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Adding Limit Points to Bass-Serre Graphs of Groups

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ABSTRACT

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We give a brief overview of Bass-Serre theory and introduce a method of adding a limit point to graphs of groups. We explore a basic example of this method, and find that while the fundamental theorem of Bass-Serre theory no longer applies in this case we still recover a group action on a covering space of sorts with a subgroup isomorphic to the fundamental group of our new base space with added limit point. We also quantify how much larger the fundamental group of a graph of groups becomes after this construction, and discuss the effects of adding and identifying together such limit points in more general graphs of groups. We conclude with a theorem stating that the cokernel of the map on fundamental groups induced by collapsing an arc between two limit points contains a certain fundamental group of a double cone of graphs of groups, and we conjecture that this cokernel is isomorphic to this double cone group.

Keywords: covering space, inverse limit, Bass-Serre

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Chapter 1. Introduction

1.1 Introduction

Bass-Serre theory was first developed by Jean-Pierre Serre, compiled into the book Trees [1] in collaboration with Hyman Bass, who later made substantial contributions on his own to the basics of the theory. The theory concerns itself with decompositions of groups using free products with amalgamation and HNN extensions, and is an important tool in fields such as geometric group theory. While largely algebraic treatments of the theory are available (see [1], for instance), we will focus on obtaining topological intuition for the theory and building from that point.

Graphs of groups are defined by choosing a graph and assigning groups to each edge and vertex, together with inclusions between groups when necessary. Our work explores the effect of moving one step away from simplicial graphs by introducing a limit point to a graph.

1.2 Outline

In Chapter 2 we will give an introduction to Bass-Serre theory. We will discuss the building blocks of the theory (free products with amalgamation and HNN extensions), give a topological interpretation of those operations, and show how they fit together to define a fundamental group of a graph of groups. We will also introduce the fundamental theorem of Bass-Serre theory in this chapter.

In Chapter 3 we will introduce inverse limits, the main tool used to coherently add a limit point to our graphs. We will then work out the details of obtaining $\{0\} \cup \{\frac{1}{n} | n \in \mathbb{N}\}$ with the standard topology as an inverse limit of finite point sets in preparation for a more complicated inverse limit in chapter 4.

In Chapter 4 we will work out a basic example of adding a limit point to a graph of groups. We will show that while it does not have a universal covering space we can still find a cover of sorts via an inverse limit of covering spaces, and we explore properties of this pseudo-cover.

Finally, in Chapter 5 we will discuss how our method of adding a limit point creates a much larger group out of the original fundamental group of a graph of groups. We will further explore adding limit points to more general graphs of groups than covered in chapter 4, and end with a theorem and a conjecture regarding the identification of limit points.

Chapter 2. Bass-Serre Theory

Our treatment of Bass-Serre Theory will come largely from [2]. As this will necessarily be a brief overview, the reader interested in a fuller treatment is invited to study [2]. Any theorems stated without proof in this chapter can be found in [2]. Another excellent introduction to this theory is [1], though his treatment is algebraic as opposed to topological.

We start by defining our main object, a *graph of groups*.

Definition 2.1. An *abstract graph* Γ is given by sets $E(\Gamma)$ and $V(\Gamma)$, called the edges and vertices of Γ, a fixed point free involution on $E(\Gamma)$, and a map $\partial_0 : E(\Gamma) \to V(\Gamma)$. We will denote the image of e under the involution by $\bar{e}$. We define $\partial_1 e = \partial_0 \bar{e}$ and say that e joins $\partial_0 e$ to $\partial_1 e$.

Definition 2.2. Given an abstract graph Γ, we denote the *realization of* Γ by $|\Gamma|$ and define it to be the topological graph with vertices $V(\Gamma)$, an edge for each element in the set $\{\{e, \bar{e}\} | e \in E(\Gamma)\}$, and edges attaching to vertices according to the map ∂_0.

Definition 2.3. A *graph of groups* $\mathcal{G}$ consists of an abstract graph Γ (whose realization we will assume is connected), together with a function assigning to each vertex v of Γ a group G_v and to each edge e a group G_e and an injective homomorphism $f_e : G_e \to G_{\partial_0 e}$, where we insist that $G_e = G_{\bar{e}}$.

These graphs of groups are the main objects of Bass-Serre theory. In the remainder of this chapter we will discuss free products with amalgamation and HNN extensions, and show both algebraically and topologically how graphs of groups are essentially the result of successive applications of these operations.

2.1 Free Products with Amalgamation

Definition 2.4. Let A and B be groups. A *word* in A and B is a string $p_1 p_2 ... p_n$, where each p_i is a group element of A or B. A word $p = p_1 p_2 ... p_n$ can be *reduced* by either

- removing some p_i from p if p_i is the identity of A or B, or
- replacing a product $p_i p_{i+1}$ by its product in A or B if p_i and p_{i+1} are both in A or both in B.

A word is called a *reduced word* if it cannot be reduced.

Definition 2.5. Let A and B be groups. The *free product* of A and B, denoted $A * B$, is the group of reduced words in A and B together with the empty set (which acts as the identity), where the group operation is concatenation followed by reduction to arrive at a reduced word.

Definition 2.6. Let A, B, and C be groups, with injective group homomorphisms $\alpha_1 : C \to A$ and $\alpha_2 : C \to B$. Let H be the subgroup of $A * B$ generated by the elements $\alpha_1(c)\alpha_2^{-1}(c)$ for all $c \in C$. The *free product with amalgamation* $A *_C B$ is defined by

$$A *_C B = A * B / \langle\langle H \rangle\rangle,$$

where $\langle\langle H \rangle\rangle$ is the normal closure of H in $A * B$.

This free product with amalgamation arises naturally when gluing two topological spaces together via path connected subspaces, as summarized in the famous Seifert van Kampen theorem (see [3], Theorems 70.1 and 70.2), listed below. We note, however, that the Seifert van Kampen does not require the maps i_1, i_2 as defined in the theorem below to be injective; free products with amalgamation thus arise as a special case of the Seifert van Kampen theorem.

Theorem 2.7. *Let $X = U \cup V$ be a topological space, where U and V are open in X; assume U, V, and $U \cap V$ are path connected; let $x_0 \in U \cap V$. Let $i_1 : \pi_1(U \cap V, x_0) \to \pi_1(U, x_0), i_2 : \pi_1(U \cap V, x_0) \to \pi_1(V, x_0), j_1 : \pi_1(U, x_0) \to \pi_1(X, x_0)$ and $j_2 : \pi_1(V, x_0) \to \pi_1(X, x_0)$ be* *homomorphisms induced by the the respective inclusions of topological spaces. Let*

$$j : \pi_1(U, x_0) * \pi_1(V, x_0) \to \pi_1(X, x_0)$$

be the homomorphism extending j_1 *and* j_2. *Then,* j *is surjective, and its kernel is the least normal subgroup* N *of the free product that contains all elements represented by words of the form* $(i_1(g)^{-1}i_2(g))$, *for* $g \in \pi_1(U \cap V, x_0)$.

We can also write $A *_C B$ as a pushout in the category of groups, defined below categorically (see [4]):

Definition 2.8. In any category, given a pair $f : a \to b, g : a \to c$ of arrows with common domain a, a *pushout* of f and g is a commutative square

$$\begin{array}{ccc} a & \xrightarrow{f} & b \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle u} \\ c & \xrightarrow{v} & r \end{array}$$

such that for any other commutative square built from f and g as shown below,

$$\begin{array}{ccc} a & \xrightarrow{f} & b \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\ c & \xrightarrow{k} & s \end{array}$$

there is a unique arrow $t : r \to s$ such that $t \circ u = h$ and $t \circ v = k$, i.e., all commutative squares built from f and g factor through the pushout.

$A *_C B$ is the pushout of α_1 and α_2 in the category of groups, as pictured below, where β_1 and β_2 are the obvious inclusions of A and B into $A *_C B$, looking it as a quotient of $A * B$.

$$\begin{array}{ccc} C & \xrightarrow{\alpha_1} & A \\ \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \beta_1} \\ B & \xrightarrow{\beta_2} & A *_C B \end{array}$$

Again hearkening back to definition 2.6, it is easy to see that any element of $A *_C B$ can be written $a_1 b_1 a_2 b_2 ... a_n b_n, a_i \in \beta_1(A), b_i \in \beta_2(B)$. Further, each element has a reduced representative, as follows:

Write each α_i as an inclusion, so that $C \subset A$ and $C \subset B$. Pick representatives $a_i \in A$ for right cosets $a_i C$ of C, giving a section of the projection $A \to A/C$, and do the same for B. We impose the restriction that the identity coset C is represented by the identity element. Then, a reduced representative is a sequence $a_1 b_1 ... a_n b_n c$ such that $c \in C$ and each a_i and b_i is one of the representatives chosen above.

We conclude this section with the following result, again due to [2]:

Theorem 2.9. *The maps $A \to A *_C B$ and $B \to A *_C B$ are injective, and each element of $A *_C B$ has a unique reduced representative.*

2.2 HNN Extensions

HNN Extensions are extremely similar to free products with amalgamation. They can also be defined topologically and as a pushout, and their elements have similarly defined reduced representatives.

Definition 2.10. Let A be a group with presentation $A = \langle S | R \rangle$, and let $\alpha_1 : C \to A$ and $\alpha_2 : C \to A$ be injective homomorphisms. Let t be a symbol not in S. Then, the HNN extension of A relative to α_1 and α_2 is given by:

$$A*_C = \langle S, t | R, t\alpha_2(c)t^{-1} = \alpha_1(c) \ \forall c \in C \rangle$$

Topologically, an HNN extension corresponds to the following situation. Suppose we have a space X with $\pi_1(X) = A$; pointed subspaces (Y_1, y_1) and (Y_2, y_2) such that the inclusion maps $Y_1 \hookrightarrow X$ and $Y_2 \hookrightarrow X$ induce inclusions $\pi_1(Y_1, y_1) \subset \pi_1(X, y_1)$ and $\pi_1(Y_2, y_2) \subset \pi_1(X, y_2)$; a homeomorphism $h : Y_1 \to Y_2$ with $h(y_1) = y_2$; and a path l from y_1 to y_2. We write $\pi_1(Y_1, y_1) = \pi_1(Y_2, y_2) = C$, and we define α_1 and α_2 as follows. $\alpha_1 : \pi_1(Y_1, y_1) \hookrightarrow$

$\pi_1(X, y_1)$ is the inclusion on fundamental groups induced by inclusion of Y_1 into X, and $\alpha_2 : \pi_1(Y_2, y_2) \hookrightarrow \pi_1(X, y_1)$ is the composition of the inclusion of $\pi_1(Y_2, y_2)$ into $\pi_1(X, y_2)$ with the isomorphism $\pi_1(X, y_2) \to \pi_1(X, y_1)$ induced by conjugation of paths by l. Let $\tilde{X}$ be the space obtained from $X \sqcup (Y_1 \times I)$ by attaching the endpoints of $Y_1 \times I$ to Y_1 and Y_2 according to the homeomorphism h. Let t be the image of $l \cup (y_1 \times I)$ in $\pi_1(\tilde{X}, y_1)$. Then, it can be shown that $\pi_1(\tilde{X}, y_1) = A *_C$, with α_i as defined above.

In other words, HNN extensions describe how the fundamental group of a space changes when sufficiently nice homeomorphic subspaces are joined together with a tube. Details of this interpretation can be found in [2].

The initial setup described is pictured in Figure 2.1:

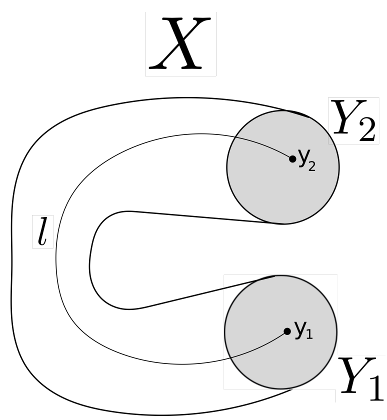

Figure 2.1: HNN Extension Setup

We can also write $A*_C$ as a pushout, just as with $A *_C B$, where β is the inclusion of A into $A*_C$:

$$\begin{array}{ccc} C & \xrightarrow{\alpha_1} & A \\ \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \beta} \\ A & \xrightarrow{\beta} & A*_C \end{array}$$

Or, more succinctly,

$$C \underset{\alpha_2}{\overset{\alpha_1}{\rightrightarrows}} A \xrightarrow{\beta} A*_C$$

Each element of $A*_C$ can be written as $a_1t^{r_1}a_2t^{r_2}...a_nt^{r_n}, a_i \in \beta(A), r_i \in \mathbb{Z}$, and has a reduced expression, as follows:

Pick right transversals T_i of $\alpha_i(C)$ in A. Our reduced expressions are defined as $a_1t^{\epsilon_1}a_2t^{\epsilon_2}...a_nt^{\epsilon_n}a_{n+1}$, where each $\epsilon = \pm 1$, $a_i \in T_1$ if $\epsilon_i = 1$, $a_i \in T_2$ if $\epsilon_i = -1$, and a_{n+1} is arbitrary.

As with section 2.1, we conclude this section with the following result (see [2] for a proof):

Theorem 2.11. *The map $\beta : A \to A*_C$ is injective, and every element of $A*_C$ has a unique reduced representative.*

2.3 Graphs of Groups

Graphs of groups were defined at the beginning of this chapter in definition 2.3. Now that we are equipped with an understanding of free products with amalgamation and HNN extensions, we will define in detail the *fundamental group of a graph of groups* and the associated topological realization of a graph of groups.

Given a graph $\mathcal{G}$ of groups, we can define an analogous graph $\mathcal{X}$ of pointed topological spaces as follows: For each G_v and G_e in $\mathcal{G}$, choose pointed topological spaces (X_v, v_0) and (X_e, e_0) such that $\pi_1(X_v, v_0) = G_v$ and $\pi_1(X_e, e_0) = G_e$, together with pointed maps $f'_e : (X_e, e_0) \to (X_{\partial_0 e}, v_0)$ inducing the associated injections in $\mathcal{G}$. Clearly such choices exist, such as via choosing Eilenberg-Maclane spaces for each X_v and X_e. We will call our (X_v, v_0) *vertex spaces* and our (X_e, e_0) *edge spaces*. Given such a graph $\mathcal{X}$, we define a *total space* X_Γ as the quotient of $\cup\{X_v | v \in V(\Gamma)\} \cup (\cup\{X_e \times I | e \in E(\Gamma)\})$ by the identifications

$$X_e \times I \to X_{\bar{e}} \times I \text{ via } (x,t) \to (x, 1-t),$$

$$X_e \times 0 \to X_{\partial_0 e} \text{ via } (x,0) \to f'_e(x).$$

Definition 2.12. The fundamental group G_Γ of the graph $\mathcal{G}$ of groups is the fundamental group of its associated total space X_Γ.

Though we do not cover the details here, it can be shown that G_Γ is independent of the choice of $\mathcal{X}$ (see [2] for details).

Consider the case when Γ has two vertices x and y, with one edge pair $\{e, \bar{e}\}$ joining them. In this case, we see from the Seifert van Kampen Theorem that $G_\Gamma = G_x *_{G_e} G_y$. In the case where Γ has one vertex x and one loop $\{e, \bar{e}\}$, we see from our discussion concerning HNN extensions in section 2.2 that $G_\Gamma = G_x *_{G_e}$. The fundamental group of a general graph of groups is then obtained simply by an iteration of free products with amalgamation and HNN extensions.

We also insert here a result about the nature of G_Γ, and point the interested reader to [2] for a proof.

Proposition 2.13. *If $\mathcal{G}$ is a graph of groups as above, then each map $G_v \to G_\Gamma$ is injective.*

This formulation of Bass-Serre theory relies heavily upon covering space theory to analyze the structure of free products with amalgamation, HNN extensions, and general graphs of groups. A consideration of covering spaces associated with subgroups of fundamental groups, for instance, gives theorem 2.14 below. We point the reader to [2] for a proof, but note that it follows immediately from treating G as the fundamental group of a graph of groups with two vertices and one edge pair and noting that the universal cover of its total space is a union of covers of its vertex spaces X_v and of its edge spaces crossed with intervals, $X_e \times I$.

Theorem 2.14. *If $G = A *_C B$ or $A*_C$ and if $H \subset G$, then H is the fundamental group of a graph of groups, where the vertex groups are subgroups of conjugates of A or B and the edge groups are subgroups of conjugates of C.*

Finally, we note that though our introduction to graphs of groups has been highly topological, the fundamental group of a graph of groups can also be defined entirely algebraically, as in Serre's original exposition [1]. While we omit the details here, we recommend this exposition to the reader interested in an algebraic construction of the fundamental group of a graph of groups using words.

2.4 Fundamental Theorem of Bass-Serre Theory

The fundamental theorem of Bass-Serre theory allows us to obtain a group acting without inversions on a graph (i.e., without sending e to $\bar{e}$) from a graph of groups and vice versa in mutually inverse operations. We will conclude our overview of Bass-Serre theory with a brief explanation of this process. We only give an outline of the constructions here, but we refer the interested reader to [2] for details.

Firstly, given a graph $\mathcal{G}$ of groups we choose a corresponding graph $\mathcal{X}$ of spaces and note that the universal cover $\tilde{X}_\Gamma$ of the total space X_Γ is a union of copies of the universal covers $\tilde{X}_v$ and $\tilde{X}_e \times I$ of X_v and $X_e \times I$. By identifying each copy of $\tilde{X}_v$ to a point and each $\tilde{X}_e \times I$ to a copy of I, we obtain a quotient space Z, which turns out to be a tree. The action of G on $\tilde{X}_\Gamma$ induces an action on Z, giving us our desired group action.

Coversely, given a group G acting on a tree Y without inversions, we first choose a connected CW complex U with fundamental group G. G then acts freely on its universal cover $\tilde{U}$, and thus also on $\tilde{U} \times Y$. We get a projection $X = (\tilde{U} \times Y)/G \to Y/G = \Gamma$, where Γ is a graph. We choose a maximal tree T in Γ and a lifting $j : T \to Y$ and write $j(T) = \tilde{T}$. Each vertex $v \in \Gamma$ has an associated vertex $\tilde{v} \in \tilde{T}$: we define G_v to be the stabilizer of $\tilde{v}$ (it turns out that this is also the isotropy group of $(\tilde{U} \times v)/G$). For each $e \in T$, we define G_e the same way. This automatically gives us the needed maps $f_e : G_e \to G_{\partial_0 e}$ for edges in T, via inclusions of isotropy groups. For each $e \notin T$, we choose an edge $\tilde{e} \in Y$ over e such that $\partial_0 \tilde{e} = (\widetilde{\partial_0 e})$ and an element $g_e \in G$ such that $\partial_1 \tilde{e} = g_e(\widetilde{\partial_1 e})$ and again let G_e be the stabilizer of $\tilde{e}$. $\alpha_0(e)$ is simply the inclusion of stabilizer groups, and $\alpha_1(e)$ is induced via conjugation by g_e. We thus obtain all the vertex groups, edge groups, maps between them, and maps from edges to vertices needed to define our desired graph of groups.

The fundamental theorem of Bass-Serre Theory says that these two constructions are essentially mutually inverse (see [2] for details):

Theorem 2.15. *The above two constructions are mutually inverse up to isomorphisms and (for graphs of groups) replacing the $\alpha_i(e)$ by conjugate homomorphisms (to account for the* *choice of maximal tree T).*

2.5 Next Step

We can visualize moving between any two graph of group decompositions of a given group via successive operations of either collapsing an edge into a vertex, expanding a vertex into two vertices and an edge, creating a loop, or collapsing a loop. These operations, as we know, corresponds algebraically to applying and reversing free products with amalgamation and HNN extensions. The simplest extension is to take a vertex and add a new edge pair $(e, \bar{e})$ and a new vertex v, where $G_e = G_v = 0$. In terms of $\mathcal{X}_\Gamma$, this would correspond to simply adding a line segment to the space.

The rest of this thesis will revolve around exploring a slight modification of this trivial extension. In particular, we will explore what occurs if this new vertex is instead a limit point of the graph of groups, by which we mean that each neighborhood of it always contains at least one other X_v.

Chapter 3. Inverse Limits

Inverse limits will be the main tool used to add a limit point to a graph of groups in a coherent fashion. The definitions and basic properties in section 3.1 come from [5] unless otherwise noted, and the reader interested in a more thorough exposition of the topic is encouraged to look at [5, Appendix 2, Section 2].

3.1 Definition and Basic Properties

Definition 3.1. A binary relation $\leq$ on a set R is a *preorder* if it is reflexive and transitive, i.e., if

(i) $a \leq a$

(ii) $a \leq b, b \leq c \implies a \leq c$

Definition 3.2. Let $\mathcal{A}$ be a preordered set and $\{Y_\alpha | \alpha \in \mathcal{A}\}$ be a family of spaces indexed by $\mathcal{A}$. For each pair of indices α, β satisfying $\alpha \prec \beta$, assume there is given a continuous map $\mu_{\beta\alpha} : Y_\beta \to Y_\alpha$ and that these maps satisfy the following condition: If $\alpha \prec \beta \prec \gamma$, then $\mu_{\gamma\alpha} = \mu_{\beta\alpha} \circ \mu_{\gamma\beta}$. Then the family $\{Y_\alpha; \mu_{\beta\alpha}\}$ is called an inverse spectrum over $\mathcal{A}$ with spaces Y_α and connecting maps $\mu_{\beta\alpha}$

In other words, an inverse spectrum is a set of spaces and maps between them in a manner compatible with their ordering. We note that though we have defined inverse limits for topological spaces above, inverse spectra are defined similarly in other categories. In particular, an inverse spectra of groups is defined by replacing spaces by groups and continuous maps by homomorphisms in the above definition.

We also note that though we have defined inverse spectra for general preordered sets above, we will only be discussing inverse spectra over linear orders in this thesis.

Definition 3.3. Let $\{Y_\alpha; \mu_{\beta\alpha}\}$ be an inverse spectrum over $\mathcal{A}$. Form $\prod\{Y_\alpha | \alpha \in \mathcal{A}\}$, and for each α, let p_α be its projection onto the α-th factor. The subspace $\{y \in \prod_\alpha Y_\alpha | \forall \alpha, \beta :$

$[\alpha \prec \beta] \implies [p_\alpha(y) = \mu_{\beta\alpha} \circ p_\beta(y)]\}$ is called the inverse limit space of the spectrum and is denoted by $\varprojlim Y_\alpha$ or Y_∞.

The above subspace can be thought of as the space of all coherent sequences in the inverse spectrum, and its topology is the subspace topology on the product topology. For each α, let $\mu_\alpha : Y_\infty \to Y_\alpha$ be the restriction of the projection map p_α to Y_α. The topology of Y_α can be expressed by a convenient basis, described in theorem 3.4 below. Since our motivating example in chapter 4 is a countable sequences of metric spaces, we also include for convenience a classical result regarding the metrizability of countable sequences of metric spaces ([6, Theorem 4.2.2]).

Theorem 3.4. *If $\mathcal{A}$ is a directed set, then the sets $\{\mu_\alpha^{-1}(U)|$ all α, all open $U \subset Y_\alpha\}$ form a basis for Y_∞.*

Theorem 3.5. *Let $\{X_i\}_{i=1}^\infty$ be a family of metrizable spaces and let ρ_i be a metric on the space X_i bounded by 1. The topology induced on the set $X = \prod_{i=1}^\infty X_i$ by the metric ρ defined as $\rho(x, y) = \sum_{i=1}^\infty \frac{1}{2^i}\rho_i(x_i, y_i)$, where $x = \{x_i\}, y = \{y_i\}$, coincides with the topology of the Cartesian product of the spaces $\{X_i\}_{i=1}^\infty$.*

We also list here some basic results about inverse limits of topological spaces (see [5] for proofs):

Theorem 3.6. *Let $\{Y_\alpha; \mu_{\beta\alpha}\}$ be an inverse spectrum over $\mathcal{A}$.*

(i) If each Y_α is Hausdorff, then Y_∞ is closed in $\prod_\alpha Y_\alpha$

(ii) If each Y_α is compact, then Y_α is compact, but possibly empty.

(iii) If $\mathcal{A}$ is a directed set, each Y_α is compact and nonempty, and for each $\alpha \in \mathcal{A}$, $\{x \in Y_\alpha | \mu_{\alpha\alpha}(x) = x\} \neq \emptyset$, then Y_∞ is nonempty.

Finally, we introduce the universal property of inverse limits (again, see [5] for a proof):

Theorem 3.7. *Let $\{h_\alpha\} : \{X_\alpha; \lambda_{\beta\alpha}\} \to \{Y_\alpha; \mu_{\beta\alpha}\}$ be a continuous map of inverse spectra. Then there exists a unique continuous $h_\infty : X_\infty \to Y_\infty$ such that for each $\alpha \in \mathcal{A}$, the following diagram commutes:*

$$\begin{array}{ccc} X_\infty & \xrightarrow{h_\infty} & Y_\infty \\ \downarrow{\scriptstyle \lambda_\alpha} & & \downarrow{\scriptstyle \mu_\alpha} \\ X_\alpha & \xrightarrow{h_\alpha} & Y_\alpha \end{array}$$

In other words, coherent maps into Y_α extend uniquely to a map into Y_∞.

3.2 $\{0\} \cup \frac{1}{n}$ as an Inverse Limit

We will now construct the subset $\{0\} \cup \{\frac{1}{n} | n \in \mathbb{N}\}$ of $\mathbb{R}$ as an inverse limit, in preparation for considering the limiting process of graphs of groups that we wish to explore.

We construct an inverse spectrum $\{X_i; \mu_{ji}\}$ and show its limit is homeomorphic to $\{0\} \cup \{\frac{1}{n} | n \in \mathbb{N}\}$. For each $i \in \mathbb{N}$, let X_i consist of i points with the discrete topology, ordered from left to right as $x_0, ..., x_{i-1}$. Define μ_{ii} to be the identity, and let $\mu_{(i+1)i}$ send x_0 to x_0 and x_k to x_{k-1} for $k \neq 0$. Define all μ_{jl} by compositions of the $\mu_{(i+1)i}$. In essence, each map $X_{i+1} \to X_i$ projects x_1 to x_0 and is the identity elsewhere. Since we have indexed our X_i by the natural numbers, we will treat points $x \in X_\infty$ (our inverse limit) as sequences. Notice that any point $x \in X_\infty$ is either of the form $\{x_0, x_0, ...\}$ or of the form $\{x_0, x_0, ..., x_0, x_1, x_2, x_3, ...\}$. Let $x^0 = \{x_0, x_0, ...\}$, and for $x \neq x^0 \in X_\infty$, if k is the minimum number for which $\mu_k(x) \neq x_0$, we will write $x = x^k$.

We will define a map $f : \{0\} \cup \{\frac{1}{n} | n \in \mathbb{N}\} \to X_\infty$ and show it is a homeomorphism. Let $f(0) = x^0$, and for each $n \in \mathbb{N}$, let $f(\frac{1}{n}) = x^{k+1}$. Clearly f is a bijection, and it immediate from theorem 3.4 that it is a continuous map. Since $\{0\} \cup \{\frac{1}{n} | n \in \mathbb{N}\}$ is compact and X_∞ is Hausdorff, f a homeomorpism and we have succeeded in constructing $\{0\} \cup \frac{1}{n}$ as an inverse limit of finite sets of points. We will state this result as a theorem:

Theorem 3.8. *The space X_∞ as defined in this section is homeomorphic to $\{0\} \cup \{\frac{1}{n} | n \in$*

$\mathbb{N}\} \subset \mathbb{R}$ *with the standard topology.*

It is also instructive to analyze X_∞ using the standard bounded metric on $\mathbb{R}$ and the metric given in theorem 3.5. We see that the further in the sequence that two sequences diverge, the smaller their maximum distance from each other becomes. Since x^0 has sequences branching off from it arbitrarily far into the sequence, it has points arbitrarily close to it and is a limit point in our space.

In the next chapter we will employ a largely similar construction to sequences of graphs of groups to obtain limit points of graph of groups.

Chapter 4. A Chain of $\mathbb{Z}$s

4.1 Inverse Limit of a Chain of $\mathbb{Z}$s

The simplest graph of groups with an infinite number of vertices is perhaps the graph of groups $\mathcal{G}$ pictured in figure 4.1: the underlying graph Γ has realization homeomorphic to the nonpositive real numbers with vertices at integer points, with $G_v = \mathbb{Z}$ for each vertex and $G_e = 0$ for each edge. As discussed earlier, we are interested in the effect of introducing a limit point to a graph of groups, and this example will serve as an appropriate starting point. We will introduce a limit point to $\mathcal{G}$ in much the same fashion as in section 3.2 above.

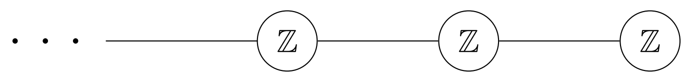

Figure 4.1: Chain of $\mathbb{Z}$s

We first construct an inverse spectrum $\{X_i; \mu_{ji}\}$ in a manner similar to section 3.2. For simplicity, we will assume that each vertex space (X_v, v_0) is $(S^1, (0, -1))$ (treating S^1 as a subset of $\mathbb{R}$), the simplest space with fundamental group $\mathbb{Z}$. For each $i \in \mathbb{N}$, let X_i be the quotient of $[0, i-1]$ with $\sqcup\{S^1_k\}_{k=1}^{i-1}$ where for each k, we identify $(0, -1) \in S^1_k$ with $k \in [0, i-1]$. We define μ_{ii} to be the identity and $\mu_{(i+1)i}$ to be the map that sends 0 to 0, $(0, 1]$ and S^1_1 to 0, and, for $k \in \mathbb{Z}^+$, sends $(k+1, k+2]$ and S^1_{k+2} to $(k, k+1]$ and S^1_{k+1} via the standard identification. In a similar fashion to our example in 3.2, we can visualize a new arc with S^1 attached appearing out of 0 every time i increases. We denote the inverse limit of this sequence by X_∞

We will now construct a space Y and show it is homeomorphic to X_∞. Let $\{S^1_j\}_{j=1}^{\infty}$ be a sequence of copies of S^1 with each S^1_j having a diameter of $\frac{1}{j}$. Let Y be the quotient of $[0, 1]$ with $\sqcup S^1_j$ defined by, for each j, attaching each $(0, -1) \in S^1_j$ to $\frac{1}{j}$.

Define maps $f_i : Y \to X_i$ as follows:

$$f_i\left(\left[0, \frac{1}{i}\right]\right) = 0$$

$$f_i\left(\left[\frac{1}{i-k}, \frac{1}{i-k-1}\right]\right) = [k, k+1] \text{ for } k \in \{0, 1, ..., i-2\} \text{ via a}$$

$$\text{linear order preserving homeomorphism.}$$

$$f_i(S_j^1) = 0 \text{ for } j \geq i$$

$$f_i(S_j^1) = S_j^1 \text{ for } j < i \text{ via the natural identification.}$$

In other words, f_i sends all but the rightmost $i-1$ arcs between adjacent points in the set $\left\{\frac{1}{n} | n \in \mathbb{N}\right\}$ and the associated copies of S^1 to 0, and identifies the remainder of Y with X_i in the obvious fashion. The f_i induce a bijective map $f : Y \to X_\infty$ as per theorem 3.7, and we see from applying theorem 3.4 that f is continuous. Thus, since f is a continuous bijection from a compact space to a Hausdorff space, f is a homeomorphism. We state this result as a theorem for emphasis:

Theorem 4.1. *Let X_∞ and Y be as defined in this section. Then X_∞ and Y are homeomorphic.*

4.2 Inverse Limit of Covers

At this point we wish to examine our construction above in 4.1 with an eye to see how the machinery of Bass-Serre theory has been affected upon moving to our inverse limit. The largest loss of this construction arises from the fact that the space X_∞ there defined is not semilocally simply connected, meaning X_∞ does not have a universal cover and making it difficult to transfer the covering space tools used in Bass-Serre theory into this new setting. That said, while X_∞ does not have a covering space, our construction in 4.1 lends itself to a natural inverse limit of covering spaces over X_∞. In fact, much of this construction of an inverse limit of covers is rather general, so we present it as such.

First, a general theorem about liftings of homotopies, the proof of which we will omit (see [3], Lemma 79.1):

Theorem 4.2. *Suppose B and E are path connected and locally path connected. Let $p : E \to B$ be a covering map; let $p(e_0) = b_0$. Let $f : Y \to B$ be a continuous map, with $f(y_0) = b_0$. Suppose Y is path connected and locally path connected. The map f can be lifted to a map $\tilde{f} : Y \to E$ such that $\tilde{f}(y_0) = e_0$ if and only if $f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(E, e_0))$. Furthermore, if such a lifting exists, it is unique.*

Let $\{(B_i, b_i) | i \in \mathbb{N}\}$ be pointed path connected and locally path connected topological spaces with universal covers $p_i : (E_i, e_i) \to (B_i, b_i)$. Suppose we have maps $f_n : (B_{n+1}, b_{n+1}) \to (B_n, b_n)$. Let $f'_n : (E_{n+1}, e_{n+1}) \to (E_n, e_n)$ be the maps induced from $f_n \circ p_n$ via Theorem 4.2, and $f_{n*} : \pi_1(B_{n+1}, b_{n+1}) \to \pi_1(B_n, b_n)$ the obvious induced map on fundamental groups. Let $B = \varprojlim_{f_i} B_i$, $E = \varprojlim_{f'_i} E_i$, and $G = \varprojlim_{f_{i*}} \pi_1(B_i, b_i)$, and let $b = (b_1, b_2, ...) \in B, e = (e_1, e_2, ...) \in E$. This setup gives the commutative diagram below:

$$\begin{array}{ccccc}
(E_1, e_1) & \xleftarrow[f'_1]{} & (E_2, e_2) & \longleftarrow & ... \\
\downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2} & & \\
(B_1, b_1) & \xleftarrow[f_1]{} & (B_2, b_2) & \longleftarrow & ...
\end{array}$$

Theorem 4.3. *If $g = (g_1, g_2, ...) \in G$ (where we treat each g_i is a covering transformation), and $x \in E_i$, $f'_{i-1}(g_i(x)) = g_{i-1}(f'_{i-1}(x))$*

Proof. Let $q : [0, 1] \to E_i$ be a path from e_i to $g_i(e_i)$.

Let $q : [0, 1] \to E_i$ be a path from e_i to $g_i(e_i)$. Let $\alpha : [0, 1] \to E_i$ be a path from x to e_i, and let $\alpha' = p_i \circ \alpha$. $\alpha * q * (g_i \circ \alpha^{-1})$ is then a path from x to $g_i(x)$, where $\alpha^{-1}(t) = \alpha(1-t)$. α' is a representative for g_i thought of as an element of $\pi_1(B_i, b_i)$, and since g is a coherent sequence, $f_{i-1}(\alpha')$ is a representative of g_{i-1} thought of as an element of $\pi_1(B_{i-1}, b_{i-1})$, and it lifts to a path in E_{i-1} from e_{i-1} to $g_{i-1}(e_{i-1})$, which equals $f'_{i-1}(g_i(e_i))$ by definition of f'_{i-1}. Similarly, $f'_{i-1}(\alpha)$ is a path from $f'_{i-1}(x)$ to e_{i_1}, and $f'_{i-1}(g_i(\alpha^{-1}))$ is a path from $g_{i-1}(e_{i-1})$ to $g_{i-1}(f'_{i-1}(x))$. We see then that $f'_{i-1}(\alpha * q * (g_i \circ \alpha^{-1}))$ begins at $f'_{i-1}(x)$ and ends at $g_{i-1}(f'_{i-1}(x))$. By definition of f'_{i-1} as the map induced from $f_{i-1} \circ p_{i-1}$, we have $f'_{i-1}(g_i(x)) = g_{i-1}(f'_{i-1}(x))$, as desired. □

Theorem 4.4. *G acts on E by homeomorphisms.*

Proof. By theorem 4.3, for any $g = (g_1, g_2, ...) \in G$, the map sending $(x_1, x_2, ...) \in E$ to $(g_1(x_1), g_2(x_2), ...)$ is a map from E to E and it is clear it is a group action. The result then follows once we can show the action by g is continuous. This, however, is immediate once we recall the basis for an inverse limit given in theorem 3.4 and note that g sends basis elements to basis elements. □

Theorem 4.5. *Let H be the subset of G that fixes the path component of E containing e. Then, H is a subgroup of G.*

Proof. H contains the identity element of G, and is clearly closed under the group operation and under taking inverses. □

This next theorem requires that $\pi_1(E, e) = 0$. It turns out that our space X_∞ from section 4.1 satisfies this property, as will be proved below.

Let $p : E \to B$ be the projection induced by the p_i. Define $h : H \to \pi_1(B, b)$ as follows. Given $g \in H$, let α be a path from e to $g(e)$. Let $h(g) = [p \circ \alpha]$. This is clearly an element of $\pi_1(B, b)$, and if we assume $\pi_1(E, e)$ is trivial, h is well-defined despite the choice made in choosing α, since all paths from e to $g(e)$ will then be path homotopic.

Similarly, we define a map $k : \pi_1(B, b) \to H$ as follows. Given a loop in B, each projection of this loop into B_i lifts to a path in E_i based at e_i, the endpoint of which defines a group element in $\pi_1(B_i, b_i)$. Since these group elements form a coherent sequence, this defines a group element in H, and this map is well defined on homotopy classes of loops in B because it is well defined in each B_i by standard arguments.

Theorem 4.6. *If $\pi_1(E, e) = 0$, the map $h : H \to \pi_1(B, b)$ is an isomorphism.*

Proof. First we show that h is a homomorphism. Since each $g \in G$ is a sequence of covering transformations, given any $x \in E$ we have $p(x) = p(g(x))$. Given $g_1, g_2 \in H$, let α_i be a path from e to $g_i(e)$. Then, $h(g_1g_2) = [p \circ (\alpha_1 * g_1(\alpha_2)]) = [p \circ \alpha_1 * p \circ g_1(\alpha_2)] = [p \circ \alpha_1] * [p \circ \alpha_2] = h(g_1)h(g_2)$, and we see that h is a homomorphism.

To show that k is also a homomorphism, we take loops α_1 and α_2 representing g_1 and $g_2 \in \pi_1(B, b)$. The projection of $k(g_1)$ into $\pi_1(B_i, b_i)$ will simply be the group element $[p_i \circ \alpha_1]$, and similarly, $k(g_2)$ projects to $[p_i \circ \alpha_2]$ and $k(g_1g_2)$ projects to $[p_i \circ (\alpha_1 * \alpha_2)]$. Since $k(g_1)k(g_2)$ agrees with $k(g_1g_2)$ in each $\pi_1(B_i, b_i)$, k is a homomorphism.

It is easy to see from the function definitions that h and k they are inverses of each other. □

Finally, we prove the following to show 4.6 applies to our space X_∞ from section 4.1.

Theorem 4.7. *Let X_i be as introduced in section 4.1, and for each i, let (E_i, e_i) be covering trees of X_i with projection maps p_i such that $p_i(e_i) = 0$. Let $f_i' : (E_{i+1}, e_{i+1}) \to (E_i, e_i)$ be as defined in this section, and let E be the inverse limit of the E_i and $e = (e_1, e_2, ...) \in E$. Then, $\pi_1(E, e) = 0$.*

Proof. Given any loop $s : [0, 1] \to T$ based at e, we will construct a contraction of a subspace of E containing s that fixes e. In other words, we will construct a sequence compatible contractions of subspaces of our E_i which contain the projections of s. We will let $s_i : [0, 1] \to E_i$ be the projection of s into E_i.

By compactness, the image of s_1 is contained in a minimal finite tree T_1. Construct a contraction $c_1 : T_1 \times I \to T_1$ as follows.

Subdivide I into n equally long segments for a sufficiently large n (such that the below construction works), and call them $I_1, I_2, ..., I_n$ in the obvious linear order. During I_1, c_1 is the constant map. During I_2, c_1 is a strong deformation retract which takes the outermost edges (i.e. the edges containing a vertex connected to only that edge) and contracts them to their inner vertex. During I_3, c_1 is the constant map. I_4 is another strong deformation retract as in I_2, and so on. This process will terminate after some finite iterations when T_1 has contracted to its vertex e_1, since T_1 is a finite tree. We insist also that I_n be a constant map.

Note that each map f_i' either sends edges homeomorphically to edges or collapses them to vertices. Thus if we choose T_2 to be the minimal subtree of E_2 containing the image of s_2 we obtain $f'(T_2) = T_1$, obtained via collapsing edges down to points. We can construct a contraction c_2 of T_2 by collapsing those edges to points during the constant portions of c_1. In general, given a contraction $c_i : T_i \times I \to T_i$, let $c_{i+1} : T_{i+1} \times I \to T_i$ be the contraction defined by setting $c_{i+1} = c_i$ during the strong deformation retract periods, and during periods where c_i is constant, subdividing those periods into segments and alternating between constant maps and strong deformation retracts in exactly the same fashion as in c_1 as necessary to collapse edges to a point if edges are present in T_{i+1} that are collapsed in T_i.

The above construction gives a coherent sequence of contractions of T_i, and thus a homotopy of s to the constant map. Thus, $\pi_1(E, e) = 0$. □

We end this chapter by noting that in the construction of X_∞ in section 4.1, the choice of S^1 for each vertex space with fundamental group $\mathbb{Z}$ was arbitrary. We will see in the next chapter that the fundamental group of the inverse limit space is independent of this choice, but it is not clear whether the associated $\pi_1(E, e)$ is also independent of this choice, leaving some uncertainty about the general applicability of theorem 4.6.

Chapter 5. More General Chains and Graphs

5.1 X_∞ as a Shrinking Wedge

Our space X_∞ defined in 4.1 is homotopy equivalent to a well-studied space known as the Hawaiian Earring. The Hawaiian Earring H is defined as the union of a countable set of circles in the plane with center $(0, \frac{1}{n})$ and radius $\frac{1}{n}$, and it is easy to see that collapsing the arc on which the S^1_j are attached in X_∞ gives us H. While H has been studied extensively, it is sufficient for our purposes at the moment to make note of the fact that $\pi_1(H)$ is much larger the fundamental group of a free group with countably many generators (i.e., the fundamental group of a wedge of countably many circles), and that this free group in fact embeds in $\pi_1(H)$ [7]. We see that process of adding a limit point to a graph of groups has caused an expansion of the fundamental group of the space. In this final section we will quantify just how much larger our group has become upon adding a limit point. We will draw heavily upon results in [8] for our results.

Recall our construction of X_∞ in section 4.1. We construct a new space Z_∞ out of spaces Z_i in exactly the same way, except that each Z_i is now the quotient of $[0, i-1]$ and $\sqcup\{X_k\}_{k=1}^{i-1}$ (as opposed to $\sqcup\{S^1_k\}_{k=1}^{i-1}$), where each X_k is now an arbitrary space. Z_∞ is then homotopy equivalent to the homotopy shrinking wedge $\oslash^H_n X_i$, defined below in definition 5.2.

Definition 5.1. Let Z be the space obtained from Z_∞ above by collapsing the arc to which each X_k is attached. We call Z a *shrinking wedge* of spaces and write $Z = \oslash_n X_i$.

Definition 5.2. Using the terminology in definition 5.1 above, let $\tilde{X}_k$ be the space obtained from X_k by attaching an arc p_k to its base point and shifting the base point to the other end of p_k. The *homotopy shrinking wedge* $\oslash^H_n X_i$ is defined by $\oslash^H_n X_i = \oslash_n \tilde{X}_i$, where the $\tilde{X}_i$ are wedged together at their new base points.

We note that $\oslash^H_n X_i$ is homotopy equivalent to Z_∞.

Definition 5.3. Given a wedge of spaces $\vee_n X_i$, we can define the *homotopy wedge* $\vee_n^H X_i$ by replacing each X_i by $\tilde{X}_i$ as in definition 5.2 above and taking their wedge. In other words, $\vee_n^H X_i = \vee_n \tilde{X}_i$.

Definition 5.4. Given a homotopy shrinking wedge $Z = \circledvee_n^H X_i$ with $\pi_1(X_i) = G_i$, $\pi_1(Z)$ is known to be a *topologist's product*, written $\circledast_n G_n$ and defined by $\circledast_n G_n = \bigcap_I (G_i * \lim_{\leftarrow n} *_{1 \leq j \leq n, j \neq i} G_j)$, a certain infinite free product where each group can only be used finitely many times (see [8] for more information).

Our X_∞ example from section 4.1 was obtained by adding a limit point to a graph of groups with a ray as its underlying graph. In this case, we can see that the inclusion of the fundamental group of our original graph of groups into $\pi_1(X_\infty)$ is induced topologically via mapping the standard wedge $\vee_n S_n^1$ to $\circledvee_n S_n^1$ in the obvious fashion, since collapsing the underlying graph in the graph of groups in this example leaves us with a wedge of spaces $\vee_n S_n^1$.

Given more arbitrary spaces X_i as in the construction for Z_∞, the inclusion of the fundamental group of the original graph of groups (i.e., the graph of groups with graph a ray and vertex spaces isomorphic to $\pi_1(X_i)$) into Z_∞ is induced via the standard map from the homotopy wedge $\vee_n^H X_i$ to $\circledvee_n^H X_i$, due again to the associated homotopy equivalences of the wedges with the original graph of groups and Z_∞ as in the above paragraph. This subsumes the case for X_∞ in the above paragraph, since it is clear that $\vee_n S_n^1$ and $\vee_n^H S_n^1$ are homotopy equivalent, and similarly that $\circledvee_n S_n^1$ and $\circledvee_n^H S_n^1$ are homotopy equivalent.

We are interested in homotopy wedges and homotopy shrinking wedges because of the equivalences $\pi_1(\vee_n^H X_n) = *_n \pi_1(X_n)$ and $\pi_1(\circledvee_n^H X_n) = \circledast_n \pi_1(X_n)$ for arbitrary spaces (see [8] for more details).

To find the difference between the fundamental group of the graph of group and the fundamental group of the space with added limit point, then, we are interested in the cokernel of the map $\pi_1(\vee_n^H X_i) \hookrightarrow \pi_1(\circledvee_n^H X_i)$, i.e., in the group $\circledast_n G_n / \langle\langle *_n G_n \rangle\rangle$. For our X_∞ example in section 4.1, we thus see that the process of adding a limit point generated a cokernel

$\circledast_n \mathbb{Z}/\langle\langle *_n \mathbb{Z} \rangle\rangle$.

Conner, Hojka, and Meilstrup proved the following in [8]:

Theorem 5.5. *(Conner, Hojka, Meilstrup) Let $\{G_n\}_{n\in\mathbb{N}}$ a collection of nontrivial countable (possibly finite) groups. If only finitely many of the G_n have elements of order 2, then*

$$\circledast_n G_n/\langle\langle *G_n \rangle\rangle \cong \circledast_n \mathbb{Z}/\langle\langle *_n \mathbb{Z} \rangle\rangle$$

If infinitely many of the G_n have elements of order 2, then

$$\circledast_n G_n/\langle\langle *G_n \rangle\rangle \cong \circledast_n \mathbb{Z}_2/\langle\langle *_n \mathbb{Z}_2 \rangle\rangle$$

Further, it is currently unclear whether $\circledast_n \mathbb{Z}/\langle\langle *_n \mathbb{Z} \rangle\rangle$ and $\circledast_n \mathbb{Z}_2/\langle\langle *_n \mathbb{Z}_2 \rangle\rangle$ are in fact different groups.

This leads immediately to the following theorem:

Theorem 5.6. *Given an arbitrary Z_∞ as described in this section, the map on fundamental groups induced by the inclusion $Z_\infty \setminus \{(0,0,...)\} \hookrightarrow Z_\infty$ has cokernel*

$$\circledast_n \mathbb{Z}/\langle\langle *_n \mathbb{Z} \rangle\rangle \text{ if only finitely many of the } \pi_1(X_i) \subset Z_\infty \text{ have elements of order two.}$$
$$\circledast_n \mathbb{Z}_2/\langle\langle *_n \mathbb{Z}_2 \rangle\rangle \text{ if infinitely many of the } \pi_1(X_i) \subset Z_\infty \text{ have elements of order two.}$$

Proof. This follows immediately from theorem 5.5 and the discussion in the paragraph immediately prior. □

In other words, in terms of the cokernel of the map on fundamental groups that arises when adding a limit point, it does not matter very much what groups are used in the original graph of groups. Our process of adding a limit point expands the fundamental group largely without regard to the groups at each vertex, leaving us with one of two (or perhaps just one) distinct cokernels.

5.2 Limit Points on More General Graphs

Up to this point we have only considered adding a limit point to a graph of groups with a ray as the underlying graph. We will now consider expanding our view to more general graphs.

In particular, we will consider three situations:

(i) The case of a general graph of groups whose realized graph contains an infinite chain of edges, the vertices of which are attached to exactly two edges each and whose edge groups are trivial.

(ii) The case of joining two limit points with an arc.

(iii) The case of identifying two limit points.

The first situation is simply the situation where we attach our space Z_∞ from section 5.1 to a generic graph of groups. We state the result of this operation below:

Theorem 5.7. *Let $\mathcal{G}$ be a graph of groups with underlying graph Γ, total space X_Γ, and fundamental group G_Γ. Choose $v \in V(\Gamma)$ and let X_v be the vertex space associated with v. Define Z_∞ as in section 5.1, with the restriction that $X_1 = X_v$. Let Y be the space obtained by identifying X_Γ and Z_∞ along X_v. Then $\pi_1(Y) = G_\Gamma *_{\pi_1(X_v)} \pi_1(Z_\infty)$, with the associated maps on π_1 induced by inclusion of X_v.*

Proof. This is a trivial application of the Seifert Van Kampen theorem, recalling that free products with amalgamation are a special case of the Seifert Van Kampen theorem where the inclusion of the shared subspace induces injective maps on fundamental groups. □

We see from this theorem that given any graph of groups with total space X_Γ, we can essentially "compactify" any number of arcs that extend forever without branching, and in the case where edge groups of the arcs extending forever are trivial, this process amounts simply to starting with the fundamental group of a subspace of X_Γ and repeatedly taking the free product with amalgamation of it with various copies of $\pi_1(Z_\infty)$.

Situation (ii) is covered by the following theorem:

Theorem 5.8. *Let Y be a space obtained from the total space X_Γ of a graph of groups $\mathcal{G}$ through multiple applications of theorem 5.7. Let x and y be two distinct points of $Y - X_\Gamma$, i.e., two distinct limit points of copies of Z_∞ as defined in section 5.1. If W is the space obtained from Y by joining x and y by an arc, then $\pi_1(W) = \pi_1(Y) * \mathbb{Z}$.*

Proof. Joining x and y together by an arc amounts to taking an HNN extension of $\pi_1(Y)$ relative to maps from the trivial group. This is equivalent to simply adding a generator to the group, hence our conclusion. □

The third and final situation deals with collapsing the arc between limit points in theorem 5.8 above to a point. We will explore this below.

Recall our discussion in 5.1 stating that our space X_∞ from section 4.1 is homotopy equivalent to the Hawaiian Earring. Let H_1 and H_2 be two copies of the Hawaiian Earring, and let $x_1 \in H_1$ and $x_2 \in H_2$ be the point in each space at which the circles are wedged together. Join x_1 and x_2 by an arc, call that space H, and denote the midpoint of the arc by x. We will consider what occurs when we collapse the arc to obtain the wedge of Hawaiian Earrings, $H_1 \vee H_2$ (denote the identified point by y).

Denote the cone of H_i by $\bar{H}_1$, and let $\bar{H}_1 \vee \bar{H}_2$ be the space obtained by identifying $x_i \in \bar{H}_i$ to a point, denoted $z \in \bar{H}_1 \vee \bar{H}_2$. Let $f : H \to H_1 \vee H_2$ be the map collapsing the arc, and let $g : H_1 \vee H_2 \to \bar{H}_1 \vee \bar{H}_2$ be the obvious inclusion. The following result makes use of basic results regarding the Hawaiian Earring; the interested reader is referred to [7] for details regarding the Hawaiian Earring.

Theorem 5.9. *Let $H_1, H_2, H,$ $H_1 \vee H_2$, f, and g be as defined above. Then $f_* : \pi_1(H, x) \to \pi_1(H_1 \vee H_2, y)$ is injective, and f and g induce a surjective map h from the cokernel of f into $\pi_1(\bar{H}_1 \vee \bar{H}_2, z)$, i.e., a map $h : \pi_1(H_1 \vee H_2, y)/\langle\langle \pi_1(H, x)\rangle\rangle \to \pi_1(\bar{H}_1 \vee \bar{H}_2, z)$.*

Proof. To show injectivity of f_*, we note that given any nulhomotopic loop in $Im(f)$, since the loop can only cross between H_1 and H_2 finitely many times it is easy to see that there is a nulhomotopy that does not pass from H_1 to H_2 or vice versa, and that this extends to a nulhomotopy in H. This gives us injectivity. We thus treat $\pi_1(H, x)$ as a subgroup of $\pi_1(H_1 \vee H_2, y)$.

Since loops in H can only cross between H_1 and H_2 finitely many times, $g \circ f$ is the trivial map. Thus, $Im(f) \subset ker(g)$ and we get an induced map from the cokernel of f_* to $\pi_1(\bar{H}_1 \vee \bar{H}_2, z)$, which is surjective because g is trivially surjective. □

The above theorem states that for the case of two copies of our space X_∞ as defined in section 4.1, collapsing an edge between limit points as in theorem 5.8 has cokernel at least as large as the wedge of the cones of the two copies of X_∞. An entirely similar argument with similar implications holds for more general Z_∞ as well.

We note that $\pi_1(\bar{H}_1 \vee \bar{H}_2, z)$ is known to be an uncountable group [7, Theorem 2.6], and that, intuitively, it consists of all loops that go between H_1 and H_2 infinitely often. We also comment that, though not proven, the induced map in the theorem above is likely injective as well, in which case the cokernel is isomorphic to $\pi_1(\bar{H}_1 \vee \bar{H}_2, z)$.

We summarize the above paragraphs with a theorem and a conjecture:

Theorem 5.10. *Let Z_∞ be a space as described in 5.1. Let X be the space obtained by taking two copies $Z_{\infty,1}, Z_{\infty,2}$ of Z_∞ and joining their limit points $z_1 = (0, 0, ...) \in Z_{\infty,1}, z_2 = (0, 0, ...) \in Z_{\infty,2}$ by an arc. Let $Z_{\infty,1} \vee Z_{\infty,2}$ be the wedge of $Z_{\infty,1}$ and $Z_{\infty,2}$ obtained by identifying z_1 and z_2 to a point z. Let Z_{DC} be the double cone of the $Z_{\infty,i}$, i.e., the space obtained by taking the cone of each $Z_{\infty,i}$ and taking their wedge at z_1 and z_2. We also call the identified point in this case z.*

Let $f : X \to Z_{\infty,1} \vee Z_{\infty,2}$ be the map that collapses the arc between z_1 and z_2, and let g be the obvious inclusion of $Z_{\infty,1} \vee Z_{\infty,2}$ into Z_{DC}. Then $f_* : \pi_1(X, z_1) \to \pi_1(Z_{\infty,1} \vee Z_{\infty,2}, z)$ is injective, and f and g induce a surjective map h from the cokernel of f_* into $\pi_1(Z_{DC}, z)$.

Proof. The proof is entirely similar to that of 5.9. □

Conjecture 5.11. *The cokernel in Theorem 5.10 above is isomorphic to the fundamental group of* Z_{DC}.

These last theorem have described operations that can be performed on graphs of groups to take limit points, connect them, and identify them. While this thesis has only covered adding limit points to graphs of groups in the case where edge groups are trivial, our inverse limit construction for adding a limit point also works when we have nontrivial edge groups. The bottleneck in this case is the lack of a clean description of the resulting space, in contrast with our situation with trivial edge groups. The addition of nontrivial edge groups may be a profitable direction to explore in future work.

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