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Dynamics for a Random Differential Equation: Invariant Manifolds, Foliations, and Smooth Conjugacy Between Center Manifolds

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ABSTRACT

Dynamics for a Random Differential Equation: Invariant Manifolds, Foliations, and Smooth Conjugacy Between Center Manifolds

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In this dissertation, we first prove that for a random differential equation with the multiplicative driving noise constructed from a Q-Wiener process and the Wiener shift, which is an approximation to a stochastic evolution equation, there exists a unique solution that generates a local dynamical system. There also exist a local center, unstable, stable, centerunstable, center-stable manifold, and a local stable foliation, an unstable foliation on the center-unstable manifold, and a stable foliation on the center-stable manifold, the smoothness of which depend on the vector fields of the equation. In the second half of the dissertation, we show that any two arbitrary local center manifolds constructed as above are conjugate. We also show the same conjugacy result holds for a stochastic evolution equation with the multiplicative Stratonovich noise term as $u \circ dW$.

Keywords: Wiener process, Wong-Zakai approximations, multiplicative noise, random dynamical systems, invariant manifolds, foliations, conjugacy

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Contents

Chapter 1. Introduction

The theory of random dynamical systems is devoted to studying how the states in a system evolve when affected by randomness. Typical examples for random dynamical systems are from the solution operators of random differential equations or stochastic differential equations. These equations arise in the modeling of many phenomena in physics, biology, climatology, economics, etc., which are often subject to uncertainty or random influences, such as stochastic forcing, uncertain parameters, random sources or inputs, and random initial and boundary conditions. For random dynamical systems generated from such equations, the stability and long term behavior of solutions are fundamental problems in the theory.

In this dissertation, we are concerned about these problems and study the dynamics for a class of random differential equations, which provides Wong-Zakai approximations of the stochastic evolution equations driven by a nonlinear multiplicative white noise in a separable Hilbert space H:

$$\mathrm{d}u = (Au + f(u))\mathrm{d}t + g(u) \circ \mathrm{d}W, \qquad u(0) = x \in H, \tag{1.1}$$

where A is a linear operator generating a strongly continuous semigroup on H, $f : H \to H$ and $g : H \to L(H_0, H)$ are nonlinear functions with H_0 being a Hilbert space, and $W(t, \omega)$ is the standard H_0-valued Wiener process of trace class, while $\circ\mathrm{d}W$ is interpreted as the Stratonovich stochastic differential.

A major difficulty in studying the sample-wise (or pathwise) dynamics of equation (1.1) is that one does not know if it generates a random dynamical system, which is a long standing open problem in the theory of random dynamical systems (see, e.g., Flandoli [FL95], Garrido-Atienza-Lu-Schmalfuss [GLS16]). So far, the pathwise dynamics such as random attractors and random invariant manifolds for equation (1.1) have been established only when g is either u and W is one-dimensional or independent of u. In the nonlinear case, some progress on the existence of unstable manifolds and random attractors has been made for a class of stochastic partial differential equations driven by a fractional Brownian motion with Hurt parameter $H > 1/2$ by using rough path analysis, see Garrido-Atienza-Lu-Schmalfuss [GLS10] and Gao-Garrido-Atienza-Schmalfuss [GGS14].

Due to the difficulty of directly dealing with stochastic partial differential equation (1.1), we propose to study the Wong-Zakai approximations of equation (1.1) by using a stationary process via the Wiener shift. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $Q : H_0 \to H_0$ be a nonnegative self-adjoint trace class operator. We consider an H_0-valued Q-Wiener process given by

$$W(t,\omega) = \sum_{k=1}^{+\infty} \psi_k(t,\omega)\sqrt{\lambda_k} e_k,$$

where $\{e_k\}$ is a complete orthonormal basis of H_0, $\lambda_k = \langle Qe_k, e_k\rangle$, and $\{\psi_k\}$ is a family of real-valued Wiener processes in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that for any $t_1 < t_2 < \cdots < t_n$, and any finite sub-collection $\{\psi_{k_1}, \ldots, \psi_{k_m}\}$,

$$\{\psi_{k_i}(t_j) - \psi_{k_i}(t_{j-1}) \mid i = 1, \ldots, m,\ j = 2, \ldots, n\}$$

is independent. Without loss of generality, we may identify each sample $\omega \in \Omega$ with the corresponding continuous sample path $W(t,\omega) \in C_0(\mathbb{R}, H_0)$. For simplicity, we also write $\omega(t) = W(t,\omega)$. Consider the Wiener shift θ_t defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ by

$$\theta_t\omega(\cdot) = \omega(t + \cdot) - \omega(t).$$

It is known that the probability measure $\mathbb{P}$ is an ergodic invariant measure for θ_t, and that $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ forms a metric dynamical system, see Arnold [AR98]. For each $\delta > 0$, let $\mathcal{G}_\delta : \Omega \to H_0$ denote the H_0 valued random variable

$$\mathcal{G}_\delta(\omega) = \frac{1}{\delta}\omega(\delta).$$

Then we have

$$\mathcal{G}_\delta(\theta_t\omega) = \frac{1}{\delta}(\omega(t+\delta) - \omega(t)). \tag{1.2}$$

From the properties of Brownian motions, it follows that $\mathcal{G}_\delta(\theta_t\omega)$ is a stationary stochastic process with a normal distribution and is unbounded in t for almost all ω. $\mathcal{G}_\delta(\theta_t\omega)$ may be viewed as an approximation of white noise in the sense

$$\lim_{\delta\to 0^+} \sup_{t\in[0,T]} \left| \int_0^t \mathcal{G}_\delta(\theta_s\omega)ds - \omega(t) \right| = 0, a.s.$$

for each $T > 0$, see [SLWZ17].

The approximation (1.2) of a white noise was studied by Lu and Wang in [LW11] in which they considered the following equation driven by it:

$$u_t = f(u) + g(u)\mathcal{G}_\delta(\theta_t\omega). \tag{1.3}$$

Assuming this equation with only drift term has a homoclinic orbit to a saddle fixed point, they proved that if the diffusion term $g(u)$ is not completely tangent to the homoclinic orbit, then for almost all sample paths of the Brownian motion, the forced equation (1.3) admits a topological horseshoe of infinitely many branches, thus is chaotic. The chaotic behavior of the same kind of equation with a heteroclinic loop was investigated by Shen, Lu, and Zhang in [SLZ13]. Lu and Wang [LW17] proved the existence of random attractors for dissipative parabolic equations driven by nonlinear multiplicative noise $g(u)\mathcal{G}_\delta(\theta_t\omega)$ and showed that the attractors converge to the attractor of the corresponding stochastic parabolic equation driven by the white noise when g is either u and W is one-dimensional or g is independent of u. Similar problems were also studied in Wang-Lu-Wang [WLW18]. Recently, Shen and Lu [SL17] studied equation (1.3) driven by an n-dimensional Brownian motion and proved that the solutions of equation (1.3) converge in the mean square to the solutions of the corresponding Stratonovich stochastic differential equation:

$$du = f(u)\mathrm{d}t + g(u) \circ \mathrm{d}W(t, \omega).$$

They also showed that for a simple multiplicative noise, the center-manifold of the Wong-Zakai approximations converges to the center-manifold of the Stratonovich stochastic differential equation. Shen, Lu, Wang and Zhao [SLWZ17] also proved that the solutions of Wong-Zakai approximations almost surely converge to the solutions of the Stratonovich stochastic evolution equation:

$$du = (Au + F(u))\mathrm{d}t + u \circ \mathrm{d}W(t, \omega).$$

They also showed that the invariant manifolds and stable foliations of the Wong-Zakai approximations converge to the invariant manifolds and stable foliations of the Stratonovich stochastic evolution equation, respectively.

The method of using deterministic differential equations to approximate stochastic differential equations was introduced by Wong and Zakai in their pioneering work [WZ65, WZ652] in which they studied both piecewise linear approximations and piecewise smooth approximations for one-dimensional Brownian motions. Their work was later extended to stochastic differential equations of higher dimensions. This was done, for example, by McShane [MS72], Stroock-Varadhan [SV72], Sussmann [SU77, SU78], Ikeda-Nakao-Yamato [INY77], Ikeda-Watanabe [IW89], and recently by Kelly-Melbourne [KM16], and Shen-Lu [SL17] in which the same approximations as in this paper were studied. The results of the Wong-Zakai approximations have also been generalized to stochastic differential equations driven by martingales and semimartingales, see for example, Nakao-Yamato [NY76], Konecny [KO83], Protter [PR85], Nakao [NA86], and Kurtz-Protter [KP91, KP91]. There are also a large number of publications on Wong-Zakai approximations of solutions for stochastic partial differential equations, see for example, Brzezniak-Capinski-Flandoli [BCF88], Gyongy [GY88, GY89], Twardowska [TW91, TW92, TW95, TW96], Bally-Millet-Sanz-Sole [BMS95], Brzezniak-Flandoli [BF95], Grecksch-Schmalfuss [GS06], Gyongy-Shmatkov [GS06], Nowak [NO06], Tessitore-Zabczyk [TZ06], Deya-Jolis-Quer-Sardanyo [DJQ13], Ganguly [GA13], and Hairer-Pardou [HP15].

In this dissertation, we consider the following Wong-Zakai approximation of equation (1.1) driven by a nonlinear multiplicative noise of $\mathcal{G}_\delta(\theta_t\omega)$, the stationary stochastic process given in (1.2):

$$u_t = Au + f(u) + g(u)\mathcal{G}_\delta(\theta_t\omega), \qquad u(0) = x \in H, \tag{1.4}$$

As a consequence, this approximated equation generates a random dynamical system. Thus one can study its sample-wise (or pathwise) dynamical properties. Among the most useful properties of dynamical systems, the invariant manifolds and their invariant foliations near an equilibrium or a periodic orbit are essential structures for describing and understanding dynamical behavior of nonlinear and random systems.

The theory of invariant manifolds dates back to the work by Hadamard [HA01], then, Lyapunov [LY47] and Perron [PE28] who used a different approach. Hadamard's graph transform method is a geometric approach, while Lyapunov-Perron method is analytic in nature. Since then, there has been an extensive literature on the stable, unstable, center, center-stable and center-unstable manifolds for both finite and infinite dimensional deterministic autonomous dynamical systems, see Pliss [PL64], Kelley [KE67], Hale [HA69], Henry [HE06], Carr [CA81], Vanderbauwhede-Van Gils [VV87], Chow-Lu [CL88, CL882], Bates-Jones [BJ89], Chow-Lin-Lu [CLL91], Chicone-Latushkin [CL97], and the references therein. When the system is given by stochastic or random differential equations, there are results for finite dimensional systems by Wanner [WA95], Arnold [AR98], Mohammed-Scheutzow [MS99] and Schmalfuss [SC98], while for infinite dimensional systems, some results can be found, for example, in Caraballo-Langa-Robinson [CLR01], Koksch-Siegmund [KS02], Duan-Lu-Schmalfuss [DLS03, DLS04], Mohammed-Zhang-Zhao [MZZ08], and Caraballo-Duan-Lu-Schmalfuss [CDLS10].

The theory of invariant foliations for deterministic dynamical systems dates back to the work by Fenichel [FE71, FE74, FE77], Hirsch-Pugh-Shub [HPS77], and Pesin [PE77]. Later works in this area can be found in Ruelle [RU82], Chow-Lin-Lu [CLL91], Bates-Lu-Zeng [BLZ98, BLZ99, BLZ00], and the references therein as well. In the case of random dynamical systems, Pesin's result was established by Liu and Qian [LQ95] for finite dimensional random dynamical systems, and by Lian and Lu [LL10] for the infinite dimensional case. The local theory of invariant foliations for stochastic partial differential equations was obtained by Lu and Schmalfuss [LS08]. Recently, Li, Lu and Bates [LLB14] proved the existence of invariant foliations of stable and unstable manifolds of a normally hyperbolic random invariant manifold, which extends Fenichel's results to finite dimensional random dynamical systems.

In this current dissertation, we prove that under certain conditions for $A,\ f,\ g$, equation (1.4) admits a smooth local center, unstable, stable, center-unstable, center-stable manifolds, a smooth local stable foliation, an unstable foliation on the center-unstable manifold, and a stable foliation on the center-stable manifold, see Theorem 3.2 and Theorem 3.3. The approach follows the standard Lyapunov-Perron technique involving the variation of constants formula. However to handle the unboundedness of the driving noise $\mathcal{G}_\delta(\omega)$, we need to use some cut-off method as in Caraballo-Duan-Lu-Schmalfuss [CDLS10], which truncates the original equation, and carefully adjust the local tempered random region we pick, so that we can construct invariant manifolds and foliations for the truncated equation and then pass it to a local result for the original system. We refer the reader to Duan-Lu-Schmalfuss [DLS04] and Chow-Lin-Lu [CLL91] for the technique we use to prove the smoothness of such structures.

In the second half part of this dissertation, we make use of the structures constructed in the first part to study a conjugacy problem for center manifolds. As suggested above, the standard method for constructing a local center manifold at a given equilibrium point is to extend the locally defined equation by a cut-off function to a globally defined one for which existence and smoothness of a unique global center manifold can be established by either Hadamard's or Perron's method. However, Sijbrand [SJ85] showed the nonuniqueness of local center manifolds resulting from the use of arbitrary cut-off functions in the construction. Bates and Jones [BJ89] proved that under certain conditions there is an exceptional case. There is little doubt that the dynamics on different local center manifolds should behave the same, but the question is then in what sense or to what degree that is so. This question has attracted a good deal of attention in the literature since the birth of the theory, and some results, which we cannot give a complete account of here, can be summarized as follows. (1) Any local center manifold of a given equilibrium point must contain all the invariant sets, such as equilibrium points, periodic, homoclinic, heteroclinic orbits, etc. near the equilibrium point; (2) the formal Taylor expansions at the equilibrium point of the vector field when restricted to different local center manifolds are exactly the same (see, e.g., Carr [CA81], Sijbrand [SJ85]). Burchard, Deng and Lu [BDL92] proved from the standpoint of smooth conjugacy that the restrictions of the equation to two arbitrary local center manifolds are actually topologically or differentiably conjugate, depending on the smoothness of the vector field. That is, the smooth conjugacy class of the restricted equations is indeed unique.

In this dissertation, we follow the geometrical proof in [BDL92] to show that the same result holds for our random differential equation (1.4) as well, provided that we put some restriction on the nonlinear term (see Hypothesis 3.3.1), which is concluded in Theorem 3.4. We also show that under a certain condition on the drift term (see Hypothesis 3.3.2), the same conjugacy result holds for a stochastic evolution equation with the multiplicative Stratonovich noise:

$$u_t = Au + f(u_{us}) + u \circ \mathrm{d}W,$$

where the driving noise W is a real-valued Wiener process.

We remark at last that invariant foliation theory has been applied to conjugacy problems by many people. For example, Anosov [AN69], Palis [PA69], Palis and Smale [PS69], and Robinson [RO75] used it to analyze structural stability of finite dimensional dynamical systems. Palis and Takens [PT77] used it to prove a result which implies that two differential equations are locally topologically conjugate if the equations when restricted to their center manifolds are topologically conjugate. Lu [LU91, LU94] used the infnite dimensional counterpart to generalize the Hartman-Grobman theorem to parabolic equations. Wanner [WA95] established the existence of invariant foliations for finite dimensional random dynamical systems in a neighborhood of a stationary solution and used the foliations to prove a Hartman-Grobman theorem for finite-dimensional random dynamical systems. Li and Lu [LL05] proved a stable and unstable foliation theorem and used it to establish a smooth linearization theorem (Sternberg type of theorem) for finite dimensional random dynamical systems. It is also very useful in other areas of study of dynamical systems. In fact, it is one of the key components for the geometric theory of singular perturbations of Fenichel [FE79] and its applications, cf. e.g., Deng [DE91]. It also plays an important role in the theory of homoclinic and heteroclinic bifurcations of Chow-Lin [CL90] and Deng [DE90].

Non-technical Overview:

Step 1: As the equation is defined in a local neighborhood of the origin, it is not guaranteed that a global solution exists for all $t \geq 0$ when starting from an arbitrary initial point in the region. However, it is necessary to have a solution existing for all time for one to look at the long-term behavior of the system. Therefore, we first introduce a cut-off technique in section 3.2, which helps establish a truncated equation that agrees with the original one within a small region, but admits better properties, as given in section 3.2. One thing to mention is that because of the non-uniform boundedness of the random driving term, the small region we choose needs to depend on ω.

Step 2: Once a truncated equation is given with a properly chosen small region we pick in the cut-off step, we prove that there is a unique solution to the truncated equation which exists for all $t \geq 0$. This is section 4.1. The approach we use is standard and analogous to the deterministic theory, that is to construct a contraction mapping from the corresponding integral equation which is derived from the variation of constants formula. Then we show in section 4.2 by basic probability theory the measurability as well as a cocycle property of the solution, thus proving that such a solution generates a random dynamical system.

Step 3: To show the existence of invariant manifolds, we follow the Lyapunov-Perron's approach. In section 5.1, we first identify such manifolds with some initial values, the solution starting from which has a proper exponential growth rate. Then we re-formulate the integral equation (given by the variation of constants formula) by using properties of the semigroup e^{At} generated by A. And we show in section 5.2 that the invariant manifolds are identified by solutions to such re-formulated integral equations. Again, as before, existence of such solutions is shown in section 5.3 by using the contraction mapping theorem on a proper continuous function space, where the metric on this space has the proper exponential growth rate. When considering the smoothness, we formally differentiate the integral equation and justify that it indeed provides the desired derivative. Conclusions on higher order regularity are given by an induction argument, where an extra spectrum gap is required. Then we prove the main theorem involving invariant manifolds in section 5.4 by restricting the global results we get on the truncated equation back to a local result for the original system.

Step 4: To show the existence of foliations, we identify its leaves still by some initial values. But this time we require that the difference between solutions starting from such initial values and a given solution has a proper exponential growth rate. Again as above, we solve a re-formulated integral equation to construct the foliation leaves for the truncated equation, and restrict them to get the local structure. This is chapter 6.

Step 5: Next we consider the conjugacy problem. Following Burchard-Deng-Lu [BDL92]'s approach, we use foliations to construct connections between local center manifolds as graphs. We prove in section 7.1 a certification for a local graph to be a invariant manifold. In section 7.2 and 7.3, using the certification and some cut-off technique, we extend an arbitrary local center manifold to a new system given by a truncated but global equation, and show that the original local center manifold is indeed contained in the unique global center manifold of the truncated equation. This equips every local center manifold with a proper foliation structure nearby. Using such structure, we show in section 7.4 that if two local center manifolds happen to share a common center-stable or center-unstable manifold, then the corresponding foliation on such manifolds gives the conjugacy. Otherwise we show that the flow structures on the two manifolds can be transformed from one to the other through the flow structure on a third local center manifold which lies in the intersection of a center-stable manifold containing one of the given center manifolds and a center-unstable manifold containing the other.

Step 6: At last in chapter 8, we consider a stochastic evolution equation but with a simpler noise term, say, the multiplicative Stratonovich differential $u \circ \mathrm{d}W$. We use the standard Ornstein-Uhlenbeck process to transfer such an equation to a random differential equation and justify the relationship between conjugacy maps of the two systems, provided one of them exists. Then we finish the proof by concluding that the conjugacy exists for the random differential equation. Although the structure is different, the approach we use is identical to that in previous chapters.

Chapter 2. Preliminaries

In this chapter, we introduce some basic concepts and results for various objects discussed in this dissertation.

2.1 Basic Probability Theory

In this section, we introduce some basic concepts and results in probability theory, any classical probability theory textbook would be a reference.

Definition 2.1.1. *A measure space is a triple* $(\Omega, \mathcal{F}, \mu)$ *consisting of:*

(i) the sample space Ω*, which is an arbitrary non-empty set;*

(ii) the σ*-algebra* $\mathcal{F} \subseteq 2^{\Omega}$*, which is a set of subsets of* Ω*, called events, such that:*

(a) $\emptyset,\ \Omega \in \mathcal{F}$*,*

(b) if $A \in \mathcal{F}$*, then* $A^c \in \mathcal{F}$*,*

(c) if $A_i \in \mathcal{F}$*,* $i = 1, 2, \ldots$*, then* $\cup_{i=1}^{+\infty} A_i \in \mathcal{F}$*;*

(iii) the measure $\mu : \mathcal{F} \to [0, +\infty)$*, which is a function defined on* $\mathcal{F}$ *such that* $\mu(\emptyset) = 0$*, and if* $\{A_i\}_{i=1}^{+\infty} \subseteq \mathcal{F}$ *is a countable collection of pairwise disjoint sets, then*

$$\mu(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{+\infty} \mu(A_i).$$

If $(\Omega, \mathcal{F}, \mathbb{P})$ *is a measure space with* $\mathbb{P}(\Omega) = 1$*, then* $\mathbb{P}$ *is called probability measure and* $(\Omega, \mathcal{F}, \mathbb{P})$ *is called a probability space.*

Remark 2.1.1.

(i) A tuple $(\Omega, \mathcal{F})$ *with (i) and (ii) in above definition satisfied is called a measurable space.*

(ii) If $\mathcal{A} \subseteq 2^{\Omega}$*, then there is a smallest* σ*-algebra* $\sigma(\mathcal{A})$ *with* $\mathcal{A} \subseteq \sigma(\mathcal{A})$*:*

$$\sigma(\mathcal{A}) = \cap_{\mathcal{F} \text{ is a } \sigma\text{-algebra},\ \mathcal{A} \subseteq \mathcal{F}} \mathcal{F}.$$

We say $\sigma(\mathcal{A})$ is the σ-algebra generated by $\mathcal{A}$.

(iii) If Ω is a topological space, then we denote by $\mathcal{B}(\Omega)$ the σ-algebra generated by all of the open sets of Ω, and call it the Borel σ-algebra.

(iv) A measure μ is said to be complete if for any $A \in \mathcal{F}$ with $\mu(A) = 0$, and $B \subseteq A$, then $B \in \mathcal{F}$. If we define

$$\tilde{\mathcal{F}} := \{A \triangle \mathcal{N} \mid A \in \mathcal{F},\ \mathcal{N} \subseteq B \text{ for some } B \in \mathcal{F} \text{ with } \mu(B) = 0\},$$

and define $\bar{\mu} : \tilde{\mathcal{F}} \to [0, 1]$ by $\bar{\mu}(A \triangle \mathcal{N}) = \mu(A)$, then $(\Omega, \tilde{\mathcal{F}}, \bar{\mu})$ is also a measure space, and $\bar{\mu}$ is complete on $\tilde{\mathcal{F}}$. We say $\bar{\mu}$ is the completion of μ, and $\tilde{\mathcal{F}}$ is the completion of $\mathcal{F}$.

Next we introduce connections between measurable spaces.

Definition 2.1.2. *Let $(\Omega, \mathcal{F})$ and $(\Omega', \mathcal{F}')$ be two measurable spaces. A function $f : \Omega \to \Omega'$ is said to be measurable if*

$$f^{-1}(A) := \{\omega \mid f(\omega) \in A\} \in \mathcal{F}, \qquad \forall A \in \mathcal{F}'.$$

We are especially interested in the case that the first space is a probability space.

Definition 2.1.3.

(i) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and $(\Omega', \mathcal{F}')$ be a measurable space. A measurable function $X : \Omega \to \Omega'$ is called a random variable with values in $(\Omega', \mathcal{F}')$.

(ii) The map $\mathbb{P}_X := \mathbb{P} \circ X^{-1}$ defines a probability on $(\Omega', \mathcal{F}')$, it is called the distribution, or law of X. We denote by $X \sim \mu$ if a measure $\mu = \mathbb{P}_X$.

(iii) A family of random variables $(X_i)_{i \in I}$ is called identically distributed if $\mathbb{P}_{X_i} = \mathbb{P}_{X_j}$, for all $i, j \in I$, where I is a subset of the natural numbers.

Remark 2.1.2.

(i) When $(\Omega', \mathcal{F}') = (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, a random variable X with values in $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is called a real random variable. For such X, the map $F_X : x \mapsto P(\{X \leq x\})$ is called the distribution

function of X.

(ii) A typical example of distributions is given in the following. Let $\mu, \sigma \in \mathbb{R}$, X be a real random variable. If

$$F_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{x} e^{-\frac{(\xi-\mu)^2}{2\sigma^2}} \mathrm{d}\xi,$$

then $\mathbb{P}_X := \mathcal{N}(\mu, \sigma^2)$ is called the Gaussian normal distribution with parameters μ and σ^2. X is then said to be normally distributed and we denote this by $X \sim \mathcal{N}(\mu, \sigma^2)$.

Now let E be a Polish space, i.e., it is a separable complete metric space. Let $\mathcal{B}(E)$ denote the Borel sets on E. Further, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and $I \subseteq \mathbb{R}$ be an arbitrary subset.

Definition 2.1.4. *A family of random variables $X = (X_i)_{i \in I}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $(E, \mathcal{B}(E))$ is called a stochastic process with index set I and range E. We also denote this by $X = X(i, \omega)$, for $i \in I$, $\omega \in \Omega$ or $X = X(i)$, for $i \in I$.*

Another important concept in probability theory is independence.

Definition 2.1.5. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $(E, \mathcal{B}(E))$ be a metric space, and $I \subseteq \mathbb{R}$ be a subset.*

(i) A family of events $\{A_i\}_{i \in I} \subseteq \mathcal{F}$ is said to be independent if for any finite subset $\{A_{i_k}\}_{k=1}^n$,

$$\mathbb{P}(\cap_{k=1}^n A_{i_k}) = \Pi_{k=1}^n \mathbb{P}(A_{i_k}). \tag{2.1}$$

(ii) A family of σ-algebras $\{\mathcal{F}_i\}_{i \in I}$, each of which is contained in $\mathcal{F}$, is said to be independent if for any finite subset $\{\mathcal{F}_i\}_{k=1}^n$, and for any $A_{i_k} \in \mathcal{F}_{i_k}$, $k = 1, \ldots, n$, (2.1) holds.

(iii) A family of random variables $\{X_i\}_{i \in I}$ with values in $(E, \mathcal{B}(E))$ is said to be independent if $\{\sigma(X_i)\}_{i \in I}$ is so, where $\sigma(X_i) := \sigma(\{X_i^{-1}(A), A \in \mathcal{B}(E)\})$ is the smallest σ-algebra such that X_i is measurable.

2.2 Wiener Process

We give the definition for Wiener process in this section. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and we first consider the case that the process has values in $\mathbb{R}$.

Definition 2.2.1. *Let $W = W(t)$, $t \in \mathbb{R}^+$ be a real-valued stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$. It is called a Wiener process (or Brownian motion) if the following are satisfied:*

(i) $W(0) = 0$ a.s., by which we mean that $W(0, \omega) = 0$ for any $\omega \in \Omega \setminus N$, where N is a given set with $\mathbb{P}(N) = 0$.

(ii) for any n and $\forall 0 \leq t_1 \leq t_2 \leq \cdots \leq t_n$, the increments $\{W(t_{i+1}) - W(t_i)\}_{i=1}^{n-1}$ as a family of random variables is independent.

(iii) for $\forall t \geq 0,\ \Delta t > 0$, the increment $W(t + \Delta t) - W(t) \sim \mathcal{N}(0, \Delta t)$.

(iv) for $\forall t \geq 0$, $W(t, \omega)$ is continuous for a.s. $\omega \in \Omega$.

It is a classical result that such a stochastic process really exists.

Next we consider the case when the Wiener process takes values in a Hilbert space. Let $(H_0, |\cdot|_{H_0})$ be a separable Hilbert space, with $\langle \cdot, \cdot \rangle$ the inner product, and let $Q : H_0 \to H_0$ be a nonnegative self-adjoint trace class operator. That is for $\{e_k\}$ a complete orthonormal basis of H_0, $\sum_{k=1}^{+\infty} \langle Q e_k, e_k \rangle := \sum_{k=1}^{+\infty} \lambda_k < +\infty$.

Definition 2.2.2. *Let $W = W(t)$, $t \in \mathbb{R}^+$ be a stochastic process with values in $(H_0, \mathcal{B}(H_0))$. It is called a Q-Wiener process if the following are satisfied:*

(i) $W(0) = 0$ a.s.

(ii) for any n and $\forall 0 \leq t_1 \leq t_2 \leq \cdots \leq t_n$, the increments $\{W(t_{i+1}) - W(t_i)\}_{i=1}^{n-1}$ as a family of random variables is independent.

(iii) for $\forall t \geq 0,\ \Delta t > 0$, the increment $W(t + \Delta t) - W(t) \sim \mathcal{N}(0, \Delta t Q)$. That is, for $\forall h \in H_0$, $\langle h, W(t + \Delta t) - W(t) \rangle \sim \mathcal{N}(0, \Delta t \langle Q h, h \rangle)$.

(iv) for $\forall t \geq 0$, $W(t, \omega)$ is continuous for a.s. $\omega \in \Omega$.

To construct such a process, we can take $\{e_k\}$ the complete orthonormal basis of H_0 given above and choose $\{\psi_k\}$ a family of real-valued Wiener processes in the probability space

$(\Omega, \mathcal{F}, \mathbb{P})$ such that for any $t_1 < t_2 < \cdots < t_n$, and any finite sub-collection $\{\psi_{k_1}, \ldots, \psi_{k_m}\}$,

$$\{\psi_{k_i}(t_j) - \psi_{k_i}(t_{j-1}) \mid i = 1, \ldots, m, \; j = 2, \ldots, n\}$$

is independent. We define

$$W(t, \omega) = \sum_{k=1}^{+\infty} \psi_k(t, \omega) \sqrt{\lambda_k} e_k.$$

W is then a desired Q-Wiener process. Similarly one can construct another Wiener process W^* which is independent from W. By setting $\bar{W}(t) := W(t)$ if $t \geq 0$ and $\bar{W}(t) := W^*(-t)$ if $t < 0$, we thus define a two-sided Q-Wiener process $\bar{W}$ whose index set is the whole $\mathbb{R}$. We will assume the Q-Wiener process we take is two-sided in the remainder of this work.

Now let $X \subseteq \Omega$ be a full measure subspace consisting of $\omega \in \Omega$ so that the sample path of such a Q-Wiener process $W(\cdot, \omega)$ is continuous. We consider the classical Wiener space $(C_0(\mathbb{R}, H_0), \mathcal{B}(C_0(\mathbb{R}, H_0)), \mu)$, where $C_0(\mathbb{R}, H_0) = \{\omega \in C(\mathbb{R}, H_0) \mid \omega(0) = 0\}$ is equipped with the open compact topology, $\mathcal{B}(C_0(\mathbb{R}, H_0))$ is the corresponding Borel σ-algebra, μ is the law of the measurable mapping

$$\mathcal{W} : X \to C_0(\mathbb{R}, H_0)$$
$$\omega \mapsto W(\cdot, \omega).$$

We then consider the stochastic process, still denoted by W, on $\mathbb{R} \times C_0(\mathbb{R}, H_0)$ given by

$$W(t)(\omega) = \omega(t), \qquad \omega \in C_0(\mathbb{R}, H_0), \; t \in \mathbb{R}. \tag{2.2}$$

This is a Wiener process in $(C_0(\mathbb{R}, H_0), \mathcal{B}(C_0(\mathbb{R}, H_0)), \mu)$, called the standard Wiener process, see Da Prato [DP06, DZ92].

2.3 Random Dynamical Systems

We introduce the definition for a random dynamical system as well as some related concepts in this section.

Definition 2.3.1. $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ *is called a metric dynamical system if*

(i) the mapping $\theta : \mathbb{R} \times \Omega \to \Omega$ *is* $(\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}, \mathcal{F})$*-measurable;*

(ii) $\theta_0 = id_\Omega$*, the identity on* Ω*,* $\theta_{t+s} = \theta_t \circ \theta_s$ *for all* $t,\ s \in \mathbb{R}$*;*

(iii) $\theta_t\mathbb{P} = \mathbb{P}$ *for all* $t \in \mathbb{R}$*.*

For $\mathbb{T} = \mathbb{Z},\ \mathbb{Z}^+,\ \mathbb{R}$ or $\mathbb{R}^+$, we have the following concept of a random dynamical system.

Definition 2.3.2. *A mapping*

$$\phi : \mathbb{T} \times \Omega \times H \to H, \qquad (t, \omega, x) \mapsto \phi(t, \omega, x)$$

is called a random dynamical system over a metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ *if*

(i) ϕ *is* $(\mathcal{B}(\mathbb{T}) \otimes \mathcal{F} \otimes \mathcal{B}(H), \mathcal{B}(H))$*-measurable;*

(ii) the mapping $\phi(t, \omega) := \phi(t, \omega, \cdot) : H \to H$ *forms a cocycle over* θ_t*:*

$$\begin{aligned} \phi(0, \omega) &= id_H, && \forall \omega \in \Omega, \\ \phi(s + t, \omega) &= \phi(t, \theta_s\omega) \circ \phi(s, \omega), && \forall s,\ t \in \mathbb{T} \text{ and } \forall \omega \in \Omega. \end{aligned} \tag{2.3}$$

ϕ *is called a* C^k *smooth random dynamical system if* ϕ *is a random dynamical system and for each* $(t, \omega) \in \mathbb{T} \times \Omega$ *the mapping*

$$\phi(t, \omega) : H \to H, \qquad x \mapsto \phi(t, \omega)x$$

is C^k*.*

We consider the Wiener shift θ_t defined on the probability space $(C_0(\mathbb{R}, H_0), \mathcal{B}(C_0(\mathbb{R}, H_0)), \mu)$ by

$$\theta_t\omega(\cdot) = \omega(t + \cdot) - \omega(t).$$

It is known that the probability measure μ is an ergodic invariant measure for θ_t. Hence,

$$(C_0(\mathbb{R}, H_0), \mathcal{B}(C_0(\mathbb{R}, H_0)), \mu, (\theta_t)_{t\in\mathbb{R}})$$

forms a metric dynamical system, see Arnold [AR98].

One typical example of a random dynamical system is the solution operator for a random differential equation driven by a real noise:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(\theta_t\omega, x),$$

where $x \in \mathbb{R}^d$, $f : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is a measurable function and $f_\omega(t, \cdot) \equiv f(\theta_t\omega, \cdot) \in L_{loc}(\mathbb{R}, C_b^{0,1})$. Another example is the solution operator for a stochastic differential equation:

$$\mathrm{d}x_t = f_0(x_t)\mathrm{d}t + \sum_{k=1}^{d} f_k(x_t) \circ \mathrm{d}W_t^k,$$

where $x \in \mathbb{R}^d$, f_k, $k = 0, \dots, d$ are smooth vector fields, and $W_t = (W_t^1, \cdots, W_t^d)$ is the standard d-dimensional Brownian motion defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\circ\mathrm{d}W_t^k$ is the Stratonovich differential. Here, $(\Omega, \mathcal{F}, \mathbb{P})$ is the classic Wiener space, i.e., $(\Omega, \mathcal{F}, \mathbb{P}) = (C_0(\mathbb{R}, \mathbb{R}^d), \mathcal{B}(C_0(\mathbb{R}, \mathbb{R}^d)), \mu)$ as given above, for details also see Arnold [AR98].

Another important concept in random dynamical systems is an invariant set. A multifunction $M = \{M(\omega)\}_{\omega\in\Omega}$ of nonempty closed sets $M(\omega)$, $\omega \in \Omega$, contained in the separable Hilbert space $(H, |\cdot|)$ is called a random set if

$$\omega \mapsto \sup_{y\in M(\omega)} |x - y|$$

is a random variable for all $x \in H$.

Definition 2.3.3. *A random set $M(\omega)$ is called an invariant set for a random dynamical system $\phi(t, \omega, x)$ if*

$$\phi(t, \omega, M(\omega)) \subseteq M(\theta_t\omega), \qquad \forall t \in \mathbb{T}.$$

Especially in the case $\mathbb{T} = \mathbb{Z}^+$ or $\mathbb{R}^+$ it is called forward invariant.

2.4 Noise Term as a Stationary Process

For each $\delta > 0$, let $\mathcal{G}_\delta : C_0(\mathbb{R}, H_0) \to H_0$ denote the random variable

$$\mathcal{G}_\delta(\omega) = \frac{1}{\delta}\omega(\delta).$$

Then we have

$$\mathcal{G}_\delta(\theta_t\omega) = \frac{1}{\delta}(\omega(t+\delta) - \omega(t)).$$

From the properties of Wiener process, it follows that $\mathcal{G}_\delta(\theta_t\omega)$ is a stationary stochastic process with a normal distribution and is unbounded in t for almost all ω. $\mathcal{G}_\delta(\theta_t\omega)$ may be viewed as an regular approximation of Wiener process in the sense that

$$\lim_{\delta\to 0^+} \sup_{t\in[0,T]} |\int_0^t \mathcal{G}_\delta(\theta_\tau\omega)\mathrm{d}\tau - \omega(t)|_{H_0} = 0 \qquad \text{a.s.}$$

for each $T > 0$, see [SLWZ17].

To introduce a proper upper bound for $\mathcal{G}_\delta$, we need the following feature for a class of random variables.

Definition 2.4.1 (Tempered random variable)**.**
(i) A random variable $R : \Omega \to (0, +\infty)$ is called tempered with respect to a metric dynamical system θ_t if

$$\lim_{t\to\pm\infty} \frac{1}{t} \log R(\theta_t\omega) = 0 \qquad a.s.$$

(ii) $R : \Omega \to [0, +\infty)$ is called tempered from above if

$$\lim_{t\to\pm\infty} \frac{1}{t} \log^+ R(\theta_t\omega) = 0 \qquad a.s.$$

(iii) $R : \Omega \to (0, +\infty]$ is called tempered from below if $1/R$ is tempered from above.

From the law of logarithms [KL73] we have

$$\lim_{s\to\pm\infty}\frac{|\omega(s)|_{H_0}}{s}=0,\qquad \text{a.s.}$$

Let $\bar{\Omega}\subseteq C_0(\mathbb{R},H_0)$ be the set of full measure on which the above holds. Let

$$C(\omega)=\sup_{s\in\mathbb{Q}}\frac{|\omega(s)|_{H_0}}{|s|+1}, \tag{2.4}$$

where $\mathbb{Q}$ is the set of rational numbers. Since for each s, $\omega(s):C_0(\mathbb{R},H_0)\to\mathbb{R}$ is measurable and the supremum is finite, $C(\omega):\bar{\Omega}\to\mathbb{R}^+$ is a measurable function and

$$|\omega(s)|_{H_0}\leq C(\omega)(|s|+1),\qquad \forall s\in\mathbb{R}.$$

It then follows that

$$C(\theta_t\omega)\leq 2C(\omega)(|t|+1),$$

and consequently it is easily shown that $C(\omega)$ is tempered from above and $C(\theta_t\omega)$ is locally integrable in t. With this we have

$$|\mathcal{G}_\delta(\omega)|_{H_0}\leq\frac{1}{\delta}|\omega(\delta)|_{H_0}\leq\frac{\delta+1}{\delta}C(\omega). \tag{2.5}$$

We now replace $\mathcal{F}$ by the trace algebra

$$\bar{\mathcal{F}}=\{\bar{\Omega}\cap A, A\in\mathcal{F}\}.$$

The probability measure on $\bar{\Omega}$ is the restriction of the Wiener measure to this new σ-algebra, which is also denoted by $\mathbb{P}$. We will restrict our study to this probability space $(\bar{\Omega},\bar{\mathcal{F}},\mathbb{P})$ and still denote it by $(\Omega,\mathcal{F},\mathbb{P})$. In the following, we will always assume that $\omega\in\Omega$.

Chapter 3. Main Results

In this chapter, we first present the hypotheses on our equation, and introduce a cut-off technique that converts the local assumptions to global ones for a truncated equation. Under the hypotheses, we provide precise descriptions for invariant manifolds and foliations, and state our main theorems. Also due to a technical requirement, we will adjust one of the hypotheses before we state the last conjugacy result.

3.1 Hypotheses

We consider equation (1.4) and for simplicity in notation rewrite it as

$$u_t = Au + F(\theta_t\omega, u), \qquad u(0) = x \in H, \tag{3.1}$$

where $F(\omega, \cdot) = f(\cdot) + g(\cdot)\mathcal{G}_\delta(\omega)$, and make the following hypotheses.

Hypothesis 3.1.1.

(1) A is a linear operator from a dense domain $D(A) \subseteq H$ into H, which generates a strongly continuous semigroup denoted by $\{e^{At}\}_{t\geq 0}$.

(2) The spectrum of A, $\sigma(A)$, splits as

$$\sigma(A) = \sigma_u \cup \sigma_c \cup \sigma_s,$$

where

$$\sigma_u = \{\lambda \in \sigma(A) \mid \mathrm{Re}\lambda > 0\},$$

$$\sigma_c = \{\lambda \in \sigma(A) \mid \mathrm{Re}\lambda = 0\},$$

$$\sigma_s = \{\lambda \in \sigma(A) \mid \mathrm{Re}\lambda < 0\},$$

and σ_u, σ_c both consist of only a finite number of isolated eigenvalues, each with a finite *dimensional generalized eigenspace. Let H^i denote the generalized eigenspace corresponding to σ_i, and $P^i : H \to H^i$ denote the corresponding projections, $i = u, c$. Further define $P^s := id_H - P^u - P^c$, then P^s is also a projection with $H^s := P^s H$.*

(3) The restriction $e^{At}|_{R(P^{cu})}$, $t \geq 0$ is an isomorphism of the range $R(P^{cu})$ of P^{cu} onto itself, and we define e^{At} for $t < 0$ as the inverse map.

(4) There exists $\beta_1 > 0$ so that

$$\inf\{|\mathrm{Re}\lambda| \mid \lambda \in \sigma_u \cup \sigma_s\} > \beta_1, \tag{3.2}$$

and for $\forall 0 < \beta_2 < \beta_1$, $\exists K_A \geq 1$ such that

$$\begin{aligned} \|e^{At}P^u\| &\leq K_A e^{\beta_1 t}, \quad \forall t \leq 0, \\ \|e^{At}P^c\| &\leq K_A e^{\beta_2 |t|}, \quad \forall t \in \mathbb{R}, \\ \|e^{At}P^s\| &\leq K_A e^{-\beta_1 t}, \quad \forall t \geq 0. \end{aligned} \tag{3.3}$$

With the decomposition given in Hypothesis 3.1.1 as

$$H = H^u \oplus H^c \oplus H^s, \tag{3.4}$$

we use for $x \in H$, $x = x_u + x_c + x_s$ with $x_i \in H^i$, $i = u, c, s$, and let $|x| = |x_u| + |x_c| + |x_s|$ give the box norm on H. Here on each restricted subspace, the norm is given by the projection of the standard inner product on H. We also simplify this notation and use A_c to denote $P^c \circ A$, etc. Notice that if A generates an analytic semigroup with spectrum decomposition described as in Hypothesis 3.1.1 (2), and there is $\beta_1 > 0$ so that (3.2) holds, then (3) and (4) in Hypothesis 3.1.1 are satisfied naturally, see Henry [HE06]. Let $U \subseteq H$ be a neighborhood of the origin $x = 0$, and let $r \geq 1$, $0 \leq \alpha \leq 1$ be given.

Hypothesis 3.1.2. *$f : U \to H$ is of class $C^{r,\alpha}$, also $f(0) = 0$ and $Df(0) = 0$.*

Hypothesis 3.1.3. *$g : U \to L(H_0, H)$ is of class C^r. Furthermore, there exist $0 < \epsilon_0 < 1$ and $R > 0$, so that*

(1) if $\alpha = 0$ and $r = 1$, there exists $M_0 > 0$, such that $\forall u,\ v \in H$, with $|u|,\ |v| \le R$,

$$|g(u) - g(v)| \le M_0(|u|^{\epsilon_0} + |v|^{\epsilon_0})|u - v|. \tag{3.5}$$

(2) If $\alpha > 0$, there exists $M_{r,R} > 0$, such that $\forall u,\ v \in H$, with $|u|,\ |v| \le R$,

$$\|D^{(r)}g(u) - D^{(r)}g(v)\| \le M_{r,R}(|u|^{\epsilon_0} + |v|^{\epsilon_0})|u - v|^{\alpha}. \tag{3.6}$$

Also $D^{(i)}g(0) = 0$, $i = 0, \dots, r$.

3.2 Cut-off Technique

With Hypotheses 3.1.2 and 3.1.3, the existence of a global solution to equation (3.1) is not guaranteed. However, if we restrict the equation on a properly chosen tempered ball

$$B_{\rho(\omega)} = \{x \in H \mid |x| < \rho(\omega)\}, \tag{3.7}$$

where $\rho(\cdot) : \Omega \to (0, +\infty)$ is a random variable tempered from below and depends on $C(\omega)$ defined in (2.4), then it is possible to study the local dynamical properties of equation (3.1).

To do this, we first introduce a cut-off function. Let $\Gamma : [0, +\infty) \to [0, +\infty)$ be a C^∞ function satisfying

$$\Gamma(t) = 1,\ \forall t \in [0, 1], \qquad \Gamma(t) = 0,\ \forall t \ge 2, \tag{3.8}$$

and $\exists K_\Gamma > 0$,

$$\sup_{0 \le t < +\infty} (|\Gamma(t)| + |\Gamma'(t)| + |\Gamma''(t)| + |\Gamma'''(t)|) \le K_\Gamma < +\infty. \tag{3.9}$$

Let $\rho > 0$ and denote $\Gamma_\rho(t) := \Gamma(\frac{t}{\rho})$. We define a cut-off function in H by

$$\zeta_\rho^H(x) = \Gamma_\rho(|x|), \qquad \forall x \in H. \tag{3.10}$$

Now let $\rho(\cdot) : \Omega \to (0, R/4]$ be a random variable tempered from below which is to be determined, where R is given by Hypothesis 3.1.3. We consider the truncated equation

$$u_t = Au + F_\rho(\theta_t\omega, u), \qquad u(0) = x \in H, \tag{3.11}$$

where $F_\rho(\omega, u) = \zeta^H_{\rho(\omega)}(u)F(\omega, u)$. Under Hypotheses 3.1.2 and 3.1.3, the truncated equation (3.11) coincides locally near the origin with equation (3.1), also the nonlinear term has better properties than that of equation (3.1). We conclude with the following lemma.

Lemma 3.1. *Assume that Hypotheses 3.1.2 and 3.1.3 hold with $r \geq 1$ and $0 \leq \alpha \leq 1$, then we have*

(i) $F_\rho(\omega, u) = F(\omega, u)$ for all $|u| \leq \rho(\omega)$.

(ii) there is a constant M_g so that for $\forall u,\ v \in H$,

$$|F_\rho(\omega, u) - F_\rho(\omega, v)| \leq 3K_\Gamma(\sup_{B_{4\rho(\omega)}} \|Df(\cdot)\| + M_g\rho(\omega)^{\epsilon_0}C(\omega))|u - v|. \tag{3.12}$$

We denote by $LipF_\rho(\omega) := 3K_\Gamma(\sup_{B_{4\rho(\omega)}} \|Df(\cdot)\| + M_g\rho(\omega)^{\epsilon_0}C(\omega))$ the Lipschitz constant for $F_\rho(\omega, \cdot)$.

(iii) Furthermore, there exist constants $M_{LipD^{(i)}F_\rho}$, $i = 1,\ 2,\ \ldots,\ r-1$, such that for $\forall u,\ v \in H$,

$$\|D^{(i)}F_\rho(\omega, u) - D^{(i)}F_\rho(\omega, v)\| \leq (O(1) + M_{LipD^{(i)}F_\rho}\rho(\omega)^{\epsilon_0}C(\omega))|u - v|,$$

for $i = 1,\ 2,\ \ldots,\ r-1$. And for $0 < \alpha \leq 1$, there exists a constant $M_{HolD^{(r)}F_\rho}$, such that for $\forall u,\ v \in H$,

$$\|D^{(r)}F_\rho(\omega, u) - D^{(r)}F_\rho(\omega, v)\| \leq (O(1) + M_{HolD^{(r)}F_\rho}\rho(\omega)^{\epsilon_0}C(\omega))|u - v|^\alpha.$$

We denote by $LipD^{(i)}F_\rho(\omega) := O(1) + M_{LipD^{(i)}F_\rho}\rho(\omega)^{\epsilon_0}C(\omega)$ for $i = 1,\ 2,\ \ldots,\ r-1$, and $HolD^{(r)}F_\rho(\omega) := O(1) + M_{HolD^{(r)}F_\rho}\rho(\omega)^{\epsilon_0}C(\omega)$, the Lipschitz constants for $D^{(i)}F_\rho(\omega, \cdot)$ and *the Hölder constant for $D^{(r)}F_\rho(\omega,\cdot)$, respectively. Here $O(1)$ is some bounded constant as $\rho(\omega)\to 0$.*

Proof. (i) As discussed above, this is directly from the definition of the cut-off functions.
(ii) Because of the cut-off function's property, we only focus on the case that $|u|,\ |v|\le 2\rho(\omega)$, the other cases are included in the inequality achieved under this situation.

$LipF$: Using Hypotheses 3.1.2 and 3.1.3, together with (2.5), we have for $\forall u,\ v\in B_{2\rho(\omega)}$,

$$\begin{aligned}
&|F(\omega,u)-F(\omega,v)|\\
\le&|f(u)-f(v)|+|g(u)-g(v)||\mathcal{G}_\delta(\omega)|_{H_0}\\
\le&(\sup_{B_{4\rho(\omega)}}\|Df(\cdot)\|+M_g\rho(\omega)^{\epsilon_0}C(\omega))|u-v|,
\end{aligned}$$

where we have used (3.5) to get

$$M_g=\begin{cases}
M_0\cdot 2^{1+\epsilon_0}\cdot\frac{\delta+1}{\delta}, & \text{if } r=1,\ \alpha=0\\
M_{1,R}\cdot 4^{\epsilon_0+\alpha}\cdot(\frac{R}{4})^\alpha\cdot\frac{\delta+1}{\delta}, & \text{if } r=1,\ \alpha>0\\
\sup_{B_{4\rho(\omega)}}\|D^{(2)}g(\cdot)\|\cdot 4\cdot(\frac{R}{4})^{1-\epsilon_0}\cdot\frac{\delta+1}{\delta}, & \text{if } r\ge 2.
\end{cases}$$

So $LipF(\omega,\cdot)\mid_{B_{2\rho(\omega)}}=\sup_{B_{4\rho(\omega)}}\|Df(\cdot)\|+M_g\rho(\omega)^{\epsilon_0}C(\omega)$, where $|_{B_{2\rho(\omega)}}$ means that the constant is given when restricted on the domain $B_{2\rho(\omega)}$.

$Lip\zeta^H_{\rho(\omega)}$: for $\forall u,\ v\in H$,

$$\begin{aligned}
&|\zeta_{\rho(\omega)}(u)-\zeta_{\rho(\omega)}(v)|\\
=&|\Gamma_{\rho(\omega)}(|u|)-\Gamma_{\rho(\omega)}(|v|)|=|\Gamma(\tfrac{|u|}{\rho(\omega)})-\Gamma(\tfrac{|v|}{\rho(\omega)})|\\
\le&\sup_{0\le t<+\infty}|\Gamma'(t)|\cdot\tfrac{|u-v|}{\rho(\omega)}\le K_\Gamma\cdot\tfrac{1}{\rho(\omega)}\cdot|u-v|,
\end{aligned}$$

so $Lip\zeta^H_{\rho(\omega)}=K_\Gamma\cdot\frac{1}{\rho(\omega)}$.

To conclude, noting that $F(\omega,0)=0$, we have for $u,\ v\in B_{2\rho(\omega)}$,

$$
\begin{aligned}
&|F_\rho(\omega,u)-F_\rho(\omega,v)|\\
\leq\,&|\zeta^H_{\rho(\omega)}(u)-\zeta^H_{\rho(\omega)}(v)||F(\omega,u)|+|\zeta^H_{\rho(\omega)}(v)||F(\omega,u)-F(\omega,v)|\\
\leq\,&Lip\zeta^H_{\rho(\omega)}|u-v|\cdot LipF(\omega,\cdot)|u|+K_\Gamma LipF(\omega,\cdot)|u-v|\\
\leq\,&3K_\Gamma(\sup\nolimits_{B_{4\rho(\omega)}}\|Df(\cdot)\|+M_g\rho(\omega)^{\epsilon_0}C(\omega))|u-v|.
\end{aligned}
$$

(iii) This follows the identical approach as (ii), for example, in the case $r=1$ and $0<\alpha\leq 1$, we first compute the Hölder or Lipschitz constants for each part involved in DF_ρ.

$HolDF$: As above, we have for $\forall u,\ v\in B_{2\rho(\omega)}$,

$$
\begin{aligned}
&\|DF(\omega,u)-DF(\omega,v)\|\\
\leq\,&\|Df(u)-Df(v)\|+\|Dg(u)-Dg(v)\||\mathcal{G}_\delta(\omega)|_{H_0}\\
\leq\,&(HolDf+M_{1,R}\cdot 2^{1+\epsilon_0}\cdot \tfrac{\delta+1}{\delta}\cdot\rho(\omega)^{\epsilon_0}C(\omega))|u-v|^\alpha,
\end{aligned}
$$

where $HolDf$ represents the Hölder constant for Df, and the rest is from (3.6). So

$$
HolDF(\omega,\cdot)\mid_{B_{2\rho(\omega)}}=HolDf+M_{1,R}\cdot 2^{1+\epsilon_0}\cdot\frac{\delta+1}{\delta}\cdot\rho(\omega)^{\epsilon_0}C(\omega).
$$

$LipD\zeta^H_{\rho(\omega)}$: First notice that $D\zeta^H_{\rho(\omega)}\mid_{B_{\rho(\omega)}}=0$ since $\zeta^H_{\rho(\omega)}\mid_{B_{\rho(\omega)}}\equiv 1$. Now in $H-\{0\}$,

$$
D\zeta^H_{\rho(\omega)}(u)=\Gamma'(\frac{|u|}{\rho(\omega)})\cdot\frac{1}{\rho(\omega)}\cdot D|\cdot|_H(u),
$$

where $D|\cdot|_H$ is the derivative of $|\cdot|$ in $H-\{0\}$ which is a bounded map as H is a Hilbert space. We let $\|D|\cdot|_H\|$ denote its norm. Then we have for $\forall u,\ v\in B_{2\rho(\omega)}$,

$$
\begin{aligned}
&\|D\zeta^H_{\rho(\omega)}(u)-D\zeta^H_{\rho(\omega)}(v)\|\\
\leq\,&K_\Gamma\tfrac{|u-v|}{\rho(\omega)}\tfrac{1}{\rho(\omega)}\|D|\cdot|_H\||u|+K_\Gamma\tfrac{1}{\rho(\omega)}\|D|\cdot|_H\||u-v|\\
\leq\,&3K_\Gamma\tfrac{1}{\rho(\omega)}\|D|\cdot|_H\||u-v|,
\end{aligned}
$$

so $LipD\zeta^H_{\rho(\omega)} = 3K_\Gamma \frac{1}{\rho(\omega)}\|D|\cdot|_H\|$.

Then a straightforward computation as in (ii) with the above two Lipschitz constants gives (iii) with $r = 0$. To be precise, we have

$$\begin{aligned}
&\|DF_\rho(\omega,u) - DF_\rho(\omega,v)\| \\
\leq &\|D\zeta^H_{\rho(\omega)}(u)F(\omega,u) - D\zeta^H_{\rho(\omega)}(v)F(\omega,v)\| \\
&+\|\zeta^H_{\rho(\omega)}(u)DF(\omega,u) - \zeta^H_{\rho(\omega)}(v)DF(\omega,v)\| \\
\leq &4R^{1-\alpha}LipD\zeta^H_{\rho(\omega)}LipF(\omega,\cdot)\mid_{B_{2\rho(\omega)}} \rho(\omega)|u-v|^\alpha \\
&+(2^{2-\alpha}+2)Lip\zeta^H_{\rho(\omega)}HolDF(\omega,\cdot)\mid_{B_{2\rho(\omega)}} \rho(\omega)|u-v|^\alpha \\
\leq &[4R^{1-\alpha}\cdot 3K_\Gamma\|D|\cdot|_H\|(\sup\nolimits_{B_{4\rho(\omega)}} Df + M_g\rho(\omega)^{\epsilon_0}C(\omega)) \\
&+(2^{2-\alpha}+2)\cdot K_\Gamma(HolDf + M_{1,R}\cdot 2^{1+\epsilon_0}\cdot \tfrac{\delta+1}{\delta}\cdot\rho(\omega)^{\epsilon_0}C(\omega))]|u-v|^\alpha.
\end{aligned}$$

The same argument works for the higher order cases. □

Remark 3.2.1. *From the above Lemma 3.1, we can make* $LipF_\rho(\omega)$ *as small as desired by choosing* $\rho(\omega)$ *to be small enough. Indeed, let* $M > 0$ *be any constant, then if we choose* $\rho(\omega)$ *to be so small that*

$$\sup_{B_{4\rho(\omega)}} \|Df(\cdot)\| \leq \frac{M}{2\cdot 3K_\Gamma},$$

and that

$$0 < \rho(\omega) < \left(\frac{M}{2\cdot 3K_\Gamma\cdot M_g C(\omega)}\right)^{\frac{1}{\epsilon_0}},$$

then we have $LipF_\rho(\omega) < M$. *This is valid since* $Df(0) = 0$, *and* $C(\omega)$ *is a random variable tempered from above by (2.5). Similarly we can make* $LipD^{(i)}F_\rho(\omega)$ *and* $HolD^{(r)}F_\rho(\omega)$ *to be bounded if we choose* $\rho(\omega)$ *properly.*

Now Hypotheses 3.1.1-3.1.3 with $r \geq 1$ guarantees the existence and uniqueness of global solutions $u(t,\omega,x)$ to equation (3.11), provided that $\rho(\omega)$ is chosen properly. We still denote by $u(t,\omega,x)$ the resulting flow. It then generates a random dynamical system over the metric dynamical system induced by the Wiener shift θ_t. For details see Chapter 4. Our interest is in the dynamical properties of such a system, and thanks to Lemma 3.1, the global phenomena for the truncated equation (3.11) restricts to the local dynamics of the original equation (3.1).

3.3 The Main Results

Recall Definition 2.3.3 for an invariant set, that is a random set $M(\omega)$ satisfying

$$\phi(t,\omega,M(\omega)) \subseteq M(\theta_t\omega)$$

for $t \in \mathbb{T}$, $\mathbb{T} = \mathbb{Z},\ \mathbb{Z}^+,\ \mathbb{R}$ or $\mathbb{R}^+$, where $\phi(t,\omega,x)$ is a random dynamical system. Now we give a definition for some more specific invariant manifolds related to the decomposition of the space H, see (3.4).

Definition 3.3.1. *Let $M(\omega)$ be an invariant set for $u(t,\omega,x)$. If we can represent $M(\omega)$ by a graph of a C^r (or Lipschitz) mapping $h^s(\omega,\cdot) : H^s \to H^{cu}$, i.e.,*

$$M(\omega) = \{\xi + h^s(\omega,\xi) \mid \xi \in H^s\},$$

with $Dh^s(\omega,0) = 0$ for $r \geq 1$, then $M(\omega)$ is called a C^r (or Lipschitz) stable manifold, and is denoted by $M^s(\omega)$. It is called a C^r (or Lipschitz) local stable manifold if the above holds in a tempered area of the origin.

Accordingly, a C^r (local) unstable (resp. center, center-stable, center-unstable) manifold $M^u(\omega)$ (resp. $M^c(\omega)$, $M^{cs}(\omega)$, $M^{cu}(\omega)$) is defined to be a graph of a C^r (or Lipschitz) mapping $h^i(\omega,\cdot) : H^i \to H^j$, $i = u$ (resp. $c,\ cs,\ cu$) and $j = cs$ (resp. $us,\ u,\ s$).

Then we can state our first result concerning the invariant manifold structures for equation (3.1).

Theorem 3.2 (Existence of local invariant manifolds)**.** *Assume that Hypotheses 3.1.1-3.1.3 for equation (3.1) hold with $r \geq 1$ and $0 \leq \alpha \leq 1$, also for β_1, β_2 given in Hypothesis 3.1.1, $(r+\alpha)\beta_2 < \beta_1$. Then there exist a $C^{r,\alpha}$ local center manifold $M^c(\omega)$, local stable mani-*

fold $M^s(\omega)$, local unstable manifold $M^u(\omega)$, local center-stable manifold $M^{cs}(\omega)$, and local center-unstable manifold $M^{cu}(\omega)$ for the random dynamical system generated by solutions to equation (3.1).

Remark 3.3.1. *Less regularity assumptions on f and g are required if one only wants to get local Lipschitz invariant manifolds. Indeed, f only needs to be of class $C^{0,1}$ with $f(0) = 0$, and satisfies the same property as stated in (3.6) with $D^{(r)}g(\cdot)$ replaced by $f(\cdot)$, i.e., there exist $0 < \epsilon_0 < 1$, $R > 0$, and $M_{r,R} > 0$, such that $\forall u,\ v \in H$, with $|u|,\ |v| \leq R$,*

$$|f(u) - f(v)| \leq M_{r,R}(|u|^{\epsilon_0} + |v|^{\epsilon_0})|u - v|.$$

If we set g to satisfy (3.5) with $\alpha = 1$, then the existence of those Lipschitz local manifolds is guaranteed with the same approach. But for the purpose of exhibiting the other part of this dissertation, we will focus on the case $r \geq 1$ for Hypotheses 3.1.2 and 3.1.3 later on.

To state the next result, we first introduce the concept of invariant foliations, which relies on the decomposition of the space H as well. Fix $\omega \in \Omega$, let $\{\mathcal{F}(\omega, x) \mid x \in H\}$ be a family of submanifolds of H parametrized by $x \in H$. $\{\mathcal{F}(\omega, x) \mid x \in H\}$ is said to be positively invariant if

$$u(t, \omega, \mathcal{F}(\omega, x)) \subseteq \mathcal{F}(\theta_t\omega, u(t, \omega, x))$$

for those $t \geq 0$. It is called a $C^{r-1,\alpha}$ family of $C^{r,\alpha}$ manifolds if the set $\{(x, y) \mid x \in H, y \in \mathcal{F}(\omega, x)\}$ is a $C^{r-1,\alpha} \times C^{r,\alpha}$ submanifold of H, where $r \geq 1$, $0 \leq \alpha \leq 1$.

Definition 3.3.2. *A family of submanifolds $\{\mathcal{F}^s(\omega, x) \mid x \in H\}$ is said to be a $C^{r-1,\alpha} \times C^{r,\alpha}$ stable foliation for H if the following conditions are satisfied:*

(i) $x \in \mathcal{F}^s(\omega, x)$ for each $x \in H$.

(ii) $\mathcal{F}^s(\omega, x)$ and $\mathcal{F}^s(\omega, \bar{x})$ are either disjoint or identical for each x and $\bar{x}$ in H.

(iii) $\mathcal{F}^s(\omega, 0)$ is tangent to H^s at the origin. Every leaf $\mathcal{F}^s(\omega, x)$ is the graph of a $C^{r,\alpha}$ function, i.e.,

$$\mathcal{F}^s(\omega, x) = \{\iota + l(\omega, \iota, x) \mid \iota \in H^s\},$$

where $l(\omega,\cdot,x) : H^s \to H^{cu}$ *is a* $C^{r,\alpha}$ *map.*

(iv) $\{\mathcal{F}^s(\omega,x) \mid x \in H\}$ *is a positively invariant* $C^{r-1,\alpha}$ *family of* $C^{r,\alpha}$ *manifolds for* H*.*

$\{\mathcal{F}^s(\omega,x) \mid x \in H\}$ is said to be a $C^{r-1,\alpha} \times C^{r,\alpha}$ local stable foliation if all the above hold in a tempered ball about the origin as described in (3.7).

Since $u(t,\omega,x)$ cannot be defined for $t \leq 0$ in general, there isn't any unstable foliation for H. However under Hypothesis 3.1.1, if we restrict ourselves to a center-unstable manifold, which is indeed finite dimensional, it can be possible to define an unstable foliation. To be precise, we consider the local case and let $M^{cu}(\omega) \subseteq V(\omega)$ be a given local center-unstable manifold, $V(\omega)$ being some tempered area about the origin as described in (3.7). Also let $r \geq 1$ and $0 \leq \alpha \leq 1$.

Definition 3.3.3. *A family of submanifolds* $\{\mathcal{F}^{cuu}(\omega,\xi) \mid \xi \in M^{cu}(\omega)\}$ *of* $M^{cu}(\omega)$ *is said to be a* $C^{r-1,\alpha} \times C^{r,\alpha}$ *local unstable foliation for* $M^{cu}(\omega)$ *if the following conditions are satisfied:*

(i) $\xi \in \mathcal{F}^{cuu}(\omega,\xi)$ *for each* $\xi \in M^{cu}(\omega)$*.*

(ii) $\mathcal{F}^{cuu}(\omega,\xi)$ *and* $\mathcal{F}^{cuu}(\omega,\bar{\xi})$ *are either disjoint or identical for each* ξ *and* $\bar{\xi}$ *in* $M^{cu}(\omega)$*.*

(iii) $\mathcal{F}^{cuu}(\omega,0)$ *is tangent to* H^u *at the origin. Every leaf* $\mathcal{F}^{cuu}(\omega,\xi)$ *is the graph of a* $C^{r,\alpha}$ *function, i.e.,*

$$\mathcal{F}^{cuu}(\omega,\xi) = \{\iota + l^u(\omega,\iota,\xi) \mid \iota \in H^u\},$$

where $l^u(\omega,\cdot,\xi) : H^u \cap V(\omega) \to H^{cs} \cap V(\omega)$ *is a* $C^{r,\alpha}$ *map.*

(iv) $\{\mathcal{F}^{cuu}(\omega,\xi) \mid \xi \in M^{cu}(\omega)\}$ *is a negatively invariant* $C^{r-1,\alpha}$ *family of* $C^{r,\alpha}$ *manifolds for* $M^{cu}(\omega)$*. By negatively invariant we mean*

$$u(t,\omega,\mathcal{F}^{cuu}(\omega,\xi)) \cap V(\theta_t\omega) \subseteq \mathcal{F}^{cuu}(\theta_t\omega, u(t,\omega,\xi)) \cap V(\theta_t\omega)$$

for those $t \leq 0$ *such that* $u(t,\omega,\xi)$ *is well-defined in* $M^{cu}(\theta_t\omega)$*, and with* $u(\tau,\omega,\xi) \in M^{cu}(\theta_\tau\omega)$ *for all* $\tau \in [t,0]$*.*

Note that we can always identify the local center-unstable manifold $M^{cu}(\omega)$ with the linear space H^{cu} locally through a function from H^{cu} to H^s whose graph is the manifold itself, and we will thus denote $\{\mathcal{F}^{cuu}(\omega,\xi) \mid \xi \in M^{cu}(\omega)\}$ by $\{\mathcal{F}^{cuu}(\omega,\xi) \mid \xi \in H^{cu}\}$ in which follows, for details see [BDL92]. Similarly, we can define a $C^{r-1,\alpha} \times C^{r,\alpha}$ stable foliation for a given center-stable manifold $M^{cs}(\omega)$. The existence of such a foliation structure for equation (3.1) is our next result.

Theorem 3.3 (Existence of foliations). *Assume that Hypotheses 3.1.1-3.1.3 for equation (3.1) hold with $r \geq 1$ and $0 \leq \alpha \leq 1$, also for β_1, β_2 given in Hypothesis 3.1.1, $(r+\alpha)\max\{r-1+\alpha, 1\}\beta_2 < \beta_1$. Then there exist a $C^{r-1,\alpha} \times C^{r,\alpha}$ local stable foliation $\mathcal{F}^s$, a $C^{r-1,\alpha} \times C^{r,\alpha}$ local stable foliation $\mathcal{F}^{css}$ for a given local center-stable manifold $M^{cs}(\omega)$ and a $C^{r-1,\alpha} \times C^{r,\alpha}$ local stable foliation $\mathcal{F}^{cuu}$ for a given local center-stable manifold $M^{cu}(\omega)$.*

At last, as discussed in the introduction, since we may choose different cut-off functions in the approach, the resulting manifold is not unique. We want to investigate the conjugacy between any two local center manifolds. Due to the regularity requirement in the time variable we meet in extending a local center manifold, we will adjust Hypothesis 3.1.3 and thus restrict the noise term to the stable and unstable subspaces of H. To be precise, we make the following hypothesis:

Hypothesis 3.3.1. *$g : U \cap H^{us} \to L(H_0, H)$ is of class C^r, where $H^{us} = H^u \oplus H^s$ is the projected space given in Hypothesis 3.1.1. And there exist $0 < \bar{\epsilon}_0 < 1$ and $\bar{R} > 0$, so that*

(1) if $\alpha = 0$, there exists $\bar{M}_0 > 0$, such that $\forall u,\ v \in H$, with $|u_{us}|,\ |v_{us}| \leq \bar{R}$,

$$|g(u_{us}) - g(v_{us})| \leq \bar{M}_0(|u_{us}|^{\epsilon_0} + |v_{us}|^{\epsilon_0})|u_{us} - v_{us}|.$$

(2) If $\alpha > 0$, there exists $M_{r,\bar{R}} > 0$, such that $\forall u,\ v \in H$, with $|u_{us}|,\ |v_{us}| \leq \bar{R}$,

$$\|D^{(r)}g(u_{us}) - D^{(r)}g(v_{us})\| \leq M_{r,\bar{R}}(|u_{us}|^{\epsilon_0} + |v_{us}|^{\epsilon_0})|u_{us} - v_{us}|^{\alpha}.$$

Also $D^{(i)}g(0) = 0$, $i = 0, \ldots, r$.

Now assume Hypotheses 3.1.1, 3.1.2 and 3.3.1 hold with $r \geq 2$, notice that changing Hypothesis 3.1.3 to 3.3.1 will not affect the results in Lemma 3.1. Hence by Theorem 3.2 there exist $C^{r,\alpha}$ local center manifolds for equation (3.1). We will show that any two of them are $C^{r-2,\alpha}$ conjugate in the sense of the following theorem. For simplicity, by a $C^{k,\alpha}$ diffeomorphism we mean a homeomorphism if $k = 0$ and $\alpha = 0$.

Theorem 3.4 (Conjugacy between center manifolds)**.** *Assume that Hypotheses 3.1.1, 3.1.2 and 3.3.1 hold with* $r \geq 2$ *and* $0 \leq \alpha \leq 1$ *for equation (3.1), also for* β_1, β_2 *given in Hypothesis 3.1.1,* $(r+\alpha)\max\{r-1+\alpha, 1\}\beta_2 < \beta_1$*. Then the local flows on two arbitrary* $C^{r,\alpha}$ *local center manifolds in* $U \subseteq H$ *are locally* $C^{r-2,\alpha}$ *conjugate. More specifically, for* $M_1^c(\omega)$ *and* $M_2^c(\omega)$ *being two such manifolds, then there is a neighborhood* $V(\omega) \subseteq H$ *of the origin and a* $C^{r-2,\alpha}$ *diffeomorphism* $\phi(\omega, \cdot) : M_1^c(\omega) \cap V(\omega) \to M_2^c(\omega) \cap V(\omega)$ *such that*

$$u(t, \omega, \phi(\omega, x)) = \phi(\theta_t \omega, u(t, \omega, x))$$

for all $x \in M_1^c(\omega) \cap V(\omega)$*, and all* t *satisfying* $u(t, \omega, x) \in M_1^c(\theta_t \omega) \cap V(\theta_t \omega)$*.*

As another application of the above approach, we will also prove the same conjugacy result for a stochastic evolution equation at the very end:

$$\mathrm{d}u = (Au + f(u_{us}))\mathrm{d}t + u \circ \mathrm{d}W,$$

where $\circ \mathrm{d}W$ is the Stratonovich differential, and $f(u_{us})$ is given by the following Hypothesis.

Hypothesis 3.3.2. *$f : U \cap H^{us} \to H$ is of class $C^{r,\alpha}$ with $r \geq 2$, and $f(0) = 0$ and $Df(0) = 0$.*

This restriction on the drift term is due to the same reason as we stated ahead of Hypothesis 3.3.1.

Chapter 4. Generation of the Systems

In this chapter, we will show that under Hypotheses 3.1.1-3.1.3 with $r \geq 1$ and $0 \leq \alpha \leq 1$, the truncated equation (3.11) admits a unique solution $u(t, \omega, x)$ for $t \geq 0$, provided that $\rho(\omega)$ is chosen properly. Also such a solution generates a random dynamical system over the metric dynamical system induced by the Wiener shift θ_t.

4.1 Existence, Uniqueness of Solutions

We first justify the existence and uniqueness of solutions to equation (3.11). We choose $\rho(\omega)$ such that the Lipschitz constant $LipF_\rho(\omega)$ given in Lemma 3.1 is restricted as

$$0 < LipF_\rho(\omega) < 1. \tag{4.1}$$

This ensures the uniform Lipschitz continuity for the nonlinear term of equation (3.11).

Proposition 4.1.1. *Assume that Hypotheses 3.1.1-3.1.3 hold with $r \geq 1$ and $0 \leq \alpha \leq 1$. If we choose $\rho(\omega)$ to be so small that (4.1) holds, then there exists a unique solution $u(t, \omega, x)$ to equation (3.11) for $t \geq 0$, such that $u(t, \omega, x)$ is Lipschitz continuous in x and measurable in (t, ω, x).*

Proof. We observe that for fixed $\omega \in \Omega$, equation (3.11) is nothing but a deterministic partial differential equation. Then we have by the variation of constants formula that a solution for (3.11) is a continuous function $u(t, \omega, x)$ satisfying

$$u(t, \omega, x) = e^{At}x + \int_0^t e^{A(t-s)} F_\rho(\theta_s\omega, u(s, \omega, x)) \mathrm{d}s. \tag{4.2}$$

Thereafter, the problem is converted to the existence and uniqueness of a solution to the integral equation (4.2).

We first prove the result for a small time period. For $T > 0$ to be determined, we define

$$\mathcal{S} := \{y \in C([0,T], H) \mid y(0) = x\}.$$

Then $\mathcal{S}$ is a complete metric space under the metric induced by the sup-norm on $C([0,T], H)$, that is for $y_1,\ y_2 \in \mathcal{S}$, the distance between them is given by

$$\|y_1 - y_2\|_{C_T} := \sup_{0 \le t \le T} |y_1(t) - y_2(t)|.$$

We define for $y \in \mathcal{S}$

$$G(y)(t) := e^{At}x + \int_0^t e^{A(t-s)} F_\rho(\theta_s\omega, y(s))\mathrm{d}s.$$

We show that G maps $\mathcal{S}$ into itself, and G is a strict contraction.

First note that it is clear from the definition of G that $G(y)(0) = x$. To see $G(y)$ is continuous for $y \in \mathcal{S}$, we pick $t \in (0,T)$ and $h > 0$ so small that $t + h \in (0,T)$. Then

$$\begin{aligned}
&|G(y)(t+h) - G(y)(t)| \\
=&|(e^{A(t+h)}x + \textstyle\int_0^{t+h} e^{A(t+h-s)} F_\rho(\theta_s\omega, y(s))\mathrm{d}s) - (e^{At}x + \int_0^t e^{A(t-s)} F_\rho(\theta_s\omega, y(s))\mathrm{d}s)| \\
\le&\|e^{At}\|\|e^{Ah} - id\||x| + \textstyle\int_0^t \|e^{A(t-s)}\|\|e^{Ah} - id\| Lip F_\rho(\theta_s\omega)|y(s)|\mathrm{d}s \\
&+ \textstyle\int_t^{t+h} \|e^{A(t+h-s)}\| Lip F_\rho(\theta_s\omega)|y(s)|\mathrm{d}s \\
=&o(1)
\end{aligned}$$

as $h \to 0$, where we have used the fact that $\|e^{At}\| \le Me^{at}$ for some $M \ge 1$ and $a \ge 0$, which is thus bounded for $t \in (0,T)$ and that $e^{Ah} \to id$ as $h \to 0$, see Pazy [PZ12]. The case that $h < 0$ and the continuity at end points $\{0, T\}$ can be shown similarly. Now take $y_1,\ y_2 \in \mathcal{S}$, for $0 \le t \le T$,

$$\begin{aligned} &|G(y_1)(t) - G(y_2)(t)| \\ &\le \int_0^t \|e^{A(t-s)}\| Lip F_\rho(\theta_s \omega) |y_1(s) - y_2(s)| \mathrm{d}s \\ &\le \int_0^t M e^{a(t-s)} \mathrm{d}s \cdot \|y_1 - y_2\|_{C_T} \\ &\le \frac{M}{a}(e^{aT} - 1) \cdot \|y_1 - y_2\|_{C_T}. \end{aligned}$$

If we choose T so small that $\frac{M}{a}(e^{aT} - 1) < \frac{1}{2}$, then we get that G is a contraction mapping from $\mathcal{S}$ into itself with

$$\|G(y_1) - G(y_2)\|_{C_T} \le \frac{1}{2}\|y_1 - y_2\|_{C_T}.$$

By the contraction mapping theorem, G has a unique fixed point in $\mathcal{S}$, we denote it by $u(t, \omega, x)$ as it depends on ω we choose at the beginning. Then $u(t, \omega, x)$ solves the integral equation (4.2).

Now since $u(\cdot, \omega, x)$ is an ω-wise limit of the iteration of contraction mapping G starting at the constant function x and mapping a $\mathcal{F}$-measurable function to a measurable function, $u(\cdot, \omega, x)$ is $\mathcal{F}$-measurable. Also it is measurable in t as the function in each iteration step is so. Furthermore, if $x_1,\ x_2 \in H$ be two initial values, then there exist solutions $u(t, \omega, x_1)$ and $u(t, \omega, x_2)$ by the above, and we have

$$|u(t, \omega, x_1) - u(t, \omega, x_2)| \le e^{At}|x_1 - x_2| + \frac{1}{2}\|u(\cdot, \omega, x_1) - u(\cdot, \omega, x_2)\|_{C_T},$$

implying that

$$\|u(\cdot, \omega, x_1) - u(\cdot, \omega, x_2)\|_{C_T} \le 2Me^{aT}|x_1 - x_2|.$$

This proves that $u(t, \omega, x)$ is Lipschitz continuous in x. Then by Lemma III.14 in Castaing and Valadier [CV77], $u(t, \omega, x)$ is measurable with respect to (t, ω, x). Combining all the above we have shown that there exists a unique solution $u(t, \omega, x)$ to equation (3.11) for $t \in [0, T]$.

At last, we show that the solution can be extended to $t = +\infty$. Suppose the solution can be only extended to a finite time interval, say, $[0, t_1)$, with $t_1 < +\infty$, but $u(t, \omega, x)$ stays bounded in H, then as our equation is defined on all H, we may just follow the approach above and extend the solution beyond t_1, which is a contradiction. Now suppose still that $t_1 < +\infty$, but there is a sequence $\{t_n\}_{n=1}^{+\infty}$ such that $t_n \to t_1^-$, and $|u(t_n, \omega, x)| \to +\infty$ as $n \to +\infty$. However, when we fix $t \in [0, t_1)$,

$$\begin{aligned} |u(t,\omega,x)| &\leq |e^{At}x| + \int_0^t \|e^{A(t-s)}\| LipF_\rho(\theta_s\omega)|u(s,\omega,x)|\mathrm{d}s \\ &\leq Me^{at_1}|x| + \int_0^t Me^{a(t-s)} LipF_\rho(\theta_s\omega)|u(s,\omega,x)|\mathrm{d}s. \end{aligned}$$

By Gronwall's inequality we have

$$|u(t,\omega,x)| \leq Me^{at_1}|x| \cdot e^{\int_0^t Me^{a(t-s)} LipF_\rho(\theta_s\omega)\mathrm{d}s} \leq Me^{at_1}|x| \cdot e^{\frac{M}{a}(e^{at_1}-1)} < +\infty,$$

which is a contradiction. Thus it should be the case that the solution can be extended to all of $[0, +\infty)$. □

4.2 Generation of Random Dynamical Systems From the Solutions

By Proposition 4.1.1 given above, we see the measurability of solution $u(t, \omega, x)$ to equation (3.11). We claim that it generates a random dynamical system over the metric dynamical system induced by the Wiener shift θ_t. It suffices to show the cocycle property (2.3) for $u(t, \omega, x)$ given by (4.2).

Let $t,\ s\in\mathbb{R}^+$, $\omega\in\Omega$. By doing a change of variable, we have

$$\begin{aligned}
&u(t,\theta_s\omega,u(s,\omega,x))\\
=&e^{At}u(s,\omega,x)+\int_0^t e^{A(t-r)}F_\rho(\theta_r(\theta_s\omega),u(r,\theta_s\omega,u(s,\omega,x)))\mathrm{d}r\\
=&e^{At}(e^{As}x+\int_0^s e^{A(s-\tau)}F_\rho(\theta_\tau\omega,u(\tau,\omega,x)))\mathrm{d}\tau))\\
&+\int_0^t e^{A(t-r)}F_\rho(\theta_r(\theta_s\omega),u(r,\theta_s\omega,u(s,\omega,x)))\mathrm{d}r\\
=&e^{A(t+s)}x+\int_0^s e^{A(t+s-\tau)}F_\rho(\theta_\tau\omega,u(\tau,\omega,x)))\mathrm{d}\tau)\\
&+\int_s^{t+s} e^{A(t+s-\sigma)}F_\rho(\theta_\sigma\omega,u(\sigma-s,\theta_s\omega,u(s,\omega,x)))\mathrm{d}\sigma
\end{aligned}$$

We thus have found that the function

$$v(r,\omega,x):=\begin{cases} u(r,\omega,x), & 0\le r\le s,\\ u(r-s,\theta_s\omega,u(s,\omega,x)), & s<r\le s+t\end{cases}$$

satisfies

$$v(s+t,\omega,x)=e^{A(t+s)}x+\int_0^{t+s} e^{A(t+s-\tau)}F_\rho(\theta_\tau\omega,v(\tau,\omega,x))\mathrm{d}\tau.$$

However, by uniqueness of solution given by Proposition 4.1.1, we have that

$$u(s+t,\omega,x)=v(s+t,\omega,x)=u(t,\theta_s\omega,u(s,\omega,x)),$$

suggesting the cocycle property as desired. Also $u(0,\omega,\cdot)=id$ follows immediately from (4.2).

Chapter 5. Invariant Manifolds

In this chapter, we will show that under Hypotheses 3.1.1-3.1.3 with $r \geq 1$ and $0 \leq \alpha \leq 1$, equation (3.1) admits all the desired local invariant manifolds, provided that we restrict the area around the origin to be small enough. To be precise, we use the cut-off function (3.10) to derive the truncated equation (3.11) and show the existence of global results for it. Then with Lemma 3.1, we can derive the local results of (3.1).

5.1 Settings and Notations

Let $\beta_2,\ \beta_1$ in (3.3) be given, we define for $\gamma_c \in (\beta_2, \beta_1)$,

$$C_{\gamma_c} := \{\phi \in C(\mathbb{R}, H) \mid \sup_{t\in\mathbb{R}} e^{-\gamma_c|t|}|\phi(t)| < +\infty\}$$

equipped with the norm

$$|\phi|_{\gamma_c} := \sup_{t\in\mathbb{R}} e^{-\gamma_c|t|}|\phi(t)|.$$

And we define for $\gamma_s \in (-\beta_1, -\beta_2)$,

$$C^+_{\gamma_s} := \{\phi \in C([0,+\infty), H) \mid \sup_{0\leq t<+\infty} e^{-\gamma_s t}|\phi(t)| < +\infty\}$$

equipped with the norm

$$|\phi|^+_{\gamma_s} := \sup_{0\leq t<+\infty} e^{-\gamma_s t}|\phi(t)|.$$

And we define for $\gamma_u \in (\beta_2, \beta_1)$,

$$C^-_{\gamma_u} := \{\phi \in C((-\infty,0], H) \mid \sup_{-\infty<t\leq 0} e^{-\gamma_u t}|\phi(t)| < +\infty\}$$

equipped with the norm

$$|\phi|^-_{\gamma_u} := \sup_{-\infty<t\leq 0} e^{-\gamma_u t}|\phi(t)|.$$

And we define for $\gamma_{cs} \in (\beta_2, \beta_1)$,

$$C^+_{\gamma_{cs}} := \{\phi \in C([0, +\infty), H) \mid \sup_{0 \le t < +\infty} e^{-\gamma_{cs} t} |\phi(t)| < +\infty\}$$

equipped with the norm

$$|\phi|^+_{\gamma_{cs}} := \sup_{0 \le t < +\infty} e^{-\gamma_{cs} t} |\phi(t)|.$$

And we define for $\gamma_{cu} \in (-\beta_1, -\beta_2)$,

$$C^-_{\gamma_{cu}} := \{\phi \in C((-\infty, 0], H) \mid \sup_{-\infty < t \le 0} e^{-\gamma_{cu} t} |\phi(t)| < +\infty\}$$

equipped with the norm

$$|\phi|^-_{\gamma_{cs}} := \sup_{-\infty < t \le 0} e^{-\gamma_{cu} t} |\phi(t)|.$$

Again let $u(t, \omega, x_0)$ denote the solution to equation (3.11) with initial value x_0, we define

$$\begin{aligned}
M^c(\omega) &:= \{x_0 \in H \mid u(\cdot, \omega, x_0) \in C_{\gamma_c}\}, \\
M^s(\omega) &:= \{x_0 \in H \mid u(\cdot, \omega, x_0) \in C^+_{\gamma_s}\}, \\
M^u(\omega) &:= \{x_0 \in H \mid u(\cdot, \omega, x_0) \in C^-_{\gamma_u}\}, \\
M^{cs}(\omega) &:= \{x_0 \in H \mid u(\cdot, \omega, x_0) \in C^+_{\gamma_{cs}}\}, \\
M^{cu}(\omega) &:= \{x_0 \in H \mid u(\cdot, \omega, x_0) \in C^-_{\gamma_{cu}}\}.
\end{aligned}$$

We will show that these are invariant manifolds for equation (3.11) which are given by graphs, provided that we properly restrict the choice of $\rho(\omega)$. As the other cases can be verified similarly, we focus on the case of center-unstable manifold in this section. To be precise, as mentioned in Remark 3.2.1, we may choose $\rho(\omega)$ so small that for $-\beta_1 < \gamma_{cu} < -\beta_2$, the Lipschitz constant $LipF_\rho(\omega)$ given in Lemma 3.1 is restricted as the following:

$$\begin{cases} K_A LipF_\rho(\omega) < 1, \\ K_A LipF_\rho(\omega)\left(-\frac{1}{\beta_2 + \gamma_{cu}} - \frac{1}{\gamma_{cu} - \beta_1} + \frac{1}{\gamma_{cu} + \beta_1}\right) < \frac{1}{6K_A}, \end{cases} \tag{5.1}$$

and as $K_A \geq 1$, we have

$$K_A LipF_\rho(\omega)(-\frac{1}{\beta_2+\gamma_{cu}} - \frac{1}{\gamma_{cu}-\beta_1} + \frac{1}{\gamma_{cu}+\beta_1}) < \frac{1}{2}$$

as well. Also for γ_{cu} satisfying

$$-\beta_1 < (r+\alpha)\cdot\gamma_{cu} < \gamma_{cu} < -\beta_2, \tag{5.2}$$

and for $\eta^* > 0$ so that

$$-\beta_1 < \gamma_{cu} + \eta^* < \gamma_{cu} + 2\eta^* < -\beta_2, \tag{5.3}$$

we can restrict $LipF_\rho(\omega)$ as the following:

$$\begin{cases} K_A LipF_\rho(\omega)\max_{j\in\{1,\ldots,r,r+\alpha\}}\sup_{0\le\eta\le 2\eta^*}(-\frac{1}{\beta_2+j\gamma_{cu,l}+\eta} - \frac{1}{j\gamma_{cu,l}+\eta-\beta_1} + \frac{1}{j\gamma_{cu,l}+\eta+\beta_1}) < \frac{1}{6K_A}, \\ \qquad LipD^{(i)}F_\rho(\omega) < +\infty,\ i=1,\ldots,r, \quad \text{and} \quad HolD^{(r)}F_\rho(\omega) < +\infty, \end{cases} \tag{5.4}$$

where $LipD^{(i)}F_\rho(\omega)$ and $HolD^{(r)}F_\rho(\omega)$ are introduced in Lemma 3.1 (iii). The choices of numbers here are not optimal, but are for the convenience.

5.2 A Lemma Directing to An Equivalent Problem

We proceed with the following two lemmas to show the existence of center-unstable manifold. For simplicity in notation, we will denote by $F_{\rho,i} := P^i \circ F_\rho$, $i = c,\ u,\ s,\ cu,\ cs$.

Lemma 5.1. *Assume that Hypotheses 3.1.1-3.1.3 hold. For any $\gamma_{cu} \in (-\beta_1, -\beta_2)$, if we choose $\rho(\omega)$ to be so small that (5.1) holds, then $x_0 \in M^{cu}(\omega)$ if and only if there exists a function $v(\cdot) \in C^-_{\gamma_{cu}}$ with the initial value $v(0) = x_0$ and satisfies*

$$v(t) = e^{At}\xi + \int_0^t e^{A(t-\tau)}F_{\rho,c}(\theta_\tau\omega, v)\mathrm{d}\tau + \int_0^t e^{A(t-\tau)}F_{\rho,u}(\theta_\tau\omega, v)\mathrm{d}\tau + \int_{-\infty}^t e^{A(t-\tau)}F_{\rho,s}(\theta_\tau\omega, v)\mathrm{d}\tau, \tag{5.5}$$

where $\xi = P^{cu}x_0$.

Proof. Let $x_0 \in M^{cu}(\omega)$. By the variation of constants formula, for $s,\ t \in \mathbb{R}$, we have for $u(t,\omega,x_0)$ the solution to equation (7.7) with initial value x_0 that

$$u(t,\omega,x_0) = e^{A(t-s)}u(s,\omega,x_0) + \int_s^t e^{A(t-\tau)}F_\rho(\theta_\tau\omega, u(\tau,\omega,x_0))\mathrm{d}\tau.$$

Restricting on H^{cu} and taking $s=0$, we have

$$\begin{aligned} P^{cu}u(t,\omega,x_0) =& e^{At}P^{cu}x_0 + \textstyle\int_0^t e^{A(t-\tau)}F_{\rho,c}(\theta_\tau\omega, u(\tau,\omega,x_0))\mathrm{d}\tau \\ &+ \textstyle\int_0^t e^{A(t-\tau)}F_{\rho,u}(\theta_\tau\omega, u(\tau,\omega,x_0))\mathrm{d}\tau. \end{aligned} \tag{5.6}$$

While on H^s we have

$$P^s u(t,\omega,x_0) = e^{A(t-s)}P^s u(s,\omega,x_0) + \int_s^t e^{A(t-\tau)}F_{\rho,s}(\theta_\tau\omega, u(\tau,\omega,x_0))\mathrm{d}\tau.$$

For $s < \min\{t,0\}$,

$$\begin{aligned} |e^{A(t-s)}P^s u(s,\omega,x_0)| &\le K_A e^{-\beta_1(t-s)}e^{\gamma_{cu}s}|u(\cdot,\omega,x_0)|^-_{\gamma_{cu}} \\ &= K_A e^{-\beta_1 t}|u(\cdot,\omega,x_0)|^-_{\gamma_{cu}} \cdot e^{(\gamma_{cu}+\beta_1)s} \to 0 \end{aligned}$$

as $s \to -\infty$ since $\gamma_{cu} + \beta_1 > 0$.

Also let us consider $t_p < t_q < \min\{t,0\}$, using (3.12) in Lemma 3.1 and the conditions on $\rho(\omega)$, we get

$$\begin{aligned} &|\textstyle\int_{t_p}^t e^{A(t-\tau)}F_{\rho,s}(\theta_\tau\omega, u(\tau,\omega,x_0))\mathrm{d}\tau - \int_{t_q}^t e^{A(t-\tau)}F_{\rho,s}(\theta_\tau\omega, u(\tau,\omega,x_0))\mathrm{d}\tau| \\ \le& \textstyle\int_{t_p}^{t_q} K_A e^{-\beta_1(t-\tau)} Lip F_\rho(\theta_\tau\omega)e^{\gamma_{cu}\tau}|u(\cdot,\omega,x_0)|^-_{\gamma_{cu}}\mathrm{d}\tau \\ \le& e^{-\beta_1 t}e^{(\gamma_{cu}+\beta_1)t_q}(1-e^{(\gamma_{cu}+\beta_1)(t_p-t_q)}) \to 0 \end{aligned}$$

as $t_p,\ t_q \to -\infty$. So if we let $s \to -\infty$, the integral

$$\int_{-\infty}^t e^{A(t-\tau)}F_{\rho,s}(\theta_\tau\omega, u(\tau,\omega,x_0))\mathrm{d}\tau \tag{5.7}$$

is well defined. Combining (5.6) and (5.7), we get (5.5). The converse direction can be verified via a direct computation. □

5.3 Solving the Equivalent Problem

Next we study the existence and smoothness of solution for equation (5.5).

Lemma 5.2. *Assume that Hypotheses 3.1.1-3.1.3 hold with $r \geq 1$ and $0 \leq \alpha \leq 1$. For any $-\beta_1 < \gamma_{cu} < -\beta_2$, if we choose $\rho(\omega)$ so small that (5.1) holds, then equation (5.5) has a unique solution $v(\cdot, \omega, \xi) \in C^-_{\gamma_{cu}}$ with $P^{cu}v(0, \omega, \xi) = \xi$ for all $\xi \in H^{cu}$ such that*

(1) $v(\cdot, \omega, \xi)$ is measurable in (ω, ξ) and is Lipschitz continuous in ξ with Lipschitz constant less than $2K_A$.

Furthermore, assume γ_{cu} satisfies (5.2), and for $\eta^* > 0$ so that (5.3) holds, if we choose $\rho(\omega)$ even smaller so that (5.4) holds, then

(2) $v(\cdot, \omega, \xi)$ is C^r from H^{cu} to $C^-_{r\gamma_{cu}+\eta}$, $\forall 0 \leq \eta \leq \eta^*$. And there exist constants $K_{i,\eta}$, $i = 1, \ldots, r$, such that for any $0 \leq \eta \leq \eta^*$,

$$\|D^{(i)}_\xi v(\cdot, \omega, \xi)\|_{L^i(H^{cu}, C^-_{i\gamma_{cu}+\eta})} \leq K_{i,\eta^*}.$$

(3) If $\alpha > 0$, $D^{(r)}_\xi v(\cdot, \omega, \xi)$ from H^{cu} to $L^r(H^{cu}, C^-_{(r+\alpha)\gamma_{cu}+\eta})$ is α-Hölder continuous in ξ for any $0 \leq \eta \leq \eta^*$.

Proof. **Step 1.** First we prove that under the conditions with our choice of $\rho(\omega)$, equation (5.5) has a unique solution $v = v(\cdot, \omega, \xi)$ which is Lipschitz continuous with $\xi \in H^{cu}$.
We define for $v \in C^-_{\gamma_{cu}}$, $\omega \in \Omega$, $\xi \in H^{cu}$

$$\begin{aligned}\mathcal{J}(v, \omega, \xi) :=& e^{At}\xi + \int_0^t e^{A(t-\tau)}F_{\rho,c}(\theta_\tau\omega, v)\mathrm{d}\tau + \int_0^t e^{A(t-\tau)}F_{\rho,u}(\theta_\tau\omega, v)\mathrm{d}\tau \\ &+ \int_{-\infty}^t e^{A(t-\tau)}F_{\rho,s}(\theta_\tau\omega, v)\mathrm{d}\tau.\end{aligned}$$

We want to show that this map has a fixed point on $C^-_{\gamma_{cu}}$. To check $\mathcal{J}$ maps $C^-_{\gamma_{cu}}$ into $C^-_{\gamma_{cu}}$, we have for $t \leq 0$,

$$
\begin{aligned}
& e^{-\gamma_{cu}t}|\mathcal{J}(v,\omega,\xi)| \\
\leq & e^{-\gamma_{cu}t}K_A e^{-\beta_2 t}|\xi| + e^{-\gamma_{cu}t}\int_t^0 K_A LipF_\rho(\theta_\tau\omega)(e^{\beta_2|t-\tau|+\gamma_{cu}\tau} + e^{\beta_1(t-\tau)+\gamma_{cu}\tau})|v|_{\gamma_{cu}}^- \mathrm{d}\tau \\
& + e^{-\gamma_{cu}t}\int_{-\infty}^t K_A LipF_\rho(\theta_\tau\omega)e^{-\beta_1(t-\tau)+\gamma_{cu}\tau}|v|_{\gamma_{cu}}^- \mathrm{d}\tau \\
\leq & K_A|\xi| + \tfrac{1}{2(-\frac{1}{\beta_2+\gamma_{cu}} - \frac{1}{\gamma_{cu}-\beta_1} + \frac{1}{\gamma_{cu}+\beta_1})} \cdot [\int_t^0 e^{(\beta_2+\gamma_{cu})(\tau-t)}\mathrm{d}\tau \\
& + \int_t^0 e^{(\gamma_{cu}-\beta_1)(\tau-t)}\mathrm{d}\tau + \int_{-\infty}^t e^{(\gamma_{cu}+\beta_1)(\tau-t)}\mathrm{d}\tau] \cdot |v|_{\gamma_{cu}}^- \\
\leq & K_A|\xi| + \tfrac{1}{2}|v|_{\gamma_{cu}}^- < +\infty.
\end{aligned}
$$

Now for each $v,\ \bar{v} \in C_{\gamma_{cu}}^-$, similarly we get

$$
\begin{aligned}
& |\mathcal{J}(v,\omega,\xi) - \mathcal{J}(\bar{v},\omega,\xi)|_{\gamma_{cu}}^- \\
\leq & \sup_{t\leq 0} e^{-\gamma_{cu}t}|\mathcal{J}(v,\omega,\xi) - \mathcal{J}(\bar{v},\omega,\xi)| \\
\leq & \tfrac{1}{2}|v-\bar{v}|_{\gamma_{cu}}^-.
\end{aligned}
$$

That is, $\mathcal{J}(\cdot,\omega,\xi)$ is a uniform contraction with respect to the parameter (ω,ξ). Using the contraction mapping principle, $\mathcal{J}(\cdot,\omega,\xi)$ has a unique fixed point $v(\cdot,\omega,\xi) \in C_{\gamma_{cu}}^-$ for each $\xi \in H^{cu}$. Clearly $v(\cdot,\omega,0) = 0$ since $F_\rho(\omega,0) = 0$. Following the same approach, for any $\xi,\ \bar{\xi} \in H^{cu}$, we have

$$
e^{-\gamma_{cu}t}|v(t,\omega,\xi) - v(t,\omega,\bar{\xi})| \leq K_A|\xi-\bar{\xi}| + \frac{1}{2}|v(\cdot,\omega,\xi) - v(\cdot,\omega,\bar{\xi})|_{\gamma_{cu}}^-,
$$

thus

$$
|v(\cdot,\omega,\xi) - v(\cdot,\omega,\bar{\xi})|_{\gamma_{cu}}^- \leq 2K_A|\xi-\bar{\xi}|. \tag{5.8}
$$

Since $v(\cdot,\omega,\xi)$ can be an ω-wise limit of the iteration of contraction mapping $\mathcal{J}$ starting at 0 and mapping an $\mathcal{F}$-measurable function to a measurable function, $v(\cdot,\omega,\xi)$ is $\mathcal{F}$-measurable. On the other hand, since $v(\cdot,\omega,\xi)$ is Lipschitz continuous in ξ, by Lemma III.14 in Castaing and Valadier [CV77], $v(\cdot,\omega,\xi)$ is measurable with respect to (ω,ξ).

Step 2. We prove $v(\cdot,\omega,\xi)$ is $C^{1,\alpha}$, that is the case $r=1$, and $0 \leq \alpha \leq 1$. By the conditions, for any $0 \leq \eta \leq 2\eta^*$, and a given $\xi_0 \in H^{cu}$, we can follow the same approach as above and show that $\mathcal{J}(\cdot,\omega,\xi_0) : C^-_{\gamma_{cu}+\eta} \to C^-_{\gamma_{cu}+\eta}$ is a contraction mapping and thus has a unique fixed point $v(\cdot,\omega,\xi_0) \in C^-_{\gamma_{cu}+\eta}$. Now for $\forall v \in C^-_{\gamma_{cu}+\eta}$, we define

$$\begin{aligned} \mathcal{S}(v)(t) = &\int_0^t e^{A(t-\tau)} D_u F_{\rho,c}(\theta_\tau\omega, v(\tau,\omega,\xi_0)) v \mathrm{d}\tau \\ &+ \int_0^t e^{A(t-\tau)} D_u F_{\rho,u}(\theta_\tau\omega, v(\tau,\omega,\xi_0)) v \mathrm{d}\tau \\ &+ \int_{-\infty}^t e^{A(t-\tau)} D_u F_{\rho,s}(\theta_\tau\omega, v(\tau,\omega,\xi_0)) v \mathrm{d}\tau. \end{aligned} \tag{5.9}$$

Note that $\|D_u F_\rho(\omega,u)\| \leq Lip F_\rho(\omega)$, so we have for any $t \leq 0$,

$$\begin{aligned} &|\mathcal{S}(v)(t)| e^{-(\gamma_{cu}+\eta)t} \\ \leq &\tfrac{1}{6K_A} \tfrac{1}{\sup_{0\leq\sigma\leq 2\eta}\left(-\frac{1}{\beta_2+\gamma_{cu}+\eta} - \frac{1}{\gamma_{cu}+\eta-\beta_1} + \frac{1}{\gamma_{cu}+\eta+\beta_1}\right)} \\ &\cdot \left[\int_t^0 e^{(\beta_2+\gamma_{cu}+\eta)(\tau-t)} \mathrm{d}\tau + \int_t^0 e^{(\gamma_{cu}+\eta-\beta_1)(\tau-t)} \mathrm{d}\tau + \int_{-\infty}^t e^{(\gamma_{cu}+\eta+\beta_1)(\tau-t)} \mathrm{d}\tau\right] \cdot |v|^-_{\gamma_{cu}+\eta} \\ \leq &\tfrac{1}{6K_A} |v|^-_{\gamma_{cu}+\eta}. \end{aligned}$$

As we have $K_A \geq 1$, so $\mathcal{S}(\cdot)$ is a bounded linear operator from $C^-_{\gamma_{cu}+\eta}$ into itself with

$$\|\mathcal{S}(\cdot)\| \leq \frac{1}{6K_A} < 1.$$

This implies that $id - \mathcal{S}$ has a bounded inverse in $C^-_{\gamma_{cu}+\eta}$.

Now let $\xi \in H^{cu}$ and $v(t,\omega,\xi)$ be the corresponding fixed point of $\mathcal{J}(\cdot,\omega,\xi)$, we define

$$\begin{aligned} \mathcal{I}(t) = &\int_0^t e^{A(t-\tau)} P^c [F_\rho(\theta_\tau\omega, v(\tau,\omega,\xi)) - F_\rho(\theta_\tau\omega, v(\tau,\omega,\xi_0)) \\ &\qquad - D_u F_\rho(\theta_\tau\omega, v(\tau,\omega,\xi_0))(v(\tau,\omega,\xi) - v(\tau,\omega,\xi_0))] \mathrm{d}\tau \\ &+ \int_0^t e^{A(t-\tau)} P^u [F_\rho(\theta_\tau\omega, v(\tau,\omega,\xi)) - F_\rho(\theta_\tau\omega, v(\tau,\omega,\xi_0)) \\ &\qquad - D_u F_\rho(\theta_\tau\omega, v(\tau,\omega,\xi_0))(v(\tau,\omega,\xi) - v(\tau,\omega,\xi_0))] \mathrm{d}\tau \\ &+ \int_{-\infty}^t e^{A(t-\tau)} P^s [F_\rho(\theta_\tau\omega, v(\tau,\omega,\xi)) - F_\rho(\theta_\tau\omega, v(\tau,\omega,\xi_0)) \\ &\qquad - D_u F_\rho(\theta_\tau\omega, v(\tau,\omega,\xi_0))(v(\tau,\omega,\xi) - v(\tau,\omega,\xi_0))] \mathrm{d}\tau. \end{aligned}$$

If for $\sigma \le \eta$ we can prove that

$$|\mathcal{I}|^-_{\gamma_{cu}+\sigma} = o(|\xi - \xi_0|), \qquad \text{as } \xi \to \xi_0, \tag{5.10}$$

then we have

$$v(t,\omega,\xi) - v(t,\omega,\xi_0) - \mathcal{S}(v(\cdot,\omega,\xi) - v(\cdot,\omega,\xi_0))(t) = e^{At}(\xi - \xi_0) + o(|\xi - \xi_0|),$$

which implies that

$$v(t,\omega,\xi) - v(t,\omega,\xi_0) = (id - \mathcal{S})^{-1} e^{At}(\xi - \xi_0) + o(|\xi - \xi_0|).$$

Thus $v(\cdot,\omega,\xi)$ is differentiable in ξ and $D_\xi v(\cdot,\omega,\xi) \in L(H^{cu}, C^-_{\gamma_{cu}+\eta})$.

To show (5.10), we split $e^{-(\gamma_{cu}+\eta)t}\mathcal{I}(t)$ into a sum of 6 terms as following,

$$e^{-(\gamma_{cu}+\eta)t}\mathcal{I}(t) = \mathcal{I}_c^1(t) + \mathcal{I}_c^2(t) + \mathcal{I}_u^1(t) + \mathcal{I}_u^2(t) + \mathcal{I}_s^1(t) + \mathcal{I}_s^2(t),$$

where for $N_c < 0$ to be determined,

$$\mathcal{I}_c^1(t) = \begin{cases} e^{-(\gamma_{cu}+\eta)t} \int_{N_c}^t e^{A(t-\tau)} P^c \left[F_\rho(\theta_\tau\omega, v(\tau,\omega,\xi)) - F_\rho(\theta_\tau\omega, v(\tau,\omega,\xi_0)) \right. \\ \qquad \left. - D_u F_\rho(\theta_\tau\omega, v(\tau,\omega,\xi_0))(v(\tau,\omega,\xi) - v(\tau,\omega,\xi_0))\right] \mathrm{d}\tau & , \text{ if } t < N_c, \\ 0 & , \text{ if } t \ge N_c. \end{cases}$$

$$\mathcal{I}_c^2(t) = \begin{cases} e^{-(\gamma_{cu}+\eta)t} \int_0^{N_c} e^{A(t-\tau)} P^c \left[F_\rho(\theta_\tau\omega, v(\tau,\omega,\xi)) - F_\rho(\theta_\tau\omega, v(\tau,\omega,\xi_0)) \right. \\ \qquad \left. - D_u F_\rho(\theta_\tau\omega, v(\tau,\omega,\xi_0))(v(\tau,\omega,\xi) - v(\tau,\omega,\xi_0))\right] \mathrm{d}\tau & , \text{ if } t < N_c, \\ e^{-(\gamma_{cu}+\eta)t} \int_0^t e^{A(t-\tau)} P^c \left[F_\rho(\theta_\tau\omega, v(\tau,\omega,\xi)) - F_\rho(\theta_\tau\omega, v(\tau,\omega,\xi_0)) \right. \\ \qquad \left. - D_u F_\rho(\theta_\tau\omega, v(\tau,\omega,\xi_0))(v(\tau,\omega,\xi) - v(\tau,\omega,\xi_0))\right] \mathrm{d}\tau & , \text{ if } t \ge N_c. \end{cases}$$

And for $N_u < 0$ to be determined,

$$\mathcal{I}_u^1(t) = \begin{cases} e^{-(\gamma_{cu}+\eta)t} \int_{N_u}^t e^{A(t-\tau)} P^u \left[F_\rho(\theta_\tau\omega, v(\tau,\omega,\xi)) - F_\rho(\theta_\tau\omega, v(\tau,\omega,\xi_0)) \right. \\ \qquad \left. - D_u F_\rho(\theta_\tau\omega, v(\tau,\omega,\xi_0))(v(\tau,\omega,\xi) - v(\tau,\omega,\xi_0))\right] \mathrm{d}\tau & , \text{ if } t < N_u, \\ 0 & , \text{ if } t \geq N_u. \end{cases}$$

$$\mathcal{I}_u^2(t) = \begin{cases} e^{-(\gamma_{cu}+\eta)t} \int_0^{N_u} e^{A(t-\tau)} P^u \left[F_\rho(\theta_\tau\omega, v(\tau,\omega,\xi)) - F_\rho(\theta_\tau\omega, v(\tau,\omega,\xi_0)) \right. \\ \qquad \left. - D_u F_\rho(\theta_\tau\omega, v(\tau,\omega,\xi_0))(v(\tau,\omega,\xi) - v(\tau,\omega,\xi_0))\right] \mathrm{d}\tau & , \text{ if } t < N_u, \\ e^{-(\gamma_{cu}+\eta)t} \int_0^{t} e^{A(t-\tau)} P^u \left[F_\rho(\theta_\tau\omega, v(\tau,\omega,\xi)) - F_\rho(\theta_\tau\omega, v(\tau,\omega,\xi_0)) \right. \\ \qquad \left. - D_u F_\rho(\theta_\tau\omega, v(\tau,\omega,\xi_0))(v(\tau,\omega,\xi) - v(\tau,\omega,\xi_0))\right] \mathrm{d}\tau & , \text{ if } t \geq N_u. \end{cases}$$

And for $N_s < 0$ to be determined,

$$\mathcal{I}_s^1(t) = \begin{cases} e^{-(\gamma_{cu}+\eta)t} \int_{N_s}^t e^{A(t-\tau)} P^s \left[F_\rho(\theta_\tau\omega, v(\tau,\omega,\xi)) - F_\rho(\theta_\tau\omega, v(\tau,\omega,\xi_0)) \right. \\ \qquad \left. - D_u F_\rho(\theta_\tau\omega, v(\tau,\omega,\xi_0))(v(\tau,\omega,\xi) - v(\tau,\omega,\xi_0))\right] \mathrm{d}\tau & , \text{ if } t > N_s, \\ 0 & , \text{ if } t \leq N_s. \end{cases}$$

$$\mathcal{I}_s^2(t) = \begin{cases} e^{-(\gamma_{cu}+\eta)t} \int_{-\infty}^{N_s} e^{A(t-\tau)} P^s \left[F_\rho(\theta_\tau\omega, v(\tau,\omega,\xi)) - F_\rho(\theta_\tau\omega, v(\tau,\omega,\xi_0)) \right. \\ \qquad \left. - D_u F_\rho(\theta_\tau\omega, v(\tau,\omega,\xi_0))(v(\tau,\omega,\xi) - v(\tau,\omega,\xi_0))\right] \mathrm{d}\tau & , \text{ if } t > N_s, \\ e^{-(\gamma_{cu}+\eta)t} \int_{-\infty}^{t} e^{A(t-\tau)} P^s \left[F_\rho(\theta_\tau\omega, v(\tau,\omega,\xi)) - F_\rho(\theta_\tau\omega, v(\tau,\omega,\xi_0)) \right. \\ \qquad \left. - D_u F_\rho(\theta_\tau\omega, v(\tau,\omega,\xi_0))(v(\tau,\omega,\xi) - v(\tau,\omega,\xi_0))\right] \mathrm{d}\tau & , \text{ if } t \leq N_s. \end{cases}$$

Now take $\eta \leq \eta^*$, for $t < N_c$, we have

$$\begin{aligned} |\mathcal{I}_c^1(t)| &\leq e^{-(\gamma_{cu}+\eta)t} \int_t^{N_c} K_A e^{\beta_2(\tau-t)} \cdot 2LipF_\rho(\theta_\tau\omega) |v(\cdot,\omega,\xi) - v(\cdot,\omega,\xi_0)|^-_{\gamma_{cu}+2\eta^*} e^{(\gamma_{cu}+2\eta^*)\tau} \mathrm{d}\tau \\ &\leq \int_t^{N_c} K_A e^{(\gamma_{cu}+2\eta^*+\beta_2)(\tau-t)} \cdot e^{(2\eta^*-\sigma)t} \cdot 2LipF_\rho(\theta_\tau\omega) \cdot 2K_A|\xi - \xi_0| \mathrm{d}\tau \\ &\leq \tfrac{2}{3} e^{\eta^* N_c} |\xi - \xi_0|. \end{aligned}$$

Note that $\eta^* > 0$, so we can choose N_c sufficiently negative so that for any given $\epsilon > 0$,

$$\sup_{t \leq 0} |\mathcal{I}_c^1(t)| \leq \frac{\epsilon}{6} |\xi - \xi_0|. \tag{5.11}$$

Fix such N_c, for $t \le 0$, we have

$$|\mathcal{I}_c^2(t)| \le \int_{N_c}^0 K_A e^{(\gamma_{cu}+\eta+\beta_2)(\tau-t)} \cdot |v(\cdot,\omega,\xi) - v(\cdot,\omega,\xi_0)|^-_{\gamma_{cu}+\sigma}$$
$$\cdot \int_0^1 \|DF_\rho(\theta_\tau\omega, \mu \cdot v(\tau,\omega,\xi) + (1-\mu)\cdot v(\tau,\omega,\xi_0)) - DF_\rho(\theta_\tau\omega, v(\tau,\omega,\xi_0))\| \mathrm{d}\mu \mathrm{d}\tau.$$

Since $DF_\rho(\theta_\tau\omega, u)$ is continuous in (τ, u) and $v(\tau,\omega,\xi)$ is continuous in (τ,ξ) for τ in a finite compact interval $[N_c, 0]$ which is independent of t, we have that

$$\|DF_\rho(\theta_\tau\omega, \mu \cdot v(\tau,\omega,\xi) + (1-\mu)\cdot v(\tau,\omega,\xi_0)) - DF_\rho(\theta_\tau\omega, v(\tau,\omega,\xi_0))\| \to 0$$

as $\xi \to \xi_0$. Then using the fact that $\|D_u F_\rho(\omega,u)\| \le LipF_\rho(\omega)$ and by the dominated convergence theorem we have for the same given ϵ, there exists a $\kappa_c > 0$ so that if $|\xi - \xi_0| < \kappa_c$, we get

$$\sup_{t\le 0} |\mathcal{I}_c^2(t)| \le \frac{\epsilon}{6}|\xi - \xi_0|. \tag{5.12}$$

Similarly, by choosing N_u to be sufficiently negative, we have that

$$\sup_{t\le 0} |\mathcal{I}_u^1(t)| \le \frac{\epsilon}{6}|\xi - \xi_0|, \tag{5.13}$$

and for such fixed N_u and the given ϵ, there exists $\kappa_u > 0$ so that if $|\xi - \xi_0| < \kappa_u$, we get

$$\sup_{t\le 0} |\mathcal{I}_u^2(t)| \le \frac{\epsilon}{6}|\xi - \xi_0|. \tag{5.14}$$

And by choosing N_s to be sufficiently negative, we have that

$$\sup_{t\le 0} |\mathcal{I}_s^2(t)| \le \frac{\epsilon}{6}|\xi - \xi_0|, \tag{5.15}$$

and for such fixed N_s and the given ϵ, there exists $\kappa_s > 0$ so that if $|\xi - \xi_0| < \kappa_s$, we get

$$\sup_{t\le 0} |\mathcal{I}_s^1(t)| \le \frac{\epsilon}{6}|\xi - \xi_0|. \tag{5.16}$$

Taking $\kappa := \min\{\kappa_c, \kappa_u, \kappa_s\}$ and combining (5.11)-(5.16), we obtain that

$$|\mathcal{I}|^{-}_{\gamma_{cu}+\eta} \le \epsilon|\xi - \xi_0|, \qquad \text{if } |\xi - \xi_0| < \kappa.$$

This implies the desired fact (5.10).

Using (5.5), $D_\xi v(t,\omega,\xi) : H^{cu} \to L(H^{cu}, C^{-}_{\gamma_{cu}+\eta})$ satisfies

$$\begin{aligned} D_\xi v(t,\omega,\xi) = e^{At}P^{cu} &+ \textstyle\int_0^t e^{A(t-\tau)} D_u F_{\rho,c}(\theta_\tau\omega, v(\tau,\omega,\xi)) D_\xi v(\tau,\omega,\xi)\mathrm{d}\tau \\ &+ \textstyle\int_0^t e^{A(t-\tau)} D_u F_{\rho,u}(\theta_\tau\omega, v(\tau,\omega,\xi)) D_\xi v(\tau,\omega,\xi)\mathrm{d}\tau \\ &+ \textstyle\int_{-\infty}^t e^{A(t-\tau)} D_u F_{\rho,s}(\theta_\tau\omega, v(\tau,\omega,\xi)) D_\xi v(\tau,\omega,\xi)\mathrm{d}\tau. \end{aligned}$$

Furthermore we can compute as above to obtain

$$e^{-(\gamma_{cu}+\eta)t}\|D_\xi v(t,\omega,\xi)\| \le K_A + \frac{1}{6K_A}\|D_\xi v(\cdot,\omega,\xi)\|_{L(H^{cu},C^{-}_{\gamma_{cu}+\eta})},$$

so

$$\|D_\xi v(\cdot,\omega,\xi)\|_{L(H^{cu},C^{-}_{\gamma_{cu}+\eta})} \le \frac{K_A}{1-\frac{1}{6K_A}} := K_{1,\eta^*}.$$

Next we prove that $D_\xi v(t,\omega,\xi)$ is continuous with respect to ξ. First notice that the above argument works if we replace the requirement $\eta \le \eta^*$ by $\eta \le \frac{3}{2}\eta^*$. Now we define for $\eta \le \frac{3}{2}\eta^*$ a map $\mathcal{S}_1 : L(H^{cu}, C^{-}_{\gamma_{cu}+\eta}) \to L(H^{cu}, C^{-}_{\gamma_{cu}+\eta})$ by the RHS of (5.9), and write

$$D_\xi v(t,\omega,\xi) - D_\xi v(t,\omega,\xi_0) = \mathcal{S}_1(D_\xi v(\cdot,\omega,\xi) - D_\xi v(\cdot,\omega,\xi_0))(t) + \mathcal{T}(t),$$

where

$$\begin{aligned} \mathcal{T}(t) = &\textstyle\int_0^t e^{A(t-\tau)} P^c[D_u F_\rho(\theta_\tau\omega, v(\tau,\omega,\xi)) - D_u F_\rho(\theta_\tau\omega, v(\tau,\omega,\xi_0))] D_\xi v(\tau,\omega,\xi)\mathrm{d}\tau \\ &+ \textstyle\int_0^t e^{A(t-\tau)} P^u[D_u F_\rho(\theta_\tau\omega, v(\tau,\omega,\xi)) - D_u F_\rho(\theta_\tau\omega, v(\tau,\omega,\xi_0))] D_\xi v(\tau,\omega,\xi)\mathrm{d}\tau \\ &+ \textstyle\int_{-\infty}^t e^{A(t-\tau)} P^s[D_u F_\rho(\theta_\tau\omega, v(\tau,\omega,\xi)) - D_u F_\rho(\theta_\tau\omega, v(\tau,\omega,\xi_0))] D_\xi v(\tau,\omega,\xi)\mathrm{d}\tau. \end{aligned}$$

Note that as computed after (5.9), $\mathcal{S}_1$ is a bounded operator with norm strictly less than 1.

Using the same approach as above we can get that $\|\mathcal{T}\|_{L(H^{cu},C^-_{\gamma_{cu}+\eta})} = o(1)$ as $\xi \to \xi_0$, for $\forall \eta \le \eta^*$. To be precise, for instance, we can deal with the first integral in formula of $\mathcal{T}$ as the following. For $t < N$, N to be determined, we have

$$
\begin{aligned}
& e^{-(\gamma_{cu}+\eta)t} |\int_0^t e^{A(t-\tau)} P^c [D_u F_\rho(\theta_\tau \omega, v(\tau,\omega,\xi)) - D_u F_\rho(\theta_\tau \omega, v(\tau,\omega,\xi_0))] D_\xi v(\tau,\omega,\xi) \mathrm{d}\tau| \\
\le & e^{-(\gamma_{cu}+\eta)t} |\int_0^N e^{A(t-\tau)} P^c [D_u F_\rho(\theta_\tau \omega, v(\tau,\omega,\xi)) - D_u F_\rho(\theta_\tau \omega, v(\tau,\omega,\xi_0))] D_\xi v(\tau,\omega,\xi) \mathrm{d}\tau| \\
& + e^{-(\gamma_{cu}+\eta)t} |\int_N^t e^{A(t-\tau)} P^c [D_u F_\rho(\theta_\tau \omega, v(\tau,\omega,\xi)) - D_u F_\rho(\theta_\tau \omega, v(\tau,\omega,\xi_0))] D_\xi v(\tau,\omega,\xi) \mathrm{d}\tau| \\
\le & \int_t^N K_A e^{(\gamma_{cu}+\frac{3}{2}\eta^*+\beta_2)(\tau-t)} \cdot 2Lip F_\rho(\theta_\tau\omega) \cdot e^{\frac{1}{2}\eta^* t} \|v(\cdot,\omega,\xi)\|_{L(H^{cu},C^-_{\gamma_{cu}+\frac{3}{2}\eta^*})} \mathrm{d}\tau \\
& + \int_N^0 K_A e^{(\gamma_{cu}+\eta+\beta_2)(\tau-t)} \|v(\cdot,\omega,\xi)\|_{L(H^{cu},C^-_{\gamma_{cu}+\sigma})} \\
& \qquad \times \int_0^1 \|D_u F_\rho(\theta_\tau\omega, \mu v(\tau,\omega,\xi) + (1-\mu) v(\tau,\omega,\xi_0)) - D_u F_\rho(\theta_\tau\omega, v(\tau,\omega,\xi_0))\| \mathrm{d}\mu \cdot \mathrm{d}\tau.
\end{aligned}
$$

Now the first term is $o(1)$ as $\xi \to \xi_0$, provided that N is chosen to be sufficiently negative. And for such N, by continuity of DF_ρ and the solution, we may confirm that the integral is also $o(1)$ using the dominated convergence theorem. The case $t \ge N$ follows similarly from the bound for the second term above. The other terms in $\mathcal{T}$ can be dealt with similarly and thus we conclude that

$$D_\xi v(t,\omega,\xi) - D_\xi v(t,\omega,\xi_0) = (id - \mathcal{S}_1)^{-1} \mathcal{T}(t) = o(1)$$

as $\xi \to \xi_0$, yielding that $D_\xi v(t,\omega,\xi)$ is continuous in ξ.

Next assume $0 < \alpha \le 1$, and we justify the Hölder continuity of $D_\xi v(\cdot,\omega,\xi)$. We need to use the boundedness of $Hol DF_\rho(\omega)$ and the extra gap in the spectrum as

$$-\beta_1 < (1+\alpha)\gamma_{cu} < (1+\alpha)\gamma_{cu} + 2\eta^* < -\beta_2.$$

We still focus on the first term of $\mathcal{T}$. For $\eta \le \eta^*$, we can choose $\eta' \le 2\eta^*$ and $\eta'' \le \frac{3}{2}\eta^*$ such that $\alpha\eta' + \eta'' = \eta$, and estimate

$$
\begin{aligned}
&e^{-((1+\alpha)\gamma_{cu}+\eta)t} \left| \int_0^t e^{A(t-\tau)} P^c [D_u F_\rho(\theta_\tau\omega, v(\tau,\omega,\xi)) - D_u F_\rho(\theta_\tau\omega, v(\tau,\omega,\xi_0))] D_\xi v(\tau,\omega,\xi) \mathrm{d}\tau \right| \\
\leq\, & e^{-((1+\alpha)\gamma_{cu}+\eta)t} \int_t^0 K_A e^{\beta_2(\tau-t)} HolDF_\rho(\theta_\tau\omega) |v(\tau,\omega,\xi) - v(\tau,\omega,\xi_0)|^\alpha \|D_\xi v(\tau,\omega,\xi)\| \mathrm{d}\tau \\
\leq\, & K_A HolDF_\rho(\cdot)(2K_A)^\alpha |\xi-\xi_0|^\alpha \int_t^0 e^{-((1+\alpha)\gamma_{cu}+\eta)t + \beta_2(\tau-t) + \alpha(\gamma_{cu}+\eta')\tau + (\gamma_{cu}+\eta'')\tau} \mathrm{d}\tau \\
\leq\, & K_A HolDF_\rho(\cdot)(2K_A)^\alpha |\xi-\xi_0|^\alpha \int_t^0 e^{((1+\alpha)\gamma_{cu}+\eta+\beta_2)(\tau-t)} \mathrm{d}\tau \\
\leq\, & K|\xi-\xi_0|^\alpha,
\end{aligned}
$$

for some constant K. We similarly bound the other two terms in $\mathcal{T}$ to conclude that

$$
\|\mathcal{T}\|_{L(H^{cu}, C^-_{(1+\alpha)\gamma_{cu}+\eta})} \leq K_{\mathcal{T}} |\xi-\xi_0|^\alpha,
$$

where $K_{\mathcal{T}}$ is a constant depending on $K_A,\ HolDF_\rho(\omega),\ \alpha,\ \gamma_{cu},\ \beta_1,\ \beta_2,\ \sigma$. This implies that $D_\xi v(t,\omega,\xi)$ is Hölder continuous in ξ.

Step 3. We prove $v(\cdot,\omega,\xi)$ is $C^{r,\alpha}$ for $r \geq 2$. We first show that it is C^r. We make the inductive assumption that $v(t,\omega,\xi)$ is C^j from H^{cu} to $C^-_{j\gamma_{cu}}$ for all $1 \leq j \leq m-1$, $2 \leq m \leq r$. Also assume that there exists coefficients $K_{j,\eta}$, $1 \leq j \leq m-1$, so that for $\eta \leq \eta^*$,

$$
\|D_\xi^{(j)} v(\cdot,\omega,\xi)\|_{L^j(H^{cu}, C^-_{j\gamma_{cu,r}+\eta})} \leq K_{j,\eta^*},
$$

and prove it for $j = m$.

By computation, we find $D_\xi^{(m-1)} v(t,\omega,\xi)$ satisfies the following equation,

$$
\begin{aligned}
D_\xi^{(m-1)} v(t,\omega,\xi) = & \int_0^t e^{A(t-\tau)} P^c D_u F_\rho(\theta_\tau\omega, v(\tau,\omega,\xi)) D_\xi^{(m-1)} v(\tau,\omega,\xi) \mathrm{d}\tau \\
& + \int_0^t e^{A(t-\tau)} P^u D_u F_\rho(\theta_\tau\omega, v(\tau,\omega,\xi)) D_\xi^{(m-1)} v(\tau,\omega,\xi) \mathrm{d}\tau \\
& + \int_{-\infty}^t e^{A(t-\tau)} P^s D_u F_\rho(\theta_\tau\omega, v(\tau,\omega,\xi)) D_\xi^{(m-1)} v(\tau,\omega,\xi) \mathrm{d}\tau \\
& + \int_0^t e^{A(t-\tau)} P^c R_{m-1}(\tau,\omega,\xi) \mathrm{d}\tau \\
& + \int_0^t e^{A(t-\tau)} P^u R_{m-1}(\tau,\omega,\xi) \mathrm{d}\tau \\
& + \int_{-\infty}^t e^{A(t-\tau)} P^s R_{m-1}(\tau,\omega,\xi) \mathrm{d}\tau,
\end{aligned}
$$

where

$$R_{m-1}(\tau,\omega,\xi)=\sum_{l=0}^{m-3}\binom{m-2}{l}D_\xi^{(m-2-l)}(D_uF_\rho(\theta_\tau\omega,v(\tau,\omega,\xi)))D_\xi^{(l+1)}v(\tau,\omega,\xi).$$

Applying the chain rule to $D_\xi^{(m-2-l)}(D_uF_\rho(\theta_\tau\omega,v(\tau,\omega,\xi)))$, we observe that each term in R_{m-1} contains factors $D_u^{(l_1)}F_\rho(\theta_\tau\omega,v(\tau,\omega,\xi))$ for some $2\le l_1\le m-1$, and at least two derivatives $D_\xi^{(l_2)}v(\tau,\omega,\xi)$ and $D_\xi^{(l_3)}v(\tau,\omega,\xi)$ for some $l_2,\ l_3\in\{1,\cdots,m-2\}$. Since $D_\xi^{(l)}v(\cdot,\omega,\xi)\in L^l(H^{cu},C^-_{l\gamma_{cu}+\eta})$ for $l=1,\cdots,m-1$, and F_ρ is C^r, we have $R_{m-1}(\tau,\omega,\xi):H^{cu}\to L^{m-1}(H^{cu},C^-_{(m-1)\gamma_{cu}+\eta})$ are C^1 in ξ. Also by (5.4) we have $LipD^{(j)}F_\rho(\omega)<+\infty$ for $j\le r$, this together with the bounds for $D_\xi^{(j)}v(\cdot,\omega,\xi)$ in the induction assumption yields the fact that $\|D_\xi R_{m-1}(\tau,\omega,\xi)\|<+\infty$.

Now we define $\mathcal{S}_{m-1}:L^{m-1}(H^{cu},C^-_{(m-1)\gamma_{cu}+\eta})\to L^{m-1}(H^{cu},C^-_{(m-1)\gamma_{cu}+\eta})$ by the RHS of (5.9), as above $\|\mathcal{S}_{m-1}\|<1$. And let $\xi,\ \xi_0\in H^{cu}$, then

$$\begin{aligned}&D_\xi^{(m-1)}v(t,\omega,\xi)-D_\xi^{(m-1)}v(t,\omega,\xi_0)-\mathcal{S}_{m-1}(D_\xi^{(m-1)}v(\cdot,\omega,\xi)-D_\xi^{(m-1)}v(\cdot,\omega,\xi_0))(t)\\&=J(\omega,\xi)-J(\omega,\xi_0)-D_\xi J(\omega,\xi_0)(\xi-\xi_0)+D_\xi J(\omega,\xi_0)(\xi-\xi_0)\\&\quad+\mathcal{I}_m^1(t)+\mathcal{I}_m^2(t)+\mathcal{I}_m^3(t)+\mathcal{I}_m^4(t)(\xi-\xi_0),\end{aligned}\tag{5.17}$$

where

$$\begin{aligned}J(\omega,\xi)=&\textstyle\int_0^t e^{A(t-\tau)}P^cR_{m-1}(\tau,\omega,\xi)\mathrm{d}\tau\\&+\textstyle\int_0^t e^{A(t-\tau)}P^uR_{m-1}(\tau,\omega,\xi)\mathrm{d}\tau\\&+\textstyle\int_{-\infty}^t e^{A(t-\tau)}P^sR_{m-1}(\tau,\omega,\xi)\mathrm{d}\tau,\end{aligned}\tag{5.18}$$

and

$$\begin{aligned}\mathcal{I}_m^1(t) = \textstyle\int_0^t e^{A(t-\tau)}P^c[&D_uF_\rho(\theta_\tau\omega, v(\tau,\omega,\xi)) - D_uF_\rho(\theta_\tau\omega, v(\tau,\omega,\xi_0))\\ &-D_u^{(2)}F(\theta_\tau\omega, v(\tau,\omega,\xi_0))(v(\tau,\omega,\xi)-v(\tau,\omega,\xi_0))]D_\xi^{(m-1)}v(\tau,\omega,\xi_0)\mathrm{d}\tau\\ +\textstyle\int_0^t e^{A(t-\tau)}P^u[&D_uF_\rho(\theta_\tau\omega, v(\tau,\omega,\xi)) - D_uF_\rho(\theta_\tau\omega, v(\tau,\omega,\xi_0))\\ &-D_u^{(2)}F(\theta_\tau\omega, v(\tau,\omega,\xi_0))(v(\tau,\omega,\xi)-v(\tau,\omega,\xi_0))]D_\xi^{(m-1)}v(\tau,\omega,\xi_0)\mathrm{d}\tau\\ +\textstyle\int_{-\infty}^t e^{A(t-\tau)}P^s[&D_uF_\rho(\theta_\tau\omega, v(\tau,\omega,\xi)) - D_uF_\rho(\theta_\tau\omega, v(\tau,\omega,\xi_0))\\ &-D_u^{(2)}F(\theta_\tau\omega, v(\tau,\omega,\xi_0))(v(\tau,\omega,\xi)-v(\tau,\omega,\xi_0))]D_\xi^{(m-1)}v(\tau,\omega,\xi_0)\mathrm{d}\tau,\end{aligned}$$

and

$$\begin{aligned}\mathcal{I}_m^2(t) = \textstyle\int_0^t e^{A(t-\tau)}P^c[&D^{(2)}F(\theta_\tau\omega, v(\tau,\omega,\xi_0))(v(\tau,\omega,\xi)-v(\tau,\omega,\xi_0))\\ &-D_u^{(2)}F(\theta_\tau\omega, v(\tau,\omega,\xi_0))D_\xi v(\tau,\omega,\xi_0)(\xi-\xi_0)]D_\xi^{(m-1)}v(\tau,\omega,\xi_0)\mathrm{d}\tau\\ +\textstyle\int_0^t e^{A(t-\tau)}P^u[&D^{(2)}F(\theta_\tau\omega, v(\tau,\omega,\xi_0))(v(\tau,\omega,\xi)-v(\tau,\omega,\xi_0))\\ &-D_u^{(2)}F(\theta_\tau\omega, v(\tau,\omega,\xi_0))D_\xi v(\tau,\omega,\xi_0)(\xi-\xi_0)]D_\xi^{(m-1)}v(\tau,\omega,\xi_0)\mathrm{d}\tau\\ +\textstyle\int_{-\infty}^t e^{A(t-\tau)}P^s[&D^{(2)}F(\theta_\tau\omega, v(\tau,\omega,\xi_0))(v(\tau,\omega,\xi)-v(\tau,\omega,\xi_0))\\ &-D_u^{(2)}F(\theta_\tau\omega, v(\tau,\omega,\xi_0))D_\xi v(\tau,\omega,\xi_0)(\xi-\xi_0)]D_\xi^{(m-1)}v(\tau,\omega,\xi_0)\mathrm{d}\tau,\end{aligned}$$

and

$$\begin{aligned}\mathcal{I}_m^3(t) = &\textstyle\int_0^t e^{A(t-\tau)}P^c[D_uF_\rho(\theta_\tau\omega, v(\tau,\omega,\xi)) - D_uF_\rho(\theta_\tau\omega, v(\tau,\omega,\xi_0))]\\ &\qquad\times(D_\xi^{(m-1)}v(\tau,\omega,\xi) - D_\xi^{(m-1)}v(\tau,\omega,\xi_0))\mathrm{d}\tau\\ &+\textstyle\int_0^t e^{A(t-\tau)}P^u[D_uF_\rho(\theta_\tau\omega, v(\tau,\omega,\xi)) - D_uF_\rho(\theta_\tau\omega, v(\tau,\omega,\xi_0))]\\ &\qquad\times(D_\xi^{(m-1)}v(\tau,\omega,\xi) - D_\xi^{(m-1)}v(\tau,\omega,\xi_0))\mathrm{d}\tau\\ &+\textstyle\int_{-\infty}^t e^{A(t-\tau)}P^s[D_uF_\rho(\theta_\tau\omega, v(\tau,\omega,\xi)) - D_uF_\rho(\theta_\tau\omega, v(\tau,\omega,\xi_0))]\\ &\qquad\times(D_\xi^{(m-1)}v(\tau,\omega,\xi) - D_\xi^{(m-1)}v(\tau,\omega,\xi_0))\mathrm{d}\tau,\end{aligned}$$

and

$$
\begin{aligned}
&\mathcal{I}_m^4(t)(\xi-\xi_0)\\
=&\int_0^t e^{A(t-\tau)}P^c[D_u^{(2)}F(\theta_\tau\omega, v(\tau,\omega,\xi_0))D_\xi v(\tau,\omega,\xi_0)(\xi-\xi_0)]D_\xi^{(m-1)}v(\tau,\omega,\xi_0)\mathrm{d}\tau\\
&+\int_0^t e^{A(t-\tau)}P^u[D_u^{(2)}F(\theta_\tau\omega, v(\tau,\omega,\xi_0))D_\xi v(\tau,\omega,\xi_0)(\xi-\xi_0)]D_\xi^{(m-1)}v(\tau,\omega,\xi_0)\mathrm{d}\tau\\
&+\int_{-\infty}^t e^{A(t-\tau)}P^s[D_u^{(2)}F(\theta_\tau\omega, v(\tau,\omega,\xi_0))D_\xi v(\tau,\omega,\xi_0)(\xi-\xi_0)]D_\xi^{(m-1)}v(\tau,\omega,\xi_0)\mathrm{d}\tau,
\end{aligned}
$$

So we can rewrite the previous identity (5.17) as

$$
\begin{aligned}
&D_\xi^{(m-1)}v(t,\omega,\xi)-D_\xi^{(m-1)}v(t,\omega,\xi_0)-(id-\mathcal{S}_{m-1})^{-1}(D_\xi J(\omega,\xi_0)+\mathcal{I}_m^4(t))(\xi-\xi_0)\\
=&\,(id-\mathcal{S}_{m-1})^{-1}\{[J(\omega,\xi)-J(\omega,\xi_0)-D_\xi J(\omega,\xi_0)(\xi-\xi_0)]+\mathcal{I}_m^1(t)+\mathcal{I}_m^2(t)+\mathcal{I}_m^2(t)\}.
\end{aligned}
\tag{5.19}
$$

Using the same approach we used in step 2, we can show that the RHS of the above equation (5.19) under the norm $\|\cdot\|_{L^m(H^{cu},C^-_{m\gamma_{cu}+\eta})}$ is equal to $o(|\xi-\xi_0|)$ as $\xi\to\xi_0$, suggesting the existence of $D_\xi^m v(t,\omega,\xi)$ (Notice that $D_\xi^{(m-1)}v(t,\omega,\xi)$ can also be shown to lie in $L^{m-1}(H^{cu},C^-_{m\gamma_{cu}+\eta})$ by a slight adjustment in the above argument). This ends the induction and we get that $v(\cdot,\omega,\xi)$ is C^r. Similarly we can follow the same approach used in step 2 and prove that the rth derivative $D_\xi^r v(t,\omega,\xi)$ is Hölder continuous. Indeed, we define $\mathcal{S}_m: L^m(H^{cu},C^-_{m\gamma_{cu}+\eta})\to L^m(H^{cu},C^-_{m\gamma_{cu}+\eta})$ by the RHS of (5.9) to derive

$$
D_\xi^{(m)}v(t,\omega,\xi)-D_\xi^{(m)}v(t,\omega,\xi_0)=(id-\mathcal{S}_m)^{-1}\{[J(\omega,\xi)-J(\omega,\xi_0)]+\mathcal{I}_r(t)\},\tag{5.20}
$$

where $J(\omega,\xi)$ is defined as in (5.18) with R_{m-1} replaced by R_r, and

$$
\begin{aligned}
\mathcal{I}_r(t)=&\int_0^t e^{A(t-\tau)}P^c(D_uF_\rho(\theta_\tau\omega,v(\tau,\omega,\xi))-D_uF_\rho(\theta_\tau\omega,v(\tau,\omega,\xi_0)))D_\xi^{(r)}v(\tau,\omega,\xi)\mathrm{d}\tau\\
&+\int_0^t e^{A(t-\tau)}P^u(D_uF_\rho(\theta_\tau\omega,v(\tau,\omega,\xi))-D_uF_\rho(\theta_\tau\omega,v(\tau,\omega,\xi_0)))D_\xi^{(r)}v(\tau,\omega,\xi)\mathrm{d}\tau\\
&+\int_{-\infty}^t e^{A(t-\tau)}P^s(D_uF_\rho(\theta_\tau\omega,v(\tau,\omega,\xi))-D_uF_\rho(\theta_\tau\omega,v(\tau,\omega,\xi_0)))D_\xi^{(r)}v(\tau,\omega,\xi)\mathrm{d}\tau.
\end{aligned}
$$

By showing that the RHS of (5.20) is bounded by $K_{I_r}|\xi-\xi_0|^\alpha$ for some constant K_{I_r} we can justify the Hölder continuity. Only notice that there is always a term involving

$$D_u^{(r)}F_\rho(\theta_\tau\omega, v(\tau,\omega,\xi))(D_\xi v(\tau,\omega,\xi))^r$$

showing up in $R_r(\tau,\omega,\xi)$, and this is why we need the Hölder bound $HolD^{(r)}F_\rho(\omega)$ and the extra spectrum gap $-\beta_1 < (r+\alpha)\gamma_{cu} < (r+\alpha)\gamma_{cu} + 2\eta^* < -\beta_2$. This completes the proof. □

5.4 Proof of the Main Theorem for Invariant Manifolds

Now let

$$h^{cu}(\omega,\xi) := P^s v(0,\omega,\xi) = \int_{-\infty}^{0} e^{-A\tau} F_{\rho,s}(\theta_\tau\omega, v(\tau,\omega,\xi))\mathrm{d}\tau, \tag{5.21}$$

where $v(\cdot,\omega,\xi)$ is the fixed point of $\mathcal{J}(\cdot,\omega,\xi)$. We now discuss the existence of center-unstable manifolds for equation (3.11).

Theorem 5.3. *Assume that Hypotheses 3.1.1-3.1.3 hold with $r \geq 1$ and $0 \leq \alpha \leq 1$. For any $-\beta_1 < \gamma_{cu} < -\beta_2$, if we choose $\rho(\omega)$ so small that (5.1) holds, then there exists a Lipschitz invariant center-unstable manifold of the random equation (3.11):*

$$M^{cu}(\omega) = \{\xi + h^{cu}(\omega,\xi) \mid \xi \in H^{cu}\},$$

where $h^{cu}(\omega,\cdot) : H^{cu} \to H^s$ is a Lipschitz continuous mapping which is given by (5.21) with $h^{cu}(\omega,0) = 0$ and $Liph^{cu}(\omega,\cdot) < \frac{1}{3}$, and h^{cu} is measurable in (ω,ξ).
Furthermore, assume γ_{cu} satisfies (5.2), and for $\eta^* > 0$ so that (5.3) holds, if we choose $\rho(\omega)$ even smaller so that (5.4) holds, then $M^{cu}(\omega)$ is a $C^{r,\alpha}$ invariant center-unstable manifold of the random equation (3.11).

Proof. By Lemma 5.1 and 5.2, $x_0 \in M^{cu}(\omega)$ if and only if there exists $\xi \in H^{cu}$ such that

$x_0 = \xi + h^{cu}(\omega, \xi)$, which implies that

$$M^{cu}(\omega) = \{\xi + h^{cu}(\omega, \xi) | \xi \in H^{cu}\}.$$

For any $\xi,\ \xi_0 \in H^{cu}$,

$$\begin{aligned} & |h^{cu}(\omega, \xi) - h^{cu}(\omega, \xi_0)| \\ \leq & \int_{-\infty}^{0} K_A e^{\beta_1 \tau} Lip F_\rho(\theta_\tau \omega) e^{\gamma_{cu} \tau} |v(\cdot, \omega, \xi) - v(\cdot, \omega, \xi_0)|_{\gamma_{cu}}^{-} \mathrm{d}\tau \\ < & \tfrac{1}{6K_A} \cdot 2K_A |\xi - \xi_0| \\ = & \tfrac{1}{3} |\xi - \xi_0|. \end{aligned}$$

Then $h^{cu}(\omega, \xi)$ is Lipschitz continuous in $\xi \in H^{cu}$ with Lipschitz constant less than $\frac{1}{3}$.

Let H_c be a countable dense set of the separable space H. For each $x \in H$, we observe that

$$\omega \mapsto \inf_{y \in H} |x - (P^{cu} y + h^{cu}(\omega, P^{cu} y))| = \inf_{y \in H_c} |x - (P^{cu} y + h^{cu}(\omega, P^{cu} y))|.$$

Using Lemma 5.2 (1) and Lemma III.14 in Castaing and Valadier [CV77], we obtain that $M^{cu}(\omega)$ is $\mathcal{F}$-measurable.

Next we show that $M^{cu}(\omega)$ is invariant, i.e., $\forall s \geq 0$,

$$u(s, \omega, M^{cu}(\omega)) \subseteq M^{cu}(\theta_s \omega),$$

We note that for each fixed $s \geq 0$ and $x_0 \in M^{cu}(\omega)$, $u(t+s, \omega, x_0)$ is a solution of the equation

$$u_t = Au + F_\rho(\theta_{t+s}\omega, u), \qquad u(0) = u(s, \omega, x_0).$$

Thus $u(t, \theta_s \omega, v(s, \omega, x_0)) = u(t+s, \omega, x_0)$. Since $u(\cdot, \omega, x_0) \in C_{\gamma_{cu}}^{-}$, so is $u(\cdot, \theta_s \omega, u(s, \omega, x_0))$. Therefore $u(s, \omega, x_0) \in M^{cu}(\theta_s \omega)$, which implies that $u(s, \omega, M^{cu}(\omega)) \subseteq M^{cu}(\theta_s \omega)$.

Finally, the $C^{r,\alpha}$-smoothness of $h^{cu}(\omega, \xi)$ is a consequence of Lemma 5.2 (2) and by the definition of h^{cu} it is clear that $Dh^{cu}(\omega, 0) = 0$. This completes the proof. □

Remark 5.4.1. *The unique existence and smoothness of center manifold $M^c(\omega)$, unstable manifold $M^u(\omega)$, stable manifold $M^s(\omega)$, and center-stable manifold $M^{cs}(\omega)$ of the equation (3.11) can be verified via the similar approach.*

Now we can prove the local result stated in Theorem 3.2.

Proof of Theorem 3.2: This now follows immediately from Theorem 5.3 as above. Indeed, by Lemma 3.1 (i) the original equation (3.1) agrees with the truncated equation (3.11) within the proper tempered ball $B_{\rho(\omega)}$, therefore, the global graph as an invariant manifold for (3.11) restricts to a local graph for (3.1), then giving the desired local invariant manifold. □

Chapter 6. Foliations

In this chapter, we will show the existence of foliation structures for equation (3.1). With the same reasoning, in order to justify the local result stated in Theorem 3.3, we actually head to the truncated equation (3.11) and show the existence of global foliation structures when $\rho(\omega)$ is properly chosen. As the other two cases can be proved similarly, say, the existence of $\{\mathcal{F}^s\}$, $\{\mathcal{F}^{css}\}$, we focus on the case of an unstable foliation $\{\mathcal{F}^{cuu}(\omega,\cdot)\}$ on a given center-unstable manifold.

6.1 Settings and Notations

By Theorem 5.3, we have that if $\rho(\omega)$ is properly chosen so that (5.1) and (5.4) hold for $-\beta_1 < \gamma_{cu} < -\beta_2$ and $\eta^* > 0$ satisfying (5.2) and (5.3), then there exists a unique smooth center-unstable manifold $M^{cu}(\omega)$ for equation (3.11) given by

$$M^{cu}(\omega) = \{\xi + h^{cu}(\omega,\xi) \mid \xi \in H^{cu}\}.$$

Then we can restrict the original truncated equation (3.11) on $M^{cu}(\omega)$ as it is invariant, and get

$$u_t^{cu} = A_{cu}u^{cu} + F_{\rho.cu}(\theta_\tau\omega, u^{cu} + h^{cu}(\omega, u^{cu})),$$

where we have used the notations $A_{cu} := P^{cu} \circ A$ and $F_{\rho,cu} := P^{cu} \circ F_\rho$. To simplify it further, we set

$$\theta^{cu}(u^{cu}) := u^{cu} + h^{cu}(\omega, u^{cu}),$$

and write the restricted equation as

$$u_t^{cu} = A_{cu}u^{cu} + F_{\rho,cu}(\theta_\tau\omega, \theta^{cu}(u^{cu})), \qquad u^{cu}(0) = \xi \in H^{cu}. \tag{6.1}$$

Let $v(t,\omega,\xi)$ denote the solution to equation (6.1). We define for $\gamma_u \in (\beta_2,\beta_1)$, $\xi \in H^{cu}$,

$$\mathcal{F}^{cuu}(\omega,\xi) := \{\tilde{\xi} \in H^{cu} \mid v(\cdot,\omega,\tilde{\xi}) - v(\cdot,\omega,\xi) \in C^-_{\gamma_u,cu}\},$$

where

$$C^-_{\gamma_u,cu} = \{\phi \in C((-\infty,0],H^{cu}) \mid \sup_{-\infty<t\le 0} e^{-\gamma_u t}|\phi(t)| < +\infty\}.$$

For simplicity of notation, we denote by $C^-_{\gamma_u}$ for $C^-_{\gamma_u,cu}$ in this section. $\mathcal{F}^{cuu}(\omega,\xi)$ is then called the unstable leaf of $\xi \in H^{cu}$ for equation (6.1). We want to prove that the unstable leaf $\mathcal{F}^{cuu}(\omega,\xi)$ is given by the graph of a $C^{r,\alpha}$ function. To ensure the existence of $M^{cu}(\omega)$, we would always assume $\rho(\omega)$ is bounded as in Theorem 5.3, and for the sake of deriving the foliation leaves, we shrink it further in the following. Notice that by the choice of $\rho(\omega)$ in Theorem 5.3 and by definition of $h^{cu}(\omega,\cdot)$ given in (5.21), we have

$$Lip\theta^{cu} \le 1 + Liph^{cu}(\omega,\cdot) < 1 + \frac{1}{3} = \frac{4}{3},$$

and that $D^{(i)}\theta^{cu} < +\infty$ for $i = 1,\ldots,r$. We may choose $\rho(\omega)$ to be so small that for $\gamma_u \in (\beta_2,\beta_1)$, and $\sigma^* > 0$ so that $\gamma_u - \sigma^* \in (\beta_2,\beta_1)$, the Lipschitz constant $LipF_\rho(\omega)$ given in Lemma 3.1 is restricted as the following:

$$\begin{cases} K_A LipF_\rho(\omega) Lip\theta^{cu} < 1, \\ K_A LipF_\rho(\omega) Lip\theta^{cu} \sup_{0\le\sigma\le\sigma^*}(\frac{1}{\gamma_u-\sigma-\beta_2} + \frac{1}{\beta_1-\gamma_u+\sigma}) < \frac{1}{4K_A}, \end{cases} \tag{6.2}$$

and as $K_A \ge 1$, we have

$$K_A LipF_\rho(\omega) Lip\theta^{cu} \sup_{0\le\sigma\le\sigma^*}(\frac{1}{\gamma_u-\sigma-\beta_2} + \frac{1}{\beta_1-\gamma_u+\sigma}) < \frac{1}{4}$$

as well. Also for γ_u satisfying

$$\beta_2 < \gamma_u < (r+\alpha)\cdot\gamma_u < \beta_1, \tag{6.3}$$

and for $\eta^* > 0$ so that

$$\beta_2 < \gamma_u - 2\eta^* < \gamma_u - \eta^* < \beta_1, \tag{6.4}$$

we can restrict $LipF_\rho(\omega)$ as the following:

$$\begin{cases} K_A LipF_\rho(\omega) \max_{j\in\{1,\dots,r,r+\alpha\}} \sup_{0\le\eta\le 2\eta^*}\left(\frac{1}{j\gamma_u-\eta-\beta_2} + \frac{1}{\beta_1 - j\gamma_u+\eta}\right) < \frac{1}{4K_A}, \\ LipD^{(i)}F_\rho(\omega) < +\infty,\ i=1,\dots,r, \quad \text{and} \quad HolD^{(r)}F_\rho(\omega) < +\infty, \end{cases} \tag{6.5}$$

where $LipD^{(i)}F_\rho(\omega)$ and $HolD^{(r)}F_\rho(\omega)$ are introduced in Lemma 3.1 (iii). The choices of numbers here are not optimal, but are for the convenience.

6.2 A Lemma Directing to An Equivalent Problem

Lemma 6.1. *Assume that Hypotheses 3.1.1-3.1.3 hold with $r \ge 1$ and $0 \le \alpha \le 1$. For $\gamma_u \in (\beta_2, \beta_1)$, and $\sigma^* > 0$ so that $\gamma_u - \sigma^* \in (\beta_2, \beta_1)$, if we choose $\rho(\omega)$ to be so small that (6.2) holds, then $\tilde{\xi} \in \mathcal{F}^{cuu}(\omega, \xi)$ if and only if $\exists z \in C^-_{\gamma_u}$ with the initial value $z(0) = \tilde{\xi} - \xi$, and satisfies for $t \le 0$,*

$$\begin{aligned} z(t) =& e^{At}\iota + \int_0^t e^{A(t-\tau)}\Delta F_{\rho,u}(\theta_\tau\omega, v(\tau,\omega,\xi), z)\mathrm{d}\tau \\ &+ \int_{-\infty}^t e^{A(t-\tau)}\Delta F_{\rho,c}(\theta_\tau\omega, v(\tau,\omega,\xi), z)\mathrm{d}\tau, \end{aligned} \tag{6.6}$$

where $\iota = P^u(\tilde{\xi} - \xi)$, and for $i = u,\ c$,

$$\Delta F_{\rho,i}(\theta_\tau\omega, v(\tau,\omega,\xi), z) = F_{\rho,i}(\theta_\tau\omega, \theta^{cu}(v(\tau,\omega,\xi) + z(\tau))) - F_{\rho,i}(\theta_\tau\omega, \theta^{cu}(v(\tau,\omega,\xi))).$$

Proof. Assume that $\tilde{\xi} \in \mathcal{F}^{cuu}(\omega, \xi)$, let $z(t) = v(t,\omega,\tilde{\xi}) - v(t,\omega,\xi)$. Using the variation of constants formula, for $t_0 < \min\{t, 0\}$, we have for $i = u,\ c$

$$P^i z(t) = e^{A(t-t_0)}P^i z(t_0) + \int_{t_0}^t e^{A(t-\tau)}\Delta F_{\rho,i}(\theta_\tau\omega, v(\tau,\omega,\xi), z(\tau))\mathrm{d}\tau. \tag{6.7}$$

If we let $t_0 = 0$ in (6.7) with $i = u$, we get

$$P^u z(t) = e^{At}\iota + \int_0^t e^{A(t-\tau)} \Delta F_{\rho,u}(\theta_\tau\omega, v(\tau,\omega,\xi), z(\tau))\mathrm{d}\tau.$$

For the case of $i = c$ for equation (6.7), wo notice that

$$\begin{aligned}
&|e^{A(t-t_0)}P^c z(t_0)| \\
\leq &|e^{A(t-t_0)}P^c(v(t_0,\omega,\tilde{\xi}) - v(t_0,\omega,\xi))| \\
\leq &K_A e^{\beta_2(t-t_0)} e^{\gamma_u t_0} |v(\cdot,\omega,\tilde{\xi}) - v(\cdot,\omega,\xi)|^-_{\gamma_u} \\
= &K_A e^{\beta_2 t} |v(\cdot,\omega,\tilde{\xi}) - v(\cdot,\omega,\xi)|^-_{\gamma_u} \cdot e^{(\gamma_u-\beta_2)t_0} \to 0
\end{aligned}$$

as $t_0 \to -\infty$.

Next we want to show that the improper integral

$$\int_{-\infty}^t e^{A(t-\tau)} \Delta F_{\rho,c}(\theta_\tau\omega, v(\tau,\omega,\xi), z(\tau))\mathrm{d}\tau$$

is well-defined. We take $t_p < t_q < \min\{t, 0\}$, then

$$\begin{aligned}
&|\textstyle\int_{t_p}^t e^{A(t-\tau)} \Delta F_{\rho,c}(\theta_\tau\omega, v(\tau,\omega,\xi), z(\tau))\mathrm{d}\tau - \int_{t_q}^t e^{A(t-\tau)} \Delta F_{\rho,c}(\theta_\tau\omega, v(\tau,\omega,\xi), z(\tau))\mathrm{d}\tau| \\
\leq &\textstyle\int_{t_p}^{t_q} K_A e^{\beta_2(t-\tau)} Lip F_\rho Lip \theta^{cu} e^{\gamma_u\tau} |z(\cdot)|^-_{\gamma_u} \mathrm{d}\tau \\
\leq &e^{\beta_2 t} e^{(\gamma_u-\beta_2)t_q}(1 - e^{(\gamma_u-\beta_2)(t_p-t_q)})|z(\cdot)|^-_{\gamma_u} \to 0,
\end{aligned}$$

as $t_p,\ t_q \to -\infty$, yielding the well-definedness of the improper integral and thus giving (6.6). The converse direction follows a straightforward computation. □

6.3 Solving the Equivalent Problem

Next we show the existence of a unique solution to equation (6.6).

Lemma 6.2. *Assume that Hypotheses 3.1.1-3.1.3 hold with $r \geq 1$ and $0 \leq \alpha \leq 1$. For*

$\gamma_u \in (\beta_2, \beta_1)$, and $\sigma^* > 0$ so that $\gamma_u - \sigma^* \in (\beta_2, \beta_1)$, if we choose $\rho(\omega)$ to be so small that (6.2) holds, then equation (6.6) has a unique solution $z(\cdot, \omega, \iota, \xi) \in C^-_{\gamma_u}$ with $z^u := P^u z(0, \omega, \iota, \xi) = \iota$, $\forall \iota \in H^u$, such that

(1) $z(\cdot, \omega, \iota, \xi)$ is Lipschitz continuous in ι, continuous in (ι, ξ), and measurable in (ω, ι, ξ).
Moreover, assume γ_u satisfies (6.3) and (6.4) for $\eta^* > 0$, if we choose $\rho(\omega)$ to be so small that (6.5) holds, then $\forall 0 \le \eta \le \eta^*$,

(2) $z(\cdot, \omega, \iota, \xi)$ is C^r in ι from H^u to $C^-_{r\gamma_u - \eta}$. And if $\alpha > 0$, $D^{(r)}_\iota z(\cdot, \omega, \iota, \xi)$ from H^u to $L^r(H^u, C^-_{(r+\alpha)\gamma_u - \eta})$ is α-Hölder continuous in ι.
If we assume further that $\gamma_u > -\gamma_{cu}$ for $r = 1$ and $\gamma_u > -(r - 1 + \alpha)\gamma_{cu}$ for $r > 1$, then $\forall 0 \le \eta \le \eta^*$,

(3) $z(\cdot, \omega, \iota, \xi)$ is $C^{r-1,\alpha}$ in ξ from H^{cu} to $C^-_{(r-1)\gamma_u - \eta}$.

Proof. We first show that for any $\iota \in H^u$, the integral equation (6.6) has a unique solution in $C^-_{\gamma_u}$. Denote the RHS of (6.6) by $\mathcal{Q}^u(z, \omega, \iota, \xi)$, we have for $t \le 0$,

$$\begin{aligned}
&e^{-\gamma_u t}|\mathcal{Q}^u(z, \omega, \iota, \xi)| \\
\le\, & e^{-\gamma_u t} K_A e^{\beta_1 t}|\iota| + e^{-\gamma_u t}\textstyle\int_t^0 K_A e^{\beta_1(t-\tau)} Lip F_\rho(\theta_\tau\omega) Lip\theta^{cu} e^{\gamma_u \tau}|z|^-_{\gamma_u} \mathrm{d}\tau \\
& + e^{-\gamma_u t}\textstyle\int_{-\infty}^t K_A e^{\beta_2(t-\tau)} Lip F_\rho(\theta_\tau\omega) Lip\theta^{cu} e^{\gamma_u \tau}|z|^-_{\gamma_u} \mathrm{d}\tau \\
\le\, & K_A|\iota| + \tfrac{1}{4}|z|^-_{\gamma_u} < +\infty.
\end{aligned}$$

Thus $\mathcal{Q}^u(z, \omega, \iota, \xi)$ maps $C^-_{\gamma_u}$ into $C^-_{\gamma_u}$. Now for $z, \bar{z} \in C^-_{\gamma_u}$, similarly we get

$$|\mathcal{Q}^u(z, \omega, \iota, \xi) - \mathcal{Q}^u(\bar{z}, \omega, \iota, \xi)|^-_{\gamma_u} \le \frac{1}{4}|z - \bar{z}|^-_{\gamma_u},$$

therefore $\mathcal{Q}^u(\cdot, \omega, \iota, \xi)$ is a uniform contraction mapping with respect to (ω, ι, ξ). By the uniform contraction mapping principle, $\mathcal{Q}^u(\cdot, \omega, \iota, \xi)$ has a unique fixed point $z(\cdot, \omega, \iota, \xi)$ in $C^-_{\gamma_u}$. Clearly $z(\cdot, \omega, 0, \xi) = 0$. And $\forall \iota,\ \bar{\iota} \in H^u$, for $t \le 0$,

$$e^{-\gamma_u t}|z(\cdot, \omega, \iota, \xi) - z(\cdot, \omega, \bar{\iota}, \xi)| \le K_A|\iota - \bar{\iota}| + \frac{1}{4}|z(\cdot, \omega, \iota, \xi) - z(\cdot, \omega, \bar{\iota}, \xi)|^-_{\gamma_u},$$

hence

$$|z(\cdot,\omega,\iota,\xi)-z(\cdot,\omega,\bar\iota,\xi)|^-_{\gamma_u}\le \tfrac{4}{3}K_A|\iota-\bar\iota|, \qquad (6.8)$$
$$|z(\cdot,\omega,\iota,\xi)|^-_{\gamma_u}\le \tfrac{4}{3}K_A|\iota|.$$

Note that for $\sigma\le\sigma^*$, $\gamma_u-\sigma\in(\beta_2,\beta_1)$ and (6.2) holds, there is a fixed point for $\mathcal{Q}^u(\cdot,\omega,\iota,\xi)$ in $C^-_{\gamma_u-\sigma}$, and (6.8) still holds with γ_u replaced by $\gamma_u-\sigma$.

Next we want to show that $z(\cdot,\omega,\iota,\xi)$ is continuous in (ι,ξ). For $\forall(\iota,\xi)$, $(\bar\iota,\bar\xi)\in H^u\times H^{cu}$, by the last discussion, there exists $z(\cdot,\omega,\iota,\xi)$, $z(\cdot,\omega,\bar\iota,\bar\xi)\in C^-_{\gamma_u-\sigma}$ (also in $C^-_{\gamma_u}$). Then for $t\le 0$,

$$z(\cdot,\omega,\iota,\xi)-z(\cdot,\omega,\bar\iota,\bar\xi)=e^{At}(\iota-\bar\iota)+\int_0^t e^{A(t-\tau)}P^uG(\tau,\omega)\mathrm{d}\tau+\int_{-\infty}^t e^{A(t-\tau)}P^cG(\tau,\omega)\mathrm{d}\tau,$$

where

$$G(\tau,\omega):=\Delta F_{\rho,cu}(\theta_\tau\omega,v(\tau,\omega,\xi),z(\tau,\omega,\iota,\xi))-\Delta F_{\rho,cu}(\theta_\tau\omega,v(\tau,\omega,\bar\xi),z(\tau,\omega,\bar\iota,\bar\xi)).$$

We set

$$I_u=e^{-\gamma t}\textstyle\int_0^t e^{A(t-\tau)}P^uG(\tau,\omega)\mathrm{d}\tau,$$
$$I_c=e^{-\gamma t}\textstyle\int_{-\infty}^t e^{A(t-\tau)}P^cG(\tau,\omega)\mathrm{d}\tau.$$

And for $M_u\le 0$ to be determined, let

$$I_u^1=\begin{cases} e^{-\gamma_u t}\int_{M_u}^t e^{A(t-\tau)}P^uG(\tau,\omega)\mathrm{d}\tau &, \text{ if } t\le M_u,\\ 0 &, \text{ if } t>M_u.\end{cases}$$

$$I_u^2=\begin{cases} e^{-\gamma_u t}\int_0^{M_u} e^{A(t-\tau)}P^uG(\tau,\omega)\mathrm{d}\tau &, \text{ if } t\le M_u,\\ e^{-\gamma_u t}\int_0^t e^{A(t-\tau)}P^uG(\tau,\omega)\mathrm{d}\tau &, \text{ if } t>M_u.\end{cases}$$

And for $M_c\le 0$ to be determined, let

$$I_c^1=\begin{cases} e^{-\gamma_u t}\int_{M_c}^t e^{A(t-\tau)}P^cG(\tau,\omega)\mathrm{d}\tau &, \text{ if } t\ge M_c,\\ 0 &, \text{ if } t<M_c.\end{cases}$$

$$I_c^2 = \begin{cases} e^{-\gamma_u t}\int_{-\infty}^{M_c} e^{A(t-\tau)}P^c G(\tau,\omega)\mathrm{d}\tau & , \text{ if } t \geq M_c, \\ e^{-\gamma_u t}\int_{-\infty}^{t} e^{A(t-\tau)}P^c G(\tau,\omega)\mathrm{d}\tau & , \text{ if } t < M_c. \end{cases}$$

Now if $t \leq M_u$, and we choose $|\iota - \bar{\iota}| < 1$, then

$$\begin{aligned} |I_u^1| &\leq e^{\sigma M_u}\int_t^{M_u} K_A e^{(\gamma_u-\sigma-\beta_1)(\tau-t)} LipF_\rho(\theta_\tau\omega)Lip\theta^{cu}\cdot[|z(\cdot,\omega,\iota,\xi)|_{\gamma_u-\sigma}^- + |z(\cdot,\omega,\bar{\iota},\bar{\xi})|_{\gamma_u-\sigma}^-]\mathrm{d}\tau \\ &\leq e^{\sigma M_u}\cdot \tfrac{2}{3}K_A(2|\bar{\iota}|+1). \end{aligned}$$

Hence for a given $\epsilon > 0$, we may choose M_u sufficiently negative that

$$\sup_{t\leq 0}|I_u^1| \leq \frac{\epsilon}{10}. \tag{6.9}$$

For such chosen M_u, for $t \leq M_u$, we have

$$\begin{aligned} |I_u^2| \leq& e^{-\gamma_u t}\int_{M_u}^0 K_A e^{\beta_1(t-\tau)} \\ &\times[|F_{\rho,cu}(\theta_\tau\omega,\theta^{cu}(v(\tau,\omega,\xi)+z(\tau,\omega,\iota,\xi))) - F_{\rho,cu}(\theta_\tau\omega,\theta^{cu}(v(\tau,\omega,\bar{\xi})+z(\tau,\omega,\bar{\iota},\bar{\xi})))| \\ &\qquad + |F_{\rho,cu}(\theta_\tau\omega,\theta^{cu}(v(\tau,\omega,\xi))) - F_{\rho,cu}(\theta_\tau\omega,\theta^{cu}(v(\tau,\omega,\bar{\xi})))|]\mathrm{d}\tau \\ \leq& e^{-\gamma_u t}\int_{M_u}^0 K_A e^{\beta_1(t-\tau)} LipF_\rho(\theta_\tau\omega)Lip\theta^{cu} \\ &\times[|z(\cdot,\omega,\iota,\xi)-z(\cdot,\omega,\bar{\iota},\bar{\xi})|_{\gamma_u}^- e^{\gamma_u\tau} + 2|v(\cdot,\omega,\xi)-v(\cdot,\omega,\bar{\xi})|]\mathrm{d}\tau. \end{aligned}$$

Since $v(\tau,\omega,\xi)$ is a solution to equation (6.1) which is continuous in ξ, there exists $\delta_u > 0$, so that if $|\xi - \bar{\xi}| < \delta_u$, we have

$$\sup_{t\in[M_u,0]}|v(t,\omega,\xi)-v(t,\omega,\bar{\xi})| \leq \frac{\epsilon}{10}\cdot\Big(\frac{\int_{Mu}^0 e^{-\beta_1\tau}\mathrm{d}\tau}{2(\frac{1}{\gamma_u-\sigma-\beta_2}+\frac{1}{\beta_1-\gamma_u+\sigma})}\Big)^{-1}.$$

Consequently we get that if $t \leq M_u$, and $|\xi-\bar{\xi}| < \delta_u$,

$$|I_u^2| \leq \frac{1}{4}|z(\cdot,\omega,\iota,\xi)-z(\cdot,\omega,\bar{\iota},\bar{\xi})|_{\gamma_u}^- + \frac{\epsilon}{10}. \tag{6.10}$$

If $t > M_u$, we can choose the same δ_u and if $|\xi - \bar{\xi}| < \delta_u$, we get

$$\begin{aligned}\sup_{\tau\in[t,0]} |v(t,\omega,\xi) - v(t,\omega,\bar{\xi})| &\le \sup_{t\in[M_u,0]} |v(t,\omega,\xi) - v(t,\omega,\bar{\xi})| \\ &\le \tfrac{\epsilon}{10} \cdot \Big(\frac{\int_{Mu}^0 e^{-\beta_1\tau}\mathrm{d}\tau}{2(\frac{1}{\gamma_u-\sigma-\beta_2}+\frac{1}{\beta_1-\gamma_u+\sigma})}\Big)^{-1} \\ &\le \tfrac{\epsilon}{10} \cdot \Big(\frac{\int_{t}^0 e^{-\beta_1\tau}\mathrm{d}\tau}{2(\frac{1}{\gamma_u-\sigma-\beta_2}+\frac{1}{\beta_1-\gamma_u+\sigma})}\Big)^{-1},\end{aligned}$$

still yielding (6.10). Similarly, we can choose M_c sufficiently negative that

$$|I_c^2| \le \frac{\epsilon}{10}, \tag{6.11}$$

and for such M_c, we may choose a $\delta_c > 0$, so that if $|\xi - \bar{\xi}| < \delta_c$,

$$|I_c^1| \le \frac{1}{4}|z(\cdot,\omega,\iota,\xi) - z(\cdot,\omega,\bar{\iota},\bar{\xi})|_{\gamma_u}^{-} + \frac{\epsilon}{10}. \tag{6.12}$$

Moreover, $\exists\delta_\iota > 0$, so that if $|\iota - \bar{\iota}| < \delta_\iota$,

$$\sup_{t\le 0} e^{-\gamma_u t}|e^{At}(\iota - \bar{\iota})| \le \sup_{t\le 0} e^{(\beta_1-\gamma_u)t}|\iota - \bar{\iota}| \le \frac{\epsilon}{10}. \tag{6.13}$$

Now taking $\delta_z := \min\{\delta_u, \delta_c, \delta_\iota, 1\}$, and combining (6.9)-(6.13), we conclude that if $|\xi - \bar{\xi}| < \delta_z$, $|\iota - \bar{\iota}| < \delta_z$, then

$$|z(\cdot,\omega,\iota,\xi) - z(\cdot,\omega,\bar{\iota},\bar{\xi})|_{\gamma_u}^{-} \le \frac{1}{2}|z(\cdot,\omega,\iota,\xi) - z(\cdot,\omega,\bar{\iota},\bar{\xi})|_{\gamma_u}^{-} + \frac{\epsilon}{2},$$

and therefore

$$|z(\cdot,\omega,\iota,\xi) - z(\cdot,\omega,\bar{\iota},\bar{\xi})|_{\gamma_u}^{-} \le \epsilon.$$

Since $z(\cdot,\omega,\iota,\xi)$ is an ω-wise limit of the iteration of contraction mapping $\mathcal{Q}^u$ starting at 0 and mapping a $\mathcal{F}$-measurable function to a measurable function, $z(\cdot,\omega,\iota,\xi)$ is $\mathcal{F}$-measurable. On the other hand, $z(\cdot,\omega,\iota,\xi)$ is continuous in (ι,ξ), by Lemma III.14 in Castaing and Valadier [CV77], $z(\cdot,\omega,\iota,\xi)$ is measurable with respect to (ω,ι,ξ).

Next, by assuming higher smoothness of F and larger spectrum gap, we can show that $z(\cdot,\omega,\iota,\xi)$ is $C^{r,\alpha}$ in ι. The proof is identical to that of Lemma 5.2 and we omit it here. However, when considering the base point variable ξ, the smoothness of $z(\cdot,\omega,\iota,\xi)$ is reduced by 1. To be precise, we will show that it is $C^{r-1,\alpha}$. As the higher order case can be carried out similarly using a induction argument, we show that for $r=1$ and $0<\alpha\leq 1$, $z(\cdot,\omega,\iota,\xi)$ is $C^{0,\alpha}$ in ξ, and for $r=2$ it is C^1 in ξ.

Let $\xi,\ \bar{\xi}\in H^{cu}$ be given as two base points, and let $\iota\in H^u$ be fixed. Recall that by (6.6) we have for $t\leq 0$

$$
\begin{aligned}
&z(t,\omega,\iota,\xi)-z(t,\omega,\iota,\bar{\xi})\\
=&\int_0^t e^{A(t-\tau)}\Delta F_{\rho,u}(\theta_\tau\omega,v(\tau,\omega,\xi),z(\tau,\omega,\iota,\xi))-\Delta F_{\rho,u}(\theta_\tau\omega,v(\tau,\omega,\bar{\xi}),z(\tau,\omega,\iota,\bar{\xi}))\mathrm{d}\tau\\
&+\int_{-\infty}^t e^{A(t-\tau)}\Delta F_{\rho,c}(\theta_\tau\omega,v(\tau,\omega,\xi),z(\tau,\omega,\iota,\xi))-\Delta F_{\rho,c}(\theta_\tau\omega,v(\tau,\omega,\bar{\xi}),z(\tau,\omega,\iota,\bar{\xi}))\mathrm{d}\tau\\
:=&I_{\Delta,u}+I_{\Delta,c}.
\end{aligned}
$$

We multiply $e^{-(\gamma_u-\eta)t}$ to $|I_{\Delta,u}|$, and by (5.8) which says that v is Lipschitz in ξ, we have

$$
\begin{aligned}
& e^{-(\gamma_u-\eta)t}|I_{\Delta,u}| \\
\leq\, & e^{-(\gamma_u-\eta)t}\int_t^0 e^{\beta_1(t-\tau)}|F_{\rho,u}(\theta_\tau\omega,\theta^{cu}(v(\tau,\omega,\xi)+z(\tau,\omega,\iota,\xi)))-F_{\rho,u}(\theta_\tau\omega,\theta^{cu}(v(\tau,\omega,\xi))) \\
& \qquad -F_{\rho,u}(\theta_\tau\omega,\theta^{cu}(v(\tau,\omega,\bar\xi)+z(\tau,\omega,\iota,\bar\xi)))+F_{\rho,u}(\theta_\tau\omega,\theta^{cu}(v(\tau,\omega,\bar\xi)))|\mathrm{d}\tau \\
\leq\, & e^{-(\gamma_u-\eta)t}\int_t^0 e^{\beta_1(t-\tau)}|F_{\rho,u}(\theta_\tau\omega,\theta^{cu}(v(\tau,\omega,\xi)+z(\tau,\omega,\iota,\xi))) \\
& \qquad\qquad -F_{\rho,u}(\theta_\tau\omega,\theta^{cu}(v(\tau,\omega,\xi)+z(\tau,\omega,\iota,\bar\xi)))|\mathrm{d}\tau \\
& +e^{-(\gamma_u-\eta)t}\int_t^0 e^{\beta_1(t-\tau)}|F_{\rho,u}(\theta_\tau\omega,\theta^{cu}(v(\tau,\omega,\xi)+z(\tau,\omega,\iota,\bar\xi)))-F_{\rho,u}(\theta_\tau\omega,\theta^{cu}(v(\tau,\omega,\xi))) \\
& \qquad -F_{\rho,u}(\theta_\tau\omega,\theta^{cu}(v(\tau,\omega,\bar\xi)+z(\tau,\omega,\iota,\bar\xi)))+F_{\rho,u}(\theta_\tau\omega,\theta^{cu}(v(\tau,\omega,\bar\xi)))|\mathrm{d}\tau \\
\leq\, & e^{-(\gamma_u-\eta)t}\int_t^0 e^{\beta_1(t-\tau)}\int_0^1\|DF_{\rho,u}(\theta_\tau\omega,r\theta^{cu}(v(\tau,\omega,\xi)+z(\tau,\omega,\iota,\xi)) \\
& \qquad +(1-r)\theta^{cu}(v(\tau,\omega,\xi)+z(\tau,\omega,\iota,\bar\xi)))\|Lip\theta^{cu}|z(\tau,\omega,\iota,\xi)-z(\tau,\omega,\iota,\bar\xi)|\mathrm{d}r\mathrm{d}\tau \\
& +e^{-(\gamma_u-\eta)t}\int_t^0 e^{\beta_1(t-\tau)}\int_0^1\|DF_{\rho,u}(\theta_\tau\omega,r\theta^{cu}(v(\tau,\omega,\xi)+z(\tau,\omega,\iota,\bar\xi))+(1-r)\theta^{cu}(v(\tau,\omega,\xi))) \\
& \qquad -DF_{\rho,u}(\theta_\tau\omega,r\theta^{cu}(v(\tau,\omega,\bar\xi)+z(\tau,\omega,\iota,\bar\xi))+(1-r)\theta^{cu}(v(\tau,\omega,\bar\xi)))\| \\
& \qquad\qquad \times Lip\theta^{cu}|z(\tau,\omega,\iota,\bar\xi))|\mathrm{d}r\mathrm{d}\tau \\
\leq\, & e^{-(\gamma_u-\eta)t}\int_t^0 e^{\beta_1(t-\tau)}\int_0^1 LipF_\rho(\theta_\tau\omega)Lip\theta^{cu}|z(\tau,\omega,\iota,\xi)-z(\tau,\omega,\iota,\bar\xi)|\mathrm{d}r\mathrm{d}\tau \\
& +e^{-(\gamma_u-\eta)t}\int_t^0 e^{\beta_1(t-\tau)}\int_0^1 HolDF_\rho(\theta_\tau\omega)(Lip\theta^{cu})^\alpha|v(\tau,\omega,\xi)-v(\tau,\omega,\bar\xi)|^\alpha \\
& \qquad\qquad \times Lip\theta^{cu}|z(\tau,\omega,\iota,\bar\xi)|\mathrm{d}r\mathrm{d}\tau \\
\leq\, & \int_t^0 e^{(\beta_1-\gamma_u+\eta)(t-\tau)}LipF_\rho(\theta_\tau\omega)Lip\theta^{cu}|z(\cdot,\omega,\iota,\xi)-z(\cdot,\omega,\iota,\bar\xi)|^-_{\gamma_u-\eta}\mathrm{d}\tau \\
& +|\xi-\bar\xi|^\alpha\int_t^0 e^{(\beta_1-\gamma_u+\eta)(t-\tau)}HolDF_\rho(\theta_\tau\omega)(Lip\theta^{cu})^{\alpha+1}|z(\cdot,\omega,\iota,\bar\xi)|^-_{(1+\alpha)\gamma_u-\eta} \\
& \qquad\qquad \times(2K_A)^\alpha e^{(\alpha\gamma_u+\alpha\gamma_{cu})\tau}\mathrm{d}\tau
\end{aligned}
$$

Notice that by the condition we have $\alpha\gamma_u+\alpha\gamma_{cu}>0$ so the last exponential term is indeed bounded by 1. Doing the same estimate for $e^{-(\gamma_u-\eta)t}|I_{\Delta,c}|$, we conclude that for some constant K,

$$
e^{-(\gamma_u-\eta)t}|z(t,\omega,\iota,\xi)-z(t,\omega,\iota,\bar\xi)|\leq\frac{1}{4}|z(\cdot,\omega,\iota,\xi)-z(\cdot,\omega,\iota,\bar\xi)|^-_{\gamma_u-\eta}+K|\xi-\bar\xi|^\alpha,
$$

thus yielding

$$|z(\cdot,\omega,\iota,\xi)-z(\cdot,\omega,\iota,\bar{\xi})|^{-}_{\gamma_u-\eta}\le\frac{4}{3}K|\xi-\bar{\xi}|^{\alpha}.$$

Now let $r=2$ and we show that $z(\cdot,\omega,\iota,\xi)$ is indeed C^1 in ξ. We first use the formula (6.6) to take the formal derivative in ξ of $z(t,\omega,\iota,\xi)$, and get

$$\begin{aligned}
&D_\xi z(t,\omega,\iota,\xi)\\
=&\int_0^t e^{A(t-\tau)}\{(DF_{\rho,u}(\theta_\tau\omega,\theta^{cu}(\cdot))\cdot D\theta^{cu})(v(\tau,\omega,\xi)+z(\tau,\omega,\iota,\xi))D_\xi z(\tau,\omega,\iota,\xi)\\
&\qquad+[(DF_{\rho,u}(\theta_\tau\omega,\theta^{cu}(\cdot))\cdot D\theta^{cu})(v(\tau,\omega,\xi)+z(\tau,\omega,\iota,\xi))\\
&\qquad\qquad-(DF_{\rho,u}(\theta_\tau\omega,\theta^{cu}(\cdot))\cdot D\theta^{cu})(v(\tau,\omega,\xi))]D_\xi v(\tau,\omega,\xi)\}\mathrm{d}\tau \qquad (6.14)\\
&+\int_{-\infty}^t e^{A(t-\tau)}\{(DF_{\rho,c}(\theta_\tau\omega,\theta^{cu}(\cdot))\cdot D\theta^{cu})(v(\tau,\omega,\xi)+z(\tau,\omega,\iota,\xi))D_\xi z(\tau,\omega,\iota,\xi)\\
&\qquad+[(DF_{\rho,c}(\theta_\tau\omega,\theta^{cu}(\cdot))\cdot D\theta^{cu})(v(\tau,\omega,\xi)+z(\tau,\omega,\iota,\xi))\\
&\qquad\qquad-(DF_{\rho,c}(\theta_\tau\omega,\theta^{cu}(\cdot))\cdot D\theta^{cu})(v(\tau,\omega,\xi))]D_\xi v(\tau,\omega,\xi)\}\mathrm{d}\tau
\end{aligned}$$

By (6.2) and the condition $\gamma_u>-(r-1+\alpha)\gamma_{cu}$, also notice that

$$\|(DF_\rho(\theta_\tau\omega,\theta^{cu}(\cdot))\cdot D\theta^{cu})(v(\tau,\omega,\xi)+z(\tau,\omega,\iota,\xi))\|\le LipF_\rho(\theta_\tau\omega)Lip\theta^{cu},$$

we can use the contraction mapping principle to get a fixed point for the above equation, which we formally denote for now by $D_\xi z(t,\omega,\iota,\xi)$. It then suffices to show that

$$z(t,\omega,\iota,\xi)-z(t,\omega,\iota,\bar{\xi})-D_\xi z(t,\omega,\iota,\bar{\xi})(\xi-\bar{\xi})=o(|\xi-\bar{\xi}|)$$

as $\xi\to\bar{\xi}$. We define a map $\mathcal{S}_f:C^-_{\gamma_u}\to C^-_{\gamma_u}$ by

$$\begin{aligned}
\mathcal{S}_f(v)(t)=&\int_0^t e^{A(t-\tau)}(DF_{\rho,u}(\theta_\tau\omega,\theta^{cu}(\cdot))\cdot D\theta^{cu})(v(\tau,\omega,\bar{\xi})+z(\tau,\omega,\iota,\bar{\xi}))v(\tau)\mathrm{d}\tau\\
&+\int_{-\infty}^t e^{A(t-\tau)}(DF_{\rho,c}(\theta_\tau\omega,\theta^{cu}(\cdot))\cdot D\theta^{cu})(v(\tau,\omega,\bar{\xi})+z(\tau,\omega,\iota,\bar{\xi}))v(\tau)\mathrm{d}\tau.
\end{aligned}$$

Notice that again using (6.2) we have $\|\mathcal{S}_f\| < 1$. Then by (6.14) we have

$$\begin{aligned} & z(t,\omega,\iota,\xi) - z(t,\omega,\iota,\bar{\xi}) - D_\xi z(t,\omega,\iota,\bar{\xi})(\xi-\bar{\xi}) \\ = & \mathcal{S}_f(z(\cdot,\omega,\iota,\xi) - z(\cdot,\omega,\iota,\bar{\xi}) - D_\xi z(\cdot,\omega,\iota,\bar{\xi})(\xi-\bar{\xi}))(t) + \mathcal{I}_f^u + \mathcal{I}_f^c, \end{aligned} \tag{6.15}$$

where

$$\begin{aligned} & \mathcal{I}_f^u \\ = & \textstyle\int_0^t e^{A(t-\tau)}\{[F_{\rho,u}(\theta_\tau\omega, \theta^{cu}(v(\tau,\omega,\xi) + z(\tau,\omega,\iota,\bar{\xi}))) - F_{\rho,u}(\theta_\tau\omega, \theta^{cu}(v(\tau,\omega,\xi))) \\ & \qquad - F_{\rho,u}(\theta_\tau\omega, \theta^{cu}(v(\tau,\omega,\bar{\xi}) + z(\tau,\omega,\iota,\bar{\xi}))) + F_{\rho,u}(\theta_\tau\omega, \theta^{cu}(v(\tau,\omega,\bar{\xi})))] \\ & \quad - [(DF_{\rho,u}(\theta_\tau\omega, \theta^{cu}(\cdot)) \cdot D\theta^{cu})(v(\tau,\omega,\bar{\xi}) + z(\tau,\omega,\iota,\bar{\xi})) \\ & \qquad - (DF_{\rho,u}(\theta_\tau\omega, \theta^{cu}(\cdot)) \cdot D\theta^{cu})(v(\tau,\omega,\bar{\xi}))]D_\xi v(\tau,\omega,\bar{\xi})(\xi-\bar{\xi})\}\mathrm{d}\tau \\ & + \textstyle\int_0^t e^{A(t-\tau)}[F_{\rho,u}(\theta_\tau\omega, \theta^{cu}(v(\tau,\omega,\xi) + z(\tau,\omega,\iota,\xi))) - F_{\rho,u}(\theta_\tau\omega, \theta^{cu}(v(\tau,\omega,\xi + z(\tau,\omega,\iota,\bar{\xi})))) \\ & \quad - (DF_{\rho,u}(\theta_\tau\omega, \theta^{cu}(\cdot)) \cdot D\theta^{cu})(v(\tau,\omega,\bar{\xi}) + z(\tau,\omega,\iota,\bar{\xi}))(z(\tau,\omega,\iota,\xi) - z(\tau,\omega,\iota,\bar{\xi}))]\mathrm{d}\tau \\ := & \mathcal{I}_f^{u,1} + \mathcal{I}_f^{u,2}, \end{aligned}$$

and $\mathcal{I}_f^c$ is given similarly by changing $\int_0^t e^{A(t-\tau)}P^u$ to $\int_{-\infty}^t e^{A(t-\tau)}P^c$ in $\mathcal{I}_f^u$. Once we show that $|\mathcal{I}_f^u + \mathcal{I}_f^c|_{\gamma_u}^- = o(|\xi-\bar{\xi}|)$ as $\xi \to \bar{\xi}$, then using (6.15), we have

$$z(t,\omega,\iota,\xi) - z(t,\omega,\iota,\bar{\xi}) - D_\xi z(t,\omega,\iota,\bar{\xi})(\xi-\bar{\xi}) = (id - \mathcal{S}_f)^{-1}(\mathcal{I}_f^u + \mathcal{I}_f^c) = o(|\xi-\bar{\xi}|),$$

which implies the actual existence of the derivative $D_\xi z(t,\omega,\iota,\xi)$.

We first rewrite $\mathcal{I}_f^{u,1}$ as

$$
\begin{aligned}
&\mathcal{I}_f^{u,1}\\
=&\int_0^t e^{A(t-\tau)}\int_0^1\{[(DF_{\rho,u}(\theta_\tau\omega,\theta^{cu}(\cdot))\cdot D\theta^{cu})(rv(\tau,\omega,\xi)+(1-r)v(\tau,\omega,\bar{\xi})+z(\tau,\omega,\iota,\bar{\xi}))\\
&\qquad-(DF_{\rho,u}(\theta_\tau\omega,\theta^{cu}(\cdot))\cdot D\theta^{cu})(rv(\tau,\omega,\xi)+(1-r)v(\tau,\omega,\bar{\xi}))]\\
&\qquad\times(v(\tau,\omega,\xi)-v(\tau,\omega,\bar{\xi}))\\
&\quad-[(DF_{\rho,u}(\theta_\tau\omega,\theta^{cu}(\cdot))\cdot D\theta^{cu})(v(\tau,\omega,\bar{\xi})+z(\tau,\omega,\iota,\bar{\xi}))\\
&\qquad-(DF_{\rho,u}(\theta_\tau\omega,\theta^{cu}(\cdot))\cdot D\theta^{cu})(v(\tau,\omega,\bar{\xi}))]D_\xi v(\tau,\omega,\bar{\xi})(\xi-\bar{\xi})\}\mathrm{d}r\mathrm{d}\tau\\
=&\int_0^t e^{A(t-\tau)}\int_0^1\int_0^1\{D((DF_{\rho,u}(\theta_\tau\omega,\theta^{cu}(\cdot))\cdot D\theta^{cu}))(rv(\tau,\omega,\xi)+(1-r)v(\tau,\omega,\bar{\xi})+sz(\tau,\omega,\iota,\bar{\xi}))\\
&\qquad\times z(\tau,\omega,\iota,\bar{\xi})(v(\tau,\omega,\xi)-v(\tau,\omega,\bar{\xi}))\\
&\quad-D((DF_{\rho,u}(\theta_\tau\omega,\theta^{cu}(\cdot))\cdot D\theta^{cu}))(v(\tau,\omega,\bar{\xi})+sz(\tau,\omega,\iota,\bar{\xi}))\\
&\qquad\times z(\tau,\omega,\iota,\bar{\xi})D_\xi v(\tau,\omega,\bar{\xi})(\xi-\bar{\xi})\}\mathrm{d}s\mathrm{d}r\mathrm{d}\tau\\
=&\int_0^t e^{A(t-\tau)}\int_0^1\int_0^1\{[D((DF_{\rho,u}(\theta_\tau\omega,\theta^{cu}(\cdot))\cdot D\theta^{cu}))(rv(\tau,\omega,\xi)+(1-r)v(\tau,\omega,\bar{\xi})+sz(\tau,\omega,\iota,\bar{\xi}))\\
&\qquad-D((DF_{\rho,u}(\theta_\tau\omega,\theta^{cu}(\cdot))\cdot D\theta^{cu}))(v(\tau,\omega,\bar{\xi})+sz(\tau,\omega,\iota,\bar{\xi}))]\\
&\quad\times z(\tau,\omega,\iota,\bar{\xi})(v(\tau,\omega,\xi)-v(\tau,\omega,\bar{\xi}))\}\mathrm{d}s\mathrm{d}r\mathrm{d}\tau\\
&+\int_0^t e^{A(t-\tau)}\int_0^1\int_0^1\{D((DF_{\rho,u}(\theta_\tau\omega,\theta^{cu}(\cdot))\cdot D\theta^{cu}))(v(\tau,\omega,\bar{\xi})+sz(\tau,\omega,\iota,\bar{\xi}))z(\tau,\omega,\iota,\bar{\xi})\\
&\quad\times[v(\tau,\omega,\xi)-v(\tau,\omega,\bar{\xi})-D_\xi v(\tau,\omega,\bar{\xi})(\xi-\bar{\xi})]\}\mathrm{d}s\mathrm{d}r\mathrm{d}\tau,
\end{aligned}
$$

where we have used that both F and $\theta^{cu}=id+h^{cu}(\omega,\cdot)$ are C^2. Now with the help of the factor $z(\tau,\omega,\iota,\bar{\xi})$ showing up in each term, and the condition $\gamma_u>-(r-1+\alpha)\gamma_{cu}$, we can decompose the integrals and estimate as what we did in step 2 of proof for Lemma 5.2, and thus show that $e^{-\gamma_u t}|\mathcal{I}_f^{u,1}|=o(|\xi-\bar{\xi}|)$. Similar work can be done on $\mathcal{I}_f^{u,2}$ and $\mathcal{I}_f^c$, also changing γ_u to $\gamma_u-\eta$ will not affect the argument. This completes the proof. □

6.4 Proof of the Main Theorem for Foliations

We define the mapping $l^u:\Omega\times H^u\times H^{cu}\to H^c$ as

$$l^u(\omega,\iota,\xi):=P^c\xi+J^u(\omega,\iota-P^u\xi,\xi),$$

where $J^u(\omega,\iota,\xi) := P^c z(0,\omega,\iota,\xi)$. From (6.6), we find that J^u satisfies

$$J^u(\omega,\iota,\xi) = \int_{-\infty}^{0} e^{-A\tau} P^c \Delta F_{\rho,cu}(\theta_\tau\omega, v(\tau,\omega,\xi), z(\tau,\omega,\iota,\xi))\mathrm{d}\tau.$$

Now we can establish the theorem showing existence of an unstable foliation for equation (6.1).

Theorem 6.3. *Assume that Hypotheses 3.1.1-3.1.3 hold with $r \geq 1$ and $0 \leq \alpha \leq 1$. For $\gamma_u \in (\beta_2, \beta_1)$, and $\sigma^* > 0$ so that $\gamma_u - \sigma^* \in (\beta_2, \beta_1)$, if we choose $\rho(\omega)$ to be so small that (6.2) holds, then there exists a Lipschitz invariant unstable foliation for the random equation (6.1) whose unstable leaf is given by*

$$\mathcal{F}^{cuu}(\omega,\xi) = \{\iota + l^u(\omega,\iota,\xi) \mid \iota \in H^u\}$$

with $Lip l^u(\omega,\cdot,\xi) \leq \frac{1}{3}$. Also, $\mathcal{F}^{cuu}(\omega,\xi)$ has a unique intersection point with $M^c(\omega)$, a center manifold contained in $M^{cu}(\omega)$ which is given by

$$M^c(\omega) = \{\eta + h^c(\omega,\eta) \mid \eta \in H^c\},$$

where $h^c(\omega,\cdot): H^c \to H^{us}$ has $Lip h^c(\omega,\cdot) < \frac{1}{3}$.
Furthermore, assume γ_u satisfies (6.3) and (6.4) for $\eta^* > 0$, if we choose $\rho(\omega)$ to be so small that (6.5) holds, then $l^u(\omega,\iota,\xi)$ is $C^{r,\alpha}$ in ι. If we assume further that $\gamma_u > -\gamma_{cu}$ for $r = 1$ and $\gamma_u > -(r-1+\alpha)\gamma_{cu}$ for $r > 1$, then $l^u(\omega,\iota,\xi)$ is $C^{r-1,\alpha}$ in ξ.

Proof. By Lemmas 6.1 and 6.2, $\tilde{\xi} \in \mathcal{F}^{cuu}(\omega,\xi)$ if and only if $\exists \iota \in H^u$, so that

$$\tilde{\xi} = \xi + \iota + J^u(\omega,\iota,\xi).$$

We observe that

$$\xi + \iota + J^u(\omega,\iota,\xi) = P^c\xi + P^u\xi + \iota + J^u(\omega, P^u\xi + \iota - P^u\xi, \xi).$$

If we replace $P^u\xi + \iota \in H^u$ by $\iota \in H^u$ in the above expression, it yields

$$\tilde{\xi} = \iota + P^c\xi + J^u(\omega, \iota - P^u\xi, \xi) = \iota + l^u(\omega, \iota, \xi),$$

hence

$$\mathcal{F}^{cuu}(\omega, \xi) = \{\iota + l^u(\omega, \iota, \xi) \mid \iota \in H^u\}.$$

Now for any $\iota,\ \bar{\iota} \in H^u$, using (6.8), we get

$$\begin{aligned}
&|J^u(\omega, \iota, \xi) - J^u(\omega, \bar{\iota}, \xi)| \\
\leq& \textstyle\int_{-\infty}^0 K_A e^{-\beta_2\tau} |[F_{\rho,c}(\theta_\tau\omega, \theta^{cu}(v(\tau,\omega,\xi) + z(\tau,\omega,\iota,\xi))) - F_{\rho,c}(\theta_\tau\omega, \theta^{cu}(v(\tau,\omega,\xi)))] \\
&-[F_{\rho,c}(\theta_\tau\omega, \theta^{cu}(v(\tau,\omega,\xi) + z(\tau,\omega,\bar{\iota},\xi))) - F_{\rho,c}(\theta_\tau\omega, \theta^{cu}(v(\tau,\omega,\xi)))]|\mathrm{d}\tau \\
\leq& \textstyle\int_{-\infty}^0 K_A e^{-\beta_2\tau} LipF_\rho(\theta_\tau\omega) Lip\theta^{cu} e^{\gamma_u\tau} |z(\cdot,\omega,\iota,\xi) - z(\cdot,\omega,\bar{\iota},\xi)|_{\gamma_u}^{-} \mathrm{d}\tau \\
\leq& \tfrac{1}{4K_A} \cdot \tfrac{4}{3} K_A |\iota - \bar{\iota}| \\
=& \tfrac{1}{3}|\iota - \bar{\iota}|.
\end{aligned}$$

This implies that $J^u(\omega, \iota, \xi)$ is Lipschitz continuous in $\iota \in H^u$, with $LipJ^u(\omega, \cdot, \xi) \leq \frac{1}{3}$. Also for $(\iota, \xi),\ (\bar{\iota}, \bar{\xi}) \in H^u \times H^{cu}$, we have

$$|J^u(\omega, \iota, \xi) - J^u(\omega, \bar{\iota}, \bar{\xi})| \leq \|P^c\| |z(\cdot,\omega,\iota,\xi) - z(\cdot,\omega,\bar{\iota},\bar{\xi})|_{\gamma_u}^{-},$$

but by the last lemma we know that $z(\cdot,\omega,\iota,\xi)$ is continuous in (ι, ξ), thus so is $J^u(\omega, \iota, \xi)$. By definition of l^u, it is also continuous in (ι, ξ) and is Lipschitz continuous in ι with

$$Lipl^u(\omega, \cdot, \xi) = LipJ^u(\omega, \cdot, \xi) \leq \frac{1}{3}.$$

Now let H_c^{cu} be a countable dense set of the separable space H^{cu}. For each $y_0 \in H^{cu}$, we have that

$$\omega \mapsto \inf_{y \in H^{cu}} |y_0 - (P^u y + l^u(\omega, P^u y, \xi))| = \inf_{y \in H_c^{cu}} |y_0 - (P^u y + l^u(\omega, P^u y, \xi))|.$$

Using Lemma 6.2 and Lemma III.14 in Castaing and Valadier [CV77], we obtain that $\mathcal{F}^{cuu}(\omega, \xi)$ is $\mathcal{F}$-measurable.

To prove the invariance of $\mathcal{F}^{cuu}(\omega, \xi)$, we show that the time τ-map $v(\tau, \omega, \cdot)$ maps $\mathcal{F}^{cuu}(\omega, \xi)$ into $\mathcal{F}^{cuu}(\theta_\tau \omega, v(\tau, \omega, \xi))$ for $\tau > 0$. Indeed, for any $\tilde{\xi} \in \mathcal{F}^{cuu}(\omega, \xi)$, we have

$$v(\cdot, \omega, \tilde{\xi}) - v(\cdot, \omega, \xi) \in C^{-}_{\gamma_u},$$

which implies that

$$v(\cdot + \tau, \omega, \tilde{\xi}) - v(\cdot + \tau, \omega, \xi) \in C^{-}_{\gamma_u}.$$

But for $x^* = \tilde{\xi}$ or ξ, we have

$$v(\cdot + \tau, \omega, x^*) = v(\cdot, \theta_\tau \omega, v(\tau, \omega, x^*)),$$

hence $v(\tau, \omega, \tilde{\xi}) \in \mathcal{F}^{cuu}(\theta_\tau \omega, v(\tau, \omega, \xi))$.

It is by definition that if $\mathcal{F}^{cuu}(\omega, \xi) \cap \mathcal{F}^{cuu}(\omega, \bar{\xi}) \neq \emptyset$, then we have $\mathcal{F}^{cuu}(\omega, \xi) = \mathcal{F}^{cuu}(\omega, \bar{\xi})$. Hence there exists an unstable foliation of $M^{cu}(\omega)$ (which is identified by H^{cu}) given by

$$M^{cu}(\omega) = \cup_{\xi \in H^{cu}} \mathcal{F}^{cuu}(\omega, \xi).$$

Next, we claim that $\mathcal{F}^{cuu}(\omega, \xi)$ has a unique intersection point with $M^c(\omega)$, which as stated is given by

$$M^c(\omega) = \{\eta + h^c(\omega, \eta) \mid \eta \in H^c\},$$

where $h^c(\omega, \cdot) : H^c \to H^{us}$ has $Liph^c(\omega, \cdot) < \frac{1}{3}$. In fact, if $\tilde{\xi} \in \mathcal{F}^{cuu}(\omega, \xi) \cap M^c(\omega)$, there exists $\iota \in H^u$ and $\eta \in H^c$ such that

$$\tilde{\xi} = \iota + l^u(\omega, \iota, \xi) = \eta + h^c(\omega, \eta).$$

Then we have

$$\begin{cases} \iota = h^c(\omega, \eta), \\ \eta = l^u(\omega, \iota, \xi). \end{cases}$$

It follows that

$$\iota = h^c(\omega, l^u(\omega, \iota, \xi)). \tag{6.16}$$

By our previous results,

$$Liph^c(\omega, \cdot) \cdot Lipl^u(\omega, \cdot, \xi) \leq \frac{1}{3} \cdot \frac{1}{3} < 1.$$

Therefore, equation (6.16) has a unique solution $\iota \in H^c$ and consequently $\tilde{\xi}$ is uniquely determined.

At last, smoothness of $l^u(\omega, \iota, \xi)$ is a direct consequence of Lemma 6.2. This completes the proof. □

Remark 6.4.1. *1. As shown above, $l^u(\omega, \iota, \xi)$ is $C^{r,\alpha}$ in ι and $C^{r-1,\alpha}$ in ξ, and according to Definition 3.3.3, we say $\mathcal{F}^{cuu}(\omega, \xi)$ is a $C^{r-1,\alpha} \times C^{r,\alpha}$ unstable foliation leaf.*
2. Similarly we can show the existence and smoothness of a $C^{r-1,\alpha} \times C^{r,\alpha}$ stable foliation $\{\mathcal{F}^{css}(\omega, \xi)\}$ for the equation (7.7) restricted on the center-stable manifold $M^{cs}(\omega)$, and a $C^{r-1,\alpha} \times C^{r,\alpha}$ stable foliation $\{\mathcal{F}^s(\omega, x)\}$ for the whole space.

Now we can prove the local result stated in Theorem 3.3.

Proof of Theorem 3.3: Similar to the proof of Theorem 3.2, Theorem 6.3 and the remark above give the global graphs as invariant leaves of foliations, by Lemma 3.1 (i) such graphs for the truncated equation (3.11) restrict to the desired local structure for the original equation (3.1). □

Chapter 7. Conjugacy Between Center Manifolds

In this chapter, we get to the conjugacy problem raised in the introduction. As mentioned before, with the cut-off approach in the last two chapters, we cannot guarantee the uniqueness of local center manifold for equation (3.1). However, we can still show that there exists some smooth conjugacy between any two such local center manifolds by following the geometrical proof carried out in [BDL92]. We first show that for an arbitrarily given local center manifold, we can extend such manifold to a local center-stable manifold with a stable foliation on it, and a local center-unstable manifold with an unstable foliation on it. Then we use two technical lemmas to show the conjugacy based on the structure we get from the extension step.

7.1 Certification for a Graph to Be Invariant

We still consider equation (3.1)

$$u_t = Au + F(\theta_t\omega, u), \qquad u(0) = x \in H,$$

with Hypotheses 3.1.1, 3.1.2 and 3.3.1 holding with $r \geq 2$ and $0 \leq \alpha \leq 1$, and start by examining a certification for a graph to be invariant for such type of equation.

Lemma 7.1. *Let $V \subseteq H^c$ be an open neighborhood of the origin. Assume $h(\omega, \cdot) : V \to H^{su}$ is of class $C^{r,\alpha}$ with $r \geq 1$ and $0 \leq \alpha \leq 1$ for all $\omega \in \Omega$. Let $W(\omega) = \mathrm{graph}(\mathrm{h}(\omega, \cdot))$. Then $W(\omega)$ is invariant for equation (3.1) if and only if $h(\omega, \cdot)$ maps V into $D(A)$, $h(\theta_t\omega, \cdot)$ is* *differentiable in t, and the following identity holds for all $x_c \in V$:*

$$A_{us}h(\omega, x_c)+F_{us}(\omega, x_c+h(\omega, x_c)) = Dh(\omega, x_c)[A_c x_c+F_c(\omega, x_c+h(\omega, x_c))]+\partial_t h(\theta_t\omega, x_c) \mid_{t=0}, \tag{7.1}$$

where $Dh(\omega, \cdot)$ denotes the derivative in the spatial variable of h, while $\partial_t h(\theta_t\omega, \cdot)$ denotes the derivative in t of $h(\theta_t\omega, \cdot)$, and A_i denotes the projection of A on H^i, $i = us,\ c$.

Proof. We first suppose that $W(\omega)$ is invariant. That is $\forall x_c \in V$, for the corresponding point $x := x_c + h(\omega, x_c) \in W(\omega)$, if $\tilde{u}(t, \omega, x)$ is the solution to equation (3.1) starting at x when $t = 0$, then we have

$$\tilde{u}_{us}(t, \omega, x) = h(\theta_t\omega, \tilde{u}_c(t, \omega, x)) \tag{7.2}$$

by the invariance of $W(\omega)$. Because $\tilde{u}(t, \omega, x)$ is a solution, we have

$$h(\omega, x_c) = h(\omega, \tilde{u}_c(0, \omega, x)) = \tilde{u}_{us}(0, \omega, x) \in D(A)$$

by definition. To see that $\partial_t h(\theta_t\omega, \cdot)$ exists, we first notice that the LHS of (7.2) is differentiable in t as it is a projection of the solution. So for $t > 0$ and $|\Delta t|$ so small, we have

$$\lim_{\Delta t\to 0} \frac{1}{\Delta t}[h(\theta_{t+\Delta t}\omega, \tilde{u}_c(t+\Delta t, \omega, x))-h(\theta_t\omega, \tilde{u}_c(t, \omega, x))] = \lim_{\Delta t\to 0} \frac{1}{\Delta t}[\tilde{u}_{us}(t+\Delta t, \omega, x)-\tilde{u}_{us}(t, \omega, x)]$$

exists. Also,

$$\begin{aligned} &h(\theta_{t+\Delta t}\omega, \tilde{u}_c(t + \Delta t, \omega, x)) - h(\theta_t\omega, \tilde{u}_c(t, \omega, x)) \\ =&h(\theta_{t+\Delta t}\omega, \tilde{u}_c(t + \Delta t, \omega, x)) - h(\theta_t\omega, \tilde{u}_c(t + \Delta t, \omega, x)) \\ &+h(\theta_t\omega, \tilde{u}_c(t + \Delta t, \omega, x)) - h(\theta_t\omega, \tilde{u}_c(t, \omega, x)). \end{aligned}$$

However,

$$\lim_{\Delta t\to 0} \frac{1}{\Delta t}[h(\theta_t\omega, \tilde{u}_c(t + \Delta t, \omega, x)) - h(\theta_t\omega, \tilde{u}_c(t, \omega, x))]$$

exists since h is differentiable in the spatial variable and $\tilde{u}_c$ as a projection of the solution is differentiable. This implies that

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t}[h(\theta_{t+\Delta t}\omega, \tilde{u}_c(t+\Delta t, \omega, x)) - h(\theta_t\omega, \tilde{u}_c(t+\Delta t, \omega, x))]$$

exists, which suggests the existence of $\partial_t h(\theta_t\omega, \cdot)$. Now we differentiate (7.2) at $t = 0$, the LHS of the equation gives

$$A_{us}\tilde{u}_{us}(t,\omega,x) \mid_{t=0} + F_{us}(\theta_t\omega, \tilde{u}(t,\omega,x)) \mid_{t=0} = A_{us}h(\omega, x_c) + F_{us}(\omega, x_c + h(\omega, x_c)),$$

while the RHS gives

$$\begin{aligned} \{Dh(\theta_t\omega, \tilde{u}_c(t,\omega,x)) \cdot [A_c x_c + F_c(\theta_t\omega, \tilde{u}(t,\omega,x))] + \partial_t h(\theta_t\omega, \tilde{u}_c(t,\omega,x))\} \mid_{t=0} \\ = Dh(\omega, x_c)[A_c x_c + F_c(\omega, x_c + h(\omega, x_c))] + \partial_t h(\theta_t\omega, x_c) \mid_{t=0} . \end{aligned}$$

Combining these two gives the identity (7.1).

Conversely, suppose that $h(\omega, \cdot)$ maps V into $D(A)$, $h(\theta_t\omega, \cdot)$ is differentiable in t, and identity (7.1) holds for all $x_c := P^c x \in V$. Then for every $x \in H$ such that $x_c \in V$, let $\tilde{u}_c(t,\omega,x_c)$ be the solution of the random ordinary differential equation

$$\dot{u}_c = A_c u_c + F_c(\theta_t\omega, u_c + h(\theta_t\omega, u_c)), \tag{7.3}$$

so that $\tilde{u}_c(t,\omega,x_c) \in V$ for small $|t|$ and $\tilde{u}_c(0,\omega,x_c) = x_c$. Now we consider

$$\tilde{u}(t,\omega) := \tilde{u}_c(t,\omega,x_c) + h(\theta_t\omega, \tilde{u}_c(t,\omega,x_c)) \in W(\theta_t\omega).$$

This gives a solution to equation (3.1) in $W(\omega)$ containing x as an interior point. Indeed, we have $\tilde{u}_{us}(t,\omega) := \tilde{u}(t,\omega) - \tilde{u}_c(t,\omega,x_c) = h(\theta_t\omega, \tilde{u}_c(t,\omega,x_c))$, and by identity (7.1),

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}\tilde{u}_{us}(t,\omega) &= \tfrac{\mathrm{d}}{\mathrm{d}t}h(\theta_t\omega, \tilde{u}_c(t,\omega,x_c)) \\ &= Dh(\theta_t\omega, \tilde{u}_c(t,\omega,x_c)) \cdot [A_c\tilde{u}_c(t,\omega,x_c) + F_c(\theta_t\omega, \tilde{u}_c(t,\omega,x_c) + h(\theta_t\omega, \tilde{u}_c(t,\omega,x_c)))] \\ &\quad + \partial_\tau h(\theta_{\tau+t}\omega, \tilde{u}_c(t,\omega,x_c)) \mid_{\tau=0} \\ &= A_{us}h(\theta_t\omega, \tilde{u}_c(t,\omega,x_c)) + F_{us}(\theta_t\omega, \tilde{u}_c(t,\omega,x_c) + h(\theta_t\omega, \tilde{u}_c(t,\omega,x_c))).\end{aligned}$$

Combining this with (7.3), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{u}(t,\omega) = A\tilde{u}(t,\omega) + F(\theta_t\omega, \tilde{u}(t,\omega)).$$

□

7.2 Cut-off Technique

Now for an arbitrary $C^{r,\alpha}$ local center manifold $W^c_{loc}(\omega) \subseteq U$, where $U \subseteq H$ is given in Hypotheses 3.1.2 and 3.3.1, by definition, we have

$$W^c_{loc}(\omega) = \mathrm{graph}(h(\omega,\cdot))$$

for some neighborhood $V \subseteq U$ of the origin and a $C^{r,\alpha}$ function $h(\omega,\cdot) : V \cap H^c \to H^{us}$ with $h(\omega,0) = 0$, $Dh(\omega,0) = 0$. With the above certification we may extend such a center manifold to a global one for some truncated equation.

Let us recall the cut-off function we defined previously. $\Gamma : [0,+\infty) \to [0,+\infty)$ is a C^∞ function satisfying (3.8) and (3.9),

$$\Gamma(t) = 1,\ \forall t \in [0,1], \qquad \Gamma(t) = 0,\ \forall t \geq 2,$$

and $\exists K_\Gamma > 0$,

$$\sup_{0\le t<+\infty}(|\Gamma(t)|+|\Gamma'(t)|+|\Gamma''(t)|+|\Gamma'''(t)|)\le K_\Gamma<+\infty.$$

For $\rho_1,\rho_2>0$, we introduce two new cut-off functions in H defined by

$$\zeta_{\rho_1}(x_c):=\Gamma(\frac{|x_c|}{\rho_1}),\ \forall x_c\in H^c,\qquad \tilde{\zeta}_{\rho_1,\rho_2}(x):=\Gamma(\frac{|x|}{\rho_1+2\rho_2}),\ \forall x\in H. \tag{7.4}$$

Let $\rho^c\in(0,1]$, and $\rho(\cdot):\Omega\to(0,\bar{R}/8]$ be a random variable tempered from below, where R is the constant given in Hypothesis 3.3.1. We can choose them to be so small that $Q_{2\rho}\subseteq V$, where for any integer $k>0$ we denote

$$Q_{k\rho}:=\{x\in H\mid |x_u|,|x_s|<k\rho(\omega),|x_c|<k\rho^c\}. \tag{7.5}$$

We define

$$\begin{aligned} h^c(\omega,x_c)&:=\zeta_{\rho^c}(x_c)h(\omega,x_c),\\ \bar{W}^c(\omega)&:=\text{graph}(h^c(\omega,\cdot)),\\ \bar{F}(\omega,x)&:=\tilde{\zeta}_{\rho^c,\rho(\omega)}(x)F(\omega,x)+G(\omega,x_c), \end{aligned} \tag{7.6}$$

where $G(\omega,\cdot):H^c\to H^{us}$ is to be determined. Then we consider the truncated equation

$$u_t=Au+\bar{F}(\theta_t\omega,u). \tag{7.7}$$

In order to have $\bar{W}^c(\omega)$ be a center manifold for equation (7.7), by lemma 7.1, we need choose a proper formula for G so that

$$\begin{aligned} A_{us}h^c(\omega,x_c)=&Dh^c(\omega,x_c)[A_cx_c+\bar{F}_c(\omega,x_c+h^c(\omega,x_c))]+\partial_t h^c(\theta_t\omega,x_c)\mid_{t=0}\\ &-\bar{F}_{us}(\omega,x_c+h^c(\omega,x_c)). \end{aligned} \tag{7.8}$$

Because the local center manifold $W^c_{loc}(\omega)$ is invariant for equation (3.1), we have by lemma 7.1 that

$$
\begin{aligned}
A_{us}h(\omega,x_c) = & Dh(\omega,x_c)[A_c x_c + F_c(\omega, x_c + h(\omega,x_c))] + \partial_t h(\theta_t\omega, x_c) \mid_{t=0} \\
& - F_{us}(\omega, x_c + h(\omega, x_c)).
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
A_{us}h^c(\omega,x_c) &= A_{us}\zeta_{\rho^c}(x_c)h(\omega,x_c) \\
&= \zeta_{\rho^c}(x_c)A_{us}h(\omega,x_c) \\
&= \zeta_{\rho^c}(x_c)\{Dh(\omega,x_c)[A_c x_c + F_c(\omega,x_c+h(\omega,x_c))] \\
&\quad + \partial_t h(\theta_t\omega,x_c) \mid_{t=0} - F_{us}(\omega, x_c + h(\omega,x_c))\}.
\end{aligned}
\tag{7.9}
$$

Also the right hand side of (7.8) reads

$$
\begin{aligned}
& Dh^c(\omega,x_c)\,[A_c x_c + \tilde{\zeta}_{\rho^c,\rho(\omega)}(x_c + h^c(\omega,x_c))F_c(\omega, x_c + h^c(\omega,x_c))] + \partial_t h^c(\theta_t\omega,x_c) \mid_{t=0} \\
& - \tilde{\zeta}_{\rho^c,\rho(\omega)}(x_c + h^c(\omega,x_c))F_{us}(\omega,x_c + h^c(\omega,x_c)) - G(\omega,x_c).
\end{aligned}
$$

Identifying the above with the RHS of (7.9) we get the formula for G as

$$
\begin{aligned}
G(\omega,x_c) = & Dh^c(\omega,x_c)[A_c x_c + \tilde{\zeta}_{\rho^c,\rho(\omega)}(x_c + h^c(\omega,x_c))F_c(\omega,x_c + h^c(\omega,x_c))] \\
& - \tilde{\zeta}_{\rho^c,\rho(\omega)}(x_c + h^c(\omega,x_c))F_{us}(\omega,x_c + h^c(\omega,x_c)) \\
& - \zeta_{\rho^c}(x_c)\{Dh(\omega,x_c)[A_c x_c + F_c(\omega,x_c + h(\omega,x_c))] - F_{us}(\omega,x_c + h(\omega,x_c))\},
\end{aligned}
\tag{7.10}
$$

where we have used the fact that $\zeta_{\rho^c}(x_c)\partial_t h(\theta_t\omega,x_c) \mid_{t=0} = \partial_t h^c(\theta_t\omega,x_c) \mid_{t=0}$ by the construction of h^c. Therefore we get (7.8) holds and thus $\bar{W}^c(\omega)$ is invariant for the truncated equation (7.7). Also by our construction we have

$$
\bar{W}^c(\omega) \cap Q_\rho = W^c_{loc}(\omega) \cap Q_\rho.
$$

Moreover, as we have

$$\zeta_{\rho^c} \mid_{Q_\rho} = \tilde{\zeta}_{\rho^c,\rho(\omega)} \mid_{Q_\rho} = 1,$$

so $G(\omega, \cdot) \mid_{Q_\rho} = 0$ and thus $\bar{F} \mid_{Q_\rho} = F \mid_{Q_\rho}$.

We define for the truncated equation (7.7) the global center manifold $W^c(\omega)$, the global center-stable manifold $W^{cs}(\omega)$, and the global center-unstable manifold $W^{cu}(\omega)$ as following (see [BDL92]):

$$\begin{aligned} W^c(\omega) &:= \{x \in H \mid \sup_{|t|<+\infty} |P^{us} u(t,\omega,x)| < +\infty\}, \\ W^{cs}(\omega) &:= \{x \in H \mid \sup_{0\le t<+\infty} |P^{u} u(t,\omega,x)| < +\infty\}, \\ W^{cu}(\omega) &:= \{x \in H \mid \sup_{-\infty< t\le 0} |P^{s} u(t,\omega,x)| < +\infty\}, \end{aligned} \tag{7.11}$$

where $u(t,\omega,x)$ is a solution to equation (7.7). Following these definitions, since $\bar{W}^c(\omega) - Q_{2\rho} = H^c - Q_{2\rho}$ by the property of cut-off function, $\bar{W}^c(\omega)$ satisfies the characterization of a global center manifold for equation (7.7).

Therefore, our next step is to show that if we choose ρ^c and $\rho(\omega)$ to be properly small, there exists a unique center manifold $W^c(\omega)$, a unique center-stable manifold $W^{cs}(\omega)$, and a unique center-unstable manifold $W^{cu}(\omega)$ for equation (7.7), furthermore we have $W^c(\omega) = W^{cs}(\omega) \cap W^{cu}(\omega)$. By the above discussion, $\bar{W}^c(\omega)$ agrees with $W^c_{loc}(\omega)$ locally, while it is the global center manifold for the truncated equation. This suggests that any local center manifold for equation (3.1) can be extended to a global center manifold for a truncated equation of the form (7.7). We will use this observation to conclude the main theorem of this part at the end of the section.

7.3 Extension of a Local Center Manifold

Now we focus on the truncated equation (7.7)

$$u_t = Au + \bar{F}(\theta_t\omega, u),$$

where

$$\bar{F}(\theta_t\omega, u) = \tilde{\zeta}_{\rho^c,\rho(\theta_t\omega)} F(\theta_t\omega, u) + G(\theta_t\omega, u_c),$$

$F(\omega, u) = f(u) + g(u)\mathcal{G}_\delta(\omega)$ is given as in equation (3.1) and $G(\omega, \cdot)$ is given in (7.10). To simplify notation we define for $u_c \in H^c$

$$\theta(\omega, u_c) := u_c + h(\omega, u_c), \qquad \theta^c(\omega, u_c) = u_c + h^c(\omega, u_c).$$

Hence G can be written as

$$\begin{aligned} G(\omega, u_c) = & Dh^c(\omega, u_c)[A_c u_c + \tilde{\zeta}_{\rho^c,\rho(\omega)}(\theta^c(\omega, u_c))F_c(\omega, \theta^c(\omega, u_c))] \\ & -\tilde{\zeta}_{\rho^c,\rho(\omega)}(\theta^c(\omega, u_c))F_{us}(\omega, \theta^c(\omega, u_c)) \\ & -\zeta_{\rho^c}(u_c)\{Dh(\omega, u_c)[A_c u_c + F_c(\omega, \theta(\omega, u_c))] - F_{us}(\omega, \theta(\omega, u_c))\} \end{aligned}$$

We first investigate some properties of $\bar{F}$. Because of the terms $Dh^c(\omega, \cdot)$ and $Dh(\omega, \cdot)$ in formula of $G(\omega, \cdot)$, we see the regularity for G is dropped by 1 from that of h.

Lemma 7.2. *Assume that Hypotheses 3.1.1, 3.1.2 and 3.3.1 hold with $r \geq 2$ and $0 \leq \alpha \leq 1$, and $\rho^c \leq 1$ and $\rho(\omega) \leq R/8$, then we have*
(i) $\bar{F} \mid_{Q_\rho} = F \mid_{Q_\rho}$.
(ii) There are constants $K_{\bar{F}}$ and $\bar{M}_g$ which are bounded, so that for $\forall u,\ v \in H$,

$$|\bar{F}(\omega, u) - \bar{F}(\omega, v)| \leq K_{\bar{F}}(\|A_c\|\rho^c + \sup_{Q_{4\rho}} \|Df(\cdot)\| + \bar{M}_g \rho(\omega)^{\bar{\epsilon}_0} C(\omega))|u - v|,$$

where $\|A_c\|$ denotes the operator norm of A_c. We denote by $Lip\bar{F}(\omega) = K_{\bar{F}}(\|A_c\|\rho^c + \sup_{Q_{4\rho}} \|Df(\cdot)\| + \bar{M}_g \rho(\omega)^{\bar{\epsilon}_0} C(\omega))$.
(iii) Furthermore, there exist constants $M_{LipD^{(i)}\bar{F}_\rho}$, $i = 1,\ 2,\ \ldots,\ r-2$, such that for $\forall u,\ v \in H$,

$$\|D^{(i)}\bar{F}_\rho(\omega, u) - D^{(i)}\bar{F}_\rho(\omega, v)\| \leq (O(1) + M_{LipD^{(i)}F_\rho} \rho(\omega)^{\bar{\epsilon}_0} C(\omega))|u - v|,$$

for $i = 1,\ 2,\ \ldots,\ r-2$. *And for* $0 < \alpha \leq 1$, *there exists a constant* $M_{HolD^{(r-1)}\bar{F}_\rho}$, *such that for* $\forall u,\ v \in H$,

$$\|D^{(r-1)}\bar{F}_\rho(\omega,u) - D^{(r-1)}\bar{F}_\rho(\omega,v)\| \leq (O(1) + M_{HolD^{(r-1)}\bar{F}_\rho}\rho(\omega)^{\bar{\epsilon}_0}C(\omega))|u-v|^\alpha.$$

We denote by $LipD^{(i)}F_\rho(\omega) = O(1) + M_{LipD^{(i)}\bar{F}_\rho}\rho(\omega)^{\bar{\epsilon}_0}C(\omega)$ *for* $i = 1,\ 2,\ \ldots,\ r-2$, *and* $HolD^{(r-1)}\bar{F}_\rho(\omega) = (O(1) + M_{HolD^{(r-1)}\bar{F}_\rho}\rho(\omega)^{\bar{\epsilon}_0}C(\omega))$. *Here* $O(1)$ *is some bounded constant as* $\rho(\omega) \to 0$.

Proof. (i) As discussed above, this is a direct consequence from the definition of the cut-off functions.

(ii) To estimate the Lipschitz constant for $\bar{F}$, we let $u,\ v \in H$, then

$$|\bar{F}(\omega,u) - \bar{F}(\omega,v)| \leq |\tilde{\zeta}_{\rho^c,\rho(\omega)}(u)F(\omega,u) - \tilde{\zeta}_{\rho^c,\rho(\omega)}(v)F(\omega,v)| + |G(\omega,u_c) - G(\omega,v_c)|.$$

If $u,\ v \in Q_{2\rho}^C = H - Q_{2\rho}$,

$$\tilde{\zeta}_{\rho^c,\rho(\omega)}(u)F(\omega,u) = \tilde{\zeta}_{\rho^c,\rho(\omega)}(v)F(\omega,v) = 0.$$

If $u \in Q_{2\rho}^C$, $v \in Q_{2\rho}$,

$$|\tilde{\zeta}_{\rho^c,\rho(\omega)}(u)F(\omega,u) - \tilde{\zeta}_{\rho^c,\rho(\omega)}(v)F(\omega,v)| = |\tilde{\zeta}_{\rho^c,\rho(\omega)}(u)F(\omega,v) - \tilde{\zeta}_{\rho^c,\rho(\omega)}(v)F(\omega,v)|.$$

And if $u,\ v \in Q_{2\rho}$,

$$\begin{aligned}
&|\tilde{\zeta}_{\rho^c,\rho(\omega)}(u)F(\omega,u) - \tilde{\zeta}_{\rho^c,\rho(\omega)}(v)F(\omega,v)| \\
\leq &|\tilde{\zeta}_{\rho^c,\rho(\omega)}(u)F(\omega,u) - \tilde{\zeta}_{\rho^c,\rho(\omega)}(u)F(\omega,v)| + |\tilde{\zeta}_{\rho^c,\rho(\omega)}(u)F(\omega,v) - \tilde{\zeta}_{\rho^c,\rho(\omega)}(v)F(\omega,v)|.
\end{aligned}$$

Observe that the other two cases follow similarly once we give a bound for the case that $u,\ v \in Q_{2\rho}$, so we focus on the third case in the following. We first make an estimate for each part of $\bar{F}$.

$LipF$: Using Hypotheses 3.1.2 and 3.3.1, and bound (2.5), we have for $\forall u,\ v \in Q_{2\rho}$,

$$\begin{aligned}&|F(\omega,u)-F(\omega,v)|\\ \leq&|f(u)-f(v)|+|g(u_{us})-g(v_{us})||\mathcal{G}_\delta(\omega)|_{H_0}\\ \leq&(\sup_{Q_{4\rho}}\|Df(\cdot)\|+\bar{M}_g\rho(\omega)^{\bar{\epsilon}_0}C(\omega))|u-v|,\end{aligned}$$

where

$$\bar{M}_g=\begin{cases} M_{0,\bar{R}}\cdot 8^{\bar{\epsilon}_0}\cdot\bar{R}\cdot\frac{\delta+1}{\delta}, & \text{if } r=0,\\ \sup_{Q_{4\rho}}\|D^{(2)}g(\cdot)\|\cdot 8\cdot(\frac{\bar{R}}{8})^{1-\bar{\epsilon}_0}\cdot\frac{\delta+1}{\delta}, & \text{if } r\geq 1.\end{cases}$$

So $LipF(\omega,\cdot)\mid_{Q_{2\rho}}=\sup_{Q_{4\rho}}\|Df(\cdot)\|+\bar{M}_g\rho(\omega)^{\bar{\epsilon}_0}C(\omega)$.

$Lip\zeta_{\rho^c}$: for $\forall u_c,\ v_c\in H^c$,

$$\begin{aligned}\zeta_{\rho^c}(u_c)-\zeta_{\rho^c}(v_c)|=&|\Gamma(\tfrac{|u_c|}{\rho^c})-\Gamma(\tfrac{|v_c|}{\rho^c})|\\ \leq&\sup_t|\Gamma'(t)|\cdot\tfrac{|u_c-v_c|}{\rho^c}\leq K_\Gamma\cdot\tfrac{1}{\rho^c}\cdot|u_c-v_c|,\end{aligned}$$

so $Lip\zeta_{\rho^c}=K_\Gamma\cdot\frac{1}{\rho^c}$.

$Lip\tilde{\zeta}_{\rho^c,\rho(\omega)}$: for $\forall u,\ v\in H$,

$$|\tilde{\zeta}_{\rho^c,\rho(\omega)}(u)-\tilde{\zeta}_{\rho^c,\rho(\omega)}(v)|=|\Gamma(\frac{|u|}{\rho^c+2\rho(\omega)})-\Gamma(\frac{|v|}{\rho^c+2\rho(\omega)})|\leq K_\Gamma\cdot\frac{|u-v|}{\rho^c+2\rho(\omega)},$$

so $Lip\tilde{\zeta}_{\rho^c,\rho(\omega)}=K_\Gamma\cdot\frac{1}{\rho^c+2\rho(\omega)}$.

$LipD\zeta_{\rho^c}$: first notice that $D\zeta_{\rho^c}\mid_{Q_\rho}=0$ since $\zeta_{\rho^c}\mid_{Q_\rho}\equiv 1$. Now in $H^c-\{0\}$,

$$D\zeta_{\rho^c}(u_c)=\Gamma'(\frac{|u_c|}{\rho^c})\cdot\frac{1}{\rho^c}\cdot D|\cdot|_c(u_c),$$

where $D|\cdot|_c$ is the derivative of $|\cdot|$ in $H^c-\{0\}$ which is a bounded constant map as H^c is finite dimensional, we let $\|D|\cdot|_c\|$ denote its norm. Then we have for $\forall u_c,\ v_c \in Q_{2\rho} \cap H^c$,

$$\begin{aligned}
&|D\zeta_{\rho^c}(u_c) - D\zeta_{\rho^c}(v_c)| \\
\leq &K_\Gamma \tfrac{|u_c - v_c|}{\rho^c} \tfrac{1}{\rho^c} \|D|\cdot|_c\| |u_c| + K_\Gamma \tfrac{1}{\rho^c} \|D|\cdot|_c\| |u_c - v_c| \\
\leq &3K_\Gamma \tfrac{1}{\rho^c} \|D|\cdot|_c\| |u_c - v_c|,
\end{aligned}$$

so $LipD\zeta_{\rho^c} = 3K_\Gamma \frac{1}{\rho^c}\|D|\cdot|_c\|$.

$Liph^c(\omega,\cdot)$: we have for $u_c,\ v_c \in Q_{2\rho} \cap H^c$,

$$\begin{aligned}
&|h^c(\omega, u_c) - h^c(\omega, v_c)| \\
= &|\zeta_{\rho^c}(u_c)h(\omega, u_c) - \zeta_{\rho^c}(v_c)h(\omega, v_c)| \\
\leq &Lip\zeta_{\rho^c}|u_c| \sup\nolimits_{Q_{4\rho}} \|Dh(\omega,\cdot)\| |u_c - v_c| + Lip\zeta_{\rho^c}|u_c - v_c| \sup\nolimits_{Q_{4\rho}} \|Dh(\omega,\cdot)\| |v_c| \\
\leq &2 \cdot K_\Gamma \cdot \tfrac{1}{\rho^c} \cdot \sup\nolimits_{Q_{4\rho}} \|Dh(\omega,\cdot)\| \cdot 2\rho^c |u_c - v_c| \\
= &4K_\Gamma \sup\nolimits_{Q_{4\rho}} \|Dh(\omega,\cdot)\| |u_c - v_c|,
\end{aligned}$$

so $Liph^c(\omega,\cdot)\ |_{Q_{2\rho}} = 4K_\Gamma \sup_{Q_{4\rho}} \|Dh(\omega,\cdot)\|$.

$LipDh^c(\omega,\cdot)$: we have by the product rule that for $u_c \in H^c$,

$$Dh^c(\omega, u_c) = D(\zeta_{\rho^c}(u_c)h(\omega, u_c)) = D\zeta_{\rho^c}(u_c)h(\omega, u_c) + \zeta_{\rho^c}(u_c)Dh(\omega, u_c).$$

Then for $u_c,\ v_c \in Q_{2\rho} \cap H^c$,

$$\begin{aligned}
&|Dh^c(\omega, u_c) - Dh^c(\omega, v_c)| \\
\leq &|D\zeta_{\rho^c}(u_c)h(\omega, u_c) - D\zeta_{\rho^c}(v_c)h(\omega, v_c)| + |\zeta_{\rho^c}(u_c)Dh(\omega, u_c) - \zeta_{\rho^c}(v_c)Dh(\omega, v_c)| \\
\leq &4\rho^c(LipD\zeta_{\rho^c} \sup\nolimits_{Q_{4\rho}} \|Dh(\omega,\cdot)\| + Lip\zeta_{\rho^c} M_h)|u_c - v_c|,
\end{aligned}$$

where

$$M_h = \begin{cases} LipDh(\omega,\cdot), & \text{if } r = 0, \\ \sup_{Q_{4\rho}} \|D^{(2)}h(\omega,\cdot)\|, & \text{if } r \geq 1. \end{cases}$$

So we have

$$
\begin{aligned}
LipDh^c(\omega,\cdot)\mid_{Q_{2\rho}} &= 4\rho^c(LipD\zeta_{\rho^c}\sup\nolimits_{Q_{4\rho}}\|Dh(\omega,\cdot)\| + Lip\zeta_{\rho^c}M_h)\\
&= 4\rho^c(3K_\Gamma\tfrac{1}{\rho^c}\|D|\cdot|_c\|\sup\nolimits_{Q_{4\rho}}\|Dh(\omega,\cdot)\| + K_\Gamma\tfrac{1}{\rho^c}M_h)\\
&= 12K_\Gamma\|D|\cdot|_c\|\sup\nolimits_{Q_{4\rho}}\|Dh(\omega,\cdot)\| + 4K_\Gamma M_h.
\end{aligned}
$$

Lipθ: we have for $u_c,\ v_c \in Q_{2\rho}\cap H^c$,

$$
|\theta(\omega,u_c)-\theta(\omega,v_c)| = |(u_c+h(\omega,u_c))-(v_c+h(\omega,v_c))| \le (1+\sup_{Q_{4\rho}}\|Dh(\omega,\cdot)\|)|u_c-v_c|,
$$

so $Lip\theta = 1+\sup_{Q_{4\rho}}\|Dh(\omega,\cdot)\|$.

Lipθ^c: we have for $u_c,\ v_c \in Q_{2\rho}\cap H^c$,

$$
|\theta^c(\omega,u_c)-\theta^c(\omega,v_c)| = |(u_c+h^c(\omega,u_c))-(v_c+h^c(\omega,v_c))| \le (1+4K_\Gamma\sup_{Q_{4\rho}}\|Dh(\omega,\cdot)\|)|u_c-v_c|,
$$

so $Lip\theta^c = 1+4K_\Gamma\sup_{Q_{4\rho}}\|Dh(\omega,\cdot)\|$.

With all above Lipschitz constants prepared, we can now estimate $Lip\bar{F}(\omega)$. For $u,\ v\in Q_{2\rho}$,

$$
\begin{aligned}
&|\tilde{\zeta}_{\rho^c,\rho(\omega)}(u)F(\omega,u)-\tilde{\zeta}_{\rho^c,\rho(\omega)}(v)F(\omega,v)|\\
\le{}&|\tilde{\zeta}_{\rho^c,\rho(\omega)}(u)F(\omega,u)-\tilde{\zeta}_{\rho^c,\rho(\omega)}(u)F(\omega,v)|+|\tilde{\zeta}_{\rho^c,\rho(\omega)}(u)F(\omega,v)-\tilde{\zeta}_{\rho^c,\rho(\omega)}(v)F(\omega,v)|\\
\le{}&K_\Gamma LipF(\omega,\cdot)\mid_{Q_{2\rho}}|u-v| + K_\Gamma\tfrac{|u-v|}{\rho^c+2\rho(\omega)}LipF(\omega,\cdot)\mid_{Q_{2\rho}}|v|\\
\le{}&3K_\Gamma LipF(\omega,\cdot)\mid_{Q_{2\rho}}|u-v|.
\end{aligned}
$$

And we can make an estimate on the other two cases similarly and conclude

$$
|\tilde{\zeta}_{\rho^c,\rho(\omega)}(u)F(\omega,u)-\tilde{\zeta}_{\rho^c,\rho(\omega)}(v)F(\omega,v)| \le 3K_\Gamma(\sup_{Q_{4\rho}}\|Df(\cdot)\|+\bar{M}_g\rho(\omega)^{\bar{\epsilon}_0}C(\omega))|u-v|. \quad (7.12)
$$

Next we look at $|G(\omega,u_c)-G(\omega,v_c)|$, as above, when $u_c,\ v_c\in Q_{2\rho}^C\cap H^c$, both of the two terms equal 0, and if one is in $Q_{2\rho}$ while the other is in $Q_{2\rho}^C$, the corresponding estimate follows similarly as that for the case $u_c,\ v_c \in Q_{2\rho} \cap H^c$. Also we know that $G\mid_{Q_\rho}= 0$, so we indeed focus on the case that $u_c,\ v_c \in (Q_{2\rho} - Q_\rho) \cap H^c$.

We split $|G(\omega, u_c) - G(\omega, v_c)|$ into 3 parts as

$$|G(\omega, u_c) - G(\omega, v_c)| \leq B_1 + B_2 + B_3,$$

where

$$\begin{aligned}
B_1 =& |Dh^c(\omega, u_c)[A_c u_c + \tilde{\zeta}_{\rho^c,\rho(\omega)}(\theta^c(\omega, u_c))F_c(\omega, \theta^c(\omega, u_c))] \\
&-Dh^c(\omega, v_c)[A_c v_c + \tilde{\zeta}_{\rho^c,\rho(\omega)}(\theta^c(\omega, v_c))F_c(\omega, \theta^c(\omega, v_c))]|, \\
B_2 =& |\tilde{\zeta}_{\rho^c,\rho(\omega)}(\theta^c(\omega, u_c))F_{us}(\omega, \theta^c(\omega, u_c)) - \tilde{\zeta}_{\rho^c,\rho(\omega)}(\theta^c(\omega, v_c))F_{us}(\omega, \theta^c(\omega, v_c))|, \\
B_3 =& |\zeta_{\rho^c}\{Dh(\omega, u_c)[A_c u_c + F_c(\omega, \theta(\omega, u_c))] - F_{us}(\omega, \theta(\omega, u_c))\} \\
&-\zeta_{\rho^c}\{Dh(\omega, v_c)[A_c v_c + F_c(\omega, \theta(\omega, v_c))] - F_{us}(\omega, \theta(\omega, v_c))\}|.
\end{aligned}$$

First we estimate B_1. It can be further decomposed as

$$\begin{aligned}
B_1 \leq& |Dh^c(\omega, u_c)A_c u_c - Dh^c(\omega, v_c)A_c v_c| \\
&+|Dh^c(\omega, u_c)\tilde{\zeta}_{\rho^c,\rho(\omega)}(\theta^c(\omega, u_c))F_c(\omega, \theta^c(\omega, u_c)) \\
&\quad -Dh^c(\omega, v_c)\tilde{\zeta}_{\rho^c,\rho(\omega)}(\theta^c(\omega, v_c))F_c(\omega, \theta^c(\omega, v_c))| \\
=:& B_{11} + B_{12},
\end{aligned}$$

where A_c is bounded as H^c is finite dimensional. We estimate B_{11} as following.

$$\begin{aligned}
B_{11} \leq& LipDh^c(\omega, \cdot)\mid_{Q_{2\rho}} |u_c|\|A_c\||u_c - v_c| + LipDh^c(\omega, \cdot)\mid_{Q_{2\rho}} |u_c - v_c|\|A_c\||v_c| \\
\leq& 4LipDh^c(\omega, \cdot)\mid_{Q_{2\rho}} \|A_c\|\rho^c|u_c - v_c|.
\end{aligned}$$

And we have

$$\begin{aligned}
B_{12} \leq& 3LipDh^c(\omega, \cdot)\mid_{Q_{2\rho}} Lip\tilde{\zeta}_{\rho^c,\rho(\omega)}(Lip\theta^c)^2 LipF(\omega, \cdot)\mid_{Q_{2\rho}} (2\rho^c)^2|u_c - v_c| \\
\leq& 12LipDh^c(\omega, \cdot)\mid_{Q_{2\rho}} K_\Gamma(Lip\theta^c)^2 LipF(\omega, \cdot)\mid_{Q_{2\rho}} \rho^c|u_c - v_c|.
\end{aligned}$$

So

$$B_1 \leq LipDh^c(\omega,\cdot)\mid_{Q_{2\rho}} (4\|A_c\|\rho^c + 12K_\Gamma(Lip\theta^c)^2\rho^c LipF(\omega,\cdot)\mid_{Q_{2\rho}})|u_c - v_c|.$$

Similarly, we can bound B_2 and B_3 as

$$\begin{aligned}
B_2 &\leq 4K_\Gamma(Lip\theta^c)^2 LipF(\omega,\cdot)\mid_{Q_{2\rho}} |u_c - v_c|,\\
B_3 &\leq (12K_\Gamma M_h\rho^c(\|A_c\| + Lip\theta LipF(\omega,\cdot)\mid_{Q_{2\rho}}) + 4K_\Gamma Lip\theta LipF(\omega,\cdot)\mid_{Q_{2\rho}})|u_c - v_c|.
\end{aligned}$$

Combining all the above yields

$$|G(\omega,u_c)-G(\omega,v_c)| \leq K(K_\Gamma, \|D|\cdot|_c\|, \sup_{Q_{4\rho}}\|Dh(\omega,\cdot)\|, M_h)(\|A_c\|\rho^c + LipF(\omega,\cdot)\mid_{Q_{2\rho}})|u_c - v_c|,$$

where $K(K_\Gamma, \|D|\cdot|_c\|, \sup_{Q_{4\rho}}\|Dh(\omega,\cdot)\|, M_h)$ is a bounded constant provided that $\rho^c \leq 1$ and $\rho(\omega) \leq \bar{R}/8$. Therefore, by combining the above and (7.12) we have

$$\begin{aligned}
&|\bar{F}(\omega,u) - \bar{F}(\omega,v)|\\
\leq\, &K(K_\Gamma, \|D|\cdot|_c\|, \sup\nolimits_{Q_{4\rho}}\|Dh(\omega,\cdot)\|, M_h)(\|A_c\|\rho^c + \sup\nolimits_{Q_{4\rho}}\|Df(\cdot)\| + \bar{M}_g\rho(\omega)^{\bar{\epsilon}_0}C(\omega))|u-v|\\
:=\, &K_{\bar{F}}(\|A_c\|\rho^c + \sup\nolimits_{Q_{4\rho}}\|Df(\cdot)\| + \bar{M}_g\rho(\omega)^{\bar{\epsilon}_0}C(\omega))|u-v|.
\end{aligned}$$

(iii) This follows a similar computation as (ii). □

Now as suggested before, with the property of $\bar{F}$, we can prove Theorem 5.3 and 6.3 for the new truncated equation (7.7). The key step is to let $Lip\bar{F}(\omega)$, $LipD^{(i)}\bar{F}(\omega)$ and $HolD^{(r-1)}\bar{F}(\omega)$ satisfy conditions given similarly as in (5.1), (5.4), (6.2) and (6.5), with r reduced to $r-1$. Indeed, we make use of the fact that

$$\|A_c\|\rho^c + \sup_{Q_{4\rho}}\|Df(\cdot)\| + \bar{M}_g\rho(\omega)^{\bar{\epsilon}_0}C(\omega) = o(1)$$

as $\rho^c \to 0$ and $\rho(\omega) \to 0$. Only notice that as we can see, the choice is made both on ρ^c and $\rho(\omega)$. By this we get the local structures that recover the originally given local center manifold. We state it precisely as the following theorem.

Theorem 7.3. *Assume that Hypotheses 3.1.1, 3.1.2 and 3.3.1 hold with* $r \geq 2$ *and* $0 \leq \alpha \leq 1$ *for equation (3.1), then for any* $C^{r,\alpha}$ *local center manifold* $W(\omega) \subseteq U$ *of the origin, there are a* $C^{r-1,\alpha}$ *local center-unstable manifold* $M^{cu}(\omega)$ *and a* $C^{r-1,\alpha}$ *local center-stable manifold* $M^{cs}(\omega)$ *in a tempered region* $Q_{\rho(\omega)} \subseteq U$ *of the origin both containing* $W(\omega)$ *as a submanifold and satisfying* $LipM^i(\omega) < \frac{1}{3}$, $i = cu,\ cs$. *Moreover, there are a* $C^{r-2,\alpha} \times C^{r-1,\alpha}$ *unstable foliation with leaves* $\mathcal{F}^{cuu}$ *on* $M^{cu}(\omega)$ *and a* $C^{r-2,\alpha} \times C^{r-1,\alpha}$ *stable foliation with leaves* $\mathcal{F}^{css}$ *on* $M^{cs}(\omega)$*, both leaves have Lipschitz constant* $\frac{1}{3}$.

Proof. For a given center manifold $W(\omega)$, we can always use (7.6) and (7.10) to construct a truncated equation (7.7) with a global center manifold coinciding with $W(\omega)$ locally in a region Q_ρ. But there is a unique center manifold as the intersection of a unique center-unstable manifold and a unique center-stable manifold if we restrict ρ^c and $\rho(\omega)$ properly according to the $Lip\bar{F}(\omega)$, $LipD^{(i)}\bar{F}(\omega)$ and $HolD^{(r-1)}\bar{F}(\omega)$ we get in Lemma 7.2, with foliation structures on those manifolds as justified in Theorem 5.3 and 6.3. Therefore, that global center manifold containing $W(\omega)$ is the center manifold for the truncated equation and all conclusions follow. □

7.4 Two Technical Lemmas

Now we can head to investigate smooth conjugacy between two arbitrary local center manifolds for the main equation (3.1). To do this, we need the following two lemmas.

Lemma 7.4. *Let* $M^{cs}(\omega)$ *be a* $C^{r-1,\alpha}$ *local center-stable manifold of the origin with* $r \geq 2$ *and* $LipM^{cs}(\omega) < \frac{1}{3}$. *Assume there is a* $C^{r-2,\alpha} \times C^{r-1,\alpha}$ *stable foliation with leaves* $\mathcal{F}^{css}(\omega, \xi)$ *on* $M^{cs}(\omega)$. *Then for two arbitrary* $C^{r-1,\alpha}$ *local center manifolds* $M_1^c(\omega)$, $M_2^c(\omega) \subseteq M^{cs}(\omega)$ *of the origin with* $LipM_i^c(\omega) < 1$, $i = 1,\ 2$, *there is a neighborhood* $V(\omega) \subseteq U$ *of the origin and a* $C^{r-2,\alpha}$ *diffeomorphism* $\phi(\omega, \cdot) : M_1^c(\omega) \cap V(\omega) \to M_2^c(\omega) \cap V(\omega)$ *such that*

$$u(t, \omega, \phi(\omega, x)) = \phi(\theta_t\omega, u(t, \omega, x)),$$

for all $x \in M_1^c(\omega) \cap V(\omega)$, and all t satisfying $u(t,\omega,x) \in M_1^c(\theta_t\omega) \cap V(\theta_t\omega)$, where $u(t,\omega,\cdot)$ is the flow induced by the solution to equation (3.1).

Proof. We only demonstrate the $C^{0,\alpha}$ case since the $C^{r-2,\alpha}$ case with $r > 2$ is simplified when the contraction mapping principle is replaced by the implicit function theorem.
We want to show that a homeomorphism is defined by $\phi(\omega,p) = q := \mathcal{F}^{css}(\omega,p) \cap M_2^c(\omega)$ for $p \in M_1^c(\omega)$ near the origin.

To do this, we begin by identifying the local center-stable manifold $M^{cs}(\omega)$ with the coordinate plane $H^{cs} \cap Q_\rho$ via a $C^{0,1}$ function $h^{cs}(\omega,\cdot)$, where $M^{cs}(\omega) = \text{graph}(h^{cs}(\omega,\cdot))$. Here Q_ρ is given as in (7.5). Now, in terms of the coordinate system for H^{cs}, let $M_i^c(\omega) = \text{graph}(h_i^c(\omega,\cdot))$ for some $C^{0,1}$ function $h_i^c(\omega,\cdot) : H^c \cap Q_\rho \to H^s \cap Q_\rho$ with $Liph_i^c(\omega,\cdot) < 1$ (projection mapping has norm less than or equal to 1), $i = 1,\ 2$. And let $\varphi(\omega,\cdot,\cdot) : \{H^s \cap Q_\rho\} \times \{H^{cs} \cap Q_\rho\} \to H^c \cap Q_\rho$ represent the stable foliation leaf on $M^{cs}(\omega)$, satisfying $\mathcal{F}^{css}(\omega,p) = \text{graph}(\varphi(\omega,\cdot,p))$, and $Lip\varphi(\omega,\cdot,p) \le \frac{1}{3}$ for all $p \in H^{cs} \cap Q_\rho$.

Let $\rho_0(\omega) > 0$ be a random variable tempered from below which is so small that the closed box

$$B_{\rho_0} := \{x \in H^{cs} \mid |x_s| \le \rho_0(\omega), |x_c| \le \rho_0(\omega)\}$$

centered at the origin is contained entirely in Q_ρ. Consider the operator $\Phi(\omega,\cdot,p)$ on B_{ρ_0} defined by

$$\begin{cases} P^c\Phi(\omega,x,p) = \varphi(\omega,x_s,p), \\ P^s\Phi(\omega,x,p) = h_2^c(\omega,x_c), \end{cases} \qquad x \in B_{\rho_0}.$$

We want to show that for some carefully chosen neighborhood $V_0(\omega) \subseteq B_{\rho_0}$ of the origin, there is a unique fixed point $q \in M_2^c(\omega) \cap V_0(\omega)$ for the operator $\Phi(\omega,\cdot,p)$, which is the intersection point of $\mathcal{F}^{css}(\omega,p)$, $p \in M_1^c(\omega) \cap V_0(\omega)$ and $M_2^c(\omega)$, solving the equation

$$x = \Phi(\omega,x,p),$$

and giving rise to the conjugating map ϕ.

First, since we have $Liph_i^c(\omega,\cdot) < 1,\ i = 1,\ 2$, and $Lip\varphi(\omega,\cdot,p) \leq \frac{1}{3}\ \forall p \in H^{cs} \cap Q_\rho$, so there exist $\rho_1(\omega) < \rho_0(\omega)$ so small and

$$B_{\rho_1} := \{x \in H^{cs} \mid |x_s| \leq \rho_1(\omega), |x_c| \leq \rho_1(\omega)\}$$

such that if we pick $p \in H^{cs} \cap B_{\rho_1}$, for $\forall x \in B_{\rho_0}$,

$$\begin{aligned}|\varphi(\omega,x_s,p)| &\leq Lip\varphi(\omega,\cdot,p)|x_s| + |\varphi(\omega,0,p) - \varphi(\omega,p_s,p)| + |\varphi(\omega,p_s,p)| \\ &\leq Lip\varphi(\omega,\cdot,p)\cdot\rho_0(\omega) + Lip\varphi(\omega,\cdot,p)\cdot\rho_1(\omega) + \rho_1(\omega) \\ &< \rho_0(\omega),\end{aligned}$$

and that

$$|h_2^c(\omega,x_c)| \leq Liph_2^c(\omega,\cdot)|x_c| < \rho_0(\omega).$$

That is, for all $p \in H^{cs} \cap B_{\rho_1}$, $\Phi(\omega,\cdot,p)$ maps B_{ρ_0} into itself. Moreover, by the box norm for the space H, we have

$$Lip\Phi(\omega,\cdot,p) = \max\{Lip\varphi(\omega,\cdot,p), Liph_2^c(\omega,\cdot)\} < 1$$

uniformly for all $p \in M_1^c(\omega) \cap B_{\rho_1}$. Therefore, by the contraction mapping principle, there is a unique fixed point $q(\omega,p) \in M_2^c(\omega) \cap B_{\rho_0}$ for every $p \in M_1^c(\omega) \cap B_{\rho_1}$. Denote by $q := \phi(\omega,p)$ the fixed point, then $\phi(\omega,\cdot)$ is $C^{0,\alpha}$, by the fact that $\varphi(\omega,x_s,p)$ is $C^{0,\alpha}$ in variable p. Arguing symmetrically, we can also show that for every point $q \in M_2^c(\omega) \cap B_{\rho_1}$ there is a unique fixed point $p \in M_1^c(\omega) \cap B_{\rho_0}$ of the operator $\bar{\Phi}(\omega,\cdot,q)$ on B_{ρ_0} defined by

$$\begin{cases} P^c\bar{\Phi}(\omega,x,q) = \varphi(\omega,x_s,q), \\ P^s\bar{\Phi}(\omega,x,q) = h_1^c(\omega,x_c), \end{cases} \qquad x \in B_{\rho_0}.$$

Hence the fixed point $p := \bar{\phi}(\omega,q)$, which is the intersection point of $\mathcal{F}^{css}(\omega,q)$ and $M_1^c(\omega) \cap B_{\rho_0}$, depends continuously on $q \in M_2^c(\omega) \cap B_{\rho_1}$.

We claim that the function ϕ is actually locally invertible with inverse $\bar{\phi}$. Let

$$q \in \phi(\omega, M_1^c(\omega) \cap B_{\rho_1}) \cap B_{\rho_1},$$

we want to show that $q = \phi(\omega, \bar{\phi}(\omega, q))$. Indeed, let $p = \bar{\phi}(\omega, q)$, by definition, we have

$$\begin{cases} p_c = \varphi(\omega, p_s, q), \\ p_s = h_1^c(\omega, p_c), \end{cases}$$

and for some $p^* \in M_1^c(\omega) \cap B_{\rho_1}$,

$$\begin{cases} q_c = \varphi(\omega, q_s, p^*), \\ q_s = h_2^c(\omega, q_c). \end{cases}$$

Then by the property of leaves of foliation, $p,\ q,\ p^*$ share the same graph of leaf, so

$$q_c = \varphi(\omega, q_s, p).$$

Thus if $p \in M_1^c(\omega) \cap B_{\rho_1}$, we have

$$\begin{cases} q_c = \varphi(\omega, q_s, p), \\ q_s = h_2^c(\omega, q_c). \end{cases}$$

That is q is the fixed point of the operator $\Phi(\omega, \cdot, p)$, then $q = \phi(\omega, p) = \phi(\omega, \bar{\phi}(\omega, q))$.

Now if $p \notin M_1^c(\omega) \cap B_{\rho_1}$, since $p^* \in M_1^c(\omega) \cap B_{\rho_1}$, we have

$$\begin{cases} p_c^* = \varphi(\omega, p_s^*, q), \\ p_s^* = h_1^c(\omega, p_c^*). \end{cases}$$

Then

$$|p_c - p_c^*| = |\varphi(\omega, p_s, q) - \varphi(\omega, p_s^*, q)| < |p_s - p_s^*|,$$

$$|p_s - p_s^*| = |h_1^c(\omega, p_c) - h_1^c(\omega, p_c^*)| < |p_c - p_c^*|,$$

yielding

$$|p - p^*| < |p - p^*|,$$

a contradiction.

Conversely, the same argument shows that if $p \in \bar{\phi}(\omega, M_2^c(\omega) \cap B_{\rho_1}) \cap B_{\rho_1}$, then $p = \bar{\phi}(\omega, \phi(\omega, p))$. To end the proof of the claim, we need to locate the region. By the above, the local region $V_0(\omega)$ should satisfy

$$V_0(\omega) \cap M_2^c(\omega) \subseteq \phi(\omega, M_1^c(\omega) \cap B_{\rho_1}) \cap B_{\rho_1},$$

$$V_0(\omega) \cap M_1^c(\omega) \subseteq \bar{\phi}(\omega, M_2^c(\omega) \cap B_{\rho_1}) \cap B_{\rho_1}.$$

We may take

$$\begin{aligned} V_0(\omega) := B_{\rho_1} - \{ &(M_2^c(\omega) - \phi(\omega, M_1^c(\omega) \cap B_{\rho_1}) \cap B_{\rho_1}) \\ &\cup (M_1^c(\omega) - \bar{\phi}(\omega, M_2^c(\omega) \cap B_{\rho_1}) \cap B_{\rho_1})\}. \end{aligned}$$

Finally, since $u(t, \omega, p) \in M_1^c(\theta_t\omega)$, $u(t, \omega, \phi(\omega, p)) \in M_1^c(\theta_t\omega)$, $\mathcal{F}^{css}(\omega, \phi(\omega, p)) = \mathcal{F}^{css}(\omega, p)$, and $u(t, \omega, \mathcal{F}^{css}(\omega, p)) \subseteq \mathcal{F}^{css}(\theta_t\omega, u(t, \omega, p))$, locally by the invariance of the center manifolds and the foliations, we have

$$\begin{aligned} u(t, \omega, \phi(\omega, p)) &= u(t, \omega, \mathcal{F}^{css}(\omega, p) \cap M_2^c(\omega)) \\ &\subseteq \mathcal{F}^{css}(\theta_t\omega, u(t, \omega, p)) \cap M_2^c(\theta_t\omega) \\ &= \phi(\theta_t\omega, u(t, \omega, p)), \end{aligned}$$

as long as $u(t, \omega, p) \in M_1^c(\theta_t\omega) \cap V_0(\theta_t\omega)$. As the two sides of the above inclusion relationship are both denoting a single point, it is indeed an identity. □

Lemma 7.5. *Let $M_1^c(\omega)$, $M_2^c(\omega)$ be two $C^{r,\alpha}$ local center manifolds of the origin. Let $M^{cs}(\omega)$ be a $C^{r-1,\alpha}$ local center-stable manifold containing $M_1^c(\omega)$, and let $M^{cu}(\omega)$ be a $C^{r-1,\alpha}$ local* *center-stable manifold containing* $M_2^c(\omega)$*, both constructed according to Theorem 7.3. Then the intersection* $M^{cs}(\omega)\cap M^{cu}(\omega)$ *is another* $C^{r-1,\alpha}$ *local center manifold* $M^c(\omega)$ *of the origin with* $Lip M^c(\omega) < 1$.

Proof. We only demonstrate that the intersection is a $C^{0,1}$ local center manifold, the higher order case again follows by replacing the contraction mapping principle used in the proof with the implicit function theorem. Since the center-stable and center-unstable manifolds are constructed according to Theorem 7.3, they satisfy

$$M^{cs}(\omega) = \mathrm{graph}(h^{cs}(\omega,\cdot)), \qquad M^{cu}(\omega) = \mathrm{graph}(h^{cu}(\omega,\cdot)),$$

for some $C^{0,1}$ functions $h^{cs}(\omega,\cdot)$, $h^{cu}(\omega,\cdot)$ defined near the origin with $Liph^i(\omega,\cdot) < \frac{1}{3}$, $i = cs,\ cu$. The intersection $M^{cs}(\omega)\cap M^{cu}(\omega)$ consists of all points $x_u + x_c + x_s \in H$ satisfying

$$\begin{cases} x_u = h^{cs}(\omega, x_c + x_s), \\ x_s = h^{cu}(\omega, x_c + x_u). \end{cases}$$

We can think of the right hand side of the above two equations as an operator parametrized by x_c and defined on H^{us}, to be precise, for $x_u + x_s \in H^{us}$, we define

$$\mathcal{H}(x_c, \omega, x_u + x_s) := h^{cs}(\omega, x_c + x_s) + h^{cu}(\omega, x_c + x_u),$$

then it is clear that

$$Lip\mathcal{H}(x_c, \omega, \cdot) \leq \max\{Liph^{cs}(\omega,\cdot), Liph^{cu}(\omega,\cdot)\} < 1,$$

uniformly for all x_c within a small enough region around the origin. Then by the contraction mapping principle, when restricted in a small region, there exists a unique solution for the equation

$$x_u + x_s = \mathcal{H}(x_c, \omega, x_u + x_s).$$

We define such a solution as $x_u + x_s := h^c(\omega, x_c)$. Now for $x_c,\ \bar{x}_c$ within that region, we have

$$\begin{aligned}
|h^c(\omega,x_c) - h^c(\omega,\bar{x}_c)| &= |(x_u+x_s) - (\bar{x}_u+\bar{x}_s)| \\
&= |h^{cs}(\omega, x_c+x_s) + h^{cu}(\omega, x_c+x_u) - h^{cs}(\omega,\bar{x}_c+\bar{x}_s) - h^{cu}(\omega,\bar{x}_c+\bar{x}_u)| \\
&\leq \max\{Liph^{cs}(\omega,\cdot), Liph^{cu}(\omega,\cdot)\} \cdot |(x_u+x_s) - (\bar{x}_u+\bar{x}_s)| \\
&\quad +(Liph^{cs}(\omega,\cdot) + Liph^{cu}(\omega,\cdot))|x_c - \bar{x}_c|,
\end{aligned}$$

implying that

$$|h^c(\omega,x_c) - h^c(\omega,\bar{x}_c)| \leq \frac{Liph^{cs}(\omega,\cdot) + Liph^{cu}(\omega,\cdot)}{1-\max\{Liph^{cs}(\omega,\cdot), Liph^{cu}(\omega,\cdot)\}}|x_c - \bar{x}_c| < |x_c - \bar{x}_c|,$$

as $Liph^i(\omega,\cdot) < \frac{1}{3}$, $i = cs,\ cu$. That is $Liph^c(\omega,\cdot) < 1$.

To show $M^c(\omega) := \text{graph}(h^c(\omega,\cdot))$ is indeed a $C^{0,1}$ local center manifold of the origin, it suffices to show that it is locally invariant. Let $p \in M^c(\omega)$, and let $u(t,\omega) \in M^{cu}(\theta_t\omega)$ be the solution curve in the center-unstable manifold containing p as an interior point. Because of the uniqueness for the initial value problem of equation (3.1) and the invariance of $M^{cs}(\omega)$, $u(t,\omega) \in M^{cs}(\theta_t\omega)$ for small $t \geq 0$. For the backward flow, since all forward solution flows starting from points on $u(t,\omega)$ with $t \leq 0$ so small, which are getting into $M^{cs}(\theta.\omega)$ after some finite time period as justified above, would satisfy the characterization of the global center-stable manifold given in (7.11), hence combining the extension theorem 7.3 we can conclude that $u(t,\omega) \in M^{cu}(\theta_t\omega)$ for all small $|t|$ for which it is defined. Therefore, $u(t,\omega) \in M^c(\theta_t\omega) = M^{cu}(\theta_t\omega) \cap M^{cs}(\theta_t\omega)$ for all the small $|t|$. □

7.5 Proof of the the Main Theorem for Conjugacy

Now we can prove our last main result Theorem 3.4.

Proof of Theorem 3.4: Let $M^{cs}(\omega)$ be a $C^{r-1,\alpha}$ local center-stable manifold containing $M_1^c(\omega)$ and $M^{cu}(\omega)$ be a $C^{r-1,\alpha}$ local center-stable manifold containing $M_2^c(\omega)$ by Theorem 7.3. Then $M_3^c(\omega) := M^{cs}(\omega) \cap M^{cu}(\omega)$ is another $C^{r-1,\alpha}$ local center manifold by Lemma 7.5.

By Theorem 7.3 again, there exist $C^{r-1,\alpha} \times C^{r-2,\alpha}$ stable and unstable foliations on $M^{cs}(\omega)$ and $M^{cu}(\omega)$ respectively. Hence the conditions of Lemma 7.4 are satisfied for $M_1^c(\omega),\ M_3^c(\omega)$ on $M^{cs}(\omega)$, and $M_2^c(\omega),\ M_3^c(\omega)$ on $M^{cu}(\omega)$, respectively. Thus the flows on $M_i^c(\omega)$ and $M_3^c(\omega)$ are C^r conjugate for $i = 1,\ 2$. Therefore, the local flows on $M_1^c(\omega)$ and $M_2^c(\omega)$ are also C^r conjugate since C^r conjugacy is an equivalence relation. □

Chapter 8. Smooth Conjugacy For a Stochastic Differential Equation

In this chapter, we consider the stochastic differential equation

$$\mathrm{d}u = (Au + f(u_{us}))\mathrm{d}t + u \circ \mathrm{d}W, \qquad u(0) = x \in H, \tag{8.1}$$

where W is a two-sided real-valued Wiener process given as in (2.2) with $H_0 = \mathbb{R}$, while $u \circ \mathrm{d}W$ is interpreted as a Stratonovich stochastic differential. And A is given in Hypothesis 3.1.1, and f is given by Hypothesis 3.3.2. We will justify that there exists smooth conjugacy between local center manifolds for such a system as well.

8.1 Ornstein-Uhlenbeck Process

Following the approach used in ([DLS03], [DLS04], [SLWZ17]), we first use a coordinate transform to convert such a stochastic equation into an infinite dimensional random dynamical system. To be precise, we considered a linear stochastic differential equation

$$\mathrm{d}z = -z\mathrm{d}t + \mathrm{d}W. \tag{8.2}$$

A solution of this equation is called an Ornstein-Uhlenbeck process. We have the following results, see [DLS03].

Lemma 8.1.

(i) There exists a $\{\theta_t\}_{t\in\mathbb{R}}$-invariant set $\Omega_1 \in \mathcal{B}(C_0(\mathbb{R}, \mathbb{R}))$ of full measure with sub-linear growth:

$$\lim_{t\to\pm\infty} \frac{\omega(t)}{|t|} = 0, \qquad \forall \omega \in \Omega_1$$

(ii) For $\omega \in \Omega_1$ the random variable

$$z(\omega) = -\int_{-\infty}^{0} e^r \omega(r)\mathrm{d}r \tag{8.3}$$

exists and generates a unique stationary solution of (8.2) given by

$$\Omega_1 \times \mathbb{R} \ni (\omega, t) \mapsto z(\theta_t\omega) = -\int_{-\infty}^{0} e^r \theta_t\omega(r)\mathrm{d}r = -\int_{-\infty}^{0} e^r \omega(r+t)\mathrm{d}r + \omega(t).$$

The mapping $t \mapsto z(\theta_t\omega)$ is continuous.

(iii) In particular,

$$\lim_{t\to\pm\infty} \frac{|z(\theta_t\omega)|}{|t|} = 0,$$

and

$$\lim_{t\to\pm\infty} \frac{1}{t}\int_0^t z(\theta_r\omega)\mathrm{d}r = 0,$$

for all $\omega \in \Omega_1$.

We now replace $\mathcal{B}(C_0(\mathbb{R},\mathbb{R}))$ by

$$\{\Omega_1 \cap F, F \in \mathcal{B}(C_0(\mathbb{R},\mathbb{R}))\}$$

for Ω_1 given above in Lemma 8.1, and still denote it as $\mathcal{F}$. We denote Ω_1 as Ω, and the probability measure is the restriction of the Wiener measure to this new σ-algebra, which is also denoted by $\mathbb{P}$. In the following we will consider the metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$. We consider the following equation with random coefficients

$$u_t = Au + z(\theta_t\omega)u + e^{-z(\theta_t\omega)} f(e^{z(\theta_t\omega)}u_{us}), \qquad u(0) = x \in H. \tag{8.4}$$

Then the solution can be interpreted in a mild sense

$$u(t,\omega,x) = e^{At+\int_0^t z(\theta_r\omega)\mathrm{d}r}x + \int_0^t e^{A(t-s)+\int_s^t z(\theta_r\omega)\mathrm{d}r} e^{-z(\theta_s\omega)} f(e^{z(\theta_s\omega)}u_{us}(s,\omega,x))\mathrm{d}s.$$

And the solution mapping $(t,\omega,x) \mapsto u(t,\omega,x)$ generates a random dynamical system. We introduce the transform

$$T(\omega,x) = xe^{-z(\omega)}$$

and its inverse transform

$$T^{-1}(\omega,x) = xe^{z(\omega)}$$

for $x \in H$ and $\omega \in \Omega$.

Lemma 8.2. *Suppose that u is the random dynamical system generated by (8.4). Then*

$$(t,\omega,x) \mapsto T^{-1}(\theta_t\omega, u(t,\omega,T(\omega,x))) =: \hat{u}(t,\omega,x) \tag{8.5}$$

is a random dynamical system. For any $x \in H$ this process $(t,\omega) \mapsto \hat{u}(t,\omega,x)$ is a solution to equation (8.1).

With the results above and the key fact that the Lipschitz constant for $f(\cdot)$ and $F(\omega,\cdot)$ are exactly the same, while the bounds for $D^{(i)}f(\cdot)$ control those of $D^{(i)}F(\omega,\cdot)$ whenever we restrict the region smaller by dividing $e^{z(\omega)}$ (which is tempered from above) from the radius, one can use the cut-off technique to show the existence of smooth local invariant manifolds for (8.1) by justifying it for the random dynamical system (8.4), see [DLS04].

8.2 Relationship Between Conjugacy Maps of two systems

Now let $\hat{M}_1^c(\omega)$, $\hat{M}_2^c(\omega)$ be two local center manifolds for equation (8.1), we want to show that there exists a conjugacy between $\hat{M}_1^c(\omega)$ and $\hat{M}_2^c(\omega)$. To do this, we first make an observation of the relationship between conjugacy maps for equation (8.1) and (8.4). Let $M_1^c(\omega)$, $M_2^c(\omega)$ be two corresponding local center manifolds for equation (8.4), and by Theorem 3.3 in [DLS04], they are given by

$$M_i^c(\omega) = T(\omega, \tilde{M}_i^c(\omega)) \qquad i = 1,2.$$

We claim that if there exists a conjugacy between $M_1^c(\omega)$ and $M_2^c(\omega)$, then there is a naturally induced one between $\hat{M}_1^c(\omega)$ and $\hat{M}_2^c(\omega)$ as well. To be precise, suppose there is a neighborhood $V(\omega) \subseteq H$ of the origin and a $C^{r-2,\alpha}$ diffeomorphism $\phi(\omega, \cdot) : M_1^c(\omega) \cap V(\omega) \to M_2^c(\omega) \cap V(\omega)$ such that

$$u(t, \omega, \phi(\omega, x)) = \phi(\theta_t \omega, u(t, \omega, x)) \tag{8.6}$$

for all $x \in M_1^c(\omega) \cap V(\omega)$, and all t satisfying $u(t, \omega, x) \in M_1^c(\theta_t \omega) \cap V(\theta_t \omega)$, where $u(t, \omega, x)$ is the solution for (8.4). Then we have the following proposition.

Proposition 8.2.1. *For equation (8.1), let $\hat{u}(t, \omega, x)$ be a solution given by (8.5), $\hat{M}_1^c(\omega)$ and $\hat{M}_2^c(\omega)$ be two local center manifolds. Let $M_i^c(\omega) = T(\omega, \hat{M}_i^c(\omega))$, $i = 1, 2$, be two corresponding local center manifolds for equation (8.4) with a conjugacy $\phi(\omega, \cdot)$ as (8.6), also $\hat{V}(\omega) = T^{-1}(\omega, V(\omega))$. Then $\psi(\omega, \cdot) : \hat{M}_1^c(\omega) \cap \hat{V}(\omega) \to \hat{M}_2^c(\omega) \cap \hat{V}(\omega)$ defined by*

$$\psi(\omega, \cdot) = T^{-1}(\omega, \phi(\omega, T(\omega, \cdot)))$$

is a $C^{r-2,\alpha}$ diffeomorphism such that

$$\hat{u}(t, \omega, \psi(\omega, \hat{x})) = \psi(\theta_t \omega, \hat{u}(t, \omega, \hat{x})),$$

for all $\hat{x} \in \hat{M}_1^c(\omega) \cap \hat{V}(\omega)$, and all t satisfying $\hat{u}(t, \omega, x) \in \hat{M}_1^c(\theta_t \omega) \cap \hat{V}(\theta_t \omega)$, where $\hat{x} = T^{-1}(\omega, x)$.

Proof. This is a straightforward computation. Let $x \in M_1^c(\omega) \cap V(\omega)$ be chosen as given in the statement, and let $\hat{x} = T^{-1}(\omega, x)$. Then by using (8.5), we have

$$\begin{aligned}\hat{u}(t,\omega,\psi(\omega,\hat{x})) &= T^{-1}(\theta_t\omega, u(t,\omega,T(\omega,\psi(\omega,\hat{x}))))\\ &= T^{-1}(\theta_t\omega, u(t,\omega,\phi(\omega,T(\omega,\hat{x}))))\\ &= T^{-1}(\theta_t\omega, u(t,\omega,\phi(\omega,x)))\\ &= T^{-1}(\theta_t\omega, \phi(\theta_t\omega, u(t,\omega,x)))\\ &= T^{-1}(\theta_t\omega, \phi(\theta_t\omega, T(\theta_t\omega, \hat{u}(t,\omega,\hat{x})))) = \psi(\theta_t\omega, \hat{u}(t,\omega,\hat{x})).\end{aligned}$$

The smoothness of $\psi(\omega,\cdot)$ is a direct consequence of that for $\phi(\omega,\cdot)$ and the fact that composition with T or T^{-1} has no contribution to the regularity. □

8.3 Existence of Conjugacy

It then suffices to show the existence of a conjugacy for the random differential equation (8.4). Following the approach used in the last two sections, we need first extend a local center manifold. Let $\hat{M}^c(\omega) = \text{graph}(h(\omega,\cdot))$ be a local center manifold of equation (8.1) for some neighborhood $V \subseteq U$ of the origin, where $h(\omega,\cdot) : V \cap H^c \to H^{us}$ is a $C^{r,\alpha}$ function with $h(\omega, 0) = 0$, $Dh(\omega, 0) = 0$. By [DLS04] Theorem 3.3, $M^c(\omega) := T(\omega, \hat{M}^c(\omega))$ is invariant for equation (8.4), whose graph is given by $T(\omega, h(\omega, T^{-1}(\omega,\cdot))) = e^{-z(\omega)}h(\omega, e^{z(\omega)}\cdot)$. We still use the cut-off function Γ defined as in (3.8) and (3.9). For $\rho > 0$, recall the functions given in (7.4), for simplicity we denote by

$$\zeta(x_c) := \zeta_\rho(x_c) := \Gamma(\frac{|x_c|}{\rho}),\ \forall x_c \in H^c, \qquad \tilde{\zeta}(x) := \tilde{\zeta}_{\rho,\frac{\rho}{e^{z(\omega)}}}(x) := \Gamma(\frac{|x|}{\rho + \frac{2\rho}{e^{z(\omega)}}}),\ \forall x \in H,$$

where $z(\omega)$ is given in (8.3). And we consider for any integer $k > 0$ the following region:

$$Q_{k\rho(\omega)} := \{x \in H \mid |x_u|, |x_s| < k\frac{\rho}{e^{z(\omega)}}, |x_c| < k\rho\}.$$

Notice that by Lemma 8.1 (iii), this region is tempered from below. Following the same argument given in section 6, we want to use the cut-off technique to make up an extension of the original manifold, that is to define

$$
\begin{aligned}
h^c(\omega, x_c) &:= \zeta(x_c) e^{-z(\omega)} h(\omega, e^{z(\omega)} x_c), \\
W^c(\omega) &:= \text{graph}(h^c(\omega, \cdot)), \\
F(\omega, x) &:= \tilde{\zeta}(x) e^{-z(\omega)} f(\omega, e^{z(\omega)} x_{us}) + G(\omega, x_c),
\end{aligned}
$$

where $x \in H$ and $G(\omega, \cdot) : H^c \to H^{us}$ is to be determined by a certification of invariance for manifolds, which is analogous to (7.1) but is slightly adjusted due to Hypothesis 3.3.2 and the $z(\omega)$ process in the equation. We state it as follows:

$$
\begin{aligned}
&(A_{us} + z(\omega)) e^{-z(\omega)} h(\omega, e^{z(\omega)} x_c) + e^{-z(\omega)} f_{us}(h(\omega, e^{z(\omega)} x_c)) \\
= &D e^{-z(\omega)} h(\omega, e^{z(\omega)} x_c)[(A_c + z(\omega)) x_c + e^{-z(\omega)} f_c(h(\omega, e^{z(\omega)} x_c))] \\
&+ \partial_t e^{-z(\theta_t \omega)} h(\theta_t \omega, e^{-z(\omega)} x_c) \mid_{t=0} .
\end{aligned}
$$

Then we can conclude that $G(\omega, \cdot)$ is given by

$$
\begin{aligned}
G(\omega, x_c) = &D h^c(\omega, x_c)[(A_c + z(\omega)) x_c + \tilde{\zeta}(x_c + h^c(\omega, x_c)) e^{-z(\omega)} f_c(e^{z(\omega)} h^c(\omega, x_c))] \\
&- \tilde{\zeta}(x_c + h^c(\omega, x_c)) e^{-z(\omega)} f_{us}(e^{z(\omega)} h^c(\omega, x_c)) \\
&- \zeta(x_c) \{ D e^{-z(\omega)} h(\omega, e^{z(\omega)} x_c)[(A_c + z(\omega)) x_c + e^{-z(\omega)} f_c(h(\omega, e^{z(\omega)} x_c))] \\
&- e^{-z(\omega)} f_{us}(h(\omega, e^{z(\omega)} x_c)) \}
\end{aligned}
$$

for $x_c \in H^c$. We claim that when we choose ρ small enough, then all the desired invariant manifolds and foliations exist for equation

$$
u_t = Au + z(\theta_t \omega) u + \tilde{\zeta}(u) e^{-z(\omega)} f(\omega, e^{z(\omega)} u_{us}) + G(\omega, u_c),
$$

and the same conclusion as in Theorem 7.3 follows immediately. Indeed, the key step is still to make a proper bound for the Lipschitz constants as given in Lemma 3.1 and 7.2. But for instance, if $u,\ v \in Q_{2\rho(\omega)}$, we have

$$\begin{aligned}
&|\tilde{\zeta}(u)e^{-z(\omega)}f(e^{z(\omega)}u_{us}) - \tilde{\zeta}(v)e^{-z(\omega)}f(e^{z(\omega)}u_{us})| \\
\leq\, & e^{-z(\omega)}[|\Gamma(\tfrac{|u|}{\rho+\frac{2\rho}{e^{z(\omega)}}})f(e^{z(\omega)}u_{us}) - \Gamma(\tfrac{|u|}{\rho+\frac{2\rho}{e^{z(\omega)}}})f(e^{z(\omega)}v_{us})| \\
&+|\Gamma(\tfrac{|u|}{\rho+\frac{2\rho}{e^{z(\omega)}}})f(e^{z(\omega)}v_{us}) - \Gamma(\tfrac{|v|}{\rho+\frac{2\rho}{e^{z(\omega)}}})f(e^{z(\omega)}v_{us})|] \\
\leq\, & 3K_\Gamma \cdot \sup_{B_{8\rho}} \|Df(\cdot)\| \cdot |u-v|,
\end{aligned}$$

where $B_{8\rho} := \{u \in H^{us} \mid |u_{us}| \leq 8\rho\}$, and $\sup_{B_{8\rho}} \|Df(\cdot)\| = o(1)$ as $\rho \to 0$. The other estimates can be computed similarly as in Lemma 3.1 and 7.2, and we omit them here. Then following the geometric argument in chapter 7, we can construct a conjugacy between local center manifolds for equation (8.4) and thus for equation (8.1) as well by applying Proposition 8.2.1.

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