2013-05-31

American Spread Option Models and Valuation

Yu Hu
Brigham Young University - Provo

American Spread Option Models and Valuation

Yu Hu

A dissertation submitted to the faculty of
Brigham Young University
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

Kening Lu, Chair
Lennard F. Bakker
Jasbir S. Chahal
Christopher P. Grant
Kenneth L. Kuttler

Department of Mathematics

Brigham Young University

June 2013

ABSTRACT

American Spread Option Models and Valuation

Yu Hu
Department of Mathematics, BYU
Doctor of Philosophy

Spread options are derivative securities, which are written on the difference between the values of two underlying market variables. They are very important tools to hedge the correlation risk. American style spread options allow the holder to exercise the option at any time up to and including maturity. Although they are widely used to hedge and speculate in financial market, the valuation of the American spread option is very challenging. Because even under the classic assumptions that the underlying assets follow the log-normal distribution, the resulting spread doesn't have a distribution with a simple closed formula. In this dissertation, we investigate the American spread option pricing problem. Several approaches for the geometric Brownian motion model and the stochastic volatility model are developed. We also implement the above models and the numerical results are compared among different approaches.

CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER 1. INTRODUCTION

Vanilla options are a category of options which have only standard terms. For example, Standard European and American options are vanilla options. Generally, one can trade vanilla options to manage the volatility risk in a single asset framework. Spread options are two-asset derivative securities, which are written on the difference between the values of two underlying market variables. Because it captures the co-movement structures between the two underlying assets, it is an important tool to hedge the correlation risk.

Spread options are widely traded in financial markets, for example, the notes over bonds spread, municipal over bonds spread and the treasury over Eurodollar spread in the US fixed income market, locational spread and produce spread in the commodity market, index spread in the equity market, the heating oil/crude oil and gasoline/crude oil crack spread and spark spread (the difference between the market price of electricity or natural gas and its production cost) in the energy market.

A lot of firms, like most manufacturers and oil refineries in [1], are involved in two markets: the raw materials the firm needs to purchase and the finished products the firm needs to sell. The price of raw materials and finished products are often subjected to different variables such as demand, supply, world economy, government regulations and other factors. As such, the firm can be at enormous risk when the price of raw materials rises while the price of finished products declines. However, the firm can use spread options to hedge the risk. Following [2] and [3], we suppose the firm has the following production schedule:

t_0: hedge decision

t_1: deadline for the production decision

t_2: purchase of the raw materials

t_3: selling of the finished products

where $t_0 < t_1 < t_2 < t_3$. Let $F_{in}(t, t_2)$ and $F_{out}(t, t_3)$ be the future prices of raw materials and finished product at time t, respectively. And suppose $K_c(t)$ is the forward price of the production cost to produce one unit output product from the one unit input raw material at time t. If the forward spread between the input price and output price exceeds the production cost at time $t < t_1$, i.e., $F_{out}(t, t_3) - F_{in}(t, t_2) > K_c(t)$, then a simple hedge strategy for the firm is to use future contracts consisting of longing the raw materials and shorting the finished products, which will lock in the forward profits : $F_{out}(t, t_3) - F_{in}(t, t_2) - K_c(t)$. However, hedging with the future contracts also sacrifices the firm's opportunity to profit from a potentially widened spread. If there is a potential increasing of the spread, the inflexibility of simple future contracts trading would be a drawback. Instead one can hedge the risk by trading with spread options. There are a lot of trading strategies by using spread options discussed in [2]. Here we look at the hedge strategy by shorting a call spread option with the strike price $K = K_c(t)$, maturity t_1 and payoff function $\max\{F_{out}(t, t_3) - F_{in}(t, t_2) - K, 0\}$. At time t_1, if the spreads exceed the strike price, i.e., $F_{out}(t, t_3) - F_{in}(t, t_2) \geq K$, the option will be exercised by the holder. As a result, the firm will purchase the raw materials at time t_2, sell finished products at time t_3 and get an income $K = K_c$. At time t_1, if $F_{out}(t, t_3) - F_{in}(t, t_2) < K$, the owner will let the option expire, and the firm will not produce. In either case, the firm will earn the option premium at time t_0 and also hedge the risk involved.

On the other hand, one can also use the spread option to speculate. Like vanilla options on a single asset, the price of vanilla options is an increasing function of the price of the underlying asset. Thus, a speculator will long a call option if he thinks the price of the underlying asset will rise. The price of the spread option is an increasing function of the spread between the prices of the two underlying assets. Suppose everything else is the same. If we believe the spread will rise, we will long the spread option. In other words, if one believes that the two underlying assets will move away from each other, the price of the spread will increase. Thus, one will long the spread option.

If we believe the spread will drop, we will short the spread option. In other words, if one believes that the two underlying assets will become more aligned with each other, the price of the spread option will decline. Hence, we will short the option.

Before we go through the techniques of pricing American style spread options, let's review the European spread options and standard American options.

The payoff function for a European spread option is $\max\{S_1(T) - S_2(T) - K, 0\}$, where S_1 and S_2 are the values of the two underlying market variables, where T is maturity and where K is the strike price. The first model for pricing the spread option is to model the resulting spread directly as a geometric Brownian motion. Then, the price of the spread option is the same as the price of the standard option on a single asset. However, this is clearly not a good model as it only permits the positive spreads, and, also, it ignores the co-movement structure between the two underlying market variables, which was pointed out in [4]. An alternative approach in [5] and [6] is to model the two underlying market variables S_1 and S_2 as the arithmetic Brownian motions, together with the constant correlation. Then, the resulting spread in this model is an arithmetic Brownian motion, and a closed formula is available. This approach has its drawback as it permits the negative values for the two underlying assets.

By going one step further, one can model each individual asset as a geometric Brownian motion and assume that the correlation between the two underlying assets is a constant ρ, which is widely studied in [7], [8], [9] and [10]. Then the resulting spread at maturity is the difference of two log-normal random variables. It doesn't have a simple distribution with a closed formula, which prevents us from deriving a closed formula solution to the price of the European spread option. However, for a special case of the European spread option, i.e., $K = 0$, which is called an exchange option, there is a closed formula called Margrabe formula that comes from Margrabe in [11]. The idea of this approach is that because the payoff function for the exchange option is $\max\{S_1(T) - S_2(T), 0\}$,

by taking out $S_2(T)$, the payoff function becomes $S_2(T) \max\{\frac{S_1(T)}{S_2(T)} - 1, 0\}$. The quotient of two log-normal random variables is still log-normal. Thus, one can derive a closed formula solution for exchange option. When $K \neq 0$, generally, there is no closed formula solution, but there are several ways to approximate it. The first one is the Kirk approximation formula which was introduced in [12]. The idea of the Kirk approximation is that when $K \ll S_2$, we may regard $S_2 + K$ as a geometric Brownian motion. Then, one can get the Kirk formula by applying the Margrabe formula. Another approach is the pseudo analytic formula due to [3] and [13]. By using the conditional distribution technique which reduces the two dimensional integrals for computing the expectation to the one dimensional integral, the pseudo analytic formula involves the one dimensional integration, which one can efficiently compute by the Gauss-Hermite quadrature method.

The Black-Scholes formula gives the option price as a function of several parameters. It is easy to get most of the parameters except the volatility. Instead of computing the option price by giving the volatility, one can observe the option prices from the market, using it to solve the inverse problem to compute the volatility. The volatility implied by the market option prices are called implied volatility. If the Black-Scholes model is perfect, the implied volatility should be a constant for all market options on the same asset. However, empirical studies in [14] and [15] have revealed that the implied volatility depends on the strike price and maturity of the options. If we plot the implied volatility as a function of its strike price (and maturity), we get the so called volatility smile (surface). On the other hand, given the volatility smile, one can determine the risk neutral probability density distribution for an asset at a future time. The risk neutral probability density distribution is called the implied distribution. Hull in [14] and Cont-Tankov in [15] showed that the implied distribution has heavier tails than the log-normal distribution and is also more peaked, which means that both large and small movements are more likely than the log-normal model distribution, and intermediate movements are less likely. One of the important reasons for that is that we assume that the volatility is a constant. Hence, in order to overcome these limitations, we study the stochastic volatility models which assume the volatility is a stochastic process.

Now we consider a model that overcomes the drawback of the constant volatility by introducing stochastic volatility, in which the volatility process is a stochastic process. It is called a stochastic volatility model, which was introduced by Henston in [16]. Hong in [13] studied the three-factor stochastic volatility model to price the European spread option. He proposed the fast Fourier transform technique to price the European spread option under the three-factor stochastic volatility model. He also compared the performance for pricing the European spread option among the fast Fourier transform technique, the Monte Carlo simulation and the partial differential equation approach. It turns out that the fast Fourier transform technique is much faster, more effective.

We know that it is not optimal to exercise an American call option on a non-dividend-paying stock before the expiration date, which means a European call option and an American call option, on the same non-dividend paying stock with the same strike and maturity, have the same value. The idea is that for a non-dividend paying stock the price of the European option is always bigger than or equal to the intrinsic value of the American option. One can regard the European option price as the future expected payoff, and the intrinsic value is the value that we receive by exercising the option immediately. As long as the future expected payoff is bigger than or equal to the intrinsic value, we will hold it until maturity. Thus, the price of American option is the same as the corresponding European option. However, it could be optimal to exercise an American put option on a non-dividend-paying stock before the expiration date. Let's consider an extreme case from Hull in [14]. Suppose the strike price of the American put option is $10 and the stock price is very close to $0. By exercising it immediately, the investor can earn $10. If the investor keeps the option, it may be less than $10 as the stock price rise, but it will never be more than $10. Furthermore, the earlier to receive $10, the more interest the investor will receive.

Thus for pricing American options, we take into account the early exercise feature, and it involves the question how to make decision about the early exercise. From [17], in terms of partial differen-

5

tial equations, the American option problem is a partial differential equation with a free boundary problem. From the theory of the partial differential equation, we know that there is no simple closed formula for the partial differential equation with a free boundary condition. Hence, we focus on the numerical methods to solve the partial differential equation.

An alternative approach to price an American option is Monte Carlo simulation introduced by Tsitsiklis-Roy in [18] and Longstaff-Schwartz in [19]. The idea is that one can choose a linear combination of basis functions of current state price to approximate the expected future payoff and to compare this value with the intrinsic value of the American option. Then, it determines early exercise time. In practice, we usually get the lower bound for American options by this method. The convergence of Longstaff-Schwartz method is showed in [20]. Roger in [21] and Haugh-Kogan in [22] proposed the dual method to compute the price of the American option. The idea of this approach is to represent the price of the American option through a minimal-maximum problem, where the minimal is taken over a class of martingales. The dual method usually leads to generate an upper bound for American options.

For pricing the American spread option, we consider the European spread option with the early exercise feature, which makes the valuations of the American spread options more challenging. In this dissertation, we investigate the American spread option pricing problem. Several approaches for the geometric Brownian motion model and the stochastic volatility model are developed, including the partial differential equation method, the Monte Carlo simulation method and the dual method. We also implement the above models and the numerical results are compared among different approaches.

The remainder of this dissertation is organized as follows. In chapter 2, we introduce the option pricing theory: stochastic calculus, martingale pricing theory, Black-Scholes model, American options as well as the models for the European spread option. In chapter 3, we present the two-

factor geometric Brownian motion model for pricing the American spread option and the different approaches for the valuation of the American spread option. Then in chapter 4, we investigate the three-factor stochastic volatility model for the American spread option by introducing the volatility as a stochastic process, where the two underlying assets share the same volatility process. The subject of chapter 5 is the numerical implementations and results for the American spread options under the two-factor geometric Brownian motion model and the three-factor stochastic volatility model. The numerical results also are compared under different approaches. We conclude with directions for future research in the last chapter.

2.1 STOCHASTIC CALCULUS

We recall basic definitions and results on stochastic calculus, which will be used for our analysis. We refer readers to [23] for more details.

Definition 2.1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $W(t) = W(t, \omega)$ be a stochastic process with $W(0) = 0$. $W(t)$ is called a Brownian motion if for all $0 < t_0 < t_1 < \cdots < t_m$ the increments

$$W(t_0) = W(t_0) - W(0), W(t_1) - W(t_0), W(t_2) - W(t_1), \cdots, W(t_m) - W(t_{m-1}) \qquad (2.1)$$

are independent and each of these increments is normally distributed with

$$\mathbb{E}[W(t_{i+1}) - W(t_i)] = 0, \qquad (2.2)$$

$$\mathrm{Var}[W(t_{i+1}) - W(t_i)] = t_{i+1} - t_i. \qquad (2.3)$$

Having the one dimensional Brownian motion, then one can define a d-dimensional Brownian motion as follows.

Definition 2.2. A d-dimensional Brownian motion is a process

$$W(t) = (W_1(t), \cdots, W_d(t)) \qquad (2.4)$$

with the following properties

(i) Each $W_i(t)(i = 1, \cdots, d)$ is a one dimensional Brownian motion.

(ii) If $i \neq j$, then the process $W_i(t)$ and $W_j(t)$ are independent.

Then we have a filtration $\mathcal{F}_t$ associated with the d-dimensional Brownian motion, such that the following holds.

(iii) For $0 \leq s < t$, $\mathcal{F}_s \subseteq \mathcal{F}_t$.

(iv) For each $t > 0$, $W(t)$ is $\mathcal{F}_t$ measurable, i.e., $W(t) \in \mathcal{F}_t$.

(v) For $0 \leq s < t$, $W(t) - W(s)$ is independent of $\mathcal{F}_s$.

The following theorems tell us how to recognize a Brownian motion.

Theorem 2.3 (Levy, One Dimension). *Let $W(t), t \geq 0$, be a martingale relative to the filtration $\mathcal{F}_t, t \geq 0$. Assume that $W(0) = 0$, $W(t)$ has continuous paths, and $[W, W](t) = t$ for all $t \geq 0$. Then $W(t)$ is a Brownian motion.*

Theorem 2.4 (Levy, Two Dimension). *Let $W_1(t)$ and $W_2(t)$, $t \geq 0$, be a martingales relative to the filtration $\mathcal{F}_t, t \geq 0$. Assume that for $i = 1, 2$, we have $W_i(0) = 0$, $W_i(t)$ has continuous paths, and $[W_i, W_i](t) = t$ for all $t \geq 0$. In addition, $[W_1, W_2] = 0$ for all $t \geq 0$. Then $W_1(t)$ and $W_2(t)$ are independent Brownian motions.*

Let $W(t)$ be a $m-$dimensional Brownian motion and let $\mathcal{F}_t$ be the association filtration. We model the n-dimensional underlying process S_t to be a $\mathcal{F}_t$ measurable Markov process in $\mathbb{R}^n$ through the stochastic differential equations

$$dS(t) = \mu(t, S(t))dt + \sigma(t, S(t))dW_t, \tag{2.5}$$

where $\mu : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^{n \times m}$ are jointly Borel measurable functions. In order to guarantee the existence and uniqueness of the solution of the stochastic differential equation (2.5), we also assume that μ and σ satisfy the Lipschitz and growth conditions, see [24]. One can also write it into a vector-matrix form as follows.

$$d \begin{bmatrix} S_1(t) \\ \vdots \\ S_n(t) \end{bmatrix} = \begin{bmatrix} \mu_1(t, S(t)) \\ \vdots \\ \mu_n(t, S(t))) \end{bmatrix} dt + \begin{bmatrix} \sigma_{11}(t, S(t)) & \cdots & \sigma_{1m}(t, S(t)) \\ \vdots & \vdots & \vdots \\ \sigma_{n1}(t, S(t)) & \cdots & \sigma_{nm}(t, S(t)) \end{bmatrix} d \begin{bmatrix} W_1(t) \\ \vdots \\ W_m(t) \end{bmatrix}, \tag{2.6}$$

where $S(t) = (S_1(t), S_2(t), \cdots, S_n(t))^T$.

Theorem 2.5 (Multi-Dimensional Ito Formula). *Let $S(t)$, $t > 0$, be the solution of the stochastic differential equations (2.6). Suppose that $f(t, x_1, \cdots, x_n)$ is a twice continuously differentiable function. Then the stochastic process $V = f(t, S_1(t), \cdots, S_n(t))$ satisfies*

$$dV = \left[f_t(t, S_1, \cdots, S_n) + \frac{1}{2} \sum_{i,j=1}^n a_{ij} f_{x_i x_j}(t, S_1, \cdots, S_n) \right] dt \tag{2.7}$$

$$+ \sum_{i=1}^n \mu_i(t, S_1, \cdots, S_n) f_{x_i} dS_t(t), \tag{2.8}$$

where

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \cdots & \sigma_{1m} \\ \vdots & \vdots & \vdots \\ \sigma_{n1} & \cdots & \sigma_{nm} \end{bmatrix} \begin{bmatrix} \sigma_{11} & \cdots & \sigma_{n1} \\ \vdots & \vdots & \vdots \\ \sigma_{1m} & \cdots & \sigma_{nm} \end{bmatrix}. \tag{2.9}$$

We will need the following theorem on Markov property, which is taken from [23].

Theorem 2.6. *Let $S(t)$, $t > 0$, be the solution of the stochastic differential equations (2.6) and $h(y)$ be a Borel-measurable function. Then, there exists a Borel-measurable function $g(t, x)$, such that*

$$\mathbb{E}\left[h(S(T)) | \mathcal{F}_t \right] = g(t, S(t)). \tag{2.10}$$

2.2 MARTINGALE PRICING THEORY

In this section, we review martingale pricing theory. Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be a probability space with filtration $\mathcal{F}_t$. We assume $\mathcal{F} = \mathcal{F}_T$. Suppose now there are $(d+1)$ traded assets in the market with their processes $S_0(t), S_1(t), \cdots, S_d(t)$. We assume that the risk free interest rate is constant r and $S_0(t)$ represents the money market account

$$dS_0(t) = rS_0(t)dt. \tag{2.11}$$

Before introducing the no-arbitrage, we first define the self-financing portfolio. For any given $d+1$ traded assets with values $\{S_0(t), S_1(t), \cdots, S_d(t)\}$ at time t, the value of a portfolio at time t is

$$X(t) = \sum_{i=0}^{n} \Delta_i(t)S_i(t), \tag{2.12}$$

where $\Delta(t) = \{\Delta_0(t), \Delta_1(t), \cdots, \Delta_d(t)\}$ represents the allocation into the corresponding assets at time t. $\{\Delta_1(t), \Delta_2(t), \cdots, \Delta_n(t)\}$ is also called an investment strategy (or trading strategy). A gain process $G(t)$ is defined by

$$G(t) = \sum_{i=0}^{d} \int_0^t \Delta_i(u)dS_i(u). \tag{2.13}$$

We assume Δ is $d+1$ dimensional predictable process and the integral in (2.13) make sense.

Definition 2.7. A portfolio $X(t)$ is called self-financing if

$$X(t) = X(0) + G(t). \tag{2.14}$$

Then, a self-financing portfolio $X(t)$ means that there is no infusion or withdrawal of money. The purchase of a new asset is financed by the sale of an old one, which means the change in the value of the portfolio is only due to changes in the value of the assets. In case of two assets, the value of portfolio is given by

$$X(t) = \Delta_1(t)S_1(t) + \Delta_2(t)S_2(t), \tag{2.15}$$

and a self-financing portfolio satisfies

$$dX(t) = \Delta_1(t)dS_1(t) + \Delta_2(t)dS_2(t). \tag{2.16}$$

This is described by the stochastic integral equation

$$X(t) = X(0) + \int_0^t \Delta_1(u)dS_1(u) + \int_0^t \Delta_2(u)dS_2(u). \tag{2.17}$$

Following [25], we say that there exists an arbitrage opportunity at $[0, T]$ for the self-financing portfolio X_t if there exists a $t \in (0, T]$, such that

$$X(0) = 0, \quad X(t) \geq 0, \quad \text{and} \quad \mathbb{P}\{X(t) > 0\} > 0. \tag{2.18}$$

And if there is no arbitrage opportunity at $[0, T]$, we say it is an arbitrage free market. The following theorems, which are taken from [25], are widely used in no arbitrage pricing models.

Theorem 2.8. *Suppose we have an arbitrage free market at the time interval $[0, T]$. For any two portfolios X_1 and X_2, if*

$$X_1(T) \geq X_2(T), \tag{2.19}$$

and

$$\mathbb{P}\{X_1(T) > X_2(T)\} > 0, \tag{2.20}$$

then, for all $t \in [0, T)$,

$$X_1(t) > X_2(t). \tag{2.21}$$

The proof of this theorem follows from the definition of no-arbitrage.

Theorem 2.9. *Suppose that we have an arbitrage free market at the time interval $[0, T]$. For any two portfolios X_1 and X_2, if*

$$X_1(T) = X_2(T), \tag{2.22}$$

then, for all $t \in [0, T]$,

$$X_1(t) = X_2(t). \tag{2.23}$$

Definition 2.10. A numeraire is a strictly positive price process $N(t) > 0$, for all $t \in [0, T]$.

One can represent the price of all other traded assets by $S_i^N(t) = \frac{S_i(t)}{N(t)} (i = 1, \cdots, d)$. Usually, we take the money market account as our numeraire, i.e. $N_t = S_0(t)$, but one can also choose other process as our numeraire, which could be a useful tool to simplify derivative pricing, see [26].

Theorem 2.11 (Numeraire Invariance Theorem). *Self-financing portfolio is still a self-financing portfolio after a numeraire change.*

Because of above theorems, one can define the discounted gain process $G^{S_0}(t)$ by

$$G^{S_0}(t) = X^{S_0}(t) - X^{S_0}(0). \tag{2.24}$$

Definition 2.12. A martingale measure (risk neutral measure) $\mathbb{Q}$ is defined as the $\mathbb{P}$ equivalent measure on $(\Omega, \mathcal{F})$ under the numeraire S_0-based prices, such that $S_i^{S_0}$ are martingales under the probability measure $\mathbb{Q}$ for all $i = 1, \cdots, d$.

Definition 2.13. A self financial trading strategy is said to be $\mathbb{Q}$-admissible if the discounted gain process $G^{S_0}(t)$ is a $\mathbb{Q}$-martingale.

Definition 2.14. A derivative V is said to be attainable if there exists at least an admissible trading strategy Δ such that at time $t = T$, $X(T) = V(T)$. In this case, we say V is replicated by X.

Theorem 2.15 (Risk Neutral Formula). *Assume that there exists an equivalent martingale measure $\mathbb{Q}$. Let V be an attainable derivative replicated by a $\mathbb{Q}$-admissible self-financing trading strategy Δ. Then for each time t, $0 \leq t \leq T$, the no-arbitrage price of V is given by*

$$V(t) = S_0(t)\mathbb{E}_Q\left[\frac{V(T)}{S_0(T)}\bigg|\mathcal{F}_t\right]. \tag{2.25}$$

The following theorem is taken from Shreve [23], which tells us how to find the equivalent probability measures.

Theorem 2.16. *(Girsanov, Multiple Dimensions). Let T be a fixed positive time, let $\Theta(t) = (\Theta_1(t), \cdots, \Theta_d(t))$ be a d-dimensional adapted process. Define*

$$Z(t) = \exp\left\{-\int_0^t \Theta(u) \cdot dW(u) - \frac{1}{2}\int_0^t \|\Theta(u)\|^2 \, du\right\}, \tag{2.26}$$

$$\widetilde{W}(t) = W(t) + \int_0^t \Theta(u)du. \tag{2.27}$$

Assume that

$$\mathbb{E} \int_0^T ||\Theta(u)||^2 Z^2(u) du < \infty. \tag{2.28}$$

Set $Z = Z(T)$, then $\mathbb{E}Z = 1$, and under the probability measure $\mathbb{Q}$ given by

$$\mathbb{Q}(A) = \int_A Z(\omega) d\mathbb{P}(\omega) \tag{2.29}$$

for all $A \in \mathcal{F}$, the process $\widetilde{W}(t)$ is a d-dimensional Brownian motion.

The Ito integral in (2.26) is

$$\int_0^t \Theta(u) \cdot dW(u) = \int_0^t \sum_{j=1}^d \Theta_j(u) dW_j(u) = \sum_{j=1}^d \int_0^t \Theta_j(u) dW_j(u), \tag{2.30}$$

$||\Theta(u)||$ denotes the Euclidean norm

$$||\Theta(u)|| = \left(\sum_{j=1}^d \Theta_j^2(u) \right)^{\frac{1}{2}}, \tag{2.31}$$

and (2.27) is shorthand notation for $\widetilde{W}(t) = (\widetilde{W}_1(t), \cdots, \widetilde{W}_2(t))$ with

$$\widetilde{W}_j(t) = W_j(t) + \int_0^t \Theta_j(u) du, j = 1, \cdots, d. \tag{2.32}$$

2.3 BLACK-SCHOLES MODEL

In this section, we review how the Black-Scholes PDE is derived. The Black-Scholes model is based on the following assumptions, which are taken from [14] and [17].

(i) There is no arbitrage opportunity, which means that all risk free portfolio earn the same return.

(ii) One can borrow and lend cash at a known constant risk-free interest rate r.

14

(iii) One can buy and sell any amount, even fractional, units of the underlying asset.

(iv) The transactions do not incur any fees or costs. Although transaction cost is a real issue, they tend not to be modeled explicitly when developing pricing models, which is pointed out by Joshi [27]. The reason is that transaction cost will not create arbitrage, in any way, if a price can not be a arbitrage price in a no transaction cost model, it will not be arbitrage price in a model with transaction cost.

(v) The underlying asset does not pay a dividend. Actually, one can drop this assumption if the dividends are known beforehand.

(vi) The price of the underlying asset follows a geometric Brownian motion with constant drift and volatility. Basically, the underlying asset is the following Stochastic Differential Equation (SDE)

$$dS_t = \mu S_t dt + \sigma S_t dW_t. \tag{2.33}$$

Here S_t is the asset value, μ is the expected return, σ is the volatility and W_t is a standard Brownian motion.

Suppose we have an option whose value is $V(S, t)$ at time t. And it is not necessary to specify whether V is a call or a put. And indeed, V could be the value of a whole portfolio of different options. By using the Ito's formula, we get

$$dV = \sigma S \frac{\partial V}{\partial S} dW_t + \left(\mu S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} \right) dt. \tag{2.34}$$

We begin by constructing a portfolio $X(t)$, in which we are long one option, $V(S(t), t)$ and short $\Delta(t)$ unit of the underlying asset $S(t)$, where $\Delta(t)$ is unknown. Thus the value of our portfolio is

$$X(t) = V(S(t), t) - \Delta(t)S(t). \tag{2.35}$$

Assuming that the portfolio is self-financing, by (2.16), we have

$$dX(t) = dV(S(t), t) - \Delta(t)dS(t). \tag{2.36}$$

Plugging (2.33) and (2.34) into (2.36), we get

$$dX(t) = \sigma S(\frac{\partial V}{\partial S} - \Delta)dW_t + (\mu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} - \mu\Delta S)dt. \tag{2.37}$$

Since the portfolio is riskless, we have

$$\frac{\partial V}{\partial S} - \Delta = 0. \tag{2.38}$$

Because the portfolio is riskless, the portfolio will earn the risk free interest. Thus, we have

$$r(V - \frac{\partial V}{\partial S}S)dt = r(V - \Delta S)dt \tag{2.39}$$

$$= rX(t)dt \tag{2.40}$$

$$= dX(t) \tag{2.41}$$

$$= (\mu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} - \mu\Delta S)dt \tag{2.42}$$

$$= (\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t})dt, \tag{2.43}$$

which is the Black-Scholes PDE for non-dividend paying asset. In finance, $\frac{\partial V}{\partial S}$ represents the option delta, $\frac{\partial V^2}{\partial S^2}$ represents option gramma and $\frac{\partial V}{\partial t}$ represents the option theta. Each of them measures the sensitivity of the value of the option to a small change in the given underlying parameter. We also call them risk measures or hedge parameters. And in mathematics, the terms with $\frac{\partial V}{\partial S}$ is called convection, and the term with $\frac{\partial V^2}{\partial S^2}$ is called diffusion and the term rV is called reaction term.

It seems that European call and put options are totally different. It turns out that they are strongly correlated by the so called put-call parity. Suppose we long one underlying asset, long one put

16

option and short one call option, where the call and put option have the same maturity T and same strike price K. We use X to denote this portfolio. Thus we get our portfolio

$$X = S + P - C. \tag{2.44}$$

Then the value of the portfolio X at maturity is

$$X(T) = S + \max(K - S, 0) - \max(S - K, 0) \equiv K. \tag{2.45}$$

By the Theorem 2.9, we have

$$X(t) = K_t = Ke^{-r(T-t)}. \tag{2.46}$$

Hence, we get

$$X(t) = S_t + P(S_t, t) - C(S_t, t) = Ke^{-r(T-t)}. \tag{2.47}$$

From the above discussion, we see if we know the price for a call option, then one can compute the price of the put option as

$$P(S_t, t) = Ke^{-r(T-t)} - S_t + C(S_t, t). \tag{2.48}$$

Having derived the Black-Scholes PDE for the value of an option, we consider the terminal and boundary conditions. Otherwise, the partial differential equation has many solutions. First, we restrict our attention to the European call option with the value $C(S, t)$, strike price K, and maturity T. The terminal condition for the European call option is the price of the call option at time $t = T$. Using the no arbitrage argument, we have

$$C(S, T) = (S - K)^+, \quad 0 < S < \infty. \tag{2.49}$$

17

From equation (2.33), if we start from $S = 0$, then the value for the underlying asset will be zero in future, which means we will not exercise the option, and will get 0 payoff at maturity. Using the boundary condition at $S = 0$, we have 0 value for the option before maturity. Hence we get

$$C(0, t) = 0, \quad 0 \leq t < T. \tag{2.50}$$

For the boundary condition at $S = \infty$, we use the put-call parity. As $S \to \infty$, the value of the put option will be zero. Hence, we get the value of the call option by the put-call parity

$$C(S, t) \approx S - Ke^{-r(T-t)}, \quad S \to \infty \text{ and } 0 \leq t < T. \tag{2.51}$$

Summarizing the above discussion, for a call option we obtain the Black-Scholes PDE

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial C^2}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0, \quad 0 \leq t < T, \tag{2.52}$$

with the terminal and boundary conditions

$$C(S, t) \approx S - Ke^{-r(T-t)}, \quad S \to \infty \text{ and } 0 \leq t < T, \tag{2.53}$$

$$C(0, t) = 0, \quad 0 \leq t < T, \tag{2.54}$$

$$C(S, T) = \max(S - K, 0), \ 0 < S < \infty. \tag{2.55}$$

Now, we consider that the underlying asset pays out a dividend and the dividend is paid continuously over the life of the option. Suppose that in time dt the underlying asset pays out a dividend $qSdt$, where q is a constant and represents the dividend yield. Then, by the no arbitrage argument, we get the Black-Scholes PDE with dividends

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial C^2}{\partial S^2} + (r - q)S\frac{\partial C}{\partial S} - rC = 0, \quad 0 \leq t < T, \tag{2.56}$$

with boundary and terminal conditions

$$C(S,t) \approx Se^{-q(T-t)} - Ke^{-r(T-t)}, \ \ S \to \infty \text{ and } 0 \le t < T, \qquad (2.57)$$

$$C(0,t) = 0, \ \ 0 \le t < T, \qquad (2.58)$$

$$C(S,T) = \max(S - K, 0), \ 0 < S < \infty. \qquad (2.59)$$

2.4 AMERICAN OPTION

In this section, we review the American option. European option can be exercised only on the expiration date, while American option can be exercised before and on maturity. Hence American option is more flexible and attractive for investors. Also, the owner of American options has more exercise opportunities than the owner of the corresponding European options. So American options are more expensive than corresponding European options. For American options, it is known that it is not optimal to exercise a call option on a non-dividend-paying stock before the expiration date, while it can be optimal to exercise a put option on a non-dividend-paying stock before the expiration date. Hence, we focus on the American put option.

We note that for a European put option, since it doesn't allow early exercise and in fact, when $S \to 0$, we have

$$P(S,t) \approx Ke^{-r(T-t)} - S \qquad (2.60)$$

$$< K - S, \qquad (2.61)$$

which means that the intrinsic value could be bigger than the expected future payoff. But this couldn't happen for an American put option. At anytime, the value of American put option is bigger than or equal to its intrinsic value $\max(K - S, 0)$. Hence, the owner needs to determine when to exercise the option, which is optimal from the holder's point of view. Thus, to determine the value of the American option is more complicated. At each time we need to determine not

only the option value but also whether or not the option should be exercised. In terms of partial differential equations, it is a free boundary problem. The free boundary divides the region into two regions: in one region one should exercise the option and in the other one should hold the option. To be more precise, let $S(t)$ be the free boundary, for $0 \leq S < b(t)$, where early exercise is optimal, we have

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial P^2}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP = -rK < 0, \ \ 0 \leq t < T, \tag{2.62}$$

$$P(S,t) = K - S, \ 0 < S < b(t). \tag{2.63}$$

In the region $b(t) < S < \infty$ where early exercise is not optimal, and we have

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial P^2}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP = 0, \ \ b(t) < S < \infty, \tag{2.64}$$

$$P(S,t) > K - S, \ b(t) < S < \infty. \tag{2.65}$$

On the boundary, we have $P(b,t) = K - b$, and the slope for the intrinsic value function is -1. We also have $\frac{\partial P}{\partial S}(b(t),t) = -1$. Therefore, to price an American put option, we solve the following free boundary problem

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial P^2}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP = 0, \ \ b(t) < S < \infty, \tag{2.66}$$

$$P(S,t) = K - S, \ 0 < S < b(t), \tag{2.67}$$

$$P(b(t),t) = K - b(t), \tag{2.68}$$

$$\frac{\partial P}{\partial S}(b(t),t) = -1 \tag{2.69}$$

$$P(S,t) = 0, \ \text{as} \ S \to \infty \tag{2.70}$$

$$P(0,t) = Ke^{-r(T-t)}, \tag{2.71}$$

$$P(S,T) = \max\{K - S, 0\}. \tag{2.72}$$

It is clear from the above discussion that the mathematical model of the American option is much

more complicated than that of the European option. It turns out that there is no closed formula to American put options as a free boundary problem.

2.5 SPREAD OPTION MODELS

In this section, we review models for the European spread option. Let S_1 and S_2 be the prices of the two underlying assets. We consider the models for pricing a call spread option with strike price K and maturity T. The payoff function is $\max(S_1(T) - S_2(T) - K, 0)$.

2.5.1 One-Factor Model. This is the simplest model. In this model, $S = S_1 - S_2$ satisfies the geometric Brownian motion

$$dS = \mu S dt + \sigma S dW_t, \tag{2.73}$$

where μ is drift term, σ is the volatility and W is a standard Brownian motion. In this case, the price of an European spread option is the same as the price of a options on single asset. The drawback of this model is that the spread is positive forever, which is not supported by empirical evidence. Furthermore, it ignores the co-movement structure between the two underlying market variables as pointed out in [4], which could lead to mis-price. An alternative approach in [5] and [6] overcoming limitation of one-factor geometric Brownian motion is to model the two underlying market variables S_1 and S_2 as the arithmetic Brownian motions, together with the constant correlation between the two market variables, then the resulting spread in this model is an arithmetic Brownian motion again and a closed formula is available. This approach has its drawback as it permits the negative values for the two underlying assets.

2.5.2 Two-Factor Geometric Brownian Motion Model. Assume that each underlying asset satisfies a geometric Brownian motion in the spirit of Black-Scholes framework, which overcomes the drawback of negative value in the asset price of the arithmetic Brownian motion. In addition, in this model the spread could be negative, which overcomes the limitation of positive spreads

21

posed by the one factor geometric Brownian motion model. Because of these, this model is widely studied in [7], [8], [9], [10] and others. Here we assume that the two underlying assets S_1 and S_2 are geometric Brownian motions with expected return μ_1 and μ_2, and we also assume the underlying asset pays a dividend. The dividend is paid continuously over the life of the option. In time dt the underlying asset pays out a dividend $q_i S_i dt (i = 1, 2)$, where $q_i (i = 1, 2)$ is a constant and stands for the dividend yield. Then the system for underlying assets is

$$dS_1(t) = (\mu_1 - q_1)S_1(t)dt + \sigma_1 S_1(t)dW_1(t), \tag{2.74}$$

$$dS_2(t) = (\mu_2 - q_2)S_2(t)dt + \sigma_2 S_2(t)dW_2(t), \tag{2.75}$$

$$dW_1(t)dW_2(t) = \rho dt, \tag{2.76}$$

where σ_1 and σ_2 are volatilities of the assets S_1 and S_2, respectively. And W_1 and W_2 are two standard Brownian motions with the correlation ρ under the probability measure $\mathbb{P}$. In this case, then the resulting spread is distributed as the difference of the two log-normal random variables. It doesn't have a simple distribution with closed formula, which prevents us from deriving a closed formula solution for the price of the European spread option. However, for a special case of European spread option, say, $K = 0$, which is called exchange option, there is a closed formula which came from Margrabe [11], called Margrabe formula. The idea of Margrabe formula is that because the payoff function for exchange option is $\max\{S_1(T) - S_2(T), 0\}$, by taking out the S_2, the payoff function becomes $S_2(T) \max\{\frac{S_1(T)}{S_2(T)} - 1, 0\}$. Since the quotient of two log-normal random variables are still log-normal, one can derive a similar closed formula solution for exchange option. When $K \neq 0$, generally, there is no simple closed formula solution, but there are several ways to approximate it. The first one is the Kirk approximation formula introduced in [12]. The idea of the Kirk approximation is that when $K \ll S_2$, we may regard $S_2 + K$ as a geometric Brownian motion. Then, one can get the Kirk formula by applying the Margrabe formula. Another Approach is the pseudo analytic formula due to [3], by using the conditional distribution technique which reduces the two dimensional integrals for computing the expectation under the risk neutral measure to one dimensional integral, and the pseudo analytic formula involves one dimensional integration, which

22

one can compute by the Gauss-Hermite quadrature method.

2.5.3 Three-Factor Stochastic Volatility Model.

The three-factor stochastic volatility model was proposed by Hong [13] to model the European spread option. Let S_1 and S_2 be two underlying assets with expected return μ_1 and μ_2, respectively. Assume the underlying assets pay a dividend. The dividend is paid continuously over the life of the option. In time dt, the underlying asset pays out a dividend $q_i S_i dt (i = 1, 2)$, where q_i is a constant and stands for the dividend yield. Then the system for underlying assets is

$$dS_1(t) = (\mu_1 - q_1)S_1(t)dt + \sigma_1\sqrt{v}S_1(t)dW_1(t), \tag{2.77}$$

$$dS_2(t) = (\mu_2 - q_2)S_2(t)dt + \sigma_2\sqrt{v}S_2(t)dW_2(t), \tag{2.78}$$

$$dv = A(\alpha - v)dt + \sigma_v\sqrt{v}dW_v, \tag{2.79}$$

where W_1, W_2 and W_v are three correlated standard Brownian motions with the following correlations under the probability measure $\mathbb{P}$

$$dW_1 dW_2 = \rho dt, \tag{2.80}$$

$$dW_1 dW_v = \rho_1 dt, \tag{2.81}$$

$$dW_2 dW_v = \rho_2 dt. \tag{2.82}$$

In this model, the variance v is a stochastic process, α is the long term mean of the variance, A is the mean reversion rate and σ_v is the volatility of volatility.

In this chapter, we consider the two-factor geometric Brownian model for the American spread option. First, we derive a model under the risk neutral measure. Then, we study three approaches for pricing American spread option, the partial differential equation, the Monte Carlo simulation and the dual method.

Let S_1 and S_2 be the prices of asset 1 and asset 2 with expected return μ_1 and μ_2. We assume that the underlying assets pay a dividend. The dividend is paid continuously over the life of the option. In time dt the underlying asset pays out a dividend $q_i S_i dt (i = 1, 2)$, where q_i is a constant and stands for the dividend yield. The system for the underlying assets is

$$dS_1(t) = (\mu_1 - q_1)S_1(t)dt + \sigma_1 S_1(t)dW_1(t), \tag{3.1}$$

$$dS_2(t) = (\mu_2 - q_2)S_2(t)dt + \sigma_2 S_2(t)dW_2(t), \tag{3.2}$$

$$dW_1(t)dW_2(t) = \rho dt, \tag{3.3}$$

where σ_1 and σ_2 are volatilities of the assets S_1 and S_2, respectively. W_1 and W_2 are two standard Brownian motions with the correlation ρ under the probability measure $\mathbb{P}$.

Generally, it is much easier to deal with independent rather than correlated Brownian motions. The following lemma allows us to transfer correlated Brownian motions into independent ones.

Lemma 3.1. *We decompose correlated Brownian motions W_1 and W_2 into two independent ones as follows*

$$\begin{bmatrix} dW_1(t) \\ dW_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{bmatrix} \begin{bmatrix} dB_1(t) \\ dB_2(t) \end{bmatrix}, \tag{3.4}$$

where $B_1(t)$ and $B_2(t)$ are two independent Brownian motions under the probability measure $\mathbb{P}$.

Proof. From (3.4) , we have

$$dB_1 = dW_1, \tag{3.5}$$

$$dB_2 = -\frac{\rho}{\sqrt{1-\rho^2}}dW_1 + \frac{1}{\sqrt{1-\rho^2}}dW_2. \tag{3.6}$$

Then by the properties of W_1 and W_2, we have

$$dB_1 dB_1 = dt, \tag{3.7}$$

$$dB_1 dB_2 = 0, \tag{3.8}$$

$$dB_2 dB_2 = dt. \tag{3.9}$$

For $i = 1, 2$, we have $B_i(0) = 0$. Also, $B_i(t)$ is a martingale and has continuous paths. Then by Theorem 2.4, we have $B_1(t)$ and $B_2(t)$ are two independent Brownian motions. $\qquad\square$

Next, we derive the corresponding system under the risk neutral measure as follows. By Lemma (3.1), our system (3.1)-(3.3) becomes

$$dS_1(t) = (u_1 - q_1)S_1(t)dt + \sigma_1 S_1(t)dB_1(t), \tag{3.10}$$

$$dS_2(t) = (u_2 - q_2)S_2(t)dt + \rho\sigma_2 S_2(t)dB_1(t) + \sqrt{1-\rho^2}\sigma_2 S_2(t)dB_2(t). \tag{3.11}$$

We define the value processes

$$\hat{S}_1 = e^{q_1 t}S_1(t), \tag{3.12}$$

$$\hat{S}_2 = e^{q_2 t}S_2(t). \tag{3.13}$$

25

Then we have

$$d\hat{S}_1 = u_1\hat{S}_1(t)dt + \sigma_1\hat{S}_1(t)dB_1(t), \tag{3.14}$$

$$d\hat{S}_2 = u_2\hat{S}_2(t)dt + \rho\sigma_2\hat{S}_2(t)dB_1(t) + \sqrt{1-\rho^2}\sigma_2\hat{S}_2(t)dB_2(t). \tag{3.15}$$

We introduce the discounted value processes $\hat{S}_1^{S_0}(t)$ and $\hat{S}_2^{S_0}(t)$ given by

$$d\hat{S}_1^{S_0}(t) = (u_1 - r)\hat{S}_1^{S_0}(t)dt + \sigma_1\hat{S}_1^{S_0}(t)dB_1(t), \tag{3.16}$$

$$d\hat{S}_2^{S_0}(t) = (u_2 - r)\hat{S}_2^{S_0}(t)dt + \rho\sigma_2\hat{S}_2^{S_0}(t)dB_1(t) + \sqrt{1-\rho^2}\sigma_2\hat{S}_2^{S_0}(t)dB_2(t). \tag{3.17}$$

Now we want to find the equivalent martingale measure $\mathbb{Q}$ under which the discounted value processes are $\mathbb{Q}$ martingales. To achieve this, we use the Girsanov theorem. Define $\theta = (\theta_1, \theta_2)^T$ by

$$A\theta = u - r,$$

where

$$A = \begin{bmatrix} \sigma_1 & 0 \\ \rho\sigma_2 & \sqrt{1-\rho^2}\sigma_2 \end{bmatrix}, \quad u - r = \begin{bmatrix} u_1 - r \\ u_2 - r \end{bmatrix}. \tag{3.18}$$

Because A is invertible, Then $A\theta = u - r$ has a unique solution θ. Now we define

$$Z(t) = \exp\left\{ -\int_0^t \theta \cdot dB(u) - \frac{1}{2}\int_0^t \|\theta\|^2 \, du \right\}, \tag{3.19}$$

and

$$w(t) = B(t) + \int_0^t \theta du, \tag{3.20}$$

where $B(t) = (B_1(t), B_2(t))^T$ and $w(t) = (w_1(t), w_2(t))^T$. By Girsanov Theorem 2.16, setting

$Z = Z(T)$, then $\mathbb{E}Z = 1$. And the probability measure $\mathbb{Q}$ is given by

$$\mathbb{Q}(A) = \int_A Z(w)d\mathbb{P}(w) \tag{3.21}$$

for all $A \in \mathcal{F}$. Note that the process $w(t) = (w_1(t), w_2(t))$ is a two-dimensional Brownian motion. From (3.20), we have

$$w_1(t) = B_1(t) + \theta_1 t, \tag{3.22}$$

$$w_2(t) = B_2(t) + \theta_2 t. \tag{3.23}$$

Plugging (3.22)-(3.23) into system (3.16)-(3.17) , we have

$$d\hat{S}_1^{S_0}(t) = \sigma_1 \hat{S}_1^{S_0}(t)dw_1(t), \tag{3.24}$$

$$d\hat{S}_2^{S_0}(t) = \rho\sigma_2 \hat{S}_2^{S_0}(t)dw_1(t) + \sqrt{1-\rho^2}\sigma_2 \hat{S}_2^{S_0}(t)dw_2(t), \tag{3.25}$$

which means that the underlying processes are martingales under probability measure $\mathbb{Q}$. Then by the definition of $\hat{S}_1(t), \hat{S}_2(t)$, we obtain

$$d\hat{S}_1(t) = r\hat{S}_1 dt + \sigma_1 \hat{S}_1 dw_1(t), \tag{3.26}$$

$$d\hat{S}_2(t) = r\hat{S}_2 dt + \rho\sigma_2 \hat{S}_2 dw_1(t) + \sqrt{1-\rho^2}\sigma_2 \hat{S}_2 dw_2(t). \tag{3.27}$$

By using $\hat{S}_i(t) = e^{q_i t}S_i(t)(i = 1, 2)$, we have the new system under risk-neutral measure $\mathbb{Q}$ as follows

$$dS_1 = (r - q_1)S_1 dt + \sigma_1 S_1 dw_1(t), \tag{3.28}$$

$$dS_2 = (r - q_2)S_2 dt + \rho\sigma_2 S_2 dw_1(t) + \sqrt{1-\rho^2}\sigma_2 S_2 dw_2(t). \tag{3.29}$$

3.1 PARTIAL DIFFERENTIAL EQUATIONS APPROACH

In this section, we derive the partial differential equation for the price of American spread option. We use the standard approach based on Ito formula. Recall that we have the following system under risk neutral probability measure $\mathbb{Q}$

$$dS_1(t) = (r - q_1)S_1(t)dt + \sigma_1 S_1(t)dw_1(t), \tag{3.30}$$

$$dS_2(t) = (r - q_2)S_2(t)dt + \sigma_2 S_2(t)\left(\rho dw_1(t) + \sqrt{1 - \rho^2}dw_2(t)\right), \tag{3.31}$$

where w_1 and w_2 are two standard independent Brownian motions under the risk neutral probability measure $\mathbb{Q}$.

The following theorem gives a partial differential equation with a free boundary condition for American spread option. Let $v(x_1, x_2, t)$ be the price of the American spread option with $x_1 = S_1$ and $x_2 = S_2$ at time t.

Theorem 3.2. $v(x_1, x_2, t)$ *is the solution of the partial differential equation*

$$0 = v_t + (r - q_1)x_1 v_{x_1} + (r - q_2)x_2 v_{x_2}$$
$$+ \frac{1}{2}\sigma_1^2 x_1^2 v_{x_1 x_1} + \frac{1}{2}\sigma_2^2 x_2^2 v_{x_2 x_2} + \rho\sigma_1\sigma_2 x_1 x_2 v_{x_1 x_2} - rv, \tag{3.32}$$

with the boundary and terminal conditions

$$v(0, x_2, t) = 0, \tag{3.33}$$

$$v(b(x_2, t), x_2, t) = b(x_2, t) - x_2 - K, \tag{3.34}$$

$$v_{x_1}(b(x_2, t), x_2, t) = 1, \tag{3.35}$$

$$v_{x_2}(b(x_2, t), x_2, t) = -1, \tag{3.36}$$

$$v(x_1, x_2, T) = \max\{x_1 - x_2 - K, 0\}, \tag{3.37}$$

28

where $x_1 = b(x_2, t)$ is the free boundary.

Proof. The payoff function for the spread call option without early exercise is

$$V(T) = \max(S_1(T) - S_2(T) - K, 0), \qquad (3.38)$$

where $V(t)$ represents the option price at time t without early exercise. By applying the risk neutral formula, we get the price at time t is

$$V(t) = e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}}[\max(S_1(T) - S_2(T) - K, 0)|\mathcal{F}_t]. \qquad (3.39)$$

Because the joint process $(S_1(t), S_2(t))$ is a Markov process, by Theorem 2.6, $V(t)$ can be written as a function of the time variable and the values of these process at time t, i.e., $V = v(x_1, x_2, t)$. Then, we have

$$v(x_1, x_2, t) = e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}}[\max(S_1(T) - S_2(T) - K, 0)|\mathcal{F}_t]. \qquad (3.40)$$

We apply Ito's formula to the function $e^{-rt} v(x_1, x_2, t)$. By Ito's formula, we have

$$d(e^{-rt} v(S_1(t), S_2(t), t)) = e^{-rt}[-rv dt + v_t dt + v_{x_1} dS_1 + v_{x_2} dS_2 + \frac{1}{2} v_{x_1 x_1} dS_1 dS_1$$
$$+ \frac{1}{2} v_{x_2 x_2} dS_2 dS_2 + v_{x_1 x_2} dS_1 dS_2]$$

Then, from system (5.1)-(5.2), we plug dS_1 and dS_2 into the above equation. We obtain

$$d(e^{-rt} v(S_1(t), S_2(t), t)) = e^{-rt}[-rv dt + v_t dt + v_{x_1}[(r - q_1)S_1 dt + \sigma_1 S_1 dw_1]$$
$$+ v_{x_2}\left[(r - q2)S_2 dt + \sigma_2 S_2(\rho dw_1 + \sqrt{1 - \rho^2} dw_2)\right]$$
$$+ \frac{1}{2}\sigma_1{}^2 S_1{}^2 v_{x_1 x_1} dt + \frac{1}{2}\sigma_1{}^2 S_1{}^2 v_{x_2 x_2} dt + \rho\sigma_1\sigma_2 S_1 S_2 v_{x_1 x_2} dt]$$

29

Next, we put deterministic and stochastic terms together, respectively. We have

$$d(e^{-rt}v(S_1(t), S_2(t), t)) = e^{-rt}[-rv + v_t + (r - q_1)S_1v_{x_1} + (r - q_2)S_2v_{x_2}$$
$$+ \frac{1}{2}\sigma_1{}^2S_1{}^2v_{x_1x_1} + \frac{1}{2}\sigma_2{}^2S_2{}^2v_{x_2x_2} + \rho\sigma_1\sigma_2S_1S_2v_{x_1x_2}]dt$$
$$+ e^{-rt}\left[\sigma_1S_1v_{x_1}dw_1 + v_{x_2}\sigma_2S_2(\rho dw_1 + \sqrt{1 - \rho^2}dw_2)\right]. \quad (3.41)$$

Because the discounted option price is a martingale under the risk neutral probability measure $\mathbb{Q}$, the dt term in the above equation is zero. Thus, we obtain the partial differential equation

$$0 = v_t + (r - q_1)x_1v_{x_1} + (r - q_2)x_2v_{x_2}$$
$$+ \frac{1}{2}\sigma_1{}^2x_1{}^2v_{x_1x_1} + \frac{1}{2}\sigma_2{}^2x_2{}^2v_{x_2x_2} + \rho\sigma_1\sigma_2x_1x_2v_{x_1x_2} - rv. \quad (3.42)$$

There are two regions for the American spread option: holding region Σ_1 and exercise region Σ_2, which are separated by the free boundary $S_1 = b(S_2, t)$. In holding region Σ_1, we have

$$v(S_1, S_2, t) > (S_1 - S_2 - K)^+, \quad (3.43)$$

and v satisfies (3.42). On the boundary, we have

$$v(0, x_2, t) = 0, \quad (3.44)$$

$$v(b(x_2, t), x_2, t) = b(x_2, t) - x_2 - K, \quad (3.45)$$

$$v_{x_1}(b(x_2, t), x_2, t) = 1, \quad (3.46)$$

$$v_{x_2}(b(x_2, t), x_2, t) = -1, \quad (3.47)$$

$$v(x_1, x_2, T) = \max\{x_1 - x_2 - K, 0\}, \quad (3.48)$$

where $x_1 = b(x_2, t)$ is the free boundary for the American spread call option. Condition (3.44) is that when the value S_1 is 0, then the value of option is also 0. Condition (3.45) is the value matching condition. Conditions (3.46) and (3.47) are smooth pasting conditions. They mean that

the derivatives of option price are continuous function on the boundary. Condition (3.48) is the terminal condition. □

3.2 MONTE CARLO SIMULATION APPROACH

In this section, we use the Monte Carlo simulation method to compute American spread option price. We have the following system under risk neutral probability measure $\mathbb{Q}$

$$dS_1(t) = (r - q_1)S_1(t)dt + \sigma_1 S_1(t)dw_1(t), \tag{3.49}$$

$$dS_2(t) = (r - q_2)S_2(t)dt + \sigma_2 S_2(t)\left(\rho dw_1(t) + \sqrt{1 - \rho^2}dw_2(t)\right), \tag{3.50}$$

where w_1 and w_2 are two standard independent Brownian motions under the risk neutral probability measure $\mathbb{Q}$. The payoff function for the American spread option is

$$V(T) = \max(S_1(T) - S_2(T) - K, 0). \tag{3.51}$$

Let $\tilde{V}(s_1, s_2, t)$ be the value of an American spread call option and $\tilde{h}(s_1, s_2, t)$ be the intrinsic value at time t with $S_1(t) = s_1$ and $S_2(t) = s_2$. The value of the American spread call option is the optimal expected future payoff

$$\tilde{V}(s_1, s_2, t) = \sup_{\tau \in T_t} \mathbb{E}_{\mathbb{Q}}\left[e^{-r\tau}\tilde{h}(s_1, s_2, \tau)\right]. \tag{3.52}$$

In order to do the Monte Carlo simulation for the American spread option, we consider exercise times: $0 = t_0 < t_1 < t_2 < s < t_m = T$. We use $\tilde{V}_i(s_1, s_2)$ to represent $\tilde{V}(s_1, s_2, t_i)$ and $\tilde{h}_i(s_1, s_2)$

31

to represent $\tilde{h}(s_1, s_2, t_i)$. Then for the American spread option, we have

$$\tilde{V}_m(s_1, s_2) = \tilde{h}_m(s_1, s_2), \tag{3.53}$$

$$\tilde{V}_{i-1}(s_1, s_2) = \max\left\{\tilde{h}_{i-1}(s_1, s_2), \tilde{c}_{i-1}(s_1, s_2)\right\}, i = 1, \cdots, m, \tag{3.54}$$

where $D_{i-1,i} = e^{-r(t_i - t_{i-1})}$ and

$$\tilde{c}_m(s_1, s_2) = 0, \tag{3.55}$$

$$\tilde{c}_{i-1}(s_1, s_2) = \mathbb{E}_{\mathbb{Q}}\left[D_{i-1,i}\tilde{V}_i(S_1(t_i), S_2(t_i))|S_1(t_{i-1}) = s_1, S_2(t_{i-1}) = s_2\right], i = 1, \cdots, m. \tag{3.56}$$

To compute the American spread call option price, first, we get the value at time T from equation (3.53). Then, we use the equation (3.54) to compare the intrinsic value $\tilde{h}_{m-1}$ and the expected future payoff $\tilde{c}_{m-1}$. After that, we have the option value at time t_{m-1}. Then we use equation (3.54) again, to get the option value at time t_{m-2}. Repeating this, one can get the option value at time 0. It is more convenient that we discount all of them to the time at 0. We define the $h_i(s_1, s_2)$ and $V_i(s_1, s_2)$ by

$$h_i(s_1, s_2) = D_{0,i}\tilde{h}_i(s_1, s_2), \tag{3.57}$$

$$V_i(s_1, s_2) = D_{0,i}\tilde{V}_i(s_1, s_2), i = 1, \cdots, m. \tag{3.58}$$

Plugging $h_i(s_1, s_2)$ and $V_i(s_1, s_2)$ into system (3.53)-(3.54), we have

$$V_m(s_1, s_2) = h_m(s_1, s_2), \tag{3.59}$$

$$V_{i-1}(s_1, s_2) = \max\left\{h_{i-1}(s_1, s_2), c_{i-1}(s_1, s_2)\right\}, i = 1, \cdots, m, \tag{3.60}$$

where the expected future payoff values c_i are given

$$c_m(s_1, s_2) = 0, \tag{3.61}$$

$$c_{i-1}(s_1, s_2) = \mathbb{E}_{\mathbb{Q}}[V_i(S_1(t_i), S_2(t_i))|S_1(t_{i-1}) = s_1, S_2(t_{i-1}) = s_2], i = 1, \cdots, m. \tag{3.62}$$

The equation (3.61) means that the discount expected future payoff is 0. And equation (3.62) is the discount expected future payoff at time t_i.

For pricing American spread option, because one can exercise the option early, we always compare the discount expected future payoff with discount intrinsic value. If the discount intrinsic value is bigger than or equal to the discount expected future payoff, we exercise it immediately. Then, the stopping rule is

$$\tau = \min_{\tau \in \{t_1, t_2, \cdots, t_m\}} \{t_i : h_i(s_1, s_2) \geq c_i(s_1, s_2)\}. \tag{3.63}$$

Next, we approximate expected future payoff by

$$c_i(s_1, s_2) = \boldsymbol{\psi}(s_1, s_2)\boldsymbol{\beta}_i^T, i = 0, 1, \cdots, m - 1, \tag{3.64}$$

for some basis functions ψ_j and constants β_{ij}, where

$$\boldsymbol{\beta}_i = (\beta_{i1}, \beta_{i2}, \cdots, \beta_{iJ}), \tag{3.65}$$

$$\boldsymbol{\psi}(s_1, s_2) = (\psi_1(s_1, s_2), \psi_2(s_1, s_2), \cdots, \psi_J(s_1, s_2)). \tag{3.66}$$

Then, by the above approximation, we obtain

$$\boldsymbol{\beta}_i = B_{\boldsymbol{\psi}}^{-1} B_{\boldsymbol{\psi}V}, \tag{3.67}$$

where

$$B_{\boldsymbol{\psi}} = (\mathbb{E}_{\mathbb{Q}}[\boldsymbol{\psi}(S_1(t_i), S_2(t_i))\boldsymbol{\psi}(S_1(t_i), S_2(t_i))^T]), \tag{3.68}$$

$$B_{\boldsymbol{\psi}V} = \mathbb{E}_{\mathbb{Q}}[\boldsymbol{\psi}(S_1(t_i), S_2(t_i))V_{i+1}(S_1(t_{i+1}), S_2(t_{i+1}))]. \tag{3.69}$$

In practice, the values of $\boldsymbol{\beta}_i$ is approximated by Monte Carlo simulation. Given N independent paths of the prices of the underlying assets

$$\{S_1^{(n)}(t_1), S_1^{(n)}(t_2), \cdots, S_1^{(n)}(t_m)\}, n = 1, \cdots, N, \tag{3.70}$$

$$\{S_2^{(n)}(t_1), S_2^{(n)}(t_2), \cdots, S_2^{(n)}(t_m)\}, n = 1, \cdots, N, \tag{3.71}$$

we have

$$\widetilde{\boldsymbol{\beta}}_i = \widetilde{B}_{\boldsymbol{\psi}}^{-1}\widetilde{B}_{\boldsymbol{\psi}V}, \tag{3.72}$$

where

$$\widetilde{B}_{\boldsymbol{\psi}} = \frac{1}{N}\sum_{n=1}^{N}\boldsymbol{\psi}\left(S_1^{(n)}(t_i), S_2^{(n)}(t_i)\right)\left[\boldsymbol{\psi}(S_1^{(n)}(t_i), S_2^{(n)}(t_i))\right]^T, \tag{3.73}$$

$$\widetilde{B}_{\boldsymbol{\psi}V} = \frac{1}{N}\sum_{n=1}^{N}\boldsymbol{\psi}\left(S_1^{(n)}(t_i), S_2^{(n)}(t_i)\right)V_{i+1}\left(S_1^{(n)}(t_{i+1}), S_2^{(n)}(t_{i+1})\right). \tag{3.74}$$

Also, we have the estimated value $\widetilde{V}_i$ for V_i

$$\widetilde{V}_i\left(S_1^{(n)}(t_i), S_2^{(n)}(t_i)\right) = \max\left\{h_i\left(S_1^{(n)}(t_i), S_2^{(n)}(t_i)\right), c_i\left(S_1^{(n)}(t_i), S_2^{(n)}(t_i)\right)\right\}, \tag{3.75}$$

where the expected future payoff $c_i\left(S_1^{(n)}(t_i), S_2^{(n)}(t_i)\right)$ is given by

$$\widetilde{c}_i\left(S_1^{(n)}(t_i), S_2^{(n)}(t_i)\right) = \boldsymbol{\psi}\left(S_1^{(n)}(t_i), S_2^{(n)}(t_i)\right)\widetilde{\boldsymbol{\beta}}_i^T. \tag{3.76}$$

Therefore, the algorithm of Monte Carlo simulation is

(i) Generate N different paths of the asset price processes

$$\{S_1^n(t_1), S_1^n(t_2) \cdots, S_1^n(t_m)\}, n = 1, 2, \cdots N,$$

$$\{S_2^n(t_1), S_2^n(t_2) \cdots, S_2^n(t_m)\}, n = 1, 2, \cdots N.$$

(ii) At maturity, set $\tilde{v}_m = h_m$.

(iii) For $i = m - 1, \cdots, 2, 1$

    (a) Compute $\tilde{\boldsymbol{\beta}}_i = \tilde{\beta}_{\boldsymbol{\psi}}^{-1} \tilde{\beta}_{\boldsymbol{\psi}V}$, where $\tilde{\beta}_{\boldsymbol{\psi}}^{-1}$ and $\tilde{\beta}_{\boldsymbol{\psi}V}$ are given in (3.73) and (3.74).

    (b) Compute $\tilde{c}_i = \boldsymbol{\psi} \tilde{\boldsymbol{\beta}}_i^T, n = 1, \cdots, N$.

    (c) Set $\tilde{v}_i = \max\{h_i, \tilde{c}_i\}, n = 1, \cdots, N$.

(iv) Set $\tilde{v}_0^{(n)} = h_i$, where $t_i = \min\{t_i \in \{t_1, \cdots, t_m\} : h_i \geq \tilde{c}_i\}$.

(v) Take the mean of all $\tilde{v}_0^{(n)}$ to get the option value $\tilde{v}_0 = \frac{1}{N} \sum_{n=1}^{N} \tilde{v}_0^{(n)}$.

Following the idea of Longstaff-Schwartz [19] for American options, in step (iii)(c), one can use the approximation

$$\tilde{v}_i = \begin{cases} h_i, & \text{if } h_i \geq \tilde{c}_i, \\ \tilde{v}_{i+1}, & \text{if } h_i < \tilde{c}_i. \end{cases} \tag{3.77}$$

The idea for this approximate is that we use the approximated expected future values to determine the exercise times. Then, we have a new Monte Carlo simulation algorithm for the American spread option. We call this algorithm the improved Monte Carlo simulation algorithm. The algorithm is

(i) Generate N different paths of the asset price process

$$\{S_1^n(t_1), S_1^n(t_2) \cdots, S_1^n(t_m)\}, n = 1, 2, \cdots N,$$

$$\{S_2^n(t_1), S_2^n(t_2) \cdots, S_2^n(t_m)\}, n = 1, 2, \cdots N.$$

35

(ii) At maturity, set $\tilde{v}_m = h_m$.

(iii) For $i = m - 1, \cdots, 2, 1$

    (a) Compute $\tilde{\boldsymbol{\beta}}_i = \tilde{\beta}_{\boldsymbol{\psi}}^{-1} \tilde{\beta}_{\boldsymbol{\psi} V}$, where $\tilde{\beta}_{\boldsymbol{\psi}}^{-1}$ and $\tilde{\beta}_{\boldsymbol{\psi} V}$ are given in (3.73) and (3.74).

    (b) Compute $\tilde{c}_i = \boldsymbol{\psi} \tilde{\boldsymbol{\beta}}_i^T, n = 1, \cdots, N$.

    (c) Compute $\tilde{v}_i$

        • If $h_i \geq \tilde{c}_i$, $\tilde{v}_i = h_i$.

        • If $h_i < \tilde{c}_i$, $\tilde{v}_i = \tilde{v}_{i+1}$.

(iv) Set $\tilde{v}_0^{(n)} = h_i$, where $t_i = \min\{t_i \in \{t_1, \cdots, t_m\} : h_i \geq \tilde{c}_i\}$.

(v) Take the mean of all $\tilde{v}_0^{(n)}$ to get the option value $\tilde{v}_0 = \frac{1}{N} \sum_{n=1}^{N} \tilde{v}_0^{(n)}$.

3.3 DUAL METHOD APPROACH

In this section, we use the dual method to price the American spread option. Recall that we have the following system under risk neutral probability measure $\mathbb{Q}$

$$dS_1(t) = (r - q_1)S_1(t)dt + \sigma_1 S_1(t)dw_1(t), \tag{3.78}$$

$$dS_2(t) = (r - q_2)S_2(t)dt + \sigma_2 S_2(t)\left(\rho dw_1(t) + \sqrt{1 - \rho^2}dw_2(t)\right), \tag{3.79}$$

where w_1 and w_2 are two standard independent Brownian motions under the risk neutral probability measure $\mathbb{Q}$. The payoff function for the American spread option is

$$V(T) = \max(S_1(T) - S_2(T) - K, 0). \tag{3.80}$$

The dual method usually provides an upper bound for the American spread option. The idea of dual method is that we formulate American spread option problem as a min-max problem over

martingales. In the following theorem, we extend the results from [21] to two dimensional for the American spread option.

Theorem 3.3. *For any martingales* $M = \{M_{t_i}, i = 0, \cdots, m\}$ *satisfying* $M_0 = 0$, *the price of the American spread option* $V_0(s_1, s_2)$ *satisfies the following inequality*

$$V_0(s_1, s_2) \leq \inf_M \mathbb{E}_{\mathbb{Q}} \left[\max_{i=1,\cdots,m} \{h_i(s_1, s_2) - M_{t_i}\} \right]. \tag{3.81}$$

Proof. Let $M_{t_i}, i = 0, \cdots, m$ satisfies $M_0 = 0$ is a martingale. We use M_i to represent M_{t_i}. According to the optional sampling theorem of martingales, for any stopping time τ taking values in $\{t_1, \cdots, t_m\}$, we have

$$\mathbb{E}_{\mathbb{Q}}[h_\tau(s_1, s_2)] = \mathbb{E}_{\mathbb{Q}} \left[h_\tau(s_1, s_2) - M_\tau \right] \leq \mathbb{E}_{\mathbb{Q}} \left[\max_{i=1,\cdots,m} \{h_i(s_1, s_2) - M_i\} \right] \tag{3.82}$$

By taking the infimum over all the martingales M, we obtain

$$\mathbb{E}_{\mathbb{Q}}[h_\tau(s_1, s_2)] \leq \inf_M \mathbb{E}_{\mathbb{Q}} \left[\max_{i=1,\cdots,m} \{h_i(s_1, s_2) - M_i\} \right] \tag{3.83}$$

Because this inequality holds for any τ, it also should hold for the supremum over τ. Hence we have

$$\sup_\tau \mathbb{E}_{\mathbb{Q}}[h_\tau(s_1, s_2)] \leq \inf_M \mathbb{E}_{\mathbb{Q}} \left[\max_{i=1,\cdots,m} \{h_i(s_1, s_2) - M_i\} \right]. \tag{3.84}$$

We also have

$$V_0(s_1, s_2) = \sup_\tau \mathbb{E}_{\mathbb{Q}}[h_\tau(s_1, s_2)] \leq \inf_M \mathbb{E}_{\mathbb{Q}} \left[\max_{i=1,\cdots,m} \{h_i(s_1, s_2) - M_i\} \right]. \tag{3.85}$$

Thus, we get

$$V_0(s_1, s_2) \leq \inf_M \mathbb{E}_{\mathbb{Q}} \left[\max_{i=1,\cdots,m} \{h_i(s_1, s_2) - M_i\} \right]. \tag{3.86}$$

$\square$

In order to use the above theorem for pricing American spread option, one needs to find a martingale M with $M(0) = 0$. It turns that discount European spread option price is a martingale. If we shift this martingale such that it starts with 0. Then, one can use this martingale into above theorem to get an upper bound for the American spread option. Therefore, one can get an upper bound for American spread option as long as we have an effective way to find the price of corresponding European spread option. Fortunately, one can effectively compute European spread option by a pseudo-analytic formula. From [3] and [13], we have the following pseudo-analytic formula

$$c(s_1, s_2, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}v^2} f(v) dv, \tag{3.87}$$

where

$$f(v) = s_1 e^{d_3} N(d_1) - h(v) e^{-r(T-t)} N(d_2)$$

$$h(v) = K + s_2 \exp\left[(r - q_2 - \frac{1}{2}\sigma_2^2)(T - t) + \sigma_2\sqrt{(T - t)}\right]$$

$$\sigma = \sigma_1\sqrt{(1 - \rho^2)(T - t)}$$

$$d_1 = \frac{1}{\sigma}\left[\log\left(\frac{s_1}{h(v)}\right) + \left(-q_1 + (\frac{1}{2} - \rho^2)\sigma_1^2\right)(T - t)\right]$$

$$d_2 = d_1 - \sigma$$

$$d_3 = -q_1(T - t) - \frac{1}{2}\rho^2\sigma_1^2(T - t) + \rho\sigma_1\sqrt{(T - t)}v$$

where $N(\cdot)$ representing the standard normal cumulative distribution.

CHAPTER 4. THREE-FACTOR STOCHASTIC VOLATILITY MODEL

In this chapter, we consider the three-factor stochastic volatility model for the American spread option. First, we derive a model under risk neutral measure. Then, we study three approaches for pricing American spread option, the partial differential equation approach, the Monte Carlo simulation approach and the dual method approach.

Let S_1 and S_2 be the prices of asset 1 and asset 2 with expected return μ_1 and μ_2. We assume that the underlying asset pays a dividend. The dividend is paid continuously over the life of the option. In time dt the underlying asset pays out a dividend $q_i S_i dt (i = 1, 2)$, where q_i is a constant and stands for the dividend yield. The system for the underlying assets is

$$dS_1(t) = (\mu_1 - q_1)S_1(t)dt + \sigma_1\sqrt{v}S_1(t)dW_1(t), \tag{4.1}$$

$$dS_2(t) = (\mu_2 - q_2)S_2(t)dt + \sigma_2\sqrt{v}S_2(t)dW_2(t), \tag{4.2}$$

$$dv = A(\alpha - v)dt + \sigma_v\sqrt{v}dW_v. \tag{4.3}$$

In this model, the variance process v is a stochastic process, α is the long term mean of the variance, A is the mean reversion rate and σ_v is the volatility of volatility, σ_1 and σ_2 are constants, W_1, W_2 and W_v are three correlated standard Brownian motions with the following correlations under the probability measure $\mathbb{P}$

$$dW_1dW_2 = \rho dt, \tag{4.4}$$

$$dW_1dW_v = \rho_1 dt, \tag{4.5}$$

$$dW_2dW_v = \rho_2 dt. \tag{4.6}$$

Generally, it is much easier to deal with independent rather than correlated Brownian motions. The

following lemma allows us to transfer correlated Brownian motions into independent ones.

Lemma 4.1. *We decompose correlated Brownian motions* W_1, W_2 *and* W_v *into independent ones as follows*

$$\begin{bmatrix} dW_1(t) \\ dW_2(t) \\ dW_v(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \rho & \sqrt{1-\rho^2} & 0 \\ \rho_1 & \rho_3 & \rho_4 \end{bmatrix} \begin{bmatrix} dB_1(t) \\ dB_2(t) \\ dB_3(t) \end{bmatrix}, \tag{4.7}$$

where

$$\rho_3 = \frac{\rho_2 - \rho\rho_1}{\sqrt{1-\rho^2}}, \tag{4.8}$$

$$\rho_4 = \sqrt{\frac{1 - \rho^2 - \rho_1^2 - \rho_2^2 + 2\rho_1\rho_2\rho}{1 - \rho^2}}, \tag{4.9}$$

$B_1(t)$, $B_2(t)$ *and* $B_3(t)$ *are three independent standard Brownian motions under the probability measure* $\mathbb{P}$.

Proof. From (4.7), we have

$$dB_1 = dW_1, \tag{4.10}$$

$$dB_2 = -\frac{\rho}{\sqrt{1-\rho^2}}dW_1 + \frac{1}{\sqrt{1-\rho^2}}dW_2, \tag{4.11}$$

$$dB_3 = \frac{1}{\rho_4}(dW_v - \rho_1 dB_1 - \rho_3 dB_2). \tag{4.12}$$

By Lemma (3.1), we get

$$dB_1 dB_1 = dt, \tag{4.13}$$

$$dB_1 dB_2 = 0, \tag{4.14}$$

$$dB_2 dB_2 = dt. \tag{4.15}$$

40

Next, from (4.10) and (4.12), we have

$$dB_1 dB_3 = \frac{1}{\rho_4} dW_1 (dW_v - \rho_1 dB_1 - \rho_3 dB_2). \tag{4.16}$$

By $dW_1 dW_v = \rho_1 dt$, $dW_1 dB_1 = dt$ and $dW_1 dB_2 = 0$, we obtain

$$dB_1 dB_3 = \frac{1}{\rho_4}(\rho_1 - \rho_1)dt = 0. \tag{4.17}$$

From (4.11) and (4.12), we have

$$dB_2 dB_3 = \frac{1}{\rho_4} \left(-\frac{\rho}{\sqrt{1-\rho^2}} dW_1 + \frac{1}{\sqrt{1-\rho^2}} dW_2 \right)(dW_v - \rho_1 dB_1 - \rho_3 dB_2). \tag{4.18}$$

By $dW_1 dW_v = \rho_1 dt$, $dW_1 dB_1 = dt$, $dW_1 dB_2 = 0$, $dW_2 dW_v = \rho_2 dt$, $dW_2 dB_1 = dW_2 dW_1 = \rho$

and $dW_2 dB_2 = dW_2(-\frac{\rho}{\sqrt{1-\rho^2}} dW_1 + \frac{1}{\sqrt{1-\rho^2}} dW_2) = -\frac{\rho^2}{\sqrt{1-\rho^2}} + \frac{1}{\sqrt{1-\rho^2}}$, we have

$$\begin{aligned} dB_2 dB_3 = \frac{1}{\rho_4} \Bigg[&-\frac{\rho}{\sqrt{1-\rho^2}}(\rho_1 - \rho_1)dt \\ &+\frac{1}{\sqrt{1-\rho^2}} \left(\rho_2 - \rho_1\rho - \rho_3(-\frac{\rho^2}{\sqrt{1-\rho^2}} + \frac{1}{\sqrt{1-\rho^2}}) \right) dt \Bigg] \end{aligned} \tag{4.19}$$

After we simplify the above equation, we obtain

$$dB_2 dB_3 = \frac{1}{\rho_4} \left[\frac{1}{\sqrt{1-\rho^2}} \left(\rho_2 - \rho_1\rho - \rho_3\sqrt{1-\rho^2} \right) dt \right]. \tag{4.20}$$

Then by (4.8), we have

$$dB_2 dB_3 = 0. \tag{4.21}$$

From (4.12), we have

$$dB_3dB_3 = \frac{1}{\rho_4^2}(dW_v - \rho_1 dB_1 - \rho_3 dB_2)(dW_v - \rho_1 dB_1 - \rho_3 dB_2). \tag{4.22}$$

By $dW_v dW_v = dt$, $dB_1 dB_1 = dt$, $dB_2 dB_2 = dt$, $dW_v dB_1 = \rho_1 dt$ $dW_v dB_2 = dW_v(-\frac{\rho}{\sqrt{1-\rho^2}}dW_1 +$ $\frac{1}{\sqrt{1-\rho^2}}dW_2) = \rho_3 dt$, $dB_1 dB_2 = 0$, we get

$$dB_3dB_3 = \frac{1}{\rho_4^2}\left[1 + \rho_1^2 + \rho_3^2 - 2\rho_1^2 - 2\rho_3^2(-\rho_1\frac{\rho}{\sqrt{1-\rho^2}} + \frac{1}{\sqrt{1-\rho^2}})\right]dt. \tag{4.23}$$

After we simplify the above equation, we obtain

$$dB_3dB_3 = \frac{1}{\rho_4^2}(1 - \rho_1^2 - \rho_3^2)dt. \tag{4.24}$$

Then by (4.8) and (4.9), we have

$$dB_3dB_3 = dt. \tag{4.25}$$

For $i = 1, 2, 3$, we have $B_i(0) = 0$. Also, $B_i(t)$ is a martingale and has continuous paths. In addition, we have that equations (4.13)-(4.15), (4.17), (4.21) and (4.24) hold. Thus, by Theorem 2.4, we have $B_1(t)$, $B_2(t)$ and $B_3(t)$ are independent Brownian motions. $\square$

Next, we derive the corresponding system under the risk neutral measure as follows. By Lemma 4.1, system (4.1)-(4.3) becomes

$$dS_1(t) = (u_1 - q_1)S_1(t)dt + \sigma_1\sqrt{v}S_1(t)dB_1(t), \tag{4.26}$$

$$dS_2(t) = (u_2 - q_2)S_2(t)dt + \rho\sqrt{v}S_2(t)dB_1(t) + \sqrt{1-\rho^2}\sqrt{v}S_2(t)dB_2(t), \tag{4.27}$$

$$dv = A(\alpha - v)dt + \rho_1\sigma_v\sqrt{v}dB_1(t) + \rho_3\sigma_v\sqrt{v}dB_2(t) + \rho_4\sigma_v\sqrt{v}dB_3(t), \tag{4.28}$$

where

$$\rho_3 = \frac{\rho_2 - \rho\rho_1}{\sqrt{1 - \rho^2}}, \tag{4.29}$$

$$\rho_4 = \sqrt{\frac{1 - \rho^2 - \rho_1^2 - \rho_2^2 + 2\rho_1\rho_2\rho}{1 - \rho^2}}. \tag{4.30}$$

Again, we define the value processes $\hat{S}_i(t)(i = 1, 2)$ by

$$\hat{S}_1(t) = e^{q_1 t} S_1(t), \tag{4.31}$$

$$\hat{S}_2(t) = e^{q_2 t} S_2(t). \tag{4.32}$$

Then, we have

$$d\hat{S}_1(t) = u_1 \hat{S}_1(t)dt + \sigma_1\sqrt{v}\hat{S}_1(t)dB_1(t), \tag{4.33}$$

$$d\hat{S}_2(t) = u_2 \hat{S}_2(t)dt + \rho\sqrt{v}\hat{S}_2(t)dB_1(t) + \sqrt{1 - \rho^2}\sqrt{v}\hat{S}_2(t)dB_2(t), \tag{4.34}$$

$$dv = A(\alpha - v)dt + \rho_1\sigma_v\sqrt{v}dB_1(t) + \rho_3\sigma_v\sqrt{v}dB_2(t) + \rho_4\sigma_v\sqrt{v}dB_3(t). \tag{4.35}$$

We introduce the discounted value processes $\hat{S}_1^{S_0}(t)$ and $\hat{S}_2^{S_0}(t)$ by

$$d\hat{S}_1^{S_0} = (u_1 - r)\hat{S}_1^{S_0}dt + \sigma_1\sqrt{v}\hat{S}_1^{S_0}dB_1(t), \tag{4.36}$$

$$d\hat{S}_2^{S_0} = (u_2 - r)\hat{S}_2^{S_0}dt + \rho\sqrt{v}\hat{S}_2^{S_0}dB_1(t) + \sqrt{1 - \rho^2}\sqrt{v}\hat{S}_2^{S_0}dB_2(t), \tag{4.37}$$

$$dv = A(\alpha - v)dt + \rho_1\sigma_v\sqrt{v}dB_1(t) + \rho_3\sigma_v\sqrt{v}dB_2(t) + \rho_4\sigma_v\sqrt{v}dB_3(t). \tag{4.38}$$

Now we want to find the equivalent martingale measure $\mathbb{Q}$ under which the discounted value processes are $\mathbb{Q}$ martingales. To achieve this, we use the Girsanov Theorem. We define $\theta = (\theta_1, \theta_2, \theta_3)^T$ by

$$A\theta = u - r, \tag{4.39}$$

where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ \rho & \sqrt{1-\rho^2} & 0 \\ \rho_1 & \rho_3 & \rho_4 \end{bmatrix}, u - r = \begin{bmatrix} u_1 - r \\ u_2 - r \end{bmatrix}. \tag{4.40}$$

As we see, we have two equations, but there are three unknowns $\theta_i (i = 1, 2, 3)$. Thus, we have many solutions. We fix $\theta_3 = \lambda$ and define

$$Z(t) = \exp \left\{ -\int_0^t \theta \cdot dB(u) - \frac{1}{2} \int_0^t ||\theta||^2 du \right\}, \tag{4.41}$$

and

$$w(t) = B(t) + \int_0^t \theta du, \tag{4.42}$$

where $B(t) = (B_1(t), B_2(t), B_3(t))^T$ and $w(t) = (w_1(t), w_2(t), w_3(t)^T$. By Girsanov Theorem 2.16, setting $Z = Z(T)$, then $\mathbb{E}Z = 1$. And the probability measure $\mathbb{Q}$ is given by

$$Q(A) = \int_A Z(w) d\mathbb{P}(w), \tag{4.43}$$

for all $A \in \mathcal{F}$. Note that the process $w(t)$ is a three-dimensional Brownian motions under $\mathbb{Q}$. By (4.36)-(4.38) and (4.42), under the equivalent probability measure $\mathbb{Q}$, we have

$$d\hat{S}_1^{S_0}(t) = \sigma_1 \hat{S}_1^{S_0}(t) \sqrt{v} dw_1(t), \tag{4.44}$$

$$d\hat{S}_2^{S_0}(t) = \rho \sigma_2 \hat{S}_2^{S_0}(t) \sqrt{v} dw_1(t) + \sqrt{1-\rho^2} \sigma_2 \hat{S}_2^{S_0}(t) \sqrt{v} dw_2(t), \tag{4.45}$$

$$dv = A(\alpha - v)dt - \lambda \sigma_v \sqrt{v} dt + \rho_1 \sigma_v \sqrt{v} dw_1(t) + \rho_3 \sigma_v \sqrt{v} dw_2(t) + \rho_4 \sigma_v \sqrt{v} dw_3(t). \tag{4.46}$$

Then by $\hat{S}_i(t) = S_0(t)S_i^{S_0}(t)(i = 1, 2)$, we obtain

$$d\hat{S}_1(t) = r\hat{S}_1 dt + \sigma_1\sqrt{v}\hat{S}_1 dw_1(t), \tag{4.47}$$

$$d\hat{S}_2(t) = r\hat{S}_2 dt + \rho\sigma_2\sqrt{v}\hat{S}_2 dw_1(t) + \sqrt{1-\rho^2}\sigma_2\hat{S}_2 dw_2(t), \tag{4.48}$$

$$dv = A(\alpha - v)dt - \lambda\sigma_v\sqrt{v}dt + \rho_1\sigma_v\sqrt{v}dw_1(t) + \rho_3\sigma_v\sqrt{v}dw_2(t) + \rho_4\sigma_v\sqrt{v}dw_3(t). \tag{4.49}$$

In addition, by $\hat{S}_i(t) = e^{q_i t}S_i(t)(i = 1, 2)$, we get

$$dS_1(t) = (r - q_1)S_1(t)dt + \sigma_1 S_1(t)dw_1, \tag{4.50}$$

$$dS_2(t) = (r - q_2)S_2(t)dt + \rho\sigma_2\sqrt{v}S_2(t)dw_1(t) + \sqrt{1-\rho^2}\sigma_2\sqrt{v}S_2(t)dw_2(t), \tag{4.51}$$

$$dv = A(\alpha - v)dt - \lambda\sigma_v\sqrt{v}dt + \rho_1\sigma_v\sqrt{v}dw_1(t) + \rho_3\sigma_v\sqrt{v}dw_2(t) + \rho_4\sigma_v\sqrt{v}dw_3(t). \tag{4.52}$$

Then we assume that λ is a linear function of $\sqrt{v}$, i.e., $\lambda = c\sqrt{v}$. We have

$$dv = A(\alpha - v)dt - cv\sigma_v dt + \rho_1\sigma_v\sqrt{v}dw_1(t) + \rho_3\sigma_v\sqrt{v}dw_2(t) + \rho_4\sigma_v\sqrt{v}dw_3(t). \tag{4.53}$$

By taking $k = A + c\sigma_v$, $u = \dfrac{A\alpha}{A + \sigma_v}$, we get

$$dv = k(u - v)dt + \rho_1\sigma_v\sqrt{v}dw_1(t) + \rho_3\sigma_v\sqrt{v}dw_2(t) + \rho_4\sigma_v\sqrt{v}dw_3(t). \tag{4.54}$$

Therefore, we get the system under the risk neutral measure Q

$$dS_1(t) = (r - q_1)S_1(t)dt + \sigma_1\sqrt{v}S_1(t)dw_1(t), \tag{4.55}$$

$$dS_2(t) = (r - q_2)S_2(t)dt + \sigma_2\sqrt{v}S_2(t)\left(\rho dw_1(t) + \sqrt{1-\rho^2}dw_2(t)\right), \tag{4.56}$$

$$dv = k(\mu - v)dt + \sigma_v\sqrt{v}\left(\rho_1 dw_1(t) + \rho_3 dw_2(t) + \rho_4 dw_3\right). \tag{4.57}$$

4.1 PARTIAL DIFFERENTIAL EQUATIONS APPROACH

In this section, we derive the partial differential equation for pricing the American spread option under the three-factor stochastic volatility model. We use the standard approach based on Ito formula. Recall that we have the system under the risk neutral measure $\mathbb{Q}$

$$dS_1(t) = (r - q_1)S_1(t)dt + \sigma_1\sqrt{v}S_1(t)dw_1(t), \tag{4.58}$$

$$dS_2(t) = (r - q_2)S_2(t)dt + \sigma_2\sqrt{v}S_2(t)\left(\rho dw_1(t) + \sqrt{1-\rho^2}dw_2(t)\right), \tag{4.59}$$

$$dv = k(\mu - v)dt + \sigma_v\sqrt{v}\left(\rho_1 dw_1(t) + \rho_3 dw_2(t) + \rho_4 dw_3\right), \tag{4.60}$$

where

$$\rho_3 = \frac{\rho_2 - \rho\rho_1}{\sqrt{1-\rho^2}}, \tag{4.61}$$

$$\rho_4 = \sqrt{\frac{1 - \rho^2 - \rho_1^2 - \rho_2^2 + 2\rho_1\rho_2\rho}{1-\rho^2}}, \tag{4.62}$$

and w_1, w_2 and w_3 are three independent standard Brownian motions under the risk neutral probability measure $\mathbb{Q}$.

The following theorem gives a partial differential equation with a free boundary condition for the American spread option. Let $v(x_1, x_2, x_3, t)$ be the price of the American spread option with $x_1 = S_1(t)$, $x_2 = S_2(t)$ and $x_3 = v(t)$ at time t.

Theorem 4.2. $v(x_1, x_2, x_3, t)$ *is the solution of the PDE*

$$
\begin{aligned}
0 = {}& -rv + v_t + (r - q_1)x_1 v_{x_1} + (r - q_2)x_2 v_{x_2} + k(\mu - x_3)v_{x_3} \\
& + \frac{1}{2}\sigma_1{}^2 x_1{}^2 x_3 v_{x_1 x_1} + \frac{1}{2}\sigma_2{}^2 x_2{}^2 x_3 v_{x_2 x_2} + \frac{1}{2}\sigma_v{}^2 x_3 v_{x_3 x_3} \\
& + \rho\sigma_1\sigma_2 x_1 x_2 x_3 v_{x_1 x_2} + \rho_1\sigma_1\sigma_v x_1 x_3 v_{x_1 x_3} + \rho_2\sigma_2\sigma_v x_2 x_3 v_{x_1 x_2}
\end{aligned}
\tag{4.63}
$$

with the boundary and terminal conditions

$$v(0, x_2, x_3, t) = 0, \tag{4.64}$$

$$v(b(x_2, x_3, t), x_2, x_3, t) = b(x_2, x_3, t) - x_2 - K, \tag{4.65}$$

$$v_{x_1}(b(x_2, x_3, t), x_2, x_3, t) = 1, \tag{4.66}$$

$$v_{x_2}(b(x_2, x_3, t), x_2, x_3, t) = -1, \tag{4.67}$$

$$v_{x_3}(b(x_2, x_3, t), x_2, x_3, t) = 0, \tag{4.68}$$

$$v(x_1, x_2, x_3, T) = (x_1 - x_2 - K)^+, \tag{4.69}$$

where $x_1 = b(x_2, x_3, t)$ is the free boundary.

Proof. The payoff function for the spread call option without early exercise is

$$V(T) = \max(S_1(T) - S_2(T) - K, 0), \tag{4.70}$$

where $V(t)$ represents the option price without early exercise at time t. By applying the risk neutral formula, we get the price at time t is

$$V(t) = e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}[\max(S_1(T) - S_2(T) - K, 0)|\mathcal{F}_t]. \tag{4.71}$$

Because the joint processes $(S_1(t), S_2(t), v(t))$ is a Markov process, $V(t)$ can be written as a function of the time variable and the values of these processes at time t, i.e., $V = v(x_1, x_2, x_3, t)$. Then, we have

$$v(x_1, x_2, x_3, t) = e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}[\max(S_1(T) - S_2(T) - K, 0)|\mathcal{F}_t]. \tag{4.72}$$

We apply Ito's formula to the function $e^{-rt}v(x_1, x_2, x_3, t)$. By Ito's formula, we have

$$d(e^{-rt}v(S_1, S_2, v, t)) = e^{-rt}[-rvdt + v_t dt + v_{x_1}dS_1 + v_{x_2}dS_2 + v_{x_3}dv$$
$$\frac{1}{2}v_{x_1x_1}dS_1dS_1 + \frac{1}{2}v_{x_2x_2}dS_2dS_2 + \frac{1}{2}v_{x_3x_3}dvdv$$
$$v_{x_1x_2}dS_1dS_2 + v_{x_1x_3}dS_1dv + v_{x_2x_3}dS_2dv].$$

Then, from system (4.58)-(4.60), we plug dS_1, dS_2 and dv into the above equation. We obtain

$$d(e^{-rt}v(S_1, S_2, v, t)) = e^{-rt}[-rvdt + v_t dt + v_{x_1}((r - q_1)S_1dt + \sigma_1 S_1 dw_1)$$
$$+ v_{x_2}\left[(r - q_2)S_2dt + \sigma_2 S_2(\rho dw_1 + \sqrt{1 - \rho^2}dw_2)\right]$$
$$+ v_{x_3}\left[k(\mu - v)dt + \sigma_v\sqrt{v}(\rho_1 dw_1 + \rho_3 dw_2 + \rho_4 dw_3)\right]$$
$$+ \frac{1}{2}\sigma_1^2 S_1^2 vv_{x_1x_1}dt + \frac{1}{2}\sigma_2^2 S_2^2 vv_{x_2x_2}dt + \frac{1}{2}\sigma_v^2 vv_{x_3x_3}dt$$
$$+ \rho\sigma_1\sigma_2 S_1 S_2 vv_{x_1x_2}dt + \rho_1\sigma_1\sigma_v S_1 vv_{x_1x_3}dt + \rho_2\sigma_2\sigma_v S_2 vv_{x_1x_2}dt].$$

Next, we put deterministic and stochastic terms together, respectively. We have

$$d(e^{-rt}v(S_1, S_2, v, t)) = e^{-rt}[-rv + v_t + (r - q_1)S_1 v_{x_1} + (r - q_2)S_2 v_{x_2} + k(\mu - v)v_{x_3}$$
$$+ \frac{1}{2}\sigma_1^2 S_1^2 vv_{x_1x_1} + \frac{1}{2}\sigma_2^2 S_2^2 vv_{x_2x_2} + \frac{1}{2}\sigma_v^2 vv_{x_3x_3}$$
$$+ \rho\sigma_1\sigma_2 S_1 S_2 vv_{x_1x_2} + \rho_1\sigma_1\sigma_v S_1 vv_{x_1x_3} + \rho_2\sigma_2\sigma_v S_2 vv_{x_1x_2}]dt$$
$$+ e^{-rt}[\sigma_1 S_1 v_{x_1}dw_1 + \sigma_2 S_2(\rho dw_1 + \sqrt{1 - \rho^2}dw_2)v_{x_2}$$
$$+ \sigma_v\sqrt{v}(\rho_1 dw_1 + \rho_3 dw_2 + \rho_4 dw_3)v_{x_3}]. \tag{4.73}$$

Because the discounted option price is a martingale under the risk neutral probability measure $\mathbb{Q}$,

the dt term in the above equation is zero. Thus, we obtain the partial differential equation

$$0 = -rv + v_t + (r - q_1)x_1 v_{x_1} + (r - q_2)x_2 v_{x_2} + k(\mu - x_3)v_{x_3}$$
$$+ \frac{1}{2}\sigma_1{}^2 x_1{}^2 x_3 v_{x_1 x_1} + \frac{1}{2}\sigma_2{}^2 x_2{}^2 x_3 v_{x_2 x_2} + \frac{1}{2}\sigma_v{}^2 x_3 v_{x_3 x_3}$$
$$+ \rho\sigma_1\sigma_2 x_1 x_2 x_3 v_{x_1 x_2} + \rho_1\sigma_1\sigma_v x_1 x_3 v_{x_1 x_3} + \rho_2\sigma_2\sigma_v x_2 x_3 v_{x_1 x_2}. \tag{4.74}$$

There are two regions for American spread option: holding region Σ_1 and exercise region Σ_2, which are separated by the free boundary $S_1 = b(S_2, v, t)$. In holding region Σ_1, we have

$$v(S_1, S_2, v, t) > (S_1 - S_2 - K)^+, \tag{4.75}$$

and v satisfies (4.74). On the boundary, we have

$$v(0, x_2, x_3, t) = 0, \tag{4.76}$$

$$v(b(x_2, x_3, t), x_2, x_3, t) = b(x_2, x_3, t) - x_2 - K, \tag{4.77}$$

$$v_{x_1}(b(x_2, x_3, t), x_2, x_3, t) = 1, \tag{4.78}$$

$$v_{x_2}(b(x_2, x_3, t), x_2, x_3, t) = -1, \tag{4.79}$$

$$v_{x_3}(b(x_2, x_3, t), x_2, x_3, t) = 0, \tag{4.80}$$

$$v(x_1, x_2, x_3, T) = (x_1 - x_2 - K)^+, \tag{4.81}$$

where $x_1 = b(x_2, x_3, t)$ is the free boundary for the American spread call option. Condition (4.76) is that when the value S_1 is 0, then the value of option is also 0. Condition (4.77) is the value matching condition. Conditions (4.78), (4.79) and (4.80) are smooth pasting conditions. They mean that the derivatives of option price are continuous function on the boundary. Condition (4.81) is the terminal condition.

$\square$

4.2 MONTE CARLO SIMULATION APPROACH

In this section, we use Monte Carlo simulation method to compute the price of the American spread option under three-factor stochastic volatility model. Recall that we have the system under the risk neutral measure $\mathbb{Q}$

$$dS_1(t) = (r - q_1)S_1(t)dt + \sigma_1\sqrt{v}S_1(t)dw_1(t),$$

$$dS_2(t) = (r - q_2)S_2(t)dt + \sigma_2\sqrt{v}S_2(t)\left(\rho dw_1(t) + \sqrt{1 - \rho^2}dw_2(t)\right),$$

$$dv = k(\mu - v)dt + \sigma_v\sqrt{v}\left(\rho_1 dw_1(t) + \rho_3 dw_2(t) + \rho_4 dw_3\right),$$

where

$$\rho_3 = \frac{\rho_2 - \rho\rho_1}{\sqrt{1 - \rho^2}},$$

$$\rho_4 = \sqrt{\frac{1 - \rho^2 - \rho_1^2 - \rho_2^2 + 2\rho_1\rho_2\rho}{1 - \rho^2}},$$

and w_1, w_2 and w_3 are three independent standard Brownian motions under the risk neutral probability measure $\mathbb{Q}$.

Most of the steps are the same as the Monte Carlo simulation method to price American spread options under two-factor geometric Brownian motion model. The main difference is that when we approximate expected future payoff value, we take one more stochastic factor into account by the following

$$
\begin{aligned}
c_i(s_1, s_2, s_3) &= \mathbb{E}_{\mathbb{Q}}[V_{i+1}(S_1(t_i), S_2(t_i), v_{t_i})|S_1(t_{i-1}) = s_1, S_2(t_{i-1}) = s_2, v(t_{i-1}) = s_3] \\
&\approx \sum_{j=1}^{J} \beta_{ij}\psi_j(s_1, s_2, s_3) \\
&= \boldsymbol{\psi}(s_1, s_2, s_3) \cdot \boldsymbol{\beta}_i^T, i = 0, 1, 2, \cdots, m - 1,
\end{aligned}
\tag{4.82}
$$

where

$$\boldsymbol{\beta}_i = (\beta_{i1}, \beta_{i2}, \cdots, \beta_{iJ}), \tag{4.83}$$

$$\boldsymbol{\psi}(s_1, s_2, s_3) = (\psi_1(s_1, s_2, s_3), \psi_2(s_1, s_2, s_3), \cdots, \psi_J(s_1, s_2, s_3)). \tag{4.84}$$

4.3   DUAL METHOD APPROACH

In this section, we use the dual method to compute the price of the American spread option. Recall that we have the system under the risk neutral measure $\mathbb{Q}$

$$dS_1(t) = (r - q_1)S_1(t)dt + \sigma_1\sqrt{v}S_1(t)dw_1(t),$$

$$dS_2(t) = (r - q_2)S_2(t)dt + \sigma_2\sqrt{v}S_2(t)\left(\rho dw_1(t) + \sqrt{1 - \rho^2}dw_2(t)\right),$$

$$dv = k(\mu - v)dt + \sigma_v\sqrt{v}\left(\rho_1 dw_1(t) + \rho_3 dw_2(t) + \rho_4 dw_3\right),$$

where

$$\rho_3 = \frac{\rho_2 - \rho\rho_1}{\sqrt{1 - \rho^2}},$$

$$\rho_4 = \sqrt{\frac{1 - \rho^2 - \rho_1^2 - \rho_2^2 + 2\rho_1\rho_2\rho}{1 - \rho^2}},$$

and w_1, w_2 and w_3 are three independent standard Brownian motions under the risk neutral probability measure $\mathbb{Q}$. We do the substitution as follows

$$y_1(t) = \log(S_1(t)),$$

$$y_2(t) = \log(S_2(t)),$$

$$y_3(t) = v(t).$$

Then, we have the new system

$$dy_1(t) = (r - q_1 - \frac{1}{2}\sigma_1^2 y_3)y_1(t)dt + \sigma_1\sqrt{y_3}y_1(t)dw_1(t) \tag{4.85}$$

$$dy_2(t) = (r - q_2 - \frac{1}{2}\sigma_2^2 y_3)y_2(t)dt + \sigma_2\sqrt{y_3}y_2(t)\left(\rho dw_1(t) + \sqrt{1-\rho^2}dw_2(t)\right) \tag{4.86}$$

$$dy_3(t) = k(\mu - y_3)dt + \sigma_{y_3}\sqrt{y_3}\left(\rho_1 dw_1(t) + \rho_3 dw_2(t) + \rho_4 dw_3\right) \tag{4.87}$$

The dual method usually provides an upper bound for the American spread option. The idea of dual method is that we formulate American spread option problem as a min-max problem over martingales. In the following theorem, we extend the results from [21] to three dimensional for the American spread option.

Theorem 4.3. *For any martingales* $M = \{M_{t_i}, i = 0, \cdots, m\}$ *satisfying* $M_0 = 0$, *the price of the American spread option* $V_0(s_1, s_2, s_3)$ *satisfies the following inequality*

$$V_0(s_1, s_2, s_3) \leq \inf_M \mathbb{E}_\mathbb{Q}\left[\max_{i=1,\cdots,m}\{h_i(s_1, s_2) - M_{t_i}\}\right]. \tag{4.88}$$

The proof of the above theorem is similar to the proof of Theorem 3.3. So we omit the proof.

In order to use the above theorem for pricing American spread option, one needs to find a martingale M with $M(0) = 0$. It turns that discount European spread option price is a martingale. If we shift this martingale such that it starts with 0, then, one can use this martingale in above theorem to get an upper bound for the American spread option. Therefore, one can get an upper bound for American spread option as long as we have an effective way to find the price of corresponding European spread option. Fortunately, one can effectively compute European spread option by

characteristic function technique. From [13], we have the characteristic function

$$\phi_{3sv}(u_1, u_2) = \exp[i[y_1(0) + (r - q_1)T]u_1$$

$$+ i[y_2(0) + (r - q_2)T]u_2 + D(T)v(0) + C(T)], \tag{4.89}$$

where

$$D(T) = \frac{2\xi(1 - \exp(-\theta T))}{2\theta - (\theta - \gamma)(1 - \exp(-\theta T))}, \tag{4.90}$$

$$C(T) = -\frac{k\mu}{\sigma_v^2} \left[2\log\left(\frac{2\theta - (\theta - \gamma)(1 - \exp(-\theta T))}{2\theta}\right) + (\theta - \gamma)T \right]$$

$$+ i(r - q_1)u_1 T + i(r - q_2)u_2 T, \tag{4.91}$$

$$\theta = \sqrt{\gamma^2 - 2\sigma^2 \xi}, \tag{4.92}$$

$$\gamma = k - i(\rho_1 \sigma_1 u_1 + \rho_2 \sigma_2 u_2)\sigma_v, \tag{4.93}$$

$$\xi = -\frac{1}{2}\left[\left(\sigma_1^2 u_1^2 + \sigma_2^2 u_2^2 + 2\rho\sigma_1\sigma_2 u_1 u_2\right) + i(\sigma_1^2 u_1 + \sigma_2^2 u_2)\right]. \tag{4.94}$$

In this chapter, we investigate the numerical implementations and results for the American spread option under different models, including the two-factor geometric Brownian motion model and the three-factor stochastic volatility model. For each model, we compare the numerical results of different approaches.

5.1 TWO-FACTOR GEOMETRIC BROWNIAN MOTION MODEL

In this section, we study the numerical algorithms and results to the American spread call option under the two-factor geometric Brownian motion model. We have three approaches for pricing the American spread option: the partial differential equation approach, Monte Carlo simulation approach and the dual method approach.

Recall that we have the following system under the risk neutral measure $\mathbb{Q}$

$$dS_1(t) = (r - q_1)S_1(t)dt + \sigma_1 S_1(t)dw_1(t), \tag{5.1}$$

$$dS_2(t) = (r - q_2)S_2(t)dt + \sigma_2 S_2(t)\left(\rho dw_1(t) + \sqrt{1 - \rho^2}dw_2(t)\right), \tag{5.2}$$

where w_1 and w_2 are two standard independent Brownian motions under the risk neutral probability measure $\mathbb{Q}$. We take American spread call option as an example. The payoff function is $\max\{S_1 - S_2 - K, 0\}$. The parameters are $S_2 = 36$, $\sigma_1 = 0.4$, $\sigma_2 = 0.2$, $r = 0.06$, $\rho = 0.5$, $T = 1$, $K = 5$, $q_1 = 0.05$ and $q_2 = 0.01$.

5.1.1 Partial Differential Equations Approach. In this subsection, we use the partial differential equation method to compute the price of the American spread option under the two-factor geometric Brownian motion model.

From Chapter 3, we know that the price of the American spread option is the solution of the partial differential equation with a free boundary condition. In order to solve it numerically, we get rid of the free boundary and formulate the problem into complementarity problem. Let $v(x_1, x_2, t)$ be the price of the American spread option with $x_1 = S_1$ and $x_2 = S_2$ at time t. We use Lv to represent

$$
\begin{aligned}
Lv = {}& v_t + (r - q_1)x_1 v_{x_1} + (r - q_2)x_2 v_{x_2} \\
& + \frac{1}{2}\sigma_1^2 x_1^2 v_{x_1 x_1} + \frac{1}{2}\sigma_2^2 x_2^2 v_{x_2 x_2} + \rho\sigma_1\sigma_2 x_1 x_2 v_{x_1 x_2} - rv.
\end{aligned}
\tag{5.3}
$$

Also, we know that we have two regions for the American spread option. In one region, early exercise is optimal. We have

$$
Lv \leq 0,
\tag{5.4}
$$

$$
v = h.
\tag{5.5}
$$

The other one, early exercise is not optimal. We have

$$
Lv = 0,
\tag{5.6}
$$

$$
v \geq h.
\tag{5.7}
$$

Hence, the price is formulated as the solution of the following complementarity problem

$$
Lv \leq 0,
\tag{5.8}
$$

$$
v - h \geq 0,
\tag{5.9}
$$

$$
(Lv)(v - h) = 0,
\tag{5.10}
$$

where $h = \max(x_1 - x_2 - K, 0)$.

It is more convenient to solve the above complementarity problem (5.8)-(5.10) by doing the following substitutions

$$x_1 = e^{y_1},$$

$$x_2 = e^{y_2},$$

$$v(x_1, x_2, t) = u(y_1, y_2, t).$$

Then, we have

$$Lu \leq 0, \tag{5.11}$$

$$u - h \geq 0, \tag{5.12}$$

$$(Lu)(u - h) = 0, \tag{5.13}$$

where $h = \max(e^{y_1} - e^{y_2} - K, 0)$ and

$$Lu = u_t + \left(r - q_1 - \frac{\sigma_1^2}{2}\right)u_{y_1} + \left(r - q_2 - \frac{\sigma_2^2}{2}\right)u_{y_2}$$

$$+ \frac{\sigma_1^2}{2}u_{y_1 y_1} + \frac{\sigma_2^2}{2}u_{y_2 y_2} + \rho\sigma_1\sigma_2 u_{y_1 y_2} - ru. \tag{5.14}$$

Then we use the finite difference method to solve the above complementarity problem. First, we divide the life of the option T into N equally spaced intervals of the length $\Delta t = \frac{T}{N}$. Then a total of $N + 1$ times are therefore considered: $0,\ \Delta t,\ 2\Delta t,\ \cdots,\ T$. Because the range of $y_i(i = 1, 2)$ is $(-\infty, +\infty)$, we restrict y_i by taking a very large positive number $\overline{y}_i$ and a very large negative number $\underline{y}_i$, such that $\underline{y}_i \leq y_i \leq \overline{y}_i$. Then one can divide y_i into M equally spaced intervals of length $\Delta y_i = \frac{\overline{y}_i - \underline{y}_i}{M}$. The space and time variables define a grid consisting of a total of $(M + 1)(M + 1)(N + 1)$ points. We use the variable $u_{i,j}^k$ to represent $u(y_{1i}, y_{2j}, t_k)$ with

$y_{1i} = \underline{y}_1 + (i-1)\Delta y_1$, $y_{2j} = \underline{y}_2 + (j-1)\Delta y_2$ and $t_k = (k-1)\Delta t$. For the point (y_{1i}, y_{2j}, t_k), we have the following finite differences

$$u_t = \frac{u_{i,j}^k - u_{i,j}^{k-1}}{\Delta t} + O(\Delta t), \tag{5.15}$$

$$u_{y_1} = \frac{u_{i+1,j}^k - u_{i-1,j}^k}{2\Delta y_1} + O\left[(\Delta y_1)^2\right], \tag{5.16}$$

$$u_{y_2} = \frac{u_{i,j+1}^k - u_{i,j-1}^k}{2\Delta y_2} + O\left[(\Delta y_2)^2\right], \tag{5.17}$$

$$u_{y_1 y_1} = \frac{u_{i+1,j}^k + u_{i-1,j}^k - 2u_{i,j}^k}{(\Delta y_1)^2} + O\left[(\Delta y_1)^2\right], \tag{5.18}$$

$$u_{y_2 y_2} = \frac{u_{i,j+1}^k + u_{i,j-1}^k - 2u_{i,j}^k}{(\Delta y_2)^2} + O\left[(\Delta y_2)^2\right], \tag{5.19}$$

$$u_{y_1 y_2} = \frac{u_{i+1,j+1}^k + u_{i-1,j-1}^k - u_{i-1,j+1}^k - u_{i+1,j-1}^k}{4\Delta y_1 \Delta y_2} + O\left[(\Delta y_1)^2 + (\Delta y_2)^2\right]. \tag{5.20}$$

Then we incorporate all of the above finite differences into equation (5.14). We get

$$\begin{aligned}
Lu_{i,j}^k = {} & \frac{u_{i,j}^k - u_{i,j}^{k-1}}{\Delta t} + \left(r - q_1 - \frac{1}{2}\sigma_1^2\right)\left(\frac{u_{i+1,j}^k - u_{i-1,j}^k}{2\Delta y_1} + O\left[(\Delta y_1)^2\right]\right) \\
& + \left(r - q_2 - \frac{1}{2}\sigma_2^2\right)\left(\frac{u_{i,j+1}^k - u_{i,j-1}^k}{2\Delta y_2} + O\left[(\Delta y_2)^2\right]\right) \\
& + \frac{1}{2}\sigma_1^2\left(\frac{u_{i+1,j}^k + u_{i-1,j}^k - 2u_{i,j}^k}{(\Delta y_1)^2} + O\left[(\Delta y_1)^2\right]\right) \\
& + \frac{1}{2}\sigma_2^2\left(\frac{u_{i,j+1}^k + u_{i,j-1}^k - 2u_{i,j}^k}{(\Delta y_2)^2} + O\left[(\Delta y_2)^2\right]\right) \\
& + \rho\sigma_1\sigma_2\left(\frac{u_{i+1,j+1}^k + u_{i-1,j-1}^k - u_{i-1,j+1}^k - u_{i+1,j-1}^k}{4\Delta y_1 \Delta y_2} + O\left[(\Delta y_1)^2 + (\Delta y_2)^2\right]\right) \\
& - ru_{i,j}^k. \tag{5.21}
\end{aligned}$$

After we simplify it and we ignore the higher order term, we obtain

$$(\Delta t) * Lu_{i,j}^k = -u_{i,j}^{k-1} + A * u_{i+1,j+1}^k + B * u_{i+1,j}^k + C * u_{i+1,j-1}^k$$

$$+ D * u_{i,j+1}^k + E * u_{i,j}^k + F * u_{i,j-1}^k$$

$$+ G * u_{i-1,j+1}^k + H * u_{i-1,j}^k + I * u_{i-1,j-1}^k, \tag{5.22}$$

where

$$a_1 = r - q_1 - \frac{1}{2}\sigma_1^2, \tag{5.23}$$

$$a_2 = r - q_2 - \frac{1}{2}\sigma_2^2, \tag{5.24}$$

$$A = \frac{\rho\sigma_1\sigma_2\Delta t}{4\Delta y_1\Delta y_2}, \tag{5.25}$$

$$B = \frac{\sigma_1^2\Delta t}{2(\Delta y_1)^2} + \frac{a_1\Delta t}{2\Delta y_1}, \tag{5.26}$$

$$C = -\frac{\rho\sigma_1\sigma_2\Delta t}{4\Delta y_1\Delta y_2}, \tag{5.27}$$

$$D = \frac{\sigma_2^2\Delta t}{2(\Delta y_2)^2} + \frac{a_2\Delta t}{2\Delta y_2}, \tag{5.28}$$

$$E = 1 - r\Delta t - \frac{\sigma_1^2\Delta t}{(\Delta y_1)^2} - \frac{\sigma_2^2\Delta t}{(\Delta y_2)^2}, \tag{5.29}$$

$$F = \frac{\sigma_2^2\Delta t}{2(\Delta y_2)^2} - \frac{a_2\Delta t}{2\Delta y_2}, \tag{5.30}$$

$$G = C, \tag{5.31}$$

$$H = \frac{\sigma_1^2\Delta t}{2(\Delta y_1)^2} - \frac{a_1\Delta t}{2(\Delta y_1)}, \tag{5.32}$$

$$I = A. \tag{5.33}$$

For the complementarity problem (5.11)-(5.13), we have

$$\min\{-Lu, u - h\} = 0, \tag{5.34}$$

Then by (5.22), for the point (y_{1i}, y_{2j}, t_k), we have

$$\min\{(\Delta t) * Lu_{i,j}^k, u_{i,j}^{k-1} - h_{i,j}\} = 0, \tag{5.35}$$

where $h_{i,j} = (e^{y_{1i}} - e^{y_{2j}} - K)^+$. Next, by $\min\{X - Y, X - Z\} = 0 \Leftrightarrow X = \max\{Y, Z\}$ and (5.22), we have the finite difference method for the American spread option as follows

$$
\begin{aligned}
u_{i,j}^{k-1} = \max\{ & A * u_{i+1,j+1}^k + B * u_{i+1,j}^k + C * u_{i+1,j-1}^k \\
& + D * u_{i,j+1}^k + E * u_{i,j}^k + F * u_{i,j-1}^k \\
& + G * u_{i-1,j+1}^k + H * u_{i-1,j}^k + I * u_{i-1,j-1}^k, h_{i,j}\},
\end{aligned}
\tag{5.36}
$$

with boundary and terminal conditions

$$u(\bar{y}_1, y_2, t) = \text{AMPut}(e^{y_2}, K_1, T - t, r, \sigma_2, q_2), \tag{5.37}$$

$$u(y_1, \bar{y}_2, t) = 0, \tag{5.38}$$

$$u(\underline{y}_1, y_2, t) = 0, \tag{5.39}$$

$$u(y_1, \underline{y}_2, t) = \text{AMCall}(e^{y_1}, K, T - t, r, \sigma_1, q_1), \tag{5.40}$$

$$u(y_1, y_2, T) = \max\{e^{y_1} - e^{y_2} - K, 0\}. \tag{5.41}$$

where $K_1 = e^{\bar{y}_1} - K$. $\text{AMPut}(S, K, T, r, \sigma, q)$ and $\text{AMCall}(S, K, T, r, \sigma, q)$ represent the prices of standard American put and call options, respectively. Condition (5.37) means that for the American spread option at time t, when $e^{\bar{y}_1}$ is constant and very large, we regard it as the corresponding American put option on the second underlying asset with strike $K_1 = e^{\bar{y}_1} - K$. The reason is that we have

$$(e^{\bar{y}_1} - e^{y_2} - K)^+ = ((e^{\bar{y}_1} - K) - e^{y_2})^+.$$

Because we don't have a closed formula for the American option, in order to make our computation faster, in practice, we use the corresponding European option to approximate it, since we have closed formula for the European option. Condition (5.38) means that when the price of the second underlying is a very large number, we have that the option price is 0. Condition (5.39) means that when the price of the first underlying is a very small number, we have that the option price is 0. Condition (5.40) means that for the American spread option at time t, when $e^{\bar{y}_2}$ is constant and very small, we regard it as the corresponding American call option on the first underlying asset with strike K. Condition (5.41) is the terminal condition. The above method is an explicit finite difference method for the complementarity problem.

It turns out that one can get the corresponding European spread option price by setting $h_{i,j} = -\infty$ in (5.36), which means we have the finite difference method for the corresponding European spread option as follows

$$
\begin{aligned}
u_{i,j}^{k-1} = {} & A * u_{i+1,j+1}^{k} + B * u_{i+1,j}^{k} + C * u_{i+1,j-1}^{k} \\
& + D * u_{i,j+1}^{k} + E * u_{i,j}^{k} + F * u_{i,j-1}^{k} \\
& + G * u_{i-1,j+1}^{k} + H * u_{i-1,j}^{k} + I * u_{i-1,j-1}^{k},
\end{aligned}
\tag{5.42}
$$

with boundary and terminal conditions

$$
u(\overline{y}_1, y_2, t) = \text{BSPut}(e^{y_2}, K_1, T - t, r, \sigma_2, q_2),
\tag{5.43}
$$

$$
u(y_1, \overline{y}_2, t) = 0,
\tag{5.44}
$$

$$
u(\underline{y}_1, y_2, t) = 0,
\tag{5.45}
$$

$$
u(y_1, \underline{y}_2, t) = \text{BSCall}(e^{y_1}, K, T - t, r, \sigma_1, q_1),
\tag{5.46}
$$

$$
u(y_1, y_2, T) = \max\left\{ e^{y_1} - e^{y_2} - K, 0 \right\},
\tag{5.47}
$$

where BSPut$(e^{y_2}, K_1, T - t, r, \sigma_2, q_2)$ and $=$ BSCall$(e^{y_1}, K, T - t, r, \sigma_1, q_1)$ represents the prices of the standard European put option and call option. Because we have the pseudo analytic formula to compute the price of the European spread option, one can use the European spread option as the benchmark to test our finite difference approximations.

Table A.1 (Figures 5.1 and 5.2) displays the numerical results obtained by applying the explicit finite difference method and pseudo analytic formula to the corresponding European spread option, varying the first underlying assets S_1 from 30 to 50. We take the numbers of space steps and of time steps are 50×100, 80×400, 100×800, 120×1000, 200×1200, 300×3000 and 400×4000, respectively. From Table A.1 (Figures 5.1 and 5.2), we note that when the number of steps of finite difference method are larger, we get the closer results to the pseudo analytic formula. This means there ia a good agreement between these two methods.

It turns out that one can eliminate the mixed derivative term in the equation (5.3)

$$Lv = v_t + (r - q_1 - \frac{1}{2}\sigma_1^2)v_{x_1} + (r - q_2 - \frac{1}{2}\sigma_2^2)v_{x_2}$$
$$+ \frac{1}{2}\sigma_1{}^2 v_{x_1 x_1} + \frac{1}{2}\sigma_2{}^2 v_{x_2 x_2} + \rho\sigma_1\sigma_2 v_{x_1 x_2} - rv.$$

Because the cross derivative term introduces 4 more points on our finite difference grid, which could bring more errors in our computations. In fact, one can do the following transformations to eliminate the cross derivative term. We do the following change of variables

$$y_1 = \frac{\log(x_1)}{\sigma_1}, \tag{5.48}$$

$$y_2 = \frac{\log(x_2)\sigma_1 - \rho\sigma_2\log(x_1)}{\sqrt{1 - \rho^2}\sigma_1\sigma_2}, \tag{5.49}$$
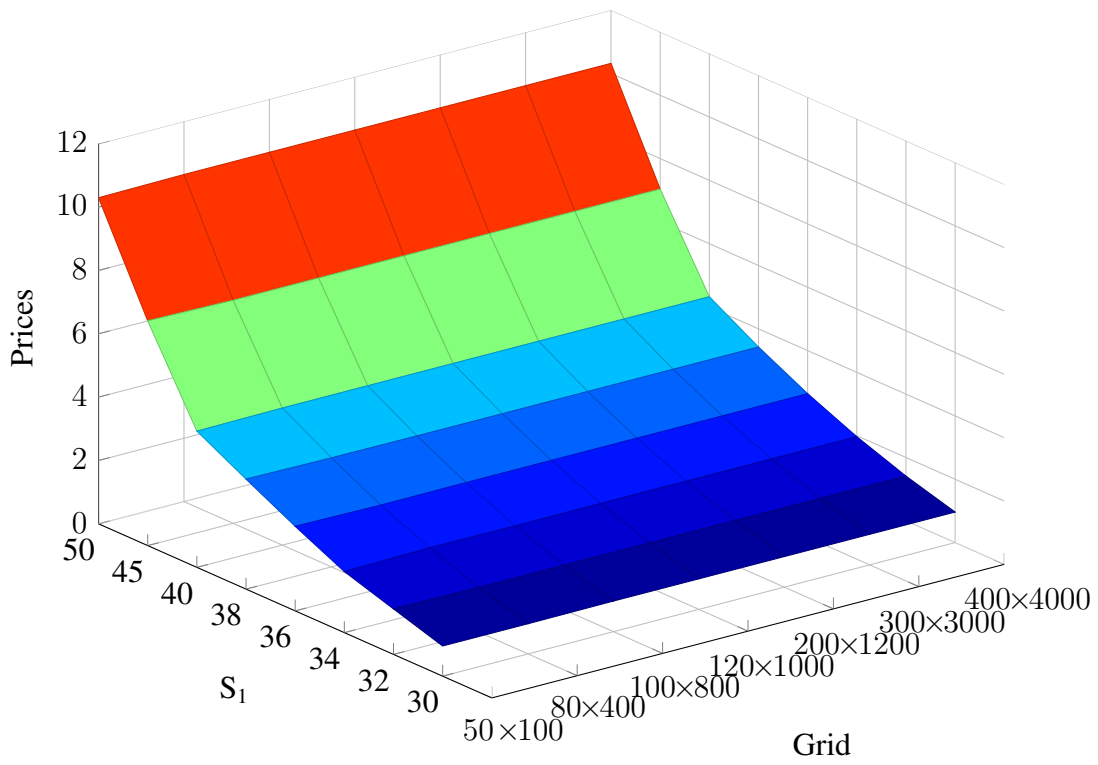
$$u(y_1, y_2, t) = v(x_1, x_2, t). \tag{5.50}$$

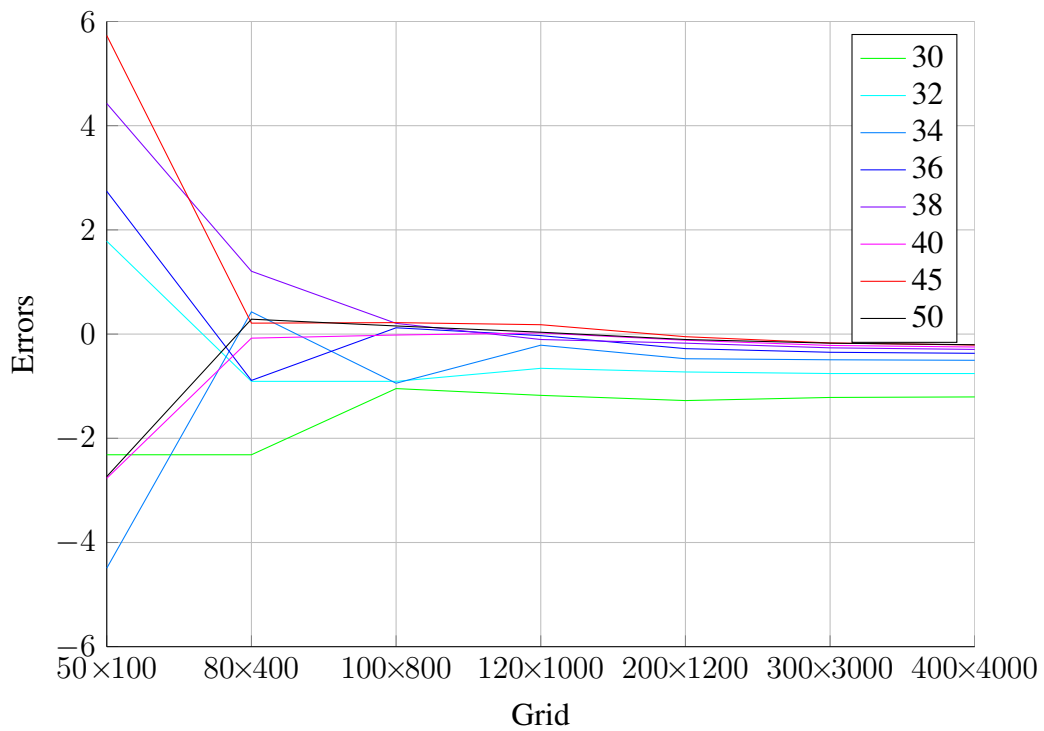Figure 5.1: Explicit Finite Difference Method for European Spread Options(Prices)

Figure 5.2: Explicit Finite Difference Method for European Spread Options(Errors)

Then we get

$$\frac{\partial v}{\partial x_1} = \frac{1}{\sigma_1}\frac{\partial u}{\partial y_1} + \frac{-\rho}{\sqrt{1-\rho^2}\sigma_1}\frac{\partial u}{\partial y_2}, \tag{5.51}$$

$$\frac{\partial v}{\partial x_2} = \frac{1}{\sqrt{1-\rho^2}\sigma_2}\frac{\partial u}{\partial y_2}, \tag{5.52}$$

$$\frac{\partial^2 v}{\partial x_1^2} = \frac{1}{\sigma_1^2}\frac{\partial^2 u}{\partial y_1^2} + \frac{-2\rho}{\sqrt{1-\rho^2}\sigma_1^2}\frac{\partial^2 u}{\partial y_1 \partial y_2} + \frac{\rho^2}{(1-\rho^2)\sigma_1^2}\frac{\partial^2 u}{\partial y_2^2}, \tag{5.53}$$

$$\frac{\partial^2 v}{\partial x_2^2} = \frac{1}{(1-\rho^2)\sigma_2^2}\frac{\partial^2 u}{\partial y_2^2}, \tag{5.54}$$

$$\frac{\partial^2 v}{\partial x_2 \partial x_1} = \frac{-\rho}{(1-\rho^2)\sigma_1\sigma_2}\frac{\partial^2 u}{\partial y_2^2} + \frac{1}{\sqrt{1-\rho^2}\sigma_1\sigma_2}\frac{\partial^2 u}{\partial y_2 \partial y_1}. \tag{5.55}$$

Then, we incorporate all of the above terms into the equation (5.3). We have

$$
\begin{aligned}
Lu = & \frac{\partial u}{\partial t} + (r - q_1 - \frac{1}{2}\sigma_1^2)\frac{1}{\sigma_1}\frac{\partial u}{\partial y_1} \\
& + \frac{-\rho(r - q_1 - \frac{1}{2}\sigma_1^2)}{\sqrt{1-\rho^2}\sigma_1}\frac{\partial u}{\partial y_2} + (r - q_2 - \frac{1}{2}\sigma_2^2)\frac{1}{\sqrt{1-\rho^2}\sigma_2}\frac{\partial u}{\partial y_2} \\
& + \frac{1}{2}\sigma_1{}^2\left[\frac{1}{\sigma_1^2}\frac{\partial^2 u}{\partial y_1^2} + \frac{-2\rho}{\sqrt{1-\rho^2}\sigma_1^2}\frac{\partial^2 u}{\partial y_1 \partial y_2} + \frac{\rho^2}{(1-\rho^2)\sigma_1^2}\frac{\partial^2 u}{\partial y_2^2}\right] \\
& + \frac{1}{2}\sigma_2{}^2\left[\frac{1}{(1-\rho^2)\sigma_2^2}\frac{\partial^2 u}{\partial y_2^2}\right] \\
& + \rho\sigma_1\sigma_2\left[\frac{-\rho}{(1-\rho^2)\sigma_1\sigma_2}\frac{\partial^2 u}{\partial y_2^2} + \frac{1}{\sqrt{1-\rho^2}\sigma_1\sigma_2}\frac{\partial^2 u}{\partial y_2 \partial y_1}\right] - ru. \tag{5.56}
\end{aligned}
$$

After we simplify the above equation, we get

$$
\begin{aligned}
Lu = & \frac{\partial u}{\partial t} + (r - q_1 - \frac{1}{2}\sigma_1^2)\frac{1}{\sigma_1}\frac{\partial u}{\partial y_1} \\
& + \frac{1}{\sqrt{1-\rho^2}}\left[\frac{(r - q_2 - \frac{1}{2}\sigma_2^2)}{\sigma_2} - \frac{\rho(r - q_1 - \frac{1}{2}\sigma_1^2)}{\sigma_1}\right]\frac{\partial u}{\partial y_2} + \frac{1}{2}\left[\frac{\partial^2 u}{\partial y_1^2} + \frac{\partial^2 u}{\partial y_2^2}\right] - ru. \tag{5.57}
\end{aligned}
$$

Hence we see that for the equation (5.57), there is no cross derivative term. Again, we use the finite difference method to solve the complementarity problem. Similarly, we incorporate all of

the finite differences into the equation (5.57). We get

$$Lu_{i,j}^k = \frac{u_{i,j}^k - u_{i,j}^{k-1}}{\Delta t} + a_1 \left(\frac{u_{i+1,j}^k - u_{i-1,j}^k}{2\Delta y_1} + O\left[(\Delta y_1)^2\right] \right)$$

$$+ a_2 \left(\frac{u_{i,j+1}^k - u_{i,j-1}^k}{2\Delta y_2} + O\left[(\Delta y_2)^2\right] \right)$$

$$+ \frac{1}{2} \left(\frac{u_{i+1,j}^k + u_{i-1,j}^k - 2u_{i,j}^k}{(\Delta y_1)^2} + O\left[(\Delta y_1)^2\right] \right)$$

$$+ \frac{1}{2} \left(\frac{u_{i,j+1}^k + u_{i,j-1}^k - 2u_{i,j}^k}{(\Delta y_2)^2} + O\left[(\Delta y_2^2)\right] \right)$$

$$- ru_{i,j}^k. \tag{5.58}$$

After we simplify it and ignore the higher order term, we have

$$(\Delta t) * Lu_{i,j}^k = -u_{i,j}^{k-1} + A * u_{i+1,j}^k + B * u_{i,j+1}^k + C * u_{i,j}^k$$

$$+ D * u_{i,j-1}^k + E * u_{i-1,j}^k, \tag{5.59}$$

where

$$a_1 = \frac{(r - q_1 - \frac{1}{2}\sigma_1^2)}{\sigma_1}, \tag{5.60}$$

$$a_2 = \frac{1}{\sqrt{1-\rho^2}} \left[\frac{(r - q_2 - \frac{1}{2}\sigma_2^2)}{\sigma_2} - \frac{\rho(r - q_1 - \frac{1}{2}\sigma_1^2)}{\sigma_1} \right], \tag{5.61}$$

$$A = \frac{\Delta t}{2(\Delta y_1)^2} + \frac{a_1 \Delta t}{2\Delta y_1}, \tag{5.62}$$

$$B = \frac{\Delta t}{2(\Delta y_2)^2} + \frac{a_2 \Delta t}{2\Delta y_2}, \tag{5.63}$$

$$C = 1 - r\Delta t - \frac{\Delta t}{(\Delta y_1)^2} - \frac{\Delta t}{(\Delta y_2)^2}, \tag{5.64}$$

$$D = \frac{\Delta t}{2(\Delta y_2)^2} - \frac{a_2 \Delta t}{2\Delta y_2}, \tag{5.65}$$

$$E = \frac{\Delta t}{2(\Delta y_1)^2} - \frac{a_1 \Delta t}{2\Delta y_1}. \tag{5.66}$$

Here we use 5 points on the grid rather than 9 points with the cross derivative term, which simplifies

our implementation significantly. Similarly, we have the finite difference method for the American spread option as follows

$$
\begin{aligned}
u_{i,j}^{k-1} = \max\{ & A * u_{i+1,j}^k + B * u_{i,j+1}^k + C * u_{i,j}^k \\
& + D * u_{i,j-1}^k + E * u_{i-1,j}^k, h_{i,j}\},
\end{aligned} \tag{5.67}
$$

with terminal condition

$$
u(y_1, y_2, T) = \max\left\{ e^{\sigma_1 y_1} - e^{\rho\sigma_2 y_1 + \sqrt{1-\rho^2}\sigma_2 y_2} - K, 0 \right\}, \tag{5.68}
$$

and boundary conditions

$$
u(\overline{y}_1, y_2, t) = \text{AMPut}(e^{\rho\sigma_2\overline{y}_1 + \sqrt{1-\rho^2}\sigma_2 y_2}, K_1, T-t, r, \sigma_2, q_2), \tag{5.69}
$$

$$
u(y_1, \overline{y}_2, t) = 0, \tag{5.70}
$$

$$
u(\underline{y}_1, y_2, t) = 0, \tag{5.71}
$$

$$
u(y_1, \underline{y}_2, t) = \text{AMCall}(e^{\sigma_1 y_1}, K, T-t, r, \sigma_1, q_1), \tag{5.72}
$$

$$
u(y_1, y_2, T) = \max\left\{ e^{\sigma_1 y_1} - e^{\rho\sigma_2\overline{y}_1 + \sqrt{1-\rho^2}\sigma_2 y_2} - K, 0 \right\}, \tag{5.73}
$$

where $K_1 = e^{\sigma_1 y_1} - K$. This is the explicit finite difference method for the complementarity problem without the cross derivative term.

Again, because we have the analytic formula to compute the price of the European spread option, one can use the European spread option as the benchmark to test our finite difference approximations for the partial differential equation without cross derivative term. And we have the finite difference method for the European spread option as follows

$$
\begin{aligned}
u_{i,j}^{k-1} = & A * u_{i+1,j}^k + B * u_{i,j+1}^k + C * u_{i,j}^k \\
& + D * u_{i,j-1}^k + E * u_{i-1,j}^k,
\end{aligned} \tag{5.74}
$$
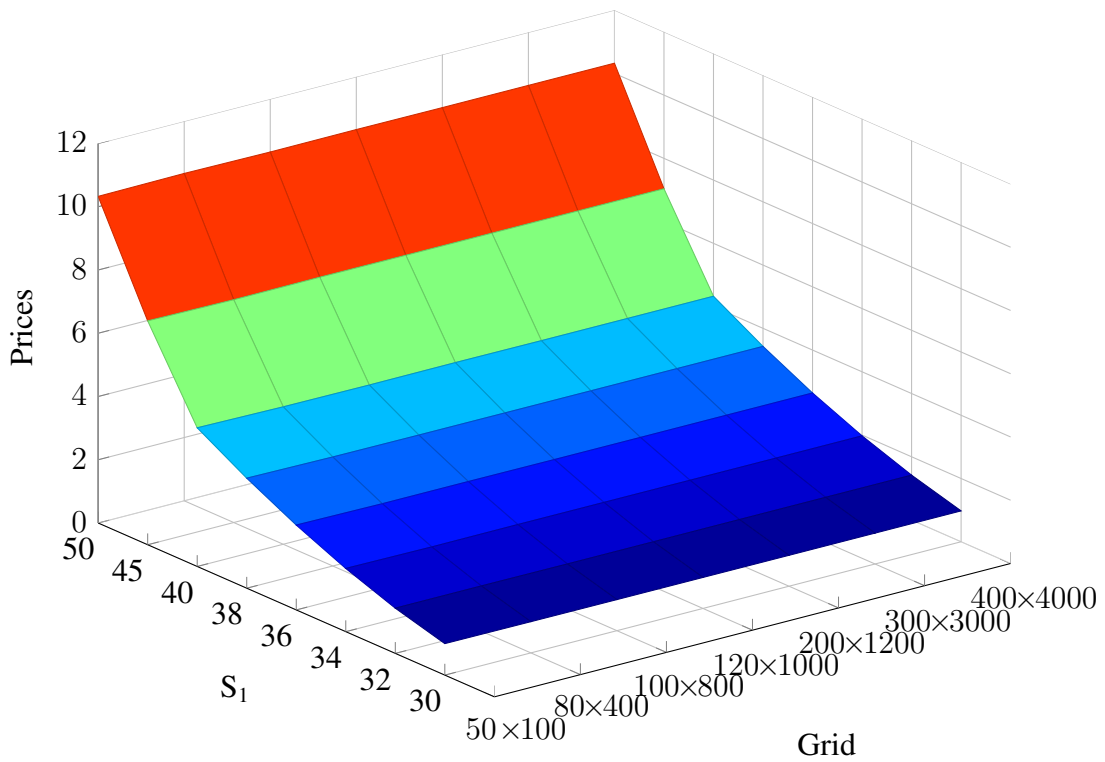
Figure 5.3: Explicit Finite Difference Method to Improved Partial Differential Equation for European Spread Options(Prices)
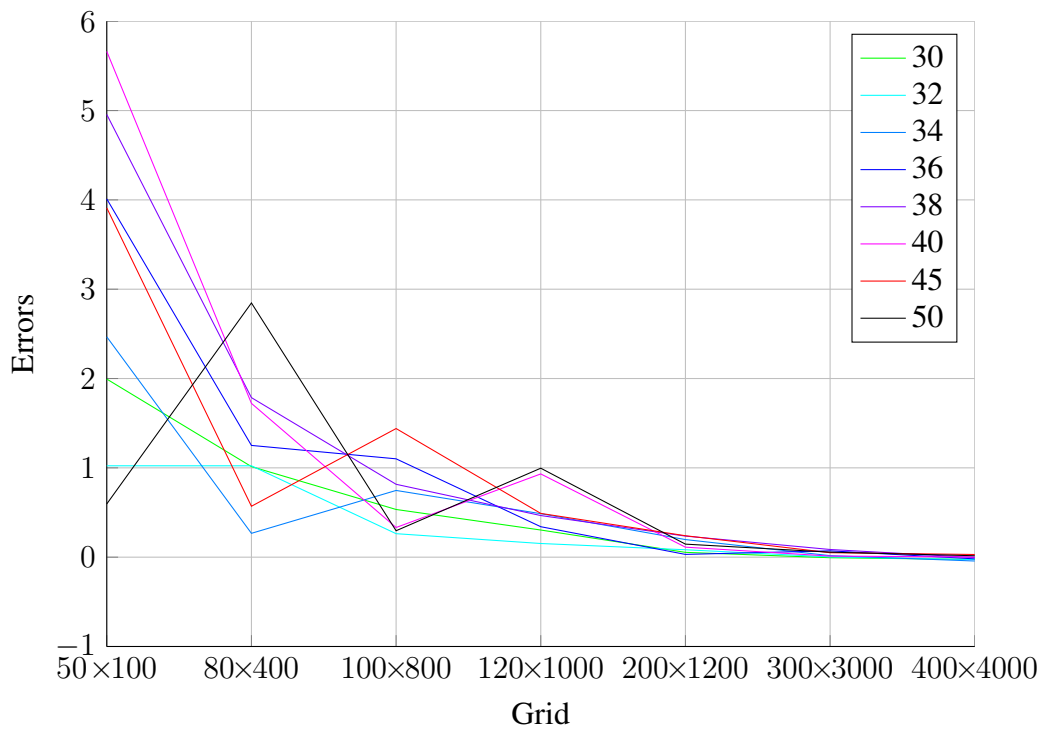
Figure 5.4: Explicit Finite Difference Method to Improved Partial Differential Equation for European Spread Options(Errors)

with terminal condition

$$u(y_1, y_2, T) = \max\left\{e^{\sigma_1 y_1} - e^{\rho\sigma_2 y_1 + \sqrt{1-\rho^2}\sigma_2 y_2} - K, 0\right\}, \tag{5.75}$$

and boundary conditions

$$u(\overline{y}_1, y_2, t) = BSPut(e^{\rho\sigma_2\overline{y}_1 + \sqrt{1-\rho^2}\sigma_2 y_2}, K_1, T - t, r, \sigma_2, q_2), \tag{5.76}$$

$$u(y_1, \overline{y}_2, t) = 0, \tag{5.77}$$

$$u(\underline{y}_1, y_2, t) = 0, \tag{5.78}$$

$$u(y_1, \underline{y}_2, t) = BSCall(e^{\sigma_1 y_1}, K, T - t, r, \sigma_1, q_1), \tag{5.79}$$

$$u(y_1, y_2, T) = \max\left\{e^{\sigma_1 y_1} - e^{\rho\sigma_2\overline{y}_1 + \sqrt{1-\rho^2}\sigma_2 y_2} - K, 0\right\}, \tag{5.80}$$

where $K_1 = e^{\sigma_1 y_1} - K$. This is the explicit finite difference method for the partial differential equation without the cross derivative term.

Table A.2 (Figures 5.3 and 5.4) documents the numerical results obtained by the explicit finite difference method for the equation (5.57) and pseudo analytic formula to the corresponding European spread option. We call this finite difference method as the improved partial differential equation approach(IPDE). In order to compare to the numerical results obtained by the Table A.1. We take the numbers of space steps and of time steps the same as in the Table A.1, which are 50×100, 80×400, 100×800, 120×1000, 200×1200, 300×3000 and 400×4000, respectively.

From Table A.1 and A.2, we note that the results obtained by improved partial differential equation approach is much better than the one obtained by original partial differential approach. When we take numbers of space steps and of time steps are 400×4000, the maximum error in the Table A.1 is about 100 basis points, while it is about 4 basis points in Table A.2. Thus we take the improved partial differential equation approach as our benchmark for the American spread option under the
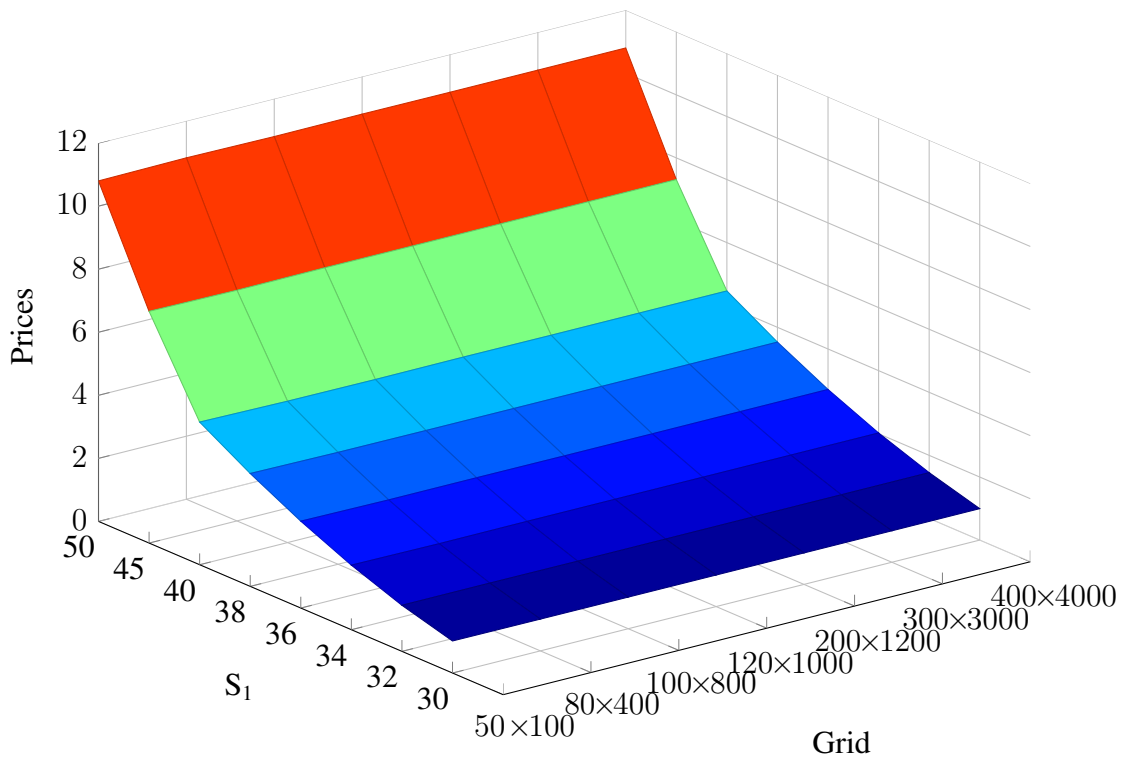
Figure 5.5: Explicit Finite Difference Method to IPDE for American Spread Options(Prices)
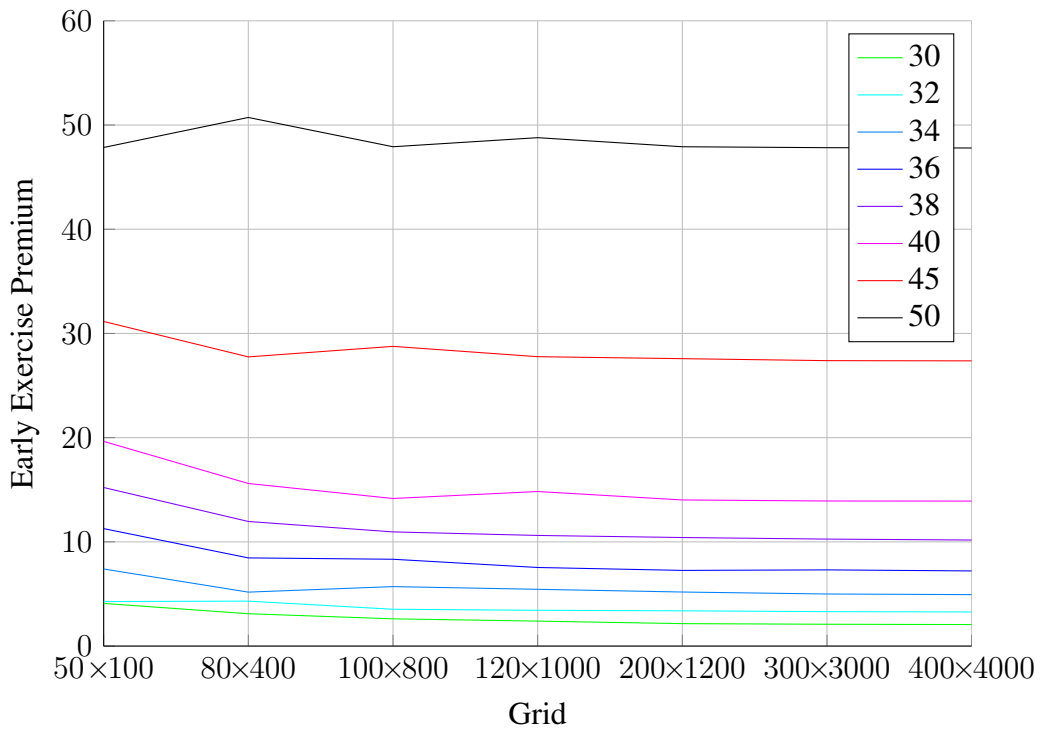
Figure 5.6: Explicit Finite Difference Method to IPDE for American Spread Options(Early Exercise Premiums)

two-factor geometric Brownian motion model. The numerical results obtained by this method for the American spread option are listed in Table A.3 (Figures 5.5 and 5.6). Then we use this method to test our explicit finite difference method to solve the original equation (5.14). The numerical results are shown in Table A.4 (Figures 5.7 and 5.8).

5.1.2 Monte Carlo Simulation Approach.

In this subsection, we use the Monte Carlo simulation method to compute the price of the American spread option under the two-factor geometric Brownian motion model.

From Chapter 3, we have the following Monte Carlo simulation algorithm for pricing the American spread option. The algorithm is

(i) Generate N different paths of the asset price processes

$$\{S_1^n(t_1), S_1^n(t_2) \cdots , S_1^n(t_m)\}, n = 1, 2, \cdots N,$$

$$\{S_2^n(t_1), S_2^n(t_2) \cdots , S_2^n(t_m)\}, n = 1, 2, \cdots N.$$

(ii) At maturity, set $\tilde{v}_m = h_m$.

(iii) For $i = m - 1, \cdots , 2, 1$

    (a) Compute $\tilde{\boldsymbol{\beta}}_i = \tilde{\beta}_\psi^{-1} \tilde{\beta}_{\psi V}$, where $\tilde{\beta}_\psi^{-1}$ and $\tilde{\beta}_{\psi V}$ are given in (3.73)and (3.74).

    (b) Compute $\tilde{c}_i = \boldsymbol{\psi} \tilde{\boldsymbol{\beta}}_i^T, n = 1, \cdots , N$.

    (c) Set $\tilde{v}_i = \max\{h_i, \tilde{c}_i\}, n = 1, \cdots , N$.

(iv) Set $\tilde{v}_0^{(n)} = h_i$, where $t_i = \min\{t_i \in \{t_1, \cdots , t_m\} : h_i \geq \tilde{c}_i\}$.

(v) Take the mean of all $\tilde{v}_0^{(n)}$ to get the option value $\tilde{v}_0 = \frac{1}{N} \sum_{n=1}^N \tilde{v}_0^{(n)}$.

After we implement the finite difference method partial differential equation approach, we are

Figure 5.7: Explicit Finite Difference Method to Original Partial Differential Equation for American Spread Options(Prices)

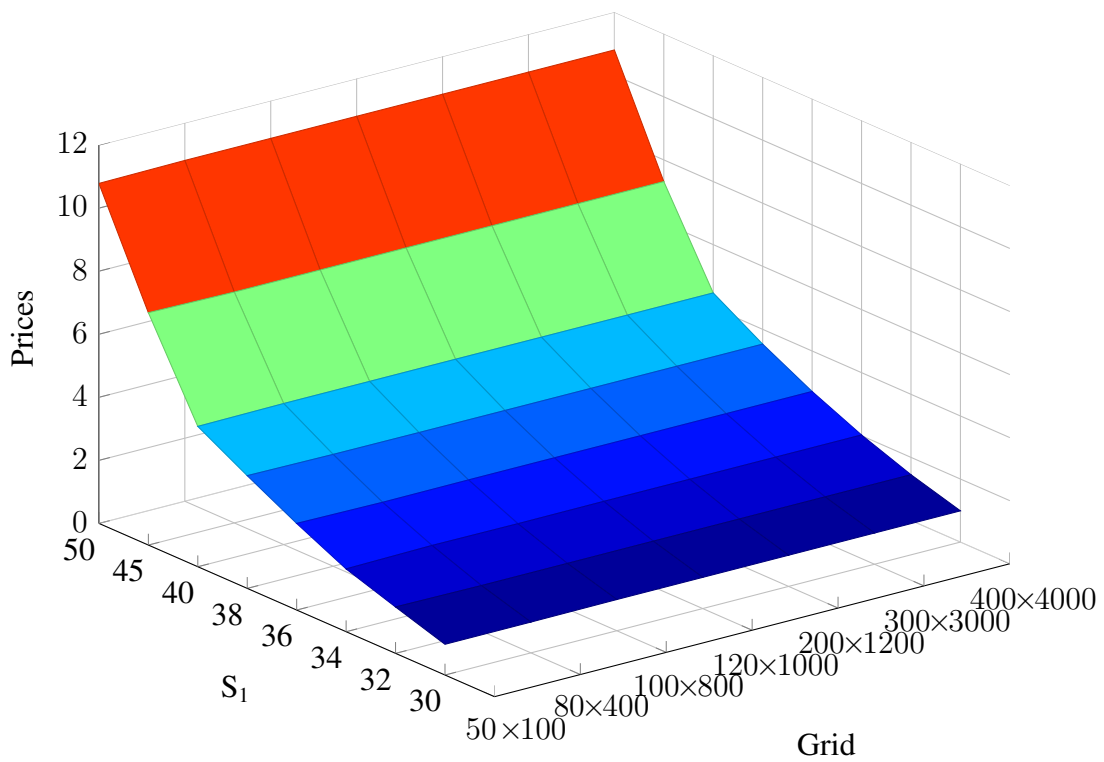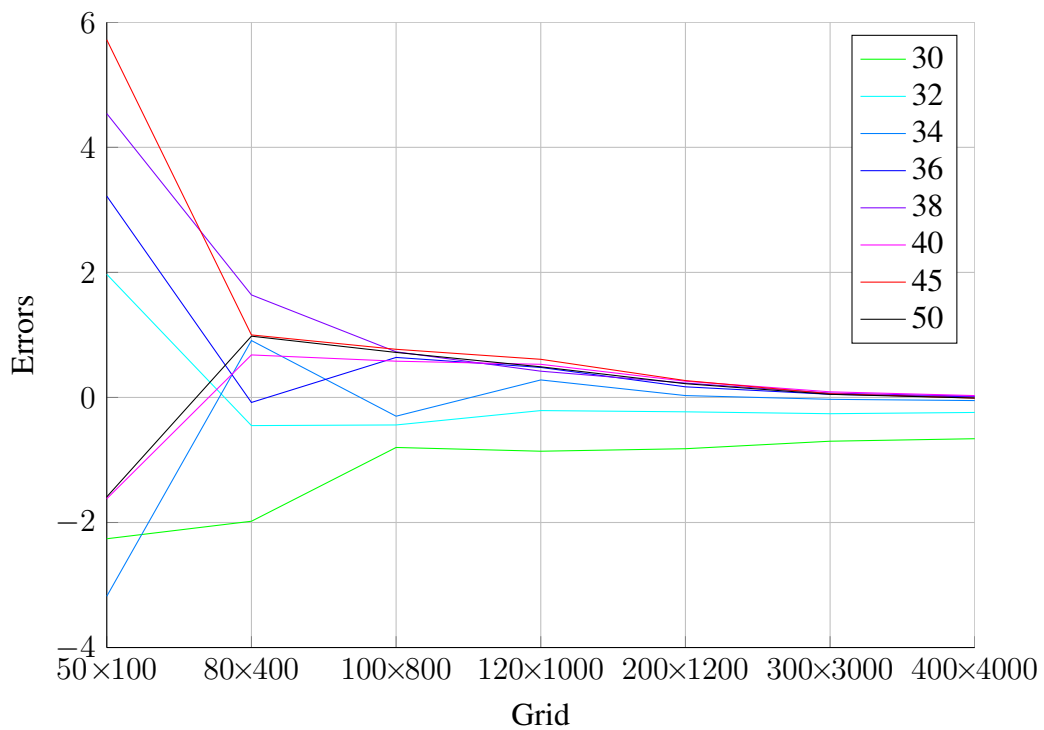Figure 5.8: Explicit Finite Difference Method to Original Partial Differential Equation for American Spread Options(Errors)

ready to compare the numerical results with the Monte Carlo simulation method under the two-factor geometric Brownian motion model. The numerical results obtained by Monte Carlo simulation method are documented in Table A.5 (Figures 5.9 and 5.10). The paths of Monte Carlo simulation are 10000, 20000, 50000, 100000 and 200000, respectively. The basis functions are standard basis functions $1, S_1, S_1^2, S_1^3, S_2, S_2^2, S_2^3$. We fix the number of time steps $m = 50$ and vary the first underlying asset price S_1 from 30 to 50. We note that when the number of paths are larger, we get the closer results compared to the numerical results obtained by the partial differential equations in the previous section. And the number in the parentheses represents the standard errors, when we increase the number of paths, it is much smaller, which is a good agreement with the method of Monte Carlo simulation.

In order to investigate the effects of the basis functions, then we include the cross product term $S_1 S_2$ into our standard basis functions $1, S_1, S_1^2, S_1^3, S_2, S_2^2, S_2^3$. The numerical results are then shown in Table A.6 (Figures 5.11 and 5.12). When we compare it to the numerical results in Table A.5 obtained by the standard basis functions, it's improved a little. Next, we include the payoff function $\max\{S_1 - S_2 - K, 0\}$ into our standard basis functions $1, S_1, S_1^2, S_1^3, S_2, S_2^2, S_2^3$, the numerical results are shown in Table A.7 (Figures 5.13 and 5.14). In this case, the numerical results are improved a lot when we compare it to the results obtained by the partial differential approaches. When we simulate the paths for far out of the money options, there are possibilities that for all paths, we have $S_1 < S_2 + K$. Therefore, in order to exclude these case, we consider relative out of money option paths. We vary S_1 from 38 to 50. Similarly, we vary S_1 from 38 to 50 for the improved Monte Carlo simulation method where we take in the money paths to approximate β_i.

Lastly, we incorporate both cross product term $S_1 S_2$ and the payoff function $\max\{S_1 - S_2 - K, 0\}$ into our standard basis functions $1, S_1, S_1^2, S_1^3, S_2, S_2^2, S_2^3$. The numerical results are in Table A.8 (Figures 5.15 and 5.16). The results are very close to the results obtained by applying the finite difference method partial different equation approach. As we see, when we take 200000 paths,

71

Figure 5.9: Monte Carlo Simulation for American Spread Options(Prices)
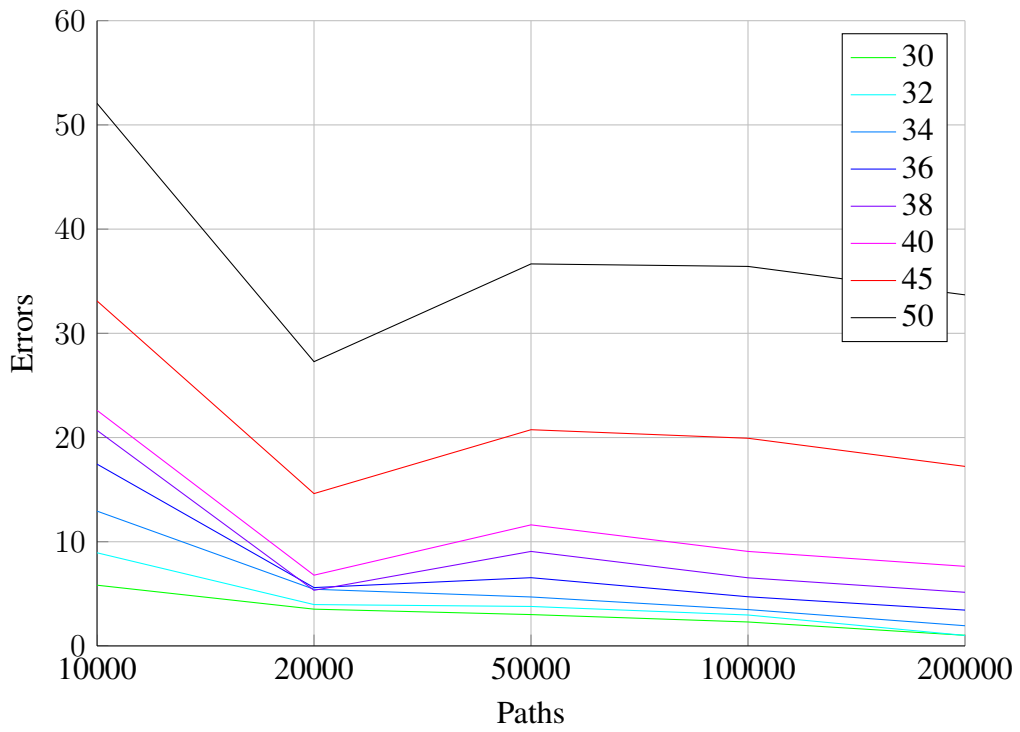
Figure 5.10: Monte Carlo Simulation for American Spread Options(Errors)

the maximum error from the partial differential equation is less than 1 percent, which is pretty agreement between these two methods.

From tables A.5, A.6, A.7 and A.8, another noticeable fact is that the results obtained by Monte Carlo simulation method are always less than the results by the partial differential equation method. It is in agreement with the argument that we usually get a lower bound from the Monte Carlo simulation method, because the Monte Carlo simulation method usually leads a sub-optimal exercise strategy.

So far, for the Monte Carlo simulation, we have used all paths to approximate β_i, while improved Monte Carlo simulation method usually use in the money paths. It usually save a lot of computational time. Recall that we have the following improved Monte Carlo simulation algorithm

(i) Generate N different paths of the asset price process

$$\{S_1^n(t_1), S_1^n(t_2) \cdots , S_1^n(t_m)\}, n = 1, 2, \cdots N,$$
$$\{S_2^n(t_1), S_2^n(t_2) \cdots , S_2^n(t_m)\}, n = 1, 2, \cdots N.$$

(ii) At maturity, set $\tilde{v}_m = h_m$.

(iii) For $i = m - 1, \cdots , 2, 1$

(a) Compute $\tilde{\boldsymbol{\beta}}_i = \tilde{\beta}_{\boldsymbol{\psi}}^{-1} \tilde{\beta}_{\boldsymbol{\psi}V}$, where $\tilde{\beta}_{\boldsymbol{\psi}}^{-1}$ and $\tilde{\beta}_{\boldsymbol{\psi}V}$ are given in (3.73)and (3.74).

(b) Compute $\tilde{c}_i = \boldsymbol{\psi}\tilde{\boldsymbol{\beta}}_i^T, n = 1, \cdots , N$.

(c) Compute $\tilde{v}_i$

- If $h_i \geq \tilde{c}_i$, $\tilde{v}_i = h_i$.

- If $h_i < \tilde{c}_i$, $\tilde{v}_i = \tilde{v}_{i+1}$.

(iv) Set $\tilde{v}_0^{(n)} = h_i$, where $t_i = \min\{t_i \in \{t_1, \cdots , t_m\} : h_i \geq \tilde{c}_i\}$.

73

Figure 5.11: Monte Carlo Simulation for American Spread Options-Standard Basis Functions with Cross Product Term(Prices)

Figure 5.12: Monte Carlo Simulation for American Spread Options-Standard Basis Functions with Cross Product Term(Errors)

74
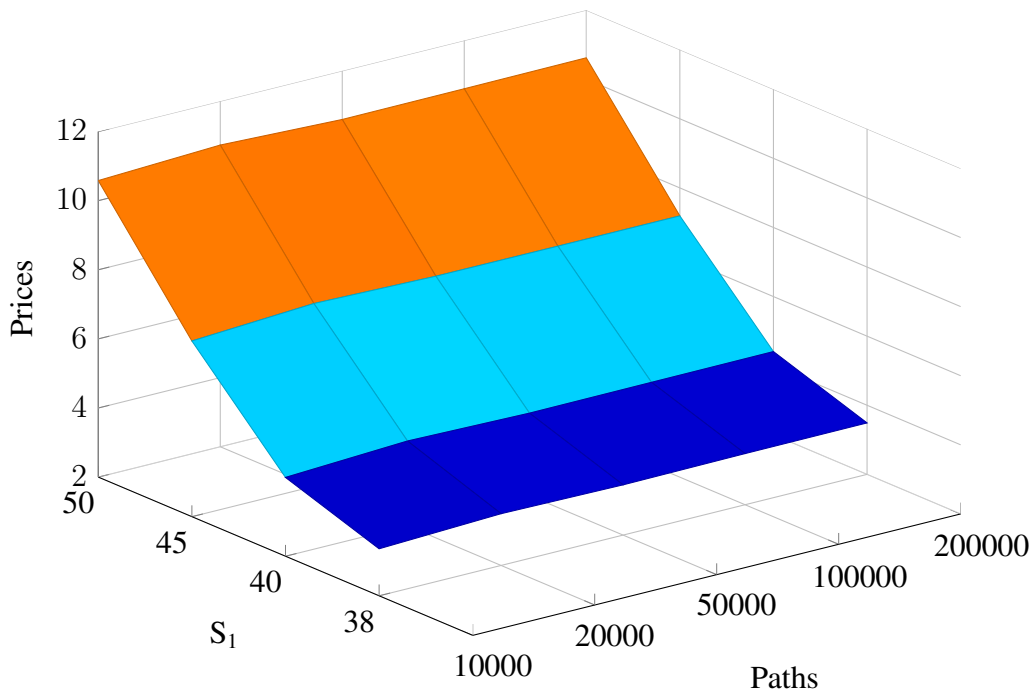
Figure 5.13: Monte Carlo Simulation for American Spread Options-Standard Basis Functions with Payoff Function Term(Prices)
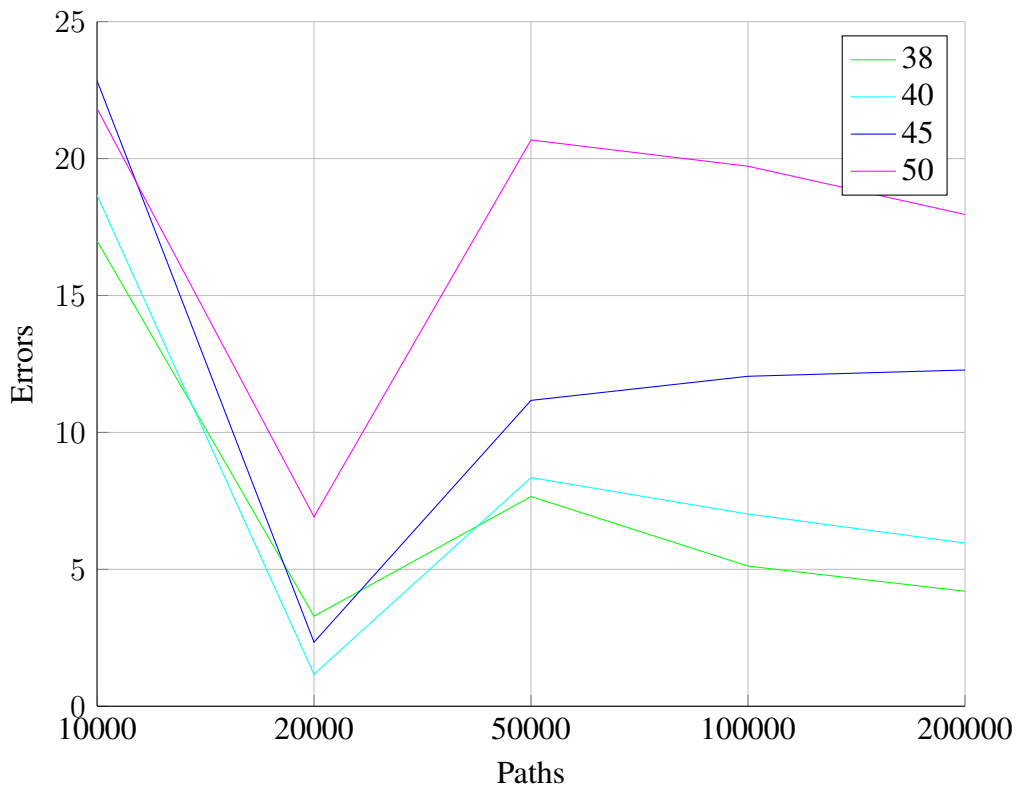
Figure 5.14: Monte Carlo Simulation for American Spread Options-Standard Basis Functions with Payoff Function Term(Errors)
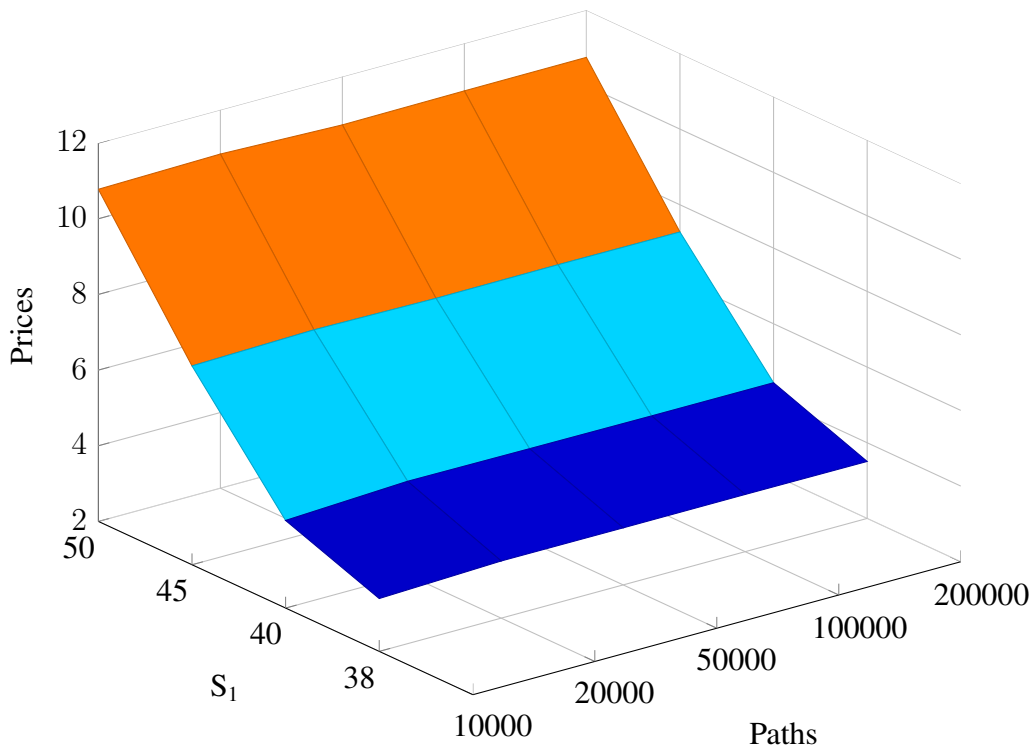
Figure 5.15: Monte Carlo Simulation for American Spread Options-Standard Basis Functions with both Cross Production and Payoff Function Term(Prices)
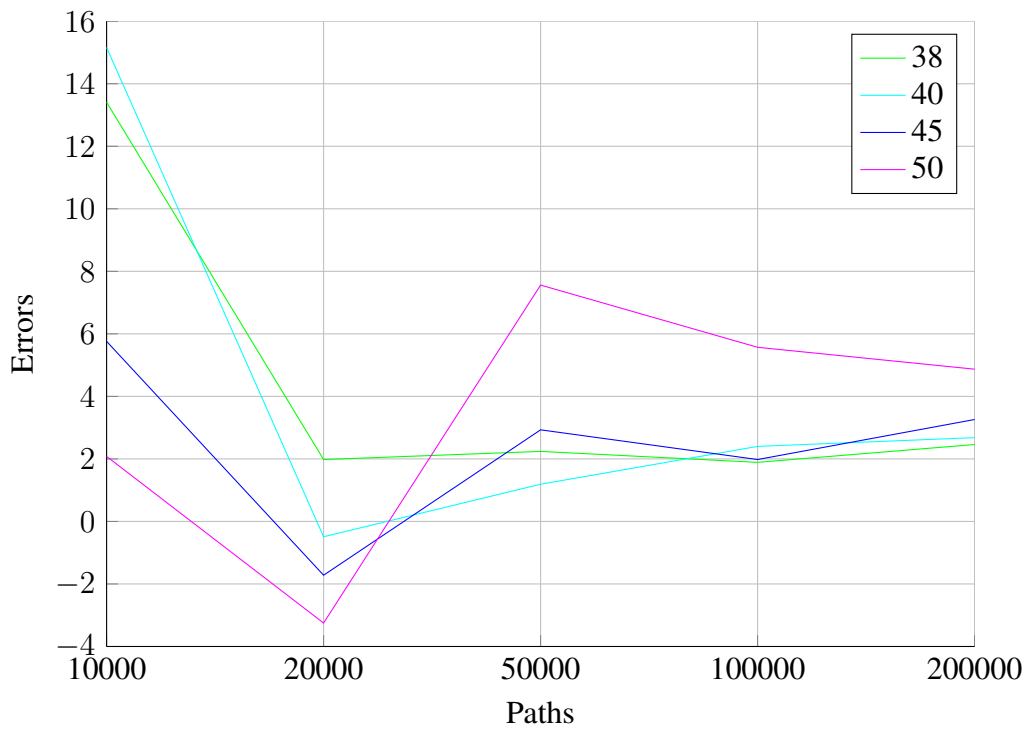
Figure 5.16: Monte Carlo Simulation for American Spread Options-Standard Basis Functions with both Cross Production and Payoff Function Term(Errors)
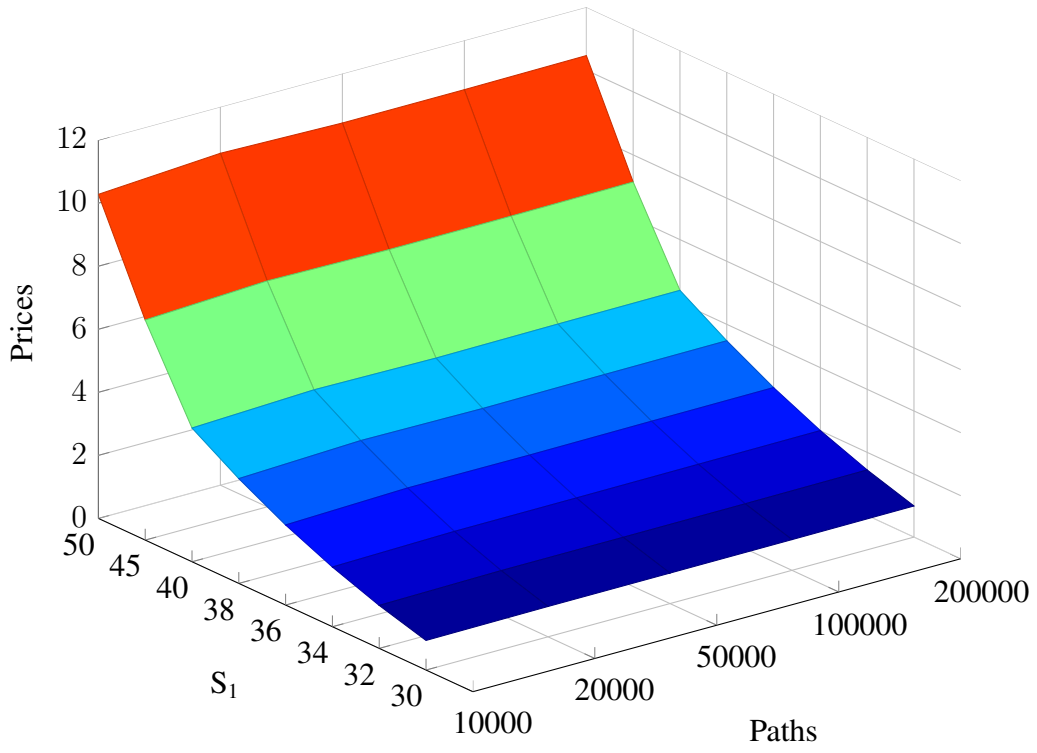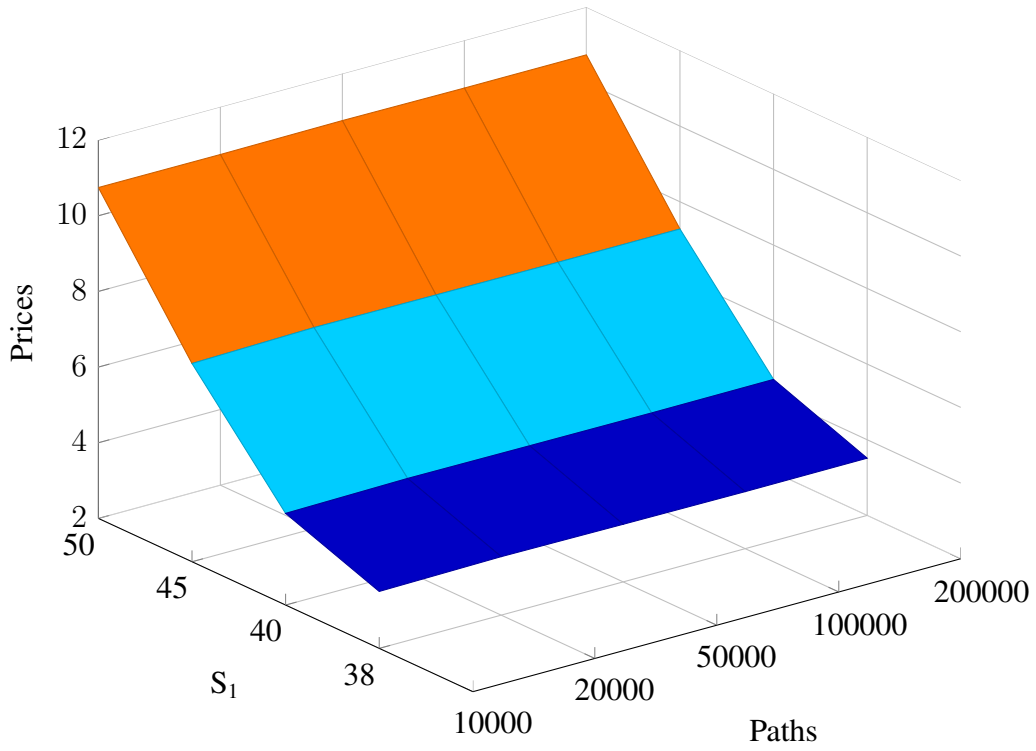
Figure 5.17: Improved Monte Carlo Simulation for American Spread Options(Prices)
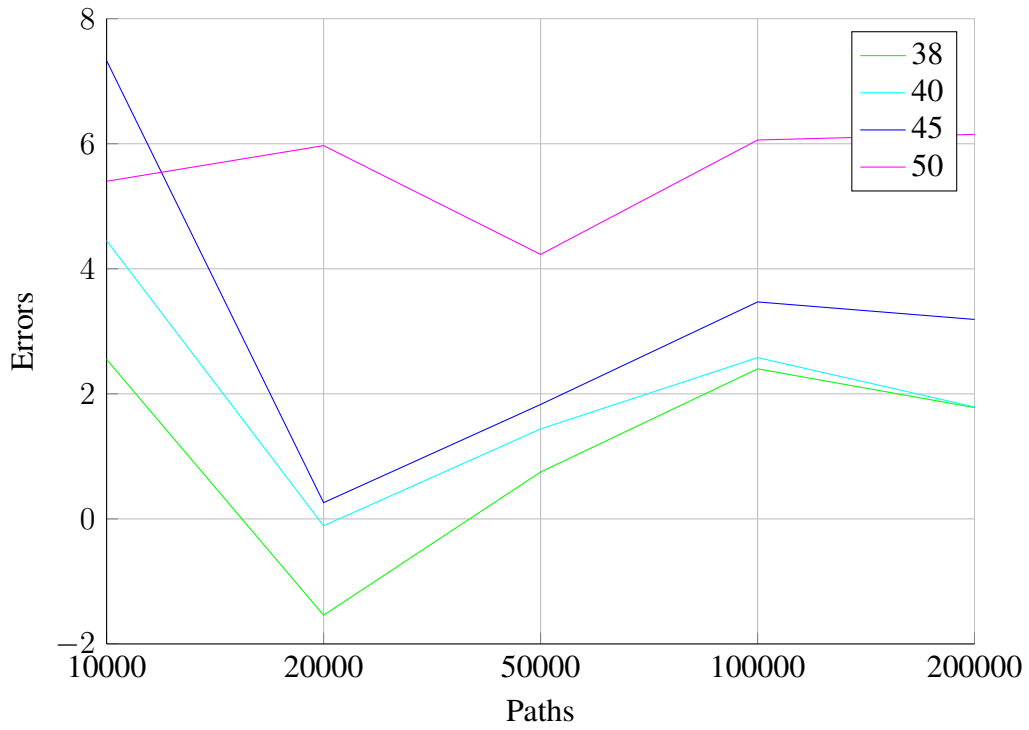
Figure 5.18: Improved Monte Carlo Simulation for American Spread Options(Errors)

(v) Take the mean of all $\tilde{v}_0^{(n)}$ to get the option value $\tilde{v}_0 = \frac{1}{N} \sum_{n=1}^{N} \tilde{v}_0^{(n)}$.

Then, we study the numerical results of the price of the American spread option by the improved Monte Carlo simulation method. First, we take the standard basis functions as our basis functions. The numerical results are listed in Table A.9 (Figures 5.17 and 5.18). Then, we incorporate cross product term $S_1 S_2$ into our standard basis functions as our basis functions. The numerical results are listed in Table A.10.

5.1.3 Dual method Approach.

In this subsection, we use the dual method to compute the price of the American spread option under the two-factor geometric Brownian motion model.

In the previous two subsections, we have examined the numerical results obtained by the partial differential equation method and Monte Carlo simulation method. We know that the Monte Carlo simulation often generates a lower bound of the option price. In this subsection, we combine the pseudo analytic formula and dual method to generate an upper bound of the option price.

We apply the Gauss-Hermite quadrature method to compute the integration in the pseudo analytic formula. Then, we apply the dual method Theorem 3.3 to get the upper bound of the American spread option.

First, we investigate into the numerical results when we take large number of paths for computing the expectation in the Theorem 3.3. The results are in Table A.11 (Figures 5.21 and 5.22). It turns out that our results are very stable. When we increase the number of paths 20 times from 10000 to 200000, the maximum standard error reduce from 0.007 to 0.002. This implies that we may compute a very small amount of paths and the results are listed in Table A.12 (Figures 5.23 and 5.24). As we see, even when we just take 500 paths, the maximum standard error is about 3 cents. It is a very good method to get a tight upper bound for the American spread option, because we compute a very small number of paths.
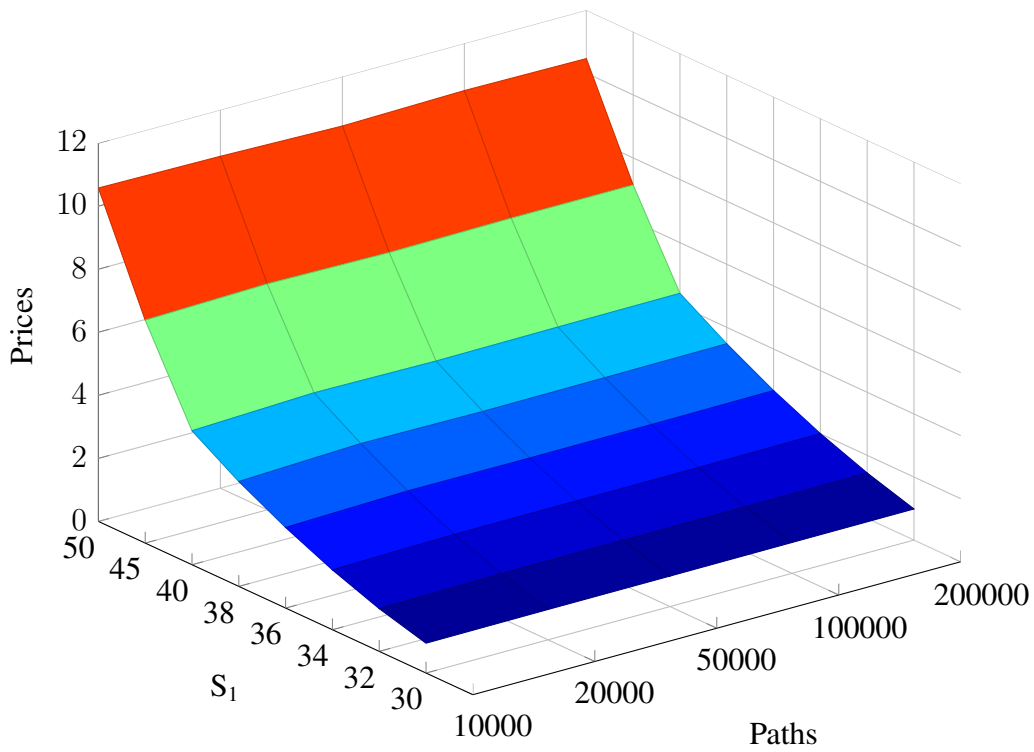
78
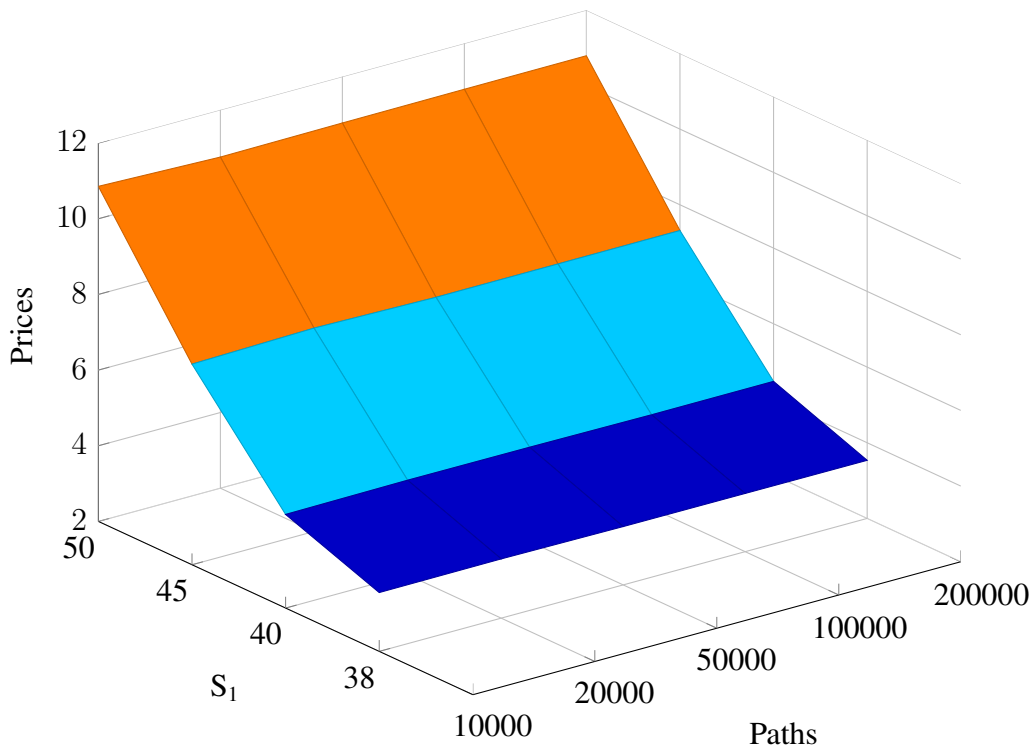
Figure 5.19: Improved Monte Carlo Simulation for American Spread Options-Standard Basis Functions with Cross Product Term(Prices)
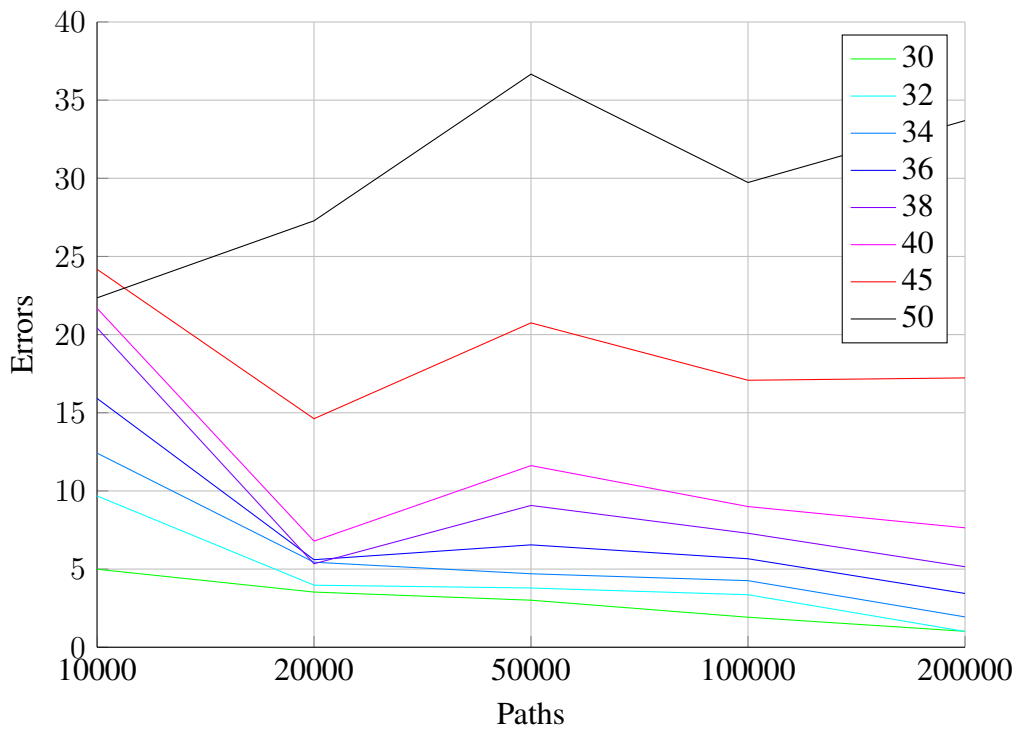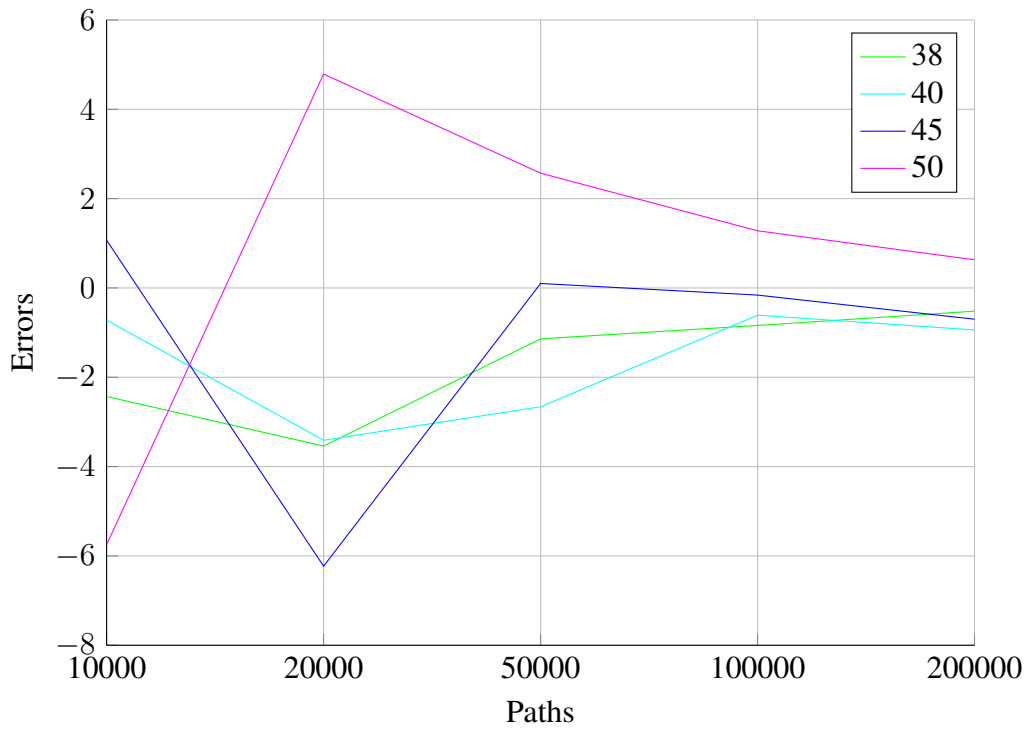

Figure 5.20: Improved Monte Carlo Simulation for American Spread Options-Standard Basis Functions with Cross Product Term(Errors)
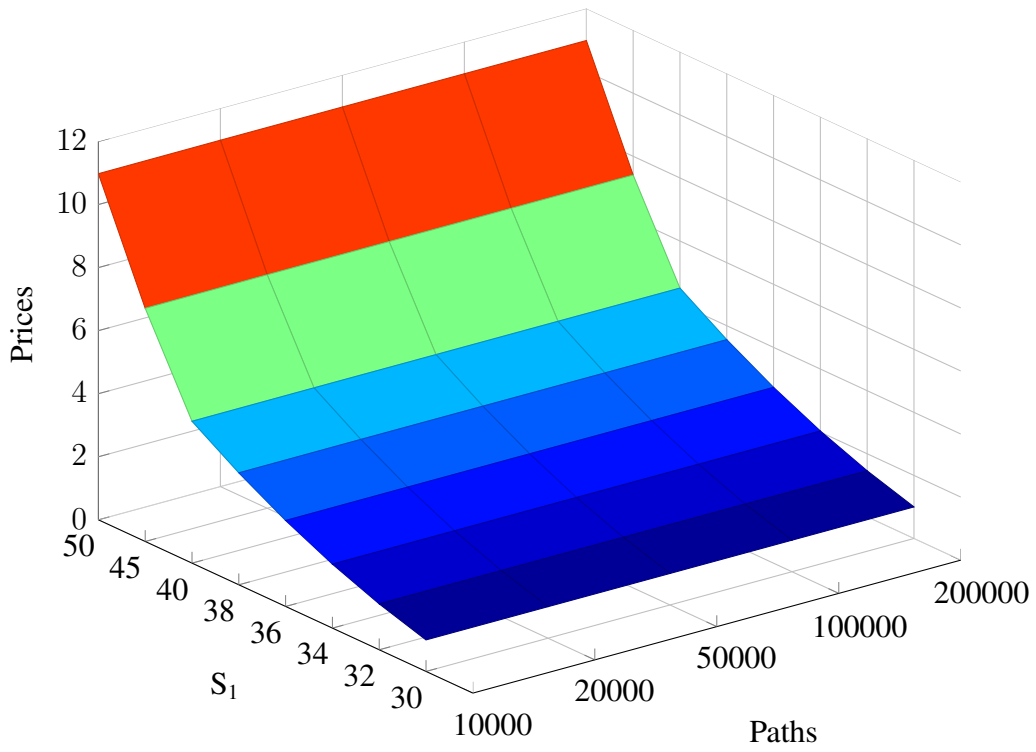
Figure 5.21: Dual Method for American Spread Options(Prices)
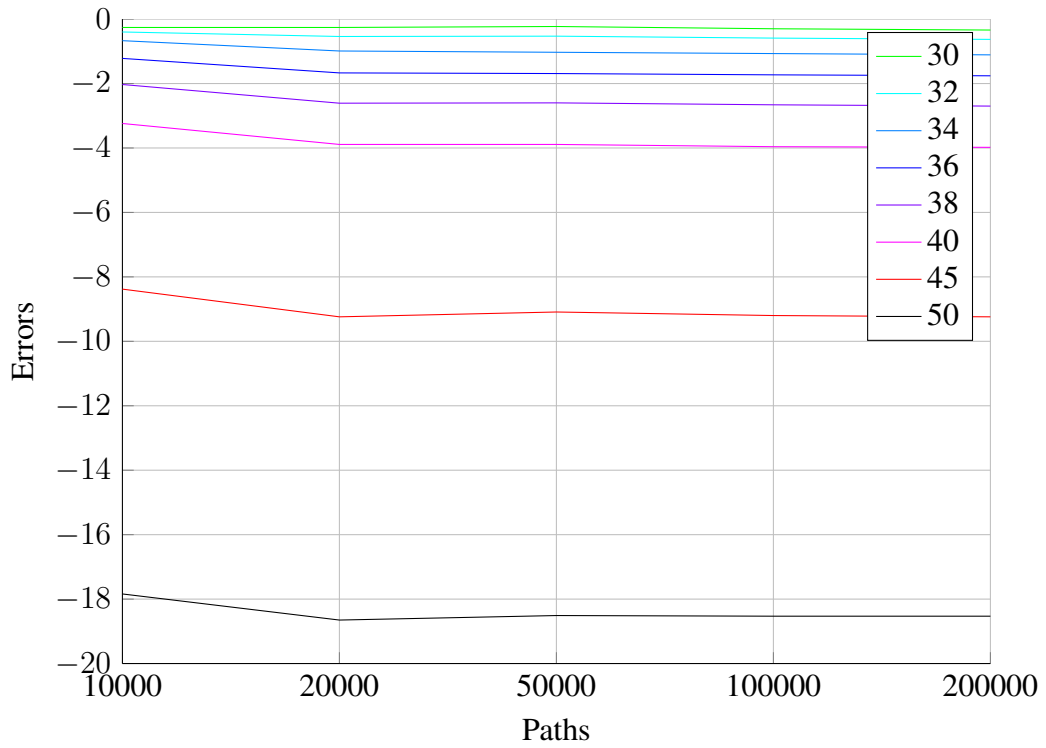
Figure 5.22: Dual Method for American Spread Options(Errors)

We also notice that the most of results in tables A.11 and A.12 are bigger than the results obtained by the partial differential approach. This means we get an upper bound of the option price, this is in agreement with the dual method Theorem 3.3.

So far, we have the numerical results of three methods for the American spread option under the two-factor geometric Brownian motion model. We compare the differences between them. The results are listed in Figure 5.25. The differences are about 1 or 2 percent, which means that the numerical results are very good.
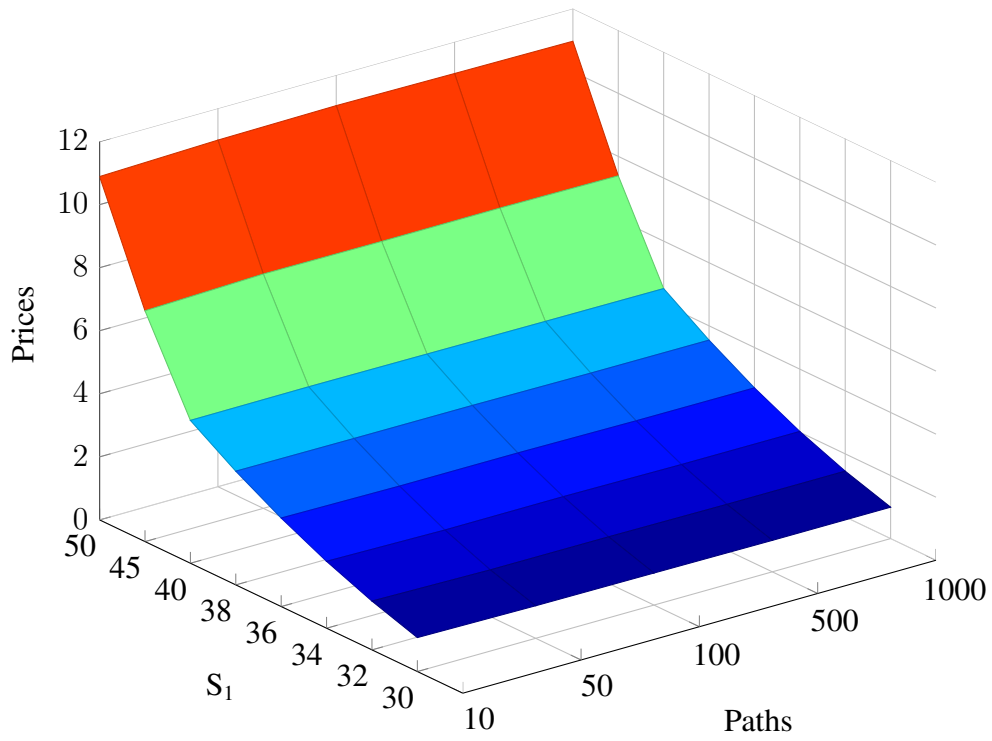
Figure 5.23: Dual Method for American Spread Options-Small Number of Paths(Prices)
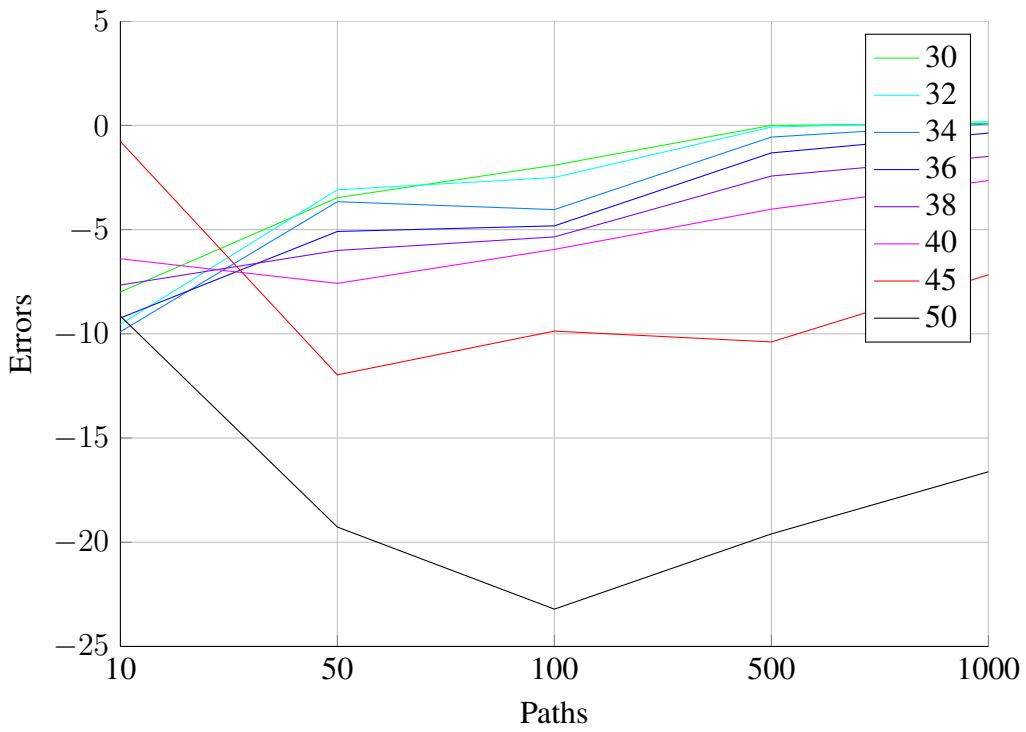
Figure 5.24: Dual Method for American Spread Options-Small Number of Paths(Errors)
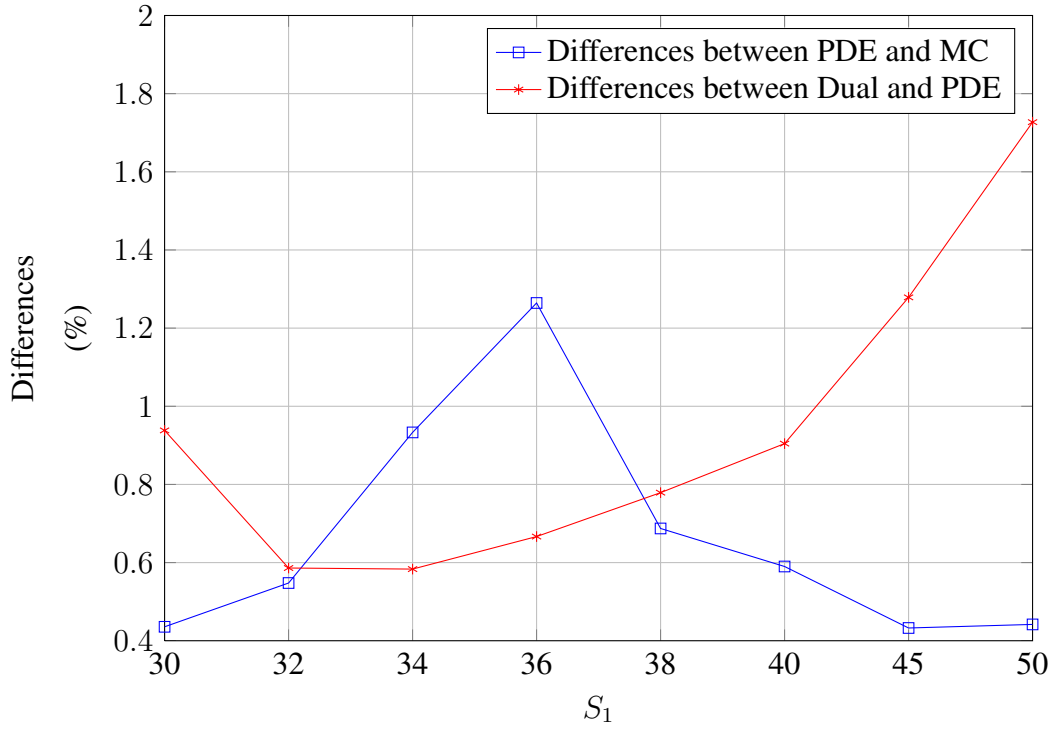
Figure 5.25: Comparison of Numerical Results between Three Methods(MC, PDE, Dual) under Two-Factor Geometric Brownian Motion Model

5.2 THREE-FACTOR STOCHASTIC VOLATILITY MODEL

In this section, we study the numerical implementations and results for pricing American spread call option under three-factor stochastic volatility model. We have three approaches for pricing the American spread option: the partial differential equation approach, Monte Carlo simulation approach and the dual method approach.

Recall that we have the system under the risk neutral measure $\mathbb{Q}$

$$dS_1(t) = (r - q_1)S_1(t)dt + \sigma_1\sqrt{v}S_1(t)dw_1(t), \tag{5.81}$$

$$dS_2(t) = (r - q_2)S_2(t)dt + \sigma_2\sqrt{v}S_2(t)\left(\rho dw_1(t) + \sqrt{1 - \rho^2}dw_2(t)\right), \tag{5.82}$$

$$dv = k(\mu - v)dt + \sigma_v\sqrt{v}\left(\rho_1 dw_1(t) + \rho_3 dw_2(t) + \rho_4 dw_3\right), \tag{5.83}$$

where

$$\rho_3 = \frac{\rho_2 - \rho\rho_1}{\sqrt{1 - \rho^2}}, \tag{5.84}$$

$$\rho_4 = \sqrt{\frac{1 - \rho^2 - \rho_1^2 - \rho_2^2 + 2\rho_1\rho_2\rho}{1 - \rho^2}}, \tag{5.85}$$

and w_1, w_2 and w_3 are three independent standard Brownian motions under the risk neutral proba-bility measure $\mathbb{Q}$. We choose the American call spread option as an example. The payoff function is $\max\{S_1 - S_2 - K, 0\}$. The parameters are $S_2 = 36$, $\sigma_1 = 1$, $\sigma_2 = 0.5$, $r = 0.06$, $\rho = 0.5$, $T = 1$, $q_1 = 0.05$, $q_2 = 0.01$, $v(0) = 0.16$, $\mu = 0.16$, $k = 1$, $\sigma_v = 0.2$, $\rho_1 = 0$, $\rho_2 = 0$ and $K = 5$.

5.2.1 Partial Differential Equations Approach.

In this subsection, we use the partial differ-ential equation method to compute the price of the American spread option under the three-factor stochastic volatility model.

From Chapter 4, we know that the price of the American spread option is the solution of the par-tial differential equation with a free boundary condition. We want to get rid of the free boundary and formulate the problem into complementarity problem. Let $v(x_1, x_2, x_3, t)$ be the price of the American spread option with $x_1 = S_1(t)$, $x_2 = S_2(t)$ and $x_3 = v(t)$ at time t. We use Lv to represent

$$\begin{aligned} Lv = &-rv + v_t + (r - q_1)x_1 v_{x_1} + (r - q_2)x_2 v_{x_2} + k(\mu - x_3)v_{x_3} \\ &+ \frac{1}{2}\sigma_1{}^2 x_1{}^2 x_3 v_{x_1 x_1} + \frac{1}{2}\sigma_2{}^2 x_2{}^2 x_3 v_{x_2 x_2} + \frac{1}{2}\sigma_v{}^2 x_3 v_{x_3 x_3} \\ &+ \rho\sigma_1\sigma_2 x_1 x_2 x_3 v_{x_1 x_2} + \rho_1\sigma_1\sigma_v x_1 x_3 v_{x_1 x_3} + \rho_2\sigma_2\sigma_v x_2 x_3 v_{x_1 x_2}. \end{aligned} \tag{5.86}$$

Also, we know that we have two regions for the American spread option. In one region, early exercise is optimal. We have

$$Lv \leq 0, \tag{5.87}$$

$$v = h. \tag{5.88}$$

The other one, early exercise is not optimal. We have

$$Lv = 0, \tag{5.89}$$

$$v \geq h. \tag{5.90}$$

Hence, the price is formulated as the solution of the following complementarity problem

$$Lv \leq 0, \tag{5.91}$$

$$v - h \geq 0, \tag{5.92}$$

$$(Lv)(v - h) = 0, \tag{5.93}$$

where $h = \max(x_1 - x_2 - K, 0)$.

It is more convenient to solve the above complementarity problem (5.8)-(5.10) by doing the following substitutions

$$x_1 = e^{y_1},$$
$$x_2 = e^{y_2},$$
$$x_3 = y_3,$$
$$v(x_1, x_2, x_3, t) = u(y_1, y_2, y_3, t).$$

Then, we have

$$Lu \leq 0, \tag{5.94}$$

$$u - h \geq 0, \tag{5.95}$$

$$(Lu)(u - h) = 0, \tag{5.96}$$

where $h = \max(e^{y_1} - e^{y_2} - K, 0)$ and

$$
\begin{aligned}
Lu = {} & -ru + u_t + (r - q_1 - \frac{1}{2}\sigma_1^2 y_3)u_{y_1} + (r - q_1 - \frac{1}{2}\sigma_3^2 y_3)u_{y_2} \\
& + k(\mu - y_3)u_{y_3} + \frac{1}{2}\sigma_1{}^2 y_3 u_{y_1 y_1} + \frac{1}{2}\sigma_2{}^2 y_3 u_{y_2 y_2} + \frac{1}{2}\sigma_u{}^2 y_3 u_{y_3 y_3} \\
& + \rho\sigma_1\sigma_2 y_3 u_{y_1 y_2} + \rho_1\sigma_1\sigma_u y_3 u_{y_1 y_3} + \rho_2\sigma_2\sigma_u y_3 u_{y_1 y_2}.
\end{aligned}
\tag{5.97}
$$

Then we use the finite difference method to solve the above complementarity problem. First, we divide the life of the option T into N equally spaced intervals of length $\Delta t = \frac{T}{N}$. Then a total of $N + 1$ times are therefore considered: $0,\ \Delta t,\ 2\Delta t,\ \cdots,\ T$. Then we restrict $y_i (i = 1, 2, 3)$ by $\underline{y}_i \leq y_i \leq \overline{y}_i$, we divide y_i into M equally spaced intervals of length $\Delta y_i = \frac{\overline{y}_i - \underline{y}_i}{M}$. The space and time variables define a grid consisting of a total of $(M + 1)(M + 1)(M + 1)(N + 1)$ points.

We use $u_{m,n,p}^k$ to represent $u(y_{1m}, y_{2n}, y_{3p}, t_k)$ with $y_{1m} = \underline{y}_1 + (m-1)\Delta y_1$, $y_{2n} = \underline{y}_2 + (n-1)\Delta y_2$, $y_{3p} = \underline{y}_3 + (p-1)\Delta y_3$ and $t_k = (k-1)\Delta t$. For the point $(y_{1m}, y_{2n}, y_{3p}, t_k)$, we have the following

finite differences

$$u_t = \frac{u_{m,n,p}^k - u_{m,n,p}^{k-1}}{\Delta t} + O(\Delta t), \tag{5.98}$$

$$u_{y_1} = \frac{u_{m+1,n,p}^k - u_{m-1,n,p}^k}{2\Delta y_1} + O\left[(\Delta y_1)^2\right], \tag{5.99}$$

$$u_{y_2} = \frac{u_{m,n+1,p}^k - u_{m,n-1,p}^k}{2\Delta y_2} + O\left[(\Delta y_2)^2\right], \tag{5.100}$$

$$u_{y_3} = \frac{u_{m,n,p+1}^k - u_{m,n,p-1}^k}{2\Delta y_3} + O\left[(\Delta y_2)^2\right], \tag{5.101}$$

$$u_{y_1 y_1} = \frac{u_{m+1,n,p}^k + u_{m-1,n,p}^k - 2u_{m,n,p}^k}{(\Delta y_1)^2} + O\left[(\Delta y_1)^2\right], \tag{5.102}$$

$$u_{y_2 y_2} = \frac{u_{m,n+1,p}^k + u_{m,n-1,p}^k - 2u_{m,n,p}^k}{(\Delta y_2)^2} + O\left[(\Delta y_2)^2\right], \tag{5.103}$$

$$u_{y_3 y_3} = \frac{u_{m,n,p+1}^k + u_{m,n,p-1}^k - 2u_{m,n,p}^k}{(\Delta y_3)^2} + O\left[(\Delta y_3)^2\right], \tag{5.104}$$

$$u_{y_1 y_2} = \frac{u_{m+1,n+1,p}^k + u_{m-1,n-1,p}^k - u_{m-1,n+1,p}^k - u_{m+1,n-1,p}^k}{4\Delta y_1 \Delta y_2} + O\left[(\Delta y_1)^2 + (\Delta y_2)^2\right], \tag{5.105}$$

$$u_{y_1 y_3} = \frac{u_{m+1,n,p+1}^k + u_{m-1,n,p-1}^k - u_{m-1,n,p+1}^k - u_{m+1,n,p-1}^k}{4\Delta y_1 \Delta y_3} + O\left[(\Delta y_1)^2 + (\Delta y_3)^2\right], \tag{5.106}$$

$$u_{y_2 y_3} = \frac{u_{m,n+1,p+1}^k + u_{m,n-1,p-1}^k - u_{m,n+1,p-1}^k - u_{m,n-1,p+1}^k}{4\Delta y_2 \Delta y_3} + O\left[(\Delta y_2)^2 + (\Delta y_3)^2\right]. \tag{5.107}$$

Then, we incorporate all of the above finite differences into the equation (5.97). After we simplify it and ignore the big O-terms, we obtain

$$\begin{aligned}
Lu_{m,n,p}^k =& -\frac{u_{m,n,p}^{k-1}}{\Delta t} + C_1 u_{m+1,n+1,p}^k + C_2 u_{m+1,n,p+1}^k + (A_1 + B_1)u_{m+1,n,p}^k - C_2 u_{m+1,n,p-1}^k \\
& - C_1 u_{m+1,n-1,p}^k + C_3 u_{m,n+1,p+1}^k + (A_2 + B_2)u_{m,n+1,p}^k - C_3 u_{m,n+1,p-1}^k \\
& - C_3 u_{m,n-1,p+1}^k + (-A_2 + B_2)u_{m,n-1,p}^k + C_3 u_{m,n-1,p-1}^k - C_1 u_{m-1,n+1,p}^k \\
& - C_2 u_{m-1,n,p+1}^k + (-A_1 + B_1)u_{m-1,n,p}^k + C_2 u_{m-1,n,p-1}^k + C_1 u_{m-1,n-1,p}^k \\
& + (A_3 + B_3)u_{m,n,p}^k + \left(\frac{1}{\Delta t} - r - 2(B_1 + B_2 + B_3)\right) u_{m,n,p}^k \\
& + (B_3 - A_3)u_{m,n,p-1}^k, \tag{5.108}
\end{aligned}$$

where

$$A_1 = \frac{r - q_1 - \frac{1}{2}\sigma_1^2(\underline{y}_3 + (p-1)\Delta y_3)}{2\Delta y_1}, \tag{5.109}$$

$$A_2 = \frac{r - q_2 - \frac{1}{2}\sigma_2^2(\underline{y}_3 + (p-1)\Delta y_3)}{2\Delta y_2}, \tag{5.110}$$

$$A_3 = \frac{k(\mu - (\underline{y}_3 + (p-1)\Delta y_3))}{2\Delta y_3}, \tag{5.111}$$

$$B_1 = \frac{\frac{1}{2}\sigma_1^2(\underline{y}_3 + (p-1)\Delta y_3)}{(\Delta y_1)^2}, \tag{5.112}$$

$$B_2 = \frac{\frac{1}{2}\sigma_2^2(\underline{y}_3 + (p-1)\Delta y_3)}{(\Delta y_2)^2}, \tag{5.113}$$

$$B_3 = \frac{\frac{1}{2}\sigma_v^2(\underline{y}_3 + (p-1)\Delta y_3)}{(\Delta y_3)^2}, \tag{5.114}$$

$$C_1 = \frac{\rho\sigma_1\sigma_2(\underline{y}_3 + (p-1)\Delta y_3)}{4\Delta y_1\Delta y_2}, \tag{5.115}$$

$$C_2 = \frac{\rho_1\sigma_1\sigma_3(\underline{y}_3 + (p-1)\Delta y_3)}{4\Delta y_1\Delta y_3}, \tag{5.116}$$

$$C_3 = \frac{\rho_2\sigma_2\sigma_3(\underline{y}_3 + (p-1)\Delta y_3)}{4\Delta y_2\Delta y_3}. \tag{5.117}$$

For the complementarity problem (5.94)-(5.96), we have

$$\min\{-Lu, u - h\} = 0. \tag{5.118}$$

Then, by (5.108), for the point $(y_{1m}, y_{2n}, y_{3p}, t_k)$, we have

$$\min\{-Lu_{m,n,p}^k, u_{m,n,p}^{k-1} - h_{m,n}\} = 0, \tag{5.119}$$

where $h_{m,n} = (e^{y_{1m}} - e^{y_{2n}} - K)^+$. Next, by $\min\{X - Y, X - Z\} = 0 \Leftrightarrow X = \max\{Y, Z\}$ and

(5.22), we have the finite difference method for the American spread option as follows

$$
\begin{aligned}
u_{m,n,p}^{k-1} = \max\{ & [C_1 u_{m+1,n+1,p}^k + C_2 u_{m+1,n,p+1}^k + (A_1 + B_1)u_{m+1,n,p}^k - C_2 u_{m+1,n,p-1}^k \\
& - C_1 u_{m+1,n-1,p}^k + C_3 u_{m,n+1,p+1}^k + (A_2 + B_2)u_{m,n+1,p}^k - C_3 u_{m,n+1,p-1}^k \\
& - C_3 u_{m,n-1,p+1}^k + (-A_2 + B_2)u_{m,n-1,p}^k + C_3 u_{m,n-1,p-1}^k - C_1 u_{m-1,n+1,p}^k \\
& - C_2 u_{m-1,n,p+1}^k + (-A_1 + B_1)u_{m-1,n,p}^k + C_2 u_{m-1,n,p-1}^k + C_1 u_{m-1,n-1,p}^k \\
& + (A_3 + B_3)u_{m,n,p}^k + \left(\frac{1}{\Delta t} - r - 2(B_1 + B_2 + B_3) \right) u_{m,n,p}^k \\
& + (B_3 - A_3)u_{m,n,p-1}^k] * \Delta t, h_{m,n} \},
\end{aligned}
\tag{5.120}
$$

with the approximated boundary and terminal conditions

$$
v(\underline{y}_1, y_2, y_3, t) = 0, \tag{5.121}
$$

$$
v(\overline{y}_1, y_2, y_3, t) = e^{-r(T-t)}(e^{\overline{y}_1} - e^{y_2} - K)^+, \tag{5.122}
$$

$$
v(y_1, \underline{y}_2, y_3, t), = e^{-r(T-t)}(e^{y_1} - e^{\underline{y}_2} - K)^+, \tag{5.123}
$$

$$
v(y_1, \overline{y}_2, y_3, t) = 0, \tag{5.124}
$$

$$
v(y_1, y_2, \underline{y}_3, t) = e^{-r(T-t)}(e^{y_1} - e^{y_2} - K)^+, \tag{5.125}
$$

$$
v(y_1, y_2, \overline{y}_3, t) = e^{-r(T-t)}(e^{y_1} - e^{y_2} - K)^+, \tag{5.126}
$$

$$
v(y_1, y_2, y_3, T) = (e^{y_1} - e^{y_2} - K)^+. \tag{5.127}
$$

Because we have the fast Fourier transformation technique for the corresponding European spread option under the three-factor stochastic volatility model. One can use this as a benchmark to test our finite difference approximations. The numerical results are shown in Table B.1 (Figures 5.26 and 5.27). We take the numbers of space steps and of time steps are 30×100, 40×200, 50×300, 60×400, 70×600 and 80×800, respectively. Then, we use these finite difference approximations to compute the price of the corresponding American call spread option. We get the results in Table B.2 (Figures 5.28 and 5.29). We take values in Table B.2 with numbers of space and time steps
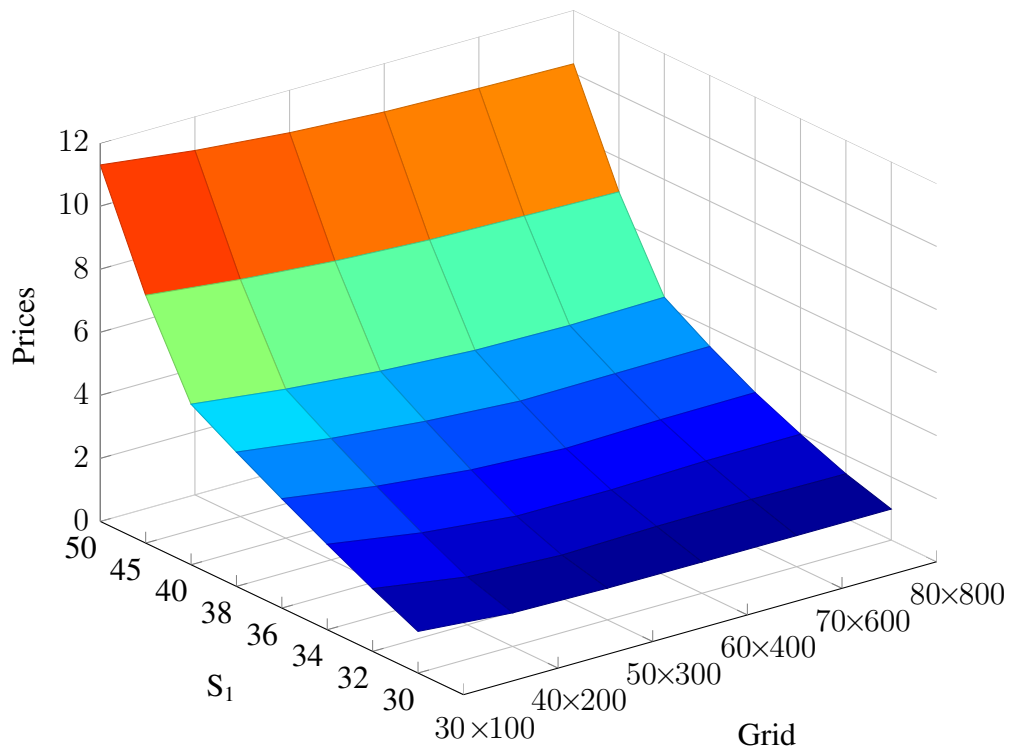
Figure 5.26: Explicit Finite Difference Method for European Spread Options with Stochastic Volatility(Prices)
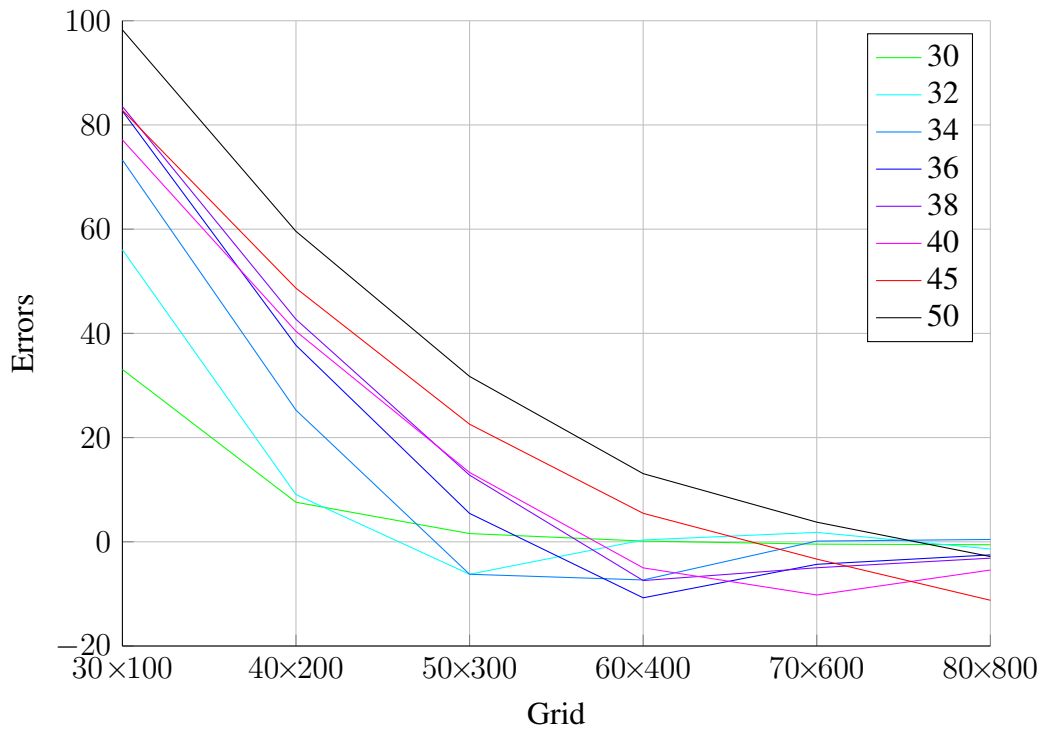
Figure 5.27: Explicit Finite Difference Method for European Spread Options with Stochastic Volatility(Errors)

80×800 as benchmark to make comparison to the Monte Carlo simulation and the dual method, which we consider in the subsequent sections.

5.2.2 Monte Carlo Simulation Approach.

In this subsection, we use the Monte Carlo simulation method to compute the price of the American spread option under the three-factor stochastic volatility model.

After we implement the finite difference method of partial differential equation approach, we are now ready to compare the numerical results of the Monte Carlo simulation method. We take the numbers of paths are $10000, 20000, 50000, 100000$ and 200000, respectively. The results are in Table B.3 (Figures 5.30 and 5.31). The basis functions are $1, S_1, S_1^2, S_1^3, S_2, S_2^2, S_2^3, S_3, S_3^2$ and the number of time steps are $m = 50$. We notice that when the number of paths are larger, we get the closer results compared to the numerical results obtained by the partial differential equations in the previous subsection. And the numbers in the parentheses represent the standard errors, when we increase the number of paths, it is much smaller, which is good agreement with the method of Monte Carlo simulation.

Next, we consider improved Monte Carlo simulation method with the following basis functions $1, S_1, S_1^2, S_1^3, S_2, S_2^2, S_2^3, S_3, S_3^2$. The numerical results are listed in Table B.4 (Figures 5.32 and 5.33).

5.2.3 Dual Method Approach.

In this subsection, we use the dual method to compute the price of the American spread option under the three-factor stochastic volatility model.

In the previous two subsections, we have examined the numerical results obtained by the finite

91
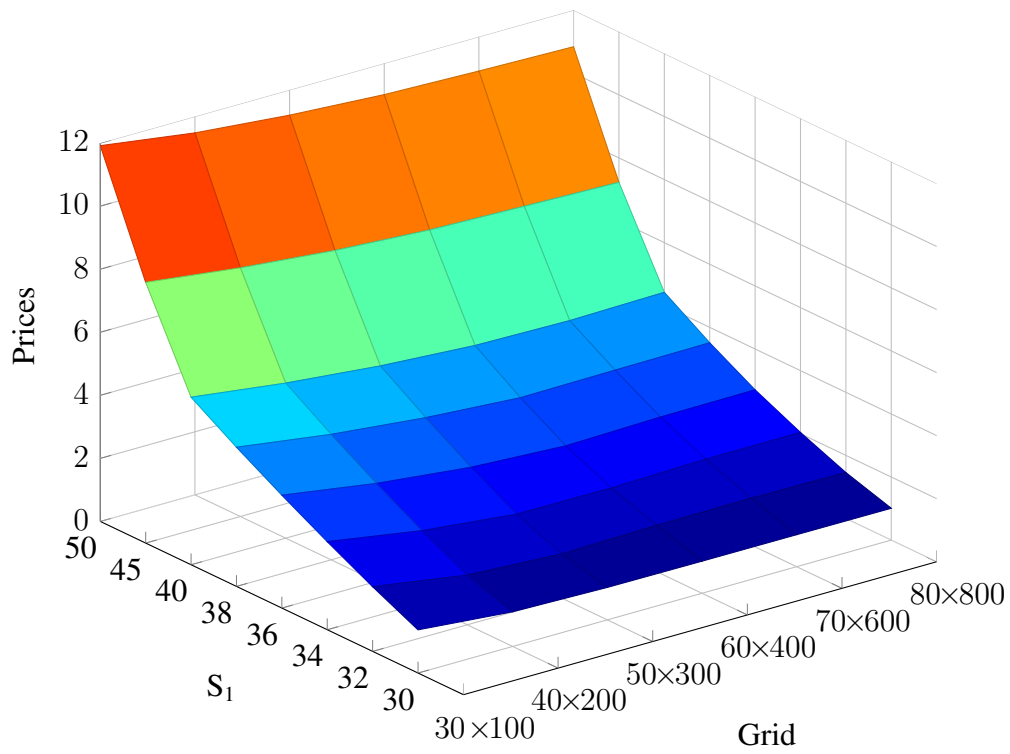
Figure 5.28: Explicit Finite Difference Method for American Spread Options with Stochastic Volatility(Prices)
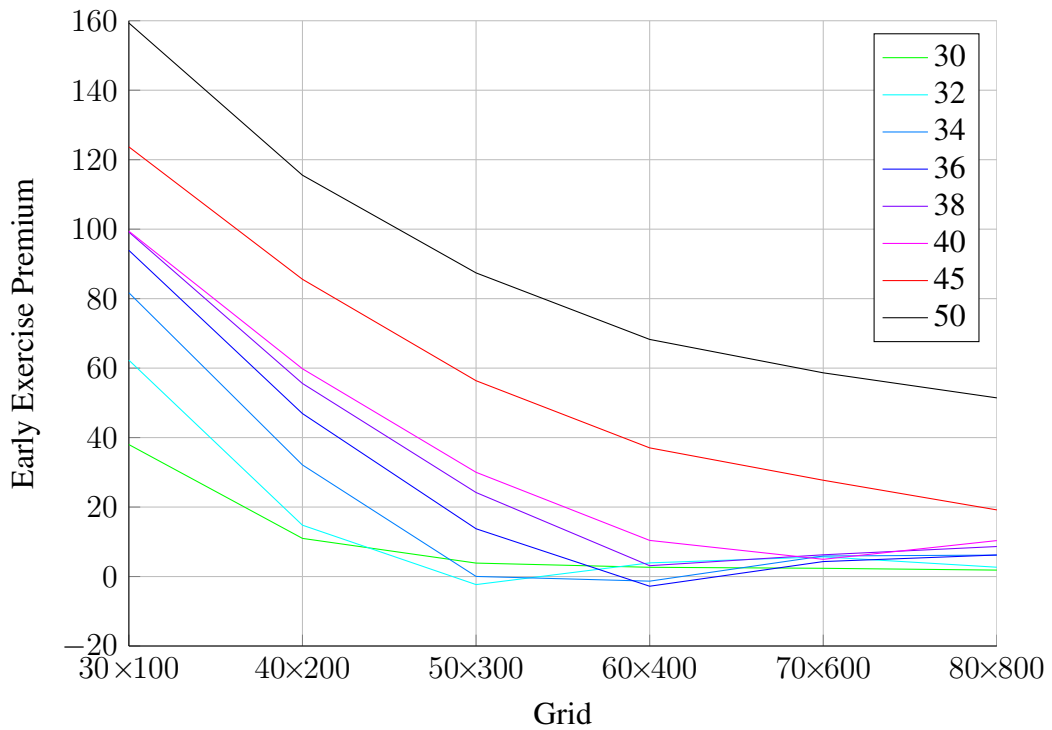
Figure 5.29: Explicit Finite Difference Method for American Spread Options with Stochastic Volatility(Early Exercise Premiums)

92
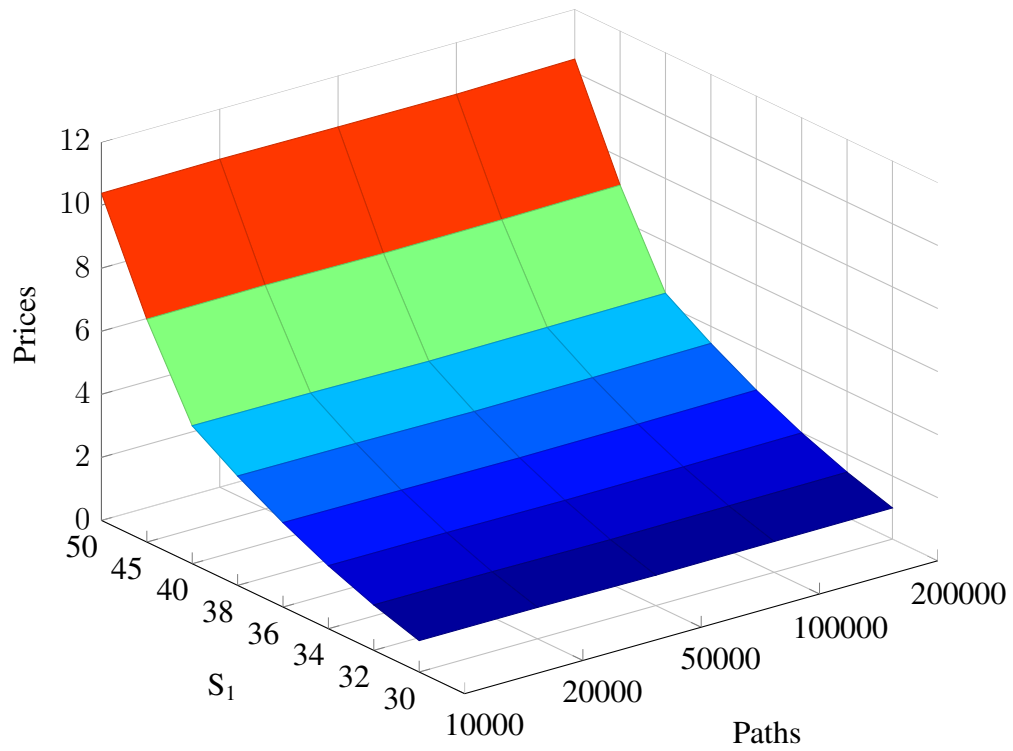
Figure 5.30: Monte Carlo Simulation for American Spread Options(Prices)
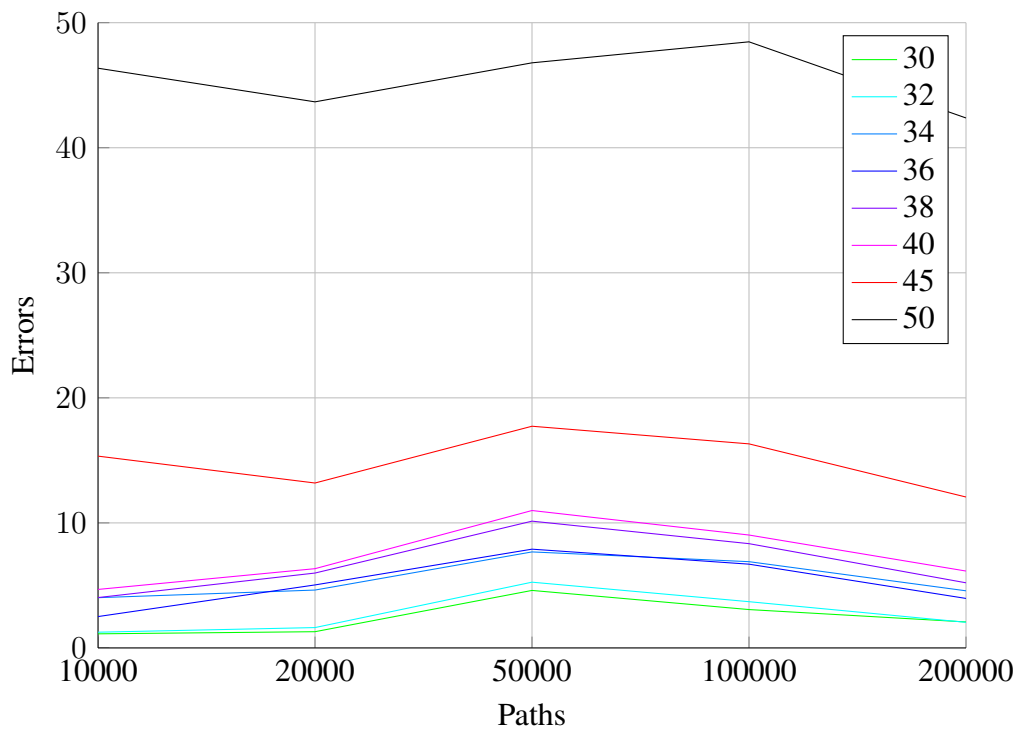
Figure 5.31: Monte Carlo Simulation for American Spread Options(Errors)
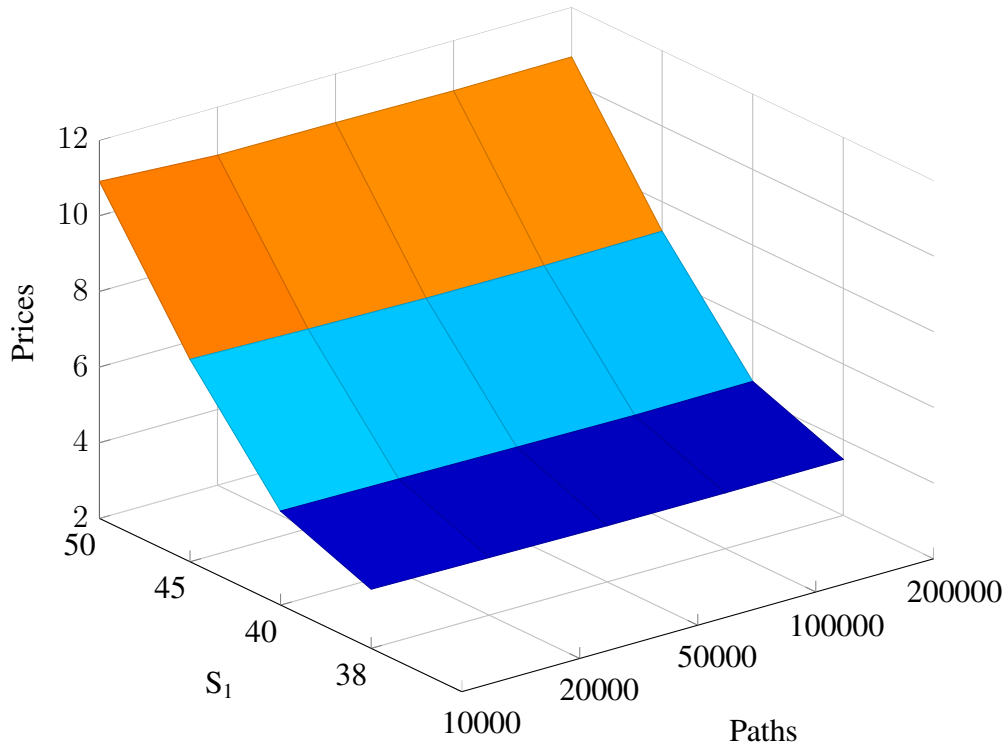
Figure 5.32: Improved Monte Carlo Simulation for American Spread Options(Prices)
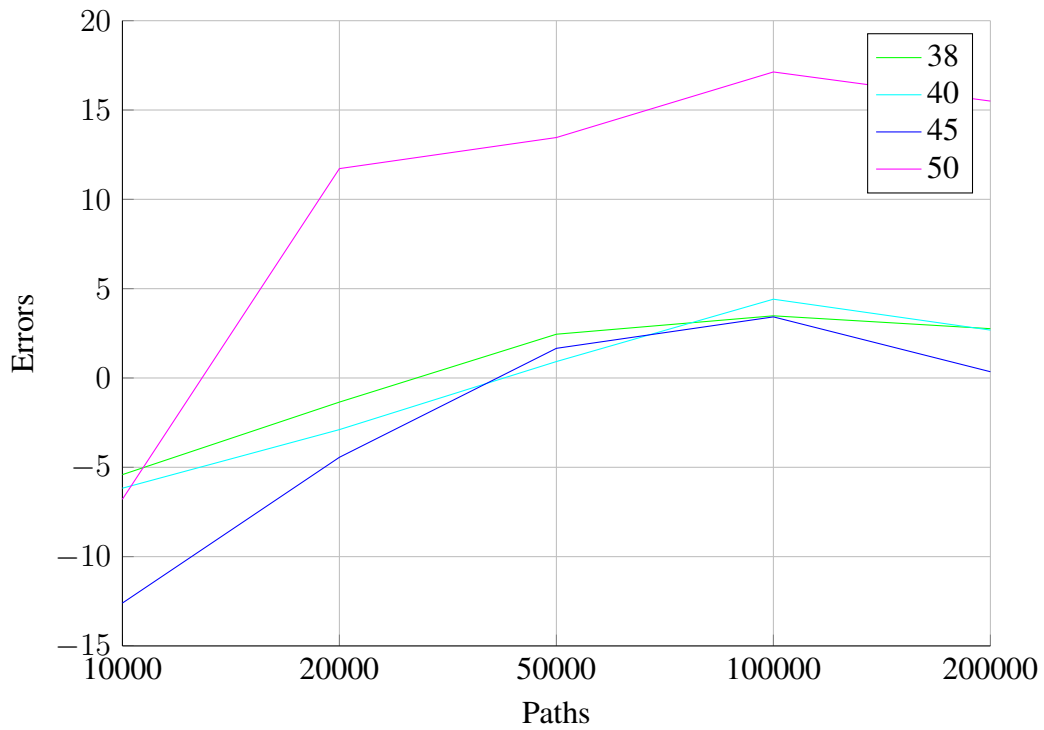
Figure 5.33: Improved Monte Carlo Simulation for American Spread Options(Errors)
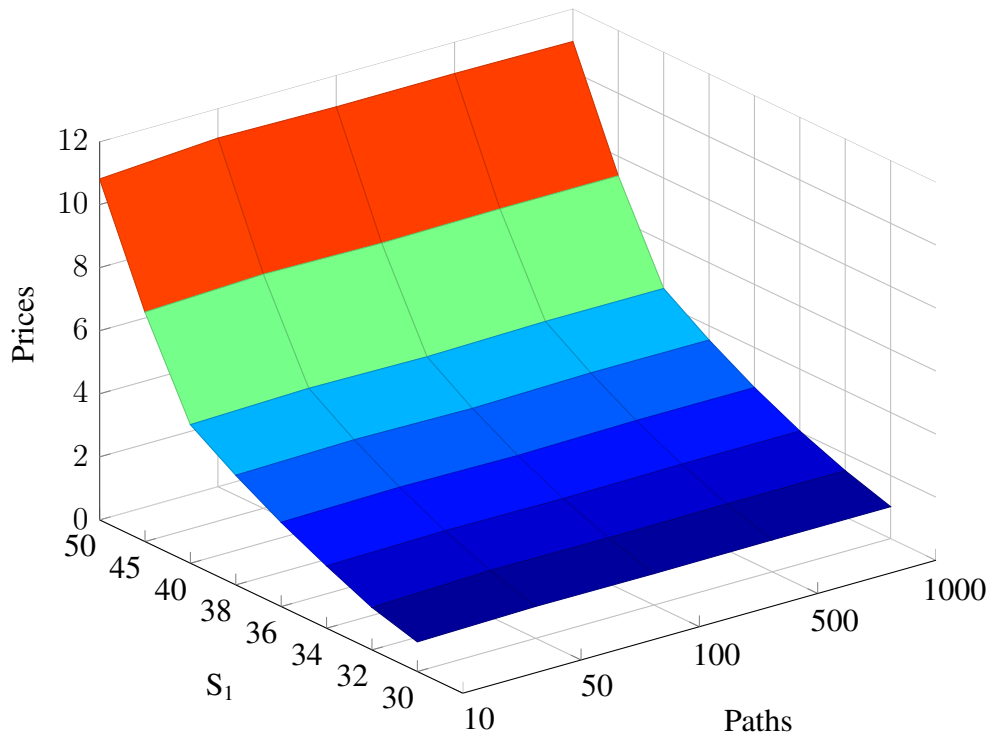
Figure 5.34: Dual Method for American Spread Options with Stochastic Volatility(Prices)
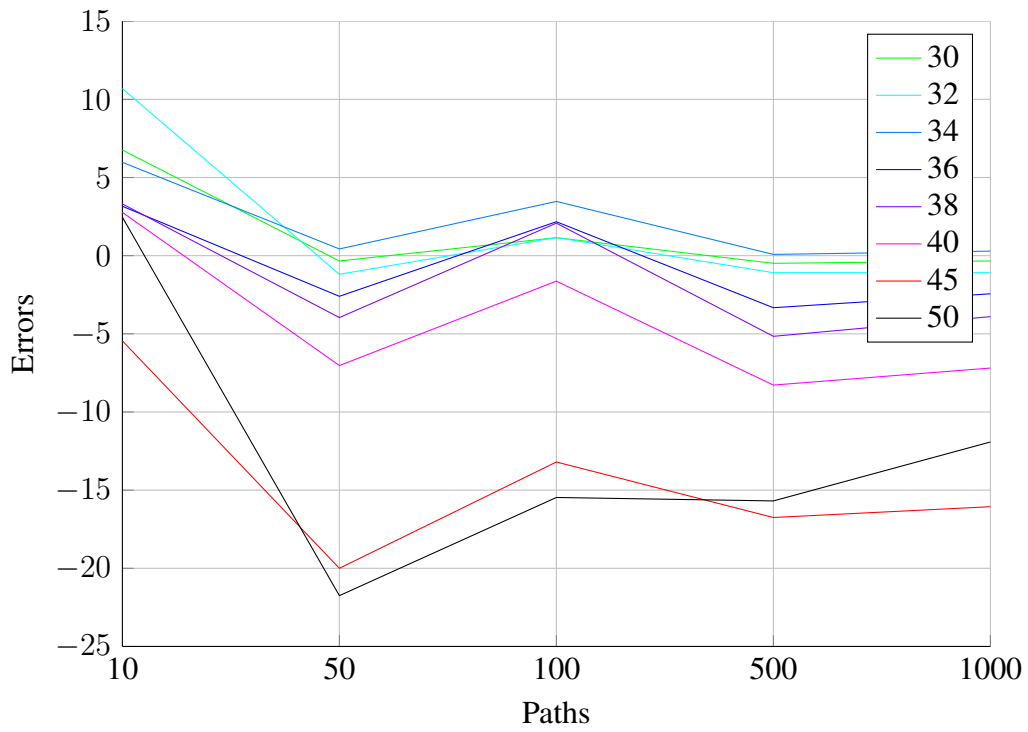
Figure 5.35: Dual method for American Spread Options with Stochastic Volatility(Errors)

difference method and Monte Carlo simulation method. We know that the Monte Carlo simulation often generates a lower bound of the option price. In this subsection, we combine the fast Fourier transformation technique and the dual method to generate an upper bound of the option price. We first compute the martingale by the fast Fourier transformation technique. Then, we use the dual method theorem to get the upper bound price of the American spread option. The numerical results are listed in Table B.5 (Figures 5.34 and 5.35).

So far, we have the numerical results of three methods under the three-factor stochastic volatility model for pricing American spread option. We compare the differences between them. The results are listed in Figure 5.36 for the American spread option.
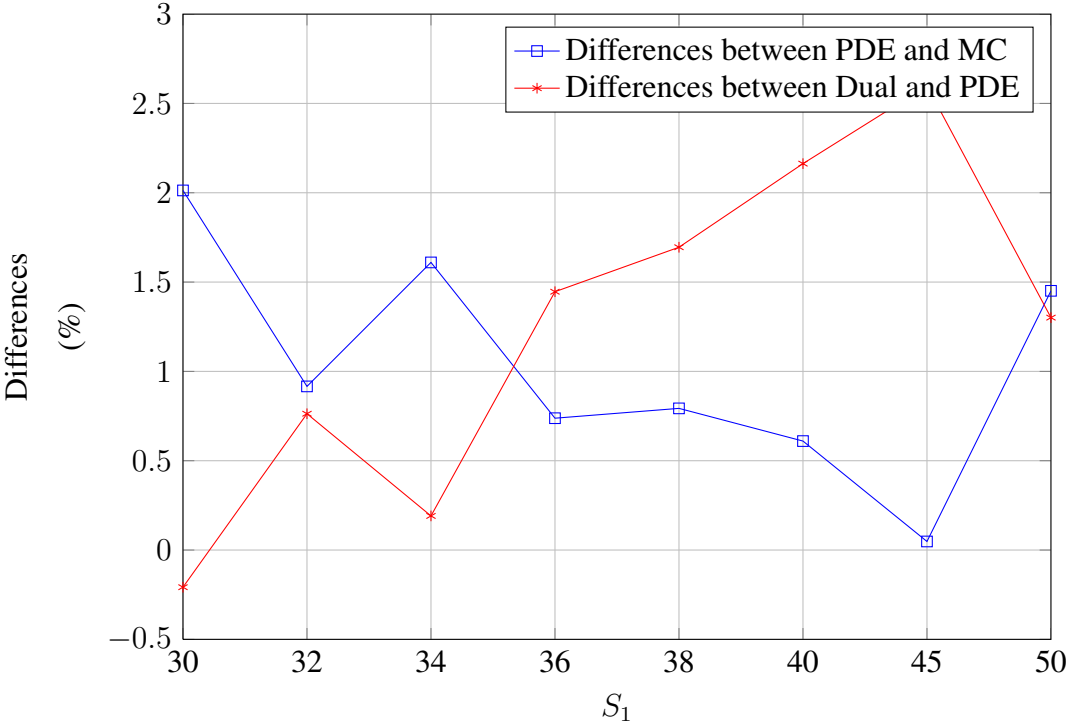
Figure 5.36: Comparison of Numerical Results between Three Methods(MC, PDE, Dual) under the Three-Factor Stochastic Volatility Model

CHAPTER 6. CONCLUSION AND FUTURE RESEARCH

6.1 CONCLUSION

We have studied the American spread option under both the two-factor geometric Brownian motion model and the three-factor stochastic volatility model. For two-factor model geometric Brownian motion model, we first considered the partial differential equation approach. We also solved it by finite difference method. But this method is slow. In order to obtain more accuracy results, it takes more computational time. Then, we considered the Monte Carlo simulation method. It is very easy to implement and extend the model to the three factor stochastic volatility model. However, it may be hard to get a very tight lower bound, because we have several parameters to choose. One is how do we choose the basis functions. Another one is how many basis functions we need to choose. These are very subjective. After we test with some parameters, we obtain a tight lower bound. Lastly, we studied the dual method for pricing the American spread option. We obtain a tight upper bound for American spread option with a very small amount of paths.

For the three-factor stochastic volatility model, first, we compute the price of the American spread option by the partial differential equation approach. However, in this approach, the dimension of the partial differential equation is high, so it takes more computational time to obtain more accuracy results. For the Monte Carlo simulation approach, we get a tight lower bound. For the dual method, we have an effective way to compute the martingale by fast Fourier transformation technique. Then, we get a tight upper bound for the price of the American spread option.

6.2 FURTHER RESEARCH

There are many exciting directions for the further research. We have studied the American spread option under the two-factor geometric Brownian motion model and three-factor stochastic volatility model. We made the assumption that the underlying assets are continuous functions, how about when there are jumps in our model?

Another direction is that we only study the American spread options. How about other American style exotic options? Such as American style barrier, American style look-back and American style Asian spread option?

Lastly, we assume that interest rates and corrections are constants for our models. However, empirical research has shown that interest rate are changing over time. Actually, there are a lot of stochastic models for interests rate, such as Hull-White model [28], Cox-Ingersoll-Ross(CIR) model [29], Heath-Jarrow-Morton (HJM) framework [30] and others. The correlations between the two underlying assets could also change as a response to the uncertain market conditions over time. And from [31], we know that the implied correlation often has a frown feature rather than smiles for implied volatility. Therefore the constant correlation assumption could lead to mis-pricing the spread option the same as the constant volatility Black-Scholes model which generates price bias in the vanilla options. An alternative approach is that we assume that both of them are stochastic. Then, how to price the options with so many stochastic factors?

APPENDIX A. TWO-FACTOR GEOMETRIC BROWNIAN MOTION MODEL

S_1	Analytic	Explicit Finite Difference						
		50×100	80×400	100×800	120×1000	200×1200	300×3000	400×4000
30	0.976	0.953	0.953	0.966	0.965	0.964	0.964	0.964
32	1.420	1.438	1.411	1.411	1.413	1.412	1.412	1.412
34	1.974	1.929	1.978	1.965	1.972	1.969	1.969	1.969
36	2.644	2.671	2.635	2.645	2.643	2.641	2.640	2.640
38	3.429	3.473	3.441	3.431	3.428	3.427	3.426	3.426
40	4.328	4.301	4.327	4.328	4.328	4.327	4.326	4.326
45	7.040	7.097	7.042	7.042	7.041	7.039	7.038	7.038
50	10.322	10.295	10.325	10.324	10.322	10.321	10.320	10.320
S_1	Analytic	Errors(100 Basis Points)						
30	0.976	-2.316	-2.316	-1.046	-1.176	-1.276	-1.216	-1.206
32	1.420	1.783	-0.907	-0.907	-0.657	-0.727	-0.757	-0.757
34	1.974	-4.493	0.427	-0.943	-0.213	-0.473	-0.493	-0.503
36	2.644	2.741	-0.889	0.121	-0.029	-0.279	-0.349	-0.369
38	3.429	4.426	1.206	0.206	-0.104	-0.174	-0.264	-0.294
40	4.328	-2.768	-0.078	-0.018	0.022	-0.118	-0.218	-0.248
45	7.040	5.730	0.210	0.220	0.180	-0.050	-0.170	-0.210
50	10.322	-2.734	0.286	0.156	0.036	-0.104	-0.174	-0.204

Table A.1: Explicit Finite Difference Method for European Spread Options

S_1	Analytic	Explicit Finite Difference to IPDE						
		50×100	80×400	100×800	120×1000	200×1200	300×3000	400×4000
30	0.976	0.996	0.987	0.982	0.980	0.977	0.976	0.976
32	1.420	1.430	1.430	1.422	1.421	1.421	1.420	1.419
34	1.974	1.999	1.977	1.982	1.979	1.976	1.974	1.974
36	2.644	2.684	2.656	2.655	2.647	2.644	2.644	2.643
38	3.429	3.479	3.447	3.437	3.434	3.431	3.430	3.429
40	4.328	4.385	4.345	4.332	4.338	4.329	4.328	4.328
45	7.040	7.079	7.045	7.054	7.045	7.042	7.040	7.040
50	10.322	10.328	10.350	10.325	10.332	10.323	10.323	10.322
S_1	Analytic	Errors(100 Basis Points)						
30	0.976	1.994	1.014	0.534	0.304	0.054	-0.006	-0.026
32	1.420	1.023	1.023	0.263	0.153	0.083	0.003	-0.027
34	1.974	2.467	0.267	0.747	0.487	0.197	0.017	-0.043
36	2.644	4.011	1.251	1.101	0.341	0.031	0.071	-0.019
38	3.429	4.956	1.786	0.816	0.466	0.236	0.086	-0.004
40	4.328	5.662	1.722	0.332	0.932	0.112	0.012	0.002
45	7.040	3.910	0.570	1.440	0.490	0.240	0.050	0.030
50	10.322	0.596	2.846	0.296	0.996	0.146	0.056	0.016

Table A.2: Explicit Finite Difference Method to Improved Partial Differential Equation for European Spread Options

S_1	Analytic	Explicit Finite Difference						
		50×100	80×400	100×800	120×1000	200×1200	300×3000	400×4000
30	0.976	1.017	1.008	1.003	1.000	0.998	0.997	0.997
32	1.420	1.462	1.463	1.455	1.454	1.454	1.453	1.452
34	1.974	2.048	2.026	2.031	2.029	2.026	2.024	2.023
36	2.644	2.756	2.728	2.727	2.719	2.716	2.717	2.716
38	3.429	3.581	3.549	3.539	3.535	3.533	3.532	3.531
40	4.328	4.525	4.484	4.470	4.477	4.468	4.467	4.467
45	7.040	7.351	7.317	7.327	7.317	7.315	7.314	7.313
50	10.322	10.800	10.829	10.801	10.810	10.801	10.800	10.800
S_1	Analytic	Early Exercise Premium(100 Basis Points)						
30	0.976	4.094	3.104	2.614	2.394	2.154	2.094	2.064
32	1.420	4.273	4.313	3.533	3.433	3.383	3.303	3.273
34	1.974	7.397	5.177	5.707	5.447	5.187	4.997	4.937
36	2.644	11.271	8.461	8.331	7.541	7.261	7.311	7.211
38	3.429	15.216	11.956	10.956	10.616	10.416	10.266	10.176
40	4.328	19.642	15.602	14.162	14.832	14.022	13.922	13.912
45	7.040	31.150	27.750	28.760	27.770	27.580	27.390	27.370
50	10.322	47.846	50.726	47.916	48.786	47.916	47.826	47.796

Table A.3: Explicit Finite Difference Method to Improved Partial Differential Equation for American Spread Options

S_1	IPDE	Explicit Finite Difference						
		50×100	80×400	100×800	120×1000	200×1200	300×3000	400×4000
30	0.997	0.975	0.977	0.989	0.989	0.989	0.990	0.991
32	1.452	1.472	1.448	1.448	1.450	1.450	1.450	1.450
34	2.023	1.992	2.033	2.020	2.026	2.024	2.023	2.023
36	2.716	2.748	2.715	2.722	2.721	2.717	2.716	2.716
38	3.531	3.576	3.547	3.538	3.535	3.533	3.532	3.531
40	4.467	4.451	4.474	4.473	4.473	4.470	4.468	4.468
45	7.313	7.371	7.323	7.321	7.319	7.316	7.314	7.313
50	10.800	10.784	10.810	10.807	10.805	10.802	10.800	10.800
S_1	IPDE	Errors(100 Basis Points)						
30	0.997	-2.260	-1.980	-0.800	-0.860	-0.820	-0.700	-0.660
32	1.452	1.970	-0.450	-0.440	-0.210	-0.230	-0.260	-0.240
34	2.023	-3.180	0.910	-0.300	0.280	0.030	-0.030	-0.050
36	2.716	3.220	-0.080	0.640	0.480	0.170	0.050	0.010
38	3.531	4.540	1.640	0.730	0.420	0.230	0.070	0.020
40	4.467	-1.620	0.680	0.580	0.530	0.260	0.090	0.030
45	7.313	5.720	1.000	0.770	0.610	0.270	0.060	-0.010
50	10.800	-1.590	0.980	0.720	0.490	0.220	0.050	-0.010

Table A.4: Explicit Finite Difference Method to Original Partial Differential Equation for American Spread Options

S_1	PDE	Monte Carlo Simulation for American Spread				
		10000	20000	50000	100000	200000
30	0.997	0.939 (0.037)	0.962 (0.026)	0.967 (0.016)	0.974 (0.011)	0.987 (0.008)
32	1.452	1.363 (0.046)	1.413 (0.032)	1.415 (0.020)	1.423 (0.014)	1.442 (0.010)
34	2.023	1.894 (0.055)	1.969 (0.039)	1.976 (0.024)	1.989 (0.017)	2.004 (0.012)
36	2.716	2.541 (0.065)	2.660 (0.046)	2.650 (0.029)	2.669 (0.021)	2.681 (0.015)
38	3.531	3.324 (0.075)	3.477 (0.054)	3.440 (0.033)	3.465 (0.024)	3.479 (0.017)
40	4.467	4.241 (0.085)	4.399 (0.061)	4.351 (0.038)	4.377 (0.027)	4.391 (0.019)
45	7.313	6.982 (0.112)	7.167 (0.080)	7.106 (0.050)	7.114 (0.036)	7.141 (0.025)
50	10.800	10.279 (0.137)	10.527 (0.099)	10.433 (0.062)	10.436 (0.044)	10.463 (0.031)
S_1	PDE	Errors(100 Basis Points)				
30	0.997	5.830	3.530	3.010	2.300	1.020
32	1.452	8.940	3.970	3.790	2.970	1.010
34	2.023	12.940	5.440	4.700	3.490	1.940
36	2.716	17.450	5.600	6.550	4.720	3.440
38	3.531	20.690	5.340	9.070	6.540	5.150
40	4.467	22.610	6.790	11.620	9.070	7.640
45	7.313	33.110	14.620	20.750	19.930	17.230
50	10.800	52.070	27.280	36.660	36.420	33.690

Table A.5: Monte Carlo Simulation for American Spread Options

S_1	PDE	Monte Carlo Simulation for American Spread				
		10000	20000	50000	100000	200000
30	0.997	0.947 (0.038)	0.962 (0.026)	0.967 (0.016)	0.978 (0.012)	0.992 (0.008)
32	1.452	1.356 (0.046)	1.413 (0.032)	1.415 (0.020)	1.419 (0.014)	1.440 (0.010)
34	2.023	1.899 (0.055)	1.969 (0.039)	1.976 (0.024)	1.981 (0.017)	1.999 (0.013)
36	2.716	2.557 (0.065)	2.660 (0.046)	2.650 (0.029)	2.659 (0.021)	2.675 (0.015)
38	3.531	3.327 (0.075)	3.477 (0.054)	3.440 (0.033)	3.458 (0.024)	3.474 (0.017)
40	4.467	4.250 (0.085)	4.399 (0.061)	4.351 (0.038)	4.377 (0.027)	4.388 (0.019)
45	7.313	7.072 (0.111)	7.167 (0.080)	7.106 (0.050)	7.143 (0.035)	7.158 (0.025)
50	10.800	10.576 (0.129)	10.527 (0.099)	10.433 (0.062)	10.503 (0.042)	10.519 (0.030)
S_1	PDE	Errors(100 Basis Points)				
30	0.997	4.990	3.530	3.010	1.920	0.470
32	1.452	9.680	3.970	3.790	3.360	1.220
34	2.023	12.420	5.440	4.700	4.260	2.470
36	2.716	15.920	5.600	6.550	5.660	4.060
38	3.531	20.430	5.340	9.070	7.290	5.720
40	4.467	21.690	6.790	11.620	9.000	7.930
45	7.313	24.180	14.620	20.750	17.080	15.510
50	10.800	22.350	27.280	36.660	29.730	28.130

Table A.6: Monte Carlo Simulation for American Spread Options-Standard Basis Functions with Cross Product Term

S_1	PDE	Monte Carlo Simulation for American Spread				
		10000	20000	50000	100000	200000
38	3.531	3.361 (0.071)	3.498 (0.052)	3.454 (0.032)	3.480 (0.023)	3.489 (0.017)
40	4.467	4.281 (0.081)	4.456 (0.059)	4.384 (0.037)	4.397 (0.026)	4.408 (0.019)
45	7.313	7.085 (0.102)	7.290 (0.075)	7.202 (0.047)	7.193 (0.034)	7.191 (0.024)
50	10.800	10.582 (0.120)	10.731 (0.089)	10.593 (0.056)	10.603 (0.040)	10.620 (0.029)
S_1	PDE	Errors(100 Basis Points)				
38	3.531	17.000	3.290	7.660	5.120	4.200
40	4.467	18.670	1.170	8.350	7.020	5.960
45	7.313	22.850	2.340	11.170	12.050	12.280
50	10.800	21.830	6.920	20.680	19.720	17.960

Table A.7: Monte Carlo Simulation for American Spread Options-Standard Basis Functions with Payoff Function Term

S_1	PDE	Monte Carlo Simulation for American Spread				
		10000	20000	50000	100000	200000
38	3.531	3.397 (0.068)	3.511 (0.049)	3.508 (0.030)	3.512 (0.022)	3.506 (0.016)
40	4.467	4.316 (0.074)	4.472 (0.054)	4.455 (0.033)	4.443 (0.024)	4.441 (0.017)
45	7.313	7.256 (0.093)	7.331 (0.067)	7.284 (0.042)	7.294 (0.030)	7.281 (0.022)
50	10.800	10.779 (0.106)	10.832 (0.078)	10.724 (0.049)	10.744 (0.035)	10.751 (0.025)
S_1	PDE	Errors(100 Basis Points)				
38	3.531	13.410	1.980	2.240	1.890	2.460
40	4.467	15.160	-0.490	1.190	2.400	2.680
45	7.313	5.760	-1.720	2.930	1.980	3.260
50	10.800	2.080	-3.250	7.560	5.570	4.870

Table A.8: Monte Carlo Simulation for American Spread Options-Standard Basis Functions with both Cross Production and Payoff Function Term

S_1	PDE	Monte Carlo Simulation for American Spread				
		10000	20000	50000	100000	200000
38	3.531	3.505 (0.068)	3.546 (0.048)	3.523 (0.029)	3.507 (0.021)	3.513 (0.015)
40	4.467	4.423 (0.077)	4.468 (0.054)	4.453 (0.033)	4.442 (0.023)	4.449 (0.016)
45	7.313	7.240 (0.092)	7.311 (0.069)	7.295 (0.041)	7.279 (0.029)	7.281 (0.020)
50	10.800	10.746 (0.103)	10.740 (0.073)	10.758 (0.046)	10.739 (0.033)	10.738 (0.023)
S_1	PDE	Errors(100 Basis Points)				
38	3.531	2.550	-1.540	0.750	2.400	1.780
40	4.467	4.450	-0.110	1.440	2.580	1.790
45	7.313	7.330	0.260	1.830	3.470	3.190
50	10.800	5.400	5.970	4.230	6.060	6.150

Table A.9: Improved Monte Carlo Simulation for American Spread Options

S_1	PDE	Monte Carlo Simulation for American Spread				
		10000	20000	50000	100000	200000
38	3.531	3.555 (0.068)	3.566 (0.048)	3.542 (0.029)	3.539 (0.020)	3.536 (0.014)
40	4.467	4.475 (0.074)	4.501 (0.053)	4.494 (0.032)	4.473 (0.023)	4.477 (0.016)
45	7.313	7.303 (0.088)	7.376 (0.068)	7.312 (0.040)	7.315 (0.028)	7.320 (0.020)
50	10.800	10.857 (0.102)	10.752 (0.066)	10.774 (0.044)	10.787 (0.031)	10.794 (0.022)
S_1	PDE	Errors(100 Basis Points)				
38	3.531	-2.430	-3.540	-1.140	-0.840	-0.520
40	4.467	-0.720	-3.410	-2.660	-0.610	-0.940
45	7.313	1.070	-6.230	0.100	-0.160	-0.700
50	10.800	-5.750	4.790	2.570	1.280	0.630

Table A.10: Improved Monte Carlo Simulation for American Spread Options-Standard Basis Functions with Cross Product Term

S_1	PDE	Dual method				
		10000	20000	50000	100000	200000
30	0.997	1.000 (0.001)	1.000 (0.001)	0.999 (0.001)	1.000 (0.000)	1.001 (0.000)
32	1.452	1.456 (0.002)	1.458 (0.001)	1.458 (0.001)	1.458 (0.001)	1.459 (0.000)
34	2.023	2.030 (0.002)	2.033 (0.001)	2.034 (0.001)	2.034 (0.001)	2.035 (0.000)
36	2.716	2.728 (0.002)	2.732 (0.002)	2.733 (0.001)	2.733 (0.001)	2.733 (0.001)
38	3.531	3.551 (0.003)	3.557 (0.002)	3.557 (0.001)	3.557 (0.001)	3.558 (0.001)
40	4.467	4.500 (0.004)	4.506 (0.003)	4.506 (0.002)	4.507 (0.001)	4.507 (0.001)
45	7.313	7.397 (0.005)	7.406 (0.004)	7.404 (0.002)	7.405 (0.002)	7.406 (0.001)
50	10.800	10.978 (0.007)	10.986 (0.005)	10.985 (0.003)	10.985 (0.002)	10.985 (0.002)
S_1	PDE	Errors(100 Basis Points)				
30	0.997	-0.260	-0.260	-0.230	-0.300	-0.340
32	1.452	-0.400	-0.540	-0.530	-0.590	-0.630
34	2.023	-0.670	-0.990	-1.030	-1.070	-1.110
36	2.716	-1.220	-1.670	-1.690	-1.730	-1.760
38	3.531	-2.030	-2.610	-2.600	-2.660	-2.700
40	4.467	-3.240	-3.890	-3.890	-3.960	-3.980
45	7.313	-8.380	-9.240	-9.090	-9.200	-9.240
50	10.800	-17.840	-18.650	-18.510	-18.530	-18.530

Table A.11: Dual method for American Spread Options

S_1	PDE	Dual method				
		10	50	100	500	1000
30	0.997	1.077 (0.101)	1.032 (0.035)	1.016 (0.015)	0.997 (0.005)	0.996 (0.003)
32	1.452	1.548 (0.128)	1.483 (0.039)	1.477 (0.019)	1.453 (0.006)	1.451 (0.004)
34	2.023	2.122 (0.148)	2.060 (0.044)	2.064 (0.024)	2.029 (0.008)	2.023 (0.006)
36	2.716	2.808 (0.165)	2.767 (0.050)	2.764 (0.030)	2.729 (0.011)	2.719 (0.007)
38	3.531	3.607 (0.178)	3.591 (0.058)	3.584 (0.036)	3.555 (0.013)	3.546 (0.009)
40	4.467	4.531 (0.190)	4.543 (0.067)	4.527 (0.042)	4.508 (0.016)	4.494 (0.011)
45	7.313	7.321 (0.218)	7.433 (0.085)	7.412 (0.056)	7.417 (0.023)	7.385 (0.016)
50	10.800	10.891 (0.246)	10.993 (0.102)	11.032 (0.066)	10.996 (0.030)	10.966 (0.021)
S_1	PDE	Errors(100 Basis Points)				
30	0.997	-7.990	-3.470	-1.910	0.000	0.100
32	1.452	-9.540	-3.090	-2.500	-0.080	0.180
34	2.023	-9.900	-3.660	-4.040	-0.560	0.070
36	2.716	-9.240	-5.090	-4.820	-1.320	-0.370
38	3.531	-7.660	-6.000	-5.350	-2.430	-1.490
40	4.467	-6.400	-7.580	-5.950	-4.020	-2.650
45	7.313	-0.770	-11.970	-9.870	-10.390	-7.170
50	10.800	-9.150	-19.270	-23.210	-19.600	-16.620

Table A.12: Dual method for American Spread Options-Small Number of Paths

S_1	FFT	Explicit Finite Difference					
		30×100	40×200	50×300	60×400	70×600	80×800
30	0.990	1.321	1.066	1.006	0.991	0.986	0.984
32	1.428	1.989	1.519	1.366	1.432	1.446	1.414
34	1.977	2.710	2.230	1.915	1.904	1.979	1.982
36	2.642	3.469	3.019	2.696	2.534	2.599	2.617
38	3.423	4.259	3.850	3.551	3.349	3.374	3.392
40	4.320	5.092	4.724	4.453	4.270	4.218	4.266
45	7.033	7.861	7.519	7.258	7.087	7.000	6.921
50	10.325	11.307	10.921	10.642	10.456	10.362	10.296
S_1	FFT	Errors(100 Basis Points)					
30	0.990	33.080	7.590	1.580	0.140	-0.450	-0.560
32	1.428	56.050	9.060	-6.230	0.350	1.810	-1.380
34	1.977	73.290	25.270	-6.230	-7.320	0.140	0.450
36	2.642	82.680	37.710	5.450	-10.750	-4.320	-2.510
38	3.423	83.540	42.700	12.790	-7.440	-4.980	-3.140
40	4.320	77.140	40.400	13.310	-5.000	-10.210	-5.420
45	7.033	82.840	48.670	22.570	5.470	-3.320	-11.220
50	10.325	98.250	59.580	31.740	13.080	3.770	-2.830

Table B.1: Explicit Finite Difference Method for European Spread Options with Stochastic Volatility

S_1	FFT	Explicit Finite Difference					
		30×100	40×200	50×300	60×400	70×600	80×800
30	0.990	1.370	1.100	1.029	1.017	1.014	1.009
32	1.428	2.051	1.576	1.405	1.468	1.485	1.455
34	1.977	2.794	2.298	1.978	1.964	2.036	2.039
36	2.642	3.581	3.111	2.779	2.614	2.685	2.704
38	3.423	4.415	3.979	3.665	3.455	3.486	3.510
40	4.320	5.314	4.918	4.620	4.424	4.370	4.424
45	7.033	8.270	7.888	7.596	7.403	7.310	7.225
50	10.325	11.918	11.480	11.199	11.007	10.911	10.839
S_1	FFT	Early Exercise Premium(100 Basis Points)					
30	0.990	37.990	10.980	3.870	2.660	2.370	1.850
32	1.428	62.290	14.780	-2.320	3.930	5.670	2.680
34	1.977	81.700	32.120	0.030	-1.330	5.910	6.170
36	2.642	93.900	46.910	13.750	-2.790	4.310	6.200
38	3.423	99.170	55.580	24.190	3.120	6.260	8.640
40	4.320	99.410	59.810	30.010	10.400	4.960	10.350
45	7.033	123.690	85.570	56.370	37.040	27.720	19.180
50	10.325	159.350	115.530	87.410	68.260	58.640	51.430

Table B.2: Explicit Finite Difference Method for American Spread Options with Stochastic Volatility

S_1	PDE	Monte Carlo Simulation for American Spread				
		10000	20000	50000	100000	200000
30	1.009	0.997 (0.038)	0.996 (0.027)	0.963 (0.016)	0.978 (0.012)	0.988 (0.008)
32	1.455	1.442 (0.047)	1.439 (0.034)	1.402 (0.020)	1.418 (0.015)	1.435 (0.010)
34	2.039	1.999 (0.057)	1.993 (0.040)	1.962 (0.024)	1.970 (0.018)	1.993 (0.012)
36	2.704	2.679 (0.067)	2.654 (0.047)	2.625 (0.029)	2.637 (0.021)	2.664 (0.015)
38	3.510	3.470 (0.078)	3.450 (0.054)	3.408 (0.033)	3.426 (0.024)	3.458 (0.017)
40	4.424	4.377 (0.089)	4.360 (0.062)	4.314 (0.038)	4.333 (0.028)	4.362 (0.019)
45	7.225	7.071 (0.115)	7.093 (0.081)	7.047 (0.050)	7.061 (0.036)	7.104 (0.026)
50	10.839	10.375 (0.142)	10.402 (0.100)	10.371 (0.062)	10.354 (0.044)	10.415 (0.031)
S_1	PDE	Errors(100 Basis Points)				
30	1.009	1.130	1.300	4.600	3.070	2.080
32	1.455	1.260	1.630	5.260	3.700	2.050
34	2.039	4.020	4.630	7.680	6.900	4.570
36	2.704	2.510	5.040	7.900	6.700	3.960
38	3.510	4.030	5.990	10.140	8.340	5.210
40	4.424	4.680	6.340	10.990	9.030	6.150
45	7.225	15.340	13.190	17.730	16.320	12.070
50	10.839	46.360	43.670	46.790	48.470	42.380

Table B.3: Monte Carlo Simulation for American Spread Options

S_1	PDE	Monte Carlo Simulation for American Spread				
		10000	20000	50000	100000	200000
38	3.510	3.564 (0.070)	3.523 (0.050)	3.485 (0.030)	3.475 (0.022)	3.482 (0.015)
40	4.424	4.485 (0.079)	4.453 (0.055)	4.414 (0.033)	4.380 (0.024)	4.397 (0.017)
45	7.225	7.351 (0.096)	7.269 (0.067)	7.208 (0.041)	7.190 (0.029)	7.221 (0.021)
50	10.839	10.907 (0.105)	10.722 (0.073)	10.704 (0.047)	10.668 (0.033)	10.684 (0.023)
S_1	PDE	Errors(100 Basis Points)				
38	3.510	-5.410	-1.350	2.450	3.480	2.760
40	4.424	-6.170	-2.890	0.920	4.410	2.680
45	7.225	-12.600	-4.440	1.660	3.420	0.350
50	10.839	-6.790	11.720	13.460	17.130	15.500

Table B.4: Improved Monte Carlo Simulation for American Spread Options

S_1	PDE	Dual method				
		10	50	100	500	1000
30	1.009	0.941 (0.034)	1.012 (0.012)	0.997 (0.005)	1.013 (0.006)	1.012 (0.004)
32	1.455	1.348 (0.075)	1.467 (0.017)	1.443 (0.008)	1.466 (0.007)	1.466 (0.005)
34	2.039	1.979 (0.030)	2.035 (0.022)	2.004 (0.012)	2.038 (0.009)	2.036 (0.006)
36	2.704	2.672 (0.027)	2.730 (0.027)	2.682 (0.015)	2.737 (0.011)	2.728 (0.008)
38	3.510	3.477 (0.052)	3.549 (0.033)	3.489 (0.019)	3.561 (0.014)	3.549 (0.010)
40	4.424	4.396 (0.084)	4.494 (0.041)	4.440 (0.025)	4.506 (0.017)	4.496 (0.011)
45	7.225	7.279 (0.113)	7.425 (0.070)	7.357 (0.045)	7.392 (0.023)	7.385 (0.016)
50	10.839	10.814 (0.140)	11.057 (0.093)	10.994 (0.060)	10.996 (0.029)	10.958 (0.021)
S_1	PDE	Errors(100 Basis Points)				
30	1.009	6.760	-0.340	1.150	-0.490	-0.340
32	1.455	10.690	-1.190	1.160	-1.090	-1.090
34	2.039	5.970	0.430	3.470	0.080	0.290
36	2.704	3.160	-2.600	2.170	-3.330	-2.440
38	3.510	3.310	-3.960	2.080	-5.160	-3.900
40	4.424	2.770	-7.030	-1.630	-8.280	-7.190
45	7.225	-5.450	-20.000	-13.200	-16.750	-16.060
50	10.839	2.460	-21.750	-15.470	-15.690	-11.920

Table B.5: Dual Method for American Spread Options with Stochastic Volatility

BIBLIOGRAPHY

[1] CME Group. Introduction to crack spreads. pages 1–16, 2012.

[2] D.C. Shimko. Options on futures spreads: Hedging, speculation, and valuation. *Journal of Futures Markets*, 14(2):183–213, 1994.

[3] Damiano Brigo and Fabio Mercurio. *Interest rate models—theory and practice: With smile, inflation and credit*. Springer Finance. Springer-Verlag, Berlin, second edition, 2006.

[4] M. Garman. Spread the load. *Risk*, 5(11):68–84, 1992.

[5] G. Poitras. Spread options, exchange options, and arithmetic brownian motion. *Journal of Futures Markets*, 18(5):487–517, 1998.

[6] René Carmona and Valdo Durrleman. Pricing and hedging spread options. *SIAM Rev.*, 45(4):627–685 (electronic), 2003.

[7] N.D. Pearson. An efficient approach for pricing spread options. *The Journal of Derivatives*, 3(1):76–91, 1995.

[8] R. Carmona and V. Durrleman. Pricing and hedging spread options. *Siam Review*, 45(4):627–685, 2003.

[9] M. Dempster and S. Hong. Pricing spread options with the fast fourier transform. In *Mathematical Finance, Bachelier Congress. Springer-Verlag, Berlin*, pages 203–220, 2000.

[10] C. Alexander and A. Venkatramanan. Analytic approximations for spread options. *Available at SSRN 1012521*, 2007.

[11] W. Margrabe. The value of an option to exchange one asset for another. *The Journal of Finance*, 33(1):177–186, 1978.

[12] E. Kirk. Correlation in the energy markets. *Managing energy price risk*, pages 71–78, 1995.

[13] S. G. Hong. *Pricing and hedging of spread options with stochastic component correlation.* 2001. Ph.D. Thesis.

[14] John C. Hull. *Options, Furtures, and other Deritives*. Perason Prentice Hall, 7 edition, 2009.

[15] R. Cont and P. Tankov. *Financial modelling with jump processes*. Chapman & Hall/CRC, 2003.

[16] S.L. Heston. A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of financial studies*, 6(2):327–343, 1993.

[17] Paul Wilmott, Sam Howison, and Jeff Dewynne. *The mathematics of financial derivatives. A student introduction*. Cambridge University Press, Cambridge, 1995.

[18] J. N. Tsitsiklis and B. Van Roy. Optimal stopping of markov processes: Hilbert space theory, approximation algorithms, and an application to pricing high-dimensional financial derivatives. *Automatic Control, IEEE Transactions on*, 44(10):1840–1851, 1999.

[19] F. A. Longstaff and E. A. Schwartz. Valuing american options by simulation: A simple least-squares approach. *Review of Financial studies*, 14(1):113–147, 2001.

[20] L. Stentoft. Convergence of the least squares monte carlo approach to american option valuation. *Management Science*, pages 1193–1203, 2004.

[21] L. C. G. Rogers. Monte Carlo valuation of American options. *Math. Finance*, 12(3):271–286, 2002.

[22] M.B. Haugh and L. Kogan. Pricing american options: a duality approach. *Operations Research*, 52(2):258–270, 2004.

[23] Steven E. Shreve. *Stochastic calculus for finance. II Continuous-time models*. Springer Finance. Springer-Verlag, New York, 2004.

[24] B. Øksendal. *Stochastic differential equations: an introduction with applications*. Springer, 2010.

[25] Lishang Jiang. *Mathematical modeling and methods of option pricing*. World Scientific Publishing Co. Inc., River Edge, NJ, 2005. Translated from the 2003 Chinese original by Canguo Li.

[26] J. Vecer. A new pde approach for pricing arithmetic average asian options. *Journal of Computational Finance*, 4(4):105–113, 2001.

[27] M. S. Joshi. *The concepts and practice of mathematical finance*. Mathematics, Finance and Risk. Cambridge University Press, Cambridge, second edition, 2008.

[28] J. Hull and A. White. One-factor interest-rate models and the valuation of interest-rate derivative securities. *Journal of financial and quantitative analysis*, 28(2), 1993.

[29] J.C. Cox, J.E. Ingersoll Jr, and S.A. Ross. An intertemporal general equilibrium model of asset prices. *Econometrica: Journal of the Econometric Society*, pages 363–384, 1985.

[30] D. Heath, R. Jarrow, and A. Morton. Bond pricing and the term structure of interest rates: A new methodology for contingent claims valuation. *Econometrica: Journal of the Econometric Society*, pages 77–105, 1992.

[31] C. Alexander and A. Scourse. Bivariate normal mixture spread option valuation. *Quantitative Finance*, 4(6):637–648, 2004.