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A Class of Univalent Convolutions of Harmonic Mappings

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ABSTRACT

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A planar harmonic mapping is a complex-valued function $f : \mathbb{D} \to \mathbb{C}$ of the form $f(x+iy) = u(x, y) + iv(x, y)$, where u and v are both real harmonic. Such a function can be written as $f = h+\overline{g}$, where h and g are both analytic; the function $\omega = g'/h'$ is called the dilatation of f. This thesis considers the convolution or Hadamard product of planar harmonic mappings that are the vertical shears of the canonical half-plane mapping $\varphi(z) = z/(1 - z)$ with respective dilatations $e^{i\theta}z$ and $e^{i\rho}z$, $\theta, \rho \in \mathbb{R}$. We prove that any such convolution is univalent. We also derive a convolution identity that extends this result to shears of $\varphi(z) = z/(1 - z)$ in other directions.

Keywords: harmonic mapping, shearing, convolution, univalent, dilatation

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Chapter 1. Introduction

The main result of this thesis is the solution to an open problem about the convolution of planar harmonic mappings. We will prove that a particular class of convolutions of harmonic mappings is univalent, or one-to-one. The convolutions are formed from vertical shears of the canonical half-plane map. This result is contained in Chapter 4. In addition, Chapter 5 contains another interesting result that generalizes this and other univalence theorems for other directions of shearing. Chapter 6 contains some of the author's research on harmonic mappings with singular inner function dilatations; this is not directly related to the results of Chapters 4 and 5.

The thesis begins with an introduction to the subject in Chapters 2 and 3. It is meant to be as self-contained as possible, requiring only familiarity with basic complex analysis. The material in Chapter 2 is more general to the subject, whereas Chapter 3 focuses on some of the current research in the field that is relevant to this thesis. A more complete general survey may be found in the monograph *Harmonic Mappings in the Plane* by Peter Duren [7]. The chapter "Anamorphosis, Mapping Problems, and Harmonic Univalent Functions," in *Explorations in Complex Analysis*, by Michael Dorff [6], contains an accessible introduction to the subject at the undergraduate level, with an emphasis on research. In particular, it includes a section on convolutions of harmonic mappings.

Naturally, the majority of the material in Chapters 2 and 3 is already well-established. An exception to this is the set of examples we have included, which is worked out in much more generality than has previously been done. In addition to being useful for the results of the later chapters, these computations will also allow us to generate many computer images of the various functions considered. This provides a better geometric picture of the convolution of harmonic mappings than has previously been available.

As a matter of notation, we will throughout let $\mathbb{C}$ be the complex numbers, D any simply connected domain in $\mathbb{C}$, and $\mathbb{D}$ the unit disk $\{z : |z| < 1\}$. The terms "function" and "mapping" will also be used interchangeably, the latter term emphasizing the geometric point of view.

Chapter 2. Background

2.1 Univalent Functions

We say that a function $f : D \to \mathbb{C}$ is *univalent* if it is analytic and one-to-one. By the Riemann Mapping Theorem, any simply connected domain that is not the entire complex plane is conformally equivalent to $\mathbb{D}$. Hence it makes sense to restrict ourselves to mappings of the unit disk. We may further impose the normalization requirement that $f(0) = 0$ and $f'(0) = 1$. The collection of all normalized univalent functions $f : \mathbb{D} \to \mathbb{C}$ is denoted by S, from the German *schlicht* (meaning *simple* or *plain*). The study of the class S forms the bulk of the classical branch of mathematics known as *univalent functions* or *geometric function theory*. This branch of mathematics found its origins in 1914 with Gronwall's proof of the Area Theorem (see [12], pp. 58-59); the Bieberbach Conjecture, made in 1916, formed the cornerstone of the subject until its final resolution in the affirmative in 1984 by de Branges.

The study of univalent functions is also concerned with various subclasses of S. We shall make mention of just one of these in this thesis. We say that a function $f \in S$ is *convex* if $f(\mathbb{D})$ is a convex domain. The subclass of convex functions is denoted by K.

2.2 Univalent Harmonic Mappings

The study of univalent harmonic mappings is a more recent outgrowth of the study of univalent (analytic) functions. Here, the condition that f be analytic is replaced by the weaker condition that f be harmonic. This means that if $f(x + iy) = u(x, y) + iv(x, y)$, then u and v satisfy Laplace's equation. That is, $u_{xx} + u_{yy} = 0$ and $v_{xx} + v_{yy} = 0$. We will sometimes refer to such a function as *planar harmonic* to distinguish it from other uses of the term *harmonic*. It is a basic fact that any analytic function satisfies Laplace's equation;

hence the set of analytic functions on D is a subset of the planar harmonic mappings on D. A complex-valued function $f : D \to \mathbb{C}$ is said to be *univalent harmonic* if it is one-to-one and harmonic.

One basic fact is that any planar harmonic mapping defined on a simply connected domain can be written in the form $f(z) = h(z) + \overline{g(z)}$, where h and g are both analytic functions. We can easily verify this fact as follows. Let $f(x + iy) = u(x, y) + iv(x, y)$ satisfy Laplace's Equation. Let K be the analytic function having u as its real part, and let L be the analytic function having v as its imaginary part. It is a fact of complex analysis that such a function exists when u and v are defined on a simply connected domain (see for instance [3], p. 202). Letting $h = \dfrac{K + L}{2}$ and $g = \dfrac{K - L}{2}$, we obtain $h+\overline{g} = \dfrac{K + \overline{K}}{2} + \dfrac{L - \overline{L}}{2} = \text{Re } K + i\text{Im } L = f$, as desired. We refer to h as the *analytic part* and g as the *coanalytic* part of f. This representation is important because it provides a link between planar harmonic mappings and the theory of analytic functions. Moreover, it gives us a series representation for f in terms of its analytic and coanalytic parts, $f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n \overline{z}^n$.

In analogy to the class S of normalized (analytic) univalent mappings, we may consider the class S_H of univalent harmonic mappings $f = h+\overline{g} : \mathbb{D} \to \mathbb{C}$ subject to the normalization $h(0) = 0$, $h'(0) = 1$, and $g(0) = 0$. This was first studied by Clunie and Sheil-Small in 1984 [2]. We may also define the subclass S_H^0 of functions that also satisfy the requirement $g'(0) = 0$. Note that $S \subset S_H^0 \subset S_H$. The above series representation now takes the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n \overline{z}^n$ for $f \in S_H$ and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=2}^{\infty} b_n \overline{z}^n$ for $f \in S_H^0$. Also, as expected, K_H and K_H^0 denote the respective subclasses of functions that are also convex.

The Jacobian of a harmonic mapping $f(x+iy) = h(x+iy) + \overline{g(x+iy)}$ is given by

$$\begin{aligned} J_f(z) &= \begin{vmatrix} (\text{Re } f)_x & (\text{Re } f)_y \\ (\text{Im } f)_x & (\text{Im } f)_y \end{vmatrix} \\ &= \begin{vmatrix} (\text{Re } h)_x + (\text{Re } g)_x & (\text{Re } h)_y + (\text{Re } g)_y \\ (\text{Im } h)_x - (\text{Im } g)_x & (\text{Im } h)_y - (\text{Im } g)_y \end{vmatrix} \end{aligned}$$

By applying the Cauchy-Riemann equations, we obtain

$$\begin{aligned} J_f(z) &= \begin{vmatrix} (\text{Re } h)_x + (\text{Re } g)_x & -(\text{Im } h)_x - (\text{Im } g)_x \\ (\text{Im } h)_x - (\text{Im } g)_x & (\text{Re } h)_x - (\text{Re } g)_x \end{vmatrix} \\ &= (\text{Re } h)_x^2 - (\text{Re } g)_x^2 + (\text{Im } h)_x^2 - (\text{Im } g)_x^2 \\ &= |h'|^2 - |g'|^2. \end{aligned}$$

It is a fact from complex analysis that an analytic function is locally univalent if and only if its Jacobian is nonvanishing for all points in its domain ([1], p. 74). Lewy proved in 1936 that this result is also true for planar harmonic mappings (see [7], pp. 2, 20). Motivated by this, we say that a harmonic mapping is *sense-preserving* if $J_f(z) > 0$ and *sense-reversing* if $J_f(z) < 0$ for all z in some domain D. Geometrically, the image of a circle oriented clockwise will have a clockwise orientation under a sense-preserving mapping, and a counter-clockwise orientation under a sense-reversing mapping.

In connection with this, the *dilatation* of a planar harmonic mapping $f = h + \overline{g}$ is defined to be $\omega = g'/h'$. Lewy's Theorem then may be expressed as follows.

Theorem A. Let $f = h + \overline{g}$ be defined on the simply connected domain G. Then f is locally univalent and sense-preserving if and only if $|\omega(z)| < 1$ for all $z \in G$.

The dilatation may be interpreted geometrically to represent the "stretching" or "distortion" of f. More precisely, f will map an infinitesimal circle around the point z to an infinitesimal ellipse, with $\dfrac{1+|\omega(z)|}{1-|\omega(z)|}$ giving the ratio of the major axis to the minor axis of the ellipse. Hence a dilatation of 0 at a point corresponds to f being analytic there.

We will typically consider dilatations that are inner functions. An *inner function* is defined as an analytic function $\omega : \mathbb{D} \to \mathbb{D}$ such that for almost all $u \in \partial\mathbb{D}$, $\lim_{z\to u} \omega(z)$ exists and $\left|\lim_{z\to u} \omega(z)\right| = 1$. This is because these are the largest possible dilatations for a univalent, sense-preserving function. Moreover, the study of smaller dilatations more properly belongs to the field of *quasiconformal mappings*.[1] The dilatations that we commonly use are also *finite Blaschke products*, meaning they are of the form $e^{i\theta}\prod_{j=1}^{n}\dfrac{z-a_j}{1-\overline{a_j}z}$, where $\theta \in \mathbb{R}$ and each $|a_j| < 1$.

2.3 The Shearing Technique

Clunie and Sheil-Small introduced the so-called *shearing technique* for creating examples of planar harmonic functions with a prescribed dilatation [2]. First, we need some more terminology. For some $\alpha \in \mathbb{R}$, we say that f is *convex in the α-direction* if $\{z + te^{i\alpha} : t \in \mathbb{R}\} \cap f(\mathbb{D})$ is connected for all $z \in \mathbb{C}$. We use the terms *convex in the horizontal* and *vertical directions* in the cases that $\alpha = 0$ and $\alpha = \pi/2$, respectively. The shearing technique is based on the following theorem by Clunie and Sheil-Small. Their original theorem was stated for functions that are convex in the horizontal direction. However, it is often convenient to consider other directions when using the shearing technique, so we state it in more generality.

Theorem B (**Shearing theorem**). Let h and g be analytic functions on the unit disk $\mathbb{D}$ such that $f = h + \overline{g}$ is locally univalent. Then f is univalent and convex in the α-direction if and only if the analytic function $\varphi = h - e^{2i\alpha}g$ has these two properties.

In the horizontal case, we see that f and φ satisfy the relationship $f = \varphi + 2\text{Re } g$. Hence f and φ differ from each other by the addition of a real-valued function. This can

[1] More precisely, a quasiconformal mapping can be defined as a harmonic mapping for which $(1 + |\omega(z)|)/(1 - |\omega(z)|)$ is bounded by some constant K. This in an extensively researched field of mathematics that largely preceded the modern interest in planar harmonic mappings.

be visualized geometrically by φ being chopped into horizontal slices, each of which is then translated or scaled continuously to obtain f. This explains the use of the word "shear" in the name of the theorem.

We will often use Theorem B as follows. Suppose we are given some $\alpha \in \mathbb{R}$, a univalent analytic function φ that is convex in the α-direction, and an analytic function ω satisfying $|\omega| < 1$. We may then find functions h and g that solve the pair of equations $\varphi = h - e^{2i\alpha}g$ and $\omega = g'/h'$. Then $f = h + \overline{g}$ is a harmonic mapping with ω as its dilatation to which Theorem B applies. We will refer to this corresponding function f as the *shear* of φ in the α direction with dilatation ω. We illustrate this process with an example that will be important to our later results.

Example 1. Let $\varphi(z) = \dfrac{z}{1-z}$. Notice that φ is a Möbius transformation mapping the unit disk onto the right half-plane $\{z : \mathrm{Re}(z) > -1/2\}$. We call φ the *canonical half-plane mapping*. It is clear that $\varphi(\mathbb{D})$ is convex, and hence convex in every direction. In this case, we will shear it in the vertical direction (that is, $\alpha = \pi/2$), with dilatation $\omega(z) = g'(z)/h'(z) = e^{i\theta}z^n$. Theorem B guarantees that the functions we obtain here are univalent and convex in the vertical direction.

Then h and g satisfy $h(z) + g(z) = \dfrac{z}{1-z}$. Taking derivatives gives $\dfrac{1}{(1-z)^2} = h'(z) + g'(z) = h'(z)(1+\omega(z)) = h'(z)(1+e^{i\theta}z^n)$. We see that

$$h(z) = \int_0^z \frac{ds}{(1-s)^2(1+\omega(s))} = \int_0^z \frac{ds}{(1-s)^2(1+e^{i\theta}s^n)}.$$

Once we obtain $h(z)$, we can readily find $g(z)$ using the relation $g(z) = \varphi(z) - h(z)$.

The first case to consider is $n = 1$ and $\theta = \pi$. Then

$$\begin{aligned} h(z) &= \int_0^z \frac{ds}{(1-s)^3} = \frac{z - \frac{1}{2}z^2}{(1-z)^2} \\ g(z) &= \frac{z}{1-z} - \frac{z - \frac{1}{2}z^2}{(1-z)^2} = \frac{-\frac{1}{2}z^2}{(1-z)^2}. \end{aligned}$$

Next, suppose that $n = 1$ and $\theta \neq \pi$. Then

$$h(z) = \int_0^z \frac{ds}{(1-s)^2(1+e^{i\theta}s)} = \frac{1}{1+e^{i\theta}}\left(\frac{z}{1-z}\right) + \frac{e^{i\theta}}{(1+e^{i\theta})^2}\log\left(\frac{1+e^{i\theta}z}{1-z}\right) \tag{2.1}$$

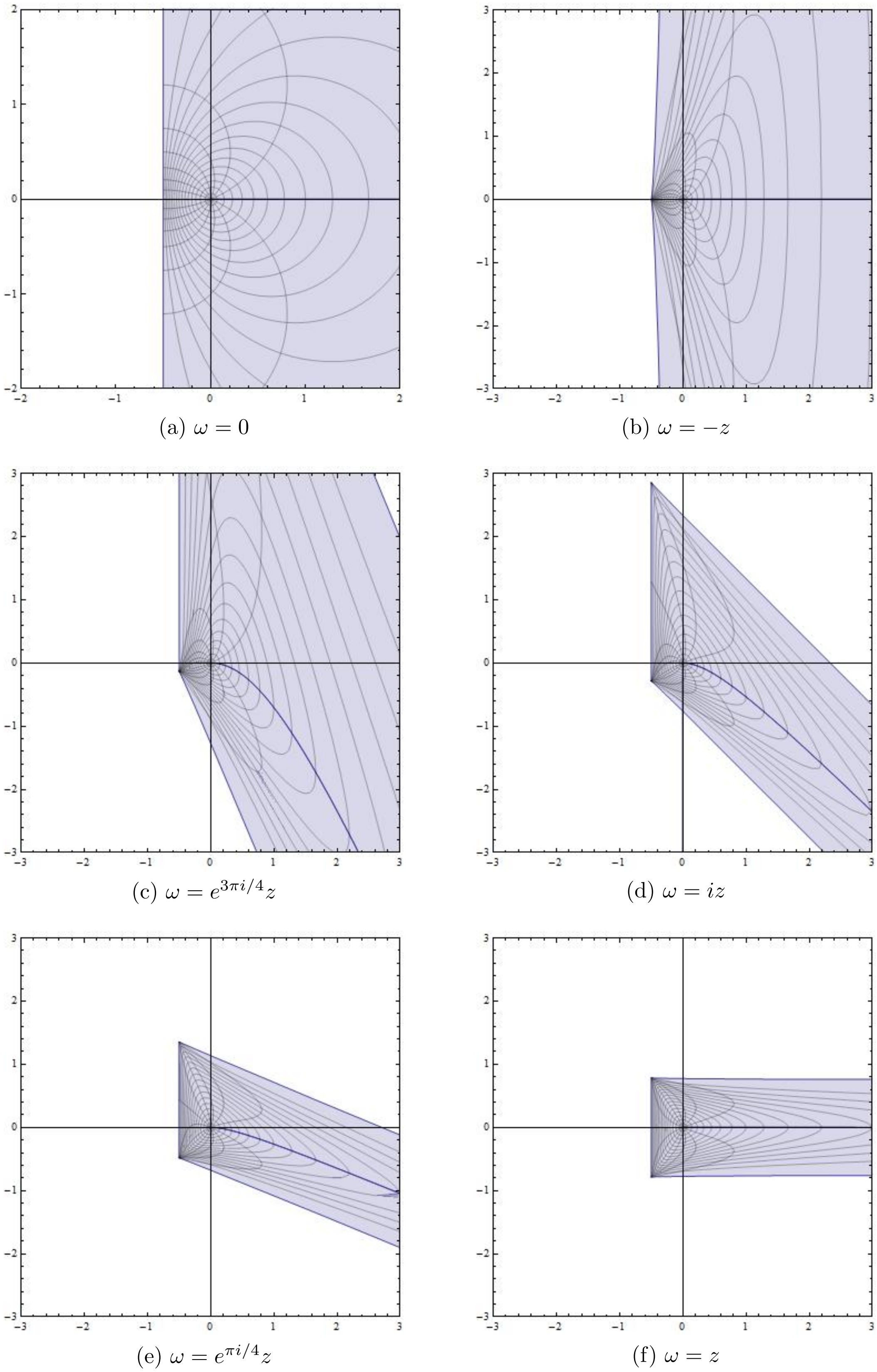
$$g(z) = \frac{e^{i\theta}}{1+e^{i\theta}}\left(\frac{z}{1-z}\right) - \frac{e^{i\theta}}{(1+e^{i\theta})^2}\log\left(\frac{1+e^{i\theta}z}{1-z}\right). \tag{2.2}$$

More generally, we will derive a formula valid for all n. We assume here that $\theta \neq \pi$. We first express h' in terms of its partial fraction decomposition.

$$\begin{aligned} h'(z) &= \frac{1}{(1-z)^2(1+e^{i\theta}z^n)} = \frac{1}{(1-z)^2\prod_{j=1}^{n}\left(1-e^{i\frac{\pi+\theta+2\pi j}{n}}z\right)} \\ &= \sum_{j=1}^{n}\frac{A_j}{1-e^{i\frac{\pi+\theta+2\pi j}{n}}z} + \frac{A_{n+1}}{1-z} + \frac{A_{n+2}}{(1-z)^2}, \end{aligned}$$

where

$$\begin{aligned} A_{n+1} &= \frac{ne^{i\theta}}{(1+e^{i\theta})^2} \qquad\qquad A_{n+2} = \frac{1}{1+e^{i\theta}} \\ A_j &= \frac{-e^{i\frac{\pi-\theta+2\pi j}{n}}}{(1-e^{i\frac{\pi-\theta+2\pi j}{n}})^2\prod_{k=1}^{n-1}(1-e^{\frac{2\pi i k}{n}})} \qquad (1 \leq j \leq n) \\ &= \frac{-e^{i\frac{\pi-\theta+2\pi j}{n}}}{(1-e^{i\frac{\pi-\theta+2\pi j}{n}})^2(n)} \\ &= \frac{1}{4n}\csc^2\left(\frac{\pi-\theta+2\pi j}{2n}\right). \end{aligned}$$



(a) $\omega = 0$

(b) $\omega = -z$

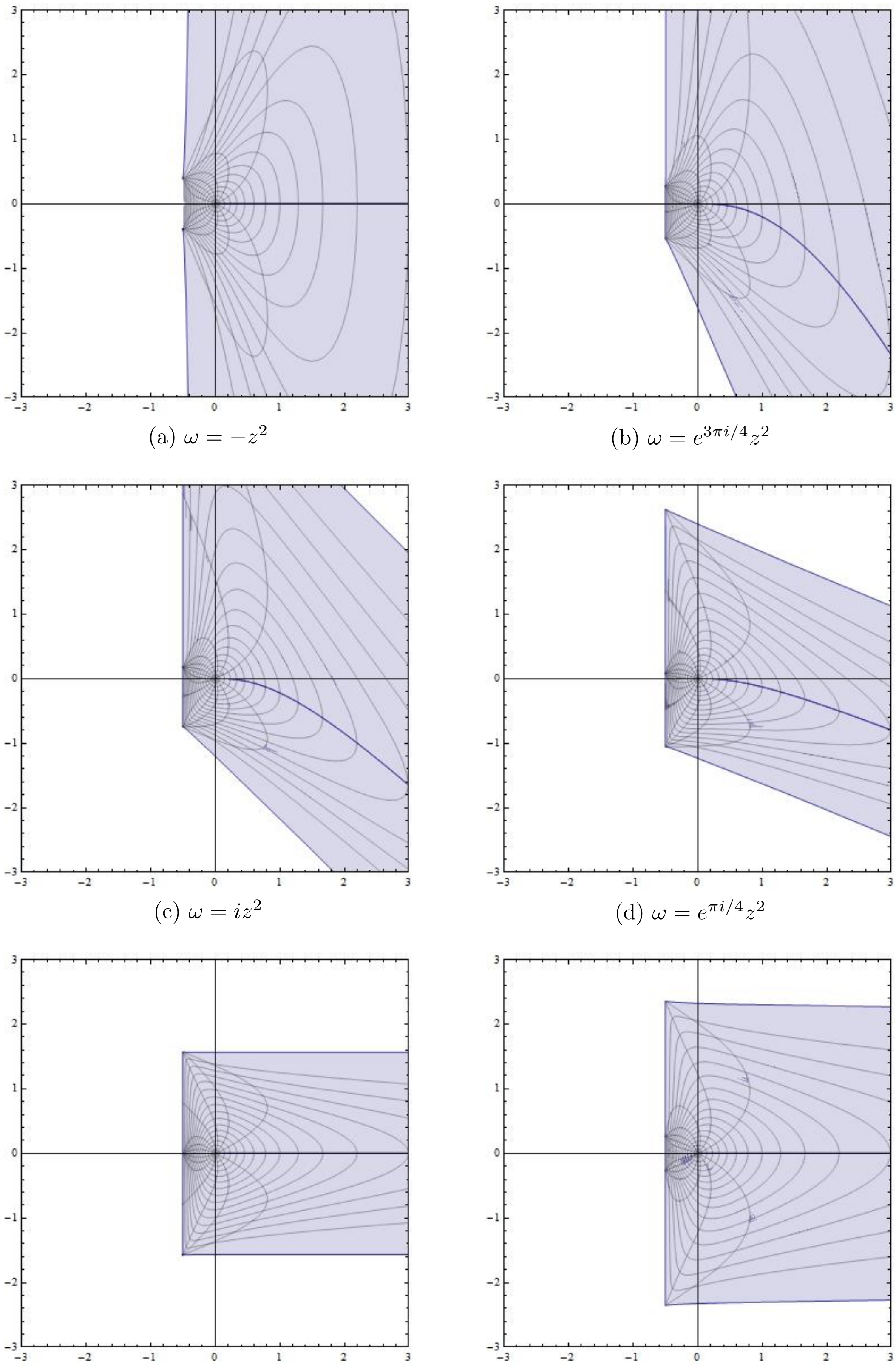
(c) $\omega = e^{3\pi i/4}z$

(d) $\omega = iz$

(e) $\omega = e^{\pi i/4}z$

(f) $\omega = z$

Figure 2.1: Images of $\mathbb{D}$ under the vertical shears of $\varphi(z) = z/(1-z)$ with dilatation ω, part (i).



(a) $\omega = -z^2$

(b) $\omega = e^{3\pi i/4} z^2$

(c) $\omega = iz^2$

(d) $\omega = e^{\pi i/4} z^2$

(e) $\omega = z^2$

(f) $\omega = z^3$

Figure 2.2: Images of $\mathbb{D}$ under the vertical shears of $\varphi(z) = z/(1-z)$ with dilatation ω, part (ii).

We see from this that $f = h + \overline{g}$ can be written as

$$
\begin{aligned}
h(z) =& \frac{1}{1+e^{i\theta}}\left(\frac{z}{1-z}\right) - \frac{e^{i\theta}n}{(1+e^{i\theta})^2}\log(1-z) \\
&+ \frac{1}{4n}\sum_{j=1}^{n}\csc^2\left(\frac{\pi-\theta+2\pi j}{2n}\right)\left(\log(1-z) + \log\left(\frac{1-e^{-\frac{i(\pi-\theta+2\pi j)}{n}}z}{1-z}\right)\right) \\
g(z) =& \frac{e^{i\theta}}{1+e^{i\theta}}\left(\frac{z}{1-z}\right) + \frac{e^{i\theta}n}{(1+e^{i\theta})^2}\log(1-z) \\
&- \frac{1}{4n}\sum_{j=1}^{n}\csc^2\left(\frac{\pi-\theta+2\pi j}{2n}\right)\left(\log(1-z) + \log\left(\frac{1-e^{-\frac{i(\pi-\theta+2\pi j)}{n}}z}{1-z}\right)\right).
\end{aligned}
$$

Note that we have written the logarithms in such a way that each has a branch cut along the negative real axis. This consistency among the logarithm terms is necessary for creating correct computer images.

In the case that $\theta = \pi$, then there is now a pole of order 3 at 1, and we must adjust our formulas correctly. We now have

$$
\begin{aligned}
h'(z) &= \frac{1}{(1-z)^2(1+e^{i\theta}z^n)} = \frac{1}{(1-z)^3\prod_{j=1}^{n-1}\left(1-e^{i\frac{2\pi j}{n}}z\right)} \\
&= \sum_{j=1}^{n-1}\frac{A_j}{1-e^{i\frac{2\pi j}{n}}z} + \frac{A_n}{1-z} + \frac{A_{n+1}}{(1-z)^2} + \frac{A_{n+2}}{(1-z)^3},
\end{aligned}
$$

where

$$
\begin{aligned}
A_n &= -\frac{n^2-1}{12n} \qquad A_{n+1} = \frac{n-1}{2n} \qquad A_{n+2} = \frac{1}{n} \\
A_j &= \frac{-e^{i\frac{2\pi j}{n}}}{(1-e^{i\frac{2\pi j}{n}})^2\prod_{k=1}^{n-1}(1-e^{\frac{2\pi ik}{n}})} \qquad (1 \le j \le n-1) \\
&= \frac{1}{4n}\csc^2\left(\frac{\pi j}{n}\right).
\end{aligned}
$$

This gives

$$
\begin{aligned}
h(z) =& \frac{(n+1)z - nz^2}{2n(1-z)^2} - \frac{n^2-1}{12n}\log(1-z) \\
& + \frac{1}{4n}\sum_{j=1}^{n-1}\csc^2\left(\frac{\pi j}{n}\right)\left(\log(1-z) + \log\left(\frac{1-e^{-\frac{i(2\pi j)}{n}}z}{1-z}\right)\right) \\
g(z) =& \frac{(n-1)z - nz^2}{2n(1-z)^2} + \frac{n^2-1}{12n}\log(1-z) \\
& - \frac{1}{4n}\sum_{j=1}^{n-1}\csc^2\left(\frac{\pi j}{n}\right)\left(\log(1-z) + \log\left(\frac{1-e^{-\frac{i(2\pi j)}{n}}z}{1-z}\right)\right).
\end{aligned}
$$

Images of a few of these shears are in Figure 2.1 and Figure 2.2. Notice that the choice of θ corresponds to the angle of the slant of the image, which is an angle of $\theta/2$ downwards from the real axis. The other observation is that the boundary collapses to a finite number of points. If $\theta \neq \pi$, then there are $n+1$ such points in $\mathbb{C}$. If $\theta = \pi$, then there are n such points in $\mathbb{C}$, as well as the point at infinity. This boundary behavior occurs for any convex harmonic mapping with inner function dilatation (see [6], p. 221). Note that this boundary behavior is not possible for analytic functions.

In light of the previous example, we will let let $S_H(R)$, $S_H^0(R)$, $K_H(R)$, and $K_H^0(R)$ denote the respective subclasses of harmonic mappings $f = h + \overline{g}$ that satisfy $h(z) + g(z) = \dfrac{z}{1-z}$.

Chapter 3. Introduction to Research Question

3.1 Harmonic Convolutions

A number of research papers in recent years have examined the *convolution* or *Hadamard product* of planar harmonic mappings. Like the entire area of study itself, this has its origins in the classical field of univalent functions. Given analytic functions $f(z) = \sum_{n=1}^{\infty} a_n z^n$, $F(z) = \sum_{n=1}^{\infty} A_n z^n$, the convolution of f and F, denoted by $f * F$, is defined to be $(f * F)(z) =$

$\sum_{n=1}^{\infty} a_n A_n z^n$.[1] A natural question is, given two univalent functions f, F, under what circumstance is their convolution $f * F$ also univalent? There are a number of nice results for this question. For instance, if $f, F \in K$, then it is also the case $f * F \in K$ and is thus univalent ([8], p. 130)).

The notion of convolution can be extended to planar harmonic mappings as follows. For $f = h + \overline{g} = \sum_{n=1}^{\infty} a_n z^n + \overline{\sum_{n=1}^{\infty} b_n z^n}$ and $F = H + \overline{G} = \sum_{n=1}^{\infty} A_n z^n + \overline{\sum_{n=1}^{\infty} B_n z^n}$, their harmonic convolution is given by $(f * F)(z) = (h * H)(z) + \overline{(g * G)(z)} = \sum_{n=1}^{\infty} a_n A_n z^n + \overline{\sum_{n=1}^{\infty} b_n B_n z^n}$.

Example 2. This is a continuation of Example 1 above. We will determine the convolution of some combinations of the functions obtained there. This set of examples was proved to be univalent by Dorff, Nowak, and Woloszkiewicz [5], as we shall see in the next section.

We first do some preliminary work to simplify the computations. Let $f_0 = h_0 + \overline{g_0}$ be the vertical shear of $z/(1-z)$ with dilatation $\omega_0 = -z$, and let $F(z) = \sum_{n=1}^{\infty} a_n z^n$ be any analytic function with $F(0) = 0$. In Example 1, we showed that f_0 is given explicitly by

$$\begin{aligned} h(z) &= \frac{z - \frac{1}{2}z^2}{(1-z)^2} = \frac{1}{2}\frac{z}{1-z} + \frac{1}{2}\frac{z}{(1-z)^2} \\ g(z) &= \frac{-\frac{1}{2}z^2}{(1-z)^2} = \frac{1}{2}\frac{z}{1-z} - \frac{1}{2}\frac{z}{1-z}. \end{aligned}$$

Since $\frac{z}{1-z} = z + z^2 + z^3 + \cdots$ and $\frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + \cdots$, we obtain

$$\begin{aligned} \frac{z}{1-z} * F(z) &= F(z) \\ \frac{z}{(1-z)^2} * F(z) &= \sum_{n=1}^{\infty} n a_n z^n = z\left(\sum_{n=1}^{\infty} n a_n z^{n-1}\right) = zF'(z) \end{aligned}$$

[1] This definition is related to the better-known notion of convolution used in Fourier analysis. For functions $\widetilde{f}, \widetilde{F} : [0, 2\pi) \to \mathbb{C}$, their *convolution* is defined as $(\widetilde{f} * \widetilde{F})(t) = \frac{1}{2\pi}\int_0^{2\pi} \widetilde{f}(\tau)\widetilde{F}(t-\tau)d\tau$. Regarding $f(e^{it})$ and $F(e^{it})$ above as functions of the real variable t, i.e. taking $\widetilde{f}(t) = f(e^{it})$ and $\widetilde{F}(t) = F(e^{it})$, it can be shown that these two definitions are equivalent.

Hence $f_0 * F$ is given explicitly by

$$(h_0 * F)(z) = \frac{F(z) + zF'(z)}{2} \tag{3.1}$$

$$(g_0 * F)(z) = \frac{F(z) - zF'(z)}{2}. \tag{3.2}$$

Now, let $f = h + \overline{g}$ be the vertical shear of $z/(1-z)$ with dilatation $\omega = e^{i\theta}z$, where $\theta \neq \pi$. Recall from Equations 2.1 and 2.2 in Example 1 that

$$h(z) = \frac{1}{1+e^{i\theta}}\left(\frac{z}{1-z}\right) + \frac{e^{i\theta}}{(1+e^{i\theta})^2}\log\left(\frac{1+e^{i\theta}z}{1-z}\right)$$

$$g(z) = \frac{e^{i\theta}}{1+e^{i\theta}}\left(\frac{z}{1-z}\right) - \frac{e^{i\theta}}{(1+e^{i\theta})^2}\log\left(\frac{1+e^{i\theta}z}{1-z}\right).$$

Then by Equations 3.1 and 3.2, $f_0 * f$ is given by

$$(h_0 * h)(z) = \frac{(2+e^{i\theta})z - (1-e^{i\theta})z^2 - e^{i\theta}z^3}{2(1+e^{i\theta})(1-z)^2(1+e^{i\theta}z)} + \frac{e^{i\theta}}{2\left(1+e^{i\theta}\right)^2}\log\left(\frac{1+e^{i\theta}z}{1-z}\right)$$

$$(g_0 * g)(z) = \frac{e^{i\theta}\left(z - 2z^2 - e^{i\theta}z^3\right)}{2(1+e^{i\theta})(1-z)^2(1+e^{i\theta}z)} - \frac{e^{i\theta}}{2\left(1+e^{i\theta}\right)^2}\log\left(\frac{1+e^{i\theta}z}{1-z}\right)$$

In the case that $\theta = \pi$, then we have the convolution $f_0 * f_0$. This is given by

$$(h_0 * h_0)(z) = \frac{-z(z^2 - 3z + 4)}{4(z-1)^3}$$

$$(g_0 * g_0)(z) = \frac{z^2}{4} + \frac{-z^3(z^2 - 3z + 4)}{4(z-1)^3}$$

Next, we will let $f = h + \overline{g}$ be the vertical shear of $z/(1-z)$ with dilatation $\omega = e^{i\theta}z^2$, where again $\theta \neq \pi$. We have

$$\begin{aligned}
(h_0 * h)(z) =& \frac{1}{16}\left(\frac{8z}{(z-1)^2\left(1+z^2e^{i\theta}\right)} - \frac{8z}{(z-1)\left(1+e^{i\theta}\right)} - \frac{16e^{i\theta}\log(1-z)}{\left(1+e^{i\theta}\right)^2}\right. \\
&+ \sec^2\left(\frac{1}{4}(\theta-3\pi)\right)\left(\log(1-z)+\log\left(\frac{1+ize^{\frac{i\theta}{2}}}{1-z}\right)\right) \\
&\left.+ \sec^2\left(\frac{\theta-\pi}{4}\right)\left(\log(1-z)+\log\left(\frac{-1+ize^{\frac{i\theta}{2}}}{z-1}\right)\right)\right) \\
(g_0 * g)(z) =& \frac{1}{16}\left(8e^{i\theta}\left(\frac{z\left(\frac{1-z}{1+e^{i\theta}} - \frac{z^2}{1+z^2e^{i\theta}}\right)}{(z-1)^2} + \frac{2\log(1-z)}{\left(1+e^{i\theta}\right)^2}\right)\right. \\
&+ \sec^2\left(\frac{1}{4}(\theta-3\pi)\right)\left(-\left(\log(1-z)+\log\left(\frac{1+ize^{\frac{i\theta}{2}}}{1-z}\right)\right)\right) \\
&\left.- \sec^2\left(\frac{\theta-\pi}{4}\right)\left(\log(1-z)+\log\left(\frac{-1+ize^{\frac{i\theta}{2}}}{z-1}\right)\right)\right)
\end{aligned}$$

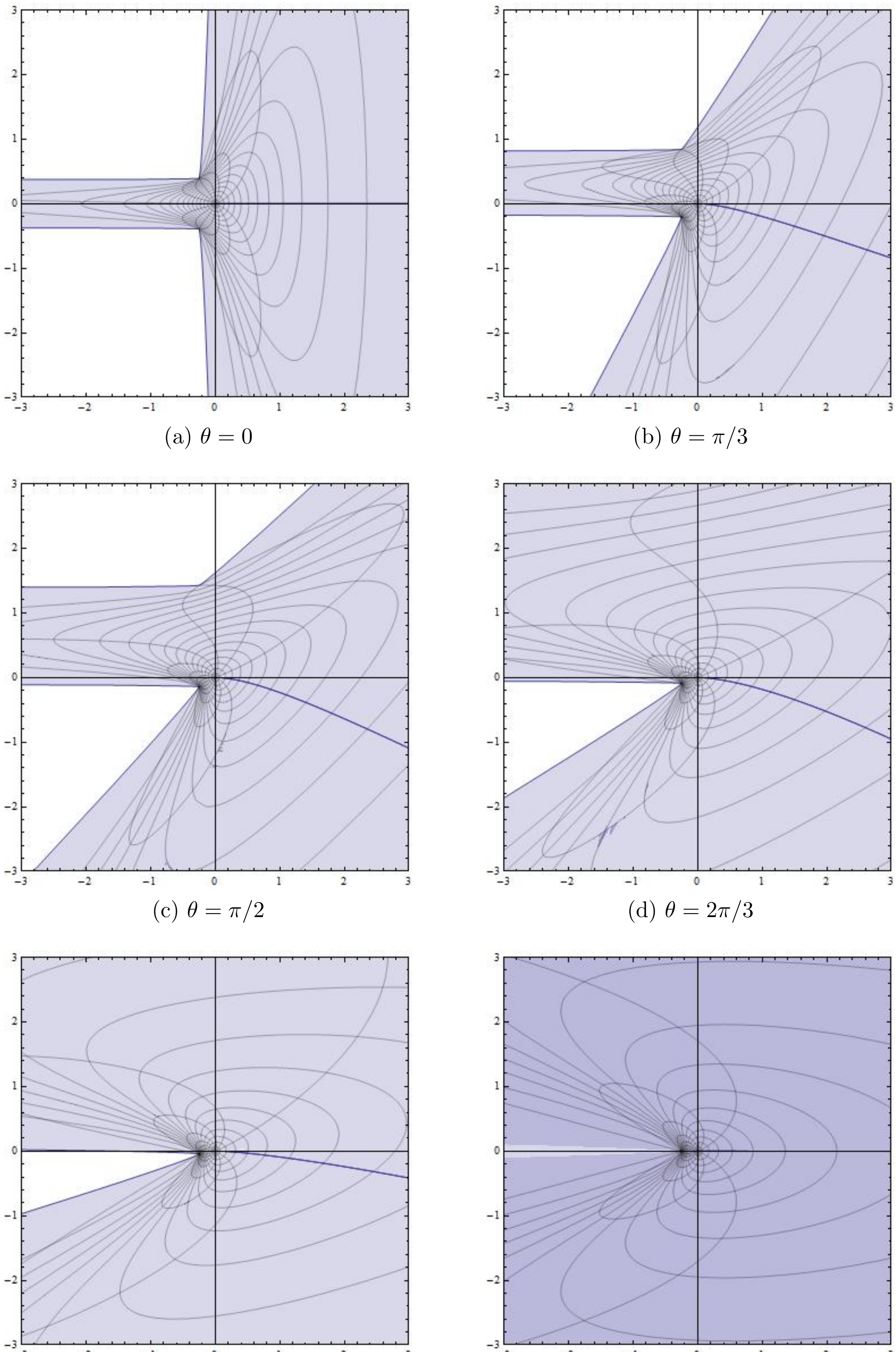
The final case is when $f = h + \overline{g}$ is the vertical shear of $z/(1-z)$ with dilatation $\omega = -z^2$. The convolution $f_0 * f$ is then given by

$$\begin{aligned}
(h_0 * h) &= \frac{z(2z^3-3z^2-2z+7)}{8(1-z)^3(1+z)} + \frac{1}{16}\log\left(\frac{1+z}{1-z}\right) \\
(g_0 * g) &= \frac{z(2z^3+3z^2-2z+1)}{8(1-z)^3(1+z)} - \frac{1}{16}\log\left(\frac{1+z}{1-z}\right)
\end{aligned}$$

Some images of these are given in Figure 3.1 and Figure 3.2. As in Example 1, we can see that the boundary collapses down to a finite number of points. This is again indicative of the dilatation being an inner function. A proof that this fact is implicit in the proof of Theorem C.

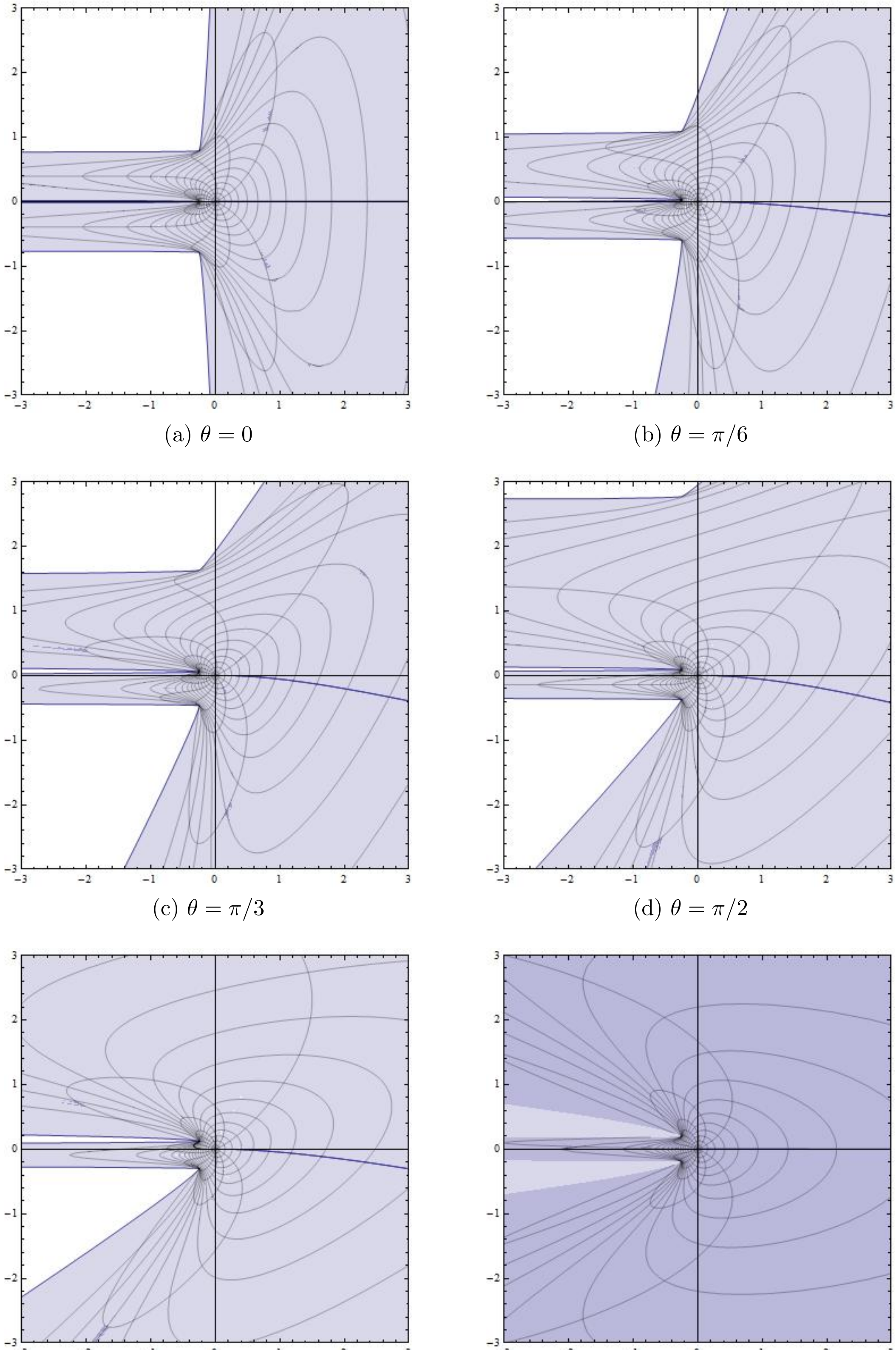
3.2 Current Research

As in the case of analytic convolutions, a major broad research problem is determining how well univalence is preserved by the harmonic convolution. Both the difficulty and interest



(a) $\theta = 0$ (b) $\theta = \pi/3$ (c) $\theta = \pi/2$ (d) $\theta = 2\pi/3$ (e) $\theta = 5\pi/6$ (f) $\theta = \pi$

Figure 3.1: Images of $\mathbb{D}$ under $f_0 * f$, where $f = h + \overline{g}$ is the vertical shear of $z/(1-z)$ with dilatation $\omega = e^{i\theta}z$.



(a) $\theta = 0$ (b) $\theta = \pi/6$

(c) $\theta = \pi/3$ (d) $\theta = \pi/2$

(e) $\theta = 2\pi/3$ (f) $\theta = \pi$

Figure 3.2: Images of $\mathbb{D}$ under $f_0 * f$, where $f = h + \overline{g}$ is the vertical shear of $z/(1-z)$ with dilatation $\omega = e^{i\theta} z^2$.

in this problem stem from the fact that most of the nice results regarding the analytic convolution are no longer true in the case of harmonic convolutions. In the absence of these, much of the research is devoted to finding more specific cases in which univalence is preserved. A summary of this research in the case of vertical shears of the canonical half-plane map, including the results of this thesis, is given in Table 7.1 in the conclusion.

One of these is the following theorem of Dorff, Nowak, and Woloszkiewicz [5].

Theorem C. Let $f_0 = h_0 + \overline{g_0} \in S_H^0(R)$ with $\omega_0(z) = -z$, and let $f = h + \overline{g} \in S_H^0(R)$ with $\omega(z) = e^{i\theta} z^n$ ($n \in \mathbb{N}$ and $\theta \in \mathbb{R}$). If $n = 1, 2$, then $f_0 * f \in S_H^0$ (i.e., is univalent) and is convex in the horizontal direction.

The fact that the convolution is convex in the horizontal direction suggests invoking the shearing theorem of Clunie and Sheil-Small in its proof. A proof along these lines would have two parts: first, one must show that $\widetilde{\omega}$, the dilatation of the convolution, satisfies $|\widetilde{\omega}| < 1$. Next, one would "unshear" the convolution back to an analytic function and justify why this unsheared function is univalent and convex in the horizontal direction. This would be reasonably straightforward, given the abundance of nice theorems for analytic functions. However, an earlier result by Dorff [4] states that we can bypass this second step altogether and need only show that $|\widetilde{\omega}(z)| < 1$.

Theorem D. Let $f_1, f_2 \in K_H(R)$. If $f_1 * f_2$ is locally univalent and sense-preserving, then $f_1 * f_2 \in S_H$ and is convex in the horizontal direction.

The proof of Theorem C is an elaborate argument that the inequality $|\widetilde{\omega}(z)| < 1$ holds for the given convolution. Their argument makes use of the following result on the roots of complex polynomials ([15], p. 375).

Theorem E. (Cohn's Rule) Given a polynomial $f(z) = a_0 + a_1 z + \cdots + a_n z^n$ of degree n, let $f^*(z) = z^n \overline{f(1/\overline{z})} = \overline{a_n} + \overline{a_{n-1}} z + \cdots \overline{a_0} z^n$. Let p be the number of zeros of f in $\mathbb{D}$ and let s be the number of zeros of f on $\partial\mathbb{D}$. If $|a_0| < |a_n|$, then $f_1(z) = \dfrac{\overline{a_n} f(z) - a_0 f^*(z)}{z}$ is of degree $n - 1$, with $p - 1$ zeros in $\mathbb{D}$ and s zeros in $\partial\mathbb{D}$.

We include here the original proof of Theorem C in [5].

Proof. Using Equations 3.1 and 3.2 and the fact that $g' = \omega h'$ and $g''(z) = \omega h'' + \omega' h'$, $\widetilde{\omega}$ can be expressed as follows.

$$\widetilde{\omega}(z) = \frac{\frac{d}{dz}\left(g(z) - zg'(z)\right)}{\frac{d}{dz}\left(h(z) + zh'(z)\right)} = \frac{-zg''(z)}{2h'(z) + zh''(z)} = \frac{-z\omega'(z)h'(z) - z\omega(z)h''(z)}{2h'(z) + zh''(z)}$$

Recall from Example 1 that $h'(z) = \dfrac{1}{(1+\omega(z))(1-z)^2}$, and thus

$$h''(z) = \frac{2(1+\omega(z)) - \omega'(z)(1-z)}{(1+\omega(z))^2(1-z)^3}.$$

Substituting h' and h'' into the equation for $\widetilde{\omega}$ gives

$$\widetilde{\omega}(z) = -z\frac{\omega^2(z) + \omega(z) - \frac{1}{2}\omega'(z)z + \frac{1}{2}\omega'(z)}{1 + \omega(z) - \frac{1}{2}\omega'(z)z + \frac{1}{2}\omega'(z)z^2}. \tag{3.3}$$

Taking now $\omega(z) = e^{i\theta}z$, we obtain

$$\widetilde{\omega}(z) = -ze^{2i\theta}\frac{z^2 + \frac{1}{2}e^{-i\theta}z + \frac{1}{2}e^{-i\theta}}{1 + \frac{1}{2}e^{i\theta}z + \frac{1}{2}e^{i\theta}z^2} = -ze^{2i\theta}\frac{p(z)}{p^*(z)},$$

where $p^*(z)$ is as defined in Theorem E. Letting b_1 and b_2 denote the two roots of $p(z)$, we see that

$$\widetilde{\omega}(z) = -ze^{2i\theta}\frac{(z-b_1)(z-b_2)}{(1-\overline{b_1}z)(1-\overline{b_2}z)}.$$

The polynomial $p_1(z) = \dfrac{\overline{a_2}p(z) - a_0p^*(z)}{z} = \dfrac{3}{4}z + \dfrac{1}{2}e^{-i\theta} - \dfrac{1}{4}$ has one zero at $z_0 = \frac{1}{3} - \frac{2}{3}e^{-i\theta}$. If $\theta \neq \pi$, then $|z_0| < 1$, and so, by Theorem E, $p(z)$ must have two zeros in $\mathbb{D}$. Hence $|b_1| < 1$ and $|b_2| < 1$. This shows that $\widetilde{\omega}(z)$ is an inner function, and thus satisfies $|\widetilde{\omega}(z)| < 1$. If $\theta = \pi$, then $b_1 = 1$ and $b_2 = -1/2$. Again, we see that $|\widetilde{\omega}(z)| < 1$. The proves the theorem in the case that $n = 1$.

In the case of $n = 2$, then Equation 3.3 now gives

$$|\widetilde{\omega}(z)| = |z^2| \left| \frac{z^3 + e^{-i\theta}}{1 + e^{i\theta} z^3} \right| = |z|^2 < 1.$$

This proves the theorem in the case that $n = 2$. □

A natural course of further investigation is to replace ω_0 and ω in Theorem C with other, more general dilatations, and likewise show that the resulting convolution is univalent. Another paper has done this, relying once again on Cohn's Rule to provide the punch line in the proof [10]. However, this argument depends on the nice representation for convolutions of h_0 and g_0 given in Equations 3.1 and 3.2, which allows $\widetilde{\omega}$ to be expressed as a rational function in z. This is no longer the case when $f_0 = h_0 + \overline{g_0}$ is replaced with other harmonic mappings. Instead, new techniques are needed.

The main result of this thesis is such a problem. Specifically, we will derive a similar theorem for the dilatations $\omega(z) = e^{i\theta} z$ and $\omega_1(z) = e^{i\rho} z$, using a different approach to show that $|\widetilde{\omega}| < 1$. The basic approach originated in an untitled draft by M. Nowak and M. Woloszkiewicz [14], in which they proved this in the case when $\theta = \rho$. Their argument generalizes, albeit with some difficulty, to the full result that we obtained.

3.3 A First Result

First, however, we will take a minor detour. It is of mention that Theorem C is no longer true when $n \geq 3$. The authors remark on this in [5], content with showing that the result is false in the case that $\theta = \pi$. As it turns out, however, even more is true—for all $\theta \in \mathbb{R}$, $f_0 * f$ will fail to be univalent. The proof of this is only slightly longer than that of the specific case $\theta = \pi$, and follows very naturally from the proof of Theorem C above.

Theorem 1. Let f_0 be the shear of $\varphi(z) = z/(1-z)$ with dilatation $\omega_0 = -z$. Let f be the shear of $\varphi(z) = z/(1-z)$ with dilatation $\omega = e^{i\theta} z^n$, $\theta \in \mathbb{R}$. Suppose that $n \geq 3$. Then $f_0 * f$ is not univalent.

Proof. Let $\widetilde{\omega}$ denote the dilatation of the convolution $f_0 * f$. Our goal is to show that $|\widetilde{\omega}(z)| \geq 1$ for some z. We begin from Equation (3.3) above.

$$\begin{aligned}
\widetilde{\omega}(z) &= -z\frac{\omega^2(z) + \omega(z) - \frac{1}{2}\omega'(z)z + \frac{1}{2}\omega'(z)}{1 + \omega(z) - \frac{1}{2}\omega'(z)z + \frac{1}{2}\omega'(z)z^2} \\
&= -z\frac{e^{2i\theta}z^{2n} + e^{i\theta}z^n - \frac{n}{2}e^{i\theta}z^n + \frac{n}{2}e^{i\theta}z^{n-1}}{1 + e^{i\theta}z^n - \frac{n}{2}e^{i\theta}z^n + \frac{n}{2}e^{i\theta}z^{n+1}} \\
&= -z^n e^{2i\theta}\frac{z^{n+1} + e^{-i\theta}(1 - \frac{n}{2})z + \frac{n}{2}e^{-i\theta}}{1 + e^{i\theta}(1 - \frac{n}{2})z^n + \frac{n}{2}e^{i\theta}z^{n+1}}
\end{aligned}$$

Let $p(z)$ be the numerator and $q(z)$ be the denominator of the previous equation. Note that $q(z) = z^{n+1}\overline{p(1/\overline{z})}$; hence a is a root of $p(z)$ if and only if $1/\overline{a}$ is a root of $q(z)$. So $\widetilde{\omega}$ can be rewritten as

$$\widetilde{\omega}(z) = -z^n e^{2i\theta}\frac{p(z)}{q(z)} = -z^n e^{2i\theta}\frac{(z - a_1)(z - a_2)\cdots(z - a_{n+1})}{(1 - \overline{a_1}z)(1 - \overline{a_2}z)\cdots(1 - \overline{a_{n+1}}z)},$$

where $a_1, a_2, \ldots, a_{n+1}$ denote the zeros of $p(z)$. Note that $\widetilde{\omega}$ has a pole at $1/\overline{a_i}$, unless it happens that $a_j = 1/\overline{a_i}$ for some j. Suppose now that $|\widetilde{\omega}| < 1$; in particular, $\widetilde{\omega}$ has no poles in $\mathbb{D}$. Hence, if $|a_i| > 1$ for some i, there must exist some j unique to that i such that $a_j = 1/\overline{a_i}$. Since $|a_i a_j| = |a_i/\overline{a_i}| = 1$, this implies immediately that $|a_1 a_2 \ldots a_{n+1}| \leq 1$. However, we also have from our equation for $\widetilde{\omega}$ that $|a_1 a_2 \ldots a_{n+1}| = |ne^{-i\theta}/2| = n/2$, which is impossible whenever $n \geq 3$. We conclude that, if $n \geq 3$, then $\widetilde{\omega}$ must have a pole in $\mathbb{D}$. This gives the result. □

One point of interest in the preceding proof is that the dilatation $\widetilde{\omega}$ of the convolution is meromorphic and not analytic. This shows that the convolution of two locally univalent harmonic mappings might not be locally univalent. (Note that any locally univalent harmonic mapping has an analytic dilatation.)

This section will be devoted to proving the following theorem.

Theorem 2. Let $f_1 = h_1 + \overline{g_1} \in S_H^0(R)$ with dilatation $\omega_1 = e^{i\theta}z$, $\theta \in \mathbb{R}$, and let $f_2 = h_2 + \overline{g_2} \in S_H^0(R)$ with dilatation $\omega_2 = e^{i\rho}z$, $\rho \in \mathbb{R}$. Then $f_1 * f_2 \in S_H^0$ and is convex in the horizontal direction.

4.1 Preliminary Results

In this section, we collect a number of preliminary results needed for the proof of Theorem 2. We begin with the Maximum Modulus Theorem, a standard theorem of complex analysis. It can be stated as follows ([3], p. 128).

Theorem F. (Maximum Modulus Theorem) Let G be a bounded open set in $\mathbb{C}$ and suppose f is a continuous function on $\overline{G}$ which is analytic in G. Then $\max\{|f(z)| : z \in \overline{G}\} = \max\{|f(z)| : z \in \partial G\}$. Moreover, if f attains its maximum at some point $z \in G$, then f is constant.

We will employ this theorem in the following modified form.

Lemma 3. Let $f : \mathbb{D} \to \mathbb{C}$ be nonconstant and analytic, where $\overline{f(\mathbb{D})}$ omits some point $w \in \{z : \operatorname{Re} z < 0\}$. Suppose that $\widehat{f}(e^{it}) = \lim\limits_{z \to e^{it}} f(z)$ exists for all $t \in \mathbb{R}$ (where possibly $\widehat{f}(e^{it}) = \infty$). If $\operatorname{Re}(\widehat{f}(e^{it})) \geq 0$ for all t such that $\widehat{f}(e^{it})$ is finite, then $\operatorname{Re}(f(z)) > 0$ for all $z \in \mathbb{D}$.

Proof. It can be shown that for any point w, where $\operatorname{Re} w < 0$, there exists some $t_1, t_2 \in \mathbb{R}$, with $0 < t_1 < t_2 < 2\pi$, such that $w = -i\left(\dfrac{e^{it_1} - e^{it_2}}{e^{it_1} - 1}\right)$. Let t_1, t_2 be such values, and let $\varphi(z) = -i\left(\dfrac{e^{it_1} - e^{it_2}}{e^{it_1} - 1}\right)\left(\dfrac{z-1}{z - e^{it_2}}\right)$. Then $\varphi(1) = 0$, $\varphi(e^{it_1}) = -i$, and $\varphi(e^{it_2}) = \infty$. This shows that $\varphi(\mathbb{D}) = \{z : \operatorname{Re} z > 0\}$ and $\varphi(\partial\mathbb{D}) = i\mathbb{R} \cup \{\infty\}$. Further note that $\varphi(\infty) = w$.

We can see that $\lim\limits_{z \to e^{it}} (\varphi^{-1} \circ f)(z)$ exists (as a finite limit) for all $t \in \mathbb{R}$. In particular, this shows that $(\varphi^{-1} \circ f)$ has a continuous extension $(\widehat{\varphi^{-1} \circ f})$ defined on $\overline{\mathbb{D}}$. From the hypothesis that $\mathrm{Re}(\widehat{f}(e^{it})) \geq 0$, we must have $(\widehat{\varphi^{-1} \circ f})(e^{it}) \leq 1$ for all $t \in \mathbb{R}$. Since f is nonconstant, Theorem F implies that $|(\varphi^{-1} \circ f)(z)| < 1$ for all $z \in \mathbb{D}$. The result follows. □

The following is another consequence of the Maximum Modulus Theorem.

Theorem G. (Schwarz's Lemma) Let $f : \mathbb{D} \to \mathbb{D}$ be analytic with $f(0) = 0$. Then $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$.

More specific to the subject of harmonic mappings is the following result of Clunie and Sheil-Small [2].

Theorem H. If $f = h + \overline{g} \in K_H$, then $\left|\dfrac{g(z_1) - g(z_2)}{h(z_1) - h(z_2)}\right| < 1$ for all $z_1, z_2 \in \mathbb{D}$.

Our application of this theorem will depend on a result of Nowak and Woloszkiewicz [14]. As this has not yet been published, we include their proof here.

Theorem I. If $f = h + \overline{g} \in S_H^0(R)$, then f is convex.

Proof. By Theorem B, it suffices to show that the function $h - e^{2i\alpha}g$ is convex in the α direction for every $\alpha \in [0, \pi)$. This is true if and only if $F_\alpha = ie^{-i\alpha}(h - e^{2i\alpha}g)$ is convex in the vertical direction.

We apply the following result of Royster and Ziegler [17].

Theorem J. Let φ be a conformal mapping that satisfies

$$\mathrm{Re}\ (-ie^{i\mu}(1 - 2\cos\nu e^{-i\mu}z + e^{-2i\mu}z^2)\varphi'(z)) \geq 0$$

for some $\mu, \nu \in [0, \pi]$. Then φ is univalent and convex in the vertical direction.

First assume that $\alpha \in [0, \pi/2]$. Taking $\mu = \nu = 0$ in Theorem J, we get

$$\begin{aligned}\mathrm{Re}\ (-iF'_\alpha(z)(1-z)^2) &= \mathrm{Re}\ (e^{-i\alpha}(h'(z) - e^{2i\alpha}g'(z))(1-z)^2) \\ &= \mathrm{Re}\ ((e^{-i\alpha}h'(z) - e^{i\alpha}g'(z))(1-z)^2) \\ &= \mathrm{Re}\ (((h'(z) - g'(z))\cos\alpha - i(h'(z) + g'(z))\sin\alpha)(1-z)^2).\end{aligned}$$

Since $h'(z) + g'(z) = \dfrac{1}{(1-z)^2}$, we get

$$\begin{aligned} \text{Re } (-iF'_\alpha(z)(1-z)^2) &= \text{Re } \left(\left(\frac{h'(z)-g'(z)}{h'(z)+g'(z)} \right) \cos\alpha - i \sin\alpha \right) \\ &= \text{Re } \left(\left(\frac{h'(z)-g'(z)}{h'(z)+g'(z)} \right) \cos\alpha \right) \end{aligned}$$

Note that $\dfrac{h'(z)-g'(z)}{h'(z)+g'(z)} = \dfrac{h'(z)(1-\omega(z))}{h'(z)(1+\omega(z))} = \dfrac{1-\omega(z)}{1+\omega(z)}$. This has positive real part for all z, since the Möbius transformation $\dfrac{1-z}{1+z}$ maps the unit disk onto the right half-plane $\{z : \text{Re } (z) > 0\}$. We may thus conclude from Theorem J that F_α is convex in the vertical direction. The same conclusion holds for $\alpha \in (\pi/2, \pi)$ by applying the same argument with $\mu = \nu = \pi$. □

We can see immediately from the formulas for f_1 and f_2 above that they are both in $S^0_H(R)$. Hence their images are convex by the preceding theorem.

4.2 Proof of Theorem

We continue now with the proof of Theorem 2. The argument follows a draft by Nowak and Woloszkiewicz in which they proved the special case of $\theta = \rho$ [14].

Proof. Theorem C proved this result in the case that $\theta = \pi$ or $\rho = \pi$. Hence we will assume that $\theta \neq \pi$ and $\rho \neq \pi$.

Note that $\log\left(\dfrac{1+e^{i\theta}z}{1-z}\right) = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n}(1+(-1)^{n-1}e^{i\theta n})z^n$. For an analytic function $F(z) = \sum_{n=1}^{\infty} a_n z^n$, we must have

$$\begin{aligned} \log\left(\frac{1+e^{i\theta}z}{1-z}\right) * F(z) &= \sum_{n=1}^{\infty} \frac{a^n}{n} z^n - \sum_{n=1}^{\infty} \frac{(-1)^n e^{i\theta n} a_n}{n} z^n \\ &= \sum_{n=1}^{\infty} \frac{a^n}{n} z^n - \sum_{n=1}^{\infty} \frac{a_n}{n} (-e^{i\theta} z)^n \\ &= \int_0^z \frac{F(u)}{u} du - \int_0^z \frac{F(-e^{i\theta}u)}{u} du = \int_0^z \frac{F(u) - F(-e^{i\theta}u)}{u} du. \end{aligned}$$

Using Equations 2.1 and 2.2 for h_1 and g_1, we obtain

$$
\begin{aligned}
(h_1 * F)(z) &= \frac{1}{1+e^{i\theta}}F(z) + \frac{e^{i\theta}}{(1+e^{i\theta})^2}\int_0^z \frac{F(u)-F(-e^{i\theta}u)}{u}du \\
(g_1 * F)(z) &= \frac{e^{i\theta}}{1+e^{i\theta}}F(z) - \frac{e^{i\theta}}{(1+e^{i\theta})^2}\int_0^z \frac{F(u)-F(-e^{i\theta}u)}{u}du.
\end{aligned}
$$

The dilatation of $f_1 * f_2$ is then

$$
\begin{aligned}
\widetilde{\omega}(z) &= \frac{\frac{d}{dz}(g_1 * g_2)(z)}{\frac{d}{dz}(h_1 * h_2)(z)} \\
&= \frac{\frac{d}{dz}\left(-\frac{e^{i\theta}}{(1+e^{i\theta})^2}\int_0^z \frac{g_2(u)-g_2(-e^{i\theta}u)}{u}du + \frac{e^{i\theta}}{1+e^{i\theta}}g_2(z)\right)}{\frac{d}{dz}\left(\frac{e^{i\theta}}{(1+e^{i\theta})^2}\int_0^z \frac{h_2(u)-h_2(-e^{i\theta}u)}{u}du + \frac{1}{1+e^{i\theta}}h_2(z)\right)} \\
&= \frac{-\frac{e^{i\theta}}{(1+e^{i\theta})^2}\left(\frac{g_2(z)-g_2(-e^{i\theta}z)}{z}\right) + \frac{e^{i\theta}}{1+e^{i\theta}}g_2'(z)}{\frac{e^{i\theta}}{(1+e^{i\theta})^2}\left(\frac{h_2(z)-h_2(-e^{i\theta}z)}{z}\right) + \frac{1}{1+e^{i\theta}}h_2'(z)} \\
&= e^{i\theta}\left(\frac{-\left(g_2(z)-g_2(-e^{i\theta}z)\right) + (1+e^{i\theta})zg_2'(z)}{e^{i\theta}\left(h_2(z)-h_2(-e^{i\theta}z)\right) + (1+e^{i\theta})zh_2'(z)}\right).
\end{aligned}
$$

By Theorem D, we can show that $f_1 * f_2 \in S_H$ and is convex in the horizontal direction and by showing that $|\widetilde{\omega}| < 1$. This is equivalent to

$$
|(g_2(-e^{i\theta}z) - g_2(z)) + (1+e^{i\theta})zg_2'(z)|^2 < |e^{i\theta}(h_2(z) - h_2(-e^{i\theta}z)) + (1+e^{i\theta})zh_2'(z)|^2.
$$

We rewrite this as

$$
\left|\frac{g_2(-e^{i\theta}z) - g_2(z)}{(1+e^{i\theta})zh_2'(z)} + \omega_2(z)\right|^2 < \left|\frac{e^{i\theta}(h_2(z) - h_2(-e^{i\theta}z))}{(1+e^{i\theta})zh_2'(z)} + 1\right|^2
$$

$$
\left|\frac{g_2(-e^{i\theta}z) - g_2(z)}{(1+e^{i\theta})z^2h_2'(z)} + e^{i\rho}\right|^2 |z|^2 < \left|\frac{(h_2(z) - h_2(-e^{i\theta}z))}{(1+e^{i\theta})zh_2'(z)} + e^{-i\theta}\right|^2.
$$

Hence it suffices to show that

$$
\left|\frac{g_2(-e^{i\theta}z) - g_2(z)}{(1+e^{i\theta})z^2h_2'(z)} + e^{i\rho}\right|^2 < \left|\frac{(h_2(z) - h_2(-e^{i\theta}z))}{(1+e^{i\theta})zh_2'(z)} + e^{-i\theta}\right|^2.
$$

We can rewrite this as

$$\left|\frac{g_2(-e^{i\theta}z)-g_2(z)}{(1+e^{i\theta})z^2h_2'(z)}\right|^2+2\mathrm{Re}\left(\frac{e^{-i\rho}(g_2(-e^{i\theta}z)-g_2(z))}{(1+e^{i\theta})z^2h_2'(z)}\right)+\left|e^{i\rho}\right|^2$$
$$<\left|\frac{(h_2(z)-h_2(-e^{i\theta}z))}{(1+e^{i\theta})zh_2'(z)}\right|^2+2\mathrm{Re}\left(\frac{e^{i\theta}(h_2(z)-h_2(-e^{i\theta}z))}{(1+e^{i\theta})zh_2'(z)}\right)+\left|e^{-i\theta}\right|^2.$$

We approach this inequality by comparing each term on the left side with the respective term on the right. For the last terms, we have $|e^{i\rho}|^2=1=\left|e^{-i\theta}\right|^2$. We consider now the first terms; that is, we want to show that

$$\left|\frac{g_2(-e^{i\theta}z)-g_2(z)}{(1+e^{i\theta})z^2h_2'(z)}\right|^2<\left|\frac{(h_2(z)-h_2(-e^{i\theta}z))}{(1+e^{i\theta})zh_2'(z)}\right|^2. \tag{4.1}$$

We know from Theorem H and Theorem I above that

$$\left|\frac{g_2(-e^{i\theta}z)-g_2(z)}{h_2(z)-h_2(-e^{i\theta}z)}\right|<1.$$

Also, since $\lim_{z\to 0}\frac{g_2(-e^{i\theta}z)-g_2(z)}{h_2(z)-h_2(-e^{i\theta}z)}=\lim_{z\to 0}\frac{-e^{i\theta}g_2'(-e^{i\theta}z)-g_2'(z)}{h_2'(z)+e^{i\theta}h_2'(-e^{i\theta}z)}=\frac{0-0}{1+e^{i\theta}}=0$, we may apply Schwarz's Lemma to conclude that $\left|\frac{g_2(-e^{i\theta}z)-g_2(z)}{h_2(z)-h_2(-e^{i\theta}z)}\right|<|z|$. This gives us the inequality in 4.1.

Next, we look at the difference of the second terms. Let

$$J(z)=\frac{e^{i\theta}(h_2(z)-h_2(-e^{i\theta}z))}{(1+e^{i\theta})zh_2'(z)}+\frac{e^{-i\rho}(g_2(z)-g_2(-e^{i\theta}z))}{(1+e^{i\theta})z^2h_2'(z)}.$$

We want to show that Re $J(z)>0$. Notice that $h_2(0)=0$, $h_2'(0)=1$, $g_2(0)=0$ and $g_2'(0)=0$. Hence $J(z)$ is analytic at $z=0$. Moreover, $h_2'(z)\neq 0$ for all $z\in\mathbb{D}$. This shows that $J(z)$ is analytic on $\mathbb{D}$.

We can now simplify $J(z)$ as follows.

$$\begin{aligned}
J(z) = {} & \frac{e^{i\theta}(1-z)^2(1+e^{i\rho}z)}{(1+e^{i\theta})z}\left(\frac{z}{(1+e^{i\rho})(1-z)}+\frac{e^{i\rho}}{(1+e^{i\rho})^2}\log\left(\frac{1+e^{i\rho}z}{1-z}\right)\right. \\
& \left.+\frac{e^{i\theta}z}{(1+e^{i\rho})(1+e^{i\theta}z)}-\frac{e^{i\rho}}{(1+e^{i\rho})^2}\log\left(\frac{1-e^{i(\theta+\rho)}z}{1+e^{i\theta}z}\right)\right) \\
& +\frac{(1-z)^2(1+e^{i\rho}z)}{(1+e^{i\theta})z^2}\left(\frac{z}{(1-z)(1+e^{i\rho})}-\frac{1}{(1+e^{i\rho})^2}\log\left(\frac{1+e^{i\rho}z}{1-z}\right)\right. \\
& \left.+\frac{e^{i\theta}z}{(1+e^{i\rho})(1+e^{i\theta}z)}+\frac{1}{(1+e^{i\rho})^2}\log\left(\frac{1-e^{i(\theta+\rho)}z}{1+e^{i\theta}z}\right)\right) \\
= {} & \frac{(1-z)^2(1+e^{i\rho}z)}{(1+e^{i\rho})(1+e^{i\theta})z}\left(\frac{e^{i\theta}z}{1-z}+\frac{e^{i(\rho+\theta)}}{1+e^{i\rho}}\log\left(\frac{1+e^{i\rho}z}{1-z}\right)\right. \\
& +\frac{e^{2i\theta}z}{1+e^{i\theta}z}-\frac{e^{i(\rho+\theta)}}{1+e^{i\rho}}\log\left(\frac{1-e^{i(\theta+\rho)}z}{1+e^{i\theta}z}\right)+\frac{1}{1-z} \\
& \left.-\frac{1}{(1+e^{i\rho})z}\log\left(\frac{1+e^{i\rho}z}{1-z}\right)+\frac{e^{i\theta}}{1+e^{i\theta}z}+\frac{1}{(1+e^{i\rho})z}\log\left(\frac{1-e^{i(\theta+\rho)}z}{1+e^{i\theta}z}\right)\right) \\
= {} & \frac{(1-z)^2(1+e^{i\rho}z)}{(1+e^{i\rho})(1+e^{i\theta})z}\left(\frac{1+e^{i\theta}z}{1-z}+\frac{e^{i\theta}+e^{2i\theta}z}{1+e^{i\theta}z}\right. \\
& \left.+\left(\frac{e^{i(\rho+\theta)}}{1+e^{i\rho}}-\frac{1}{(1+e^{i\rho})z}\right)\left(\log\left(\frac{1+e^{i\rho}z}{1-z}\right)-\log\left(\frac{1-e^{i(\theta+\rho)}z}{1+e^{i\theta}z}\right)\right)\right) \\
= {} & \frac{(1-z)^2(1+e^{i\rho}z)}{(1+e^{i\rho})(1+e^{i\theta})z}\left(\frac{1+e^{i\theta}}{1-z}-\frac{1-e^{i(\rho+\theta)}z}{(1+e^{i\rho})z}\left(\log\left(\frac{1+e^{i\rho}z}{1-z}\right)-\log\left(\frac{1-e^{i(\theta+\rho)}z}{1+e^{i\theta}z}\right)\right)\right) \\
= {} & \frac{(1-z)(1+e^{i\rho}z)}{(1+e^{i\rho})z}\left(1-\frac{(1-z)(1-e^{i(\rho+\theta)}z)}{(1+e^{i\theta})(1+e^{i\rho})z}\left(\log\left(\frac{1+e^{i\rho}z}{1-z}\right)-\log\left(\frac{1-e^{i(\theta+\rho)}z}{1+e^{i\theta}z}\right)\right)\right).
\end{aligned}$$

In order to apply Lemma 3, we need to check that $\lim\limits_{z\to e^{it}} J(z)$ exists for all $t\in\mathbb{R}$. This is clearly the case for all $t\in\mathbb{R}\setminus\{0,\pi-\rho,\pi-\theta,2\pi-\rho-\theta\}$. For the remaining values of t, we obtain the following limits.

$$\begin{aligned}
\lim_{z\to 1} J(z) &= 0 \\
\lim_{z\to e^{i(\pi-\rho)}} J(z) &= 0 \\
\lim_{z\to e^{i(2\pi-\theta-\rho)}} J(z) &= \frac{(1-e^{-i(\theta+\rho)})(1+e^{-i\theta})}{(1+e^{i\rho})e^{-i(\theta+\rho)}} = \frac{2i\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta+\rho}{2}\right)}{\cos\left(\frac{\rho}{2}\right)} \\
\lim_{z\to e^{i(\pi-\theta)}} J(z) &= \begin{cases} \infty & \text{if } \theta\neq\rho \\ 0 & \text{if } \theta=\rho \end{cases}
\end{aligned}$$

Hence it suffices to show that Re $J(e^{it})\geq 0$ for all $t\in\mathbb{R}\setminus\{0,\pi-\rho,\pi-\theta,2\pi-\rho-\theta\}$.

We have

$$J(e^{it}) = \frac{(1-e^{it})(1+e^{i(\rho+t)})}{(1+e^{i\rho})e^{it}}\left(1-\frac{(1-e^{it})(1-e^{i(\rho+\theta+t)})}{(1+e^{i\theta})(1+e^{i\rho})e^{it}}\left(\log\left(\frac{1+e^{i(\rho+t)}}{1-e^{it}}\right)-\log\left(\frac{1-e^{i(\theta+\rho+t)}}{1+e^{i(\theta+t)}}\right)\right)\right)$$
$$= \frac{-2i\sin\left(\frac{t}{2}\right)\cos\left(\frac{\rho+t}{2}\right)}{\cos\left(\frac{\rho}{2}\right)}\left(1+\frac{\sin\left(\frac{t}{2}\right)\sin\left(\frac{\theta+\rho+t}{2}\right)}{\cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\rho}{2}\right)}\left(\log\left(\frac{1+e^{i(\rho+t)}}{1-e^{it}}\right)-\log\left(\frac{1-e^{i(\theta+\rho+t)}}{1+e^{i(\theta+t)}}\right)\right)\right).$$

Hence

$$\text{Re } J(e^{it}) = \frac{2\sin^2\left(\frac{t}{2}\right)\sin\left(\frac{\theta+\rho+t}{2}\right)\cos\left(\frac{\rho+t}{2}\right)}{\cos^2\left(\frac{\rho}{2}\right)\cos\left(\frac{\theta}{2}\right)}\left(\arg\left(\frac{1+e^{i(\rho+t)}}{1-e^{it}}\right)-\arg\left(\frac{1-e^{i(\theta+\rho+t)}}{1+e^{i(\theta+t)}}\right)\right).$$

Let $A = \arg\left(\frac{1+e^{i(\rho+t)}}{1-e^{it}}\right)$ and let $B = \arg\left(\frac{1-e^{i(\theta+\rho+t)}}{1+e^{i(\theta+t)}}\right)$. We restrict ourselves now to when $0 \le \theta, \rho < \pi$. We have

$$A = \begin{cases} (\rho+\pi)/2 & \text{if } t \in (0, \pi-\rho) \\ (\rho-\pi)/2 & \text{if } t \in (\pi-\rho, 2\pi) \end{cases}$$
$$B = \begin{cases} (\rho-\pi)/2 & \text{if } t \in (0, \pi-\theta) \cup (2\pi-\theta-\rho, 2\pi) \\ (\rho+\pi)/2 & \text{if } t \in (\pi-\theta, 2\pi-\theta-\rho) \end{cases}.$$

Consider now the case that $\rho \le \theta$. Then

$$A - B = \begin{cases} \pi & \text{if } t \in (0, \pi-\theta) \\ 0 & \text{if } t \in (\pi-\theta, \pi-\rho) \cup (2\pi-\theta-\rho, 2\pi) \\ -\pi & \text{if } t \in (\pi-\rho, 2\pi-\theta-\rho) \end{cases}.$$

If $\theta < \rho$, then

$$A - B = \begin{cases} \pi & \text{if } t \in (0, \pi-\rho) \\ 0 & \text{if } t \in (\pi-\rho, \pi-\theta) \cup (2\pi-\theta-\rho, 2\pi) \\ -\pi & \text{if } t \in (\pi-\theta, 2\pi-\theta-\rho) \end{cases}.$$

Next, we consider $\dfrac{\sin\left(\frac{\theta+\rho+t}{2}\right)\cos\left(\frac{\rho+t}{2}\right)}{\cos\left(\frac{\theta}{2}\right)}$. This is positive when $t \in (0, \pi-\rho) \cup (2\pi-\rho-\theta, 2\pi)$, and negative when $t \in (\pi-\rho, 2\pi-\rho-\theta)$. A quick inspection shows that in all cases, $\operatorname{Re}(J(e^{it})) \geq 0$.

This completes the proof in the case that $\theta, \rho \in [0, \pi)$. The general result, that is, for all $\theta, \rho \in (-\pi, \pi)$, is similar. Its proof essentially amounts to relabeling the various intervals used in the case above. $\square$

It is natural to ask whether the above result is true in the case that $n > 1$. Some preliminary computations indicate that this is not the case. However, this has not yet been formally shown, and we leave it for now as an problem open for further investigation. Note that this is in contrast to Theorem C, which is true for both $n = 1$ and $n = 2$.

4.3 Formulas for the Convolutions

As in the case of Theorem C, we can compute the class of convolutions used in the previous theorem. Unlike our earlier computations, however, this will not be possible in terms of the usual elementary functions. The special function that will help us is the polylogarithm function $\operatorname{Li}_s(z)$. This is defined for all $|z| < 1$ using power series by

$$\operatorname{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}.$$

For reference, we recall equations 2.1 and 2.1 in Example 1. For $\theta \neq \pi$, the formula for $f_1 = h_1 + \overline{g_1}$ is

$$\begin{aligned} h_1(z) &= \frac{1}{1+e^{i\theta}}\left(\frac{z}{1-z}\right) + \frac{e^{i\theta}}{(1+e^{i\theta})^2}\log\left(\frac{1+e^{i\theta}z}{1-z}\right) \\ g_1(z) &= \frac{e^{i\theta}}{1+e^{i\theta}}\left(\frac{z}{1-z}\right) - \frac{e^{i\theta}}{(1+e^{i\theta})^2}\log\left(\frac{1+e^{i\theta}z}{1-z}\right). \end{aligned}$$

The formula for $f_2 = h_2 + \overline{g_2}$, $\rho \neq \pi$, is obtained by replacing θ with ρ in the preceding formula. Note that $\dfrac{z}{1-z} * F(z) = F(z)$ for any analytic function F such that $F(0) = 0$, as already shown in Example 2.

Next, we look at the convolution of the logarithm terms. The series representation for $\log\left(\dfrac{1+e^{i\theta}z}{1-z}\right)$ is

$$\log\left(\frac{1+e^{i\theta}z}{1-z}\right) = \sum_{n=1}^{\infty} \frac{1+(-1)^{n-1}e^{ni\theta}}{n} z^n.$$

We have then

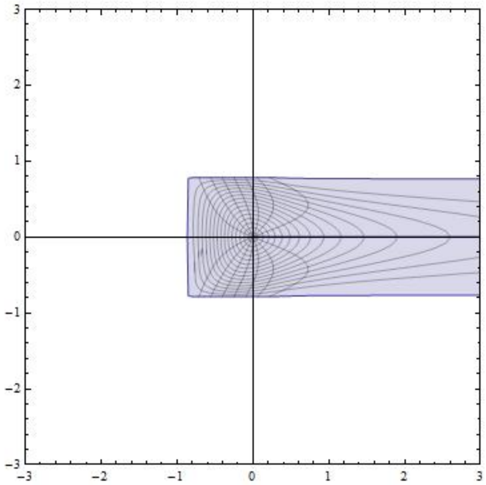
$$\begin{aligned}
\log\left(\frac{1+e^{i\theta}z}{1-z}\right) * \log\left(\frac{1+e^{i\rho}z}{1-z}\right) &= \sum_{n=1}^{\infty} \frac{(1+(-1)^{n-1}e^{ni\theta})(1+(-1)^{n-1}e^{ni\rho})}{n^2} z^n \\
&= \sum_{n=1}^{\infty} \frac{z^n}{n^2} - \sum_{n=1}^{\infty} \frac{(-e^{i\theta}z)^n}{n^2} - \sum_{n=1}^{\infty} \frac{(-e^{i\rho}z)^n}{n^2} + \sum_{n=1}^{\infty} \frac{(e^{i(\theta+\rho)}z)^n}{n^2} \\
&= \mathrm{Li}_2(z) - \mathrm{Li}_2\left(-e^{i\theta}z\right) - \mathrm{Li}_2\left(-e^{i\rho}z\right) + \mathrm{Li}_2\left(e^{i(\theta+\rho)}z\right).
\end{aligned}$$

Putting everything together, we obtain

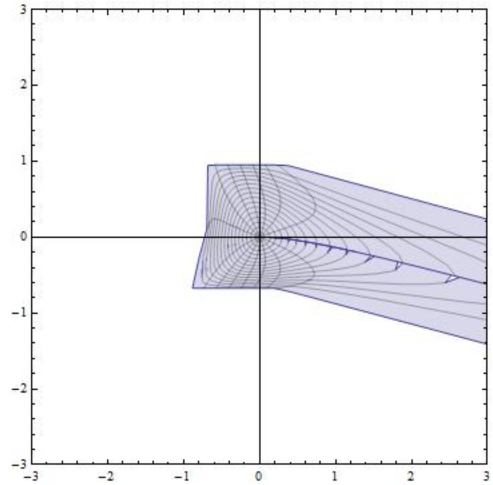
$$\begin{aligned}
(h_1 * h_2)(z) = {} & \frac{1}{16}\sec^2\left(\frac{\theta}{2}\right)\sec^2\left(\frac{\rho}{2}\right)\left(\mathrm{Li}_2\left(e^{i(\theta+\rho)}z\right) - \mathrm{Li}_2\left(-e^{i\theta}z\right) - \mathrm{Li}_2\left(-e^{i\rho}z\right) + \mathrm{Li}_2(z)\right) \\
& + \frac{z}{(1+e^{i\theta})(1+e^{i\rho})(1-z)} + \frac{\sec^2\left(\frac{\theta}{2}\right)}{4(1+e^{i\rho})}\log\left(\frac{1+e^{i\theta}z}{1-z}\right) + \frac{\sec^2\left(\frac{\rho}{2}\right)}{4(1+e^{i\theta})}\log\left(\frac{1+e^{i\rho}z}{1-z}\right) \\
(g_1 * g_2)(z) = {} & \frac{1}{16}\sec^2\left(\frac{\theta}{2}\right)\sec^2\left(\frac{\rho}{2}\right)\left(\mathrm{Li}_2\left(e^{i(\theta+\rho)}z\right) - \mathrm{Li}_2\left(-e^{i\theta}z\right) - \mathrm{Li}_2\left(-e^{i\rho}z\right) + \mathrm{Li}_2(z)\right) \\
& + \frac{e^{i(\theta+\rho)}z}{(1+e^{i\theta})(1+e^{i\rho})(1-z)} - \frac{e^{i\rho}\sec^2\left(\frac{\theta}{2}\right)}{4(1+e^{i\rho})}\log\left(\frac{1+e^{i\theta}z}{1-z}\right) - \frac{e^{i\theta}\sec^2\left(\frac{\rho}{2}\right)}{4(1+e^{i\theta})}\log\left(\frac{1+e^{i\rho}z}{1-z}\right).
\end{aligned}$$

Note that we have made these formulas slightly more compact using the fact that $\dfrac{e^{i\theta}}{(1+e^{i\theta})^2} = \sec^2\left(\dfrac{\theta}{2}\right)$ and $\dfrac{e^{i\rho}}{(1+e^{i\rho})^2} = \sec^2\left(\dfrac{\rho}{2}\right)$.

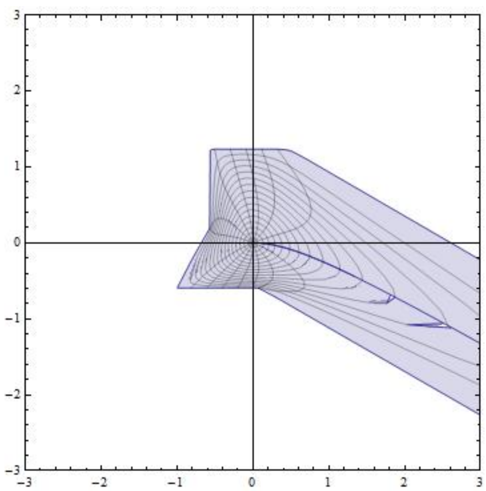
Images of the unit disk under some of these convolutions are in Figure 4.1 and in Figure 4.2. There is one point of contrast in the geometry of these compared to the previous examples, in that the boundary does not collapse to finitely many points. This implies that the dilatation of the convolution is no longer an inner function. This is reflected in the fact that Re $J(e^{it})$ is not identically zero in the proof of Theorem 2.



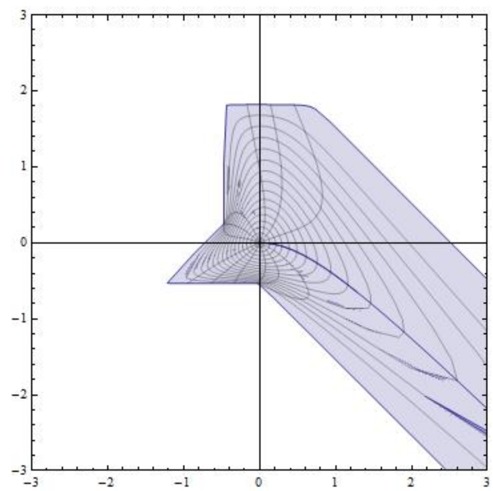
(a) $\theta = 0, \rho = 0$



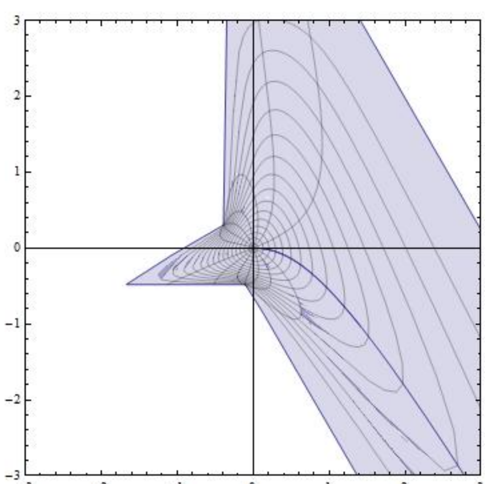
(b) $\theta = 0, \rho = \pi/6$



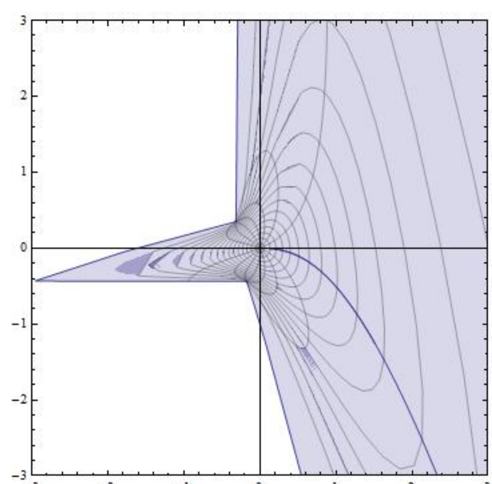
(c) $\theta = 0, \rho = \pi/3$



(d) $\theta = 0, \rho = \pi/2$

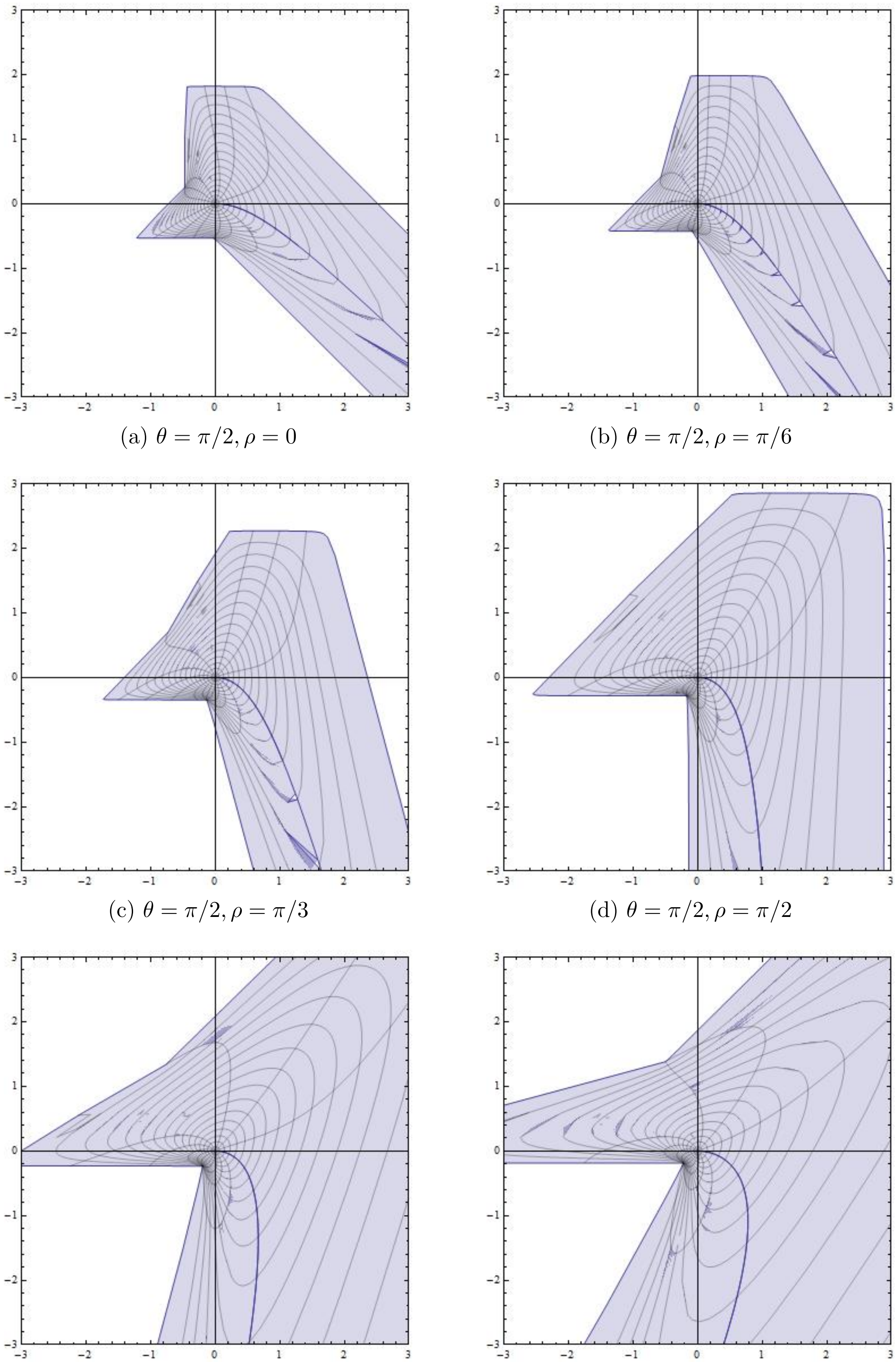


(e) $\theta = 0, \rho = 2\pi/3$



(f) $\theta = 0, \rho = 5\pi/6$

Figure 4.1: Images of $\mathbb{D}$ under $f_1 * f_2$, where f_1 is the vertical shear of $z/(1-z)$ with dilatation $\omega_1 = e^{i\theta}z$, and f_2 is the vertical shear of $z/(1-z)$ with dilatation $\omega_2 = e^{i\rho}z$, part (i).



(a) $\theta = \pi/2, \rho = 0$ (b) $\theta = \pi/2, \rho = \pi/6$

(c) $\theta = \pi/2, \rho = \pi/3$ (d) $\theta = \pi/2, \rho = \pi/2$

(e) $\theta = \pi/2, \rho = 2\pi/3$ (f) $\theta = \pi/2, \rho = 5\pi/6$

Figure 4.2: Images of $\mathbb{D}$ under $f_1 * f_2$, where f_1 is the vertical shear of $z/(1-z)$ with dilatation $\omega_1 = e^{i\theta}z$, and f_2 is the vertical shear of $z/(1-z)$ with dilatation $\omega_2 = e^{i\rho}z$, part (ii).

Chapter 5. Convolutions of Non-vertical Shears

Up to this point, we have considered only shears in the vertical direction. Another possible path to generalize Theorem C is to consider different directions of shearing. In this section we consider this problem briefly. Unfortunately, as it turns out, convolutions obtained in this manner will usually not be univalent. Even when they are, they are usually not convex in the horizontal direction, and so the techniques from above no longer apply.

The best we can do is establish the following convolution identity. While the identity applies to arbitrary directions of shearing, its real significance is the case of horizontal shears. It will establish an equivalence between convolutions of vertical shears and convolutions of horizontal shears. This will allow any result concerning vertical shearing to be immediately translated into a corresponding theorem for horizontal shears, and vice versa.

To state the identity, we must first establish some more convenient notation. We will use $f_{\alpha,\omega}$ to denote the shear of $\varphi(z) = \dfrac{z}{1-z}$ in the α direction with dilatation $e^{-2i\alpha}\omega$. That is, $f_{\alpha,\omega} = h_{\alpha,\omega} + \overline{g_{\alpha,\omega}}$, where $h_{\alpha,\omega} - e^{2i\alpha}g_{\alpha,\omega} = \dfrac{z}{1-z}$ and $g'/h' = e^{-2i\alpha}\omega$.

Lemma 4. Let $\alpha, \beta \in \mathbb{R}$, and let ω be analytic on $\mathbb{D}$ with $|\omega| < 1$. Then $h_{\alpha,\omega} = h_{\beta,\omega}$ and $g_{\alpha,\omega} = e^{2i(\alpha-\beta)}g_{\beta,\omega}$.

Proof. We examine the steps of the shearing technique, keeping track of the relationship between the two shears. Differentiating the equation $\varphi = h_{\alpha,\omega} - e^{2i\alpha}g_{\alpha,\omega}$ gives

$$\begin{aligned}\varphi' &= h'_{\alpha,\omega} - e^{2i\alpha}g'_{\alpha,\omega} \\ &= h'_{\alpha,\omega}(1 - e^{2i\alpha}(e^{-2i\alpha}\omega)) \\ &= h'_{\alpha,\omega}(1-\omega).\end{aligned}$$

Hence $h'_{\alpha,\omega} = \varphi'/(1-\omega)$. Likewise, we have $h'_{\beta,\omega} = \varphi'/(1-\omega)$. This shows that $h_{\alpha,\omega} = h_{\beta,\omega}$. This gives immediately $g_{\alpha,\omega} = e^{-2i\alpha}(h_{\alpha,\omega} - \varphi)$ and likewise $g_{\beta,\omega} = e^{-2i\beta}(h_{\beta,\omega} - \varphi)$. We conclude that $g_{\alpha,\omega} = e^{2i(\alpha-\beta)}g_{\beta,\omega}$. □

Theorem 5. Let $\alpha, \beta \in \mathbb{R}$, and let ω_1, ω_2 be analytic on $\mathbb{D}$ with $|\omega_1| < 1$ and $|\omega_2| < 1$. Then $f_{\alpha,\omega_1} * f_{-\alpha,\omega_2} = f_{\beta,\omega_1} * f_{-\beta,\omega_2}$.

Proof. This is a direct computation. We can express the various functions as follows.

$$\begin{aligned}
f_{\alpha,\omega_1}(z) &= h_{\alpha,\omega_1}(z) + \overline{g_{\alpha,\omega_1}(z)} = \sum_{n=1}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b_n}\overline{z}^n \\
f_{-\alpha,\omega_2}(z) &= h_{-\alpha,\omega_2}(z) + \overline{g_{-\alpha,\omega_2}(z)} = \sum_{n=1}^{\infty} A_n z^n + \sum_{n=1}^{\infty} \overline{B_n}\overline{z}^n \\
f_{\beta,\omega_1}(z) &= h_{\beta,\omega_1}(z) + \overline{g_{\beta,\omega_1}(z)} = \sum_{n=1}^{\infty} c_n z^n + \sum_{n=1}^{\infty} \overline{d_n}\overline{z}^n \\
f_{-\beta,\omega_2}(z) &= h_{-\beta,\omega_2}(z) + \overline{g_{-\beta,\omega_2}(z)} = \sum_{n=1}^{\infty} C_n z^n + \sum_{n=1}^{\infty} \overline{D_n}\overline{z}^n
\end{aligned}$$

We then have

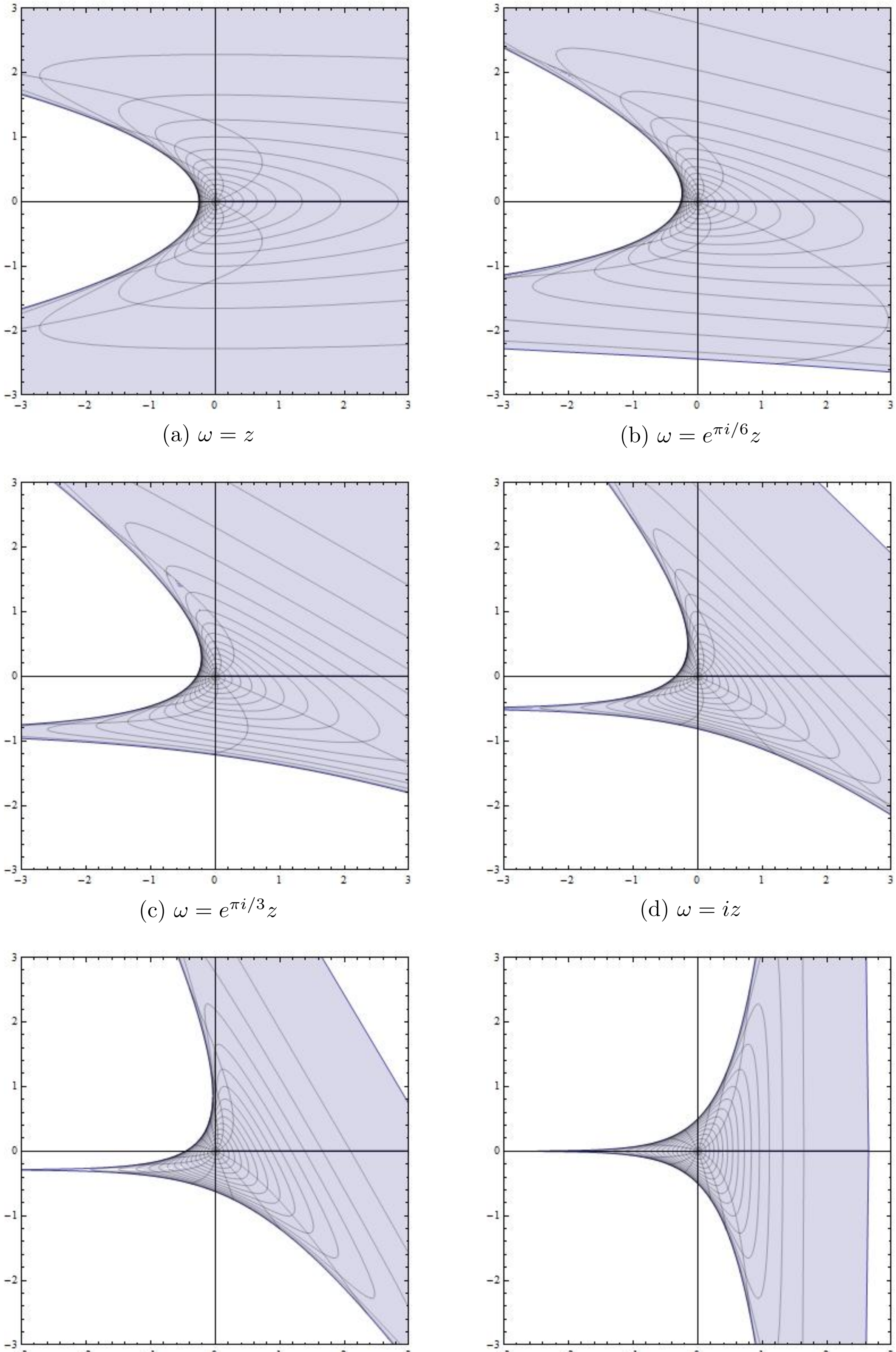
$$\begin{aligned}
(f_{\alpha,\omega_1} * f_{-\alpha,\omega_2})(z) &= \sum_{n=1}^{\infty} a_n A_n z^n + \sum_{n=1}^{\infty} \overline{b_n B_n}\overline{z}^n \\
(f_{\beta,\omega_1} * f_{-\beta,\omega_2})(z) &= \sum_{n=1}^{\infty} c_n C_n z^n + \sum_{n=1}^{\infty} \overline{d_n D_n}\overline{z}^n
\end{aligned}$$

From the Lemma we have $a_n = c_n$, $b_n = e^{2i(\alpha-\beta)} d_n$, $A_n = C_n$, and $B_n = e^{2i(-\alpha+\beta)} D_n$ for all n. Hence

$$\begin{aligned}
(f_{\beta,\omega_1} * f_{-\beta,\omega_2})(z) &= \sum_{n=1}^{\infty} a_n A_n z^n + \sum_{n=1}^{\infty} \overline{e^{2i(\alpha-\beta)} d_n e^{2i(-\alpha+\beta)} D_n}\overline{z}^n \\
&= \sum_{n=1}^{\infty} a_n A_n z^n + \sum_{n=1}^{\infty} \overline{d_n D_n}\overline{z}^n \\
&= (f_{\alpha,\omega_1} * f_{-\alpha,\omega_2})(z).
\end{aligned}$$

□

This theorem has the aesthetically displeasing aspect that the two functions being convolved are shears in different directions. The exceptions to this are the vertical and horizontal



(a) $\omega = z$

(b) $\omega = e^{\pi i/6} z$

(c) $\omega = e^{\pi i/3} z$

(d) $\omega = iz$

(e) $\omega = e^{\pi 2i/3} z$

(f) $\omega = -z$

Figure 5.1: Images of $\mathbb{D}$ under the horizontal shears of $\varphi(z) = z/(1-z)$ with dilatation ω.

directions, which is where this theorem seems most useful. We demonstrate this with the following alternate versions of Theorem D and Theorem 2.

Corollary 6. Let $f_1 = h_1 + \overline{g_1} \in S_H^0$ with $h_1 - g_1 = z/(1-z)$ and dilatation $\omega_1(z) = z$. Let $f \in h + \overline{g} \in S_H^0$ with $h - g = z/(1-z)$ and dilatation $\omega(z) = e^{i\theta}z^n$ ($n \in \mathbb{N}$ and $\theta \in \mathbb{R}$). If $n = 1, 2$, then $f_1 * f \in S_H^0$ and is convex in the horizontal direction.

Proof. In the notation of Theorem 5, we have $f_1 * f = f_{0,z} * f_{0,e^{i\theta}z^n}$. This is equal to $f_{\pi/2,z} * f_{\pi/2,e^{i\theta}z^n}$. But $f_{\pi/2,z}$ is just f_0 in the statement of Theorem C. The result follows by that theorem. □

Corollary 7. Let $f_1 = h_1 + \overline{g_1} \in S_H^0$ with $h_1 - g_1 = z/(1-z)$ and dilatation $\omega_1 = e^{i\theta}z$, $\theta \in \mathbb{R}$. Let $f_2 = h_2 + \overline{g_2} \in S_H^0$ with $h_2 - g_2 = z/(1-z)$ and dilatation $\omega_2 = e^{i\rho}z$, $\rho \in \mathbb{R}$. Then $f_1 * f_2 \in S_H^0$ and is convex in the horizontal direction.

Proof. This is similar to the first corollary. In the notation of Theorem 5, we have $f_1 * f_2 = f_{0,\omega_1} * f_{0,\omega_2}$. This is equal to $f_{\pi/2,\omega_1} * f_{\pi/2,\omega_2}$, and the result follows by Theorem 2. □

Example 3. We end this section by computing some horizontal shears of $\varphi(z) = z/(1-z)$. The convolution of any two of these functions is univalent by Corollary 7.

In slight contrast to Example 1, h and g now satisfy $h(z) - g(z) = \dfrac{z}{1-z}$. From here we get $\frac{1}{(1-z)^2} = h'(z) - g'(z) = h'(z)(1-\omega(z)) = h'(z)(1-e^{i\theta}z^n)$. So

$$h(z) = \int_0^z \frac{ds}{(1-s)^2(1-\omega(s))} = \int_0^z \frac{ds}{(1-s)^2(1-e^{i\theta}s^n)}.$$

We can then find g by using the relationship $g(z) = h(z) - \varphi(z)$.

In the case that $n = 1$ and $\theta = 0$, we have

$$\begin{aligned} h(z) &= \int_0^z \frac{ds}{(1-s)^3} = \frac{z - \frac{1}{2}z^2}{(1-z)^2} \\ g(z) &= \frac{\frac{1}{2}z^2}{(1-z)^2}. \end{aligned}$$

Next, in the case that $n = 1$ and $\theta \neq 0$, we have

$$h(z) = \int_0^z \frac{ds}{(1-s)^2(1-e^{i\theta}s)} = \frac{1}{1-e^{i\theta}}\left(\frac{z}{1-z}\right) - \frac{e^{i\theta}}{(1-e^{i\theta})^2}\log\left(\frac{1-e^{i\theta}z}{1-z}\right)$$
$$g(z) = \frac{e^{i\theta}}{1-e^{i\theta}}\left(\frac{z}{1-z}\right) - \frac{e^{i\theta}}{(1-e^{i\theta})^2}\log\left(\frac{1-e^{i\theta}z}{1-z}\right).$$

Note the similarities between these functions and those in Example 1, the exact relationship being given by Lemma 4. Some graphs of these are in Figure 5.1.

Chapter 6. Singular Inner Function Dilatation

This chapter is devoted to a research problem of a different nature from the previous two chapters. Up to this point, we have dealt almost entirely with harmonic mappings whose dilatation is an inner function. Recall that an inner function is an analytic function $\omega : \mathbb{D} \to \mathbb{D}$ such that for almost all $u \in \partial\mathbb{D}$, $\lim_{z\to u} \omega(z)$ exists and $\left|\lim_{z\to u} \omega(z)\right| = 1$. An example of an inner function is a *Blashcke product*, meaning a function $B(z) = e^{i\theta}\prod_{j=1}^{\infty}\left(\frac{z-a_j}{1-\overline{a_j}z}\right)$, where $\theta \in \mathbb{R}$ and each $|a_j| < 1$. Recall further that all the shears we have considered up this point have used a *finite* Blashcke product. Any inner function ω can be written as

$$\omega(z) = e^{i\alpha}B(z)\exp\left(-\int \frac{e^{i\theta}+z}{e^{i\theta}-z}d\mu(e^{i\theta})\right), \tag{6.1}$$

where $\alpha, \theta \in \mathbb{R}$, $B(z)$ is a Blashcke product, and μ is a singular positive measure on $\partial\mathbb{D}$ [9].

An inner function is *singular* if it has no zeros. We can see from Equation 6.1 that a singular inner function may be expressed as $e^{i\alpha}\exp\left(-\int \frac{e^{i\theta}+z}{e^{i\theta}-z}d\mu(e^{i\theta})\right)$, where again $\alpha, \theta \in \mathbb{R}$ and μ is a singular positive measure on $\partial\mathbb{D}$. Taking μ to be the unit mass measure concentrated at 1, we get a basic example of a singular inner function, $f(z) = e^{\frac{z+1}{z-1}}$.

One major difficulty in the study of harmonic mappings with singular inner function dilatation is the lack of examples. While such harmonic mappings must clearly exist, even as late as 1995 there were no known explicit examples [11]. In the following years, Weitsman gave two examples in an unpublished manuscript [18]. These, along with a small handful of other examples, can be found in [6].

In this section, we show how clever use of the shearing technique can be used to create an infinite family of harmonic mappings with singular inner function dilatation. The author's Honors Thesis [16] pursues this topic in greater detail, along with applications to minimal surfaces.

The example we will give here is based on a result of Royster and Ziegler [17]. This result has already been stated in this thesis as Theorem J in Section 4.1, but we give it again here in a slightly modified form.

Theorem K. Let φ be a conformal mapping that satisfies

$$\mathrm{Re}(-e^{i\mu}(1 - 2\cos\nu e^{-i\mu}z + e^{-2i\mu}z^2)\varphi'(z)) \geq 0$$

for some $\mu, \nu \in [0, \pi]$. Then φ is univalent and convex in the horizontal direction.

Example 4. In contrast to the previous examples, this one will be a horizontal shear. We will use as our dilatation $\omega = e^{\gamma\left(\frac{z+1}{z-1}\right)}$, where $\gamma > 0$. We will determine the analytic function $\varphi = h - g$ by setting

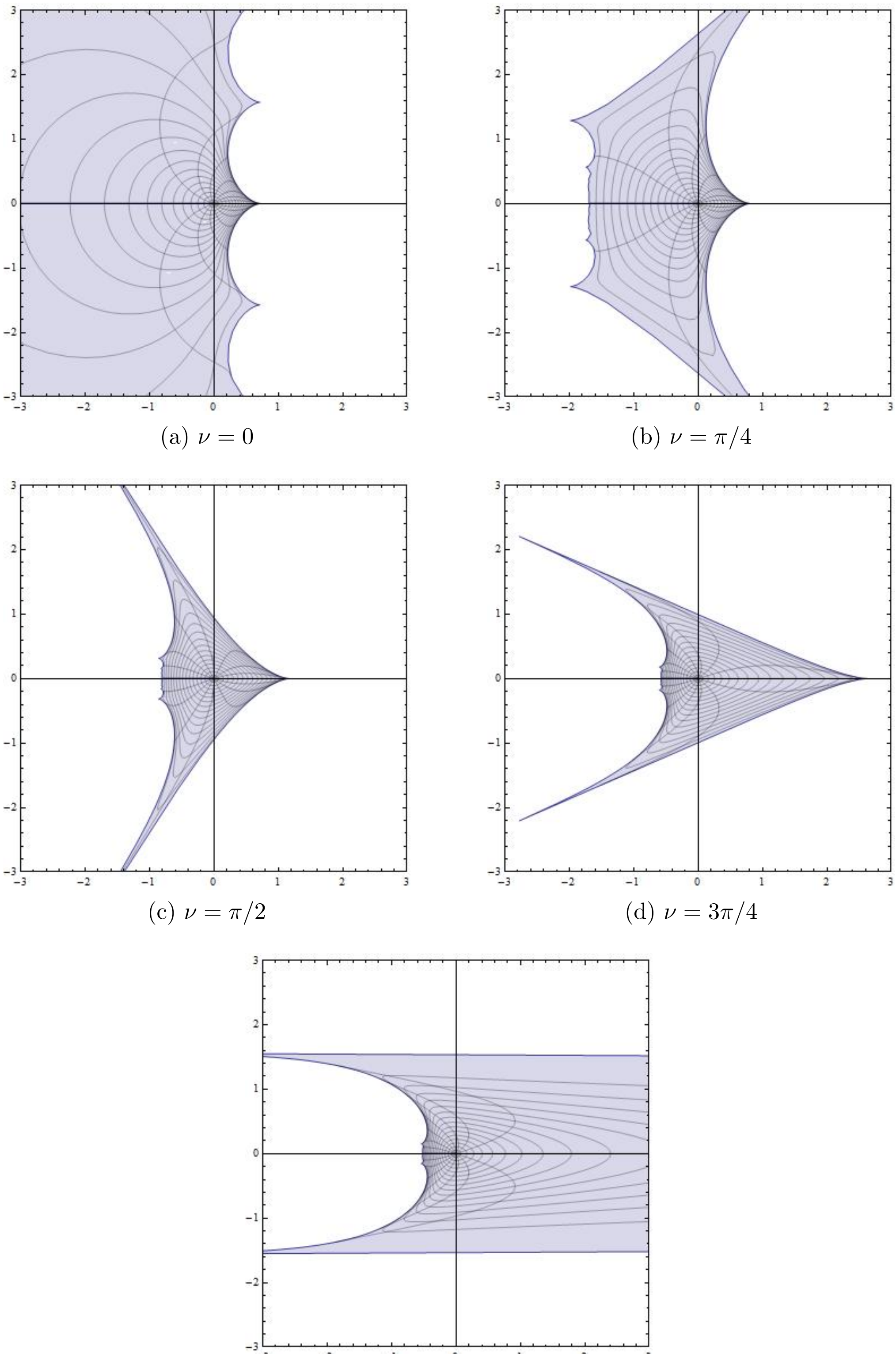
$$h'(z) = \frac{1}{-e^{i\mu}(1 - 2\cos\nu e^{-i\mu}z + e^{-2i\mu}z^2)} \quad \text{and} \quad g'(z) = h' e^{\gamma\left(\frac{z+1}{z-1}\right)}.$$

Using this choice of φ, the equation in Theorem K now takes the form

$$\mathrm{Re}\ \left(1 - e^{\gamma\left(\frac{z+1}{z-1}\right)}\right) \geq 0,$$

which is easily seen to be true. Thus φ, and the corresponding shear $f = h + \overline{g}$, are univalent and convex in the horizontal direction.

An explicit solution for f is not possible with the usual elementary functions. To remedy

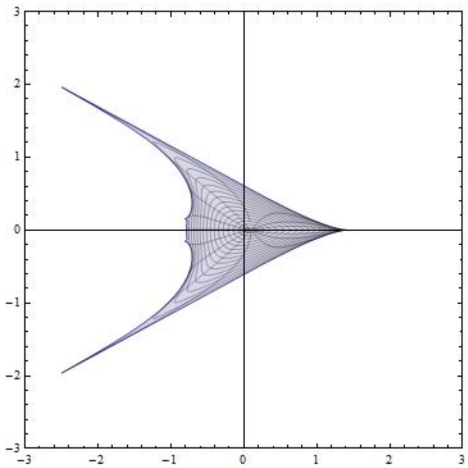


(a) $\nu = 0$ (b) $\nu = \pi/4$

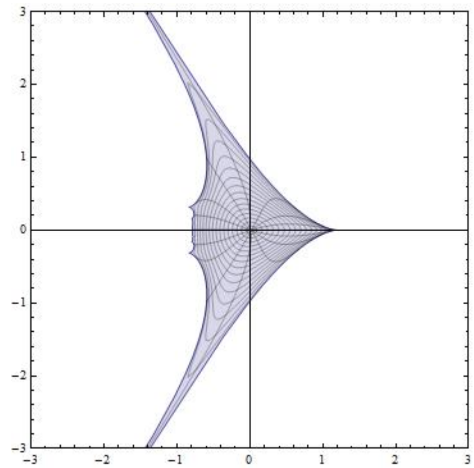
(c) $\nu = \pi/2$ (d) $\nu = 3\pi/4$

(e) $\nu = \pi$

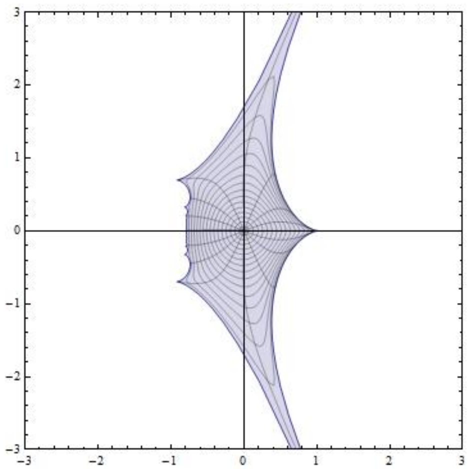
Figure 6.1: Harmonic mappings given by $h'(z) = \dfrac{1}{-e^{i\mu}(1 - 2\cos\nu e^{-i\mu}z + e^{-2i\mu}z^2)}$ and $\omega(z) = e^{\gamma\left(\frac{z+1}{z-1}\right)}$, where $\mu = 0$ and $\gamma = 2$.



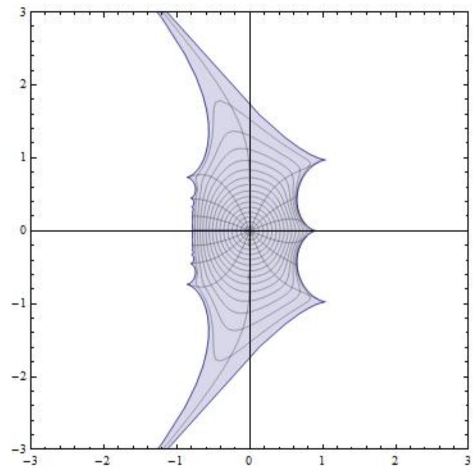
(a) $\gamma = 1$



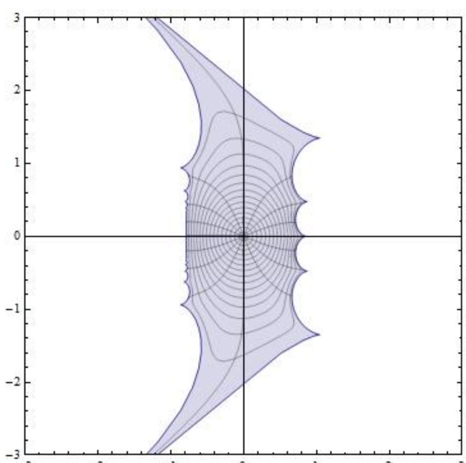
(b) $\gamma = 2$



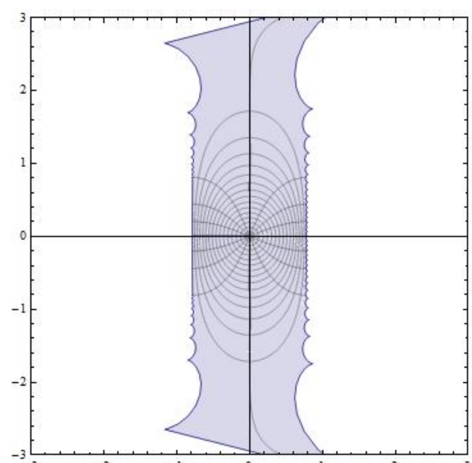
(c) $\gamma = 4$



(d) $\gamma = 8$



(e) $\gamma = 14$



(f) $\gamma = 100$

Figure 6.2: Harmonic mappings given by $h'(z) = \dfrac{1}{-e^{i\mu}(1 - 2\cos\nu e^{-i\mu}z + e^{-2i\mu}z^2)}$ and $\omega(z) = e^{\gamma\left(\frac{z+1}{z-1}\right)}$, where $\mu = 0$ and $\nu = \pi/2$.

this situation, we use the exponential integral function $E_n(z)$, defined for all $z \in \{z : \text{Re } (z) > 0\}$ by

$$E_n(z) = \int_1^\infty \frac{e^{-z\zeta}}{\zeta^n} d\zeta.$$

We simplify things here by taking $\mu = 0$; the general solution is similar but too large for inclusion here. We have then

$$\begin{aligned}
h(z) &= \int_0^z \frac{1}{-(1-2\cos\nu\zeta+\zeta^2)} d\zeta = \frac{1}{\sqrt{\alpha\beta}} \arctan\left(\frac{-z+\cos\nu}{\sqrt{\alpha\beta}}\right) \\
g(z) &= \int_0^z \frac{e^{\gamma\left(\frac{z+1}{z-1}\right)}}{-(1-2\cos\nu\zeta+\zeta^2)} d\zeta \\
&= \frac{1}{2\sqrt{-\alpha\beta}} \left(e^{-\gamma\sqrt{-\alpha/\beta}} E_1\left(-\gamma u - \gamma\sqrt{-\alpha/\beta}\right) - e^{\gamma\sqrt{-\alpha/\beta}} E_1\left(-\gamma u + \gamma\sqrt{-\alpha/\beta}\right)\right),
\end{aligned}$$

where $\alpha = 1 + \cos\nu$, $\beta = 1 - \cos\nu$, and $u = \dfrac{z+1}{z-1}$.

The interesting aspect of this example is that there are three separate parameters μ, ν, and γ, which may be varied to find a wide variety of new examples. In fact, several of the examples from [6] are special cases of Example 4. Figures 6.1 and 6.2 contain images of some of these. Notice that the boundary consists of infinitely many concave arcs, which seems typical for harmonic mappings with singular inner function dilatation.

Chapter 7. Conclusion

Here, we summarize our results on convolutions in the context of other results on univalent convolutions. This provides a more complete picture of the body of research on harmonic convolutions, as well as a list of open problems for future research. These results in the case of convolutions of vertical shears of the canonical half plane are listed in Figure 7.1.

The general Blaschke product of degree one is of the form $e^{i\theta}\dfrac{a-z}{1-\overline{a}z}$. It is of note that all the dilatations in the left column of Figure 7.1 are of this type, either taking $\theta = 0$, $\theta = \pi$, or $a = 0$. However, the various results in the table seem quite disjoint, without any obvious

Table 7.1: Convolutions of Vertical Shears of $\varphi(z) = \dfrac{z}{1-z}$

Dilatation of f_1	Dilatation of f_2	Status of $f_1 * f_2$
$-z$	$e^{i\rho}z^n$, $n = 1, 2$	Univalent for all $\rho \in \mathbb{R}$ [5].
$-z$	$e^{i\rho}z^n$, $n \geq 3$	Non-univalent for all $\rho \in \mathbb{R}$, shown here.
$e^{i\theta}z$	$e^{i\rho}z$	Univalent for all $\theta, \rho \in \mathbb{R}$, shown here.
$\dfrac{a-z}{1-\bar{a}z}$	$e^{i\theta}z^n$	If $a \in \mathbb{R}$, then univalent precisely when $a \in [(n-2)/(n+2), 1)$ [10]. Unknown for $a \notin \mathbb{R}$.
$-z$	$\dfrac{z-a}{1-\bar{a}z}$	Univalent precisely when $(\operatorname{Re} a)^2 + 9\,(\operatorname{Im} a)^2 \leq 1$ and $\operatorname{Re} a \neq \pm 1$. [13]
$\dfrac{a-z}{1-\bar{a}z}$	$\dfrac{b-z}{1-\bar{b}z}$	If $a, b \in \mathbb{R}$, univalent when $b \geq -\dfrac{1+3a}{3+a}$. [10]

connections. The major question for further research, then, would be to generalize these into a single result for all degree one Blaschke products. Given the delicate nature of preserving univalence, it is not clear even what such a generalization would be.

Presumably, Blaschke products of higher degree will result in non-univalent convolutions. Still, this is another direction that has not been explored fully. In particular, is there an efficient argument that would apply to a wide class of functions? Ideally, such a result would provide us with a complete characterization of convolutions of half-plane shears.

Another route for further research would be to derive analogous results for shears of other domains besides the canonical half plane. This was done by Dorff, M. Nowak, and M. Woloszkiewicz in [5] in the case of Theorem C. The techniques used in their proof were similar to those used to prove Theorem C, which suggests that Theorem 2 could be extended in a similar manner.

We summarize the preceding discussion with the following list of open problems.

1. Find conditions on f for $f_1 * f$ to be univalent, where f_1 is the vertical shear of $z/(1-z)$ with dilatation $e^{i\theta}\dfrac{a-z}{1-\bar{a}z}$.

2. Determine when $f_1 * f_2$ is univalent, where f_1 is the vertical shear of $z/(1-z)$ with dilatation $e^{i\theta}z, \theta \neq \pi$, and f_2 is the vertical shear of $z/(1-z)$ with dilatation $e^{i\phi}z^n$,

$\phi \neq \pi$ and $n \geq 2$.

3. Derive a result analogous to Theorem 2 for shears of other domains, such as the vertical strip given by $\varphi(z) = \dfrac{1}{2i} \log \left(\dfrac{1+iz}{1-iz} \right)$.

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