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Nonlocally Maximal Hyperbolic Sets for Flows

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ABSTRACT

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In 2004, Fisher constructed a map on a 2-disc that admitted a hyperbolic set not contained in any locally maximal hyperbolic set. Furthermore, it was shown that this was an open property, and that it was embeddable into any smooth manifold of dimension greater than one. In the present work we show that analogous results hold for flows. Specifically, on any smooth manifold with dimension greater than or equal to three there exists an open set of flows such that each flow in the open set contains a hyperbolic set that is not contained in a locally maximal one.

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Chapter 1. Introduction

A dynamical system is a way of describing the evolution of a system through time according to a fixed evolution rule. Such systems are ubiquitous in applied mathematics, from weather to fluids, electricity to populations. The main difference in dynamical systems from the study of differential equations is that dynamics concerns itself more with asymptotic tendencies of solutions instead of exactness of solutions. The theory for solvable systems became well-developed and rich, although limited in actual applications. By the end of the 19th century, Poincaré suggested using methods from topology and ergodic theory to find qualitative behavior of solutions instead of the actual solutions themselves [7, p. viii]. Precision is lost but it becomes possible to discuss behaviors of systems previously inaccessible to the classical methods of differential equations. Another equally important question is how resistant the behaviors are to small perturbations made in the evolution law. Because mathematical models are only approximations of reality, stability of the evolution law under small perturbations is an extremely important issue to address whenever one is creating a mathematical model.

By the 1930s, Birkhoff was specifically interested in hyperbolic dynamics. He was curious about the behavior of transverse homoclinic points, a phenomenon first discovered by Poincaré when working on the n-body problem. Poincaré recognized these points as having dynamical complexity, but Birkhoff made this more precise by proving that any transverse homoclinic orbit is accumulated by periodic points [7, p. viii]. When Smale introduced the horseshoe in the early 1960s [19], deeper understanding unfolded. The horseshoe as well as other maps constructed soon after were categorized by Smale's definition of uniformly hyperbolic sets [19, p. 776]: subsets of the phase space invariant under the evolution law such that the tangent space at each point splits into two subspaces – one uniformly contracted under forward iterations, and a complementary one uniformly contracted under backwards iterations.

Uniform hyperbolicity made strides in characterizing structurally stable systems – that is, systems that remained homeomorphic under sufficiently small perturbation. Andronov and Pontryagin introduced the concept in the 1930s of an orbit structure that remained the same even after the evolution law is slightly modified [2]. Uniform hyperbolicity was shown to be important in these systems, combined with a specific transversality condition, as conjectured by Palis and Smale. The broad theory of uniformly hyperbolic dynamical systems was developed mostly from the 1960s until the mid-1980s. It changed the way we look at determinism and randomness, and led to the conclusion that chaos can come from repeated applications of simple rules. While uniform hyperbolicity and structural stability were realized to be not as broadly applicable as originally hoped, and dynamics has gone beyond hyperbolicity by weakening various hypotheses, hyperbolicity still proves to be important in modern dynamics. (Much of the history above is taken from [15, p. 7-11] and [7, p. vii-ix].)

Although the evolution laws are often simple, the behavior of these systems can get extremely complicated. Those who study dynamical systems are often interested in this chaotic behavior. One way to get chaotic behavior is via the relatively simple recipe of hyperbolicity – i.e., iterated stretching and folding. In the map case, hyperbolic sets are compact sets, invariant under the discrete function, with a tangent space that at each point splits into invariant contracting and expanding directions. Uniform hyperbolicity guarantees that the contraction and expansion are globally bounded over the set, thus avoiding rates of either contraction or expansion going to zero as time increases. When dealing with flows, a compact invariant set is called hyperbolic if at each point along the trajectory the tangent space splits into invariant expanding, contracting, and flow (or center) directions. Again, uniform hyperbolicity is when the pointwise expansion and contraction in the tangent space are both globally bounded throughout the tangent bundle by a single constant.

In studying hyperbolic sets, mathematicians discovered that all known examples of hyperbolic sets were included in ones that had a property called local maximality. For M a

compact manifold and ϕ_t a flow, $\Gamma \subset M$ is said to be *locally maximal* if there exists an open neighborhood $N \subset M$ of Γ such that $\Gamma = \bigcap_{t \in \mathbb{R}} \phi_t(N)$. Locally maximal sets have many useful properties, such as local product structure and Markov partitions (see chapter 2 for definitions). At this point, they are well-understood phenomena in the study of dynamics. Because they are so well-understood, a common assumption in the statement of a result was that a hyperbolic set was contained in a locally maximal hyperbolic set in some small neighborhood of the original set. Then properties of locally maximal sets can be used to prove results about the original hyperbolic set. In [17, p. 272], Hasselblatt and Katok ask if given a hyperbolic set Λ and an open neighborhood $\mathcal{U}$ of Λ, if there is necessarily a locally maximal hyperbolic set $\tilde{\Lambda}$ such that $\Lambda \subset \tilde{\Lambda} \subset \mathcal{U}$. This question was originally asked by Alekseev [1] and Anosov [3].

In the 1980s, Fathi constructed a counterexample to the question which remained unpublished [4, p. 937]. The same example was eventually published in 2001, when Crovisier [9] constructed a diffeomorphism of the 4-torus with a hyperbolic set contained in no locally maximal hyperbolic set. However, this example was not proved to be persistent under perturbation. In 2004, Fisher [13] showed that, given any smooth manifold M of dimension greater than 2, there exists a diffeomorphism $f : M \to M$ with the desired property – and furthermore, this property is open in the function space (i.e., the property is persistent under perturbation).

In this paper we extend the result to flows.

Theorem 1.1. *Let M be a smooth compact manifold of dimension greater than or equal to three. Then there exists an open set $\mathcal{U}$ in the class of C^1 flows such that for each $\Phi \in \mathcal{U}$, there exists a hyperbolic set Λ for Φ such that Λ is not contained in a locally maximal hyperbolic set.*

Anosov closely followed these developments as they were published. Once it became known that one could find counterexamples of every positive dimension, Anosov set out to answer the same question for zero-dimensional hyperbolic sets. Consider M a smooth

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manifold, $f : M \to M$ a diffeomorphism, and Λ a zero-dimensional hyperbolic set for f. Anosov proved in [4, p. 938] that in any neighborhood of Λ there is a locally maximal invariant set Λ_1 containing Λ. If Λ_1 cannot be equal to Λ, he calls Λ *locally premaximal.* If Λ_1 can be made equal to Λ, then Λ is of course locally maximal. Anosov used shadowing to prove in a subsequent paper a necessary and sufficient condition for a hyperbolic set to be locally maximal [5, p. 23].

In chapter two we introduce the background necessary for the proof of the theorem. In chapter three we outline two critical constructions from the early-mid 2000s. In chapter four we show how those two constructions give important ideas that can be combined to prove our main theorem.

CHAPTER 2. BACKGROUND

Here we give some basic definitions and facts regarding Riemannian geometry, dynamical systems, hyperbolic sets, and locally maximal hyperbolic sets.

2.1 TOPOLOGY AND GEOMETRY

A *topological manifold* is a second-countable, locally Euclidean Hausdorff space. By definition of locally Euclidean, every point of a topological manifold M has a neighborhood homeomorphic to a subset of $\mathbb{R}^n$. For any Euclidean neighborhood U there exists a homeomorphism $\varphi : U \to \varphi(U) \subset \mathbb{R}^n$, called a *coordinate chart* on U. A set of Euclidean neighborhoods covering M, together with their respective homeomorphisms, is called an *atlas*. Given two charts φ_1 and φ_2 with overlapping neighborhoods U and V, there is a *transition function* $\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U \cap V) \to \varphi_2(U \cap V)$. By necessity each transition function is a homeomorphism between open subsets in $\mathbb{R}^n$. For $1 \leq k \leq \infty$ a C^k *(differentiable) manifold* is a topological manifold with an atlas whose transition maps are all C^k functions. For M a C^k manifold where $k \geq 1$, consider $x \in M$. Pick any chart $\varphi : U \to \mathbb{R}^n$ where $U \subset M$ is open and contains x. Suppose two curves $\gamma_1 : (-1, 1) \to M$ and $\gamma_2 : (-1, 1) \to M$ with $\gamma_1(0) = \gamma_2(0) = x$ are given such that $\varphi \circ \gamma_1$ and $\varphi \circ \gamma_2$ are both differentiable at 0. Then γ_1 and γ_2 are said to be *equivalent at 0* if the derivatives of $\varphi \circ \gamma_1$ and $\varphi \circ \gamma_2$ are identical at 0. This defines an equivalence relation on the set of curves through x, and the equivalence classes are defined to be the *tangent vectors* of M at x. The *tangent space* of M at x, denoted T_xM, is defined to be the set of all tangent vectors. Note that T_xM does not depend on the choice of the chart φ. The *tangent bundle* of M, denoted TM, is defined to be equal to $\bigcup_{x \in M} T_xM$. An element of TM can be thought of as an ordered pair (x, v) where x is a point in M and v is an element of T_xM. A *vector field* is a mapping $F : M \to TM$ so that $\pi \circ F$ is the identity mapping, where $\pi : TM \to M$ is the natural projection $(x, v) \to x$. A *Riemannian manifold* is a C^k manifold equipped with an inner product g_p on the tangent

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space T_pM such that for X and Y vector fields on M, $p \mapsto g_p(X(p), Y(p))$ is a C^k function).

A *Riemannian metric* on a C^k Riemannian manifold M is a family of positive definite inner products $g_p : T_pM \times T_pM \to \mathbb{R}, p \in M$ such that, for all differentiable vector fields X, Y on $M, p \mapsto g_p(X(p), Y(p))$ defines a C^k function $M \to \mathbb{R}$. For Riemannian manifolds M and N, a function $f : M \to N$ is *differentiable at a point p* if it is differentiable with respect to some coordinate charts defined around p and $f(p)$.

Definition 2.1. Let f be a differentiable function between differentiable manifolds M and N.

(a) If $Df_p : T_pM \to T_{f(p)}N$ is an injective function for every $p \in M$, then f is an *immersion.* Equivalently, f is an immersion if rank $Df_p = \dim M$.

(b) If $Df_p : T_pM \to T_{f(p)}N$ is a surjective function for every $p \in M$, then f is a *submersion.* Equivalently, f is a submersion if rank $Df_p = \dim N$.

(c) An *embedding* is defined to be an injective immersion which is a homeomorphism onto its image.

With these definitions in mind, we can define the following.

Definition 2.2. For M a differentiable manifold, an *immersed submanifold* is a subset S of M such that the identity map $i : S \to M$ is an immersion. If i is an embedding, then S is a *(regular) submanifold.*

Note that throughout the rest of the paper, whenever a manifold is mentioned it will be assumed to be compact, connected, and Riemannian, unless specifically stated otherwise.

2.2 Fundamental Dynamics

Definition 2.3. A *dynamical system* is a tuple (T, M, Φ) where T is an additive monoid, M is a set (called the *phase space*), and Φ is a function $\Phi : U \subset (T \times M) \to M$ where

$$\Phi(0, x) = x,$$

$$I(x) = \{t \in T : (t, x) \in U\},$$

$$\Phi(t_2, \Phi(t_1, x)) = \Phi(t_1 + t_2, x) \text{ for } t_1, t_2, t_1 + t_2 \in I(x).$$

Notationally, for any $x \in M$, we write $\Phi_t(x)$ to be the function holding x constant and we call it the *flow through x* and its graph the *trajectory through x.*

A subset P of M is called Φ-invariant if for all $x \in P$ and all $t \in T$ the statement $\Phi(t, x) \in P$ holds. Thus, we require the flow through x to be defined for all time for every element in P.

A *flow* is a dynamical system matching the definition above with T an open interval in $\mathbb{R}$, M a manifold locally diffeomorphic to a Banach space, and Φ a continuous function. If $T = \mathbb{R}$ the system is called *global*. If M is locally diffeomorphic to $\mathbb{R}^n$ the system is *finite-dimensional*; otherwise, the dynamical system is *infinite-dimensional*. We will only work with global, finite-dimensional systems here. Note that we will use the convention ϕ for a flow. Additionally note that throughout this paper, we will only use flows defined throughout all of $\mathbb{R} \times M$. Given a flow $\phi : \mathbb{R} \times M \to M$, we will use the notation $\phi(t, m) := \phi_t(m)$, and unless stated otherwise, t remains a variable.

A *map* is a dynamical system with $T = \mathbb{Z}$ and M a manifold locally diffeomorphic to a Banach space. Consider a flow $\phi : \mathbb{R} \times M \to M$, and fix some $t_0 \in \mathbb{R}$. Then $\phi_{t_0} : M \to M$ is now a map, and ϕ_{t_0} is called the *time-t_0 map* for the flow ϕ. For fixed t, then, ϕ_t is a map, but usually we will work with ϕ_t as a flow with variable t.

For a flow $\phi : \mathbb{R} \times M \to M$ we define the *orbit* of x as follows:

$$\mathcal{O}(x) := \{\phi_t(x) : t \in \mathbb{R}\}.$$

We also define the *forward orbit* of x to be $\mathcal{O}^+(x) := \{\phi_t(x) : t \in \mathbb{R}_{\geq 0}\}$.

For $f : X \to X$ a continuous function on a metric space, define the *ω-limit set* of $x \in X$

to be

$$\omega(x, f) := \{y \in X : \exists \text{ strictly increasing } (n_k) \subset \mathbb{N} \text{ s.t. } f^{n_k}(x) \to y \text{ as } k \to \infty.\}$$

For a homeomorphism f, the α-*limit set* $\alpha(x, f)$ is defined as $\omega(x, f^{-1})$.

Definition 2.4. A flow ϕ is C^1 if $\phi : \mathbb{R} \times M \to M$ is continuously differentiable.

In this paper we are working with vector fields. It is important to note that a C^1 flow generates (at the least) a C^0 vector field, and a locally Lipschitz vector field generates a flow, but a C^0 vector field doesn't necessarily generate a flow (although a C^1 vector field does). We will look at C^1 flows here.

Definition 2.5. Let M be a smooth manifold, with $\phi : \mathbb{R} \times M \to M$ a C^1 flow and $\Gamma \subset M$ a compact ϕ_t-invariant set. The set Γ is said to be a *(uniformly) hyperbolic set for the flow* ϕ_t if there exists a $\mu \in (0, 1)$ and a $C > 0$ such that for all $x \in \Gamma$ there is a decomposition $T_x M = \mathbb{E}^s(x) \oplus \mathbb{E}^u(x) \oplus \mathbb{E}^c(x)$ satisfying all of the following:

- $\frac{d}{dt} \big|_{t=0} \phi_t(x) \in \mathbb{E}^c(x) \setminus \{0\}$

- $\dim \mathbb{E}^c(x) = 1$

- $D\phi_t|_{t=0} \mathbb{E}^\alpha(x) = \mathbb{E}^\alpha(\phi_t(x))$ for $\alpha \in \{s, u\}$

- For $t > 0$ and $v \in \mathbb{E}^s(x)$ we have $\|D\phi_t(x)v\| \le C\mu^t \|v\|$

- For $t > 0$ and $w \in \mathbb{E}^u(x)$ we have $\|D\phi_{-t}(x)w\| \le C\mu^t \|w\|$.

For the above definition it is important to note the following:

- The dimensions of $\mathbb{E}^s(x), \mathbb{E}^u(x)$, and $\mathbb{E}^c(x)$ are locally constant.

- The definition is independent of the choice of the Riemannian metric on M. Furthermore, there always exists some Riemannian metric allowing $C = 1$.

- The splitting of $T_x M$ is continuously dependent on x.

It is also possible to define hyperbolicity for maps instead of flows. We will use this definition later.

Definition 2.6. For M a manifold and $f : M \to M$ a diffeomorphism, we say an f-invariant set Λ is *(uniformly) hyperbolic* if there exist constants $C > 0$ and $\lambda \in (0, 1)$ such that for all $x \in \Lambda$ there is a splitting of the tangent space $T_x M = E^s(x) \oplus E^u(x)$ such that for every $n \in \mathbb{N}$ one has

$$\|Df^n(v)\| \leq C\lambda^n \|v\| \text{ for } v \in E^s(x),$$

$$\|Df^{-n}(v)\| \leq C\lambda^n \|v\| \text{ for } v \in E^u(x).$$

Definition 2.7. Let X be a metric space and ϕ_t a continuous flow on X. Then for $x \in X$, we define the *stable set*

$$W^s(x) := \{y \in X : \lim_{t \to \infty} d(\phi_t(x), \phi_t(y)) = 0\}.$$

Further define, for $\epsilon > 0$,

$$W_\epsilon^s(x) := \{y \in W^s(x) : d(\phi_t(x), \phi_t(y)) \leq \epsilon \text{ for all } t \geq 0\}.$$

Note that the unstable sets $W^u(x)$ and $W_\epsilon^u(x)$ are defined identically under the flow ϕ_{-t}. Furthermore, we define the *center-stable set*

$$W^{cs}(x) := \phi_t(W^s(x))|_{t \in \mathbb{R}} = \bigcup_{\substack{y \in \phi_t(x) \\ t \in \mathbb{R}}} W^s(y).$$

The center-unstable set of x is defined to be the center-stable set of x under ϕ_{-t}. We will also use the notation W_{loc}^s to mean W_ϵ^s for sufficiently small ϵ (dependent on context). We use W_{loc}^u to mean a similar thing for W_ϵ^u.

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In the case where X is a manifold and ϕ_t a C^r flow, the stable set is a C^r submanifold of X. This is due to the proof of the Stable Manifold Theorem found in [17, p. 266-268]. Using the time-one map for the flow (ϕ_k for $k \in \mathbb{Z}$) and adapting Hasselblatt and Katok's intentionally-general proof, we can generalize it to the flow case. Also note that the stable manifold is an embedded copy of $\mathbb{R}^k$ where $k = \dim \mathbb{E}^s(x)$. The same applies for the unstable sets, center-stable sets, and center-unstable sets. Also note that the stable and unstable manifolds vary continuously both on each other and on the relevant point.

2.3 Properties of Hyperbolic Sets

Definition 2.8. For a metric space X and a flow ϕ, a set $\Gamma \subset X$ is said to have a *local product structure* if for all $\epsilon > 0$ there exists a $\delta > 0$ such that given $x, y \in \Gamma$ with $d(x, y) < \delta$ we have, for some real $|t| < \epsilon$, a unique point $\mathcal{S}(x, y) := b \in W_\epsilon^u(\phi_t(x)) \cap W_\epsilon^s(y)$.

A local product structure is critical for this paper although it is most often stated for maps instead of flows.

Definition 2.9. A hyperbolic set Γ has a local product structure if there exists $\delta > 0$ and $\epsilon > 0$ such that for any points $x, y \in \Gamma$ where $d(x, y) < \delta$ the set $W_\epsilon^s(x) \cap W_\epsilon^u(y)$ consists of exactly one point contained in Γ.

The following lemma is also critical to the paper. Note that this lemma is almost always stated and proved for maps, but is in fact true for flows as well (see [8, p. 1862] and [18, p. 131]).

Lemma 2.10. *A hyperbolic set Γ has a local product structure if and only if it is locally maximal.*

In the case that Γ is locally maximal and hyperbolic, then $x, y \in \Gamma$ implies $\mathcal{S}(x, y) \subset \Gamma$.

Definition 2.11. For (ϕ_t, X) a dynamical system, a subset A of X is called an *attractor* if it satisfies the following three conditions.

(i) A is forward-invariant under ϕ_t; i.e., $x \in A$ implies $\phi_t(x) \in A$ for all $t > 0$.

(ii) There exists a neighborhood of A, called the *basin of attraction* of A and denoted $\mathcal{B}(A)$, which consists of all points that tend towards A under ϕ_t as $t \to \infty$. In other words, $\mathcal{B}(A) = \{x : \text{for any open neighborhood } N \text{ of } A, \exists T > 0 \ni \phi_t(x) \in N \ \forall t > T\}$.

(iii) No proper subset of A satisfies conditions (i) and (ii).

When an attractor Λ is (uniformly) hyperbolic, it will exhibit additional properties [14]:

- Periodic points are dense in Λ.

- For $x \in \Lambda, W^{cu}(x) \subset \Lambda$.

- For x a periodic point in Λ, $\bigcup\limits_{y \in \mathcal{O}(x)} W^{cs}(y)$ is dense in $\mathcal{B}(\Lambda)$.

We will need the following technical result, known in the literature as the Inclination Lemma, or λ-lemma. The statement can be found in [6]. Note that the statement for hyperbolic periodic points would be similar. Their statement is for 3-manifolds, but the statement is identical for higher-dimensional manifolds.

Lemma 2.12 (Inclination Lemma)**.** *Let $p \in M$ be a hyperbolic fixed point for a C^r flow Φ, for $r \geq 1$, with local stable and unstable manifolds $W^s_{loc}(p)$ and $W^u_{loc}(p)$, respectively. Fix an embedded disk B in $W^u_{loc}(p)$ which is a neighborhood of p in $W^u_{loc}(p)$, and fix a neighborhood V of this disk in M. Let D be a transverse disk to $W^s_{loc}(p)$ at a point z such that D and B have the same dimension. Write D_t for the connected component of $\Phi_t(D) \cap V$ which contains $\Phi_t(z)$, for $t \geq 0$.*

Then, given $\epsilon > 0$ there exists $T > 0$ such that for all $t > T$ the disk D_t is ϵ-close to B in the C^r-topology.

We will also need a proof from Hasselblatt and Katok regarding openness of hyperbolicity in function spaces [17, p. 571]. It is first necessary to address what the authors call the

Shadowing Theorem, which is a generalization of a common result known as the Shadowing Lemma. In layman's terms, the Shadowing Lemma says that although a numerically computed chaotic trajectory diverges from the true trajectory with the same initial conditions, there exists a true trajectory with slightly perturbed initial conditions that stays uniformly close to the numerically computed one. We use Katok and Hasselblatt's statement of the generalization of that lemma here [17, p. 566-7]. We include the theorem here because it is used to show that hyperbolicity is an open condition. Both the shadowing theorem and the openness of hyperbolicity are stated for maps, but the result is similar (albeit more technical and less enlightening) for flows.

Theorem 2.13 (**Shadowing Theorem**). *Let M be a Riemannian manifold, d the natural distance function, $U \subset M$ open, $f : U \to M$ a diffeomorphism, and $\Lambda \subset U$ a compact hyperbolic set for f. Then there exists a neighborhood $U(\Lambda) \supset \Lambda$ and $\epsilon_0, \delta_0 > 0$ such that for all $\delta > 0$ there is an $\epsilon > 0$ with the following property:*

If $f' : U(\Lambda) \to M$ is a diffeomorphism ϵ_0-close to f in the C^1 topology, Y is a topological space, $g : Y \to Y$ a homeomorphism, $\alpha \in C^0(Y, U(\Lambda))$, and

$$d_{C^0}(\alpha g, f'\alpha) := \sup_{y \in Y} d(\alpha g(y), f'\alpha(y)) < \epsilon$$

then there is a $\beta \in C^0(Y, U(\Lambda))$ such that $\beta g = f'\beta$ and $d_{C^0}(\alpha, \beta) < \delta$.

Furthermore, β is locally unique: If $\overline{\beta} g = f'\overline{\beta}$ and $d_{C^0}(\alpha, \overline{\beta}) < \delta_0$, then $\overline{\beta} = \beta$.

Remark. To get the more common Shadowing Lemma take $Y = (\mathbb{Z}, \text{discrete topology}), f' = f, \epsilon_0 = 0$, and $g(n) = n + 1$ and replace $\alpha \in C^0(Y, U(\Lambda))$ by $\{x_n\}_{n \in \mathbb{Z}} \subset U(\Lambda)$ and "$\beta \in C^0(Y, U(\Lambda))$ such that $\beta g = f'\beta$" by $\{f^n(x)\}_{n \in \mathbb{Z}} \subset U(\Lambda)$. Then $d(x_n, f^n(x)) < \delta$ for all $n \in \mathbb{Z}$.

With that in mind, we prove openness of hyperbolicity in the function space [17, p. 571].

Lemma 2.14. *Let $\Lambda \subset M$ be a hyperbolic set of the diffeomorphism $f : U \to M$. Then for any open neighborhood $V \subset U$ of Λ and every $\delta > 0$ there exists $\epsilon > 0$ such that if*

$f' : U \to M$ and $d_{C^1}(f|_V, f') < \epsilon$ there is a hyperbolic set $\Lambda' = f'(\Lambda') \subset V$ for f' and a homeomorphism $h : \Lambda' \to \Lambda$ with $d_{C^0}(Id, h) + d_{C^0}(Id, h^{-1}) < \delta$ such that $h \circ f'|_{\Lambda'} = f|_\Lambda \circ h$. Moreover, h is unique when δ is sufficiently small.

Proof. In this proof we apply the Shadowing Theorem thrice. First take $\delta_0 < \delta$ as in that theorem and apply the theorem with $\epsilon < \delta_0/2, y = \Lambda, \alpha = \text{id}\,|_\Lambda$ the inclusion, and $g = f$ to obtain a unique $\beta : \Lambda \to U(\Lambda)$ such that $\beta \circ f = f' \circ \beta$. By a basic proposition from [17, p. 265], $\Lambda' := \beta(\Lambda)$ is hyperbolic.

Apply the Shadowing Theorem again to get injectivity of β by taking ϵ as above, $y = \Lambda'$, $\alpha' = \text{id}\,|_{\Lambda'}$ the inclusion, and $g = f'$ to obtain a map h such that $h \circ f' = f \circ h$. Note that we can use f' instead of f if ϵ is small enough. We claim $h \circ \beta = \text{id}$ and hence $h = \beta^{-1}$ is a homeomorphism.

Apply the uniqueness part of the Shadowing Theorem in the $f = f'$ case, when $\alpha \circ f = f \circ \alpha$ and $\overline{\beta} \circ f = f \circ \overline{\beta}$, where $\overline{\beta} := h \circ \beta$.

Since $d_{C^0}(\alpha, \overline{\beta}) = d_{C^0}(\text{id}, h \circ \beta) \leq d_{C^0}(\text{id}, \text{id} \circ \beta) + d_{C^0}(\text{id} \circ \beta, h \circ \beta) = d_{C^0}(\text{id}, \beta) + d(\text{id}, h) < \delta_0$, the uniqueness part of the Shadowing Theorem implies $\overline{\beta} = \alpha = \text{id}\,|_\Lambda$, as was claimed. $\square$

For embedding into higher dimensions, we need to mention normal hyperbolicity. A normally hyperbolic invariant manifold (NHIM) is a generalization of a hyperbolic fixed point and a hyperbolic set. A manifold M is normally hyperbolic if the dynamics on M are essentially neutral relative to the dynamics around M. They were introduced by Neil Fenichel in 1972, and were shown to possess stable and unstable manifolds [10]. Furthermore, NHIMs and their stable and unstable manifolds are persistent under perturbation [11], [12]. We define NHIMs for maps, but the definition for flows is similar (and more technical).

Definition 2.15. Let M be a compact smooth manifold and $f : M \to M$ a diffeomorphism. Then an f-invariant submanifold Λ of M is said to be a *normally hyperbolic invariant manifold* if there exist constants $0 < \mu^{-1} < \lambda < 1$ and $c > 0$ such that

- $T_\Lambda M = T\Lambda \oplus \mathbb{E}^s \oplus \mathbb{E}^u$

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- $(Df)_x \mathbb{E}^s(x) = \mathbb{E}^s(f(x))$ and $(Df)_x \mathbb{E}^u(x) = \mathbb{E}^u(f(x))$ for all $x \in \Lambda$

- $\|Df^n v\| \leq c\lambda^n \|v\|$ for all $v \in \mathbb{E}^s$ and $n > 0$,

- $\|Df^{-n} v\| \leq c\lambda^n \|v\|$ for all $v \in \mathbb{E}^u$ and $n > 0$, and

- $\|Df^n v\| \leq c\mu^{|n|} \|v\|$ for all $v \in T\Lambda$ and $n \in \mathbb{Z}$.

Adapting the above for flows gives us an important result ([10, p. 215]) which says that if a C^r vector field Y in some C^1 neighborhood of our original vector field X (equated with a flow ϕ_t, under which M is invariant) there is a C^r manifold M_Y invariant under Y and C^r diffeomorphic to M. An immediate consequence of this is that the dynamics on M_Y under the vector field Y are a perturbation of the dynamics of M under X.

Lastly, for a map on a manifold M, define the *stable distribution of* $\mathbb{E}^s$ to be $\bigcup_{x \in M} \mathbb{E}^s(x)$.

The foundation of our counterexample is the classic Plykin map. Fisher used this map with some modifications to prove the counterexample in the map case. We then elaborate on Hunt's PhD dissertation, in which he extends the Plykin map to a flow [16]. Using that flow, we apply Fisher's extension of Plykin's map to Hunt's flow and address the technicalities thereof. Each step is described in detail below.

3.1   Plykin Attractor

Theorem 1.1 uses an extension of an attracting set for the Plykin map. In order to understand the result in this paper it is vital to understand this hyperbolic attractor. The Plykin map follows the usual recipes of hyperbolicity – iterated stretching and folding – but a more detailed view is warranted. The construction below relies heavily on Hasselblatt and Katok's construction of the Plykin attractor, [17, p. 537-41].

3.1.1   DA map. Let F be the Anosov diffeomorphism of $\mathbb{T}^2$ given by $L = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix} \right)$. The matrix L has eigenvalues $\lambda_u = (1+3\sqrt{(5)})/2$ and $\lambda_s = (1-3\sqrt{(5)})/2$. Let v_u and v_s, respectively, be the corresponding normalized eigenvectors, and let e^u and e^s be the unstable and stable vector fields of v_u and v_s obtained by parallel translation. Then $\mathbb{E}^u(p) = \text{span}\{e^u(p)\}$ and $\mathbb{E}^s(p) = \text{span}\{e^s(p)\}$ and $DF_p e^u(p) = \lambda_u e^u(F(p))$ and $DF_p e^s(p) = \lambda_s e^u(F(p))$ for all $p \in \mathbb{T}^2$.

On a disk U centered at 0 introduce coordinates (x_1, x_2) s.t. $(x_1, 0) \in E^u(0)$ and $(x_2, 0) \in E^s(0)$, so $F(x_1, x_2) = (\lambda_u x_1, \lambda_s x_2)$. We now define a nonlinear diffeomorphism f on $\mathbb{T}^2$. First let $\phi : \mathbb{R} \to [0, 1]$ be a C^∞ function where $\phi(t) = \phi(-t)$ for all $t \in \mathbb{R}$, $\phi(t) = 1$ if $|t| \leq 1/8$, $\phi(t) = 0$ if $|t| \geq 1/4$, and $\phi'(t) < 0$ if $1/8 < t < 1/4$. Take $k \in \mathbb{R}$ sufficiently large – exactly

how large will be explained later – and define $f : \mathbb{T}^2 \to \mathbb{T}^2$ by $f \equiv F$ on $\mathbb{T}^2 \setminus U$ and

$$f(x_1, x_2) = F(x_1, x_2) + (0, (2 - \lambda_s)\phi(x_1)\phi(kx_2)x_2)$$

on U.

There are no fixed points of f outside U, and on U the problem reduces to

$$x_1 = \lambda_u x_1,$$

$$x_2 = \lambda_s x_2 + (2 - \lambda_s)\phi(x_1)\phi(kx_2)x_2.$$

This implies $x_1 = 0$ and then the second equation reduces to

$$0 = x_2 \left(1 - \frac{2 - \lambda_s}{1 - \lambda_s}\phi(kx_2) \right).$$

Then we have solutions $x_2 = 0, \bar{x}, -\bar{x}$ where $\bar{x}$ is such that $\phi(k\bar{x}) = \dfrac{1 - \lambda_u}{2 - \lambda_u}$. To see what type of fixed points these are, note that $Df_{(x_1, x_2)}$ is equal to

$$\begin{pmatrix} \lambda_u & 0 \\ (2 - \lambda_s)\phi'(x_1)\phi(kx_2)x_2 & h(x_1, kx_2) \end{pmatrix}$$

where $h(x_1, kx_2) := \lambda_s + (2 - \lambda_s)\phi(x_1)(\phi'(kx_2)kx_2 + \phi(kx_2))$. In particular, $Df_0 = \left(\begin{smallmatrix} \lambda_u & 0 \\ 0 & 2 \end{smallmatrix} \right)$ with 0 a repelling fixed point. By further calculation, $(0, \bar{x})$ and $(0, -\bar{x})$ are hyperbolic fixed points.

Note also that f preserves $W^s(0)$ and that Df preserves the stable distribution of $\mathbb{E}^s$ for F although it may not contract vectors in $\mathbb{E}^s$ everywhere, and in fact permutes the stable manifolds for F in the same way as F does. Consider now the set $W = W^u(0) = \{p \in \mathbb{T}^2 : \alpha(p) = \{0\}\} = \bigcup_{n \in \mathbb{N}} f^n(U_0)$ for a sufficiently smaller neighborhood U_0 of 0. It is open and we will show later that it is dense. It is an attractor by definition.

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Hasselblatt and Katok go on to prove that for sufficiently large k, $\Lambda := \mathbb{T}^2 \setminus W$ is a hyperbolic set.

We now show that $W = \mathbb{T}^2 \setminus \Lambda$ is dense in $\mathbb{T}^2$. Consider $p \in \Lambda$ and any open neighborhood U_p of p. Then there is a point $q \in U_p$ that is periodic for F for some period n. The stable manifold H of q (under F) is thus f^n-invariant and dense. Density of W follows if we can find $N \in \mathbb{N}$ s.t. $f^{-Nn}(L^1) \cap W \neq \emptyset$, where $H^1 = H \cap U_p$. This is necessarily the case, however, since otherwise $H_f := \bigcup_{n \in \mathbb{N}} f^{-Nn}(H^1) \subset \Lambda$ and by hyperbolicity f^{-n} expands H_f so $H_f = H$. But H is dense in $\mathbb{T}^2$ so we would have $\Lambda = \mathbb{T}^2$, a contradiction. Thus Λ is the complement of an open dense set.

3.1.2 Plykin attractor. To get a hyperbolic attractor on S^2 let $J : \mathbb{T}^2 \to \mathbb{T}^2$, $J(x) = -x$ (mod 1) and note that construction of the DA map is J-invariant – i.e., $f \circ J = J \circ f$. Also note that $(1/2,1/2)$ is a periodic point for f since $f(1/2, 1/2) = F(1/2, 1/2) = (1/2, 0)$, $f(1/2, 0) = (0, 1/2)$, and $f(0, 1/2) = (1/2, 1/2)$. Now we replace F by F^3 and note that F^3 fixes these four fixed points of J and perform the construction described in the above section simultaneously around the four fixed points of F^3. Thus we have a map $f : \mathbb{T}^2 \to \mathbb{T}^2$ which commutes with J, has four fixed points as repelling fixed points, and has a hyperbolic attractor Λ.

Note that on $\mathbb{T}^2$ we have

$$-\left(\frac{1}{2}, \frac{1}{2}\right) = \left(\frac{1}{2}, \frac{1}{2}\right), \; -\left(\frac{1}{2}, 0\right) = \left(\frac{1}{2}, 0\right), \; -\left(0, \frac{1}{2}\right) = \left(0, \frac{1}{2}\right).$$

Thus if V_i, $i = 1, ..., 4$, are disks around $(0, 0), (1/2, 1/2), (1/2, 0), (0, 1/2)$, respectively, contained in $\mathbb{T}^2 \setminus \Lambda$, then $M = (\mathbb{T}^2 \setminus \bigcup_{i=1}^{4} V_i)/(x \sim -x)$ is a smooth manifold. It is not hard to see that M is a 2-sphere with four holes. Since $f(-x) = -f(x)$ we obtained an induced map $f' : M \to M$ which is smooth and injective. Filling $S^2 \setminus M$ with four repellers (one fixed and one period-3 cycle) gives a diffeomorphism $\tilde{f} : S^2 \to S^2$ with a hyperbolic attractor (obtained by projecting Λ onto M). This is the *Plykin attractor,* shown in Figure 3.1.
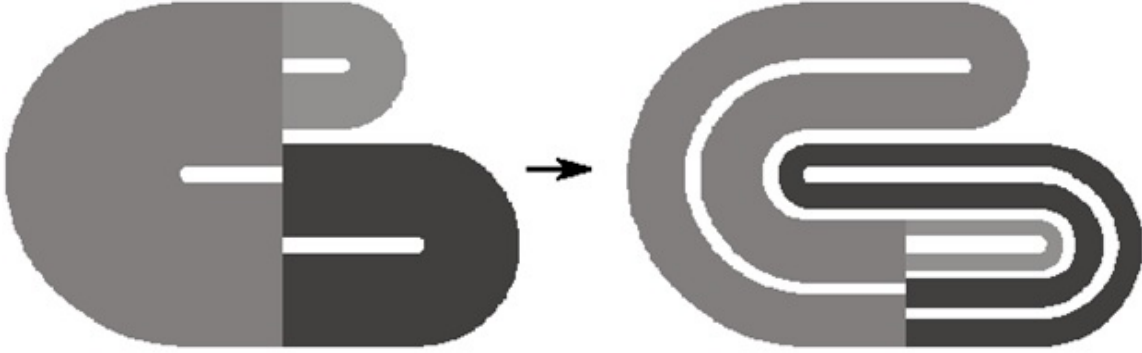
Figure 3.1: The Plykin map.

3.2 HUNT'S FLOW

The Poincare-Bendixson Theorem implies that neither diffeomorphisms of a one-manifold nor flows on a two-manifold can display chaotic behavior. Additionally, Plykin proved that if a diffeomorphism of a compact surface has a uniformly hyperbolic attractor, then the attractor must have at least four "holes" containing repelling sets. The Plykin attractor, constructed above, has four repelling sets, so it is one of the simplest examples of a uniformly hyperbolic attractor for maps. Hunt starts the Plykin attractor construction over from the beginning in the flow case. By discussion in the preceding paragraph, it follows that Hunt's Plykin attractor in a solid 2-torus is one of the simplest examples of a uniformly hyperbolic attractor in the flow case. The construction is outlined below, but the overarching idea is to view each iterate of the map as a cross-section of a solid 2-torus and connecting each point x to its image $f(x)$ via a continuous path around the torus. See Figure 3.2.

We now include a brief summary of Hunt's construction of the flow [16, p. 53-67]. The first and most important step is building the right coordinate system. He takes a square in $\mathbb{R}^2$ and changes the right half into a semicircle while keeping the left half the same. The semicircle piece isn't in polar coordinates because the construction needs to be C^1 at the least. Using regular polar coordinates, x_θ would be discontinuous on the y-axis. However, by forcing the coordinate change to map $[-\pi/2, \pi/2]$ to itself diffeomorphically, making it an odd function, and forcing the derivative values to be continuous, this complication is
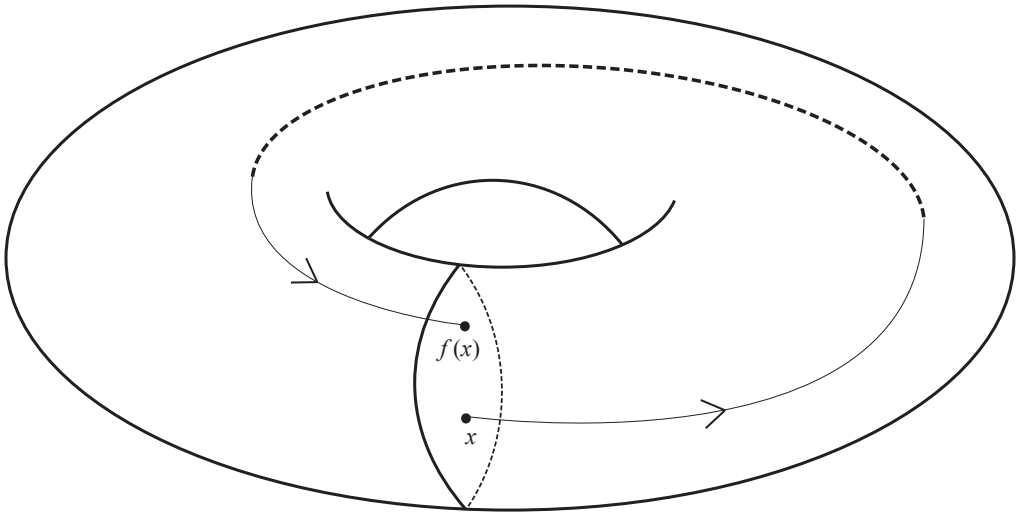
Figure 3.2: Extending a general map to a flow.

addressed. Now, using this altered-coordinate piece as a building block, he glues three of these pieces together to get the basic shape – the left-hand side of 3.1. Region 1 is the left-side building block, region 3 is the top right (the lightest shade in 3.1), and region 2 is the lower right (the darkest shade). Gluing these regions together correctly places natural constraints on the centering and size of the three regions, which constraints will be used later.

He then breaks the construction into three stages. The first is the stretching stage, then the first fold back over, then the second fold. The explicit 2π-periodic formula for the flow performs these three stages in turn, with smooth transitions between them.

The first stage involves squashing and stretching, which is what gives the flow hyperbolicity. The squashing factor is $\mu = \frac{3-\sqrt{(5)}}{2}$ and the stretching factor is $\lambda = \frac{3+\sqrt{(5)}}{2}$. The primary equation for producing the flow through this stage is $(\dot{r}, \dot{\theta}) = (\log(\mu)r, \log(\lambda)\theta)$. The solution to such an equation is $(r, \theta) = (c_1 \mu^t, c_2 \lambda^t)$. Since after one time unit we want the picture expanded, contracted, and folded as outlined above, $c_1 = c_2 = 1$. At this point it is necessary to shift the picture to ensure the original set A flows back into itself. By forcing the flow to satisfy $\phi_1(A) \subsetneq A$, there are, again, natural constraints placed on the center coordinates of the region 1, region 2, and region 3 building blocks, since region 3 needs to flow into the middle of the former region 2.

The second stage is to fold the stretched-out region 2 back into the desired location. Again, this places constraints on the coordinates of the different regions. The third stage is similar to the second, to fold the stretched-out region 1 back on top, placing further constraints on the coordinates.

Even after a total of ten constraints have been placed on the region's centers and horizontal and vertical shifts, there are still two free variables. What this ends up being equivalent to is that the total horizontal and vertical lengths of the attractor can be arbitrarily decided. The only constraint is that the x-coordinate of region 2 needs to be bigger than $\frac{(1+\sqrt{(5)})\pi}{4} = 2.54\ldots$ to ensure that the x-coordinates of the other two regions are positive.

He sets this coordinate equal to 3.

The final issue to deal with is incompatible flows in different regions. The flow is well-defined on the entirety of $\mathbb{R}^2$, but smoothness is not yet guaranteed. This boils down to using a weighted average function. What is needed is a weight function w that is 1 in region 2, 0 in region 3, and smoothly varies between them. Then in rough terms the flow can be defined as $w \times$ (region 2 flow) $+ (1 - w) \times$ (region 3 flow). He then defines two distance functions, dependent on two space variables and the flow (time) variable. One of these, called d_2, is a measure of the distance from the input point to region 2, and the other, called d_3, a measure of the distance from the input point to region 3. The weighting function, then, is defined to be

$$w(d_2, d_3) = \sin^2\left(\frac{\pi d_3}{2(d_2 + d_3)}\right).$$

He then mentions this is similar to a result from the following exercise, often given in undergraduate or beginning graduate analysis classes: "Given disjoint closed subsets A and B of a metric space (X, d), construct a continuous function $f : X \to [0, 1]$ such that $f^{-1}(\{1\}) = A$ and $f^{-1}(\{0\}) = B$," which answer is $f(x) = \frac{d(x, B)}{d(x, A) + d(x, B)}$. The difference between Hunt's function and this exercise is the $\sin^2$ modification to guarantee it is C^1.

Chapters 1-3 were focused on reviewing known results and establishing details regarding necessary constructions. We now move to our new results. In Section 1 we combine Fisher's map construction with Hunt's flow to use Fisher's ideas in a flow setting. We establish hyperbolicity of the relevant set in Section 2. In Section 3 we prove Theorem 1.1.

4.1 MODIFIED PLYKIN ATTRACTOR

In this section we take Hunt's Plykin flow on the solid 2-torus and modify it using Fisher's alterations of the Plykin map.

Hunt's flow ϕ_t maps a closed set A (containing the Plykin attractor Λ_a) strictly into itself – specifically, $\phi_t(A) \subset \text{int}(A)$ for all $t > 0$ [16, p. 64]. Embed this set A into a solid closed 2-torus, T, so $A \subset \text{int}(T)$. Extend the flow ϕ_t to contract in $\text{int}(T \setminus A)$ such that $\phi_t(x) \to A$ as $t \to \infty$ for all $x \in \text{int}(T \setminus A)$, and so that every $x \in \partial T$ is period 1.

We now have a solid closed 2-torus T where $\Lambda_a \subsetneq A \subsetneq T$, where every point in $\text{int}(T)$ asymptotically approaches Λ_a in forward time, and every point in ∂T is period 1. To later embed this system into a larger manifold while maintaining smoothness, we must enclose T in a strictly larger solid closed 2-torus, T_2. We need to extend ϕ_t so that every point on ∂T_2 is fixed and the flow is still C^2. For arbitrary $r \in \mathbb{N}$, this extension is possible in a C^r fashion. First set the boundary conditions: $\phi_t(x) = x$ for all x in a neighborhood of $\partial T_2, t \in \mathbb{R}$, and $\phi_k(x) = x$ for all $x \in \partial T, k \in \mathbb{N}$. In $T_2 \setminus T$, in between the boundaries, smoothly vary from fixed points on T_2 to period-1 points on T. This maintains the smoothness we need while gaining the properties we want.

Fix some $p \in \partial T$. Take an open neighborhood U of $\mathcal{O}(p)$, small enough to be disjoint from ∂T_2 and the attractor, and alter ϕ_t in the following way to make p a hyperbolic periodic saddle point: alter ϕ_t on $U \cap \partial T$ such that $x \in U \cap \partial T$ implies $\phi_t(x) \to \mathcal{O}(p)$ as $t \to \infty$ (in the sense that for any $\epsilon > 0$ there exists some τ such that $t \geq \tau$ implies $d(\phi_t(x), y) < \epsilon$ for
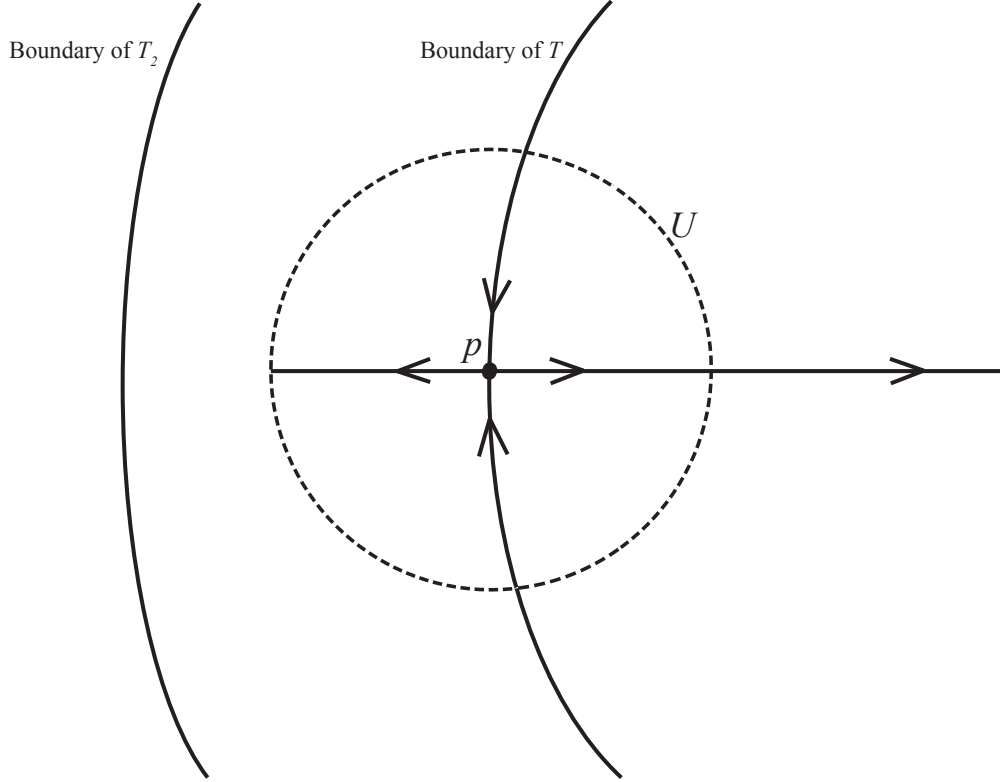
Figure 4.1: View of a cross section at p.

some $y \in \mathcal{O}(p)$. Note that this works because p is periodic). Thus, $W^s(p) = U \cap \partial T$. Again in a smooth way, change ϕ_t in $U \setminus \partial T$ to give p an unstable manifold, as in Figure 4.1. Note at this point it is clear to see that p satisfies all aspects of Definition 2.5, so it is a hyperbolic point of period 1. The center manifold of p is isometric to S^1 and each unstable manifold along $\mathcal{O}(p)$ is diffeomorphic to a line segment. Thus we have that $W^{cu}_{loc}(p)$ is diffeomorphic to a cylinder. Consider $W^{cu}(p) \setminus \mathcal{O}(p)$, which is now (locally) a disjoint union of two cylinders. One of these two components is entirely inside the basin of attraction, and one is entirely outside (reference Figure 4.1). Label as $W^*(p)$ the component of $W^{cu}(p) \setminus \mathcal{O}(p)$ which lies entirely inside the basin of attraction (equivalently, $W^*(p) = W^{cu}(p) \cap \mathrm{int}(T)$). Label as $W_0^*(p)$ the component of $W^u(p)$ which lies entirely inside the basin of attraction. Note that for all $x \in W^*(p)$ we have $\mathcal{O}(x) \subset \mathrm{int}(T)$.

We know that $\overline{W^{cs}(x)} = \overline{W^s(\Lambda_a)}$ for all $x \in \Lambda_a$ (section 2.3). By above we have that

$W^*(p) \subset \text{int}(T)$, and that $\text{int}(T) = W^s(\Lambda_a)$. Consider some periodic $q_0 \in \Lambda_a$. Then $W_0^*(p) \subset$

$W^*(p) \subset W^s(\Lambda_a) \subset \overline{W^{cs}(q_0)}$. Given any point of $W_0^*(p)$, then, there must exist some point

in $W^{cs}(q_0)$ arbitrarily close to it. Consider some $z \in W_0^*(p)$. Since q_0 is period-R for some

$R \in \mathbb{N}$ there exists an $R \in \mathbb{R}$ such that for all $\epsilon > 0$ there exists a $t \in [0, R]$ such that

$d(\phi_t(q_0), z) < \epsilon$. If $z \in W^{cs}(q_0)$ then we're done since then $z \in W^s(\phi_t(q_0))$ for some $t \in [0, R]$,

so we can assume $z \notin W^{cs}(q_0)$. Then z must be a limit point of $W^{cs}(q_0)$, by the first line

in this paragraph. Perturb the flow in a neighborhood of z (analogously to what is done in

Figure 4.2) so that $z \in W^{cs}(q_0)$.

If q is periodic with period $R \in \mathbb{N}$, then parametrize time so that q is period-1. This

is possible since q will be period-1 under the flow ϕ_{Rt}. Given $z \in W^u(p) \cap W^s(q)$ for

some period-1 $q \in \Lambda_a$, the manifolds $W^u(p)$ and $W^s(q)$ can have a transverse intersection,

after another perturbation, and we justify this as follows: by construction, we already know

that $z \in W^u(p) \cap W^s(q)$. In [13, p. 1508], a perturbation of the map is made in a small

neighborhood of $f^{-1}(z)$, which, since the map is continuously differentiable, ensures that

$z \in W^u(p) \pitchfork W^s(q)$ in the two-dimensional case. Now in the three-dimensional case, extend

this perturbation to the solid 2-torus by perturbing ϕ in a sufficiently small neighborhood of

$\phi_{-1}(z)$ (analogously to what is done in Figure 4.2) so that $W^u(p) \cap W^s(q)$ transversally at

some time t.

Here we will need two definitions (see [13, p. 1495]). A hyperbolic set Λ for a C^1 flow

has a *heteroclinic tangency* if there exist $x, y \in \Lambda$ such that $W^s(x) \cap W^u(y)$ contains a point

of tangency. A point of *quadratic tangency* for a C^2 flow is defined as a point of heteroclinic

tangency where the curvature of the stable and unstable manifolds differs at the point of

tangency.

By transversality and continuity, as well as the fact that Hunt's flow can be adapted to

be C^2 as elucidated in [16, p. 67,70], there must exist a neighborhood $J_0 \subset W_{\text{loc}}^u(q)$ of q and

a neighborhood $I_0 \subset W^u(p)$ of z such that for each $x \in I_0$ we have that $x \in W_{\text{loc}}^s(y)$ for

some $y \in J_0$. Now at some $z' \in I_0 \setminus \{z\}$, deform the flow in a sufficiently small neighborhood
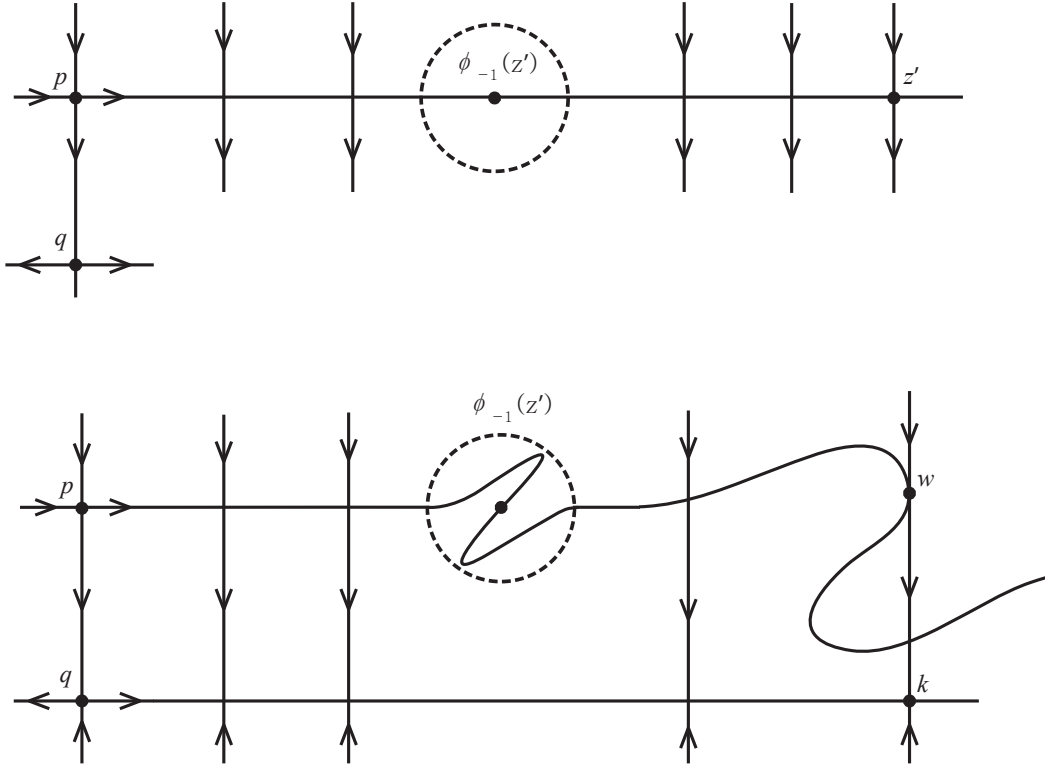
Figure 4.2: Deforming to obtain w.

of $\phi_{-1}(z')$ to create a point of quadratic tangency, $w \in I_0$, between $W^u(p)$ and $W^{cs}(k)$ for some $k \in J_0$ (see Figure 4.2). Let I be the segment of $W^u(p)$ from z to w, and let J be the segment of $W^u(q)$ from q to k.

4.1.1 The $(n \geq 3)$-dimensional Case. We have a flow on a solid torus that is the identity on ∂T_2, the solid 2-torus from the construction. To embed this system into any smooth manifold of dimension 3, first embed T_2 into a solid sphere S_3, and set $\phi_t(x) = x$ for all $x \in S_3 \setminus T_2, t \in \mathbb{R}$. Scale S_3 to be as small as necessary, set $\phi_t(x) = x$ for all $x \in M \setminus S_3, t \in \mathbb{R}$, and now the example extends to any smooth 3-manifold.

Using normal hyperbolicity (see Definition 2.15 and the remarks immediately following), we can embed our example into any smooth manifold M of dimension greater than 3, as follows: first take the solid sphere S_3 from above, such that ϕ_t is the identity on ∂S_3. For a

manifold M of dimension $n > 3$, embed S_3 into a solid n-sphere S_n. In $\text{int}(S_n \setminus S_3)$, extend ϕ_t so that it contracts sufficiently strongly in all directions towards S_3 to make S_3 a normally hyperbolic invariant manifold with respect to the flow ϕ_t. We need ϕ_t to fix every point in ∂S_n to ensure the system is easily embeddable into larger manifolds, but we simultaneously need ϕ_t to contract sufficiently strongly to make S_3 a normally hyperbolic invariant manifold. Consider a small neighborhood N of ∂S_n. In $N \cap S_n$, use a smooth bump function to let ϕ_t smoothly vary from sufficiently strong contraction (towards S_3, as previously mentioned) in $S_n \setminus N$ to fixing every point $x \in \partial S_n$ Lastly, define ϕ_t to fix every point in $M \setminus S_n$. Our example is now embeddable into M.

4.2 ESTABLISHING HYPERBOLICITY

Now that we have combined Hunt's and Fisher's constructions, we need to prove hyperbolicity of the relevant set.

We will show $\Lambda = \Lambda_a \cup \mathcal{O}(p) \cup \mathcal{O}(z)$ is hyperbolic under ϕ. Certainly $\mathcal{O}(p)$, $\mathcal{O}(z)$, and Λ_a are (at least forward-) invariant under ϕ, by definition. By construction, $\phi_t(z)$ converges to $\mathcal{O}(q)$ as $t \to \infty$ and converges to $\mathcal{O}(p)$ as $t \to -\infty$. Since these are both in Λ, and we already know Λ_a is closed, we have that Λ is closed.

Let $\lambda_p, \lambda_q \in (0, 1)$ be constants that guarantee hyperbolic behavior in $\mathcal{O}(p)$ and Λ_a, respectively. Let y be an arbitrary element of $\mathcal{O}(p)$. We assume an adapted metric, so for $t > 0$ and $v \in \mathbb{E}^s(y)$, $\|D\phi_t(y)v\| \leq \lambda_p^t\|v\|$, and for $t > 0$ and $r \in \mathbb{E}^u(y)$, $\|D\phi_{-t}(y)r\| \leq \lambda_p^t\|r\|$, and similarly for $\mathcal{O}(q)$. Let $\lambda_{max} = \max(\lambda_p, \lambda_q)$. Then certainly the constant λ_{max} guarantees hyperbolicity over $\mathcal{O}(p)$ and Λ_a, using the above definitions. Now select any $\lambda \in (\lambda_{max}, 1)$. Fix $t > 0$. By continuity and the Inclination Lemma (Lemma 2.12), there exists an $\varepsilon > 0$ such that for all x where $\text{d}(\phi_t(x), \mathcal{O}(p)) < \varepsilon$, we have $v \in \mathbb{E}^s(x)$ implying $\|D\phi_t(x)v\| \leq \lambda^t\|v\|$ (similarly for $\mathbb{E}^u(x)$ as well as identical cases for $\mathcal{O}(q)$). In other words, ϕ is hyperbolic with constant λ for any point ε-close to either $\mathcal{O}(p)$ or $\mathcal{O}(q)$.

Since $\lim_{t\to\infty}\phi_t(z) = \mathcal{O}(q)$ and $\lim_{t\to\infty}\phi_{-t}(z) = \mathcal{O}(p)$, there exists a $T > 0$ such that $\phi_T(z)$ is ε-close to $\mathcal{O}(q)$ and $\phi^{-T}(z)$ is ε-close to $\mathcal{O}(p)$. By the previous paragraph, we can guarantee hyperbolicity in ε-neighborhoods of $\mathcal{O}(p)$ and $\mathcal{O}(q)$. Now we only need guarantee hyperbolicity outside of those neighborhoods. Acting over the domains $t \in [-T, T]$ and $v \in \{a \in \mathbb{R}^3 : \|a\| = 1\}$, the function $\|D\phi_t(z)v\|$ is bounded because it is a continuous function on a (union of) compact domain(s).

Therefore there exists a $C \geq 1$ such that, for any $x \in \Lambda$ and $t > 0$, we have $\|D\phi_t(x)v\| \leq C\lambda^t\|v\|$ for all $v \in \mathbb{E}^s(x)$, and $\|D\phi_{-t}(x)w\| \leq C\lambda^t\|w\|$ for all $w \in \mathbb{E}^u(x)$. By definition, Λ is hyperbolic.

4.3 Proof of Main Theorem

Now that we have shown the system satisfies the relevant properties, we prove Theorem 1.1. As previously mentioned, our flow ϕ_t and Fisher's map f coincide at integer values of time, due to the constant roof function. In other words, $\phi_n(x) = f^n(x)$ for all $n \in \mathbb{Z}, x \in M$ where M is the manifold. Figure 4.3 is helpful to keep in mind throughout the argument.

We have a hyperbolic set Λ as a subset of a three-dimensional manifold. Since the original diffeomorphism acts on the unit disk, this three-dimensional flow lies in the solid 2-torus. Now suppose $\Lambda \subset \Lambda'$, where Λ' is a locally maximal hyperbolic set. It is sufficient to show that some point in $\mathcal{O}(w) \subset \Lambda'$ fails to exhibit continuous splitting of the tangent space, since the quadratic tangency persists in time. Now fix δ and ϵ to satisfy the local product structure.

Define I to be the closed interval from z to w along $W^u(p)$. By construction, every point in I is in the stable manifold of exactly one point in $W^u(q)$. Pick $r \in \mathbb{Z}$ such that $t \geq r$ implies $d(a, b) < \delta/2$ for all $a \in \phi_t(I), b \in W^{cu}(q)$. Let $J \subset W^u(q)$ be the set of all points $x \in W^u(q)$ such that $x \in W^s(\gamma)$ for some $\gamma \in \phi_r(I)$. The stable manifolds connecting I to J depend continuously on points in I, by hyperbolicity. Since ϕ is the continuous suspension of a diffeomorphism there exists a homeomorphism $\beta : [0, 1] \to \phi_r(I)$. By above, there exists
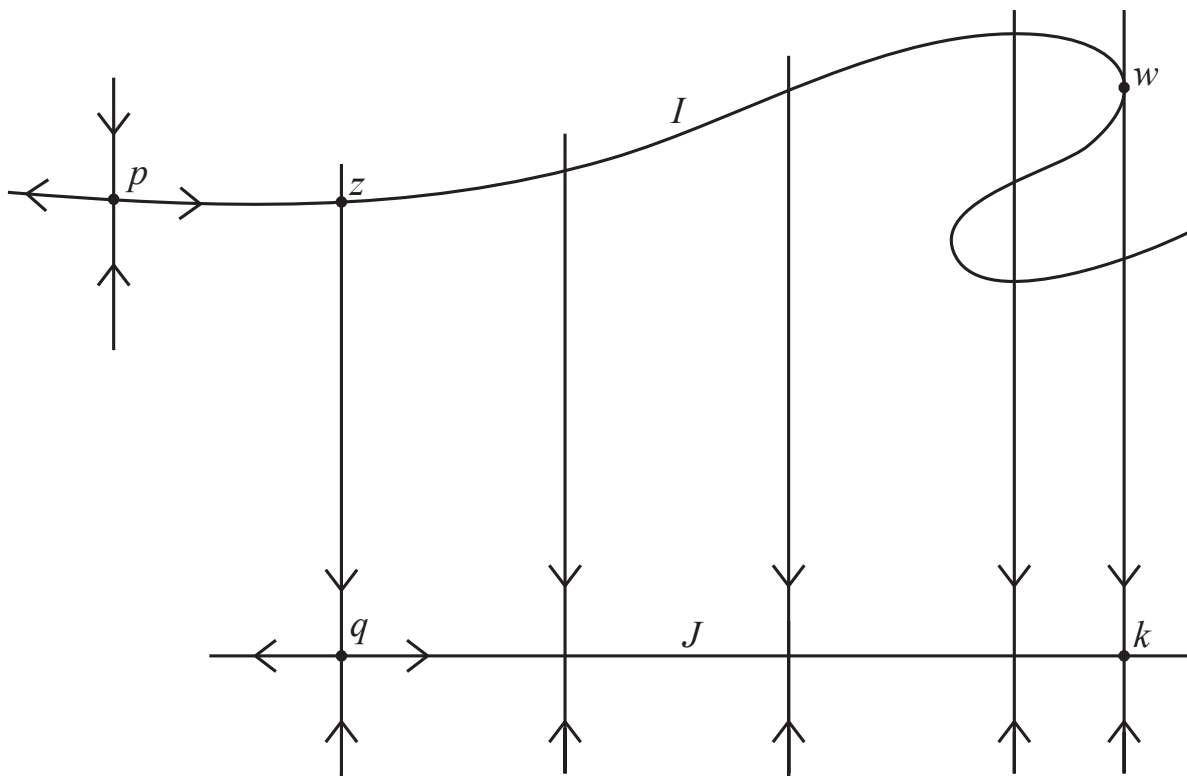
Figure 4.3: Intervals I and J.

a homeomorphism $\sigma : \phi_r(I) \to J$. Therefore $\sigma \circ \beta : [0,1] \to J$ is a homeomorphism. In other words, since $\phi_r(I)$ is a path-connected interval, J is also a path-connected interval.

Certainly $\phi_r(z) \in \phi_r(I) \cap \Lambda'$ since $z \in I$ and $\mathcal{O}(z) \subset \Lambda \subset \Lambda'$. Consider some $\alpha \in J$ such that $d(\alpha, q) < \delta/2$ (guaranteed to exist since J is path-connected). Since $d(\phi_r(z), q) < \delta/2$, then $d(\phi_r(z), \alpha) < \delta$ by the triangle inequality. By the local product structure, there exists exactly one point $\mathcal{S}(\phi_r(z), \alpha) \in W_\epsilon^u(\phi_r(z)) \cap W_\epsilon^s(\alpha) \subset \Lambda'$. Since we're adapting Fisher's flow to the solid 2-torus T with a constant roof function – $\phi_n(x) = f^n(x)$ for all $n \in \mathbb{Z}, x \in T$ – there need be no ϵ time shift as in the definition of local product structure. In this case, then, our definition for local product structure for flows can be taken to be that for all $\epsilon > 0$ there is a $\delta > 0$ such that given $x, y \in \Gamma$ with $d(x, y) < \delta$ then $\mathcal{S}(x, y) = W^u(x) \cap W^s(y) = \{b\} \subset \Gamma$. (Compare with Definition 2.8.) Note that in the perturbed case, the ϵ time shift in the flow definition of local product structure is indeed necessary but the argument still works since the unstable and stable manifolds depend continuously on each other, by hyperbolicity.

Take any point e along the interval lying in J from q to α. Thus $d(e, q) < d(\alpha, q) < \delta/2$ so similarly we get a unique $\mathcal{S}(\phi_r(z), e) \in \phi_r(I) \cap \Lambda'$. Since e was arbitrary, we now have a closed interval of points from $\phi_r(z)$ to $\mathcal{S}(\phi_r(z), \alpha)$ along $\phi_r(I) \cap \Lambda'$.

Thus $z' := \mathcal{S}(\phi_r(z), \alpha) \in \phi_r(I) \cap \Lambda'$. Inductively repeat the above process along J to see that $\phi_r(I) \subset \Lambda'$. Thus we see that the endpoint $\phi_r(w)$ of the interval $\phi_r(I)$ is also a point in $\mathcal{O}(w)$ where the hyperbolic splitting of the tangent space fails to continuously extend (see the notes immediately following Definition 2.5). We thus have found that Λ cannot be contained in a locally maximal hyperbolic set, which proves the theorem.

4.4   Robust Under Perturbation

We will see that every aspect of the system is robust under perturbation, so the entire system is as well. For the perturbed system we will use $\tilde{p}$ to denote the continuation of p, and we will similarly denote the continuations of the other aspects of the construction. We will first prove openness for 3-manifolds, and then for the n-dimensional case. Since

transversality is trivially open, and hyperbolicity is open by Lemma 2.14, it is sufficient to show that there remains a point $\tilde{w} \in W^u(\tilde{p}) \cap W^s(\tilde{u})$ for some $\tilde{u} \in W^{cu}_{\text{loc}}(\tilde{q})$. Under a small perturbation, Figure 4.2 remains identical in effect. The reasoning that gave us the quadratic tangency originally will still hold for a small C^1 perturbation of the system, as follows: by construction, the stable manifolds for all the $x \in W^{cu}_{\text{loc}}(\tilde{q})$ locally foliate the region, so there must exist a point $\tilde{u} \in W^{cu}_{\text{loc}}(\tilde{q})$ and a point $\tilde{w} \in W^{cs}(\tilde{u}) \cap W^u(\tilde{p})$ such that the one-dimensional path $W^u(\tilde{p})$ remains tangent to the two-dimensional plane $W^{cs}(\tilde{u})$ at $\tilde{w}$ – specifically, $T_{\tilde{w}} W^u(\tilde{p}) \subsetneq T_{\tilde{w}} W^{cs}(\tilde{u})$.

Since every other part of the system is known to be open, and we have just shown that the curve through w must remain tangent to some center-stable manifold even after perturbation, we have that the entire flow is open in the function space for 3-manifolds. Once there is a perturbation made in the time direction, the roof function is no longer constant so using the local product structure to show $\tilde{w} \in \widetilde{\Lambda}'$ requires use of the time-shift. The time-shift must be bounded by the ϵ from the local product structure, but for a sufficiently small perturbation this isn't an issue. The argument then works similarly: pick $\tilde{\alpha} \in W^u(\tilde{q})$ to be δ-close to $\phi_r(\tilde{z})$ for r sufficiently large. For some t where $|t| < \epsilon$ we will get $\mathcal{S}(\phi_r(\tilde{z}), \tilde{\alpha}) \in \Lambda'$. Continue along $\phi_r(\tilde{I})$ as before, using possibly different time-shifts at every iteration, to see $\tilde{w} \in \widetilde{\Lambda}'$.

For the n-dimensional case the only change made in the argument is with regards to the dynamics in S_n, the solid n-sphere from the construction in which the invariant set $S_3 \supset T_2$ was embedded. Using [10, p. 205], make the contraction in $S_n \setminus S_3$ sufficiently strong so that for a C^1 perturbation made to our flow ϕ_t, there remains some invariant manifold $\tilde{S}_3$ diffeomorphic to S_3. This means that there is a normally hyperbolic invariant manifold diffeomorphic to the previous one, so the flow $\tilde{\phi}_t$ restricted to $\tilde{S}_3$ is a small perturbation of ϕ_t restricted to S_3.

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