and that $d(y,w) \leq$

$d(y,x)+d(x,w) < 2(D_k/2) = D_k$. By Proposition 1.3.8 (2), there is a unique geodesic segment joining y and w, so in fact, z must lie in Γ. We know that

$$d(z,x) \geq d(x,y)$$

because y realizes the minimum distance from x to Γ. Also,

$$d(z,x) \leq d_k(z',x')$$

by the expanding nature of the map h, which takes x to x' and z to z'.

Case 1: Suppose $k > 0$. In this case, M_k is the 2-sphere of radius $1/\sqrt{k}$. Since $d(y,x) = d(w,x) < D_k/2$ and $\ell < D_k$, we may apply Lemma 3.1.3 to see that

$$d_k(z',x') < d_k(y',x').$$

Since

$$d_k(y',x') = d(y,x),$$

we conclude that

$$d(z,x) \leq d_k(z',x') < d_k(y',x') = d(y,x).$$

Therefore,

$$d(z,x) < d(y,x).$$

Case 2: If $k \leq 0$, by Proposition 1.3.8 (1), we know that (X,d) is also a CAT$(\hat{k})$ space for all $\hat{k} \geq k$. Choose $\hat{k}$ to be a positive number sufficiently small that $d(y,x) = d(w,x) < D_{\hat{k}}/2$. Then (X,d) is a CAT(k') space, and we can reduce to Case 1.

In both cases we arrive at the contradiction that $d(z,x) < d(y,x)$, so we must conclude that y is the unique point satisfying

$$d(x,y) = d(x,\Gamma).$$

□

Let (X,d) be a CAT(k) Busemann G-space, and let $\Gamma \subset X$ be a geodesic segment. With Proposition 3.2.1 in hand, we recognize the significance of the neighborhood

$$B_{D_k/2}(\Gamma) = \{x \in X : d(x,\Gamma) < D_k/2\}.$$

For we can define a function $p : B_{D_k/2}(\Gamma) \to \Gamma$ by $p(x)$ as the unique closest point in Γ to x. As the following proposition demonstrates, this function is a *closest-point projection*.

Proposition 3.2.2. *Let (X,d) be a CAT(k) Busemann G-space, and let $\Gamma \subset X$ be a geodesic segment. The function $p : B_{D_k/2}(\Gamma) \to \Gamma$ which takes each point in $B_{D_k/2}(\Gamma)$ to the nearest point in Γ is a projection in the sense that*

(i) It is continuous, and

(ii) $p \circ p = p$.

Proof. (i) Suppose that p is not continuous. Then there exists a point $x \in B_{D_k/2}(\Gamma)$ with a sequence $(x_n) \subset U$ converging to x but $(p(x_n))$ does not converge to $p(x)$. Since $(p(x_n))$ does not converge to $p(x)$, there are an infinite number of terms of $(p(x_n))$ lying outside of some open neighborhood G of $p(x)$. These terms constitute a subsequence $(p(x_{n_m}))$. Since the sequence $(p(x_{n_m}))$ lies in the compact set $\Gamma \setminus G$, there exists a convergent subsequence which we will also call $(p(x_{n_m}))$ which converges to some point $y \in \Gamma \setminus G$. By the triangle inequality,

$$d(x,y) \leq d(x,x_{n_m}) + d(x_{n_m},y) \leq d(x,x_{n_m}) + d(x_{n_m},p(x_{n_m})) + d(p(x_{n_m}),y).$$

By the fact that $d(x_{n_m},p(x_{n_m}))$ is the minimal distance from x_{n_m} to any point in Γ, we may also write

$$d(x,y) \leq d(x,x_{n_m}) + d(x_{n_m},p(x)) + d(p(x_{n_m}),y).$$

Since this is true for any point x_{n_m}, we may take a limit as $m \to \infty$ to see that

$$d(x,y) \leq 0 + d(x,p(x)) + 0 = d(x,p(x)).$$

But $p(x)$ is the *unique* point realizing the shortest distance from x to any point in Γ, so this is a contradiction. Thus, $(p(x_n))$ converges to $p(x)$ and p is a continuous function.

(ii) For points $x \in \Gamma$, x itself realizes the distance from x to Γ (as that distance is zero), thus, $p(x) = x$. For any point $y \in B_{D_k/2}(\Gamma)$, $p(y) \in \Gamma$, so $p(p(y)) = p(y)$. Thus, $p \circ p = p$. $\square$

Recall the definition of a retract:

Definition 3.2.3. Let X be a topological space. A subset A of X is a *retract* of X if there is a continuous map $f : X \to A$ such that f restricted to A is the identity. Such a function f is called a *retraction* of X onto A.

Finally, this projection gives way to an interesting result regarding retracts and retractions.

Corollary 3.2.4. *Let (X,d) be a CAT(k) Busemann G-space, and let $\Gamma \subset X$ be a geodesic segment. Then Γ is a retract of $B_{D_k/2}(\Gamma)$. In particular, if $k \leq 0$, Γ is a retract of X.*

Proof. The shortest-point projection $p : B_{D_k/2}(\Gamma) \to \Gamma$ realizes the retraction. If $k \leq 0$, then $B_{D_k/2}(\Gamma) = X$, so we are done. $\square$

Chapter 4

Busemann G-Spaces and the Disjoint $(0,n)$-Cells Property

In Chapter two, we proved that every Riemannian manifold is a Busemann G-space. By Proposition 1.4.2, a manifold of dimension $n+1$ also has the disjoint $(0,n)$-cells property. Therefore, for each $n \in \mathbb{N}$, we have examples of Busemann G-spaces which have the the disjoint $(0,n)$-cells property. What about a Busemann G-space which has the disjoint $(0,n)$-cells property for all $n \in \mathbb{N}$? There are no known examples of such a Busemann G-space, and we conjecture that no such space exists. In this chapter, we will determine what properties must be possessed of Busemann G-spaces with the disjoint $(0,n)$-cells property for all values of n, if any such spaces exist.

4.1 Local Properties of Balls and Spheres

We begin this chapter by examining some structure which exists locally in every Busemann G-space.

Let (X,d) be a Busemann G-space. Let $c \in X$. We have already seen the importance of small closed neighborhoods of c in which, for instance, any two points are joined by a unique geodesic.

In this chapter, we will examine neighborhoods of the form

$$\mathscr{B}_c = \overline{B_{\varepsilon_c/4}(c)} = \{x \in X : d(x,c) \leq \varepsilon_c/4\},$$

and their boundaries which we will denote

$$\mathscr{S}_c = S_{\varepsilon_c/4}(c) = \{x \in X : d(x,c) = \varepsilon_c/4\}.$$

The first local property we will consider involves cone structures.

Definition 4.1.1. Let X be a topological space. Let $P = X \times \{0\} \subset X \times [0,1]$. The *cone* over X, denoted CX is

$$CX = X \times [0,1]/P.$$

Elements in the cone CX are denoted $[x,t]$ where $x \in X$ and $t \in [0,1]$. Note that $[x,0] = [y,0]$ for all $x, y \in X$. This definition leads us to a theorem of Thurston [2].

Theorem 4.1.2. *(Thurston) Let (X,d) be a Busemann G-space, and let $c \in X$. The following hold:*

1. *For every point x of $\mathscr{B}_c \setminus \{c\}$, there is a unique point $s \in \mathscr{S}_c$ so that x lies on the geodesic segment from c to s.*

2. *$\mathscr{B}_c$ is homeormorphic to $C\mathscr{S}_c$.*

3. *Let $s \in \mathscr{S}_c$, and x be the point in $\mathscr{B}_c$ which lies on the geodesic segment joining c with s. The homeomorphism $h : \mathscr{B}_c \to C\mathscr{S}_c$ can be realized by $h(x) = [s, d(c,x)/d(s,x)]$.*

The theorem as stated above is stronger than the statement in Thurston's thesis [2]; it has been augmented with useful intermediate results from the proof of the theorem.

We will examine two functions on sets of the form $\mathscr{B}_c$ and $\mathscr{S}_c$. The first generalizes the concept of antipodal points on a sphere. For a standard n-sphere in $\mathbb{R}^{n+1}$ containing the point x, we can find its antipodal point by extending the line segment from x to the sphere's center until it again intersects the sphere. The second point of intersection is the antipodal point to x. We will construct the antipodal point to $y \in \mathscr{S}_c$ in an analogous way.

Proposition 4.1.3. *Let (X,d) be a Busemann G-space with $c \in X$. If $x \in \mathscr{S}_c$, there is a unique geodesic path $\psi : [0, \varepsilon_c/2] \to \mathscr{B}_c$ such that $\psi(0) = x$ and $\psi(\varepsilon_c/4) = c$. Moreover, $\psi(\varepsilon_c/2) \in \mathscr{S}_c$.*

Proof. We know there exists a unique geodesic path $\gamma : [0, \varepsilon_c/4] \to X$ from x to c, and that γ extends uniquely to a geodesic $\hat{\gamma} : \mathbb{R} \to X$. Define $\psi = \hat{\gamma}|_{[0,\varepsilon_c/2]}$. Since $\hat{\gamma}(\varepsilon_c/4) = c$, we conclude that the image of ψ must be contained in $B_{\varepsilon_c}(c)$. By Theorem 1.2.2 (5), ψ is an isometry, so it is a geodesic path, and $z = \psi(\varepsilon_c/2)$ a distance $\varepsilon_c/4$ from c, so $z \in \mathscr{S}_c$. Moreover, x and z are the only two points in the image of ψ in $\mathscr{S}_c$ because for all $t \in (0, \varepsilon_c/2)$,

$$d(\psi(t), c) = |t - \varepsilon_c/4| < \varepsilon_c/4.$$

We know that $d(x,z) = d(x,c) + d(c,z)$ because x, c, and z all lie on a geodesic segment. Suppose that $\psi' : [0, \varepsilon_c/2] \to X$ is another geodesic path with $\psi'(0) = x$ and $\psi'(\varepsilon_c/4) = c$. Then the uniqueness of extension axiom of Busemann G-spaces (Definition 1.2.1 (4)) tells us that $\psi'(\varepsilon_c/2) = z$. Since $d(x,z) < \varepsilon_c$, there is a unique geodesic path from x to z. Therefore, $= \psi' = \psi$ and ψ is the unique geodesic path whose domain is $[0, \varepsilon_c/2]$ such that $\psi(0) = x$ and $\psi(\varepsilon_c/4) = c$. $\square$

With the result of the above proposition in mind, we can define an antipode in a set of the form $\mathscr{S}_c$.

Definition 4.1.4. Let (X,d) be a Busemann G-space with $c \in X$. Let $x \in S_c$ and let $\psi : [0, \varepsilon_c/2]$ be the unique geodesic path such that $\psi(0) = x$ and $\psi(\varepsilon_c/4) = c$. The *antipode* of x with respect to c is $\psi(\varepsilon_c/2)$.

It is a simple exercise to show that the antipode of the antipode of x is x. We will commonly denote the antipode of x by x'. Indeed, for each c in a Busemann G-space, there exists an antipode map $\alpha : \mathscr{S}_c \to \mathscr{S}_c$ such that $\alpha(x) = x'$. Indeed, the antipode map is continuous, which fact we prove here:

Proposition 4.1.5. *The antipode map α is continuous.*

Proof. Choose $x \in \mathscr{S}_c$. Let $(x_n) \subset \mathscr{S}_c$ be a sequence converging to x. Let $y_n = \alpha(x_n)$. We claim that (y_n) converges to $\alpha(x)$. Suppose, by way of contradiction, that it does not. Then there is a neighborhood U of $\alpha(x)$ whose complement in $\mathscr{S}_c$ contains an infinite number of terms of (y_n), which constitute a subsequence (y_{n_k}). Since the sequence (y_{n_k}) lies in the bounded subset $\mathscr{S}_c$ of a Busemann G-space, it must have a convergent subsequence, which we will also call (y_{n_k}). We will call its limit point y. For each k, we have

$$d(x_{n_k}, c) + d(c, y_{n_k}) = d(x_{n_k}, y_{n_k}).$$

In the limit as $k \to \infty$, we have $d(x,c) + d(c,y) = d(x,y)$. By the uniqueness of extension property of Busemann G-spaces (Definition 1.2.1 (4)), $\alpha(x)$ is the unique point in $\mathscr{S}_c$ with the property that $d(x,c) + d(c,\alpha(x)) = d(x,\alpha(x))$. We conclude that $y = \alpha(x)$. This is a contradiction, so (y_n) converges to $\alpha(x)$. Therefore, α is a continuous function. $\square$

Another local function with which we will concern ourselves is the local spherical projection map. For a Busemann G-space (X,d) with $c \in X$, we define this local radial projection for points $x \in \mathscr{B}_c \setminus \{c\}$. By Theorem 4.1.2, there is a unique point $s \in \mathscr{S}_c$ so that x lies on a geodesic segment from c to s. The local radial projection on $\mathscr{B}_c$ is the map $\pi : \mathscr{B}_c \setminus \{c\} \to \mathscr{S}_c$ defined by $\pi(x) = s$.

Proposition 4.1.6. *The radial projection $\pi(x)$ is continuous*

Proof. Choose $x \in \mathscr{B}_c \setminus \{c\}$. Let $(x_n) \subset \mathscr{B}_c \setminus \{c\}$ be a sequence converging to x. Let $y_n = \pi(x_n)$. We claim that (y_n) converges to $\pi(x)$. Suppose, by way of contradiction, that it does not. Then there is a neighborhood U of $\pi(x)$ whose complement in $\mathscr{S}_c$ contains an infinite number of terms of (y_n), which constitute a subsequence (y_{n_k}). Since the sequence (y_{n_k}) lies in the bounded subset

$\mathscr{S}_c$ of a Busemann G-space, it must have a convergent subsequence, which we will also call (y_{n_k}). We will call its limit point y. For each k, we have

$$d(c, x_{n_k}) + d(x_{n_k}, y_{n_k}) = d(c, y_{n_k}).$$

In the limit as $k \to \infty$, we have $d(c,x) + d(x,y) = d(c,y)$. By the uniqueness of extension property of Busemann G-spaces (Definition 1.2.1 (4)), $\pi(x)$ is the unique point in $\mathscr{S}_c$ with the property that $d(c,x) + d(x,\pi(x)) = d(c,\pi(x))$. We conclude that $y = \pi(x)$. This is a contradiction, so (y_n) converges to $\pi(x)$. Therefore, π is a continuous function. $\square$

In fact, the projection can be extended to all of $B_{\varepsilon_c}(c) \setminus \{c\}$, as we show here:

Proposition 4.1.7. *Let (X,d) be a Busemann G-space with $c \in X$. The function $\Pi : B_{\varepsilon_c}(c) \setminus \{c\} \to \mathscr{S}_c$ which maps each point x to the point on the geodesic segment joining x and c which is a distance $\varepsilon_c/4$ from c is continuous, and agrees with π on $\mathscr{B}_c \setminus \{c\}$.*

Proof. Let

$$M = \{x \in B_{\varepsilon_c}(c) : \varepsilon_c/4 \leq d(x,c)\}.$$

For each $x \in M$, let $\gamma_x : [0, d(c,x)] \to X$ be the unique geodesic path from c to x. Define $\psi : M \to \mathscr{S}_c$ by $\psi(x) = \gamma_x(\varepsilon_c/4)$. The map ψ is continuous by Propostion 1.2.3. Note that the domains of the maps π and ψ intersect on exactly the set

$$\mathscr{S}_c = \{x \in X : d(x,c) = \varepsilon_c/4\},$$

and that on this set, both π and ψ are equal to the identity. Therefore, the map $\Pi : \mathscr{B}_{\varepsilon_c} \setminus \{c\} \to \mathscr{S}_c$ defined by

$$\Pi(x) = \begin{cases} \pi(x) & d(x,c) \leq \varepsilon_c/4 \\ \psi(x) & d(x,c) > \varepsilon_c/4 \end{cases}$$

is contntinuous by the pasting lemma. By construction, Π extends π. $\square$

The maps that we have defined allow us to examine local topological properties of Busemann G-spaces.

Proposition 4.1.8. *Let (X,d) be a Busemann G-space with $c \in X$. If X has the disjoint $(0,1)$-cells property, then $\mathscr{S}_c$ is path connected.*

Proof. If $\mathscr{S}_c$ is empty, then we are done. Otherwise, choose two points x and y in $\mathscr{S}_c$. There is a scaled geodesic path $\gamma : [0,1] \to X$ from x to y in X, which exists because any two points in a Busemann G-space are joined by a geodesic segment. The length of the segment is at most $\varepsilon_c/2$, so we know that the image of γ must be contained in $B_{\varepsilon_c}(c)$. By Proposition 1.4.5, there is a map $\tilde{\gamma} : [0,1] \to X$ such that the image of $\tilde{\gamma}$ does not contain c, $d(\gamma(t), \tilde{\gamma}(t)) < \varepsilon_c/8$ and if $d(c, \gamma(t)) \geq \varepsilon_c/8$, then $\gamma(t) = \tilde{\gamma}(t)$. Thus, $\tilde{\gamma}(0) = \gamma(0) = x$ and $\tilde{\gamma}(1) = \gamma(1) = y$ and for any $t \in [0,1]$, we have either $\tilde{\gamma}(t) = \gamma(t)$ or

$$d(\tilde{\gamma}(t), c) \leq d(\tilde{\gamma}(t), \gamma(t)) + d(\gamma(t), c) < 2(\varepsilon_c/8) = \varepsilon_c/4,$$

so the image of $\tilde{\gamma}$ is in $B_{\varepsilon_c}(c)$. Thus, $\pi \circ \tilde{\gamma}$ is a path in $\mathscr{S}_c$ from x to y. $\square$

Proposition 4.1.9. *Let (X,d) be a Busemann G-space with $c \in X$. The set $\mathscr{B}_c$ is contractible.*

Proof. It suffices to show that B deformation retracts in itself to c. For each $x \in \mathscr{B}_c$, let $\gamma_x : I \to X$ be the scaled geodesic path from x to c. Define $\Gamma : \mathscr{B}_c \times I \to \mathscr{B}_c$ by $\Gamma(x,t) = \gamma_x(t)$. Then Γ is continuous as shown in Proposition 1.2.3. Since $\Gamma(x,0) = x$ and $\Gamma(x,1) = c$, we conclude that B admits a deformation retraction to c, so B is contractible. $\square$

4.2 The Symmetric Property

In this section, we consider a new property we may apply to Busemann G-spaces.

Definition 4.2.1. Let (X,d) be a Busemann G-space. An open set $U \subset X$ has the *symmetric property* if the following properties hold:

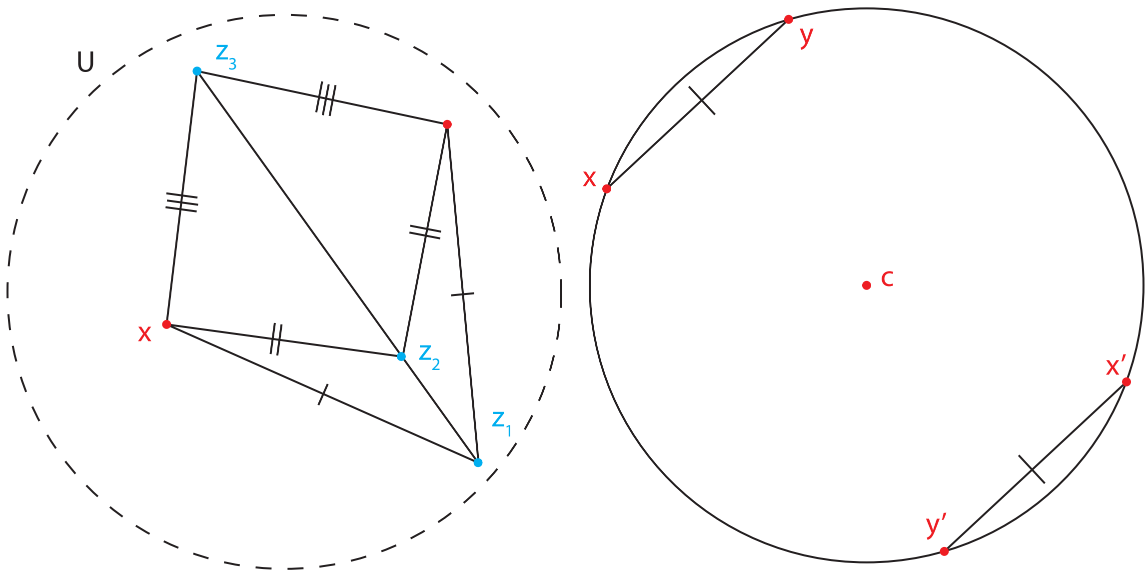

Figure 4.1 The figure on the left demonstrates property (1) of the symmetric property. The figure on the right demonstrates property (2).

1. For every choice of five distinct points x, y, z_1, z_2, and z_3 in U with $d(z_1,z_2)+d(z_2,z_3) = d(z_1,z_3)$ such that $d(x,z_i) = d(y,z_i)$ for two values of $i \in \{1,2,3\}$, then $d(x,z_i) = d(y,z_i)$ also holds for the third value.

2. For each $c \in U$ such that $\mathscr{S}_c \subset U$, if $x, y \in \mathscr{S}_c$, then $d(x,y) = d(x',y')$.

3. Sets of the form $\mathscr{B}_c$ contained in U are geodesically convex. That is, a geodesic segment joining two points in $\mathscr{B}_c$ is itself contained in $\mathscr{B}_c$.

See Figure 4.1 for diagrams.

Definition 4.2.2. A Busemann G-space is said to have *the symmetric property near a point c* if c is contained in a symmetric neighborhood.

Definition 4.2.3. A Busemann G-space has the *local symmetric property* if it has the symmetric property near every point.

The symmetric property is a desirable property for a Busemann G-space because it allows us to predict the behavior of a geodesic segment based on distance relations of just two of its points. For example, if the endpoints of a geodesic segment in a symmetric neighborhood are both equidistant from two points x and y, then every point on the geodesic segment must be the same distance from x as it is from y.

The Busemann G-spaces $\mathbb{R}^n$ and S^n have the local symmetric property.

We have two preliminary results regarding the symmetric property.

Proposition 4.2.4. *Let (X,d) be a Busemann G-space which has a point c such that $\mathscr{S}_c$ is contained in a neighborhood U with the symmetric property. If a point $y \in \mathscr{S}_c$ is equidistant from points x and x' in $\mathscr{S}_c$, then y' is also equidistant from x and x'*

Proof. Since $d(x,y) = d(x',y)$, $d(x,c) = d(x',c) = 1$ and $d(x,c) + d(c,x') = d(x,x')$, the symmetric property gives us $d(x,y') = d(x',y')$. □

The next result concerns the following definition:

Definition 4.2.5. Let (X,d) be a Busemann G-space with $c \in X$. If $z_1, z_2 \in \mathscr{B}_c$, span$(z_1,z_2)$ is the unique geodesic segment joining z_1 and z_2.

If $z_1, \cdots, z_n \in \mathscr{B}_c$, span$(z_1,\cdots,z_n)$ is the union of all geodesic segments joining z_n with points in the span$(z_1,\cdots,z_{n-1})$.

It is important to note that if $(z_{k_1},\cdots,z_{k_n})$ is some reordering of $(z_1,\cdots,z_n)$ with $n > 2$, then span$(z_{k_1},\cdots,z_{k_n})$ may not be equal to span$(z_1,\cdots,z_n)$.

Proposition 4.2.6. *Let (X,d) be a Busemann G-space which has a point c contained in a neighborhood U with the symmetric property. If $x,y,z_1,\cdots,z_n$ are distinct points contained in $\mathscr{B}_c \cap U$ with $d(x,z_i) = d(y,z_i)$ for each $i = 1,\cdots,n$, then any point z in span$(z_1,\cdots,z_n)$ satisfies $d(x,z) = d(y,z)$.*

Proof. We proceed by induction. If $n = 2$, then the result follows immediately from the symmetric property. If the result holds for some integer $m > n$, then z lies on a geodesic segment joining x_n with some point z' in $\text{span}(z_1, \cdots, z_{n-1})$. By inductive hypothesis, $d(x,z') = d(y,z')$ and by the Proposition's hypothesis, $d(x,z_n) = d(y,z_n)$. So by the symmetric property, $d(x,z) = d(y,z)$. $\square$

4.3 The Orthoplex

The proof of this chapter's major theorem involves a simple object from Euclidean geometry whose name is nevertheless a bit obscure. The object is the *n-orthoplex*, which we define using the following definitions:

Definition 4.3.1. Let $A \subset \mathbb{R}^n$. The *convex hull* of A is the intersection of all convex sets containing A.

The convex hull of a set A is the smallest convex set containing A.

Definition 4.3.2. The *n-orthoplex* is the convex hull of all $2n$ points which are given by permutations of the coordinates of the point $(\pm 1, 0, 0, \cdots, 0)$ in $\mathbb{R}^n$. The n-orthoplex is denoted Ω_n.

Other terms for the n-orthoplex include the *n-cross-polytope* and the *n-hyperoctahedron*. The 1-orthoplex is the interval $[-1,1]$ in $\mathbb{R}$, the 2-orthoplex is a square region in $\mathbb{R}^2$ with vertices $\{(-1,0),(0,-1),(1,0),(0,1)\}$. The 3-orthoplex is a regular octahedron in $\mathbb{R}^3$. In general the vertices of an orthoplex are those points in $\mathbb{R}^n$ with one coordinate equal to ± 1 and the rest equal to 0. We will denote the vertex with the kth coordinate equal to 1 as p_k, which has an opposing vertex $-p_k$ where the kth coordinate is equal to -1.

A fact of which we shall make use later on is that if $(t_1, \cdots, t_n)$ lies in the $n+1$-orthoplex and is not equal to $\pm p_n$, it lies on the unique line segment which starts at the point $\pm p_n$, passes through $(t_1, \cdots, t_n)$, and then ends at the point

$$(t_1/(1-|t_n|), t_2/(1-|t_n|), \cdots, t_{n-1}/(1-|t_n|), 0).$$

So as to not have to write this rather complicated form, we define the function $\tau : \Omega_n \setminus \{p_n, -p_n\} \to \Omega_n$ by

$$\tau(t_1, \cdots, t_n) = (t_1/(1-|t_n|), t_2/(1-|t_n|), \cdots, t_{n-1}/(1-|t_n|), 0).$$

By deleting the last coordinate, we obtain a similar map $\hat{\tau} : \Omega_n \setminus \{p_n, -p_n\} \to \Omega_{n-1}$ defined by

$$\hat{\tau}(t_1, \cdots, t_n) = (t_1/(1-|t_n|), t_2/(1-|t_n|), \cdots, t_{n-1}/(1-|t_n|)).$$

Clearly, τ and $\hat{\tau}$ are continuous.

4.4 Essential Families

There is a rich topic known as *separation dimension* from which we will need only a single result [12, 13]. First, recall the definition of a separating set:

Definition 4.4.1. If X is a connected topological space containing nonempty closed sets C_1 and C_2, the set $Y \subset X$ is said to *separate* C_1 and C_2 if $X \setminus Y$ can be written as the union of two disjoint open sets, one containing C_1 and the other containing C_2.

The result has to do with the n-dimensional cube I^n. The n dimensional cube has n pairs of opposing faces which we will call A_k and B_k for $k = 1, \cdots, n$.

Theorem 4.4.2. *If $\{C_k\}_{k=1}^n$ is a family of closed subsets of I^n such that C_k separates A_k and B_k, then*

$$\bigcap_{k=1}^{n} C_k \neq \emptyset.$$

In the language of separating dimension, the set of pairs $\{(A_k, B_k) : k = 1, \cdots, n\}$ is an *essential family* of I^n.

4.5 Busemann G-Spaces with the Disjoint $(0,n)$-Cells Property for Each n

The central theorem of this chapter is a step towards proving that Busemann G-spaces cannot have the disjoint $(0,n)$-cells property for every value of n.

Theorem 4.5.1. *A Busemann G-space which has the disjoint $(0,n)$-cells property for each n cannot have the symmetric property near any point.*

Proof. Let (X,d) be a Busemann G-space with the $(0,n)$-cells property for each n. Suppose, by way of contradiction, that X has the symmetric property near a point c, and that c is contained in a symmetric neighborhood U. Without loss of generality, we may assume that ε_c is sufficiently small that $B_{\varepsilon_c}(c) \subset U$. For convenience, we rescale the metric so that $B = \mathscr{B}_c$ has radius 1. Let $S = \mathscr{S}_c$. We will construct a sequence (x_n) with two properties:

(a) If $n,m \in \mathbb{N}$ with $n \neq m$, then $d(x_n,x_m) = d(x_n,x'_m)$,

(b) If $n,m \in \mathbb{N}$ with $n \neq m$, then $d(x_n,x_m) \geq 1$.

The second property actually follows from the first, for if $d(x_n,x_m) = d(x_n,x'_m)$, then

$$d(x_n,x_m) = \frac{1}{2}(d(x_n,x_m)+d(x_n,x'_m)) \geq \frac{1}{2}d(x_m,x'_m) = 1.$$

From property (b), we will conclude that (x_n) is an infinite set with no accumulation point, a contradiction to the finite compactness property of Busemann G-spaces (Definition 1.2.1).

Let x_1 be any point in S. Let x'_1 be the antipode of x_1. We will construct the sequence inductively.

Suppose that $x_1,\cdots,x_n$ have been chosen. We will define a map $\gamma_n : \Omega_n \to B$, where Ω_n is the n-orthoplex, which will be helpful in choosing x_{n+1}. The construction of γ_n is itself an inductive process. This map will have four specific properties:

1. γ_n is continuous,

2. $\gamma_n(p_k) = x_k$ and $\gamma_n(-p_k) = x'_k$ (Recall that p_k is the vertex in Ω_n with its kth coordinate equal to 1.) ,

3. $\gamma_n(t) = c$ if and only if $t = (0,\cdots,0)$,

4. c is the only point in the image of γ_n that is equidistant from each of the pairs x_i and x'_i, $i = 1,\cdots,n$,

5. the image of γ_n is contained in $B_4(c)$.

We define $\gamma_1 : \Omega_1 = [-1,1] \to X$ by the geodesic path from x'_1 to x_1. Clearly properties 1-4 are satisfied for γ_1.

Suppose, as the inductive hypothesis, that γ_{n-1} has been defined, and that it possesses the required properties. We will now define two families of maps, indexed by subscripts in Ω_{n-1}. Let $g_{(t_1,\cdots,t_{n-1})} : [0,1] \to B$ be the scaled geodesic path from x_n to $\gamma_{n-1}(t_1,\cdots,t_{n-1})$, and let $g'_{(t_1,\cdots,t_{n-1})} : [0,1] \to B$ be the scaled geodesic path from x'_n to $\gamma_{n-1}(t_1,\cdots,t_{n-1})$. Since γ_{n-1} is continuous (by the inductive hypothesis), and by Proposition 1.2.3, each of these maps are continuous, both in their subscripts and in their input variable. Moreover, $g_{(t_1,\cdots,t_{n-1})}(1) = g'_{(t_1,\cdots,t_{n-1})}(1)$ for any $(t_1,\cdots,t_{n-1}) \in \Omega_{n-1}$. Therefore, we can define $\gamma_n : \Omega_n \to B$ by

$$\gamma_n(t_1,\cdots,t_n) = \begin{cases} x'_n & t_n = -1 \\ g'_{\hat{\tau}(t_1,\cdots,t_n)}(1+t_n) & -1 < t_n < 0 \\ g_{\hat{\tau}(t_1,\cdots,t_n)}(1-t_n) & 0 \leq t_n < 1 \\ x_n & t_n = 1. \end{cases}$$

Recall that the map $\hat{\tau} : \Omega_n \setminus \{-p_n, p_n\} \to \Omega_{n-1}$ is defined in section 4.3.

We will check that each of the four desired properties of γ_n are satisfied.

1. By the pasting lemma, γ_n is continuous on

$$\{(t_1,\cdots,t_n : -1 < t_n < 1\}.$$

For the points (vertices) where $t_n = 1$ or $t_n = -1$, the argument for continuity is the same, so without loss of generality, we will suppose that $t_n = 1$. Therefore, we are considering continuity at the point $(0,\cdots,0,1)$. Let $\varepsilon > 0$ and let $\delta < \min\{\varepsilon/2, 1\}$. Choose $(s_1,\cdots,s_n) \in \Omega_n$ such that

$$\|(s_1,\cdots,s_{n-1},s_n) - (0,\cdots,0,1)\|_\infty < \delta.$$

If $(s_1,\cdots,s_n) = (0,\cdots,0,1)$, then

$$d(\gamma_n(s_1,\cdots,s_n),\gamma_n(0,\cdots,0,1)) = 0 < \varepsilon,$$

so we are done. Otherwise, $\gamma_n(s_1,\cdots,s_n)$ lies on the image of a scaled geodesic path starting at x_n and ending at $\gamma_{n-1}(\hat{\tau}(s_1,\cdots,s_n))$, so we can calculate its exact distance from x_n, which is

$$d(\gamma_n(s_1,\cdots,s_n),x_n) = d(x_n,\gamma_{n-1}(\hat{\tau}(s_1,\cdots,s_n)))|s_n - 1|.$$

By inductive hypothesis, $\gamma_{n-1}(\hat{\tau}(s_1,\cdots,s_n)) \in \mathscr{B}_c$. No two points contained in $\mathscr{B}_c$ can be separated by a distance of more than 2. Therefore,

$$d(x_n,\gamma_{n-1}(\hat{\tau}(s_1,\cdots,s_n))) < 2.$$

Furthermore, since

$$\|(s_1,\cdots,s_{n-1},s_n - 1)\|_\infty < \delta$$

, we have that $|s_n - 1| < \delta$. Therefore,

$$d(\gamma_n(s_1,\cdots,s_n),x_n) = d(\gamma_n(s_1,\cdots,s_n),\gamma_n(0,\cdots,0,1)) < 2\delta < \varepsilon.$$

Thus, γ_n is a continuous function.

2. Observe that

$$\gamma_n(0,\cdots,0,1) = g_{(0,\cdots,0)}(0) = x_n$$

and

$$\gamma_n(0,\cdots,0,-1) = g'_{(0,\cdots,0)}(0) = x'_n.$$

Furthermore, if $1 \le k \le n-1$, we denote the projection of p_k into Ω_{n-1} by $\hat{p}_k$ (Since $k < n$, the projection is realized trivially by removing the last coordinate of p_k, which is zero). Then

$$\gamma_n(p_k) = g_{\hat{\tau}(p_k)}(1) = \gamma_{n-1}(\hat{p}_k) = x_k$$

and

$$\gamma_n(-\hat{p}_k) = g_{\hat{\tau}(-p_k)}(1) = \gamma_{n-1}(-p'_k) = x'_k.$$

3. Suppose that $\gamma_n(t_1,\cdots,t_n) = c$. Without loss of generality, $-1 \le t_n \le 0$. Let $\gamma_n(t_1,\cdots,t_{n-1},0) = y$. Then we have that c lies on the geodesic joining x'_n and y. By property (a) of the sequence (x_n), x_n is equidistant from each of the pairs x_i and x'_i for $i = 1,\cdots,n-1$. The point c has the same property since $d(x_i,c) = d(x'_i,c) = 1$ for each i. By the symmetric property, y is also equidistant from each of these pairs. But y is also contained in the image of γ_{n-1} which by property (4) implies that $y = c$. By property (3), this implies that $t_1,\cdots,t_{n-1} = 0$. By the fact that $\gamma_n(0,\cdots,0,t_n)$ sweeps out a geodesic, it can only pass through c for one value of t_n. Since it does so for $t_n = 0$, we conclude that $(t_1,\cdots,t_n)$ is the origin.

   Conversely, if $(t_1,\cdots,t_n) = (0,\cdots,0)$ it is simple to observe that $\gamma_n(t_1,\cdots,t_n) = c$.

4. Let y be in the image of γ_n where $y = \gamma_n(t_1,\cdots,t_n)$. Suppose that y is equidistant from each of the pairs x_i and x'_i for $i = 1,\cdots,n$. Clearly, $y \notin \{x_n, x'_n\}$. We know that y lies on the geodesic segment joining x_n or x'_n with $\gamma_{n-1}(\hat{\tau}(t_1,\cdots,t_n))$. By property (a) of the sequence (x_n), we have that x_n is equidistant from each of the pairs x_i and x'_i for $i = 1,\cdots,n-1$. By Proposition 4.2.4, x'_n is also equidistant from each of the pairs x_i and x'_i for $i = 1,\cdots,n-1$. By the

symmetric property, $\gamma_{n-1}(\hat{\tau}(t_1,\cdots,t_n))$ is also equidistant from each of the pairs x_i and x_i' for $i=1,\cdots,n-1$. By the inductive hypothesis, we conclude that $\gamma_{n-1}(\hat{\tau}(t_1,\cdots,t_n))=c$. Thus, y lies on the geodesic segment joining x_n or x_n' with c, and is equidistant from both x_n and x_n'. The only point satisfying this is c, so $y=c$.

5. This follows simply from the symmetric property, which includes as part of the definition that neighborhoods of the form $\mathscr{B}_c$ contained in symmetric neighborhoods are geodesically convex, meaning that a geodesic segment joining two points in $\mathscr{B}_c$ is itself contained in $\mathscr{B}_c$. Each point in the image of γ_n is contained in a geodesic segment joining x_n or x_n' to a point in the image of γ_{n-1}. Since the image of γ_{n-1} is in $\mathscr{B}_c$ by inductive hypothesis, and since x_n and x_n' are in $\mathscr{B}_c$, the result follows.

Thus, γ_n exists and is a continuous map with the desired properties.

The image of γ_n is a singular n-cell in $B_4(c)$, on which the projection map π may extend (recall that since we have rescaled the metric, $\varepsilon_c=4$). Since $B_4(c)$ has the disjoint $(0,n)$-cells property, there is a map $\tilde{\gamma}_n$ whose image misses c, and which agrees with γ_n outside of some neighborhood of $\gamma_n^{-1}(c)=(0,\cdots,0)$. Since $\tilde{\gamma}_n$ misses c, it can be composed with the projection π to create a map

$$\pi\circ\tilde{\gamma}_n:\Omega_n\to S.$$

The remainder of this proof will focus on identifying a point in the image of $\pi\circ\tilde{\gamma}_n$ which is equidistant from x_k and x_k' for each $k=1,\cdots,n$. To do this, we will examine properties of points in the orthoplex Ω_n which map to points in $\mathscr{S}_c$ under $\pi\circ\tilde{\gamma}_n$ which are equidistant from x_k and x_k' for some particular value of k. To this end, we define another map, and prove a lemma.

Define the maps $r_k:B\to\mathbb{R}$ by $r_k(x)=d(x_k',x)-d(x_k,x)$ for $k=1,\cdots,n$. We will prove the following lemma:

Lemma 4.5.2. *Fix $k\in\{1,\cdots,n\}$, and let $(t_1,\cdots,t_n)$ be a point in the boundary of Ω_n. Then $r_k\circ\pi\circ\tilde{\gamma}_n(t_1,\cdots,t_n)=0$ if and only if $t_k=0$.*

Proof. First, it will be useful to note that since $(t_1,\cdots,t_n)$ is on the boundary of Ω_n, and since γ_n and $\tilde{\gamma}_n$ agree off of a small neighborhood of c, we can assume that the neighborhood we have chosen is sufficiently small that $\gamma_n(t_1,\cdots,t_n)=\tilde{\gamma}_n(t_1,\cdots,t_n)$. Furthermore, $r_k\circ\pi\circ\gamma_n(t_1,\cdots,t_n)=0$ if and only if $\pi\circ\gamma_n(t_1,\cdots,t_n)$ is equidistant from x_k and x'_k. Finally, since $\gamma_n(t_1,\cdots,t_n)$ lies on the geodesic joining c with $\pi\circ\gamma_n(t_1,\cdots,t_n)$, we know that $\pi\circ\gamma_n(t_1,\cdots,t_n)$ is equidistant from x_k and x'_k if and only if $\gamma_n(t_1,\cdots,t_n)$ is equidistant from x_k and x'_k. Therefore, we need only show that $\gamma_n(t_1,\cdots,t_n)$ is equidistant from x_k and x'_k if and only if $t_k=0$.

For the forward direction, suppose that $\gamma_n(t_1,\cdots,t_n)$ is equidistant from x_k and x'_k. Without loss of generality, $t_i\geq 0$ for $i=1,\cdots,n$. We will proceed by induction on n, where $k\in\{1,\cdots,n\}$.

The result holds in the case $n=1$ because the points on the boundary of Ω_1 map to x_1 and x'_1, neither of which is equidistant from x_1 to x'_1.

Now suppose that the result is true for γ_{n-1}. We will consider two cases: $k\neq n$ and $k=n$.

Case 1: If $k\neq n$, then either $\gamma_n(t_1,\cdots,t_n)=x_n$, and the result holds, or $\gamma_n(t_1,\cdots,t_n)$ lies on the geodesic segment joining x_n with $\gamma_n(\tau(t_1,\cdots,t_n))=\gamma_{n-1}(\hat{\tau}(t_1,\cdots,t_n))$. Since x_n is also equidistant from x_k and x'_k by inductive hypothesis, $\gamma_{n-1}(\hat{\tau}(t_1,\cdots,t_n))$ is equidistant from x_k and x'_k by the symmetric property. This means that $r_k\circ\pi\circ\gamma_{n-1}(\tau(t_1,\cdots,t_{n-1}))=0$. By the inductive hypothesis, we conclude that $t_k=0$.

Case 2: Now suppose that $k=n$. If $t_n=1$, we have that $\gamma_n(t_1,\cdots,t_n)=x_n$, which is not equidistant from x_n and x'_n. Therefore, $0\leq t_n<1$. We know that $\gamma_n(t_1,\cdots,t_n)$ lies on the geodesic segment joining x_n with $\gamma_n(\tau(t_1,\cdots,t_n))$. We have assumed that $t_i\geq 0$ for all $i=1,\cdots,n$, so we conclude that $\gamma_n(\tau(t_1,\cdots,t_n))$ is in $\mathrm{span}(c,x_1,\cdots,x_{n-1})$. Since $d(x_i,x_n)=d(x_i,x'_n)$ for each $i=1,\cdots,n-1$, and referring to Proposition 4.2.6, we have that both $\gamma_n(t_1,\cdots,t_n)$ and $\gamma_n(\tau(t_1,\cdots,t_n))$ are equidistant from x_n and x'_n. We know that x_n cannot be equidistant from x_n and x'_n, so we must have that $\gamma_n(t_1,\cdots,t_n)=\gamma_n(\tau(t_1,\cdots,t_n))$. The points $(t_1,\cdots,t_n)$ and $\tau(t_1,\cdots,t_n)$ both lie on a line segment in Ω_n from the point $(0,\cdots,0,1)$ to $\tau(t_1,\cdots,t_n)$. This line segment gets mapped as a scaled geodesic path to the geodesic segment from x_n to $\gamma_n(\tau(t_1,\cdots,t_n))$. The scaled geodesic path must be injective, so we conclude that

$$(t_1,\cdots,t_n) = \tau(t_1,\cdots,t_n) = \left(\frac{t_1}{1-|t_n|},\cdots,\frac{t_{n-1}}{1-|t_n|},0\right),$$

so $t_n = 0$.

Now, for the reverse direction, suppose that $t_k = 0$. Once again, we proceed by induction on n. If $n = 1$, there is no point on the boundary of Ω_1 with $t_1 = 0$, so the result holds.

Now suppose that the result holds for γ_{n-1}. We have two cases: $k \neq n$ and $k = n$.

Case 1: If $k \neq n$, then either $\gamma_n(t_1,\cdots,t_n) = x_n$, and the result holds, or $\gamma_n(t_1,\cdots,t_n)$ lies on the geodesic joining x_n with $\gamma_n(\tau(t_1,\cdots,t_n)) = \gamma_{n-1}(\hat{\tau}(t_1,\cdots,t_n))$. By the inductive hypothesis, $\gamma_{n-1}(\hat{\tau}(t_1,\cdots,t_n))$ is equidistant from x_k and x'_k, as is x_n, Hence, by the symmetric property, $\gamma_n(t_1,\cdots,t_n)$ is equidistant from x_k and x'_k.

Case 2: Finally, suppose $k = n$. We know that $\gamma_n(t_1,\cdots,t_{n-1},0)$ lies on $\text{span}(c,x_1,\cdots,x_{n-1})$, that $x_1,\cdots,x_{n-1}$ are each individually equidistant from x_n and x'_n. Therefore, by Proposition 4.2.6, we have that $\gamma_n(t_1,\cdots,t_{n-1},0)$ is equidistant from x_n and x'_n.

Therefore, we may conclude that $r_k \circ \pi \circ \tilde{\gamma}_n(t_1,\cdots,t_n) = 0$ if and only if $t_k = 0$. □

We have now classified those points in the boundary of Ω_n which map under $\pi \circ \tilde{\gamma}_n$ to points in $\mathscr{S}_c$ equidistant from x_k and x'_k for some value of k. But we need a point in Ω_n for which maps to a point equidistant from x_k and x'_k for all values $k = 1,\cdots,n$. In this last step in our proof, we will identify such a point. To begin the last step in our proof, we will define a homeomorphism from the cube $[-1,1]^n$ to the orthoplex Ω_n. The definition is quite simple:

$$h(x) = \begin{cases} x\frac{\|x\|_\infty}{\|x\|_1} & x \neq (0,\cdots,0) \\ (0,\cdots,0) & x = (0,\cdots,0). \end{cases}$$

Since Ω_n is precisely those points x in $\mathbb{R}_n$ such that $\|x\|_1 \leq 1$, it is easy to see that this map has codomain Ω_n. We will prove that this map is a homeomorphism.

First, we will show that it is injective. Suppose that $h(x) = h(y)$. If $h(x) = h(y) = (0,\cdots,0)$, then it is clear that $x = y = (0,\cdots,0)$. Otherwise, we see that

$$\begin{aligned}\|h(x)\|_1 &= \left\| x\frac{\|x\|_\infty}{\|x\|_1} \right\| \\ &= \frac{\|x\|_\infty}{\|x\|_1}\|x\|_1 \\ &= \|x\|_\infty,\end{aligned}$$

and similarly $\|h(y)\|_1 = \|y\|_\infty$. Therefore, $\|x\|_\infty = \|y\|_\infty$. Hence, noting that $h(x) = h(y)$, we have

$$x\frac{\|x\|_\infty}{\|x\|_1} = y\frac{\|y\|_\infty}{\|y\|_1},$$

or, in other words,

$$x = y\frac{\|y\|_\infty}{\|y\|_1}\frac{\|x\|_1}{\|x\|_\infty}.$$

Thus, x is a positive multiple of y, and the two have the same norm. We conclude that $x = y$.

Next, we will show that h is surjective. Suppose that $y \in \Omega_n$. If $y = (0,\cdots,0)$, observe that $h(0,\cdots,0) = y$. Otherwise, let

$$x = y\frac{\|y\|_1}{\|y\|_\infty}.$$

Since

$$\|x\|_\infty = \left\| y\frac{\|y\|_1}{\|y\|_\infty} \right\|_\infty = \|y\|_1 \leq 1,$$

we see that $x \in [-1,1]^n$. Evaluating h at x we get

$$h(x) = h\left(y\frac{\|y\|_1}{\|y\|_\infty}\right) = y\frac{\|y\|_1}{\|y\|_\infty}\frac{\left\| y\frac{\|y\|_1}{\|y\|_\infty} \right\|_\infty}{\left\| y\frac{\|y\|_1}{\|y\|_\infty} \right\|_1} = y\frac{\|y\|_1}{\|y\|_\infty}\frac{\|y\|_\infty}{\|y\|_1} = y$$

Last, we will show that h is continuous. It is clearly continuous away from zero, so we will only test continuity at the origin. Let $\varepsilon > 0$, and let $\delta < \varepsilon$. Then if $\|x\|_\infty < \delta$, we have that

$$\|h(x) - h(0,\cdots,0)\|_1 = \|h(x)\|_1 = \|x\|_\infty < \delta < \varepsilon.$$

By the topological equivalence of the norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$ on $\mathbb{R}^n$, we conclude that h is continuous. Since h is a continuous bijection on a compact set, its inverse is also continuous. Therefore, h is a homeomorphism.

We now have a method for mapping the cube into $\mathscr{S}_c$ via the orthoplex Ω_n. The mapping is continuous, and we claim that the image of the map contains a point x_{n+1} such that $d(x_k,x_{n+1}) = d(x'_k,x_{n+1})$ and $d(x_k,x_{n+1}) = d(x_k,x'_{n+1})$ for all $k = 1,\cdots,n$. In order to find such a point, we will use the property of the cube stated in Theorem 4.4.2.

Let $A_{n,k} = \{(t_1,\cdots,t_n) \in [-1,1]^n : t_k = 1\}$ and $B_{n,k} = \{(t_1,\cdots,t_n) \in [-1,1]^n : t_k = -1\}$. Theorem 4.4.2 states that the set $\{(A_{n,k},B_{n,k}) : 1 \le k \le n\}$ is an essential family of $[-1,1]^n$. This means that if $\Gamma_{n,k}$ separates $A_{n,k}$ and $B_{n,k}$, then

$$\bigcap_{k=1}^{n} \Gamma_{n,k} \neq \emptyset.$$

Figure 4.2 illustrates our strategy from this point. We will identify subsets of the cube represented by the blue surface in Figure 4.2 which map under $\pi \circ \tilde{\gamma} \circ h$ to points which are equidistant from x_k and x'_k for each $k = 1,\cdots,n$. We will demonstrate that these surfaces separate the faces of the cube containing p_k and p'_k, which are respectively A_k and B_k. Therefore, the intersection over all k of each of these surfaces contains a point which is equidistant from x_k and x'_k for each value of k.

Define $\xi_{n,k} : [-1,1]^n \to \mathbb{R}$ by $\xi_{n,k} = r_k \circ \pi \circ \tilde{\gamma} \circ h$. Let $\Gamma_{n,k} = \xi_{n,k}^{-1}(0)$. Let $p_k \in \mathbb{R}^n$ be the point which is zero in each coordinate except 1 in the kth coordinate. Since p_k and $-p_k$ are fixed by h, we have that $\xi_{n,k}(p_k) = 2$ and $\xi_{n,k}(-p_k) = -2$. Therefore, $[-1,1]^n \setminus \Gamma_{n,k}$ can be written as two disjoint sets, one containing p_k and one containing $-p_k$, or in other words, $\Gamma_{n,k}$ separates $p_k \in A_k$ and $-p_k \in B_k$. Furthermore, if $(t_1,\cdots,t_n) \in \Gamma_{n,k}$ lies in the boundary of $[-1,1]^n$, then $h(t_1,\cdots,t_n)$ lies in the boundary of Ω_n (since homeomorphisms preserve boundary), and has the property that $\gamma_n \circ h(t_1,\cdots,t_n)$ is equidistant from x_k and x'_k. This means that the kth coordinate of $h(t_1,\cdots,t_n)$ is zero. But observe that h preserves the nonzero coordinates. Therefore, $t_k = 0$, and thus $\Gamma_{n,k}$

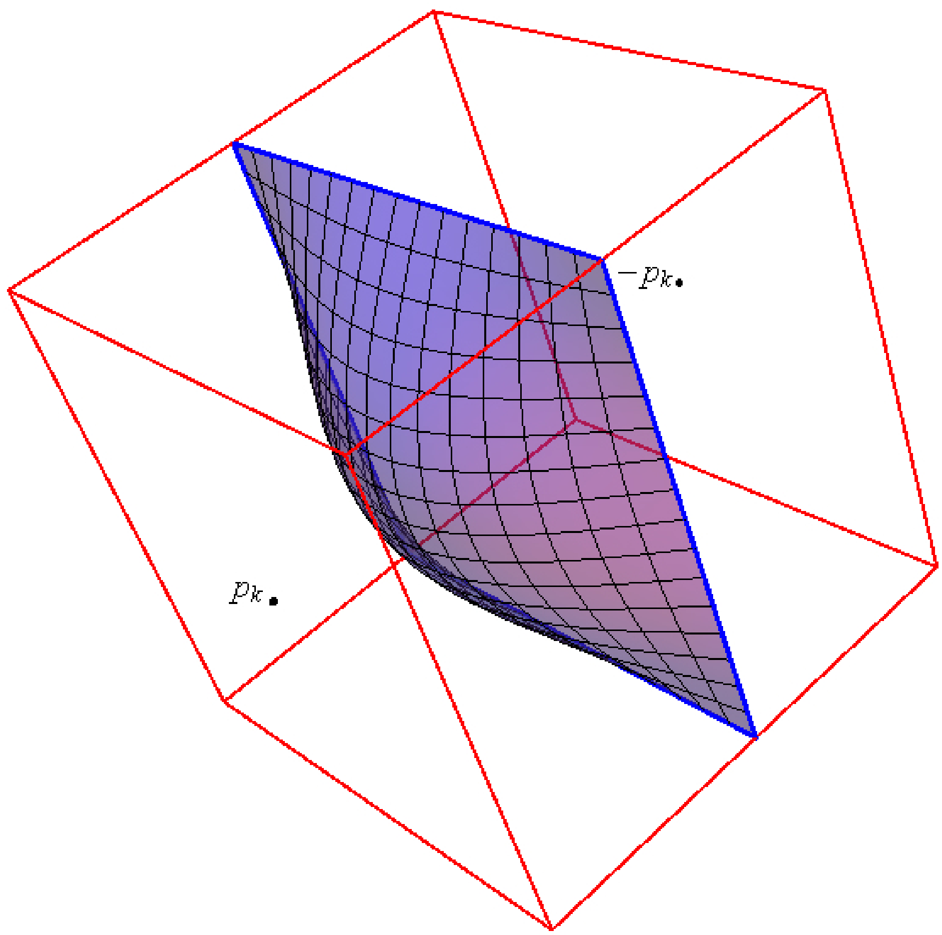

Figure 4.2 The blue surface maps under $\pi \circ \tilde{\gamma} \circ h$ to a set of points which are all equidistant from x_k and x'_k. Notice that the blue surface intersects the boundary of the cube only in the faces which do not contain p_k and $-p_k$. This is important in establishing that the blue surface separates the face containing p_k and the face containing $-p_k$.

intersects the boundary of $[-1,1]^n$ only at points where the kth coordinate is zero. The sets $A_{n,k}$ and $B_{n,k}$ both consist entirely of boundary points where the kth coordinate is not 0. Therefore, $\Gamma_{n,k}$ must separate $A_{n,k}$, and $B_{n,k}$, and the intersection

$$\bigcap_{k=1}^{n} \Gamma_{n,k}$$

contains a point $(s_1, \cdots, s_n)$, and that $\xi_{n,k}(s_1, \cdots, s_n) = 0$ for each $k = 1, \cdots, n$. Therefore, we let $x_{n+1} = \pi \circ \tilde{\gamma}_n \circ h(s_1, \cdots, s_n)$. Then x_{n+1} satisfies

$$d(x_{n+1}, x_i) = d(x_{n+1}, x_i')$$

for each $i = 1, \cdots, n$. By the symmetric property, we also have $d(x_{n+1}, x_i) = d(x_{n+1}', x_i')$, and $d(x_{n+1}, x_i') = d(x_{n+1}', x_i)$ therefore,

$$d(x_i, x_{n+1}) = d(x_i, x_{n+1}').$$

Now that we have constructed our sequence (x_n), where for each n, m where $n \neq m$, the point x_n is equidistant from x_m and x_m', we can observe that

$$d(x_n, x_m) = \frac{1}{2}(d(x_n, x_m) + d(x_n, x_m)) = \frac{1}{2}(d(x_n, x_m) + d(x_n, x_m')) \geq \frac{1}{2} d(x_m, x_m') = 1.$$

This means that there is not limit point for the set of the terms of this sequence, and yet this sequence is infinite. Therefore, X contains an infinite bounded set with no limit point, a contradiction to the finite compactness property of Busemann G-spaces. $\square$

Chapter 5

Conclusion

In this thesis, we focused on three major problems involving Busemann G-spaces. Each problem helped us develop a deeper understanding of the properties and behaviors of Busemann G-spaces.

First, we proved that closed Riemannian manifolds are Busemann G-spaces. This is an important fact to establish because it shows that the class of Busemann G-spaces includes interesting examples. It also supports the idea that there is a connection between Busemann G-spaces and manifolds. What exactly this connection might be is made more precise by the Busemann conjecture–that all Busemann G-spaces are in fact manifolds.

Next, we explored the properties of Busemann G-spaces that also have CAT(k) curvature. We showed that this is a useful property for Busemann G-spaces to have as it results in neighborhoods of geodesic segments on which we may define a closest-point projection.

Finally, we considered how the disjoint $(0,n)$-cells property plays a role in the structure of Busemann G-spaces. First, by laying out several local properties of Busemann G-spaces, and then by showing how those properties together with the $(0,n)$-cells property for each value of n lead to a contradiction. Because the $(0,n)$-cells property is related to dimension, this adds support to the conjecture that there are no infinite dimensional Busemann G-spaces.

Busemann's conjecture that all Busemann G-spaces are manifolds remains open, but we now have a greater knowledge of the properties and behavior of Busemann G-spaces.

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