2019-12-01

An Equivalence of Shape and Deck Groups; Further Classification of Sharkovskii Groups

Tyler Willes Hills
Brigham Young University

An Equivalence of Shape and Deck Groups; Further Classification of Sharkovskii Groups

Tyler Willes Hills

A dissertation submitted to the faculty of
Brigham Young University
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

Department of Mathematics

Brigham Young University

ABSTRACT

An Equivalence of Shape and Deck Groups; Further Classification of Sharkovskii Groups

Tyler Willes Hills
Department of Mathematics, BYU
Doctor of Philosophy

In part one we show that for a compact, metric, locally path-connected topological space X, the shape group of X - as defined in Foundations of Shape Theory by Mardesic and Segal - is isomorphic to the inverse limit of discrete homotopy groups introduced by Conrad Plaut and Valera Berestovskii. We begin by providing the reader preliminary definitions of the fundamental group of a topological space, inverse systems and inverse limits, the Shape Category, discrete homotopy groups, and culminate by providing an isomorphism of the shape and deck groups for peano continua. In part two we develop work and provide further classification of Sharkovskii topological groups, which we call Sharkovskii Groups. We culminate in proving the fact that a locally compact Sharkovskii group must either be $\mathbb{R}$ if it is not compact, or a torsion-free solenoid if it is compact.

Keywords: fundamental group, discrete homotopy group, inverse system, inverse limit, shape category, shape group

Acknowledgements

Special thanks to my advisor, Greg Conner, and all the help he has provided. Thanks to Curt Kent and Wolfgang Herfort for their help.

Contents

Part I

An Equivalence of the Shape and Deck Groups

1.1 Homology Theory

One encounters homology and homotopy theories very early on in a course in algebraic topology. These theories play an important and vital role in the development, study, and usefulness of algebraic topology, and in order to understand the contents of this paper, we need to understand the purpose, goals, and shortcomings, of both homology theory and homotopy theory.

The origins of homology theory can be traced back to the mid-to-late 1800s with the work of Euler, Riemann, and Betti. The theory was developed in order to study and classify manifolds by identifying, in a certain sense, the various type of "holes" within the space - if two spaces have different types of "holes", then they are fundamentally different in a topological sense. For example, the 1-dimensional sphere S^1 is topologically a different manifold than an arc, since S^1 has a "hole" while an arc does not; the plane $\mathbb{R}^2$ is a different manifold than the 2-dimensional sphere S^2 since the latter has a "hole" while the former does not – and both spaces are different than the 2-dimensional torus $\mathbb{T}^2$ since the torus has two "holes".

Homology theory makes the idea of counting and classifying the holes within a space precise by considering equivalence classes of cycles of various dimension within the space, where two cycles of the same dimension are considered equivalent if one can be continuously deformed to the another. A cycle that can be continuously deformed to a point is called nulhomotopic and is considered trivial. For example, a cycle that goes around the circle S^1 is not nulhomotopic, while any cycle in an arc is. In both the sphere S^2 and the plane $\mathbb{R}^2$, any 1-dimensional loop is nulhomotopic; however, S^2 has a non-nulhomotopic 2-dimensional cycle - perhaps better thought of as a balloon - that goes around the "hole" of the sphere, while any 2-dimensional cycle in $\mathbb{R}^2$ is, as in the 1-dimensional case, nulhomotopic.

The non-nulhomotopic n-dimensional equivalence classes of cycles generate an algebraic group, called the n-th homology group of the space, which we associate to the space. The usefulness of these groups in classifying and distinguishing spaces is due to the fact that these groups are homotopy invariants. That is, if two spaces have the same homotopy type, then they have the same homology groups; put another way, if two spaces do not have the same homology groups, then they do not have the same homotopy type - in particular they are not homeomorphic. Thus, homology theory provides a tool which can be used to distinguish if two spaces are not homeomorphic. What's more, the homology groups of a space have the benefit of ease of computability, due in a large part to the famous theorem by Mayer and Vietoris and the use of exact sequences.

1.2 Homotopy Groups and Covering Space Theory

Similar to the homology groups of a space, we can associate other homotopy invariant algebraic groups, homotopy groups, to a pointed space - a pointed space is simply a space with a chosen basepoint. The first homotopy group, called the fundamental group or π_1, of a space was first introduced and defined by Poincare in 1894 as an alternative to homology theory. The fundamental group had several advantages over homology and was generalized to higher homotopy groups denoted by π_n for a natural number n. Intuitively, the n-th homotopy group of a space consists of equivalence classes of maps from the sphere S^n into the space, where as in the case of homology, two maps are considered equivalent if one can be continuously deformed into the other.

One of the uses of the fundamental group of a space is its integral role in the theory of covering spaces. For certain nice spaces there is a Galois correspondence between subgroups of the space's fundamental group and its covering spaces. Related to this correspondence is the fact that for this same class of spaces, a space's fundamental group is isomorphic to the group of deck transformations of the space's universal cover. As mentioned, the connections between the fundamental group and covering space theory hold for a certain class of nice spaces, and generalizing these connections to other, somewhat more pathological, spaces and a weaker notion of cover is what initially motivated Plaut and Berestovskii to introduce discrete homotopy groups in their paper [7, p.1750], a topic we will visit in a later section.

1.3 Shape Theory

Despite the incredible usefulness of homotopy groups, there are some insufficiencies, however. The Warsaw Circle, for example, does not have the homotopy type of a point, yet all of its homotopy groups are 0 [20, p.xiii-xiv]. Similarly, the spaces $S^2 \times \mathbb{RP}^3$ and $\mathbb{RP}^2 \times S^3$ have isomorphic homotopy groups but are not homotopy equivalent. This is because homotopy theory is suited best for spaces which behave nice locally. in 1968, hoping to develop tools and a theory which would be better suited for spaces which do not behave well locally, Karol Borsuk introduced shape theory.

The fundamental group considers maps of S^1 into the space, whereas Borsuk considered embedding a pathological space in a larger, more well-behaved space like an absolute neighborhood retract of a metric space such as the Hilbert cube. This enabled him to consider a decreasing sequence of neighborhoods $\{U_n\}$ in the larger space whose intersection, $\bigcap U_n$, was the embedded original space. He then considered a sequence $\{f_n\}$ of maps of S^1 into the decreasing sequence - $f_n : S^1 \to U_n$ [20, p.xiv-xv]. This approach led Borsuk to develop a theory that agreed with homotopy theory on ANRs but generalized to more pathological spaces.

A few years later, Mardesic and Segal took a different approach to shape theory, which is the approach we take in this paper. This approach relies on nerves of covers of the space in question, which provides the benefit of not requiring an embedding into a larger space. Essentially, this approach looks at the inverse limit of an inverse system of homotopy groups of a sequence of nerves of refining open covers whose mesh goes to 0.

1.4 Discrete Homotopy Groups

As was mentioned in the section on homotopy groups and covering space theory, for a certain class of nice spaces, there are intimate connections between the fundamental group of a space and the space's covers. This class of spaces includes those which are connected, locally path-connected, and semilocally simply-connected. In 2007 Plaut and Berestovskii introduced a weaker notion of covering space for spaces which they called *coverable* [7, p.1748]. These coverable spaces include geodesic metric spaces, connected and locally path-connected compact spaces, Peano continua, and more pathological spaces like the topologist's sine curve as well as totally disconnected spaces. Associated with each coverable space is an analogue of the classical universal cover called the uniform universal cover, to which is associated a group which Plaut and Berestovskii call the *deck group* [7, p.1750].

In classical covering space theory one considers the fundamental group of the space. In coverable space theory, rather than maps of S^1 into the space, one instead considers algebraic groups, called discrete homotopy groups, consisting of classes of discrete loops, i.e. a finite set of points in the space satisfying certain conditions involving what Plaut and Berestovskii call an entourage [7, p.1749]. One then consider an inverse system, called the fundamental inverse system [7, p.1750], of these groups, the inverse limit of which is the deck group of the space.

The main result of part 1 of this dissertation is the following, see Theorem 28.

Theorem 1. *Let (X, x_0) be a compact, connected locally path-connected, pointed, metric space, then the shape group of X is isomorphic to the deck group of X.*

Chapter 2. Inverse Systems, Inverse Limits, and a few Categories

2.1 Inverse Systems, Inverse Limits, and Category Pro

Much of the content in this section can be found in [20, p.3-7]. Let S be a set. A *preordering* on S is a reflexive and transitive binary relation $\leq$, so that for any $s, s', s'' \in S$, we have

1. $s \leq s$
2. $s \leq s'$ and $s' \leq s''$ imply $s \leq s''$

We call S a *preordered* set if it has a preordering. By a *directed* set we mean a preordered set with the property that if $s, s' \in S$ then there exists $s'' \in S$ such that $s \leq s''$ and $s' \leq s''$. An *ordering* on S is a preordering that is also antisymmetric; that is, $s \leq s'$ and $s' \leq s$ together imply $s = s'$. A *total* or *linear* ordering is an ordering for which, given any two elements $s, s' \in S$ then either $s \leq s'$ or $s' \leq s$. Note that if an ordering is total then it is directed.

Let C be a category. An *Inverse System* in C is a triple $\mathbb{X} = (X_i, p_{i,i'}, I)$ where I is a directed set, X_i is an object in C for each $i \in I$, and $p_{i,i'} : X_{i'} \to X_i$ is a morphism in C whenever $i \leq i'$ with the conditions that $p_{i,i} : X_i \to X_i$ is the identity on X_i and if $i \leq i' \leq i''$ then $p_{i,i''} = p_{i,i'} \circ p_{i',i''}$. We call the X_i's terms and the $p_{i,i'}$'s connecting morphisms. If I is the set of natural numbers with its standard total ordering, then we often call $\mathbb{X}$ an inverse sequence and write $\mathbb{X} = (X_n, p_{n,n+1})$ since all nonconsecutive morphisms are compositions of consecutive morphisms.

If I consists of only one element, then we call the system $\mathbb{X}$ *rudimentary* and often write it, with abusive notation, simply as X where X is the single term.

If $\mathbb{X} = (X_i, p_{i,i'}, I)$ and $\mathbb{Y} = (Y_j, q_{j,j'}, J)$ are inverse systems in C, then we define a morphism between the two systems $\Psi : \mathbb{X} \to \mathbb{Y}$ to consist of a set map $\rho : J \to I$ and maps

$(f_j : X_{\rho(j)} \to Y_j)$, one such map for every $j \in J$, with the condition that each pair $j \leq j' \in J$ admits an $i \in I$ such that $\rho(j), \rho(j') \leq i$ with

$$f_j \circ p_{\rho_j, i} = q_{j,j'} \circ f_{j'} \circ p_{\rho_{j'}, i}$$

which is equivalent to the commutativity of the following diagram. Note that in such diagrams we often omit the letters p, q and their indices, since we understand the corresponding morphisms to be the connecting morphisms in their respective systems.

$$\begin{array}{ccccc} X_{\rho(j)} & \longleftarrow & X_i & \longrightarrow & X_{\rho(j')} \\ \Big\downarrow{\scriptstyle f_j} & & & & \Big\downarrow{\scriptstyle f_{j'}} \\ Y_j & \longleftarrow & \longleftarrow & \longleftarrow & Y_{j'} \end{array}$$

If $\Psi : \mathbb{X} \to \mathbb{Y}$ is a morphism between inverse systems we often write it $(f_j, \rho) : \mathbb{X} \to \mathbb{Y}$.

An example of a morphism from a rudimentary system X to an inverse sequence can be understood by a commutative diagram of the form

$$\begin{array}{ccccccc} X & & & & & & \\ \Big\downarrow{\scriptstyle f_1} & \searrow & & \overset{f_3}{\searrow} & & & \\ Y_1 & \longleftarrow & Y_2 & \longleftarrow & Y_3 & & \dots \end{array}$$

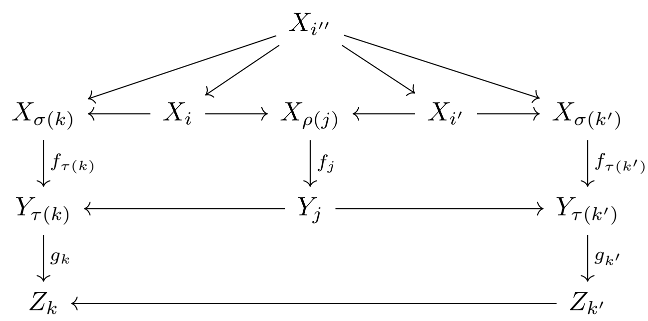
An example of a morphism from an inverse sequence to a rudimentary system consists of only one morphism f, since $\rho : J \to I$ has only one image.

$$\begin{array}{ccccccc} X_1 & \longleftarrow & X_2 & \longleftarrow & X_3 & & \dots \\ \Big\downarrow{\scriptstyle f} & & & & & & \\ Y & & & & & & \end{array}$$

Due to the prominent role they play in this area of mathematics, we include an example of a morphism between two inverse sequences.

$$\begin{array}{ccccccc} X_1 & \longleftarrow & X_2 & \longleftarrow & X_3 & & \dots \\ \Big\downarrow{\scriptstyle f_1} & & \Big\downarrow{\scriptstyle f_2} & & \Big\downarrow{\scriptstyle f_3} & & \\ Y_1 & \longleftarrow & Y_2 & \longleftarrow & Y_3 & & \dots \end{array}$$

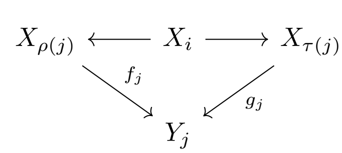
We can compose morphisms of inverse systems. Suppose $\mathbb{Z} = (Z_k, r_{k,k'}, K)$ is another system and $(g_k, \tau) : Y \to Z$ is a morphism from $\mathbb{Y}$ to $\mathbb{Z}$. We define $(h_k, \sigma)\mathbb{X} \to \mathbb{Z}$ where $\sigma = \rho \circ \tau$ and $h_k = g_k \circ f_j$. To see that this defines a morphism from $\mathbb{X}$ to $\mathbb{Z}$ consider the following diagram.



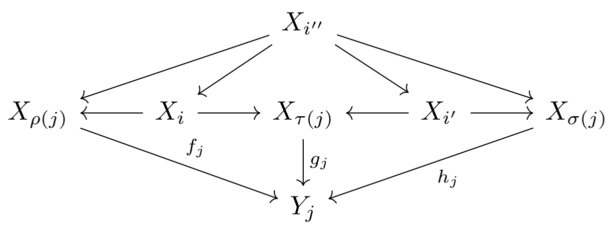
Here, $\tau(k), \tau(k') \leq j$, $\sigma(k), \rho(j) \leq i$, $\rho(j), \sigma(k') \leq i'$ where the existence of j, i, i' and the commutativity of the bottom three diagrams follow by the properties of morphisms of inverse systems. i'' is chosen by the fact that I is a directed set and the commutativity of the associated top portion of the diagram follows from the fact that $\mathbb{X}$ is an inverse system. Furthermore, function composition is clearly associative from the definition.

For an inverse system $\mathbb{X}$ we can define the identity morphism $\mathbb{I}_{\mathbb{X}} = (id_i, id_I) : \mathbb{X} \to \mathbb{X}$ where $id_I : I \to I$ is the identity function on I, and $id_i : X_i \to X_i$ for each $i \in I$. We note that $(id_i, id_I) \circ (f_j, \rho) = (f_j, \rho) \circ (id_i, id_I) = (f_j, \rho)$, and thus can define a category *inv-C* with objects inverse systems in C and morphisms the morphisms between inverse systems just defined.

If (f_j, ρ) and (g_j, τ) are two morphisms from $\mathbb{X} \to \mathbb{Y}$, we say they are equivalent, $(f_j, \rho) \sim (g_j, \tau)$, if the following holds. For any $j \in J$, there is an $i \in I$, with $\rho(j), \tau(j) \leq i$ for which the following diagram is commutative.



To see this is an equivalence relation, we trivially note that $(f_j, \rho) \sim (f_j, \rho)$, and if $(f_j, \rho) \sim (g_j, \tau)$ then $(g_j, \tau) \sim (f_j, \rho)$. If $(f_j, \rho) \sim (g_j, \tau)$ and $(g_j, \tau) \sim (h_j, \sigma)$ then the following commutative diagram suffices to show $(f_j, \rho) \sim (h_j, \sigma)$.



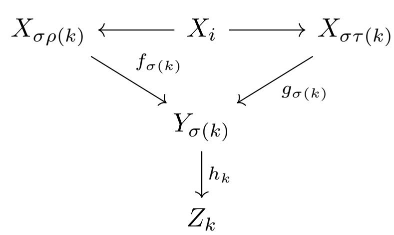
We also provide the following useful facts.

1. $(f_j, \rho) \sim (g_j, \tau)$ implies $(h_k, \sigma) \circ (f_j, \rho) \sim (h_k, \sigma) \circ (g_j, \tau)$

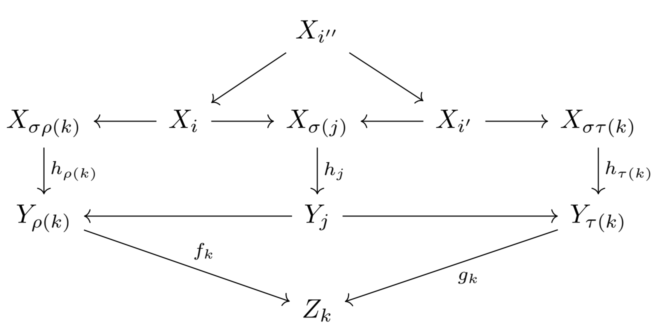
2. $(f_j, \rho) \sim (g_j, \tau)$ implies $(f_j, \rho) \circ (h_k, \sigma) \sim (g_j, \tau) \circ (h_k, \sigma)$

3. $(g_j, \tau) \circ (f_j, \rho) \sim (g'_j, \tau') \circ (f'_j, \rho')$

1) and 2) will imply 3), so we prove the first two facts. Observe that the commutativity of



implies 1), and the commutativity of



implies 2), from which 3) follows.

The discussion above provides a well-defined category *pro-C* with objects inverse systems in C and morphisms equivalence classes, with respect to the above equivalence relation, of morphisms (f_j, ρ) between inverse systems. We denote these classes by $[f_j, \rho]$. The identity morphism class is the class containing (id_i, id_I) and the composition of classes is defined by the formula $[f_j, \rho] \circ [g_k, \tau] = [f_j \circ g_k, \rho \circ \tau]$.

Definition 2. *If $\mathbb{X} = (X_i, p_{i,i'}, I)$ is an inverse system in the category C, then we understand an inverse limit of $\mathbb{X}$ to be an object X in C, together with a morphism $p : X \to \mathbb{X}$ often called the projection morphism, in pro-C with the universal mapping property.*

If $f : Y \to \mathbb{X}$ is a morphism in pro-C, there is a unique morphism $h : Y \to X$ so that $p \circ h = f$, or in other words, the following diagram is commutative in pro-C.

$$\begin{array}{ccc} & & X \\ & \overset{h}{\nearrow} & \downarrow p \\ Y & \xrightarrow{f} & \mathbb{X} \end{array}$$

Inverse limits are unique – if $q : Z \to \mathbb{X}$ is also an inverse limit of $\mathbb{X}$, then there are unique morphisms $i : X \to Z$ and $j : Z \to X$ in C so that $q \circ i = p$ and $p \circ j = q$. Then, $qij = q$ and $pji = p$, thus by uniqueness, we have $ij = id_Z$ and $ji = id_X$, so that i and j are isomorphisms in C. [20, p.54]

In the following categories every inverse system has an inverse limit: Set, Ab, Grp, Top, Cpt (compact Hausdorff spaces) [20, p.55].

2.2 The Category Shape

For additional information on this subject, one can consult [20, p.25-26]. Suppose C is a category and D a subcategory of C. Let X be an object in C, by a (C, D)-expansion, we mean a morphism $p : X \to \mathbb{X} = (X_i, p_{i,i'}, I)$ in pro-C, from X to an inverse system $\mathbb{X}$ in C, with the universal property:

If $\mathbb{Y} = (Y_j, q_{j,j'}, J)$ is an inverse system in D and $g : X \to \mathbb{Y}$ is a morphism in pro-C, then there is a unique morphism $f : \mathbb{X} \to \mathbb{Y}$ in pro-C satisfying $g = f \circ p$. In other words, we have the following commutative diagram.

$$\begin{array}{ccc} \mathbb{X} & \xleftarrow{p} & X \\ & \searrow^{f} & \big\downarrow g \\ & & \mathbb{Y} \end{array}$$

We note that if X has two (C, D)-expansions, $p : X \to \mathbb{X}$ and $p' : X \to \mathbb{X}'$. By the universal mapping property, there exist unique morphisms i and i' satisfying $ip = p'$ and $i'p' = p$. Hence, $i'ip = p$, and by uniqueness, we have $i'i = id_{\mathbb{X}}$ and $ii' = id_{\mathbb{X}'}$. Thus, $\mathbb{X}$ and $\mathbb{X}'$ are isomorphic. We call i the natural isomorphism.

A subcategory D of the category C is called dense in C if every object in C admits a (C, D)-expansion.

Suppose X and Y have (C, D)-expansions $p : X \to \mathbb{X}$, $p' : X \to \mathbb{X}'$, $q : Y \to \mathbb{Y}$ and $q' : Y \to \mathbb{Y}'$. We define morphisms $\Psi : \mathbb{X} \to \mathbb{Y}$ and $\Psi' : \mathbb{X}' \to \mathbb{Y}'$ to be equivalent if the following diagram commutes in pro-D.

$$\begin{array}{ccc} \mathbb{X} & \xrightarrow{i} & \mathbb{X}' \\ \big\downarrow \Psi & & \big\downarrow \Psi' \\ \mathbb{Y} & \xrightarrow{j} & \mathbb{Y}' \end{array}$$

With i and j being the natural isomorphisms as discussed above. It is easy to see that this defines an equivalence relation $\sim$, and so we can define the shape category for the pair (C, D), written $Sh(C, D)$. The objects of $Sh(C, D)$ are the objects of C and the morphisms are equivalences classes of morphisms in pro-C with respect to the equivalence relation just defined. Hence, a morphism in $Sh(C, D)$, $G : X \to Y$ is represented by a diagram

$$\begin{array}{ccc} \mathbb{X} & \xleftarrow[p]{} & X \\ \big\downarrow f & & \\ \mathbb{Y} & \xleftarrow[q]{} & Y \end{array}$$

Chapter 3. The Fundamental Group, Discrete Homotopy Groups, and an Equivalence of Their Inverse Limits

3.1 The Fundamental Group

The first homotopy group, the fundamental group, of a space has played a very important role in the development and application of algebraic topology. In addition to its crucial role in the development of covering space theory and the consequent galois correspondence between the covers of a space and the subgroups of the space's fundamental group, many famous results in mathematics can be easily proven using the fundamental group. These results include the fundamental theorem of Algebra [14, p.31], Brouwer's fixed point theorem for the unit disk [14, p.31], and the remarkable result that states that for any map from the unit sphere S^2 to $\mathbb{R}^2$ there exists a pair of antipodal points in S^2 which map to the same point in $\mathbb{R}^2$ [14, p.32]. This last result is both remarkable and surprising, because this means that at any given time, there exists a pair of points on the earth, directly opposite to one another, with both the same barometric pressure and temperature, for example. We also obtain rigorous proofs of somewhat more intuitive results such as surfaces with differing genus are not homeomorphic or even homotopy equivalent [14, p.51], and $\mathbb{R}^m$ is homeomorphic to $\mathbb{R}^n$ if and only if $m = n$. Furthermore, the fundamental group enables us to think of any group in a more concrete, geometric fashion; an elegant construction shows that for a given abstract group G, there exists a two-dimensional cell complex with G as its fundamental group [14, p.52].

In the introduction we described the fundamental group as essentially mapping circles into a space X. We take a slightly different but equivalent approach here by mapping in the unit interval $[0, 1]$. In this discussion, X is a topological space and all maps are assumed to be continuous. For more information on this topic, a great source is [14, p.25-28]

Definition 3. *A map $f : [0, 1] \to X$ is called a path in X. The path $\bar{f} : [0, 1] \to X$ defined by $\bar{f}(x) = f(1 - x)$ is said to be the reverse of f. If $z = f(0) = f(1)$ then the map f is said* *to be a loop based at z. A path f is said to be reduced if there does not exist a pair $a < b$ in $[0,1]$ such that $f|_{[a,b]}$ is homotopic to a constant path.*

If f is a path in X, then the path $\bar{f}$ has the same image as f but traverses the image in the reverse orientation.

Definition 4. *If f and g are two paths in a space X with basepoint z, then a homotopy between f and g, relative to the endpoints (or a relative homotopy) is a map $H : [0,1] \times [0,1] \to X$ such that $H|_{[0,1]\times\{0\}} = f$, $H|_{[0,1]\times\{1\}} = g$, $H|_{\{0\}\times[0,1]} = f(0) = g(0)$, and $H|_{\{1\}\times[0,1]} = f(1) = g(1)$. If there exists a relative homotopy between f and g, we say f and g are homotopic.*

Define the relation $f \sim g$ if there exists a relative homotopy between f and g. This relation is readily seen to be an equivalence relation, and we denote the equivalence class of f by $[f]$ and call $[f]$ the homotopy class of f.

If f and g are two paths in X with $f(1) = g(0)$, then we can concatenate these paths to form a new path $f * g : [0,1] \to X$ defined by $f * g(x) = f(2x)$ if $x \in [0, \frac{1}{2}]$ and $f * g(x) = g(2x-1)$ if $x \in [\frac{1}{2}, 1]$. It is well known that this induces a well-defined operation on the set of equivalence classes of paths in X.

If we pick a basepoint $z \in X$ and consider the set of loops starting and ending at z, we can concatenate any two loops f and g. That is, we have a well-defined operation $[f]\cdot[g] = [f*g]$. If we define the constant loop $e : [0,1] \to X$ defined by $e(x) = z$ for all $x \in [0,1]$ to be a group identity then the inverse of $[f]$ is $[f]^{-1} = [\bar{f}]$

Definition 5. *The fundamental group of a topological space X, based at x_0 and written $\pi_1(X, x_0)$, is the group whose elements are homotopy classes of loops based at x_0 with group operation $\cdot$ as defined above.*

Recall that if X and Y are spaces with map $f : X \to Y$, then the map $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is the induced homomorphism defined by $f_*[\alpha] = [f \circ \alpha]$ for $[\alpha] \in \pi_1(X, x_0)$, and also recall that for a cover $O = \{O_i\}$ of a space X, there is an associated simplicial complex,

$N(O)$, called the nerve of O. For every $O_i \in O$ there is an associated vertex, and for any n-wise intersection of open sets $O_{i_1}, \ldots, O_{i_n}$ there is an associated $n-1$-dimensional simplex with vertices the corresponding n vertices.

It is well-known that in many cases for a space X, one can take a sequence of refining open covers of X whose mesh goes to zero, so that X can be expressed as the inverse limit of the nerves of the covers. This is, in particular, true for compact metric spaces. We shall prove this as a corollary to a more general result, but in order to do this we need a few definitions and results from [20].

Recall that a if Y is a subspace of X, then Y is a retract of X if there exists a map $r : X \to Y$ such that $r \circ i : Y \to Y$ is the identity on Y, which is equivalent to $r|_Y = id_Y$. If Y is a retract of an open set U of X containing Y, then Y is said to be a neighborhood retract of X.

A class C spaces is called *weakly hereditary* if it satisfies the two conditions

1. If $X \in C$ and Y is a closed subset of X, then $Y \in C$.

2. If X and Y are homeomorphic and $X \in C$, then $Y \in C$.

Definition 6. *A space X is called an absolute (neighborhood) retract of a weakly hereditary class C, if X satisfies*

1. *$X \in C$ and*

2. *if X is homeomorphic to Y, Y is a closed subspace of Y', and $Y' \in C$, then Y is a (neighborhood) retract of Y'.*

The following is a generalization of the notion of ϵ-near for metric spaces.

Definition 7. *Suppose $f, g : X \to Y$ are maps from the space X to the space Y, and let O be an open cover of Y. f and g are O-near if given $x \in X$ there exists $U \in O$ so that $f(x)$ and $g(x)$ are both in U.*

The following definition is closely related to inverse limits, and the two coincide when working in the category pro-Cpt.

Definition 8. *Let X be a space. By a resolution of X we mean an inverse system $\mathbb{X} = (X_i, p_{i,i'}, I)$ and a morphism $p : X \to \mathbb{X}$ in pro-Top satisfying the following two conditions.*

1. *If Y is an ANR, O is an open cover of Y, and $f : X \to Y$ is a map. Then, there exists $i \in I$ and a map $f_i : X_i \to Y$ such that f and $f_i \circ p_i$ are O-near.*

2. *If Y is an ANR, O is an open cover of Y. Then there is an open cover O' of Y so that if $i \in I$ and $f, f' : X_i \to Y$ are maps so that $f \circ p_i$ and $f' \circ p_i$ are O'-near, then there is an $i' \in I$, $i' \geq i$ so that $f \circ p_{ii'}$ and $f' \circ p_{ii'}$ are O-near. If each X_i is an (polyhedron) ANR, we say that p is an (polyhedral resolution) ANR-resolution of X.*

The following theorem [20, p.74] shows that in the compact case, resolutions and inverse limits coincide.

Theorem 9. *If $p : X \to \mathbb{X}$ is a resolution in pro-Cpt. Then p is a resolution of X if and only if p is an inverse limit of X.*

Another useful theorem is the following [20, p.79].

Theorem 10. *Let $p : X \to \mathbb{X}$ be a morphism in pro-Top. If p has the following two properties*

1. *If $i \in I$ and U is an open set in X_i containing the closure of $p_i(X)$, then there exists $i' \geq i$ so that $p_{ii'}(X_{i'}) \subseteq U$.*

2. *If O is a normal open cover of X, then there exists $i \in I$ and a normal open cover O' of X_i so that $(p_i)^{-1}(O')$ is a refinement of O.*

then p is a resolution of X.

Before proving Theorem 12 we need the following definition.

Definition 11. *Let X be a space, and C an open cover of X with nerve $N(C)$. If $\{(\psi_U, U)\}$ is a partition of unity subordinate to C, then there is a canonical map $p : X \to N(C)$ defined as follows. If $U_1, \ldots, U_n$ are all the open sets containing the point $x \in X$, then $p(x) = (\psi_{U_1}(x), \ldots, \psi_{U_n}(x))$ given in barycentric coordinates with respect to the vertices corresponding to the open sets $U_1, \ldots, U_n$. The star of a vertex v is the union of the interiors of all the simplices containing v.*

We importantly note that if C is an open cover of the space X, with corresponding nerve $N(C)$ and canonical map $p : X \to N(C)$. Then the collection of all stars U_i of the nerve $N(C)$ forms an open cover of $N(C)$, and the collection $C' = \{p^{-1}(U_i) \mid U_i \in O$ is a cover of X that refines C.

We can now prove the following.

Theorem 12. *Every topological space has a polyhedral resolution.*

Proof. Denote by B the set of all open covers of the space X. Since X is compact, we can assume all covers $C \in B$ are finite, and if $C \in B$ let $\Psi_C = (\psi_U, U \in C)$ be a partition of unity subordinate to C. Denote by X_C the nerve of the cover C with canonical map $p_C : X \to X_C$ as in Definition 11.

Now, denote by Γ the set of finite subsets of B partially ordered with inclusion. If $\gamma = \{C_1, \ldots, C_m\} \in \Gamma$ we can take the open cover $C_1 * \ldots * C_m = \{U_1 \cap \ldots \cap U_m : U_i \in C_i, 1 \leq i \leq m\}$ with corresponding nerve X_γ. If $\gamma \leq \gamma\prime = \{C_1, \ldots, C_m, \ldots C_k\}$, then we define the simplicial map $q_{\gamma\gamma\prime} : X_{\gamma\prime} \to X_\gamma$ determined by sending the vertex associated with $U_1 \cap \ldots \cap U_m \cap \ldots \cap U_k$ in $X_{\gamma\prime}$ to the vertex associated with $U_1, \cap \ldots \cap U_m$ in X_γ. It is clear that, if $\gamma \leq \gamma\prime \leq \gamma\prime\prime$, then we have

$$q_{\gamma\gamma\prime} q_{\gamma\prime\gamma\prime\prime} = q_{\gamma\gamma\prime\prime}.$$

We further define maps $p_\gamma : X \to X_\gamma$ where $\gamma = \{C_1, \ldots, C_m\}$, by the following partition of unity $(\psi_{(U_1,\ldots,U_m)}, (U_1 \cap \ldots \cap U_m))$ with

$$\psi_{(U_1,\ldots,U_m)} = \psi_{U_1} \times \ldots \psi_{U_m}.$$

Observe that

$$q_{\gamma\gamma'}q_{\gamma'}(x) = \Sigma_{(U_1,\ldots,U_m,\ldots U_k)}\psi_{U_1}(x)\ldots\psi_{U_m}(x)\ldots\psi_{U_k}(x)$$

$$= \Sigma_{(U_1,\ldots,U_m)}\psi_{U_1}(x)\ldots\psi_{U_m}(x) = q_\gamma(x)$$

Thus, for $\gamma \leq \gamma'$ we have

$$q_{\gamma\gamma'}q_{\gamma'} = q_\gamma,$$

and hence $q = (q_\gamma)$ defines a morphism in pro-Top from X to $\mathbb{X}$.

We now define a system $\mathbb{Z} = (Z_j, r_{jj'}, J)$. For fixed $\alpha \in I$ choose a basis O_α of $Cl(q_\alpha(X))$ in X_α. Let J be all pairs of $j = (\alpha, B)$ with $\alpha \in I$ and $B \in O_\alpha$. Let $Z_j = B$ and $r_j : X \to Z_j$ be q_α. We define an ordering on J by $j \leq j' = (\alpha', B')$ if both $\alpha \leq \alpha'$ and $q_{\alpha\alpha'}(B') \subseteq B$, and $r_{jj'} : Z_{j'} \to Z_j$ to equal $q_{\alpha\alpha'}|_{B'}$. Therefore $r = (r_j) : X \to \mathbb{Z} = (Z_j, r_{jj'}, J)$ is a morphism in the category pro-Top.

Consider an arbitrary $j = (\alpha, B)$ and an open set V of the closure of $r_j(X) = q_\alpha(X)$ in $Z_j = B$. Because O_α constitutes a basis for the neighborhoods containing $Cl(q_\alpha(X))$ in X_α, there exists a $B' \in O_\alpha$ satisfying $B' \subseteq V$. Then, we have $j' = (\alpha, B') \in J$ and $j \leq j'$, so that $r_{jj'}(Z_{j'}) = q_{\alpha\alpha'}(B') = B' \subseteq V$. Furthermore, since open subsets of polyhedra are polyhedra, we can always choose O_α to consist of polyhedra. This completes the proof.

□

Corollary 13. *If X is a compact space, then there exists an inverse sequence of simplicial complexes and simplicial maps $\mathbb{X} = (X_n, q_{nn+1})$ and maps $q_n : X \to X_n$ for which $(X, (q_n)) = \varprojlim(X_n, q_{nn+1})$.*

Definition 14. *Let X be a space with basepoint x_0, and $\{O_n\}$ be a sequence of open covers of X such that $(X, p_n) = \varprojlim N(O_n)$ where $N(O_n)$ is the nerve of the open cover O_n. In $N(O_n)$ let $p_n(x_0)$ be the basepoint. The shape group of X is the group $Sh(X, x_0) = \varprojlim(\pi_1(N(O_n), p_n(x_0)), p_{nn+1*})$ where π_1 denotes the fundamental group and p_{*nn+1} is the induced homomorphism on fundamental groups.*

In Foundations of Shape Theory, Mardesic and Segal showed that the shape group of a space X is both well-defined and a shape invariant, that is, isomorphic objects in the category of shape have isomorphic shape groups.

3.2 Discrete Homotopy Groups

The contents of this section can be reviewed in greater generality in [7, p.1749]. This section deals specifically with metric spaces.

Let (X, d) be a metric space and fix $\varepsilon > 0$. An ε-chain is a finite, ordered, sequence of points $(x_1, x_2, \ldots, x_n)$ in X such that $d(x_i, x_{i+1}) < \varepsilon$ for each $i = 1, 2, \ldots, n-1$. The point x_1 is called the beginning (or starting) point while x_n is called the endpoint. A *basic move* is the addition or removal of a single point to an ε-chain such that the resulting set of points is again an ε-chain. Two ε-chains are said to be equivalent if one can be obtained from the other by a finite sequence of basic moves. In [7] Plaut and Berestovskii showed that is an equivalence relation among ε-chains, and the primary objects of interest are the equivalence classes of chains. We denote the equivalence class of the chain C by $[C]$.

If $C = (x_1, x_2, \ldots, x_n)$ and $D = (y_1, x_2, \ldots, y_m)$ are two ε-chains such that $d(x_n, y_1) < \varepsilon$, then we can concatenate C and D to make a new ε-chain of points $C * D = (x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m)$. It is a fact that the operation of concatenating two chains is well-defined on equivalence classes of chains. For a chain $C = (x_1, x_2, \ldots, x_n)$ and $D = (y_1, x_2, \ldots, y_m)$ we denote its equivalence class by $[C] = [x_1, x_2, \ldots, x_n)$ and $D = (y_1, x_2, \ldots, y_m]$. We observe that the operation $[C] \cdot [D] = [C * D]$ is well-defined.

We shall focus our attention on the special case where a basepoint $x_0 \in X$ is chosen and all ε-chains have beginning and endpoints equal to x_0; an ε-chain starting and ending at x_0 is called an ε-loop based at x_0. All ε-loops based at x_0 can be concatenated with each other, and thus so can all equivalence classes. It is an important fact that the set of equivalence classes of ε-loops based at x_0 forms a group under operation $\cdot$. A canonical representative of the identity element of the group is the ε-loop $e = (x_1, x_2)$, where $x_1 = x_2 = x_0$; thus we can denote the identity, or trivial, element of the group by $[e]$. Note that ε-loops always have at least 2 points, a beginning and an end point. The inverse of $[x_0, x_1, \ldots, x_n]$ is the class $[x_n, x_{n-1}, \ldots, x_1]$. We shall denote this group by $\delta_\varepsilon(X, x_0)$ and call it the discrete ε-homotopy group. We further note that each homotopy class has a representative $\{x_1, x_2, \ldots, x_n\}$ with the property that $d(x_i, x_{i+2}) \geq \varepsilon$, which representative we shall call a reduced ε-loop.

While homology and homotopy groups detect "holes" within a space, ε-homotopy groups detect holes that are, in a certain sense, "larger" than ε.

If $\varepsilon_2 < \varepsilon_1$, then any ε_2-chain is also an ε_1-chain, which induces a homomorphism $\phi_{2,1} : \delta_{\varepsilon_2}(X, x_0) \to \delta_{\varepsilon_1}(X, x_0)$. If $\{\varepsilon_n\}$ is a decreasing sequence whose limit is 0, then we obtain an inverse sequence $(\delta_{\varepsilon_n}(X, x_0), \phi_{n+1,n})$ where $\phi_{n+1,n} : \delta_{\varepsilon_{n+1}} \to \delta_{\varepsilon_n}$.

Definition 15. *If $\{\varepsilon_n\}$ is a decreasing sequence of positive numbers whose limit is 0, then the deck group of a metric space X is the group $\Delta_1(X, x_o) = \varprojlim(\delta_{\varepsilon_n}(X, x_0), q_{n,n+1})$.*

Before proving that $\Delta_1(X, x_0)$ is well-defined, we prove the following proposition.

Proposition 16. *Let $(X, p_n) = \varprojlim(X_n, p_{n,n+1})$ and $(Y_m, q_m) = \varprojlim(Y_m, q_{m,m+1})$ be inverse limits of inverse sequences in a category C. Suppose there exist subsequences $\{X_{n_i}\}$ and $\{Y_{m_j}\}$ such that $n_1 \leq m_1 \leq n_2 \leq m_2 \ldots \leq n_i \leq m_i \ldots$ and morphisms $f_k : X_{n_{k+1}} \to Y_{m_k}$, $g_k : Y_{m_k} \to X_{n_k}$, one for each natural number k, so that $p_{n_k,n_{k+1}} = g_k \circ f_k$ and $q_{m_{k-1},m_k} = f_{n_{k-1}} \circ g_k$. Then, X is isomorphic to Y.*

Proof. By assumption, the following diagrams commute.

$$\begin{array}{ccc} X_{n_{k+1}} & \xrightarrow{f_k} & Y_{m_k} \\ \downarrow & \swarrow_{g_k} & \\ X_{n_k} & & \end{array}$$

and

$$\begin{array}{ccc} & & Y_{m_k} \\ & \swarrow^{g_k} & \downarrow \\ X_{n_k} & \xrightarrow{f_{k-1}} & Y_{m_{k-1}} \end{array}$$

By composing the projection maps $p_{n_{k+1}}$ with the maps f_k we obtain a morphism $X \to (Y_m, q_{m,m+1})$, and hence a morphism $F : X \to Y$ by the universal mapping property of Y. Similarly, by composing the maps q_{m_k} with the maps g_k, we obtain a morphism $Y \to (X_n, p_{n,n+1})$, and hence a map $G : Y \to X$. However, by the commutativity of the above diagrams, we obtain $G \circ F = id_X$ and $F \circ G = id_Y$, so that F and G are isomorphisms. □

We can now prove the following.

Proposition 17. *For a pointed metric space (X, x_0), the deck group $\Delta_1(X, x_0)$ is independent of the choice of sequence $\{\varepsilon_n\}$.*

Proof. Let $\{\varepsilon_n\}$ and $\{\varepsilon'_n\}$ be decreasing sequences of positive numbers whose limit is 0. Consider $\delta_1(X, (p_n)) = \varprojlim(\delta_{\varepsilon_n}(X, x_0), p_{n,n+1})$ and $\delta'_1(X', p'_n) = \varprojlim(\delta_{\varepsilon'_n}(X, x_0), p'_{n,n+1})$. It's clear we can choose subsequences $\{\varepsilon_{n_k}\}$ and $\{\varepsilon'_{n_l}\}$ satisfying $\varepsilon_{n_1} \geq \varepsilon'_{n_1} \geq \varepsilon_{n_2} \geq \varepsilon'_{n_2} \ldots \varepsilon_{n_i} \geq \varepsilon'_{n_i} \ldots$ and define maps $g_{k'} : \delta_{\varepsilon'_{n_k}}(X, x_0) \to \delta_{\varepsilon_{n_k}}(X, x_0)$ by sending a representative loop, considered as an ε'_{n_k}-loop, to itself, considered as an ε_{n_k}-loop. Define $f_k : \delta_{\varepsilon_{n_{k+1}}}(X, x_0) \to \delta_{\varepsilon'_{n_k}}(X, x_0)$ in the same manner.

It's clear that the maps f_k and g_k satisfy the hypotheses of the proposition 16, since they are simply inclusions on ε-chains. Thus, the result follows. □

3.3 An Equivalence of the Shape Group and Deck Group

We begin this section with some definitions, lemmas, and propositions. Throughout this section let $I = [0, 1]$.

Definition 18. *Let (X, x_0) be a pointed space. We define (Ω_X, x_0) to be the set of all continuous loops in X based at x_0, that is, $(\Omega_X, x_0) = \{f : I \to X \mid f(0) = f(1) = x_0\}$*

Suppose $\varepsilon > 0$ and $f : I \to X$ is a loop based at x_0. By uniform continuity, there exists $\beta > 0$ so that whenever $|x-y| < \beta$, we have $|f(x)-f(y)| < \varepsilon$. Thus, if $0 = t_0, \ldots, t_n = 1$ is a partition of I with mesh less than β, $\big(f(t_0), \ldots, f(t_n)\big)$ is an ε-chain. We can thus associate to f the ε-chain $\tilde{h}_\varepsilon(f) = \big(f(t_0), \ldots, f(t_n)\big)$. The following lemma shows this association gives a well-defined map $\tilde{h}_\varepsilon : (\Omega_X, x_0) \to \delta_\varepsilon(X, x_0)$.

Lemma 19. *The map $\tilde{h}_\varepsilon : (\Omega_X, x_0) \to \delta_\varepsilon(X, x_0)$ defined by $\tilde{h}_\varepsilon(f) = \big[f(t_0), \ldots, f(t_n)\big]$ where $T = t_0, \ldots, t_n$ is a partition of I with mesh less than β, a uniform continuity constant corresponding to ε, is well defined.*

Proof. Let $\varepsilon > 0$ and $f : I \to X$ be a loop based at x_0. By uniform continuity, there exists $\beta > 0$ so that for a partition $T = t_0 = 0 < t_1 < \ldots < t_n = 1$ with $|t_{i+1} - t_i| < \beta$ for all $0 \leq i \leq n-1$, we have $|f(t_{i+1}) - f(t_i)| < \varepsilon$, and thus $\big(f(t_0), \ldots, f(t_n)\big)$ is an ε-loop based at x_0. We must show the choice of β, and the partition T of I does not matter. Suppose $S = s_0 = 0 < s_1 < \ldots < s_m = 1$ also satisfies $|s_{m+1} - s_m| < \beta$. We can union the sets $\{t_i\}$ and $\{s_j\}$ and order the resulting set $r_0 = 0 < r_1 < \ldots < r_k = 1$. Now, since $\operatorname{diam}\big(f([t_i, t_{i+1}])\big) < \varepsilon$, we have $\big[f(0), f(t_1), \ldots, f(t_{n-1}), f(1)\big] = \big[f(0), f(t_1), \ldots, f(t_i), f(s), f(t_{i+1}), \ldots, f(t_{n-1}), f(1)\big]$ for any $s \in [t_i, t_{i+1}]$. Repeating this argument for every point of S, we obtain $\big[f(0), f(t_1), \ldots, f(t_n)\big] = \big[f(0), f(r_1), \ldots, f(r_k)\big]$; similarly we have $\big[f(0), f(s_1), \ldots, f(s_m)\big] = \big[f(0), f(r_1), \ldots, f(r_k)\big]$. Since this is an equivalence relation, we have $\big[f(0), f(t_1), \ldots, f(t_n)\big] = \big[f(0), f(s_1), \ldots, f(s_m)\big] = \big[f(0), f(r_1), \ldots, f(r_k)\big]$. Note, this same argument also shows that the choice of β is independent. □

By a triangulation of $I \times I$, we mean a triangulation consisting of straight-edge, convex triangles in the plane.

Definition 20. *Let D_0 be a triangulation of $I \times I$. The* top chain *of D_0 is the set of vertices of D_0 contained in $I \times \{1\}$ with their natural ordering. Similarly, the* bottom chain *of D_0 is the set of vertices of D_0 contained in $I \times \{0\}$ with their natural ordering.*

Definition 21. *Let (X, x_0) be a pointed metric space, $\varepsilon > 0$, C and D be ε-loops based at x_0. An ε-homotopy between C and D consists of a triangulation D_0 of $I \times I$, and a map $H : D_0^{(0)} \to X$ such that*

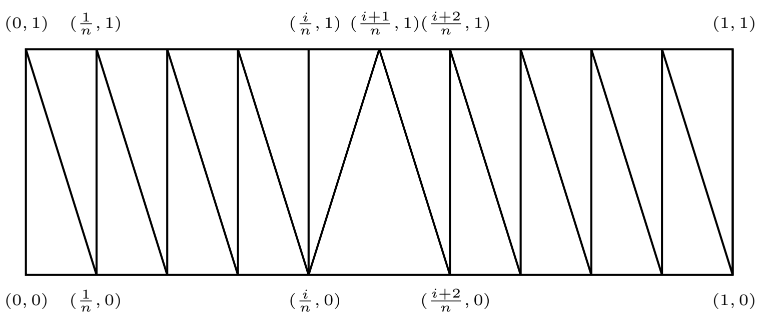
1. *$H(v) = x_0$ for all $v \in \big(\{0,1\} \times [0,1]\big) \cap D_0^{(0)}$,*
2. *$C = \big(H(v_1), \dots, H(v_n)\big)$ and $D = \big(H(w_1), \dots, H(w_m)\big)$ where $(v_1, \dots, v_n)$, $(w_1, \dots, w_m)$ are the top and bottom chains of D_0 respectively, and*
3. *$d\big(H(s), H(t)\big) < \epsilon$ for all adjacent vertices s, t of D_0.*

If D represents the trivial element in $\delta_\varepsilon(X, x_0)$, then we call the ε-homotopy an ε-nulhomotopy.

The next lemma relates basic moves in $\delta_\varepsilon(X, x_0)$ to ε-homotopies.

Lemma 22. *Let (X, x_0) be a pointed spaces and $\varepsilon > 0$. Suppose C and D are two ε-chains based at x_0. Then $[C] = [D]$ if and only if there exists an ε-homotopy between C and D.*

Proof. We can suppose C and D are reduced. First suppose that $[C] = [D]$. We show there exists an ε-homotopy between C and D in the case where C and D differ by a basic move and note that this is sufficient. To this end, suppose $C = (x_0, x_1, \dots, x_k, \dots, x_{n-1}, x_n)$ and $D = (y_0, y_1, \dots, y_{n-2}, y_{n-1})$ where $x_i = y_i$ for all $0 \le i < k$ and $x_i = y_{i-1}$ for all $k < i \le n$. Consider the set of vertices $\{(\frac{s}{n}, 1), (\frac{t}{n}, 0) \mid s \le n; t \le n, t \ne k\}$, and define $H(\frac{j}{n}, 1) = x_j$, $H(\frac{j}{n}) = y_j$ if $j \le k$ and $H(\frac{j}{n}) = y_{j-1}$ for $j \ge k+2$. The following triangulation of D_0, along with H, is the desired ε-homotopy between C and D.



Conversely, suppose D_0 and $H : D_0^{(0)} \to X$ is an ε-homotopy between C and D. We prove the following:

Claim. If $f : I \to D_0^{(1)}$ is any simple edge path from a vertex in $\{0\} \times I$ to a vertex in $\{1\} \times I$, and $(v_1, \dots, v_n)$ are the vertices, with the induced ordering, contained in the image of f, then the ε-chain $E = \big(H(v_1), \dots, H(v_n)\big)$ is equivalent to D.

Proof of claim: We induct on the number of triangles below the image of f.

Base Case. Suppose there are no triangles below the image of f. If $v_1 = (0,0)$ and $v_n = (1,0)$, then we have $E = D$; otherwise, the image of f is contained in $\big(\{0\} \times I\big) \cup \big(I \times \{0\}\big) \cup \big(\{1\} \times I\big)$, and so E is equivalent to D with E not reduced and D reduced.

Inductive step. Suppose there are $n \geq 1$ triangles below the image of f, and suppose the result holds for all simple edge paths g, beginning and ending at vertices in $\{0\} \times I$ and $\{1\} \times I$, respectively, with $n-1$ or fewer triangles below the image of g. It suffices to consider the case where g is a simple edge path identical to f, but differing by a single triangle T, hence the image of g has $n - 1 \geq 0$ triangles below it.

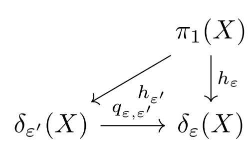
Case 1. T has a single edge J on the image of f. Thus the other two edges of T, and hence the vertex u opposite the edge J, lie on the trace of g. Since the ordered vertex set lying in the image of f is $(v_1, \dots, v_n)$, we have the ordered vertex set lying in the image of g is $(v_1, \dots, v_i, u, v_{i+1} \dots, v_n)$. Then, by property 3) in the definition of ε-homotopy, the ε-chain $E' = \big(H(v_1), \dots, H(v_i), H(u), H(v_{i+1}), \dots, H(v_n)\big)$ is equivalent to E. But by the inductive hypothesis, E' is equivalent to D; thus, E is equivalent to D.

Case 2. T has two edges, J_1 and J_2 on the image of f. This case is similar to case 1, except that the vertex set contained in the images of f has one more vertex than the image of g. □

Proposition 23. *Let (X, x_0) be a pointed, locally path-connected metric space. For each $\varepsilon > 0$ there exists a group homomorphism $h_\varepsilon : \pi_1(X, x_0) \to \delta_\varepsilon(X, x_0)$ so that if $\varepsilon' < \varepsilon$ and $q_{\varepsilon,\varepsilon'} : \delta_{\varepsilon'}(X, x_0) \to \delta_\varepsilon(X, x_0)$ then $h_\varepsilon = q_{\varepsilon,\varepsilon'} \circ h_{\varepsilon'}$.*

Proof. By Lemma 19, we have the map $\tilde{h}_\varepsilon : (\Omega_X, x_0) \to \delta_\varepsilon(X, x_0)$. We first show that this map extends to a map $h_\varepsilon : \pi_1(X, x_0) \to \delta_\varepsilon(X, x_0)$. To this end, suppose $H : D_0 = I \times I \to X$ is a homotopy between paths f and g starting and ending at x_0. By uniform continuity, there exists $\beta > 0$ so that whenever $|x - y| < \beta$ we have $|H(x) - H(y)| < \varepsilon$. Thus, we can triangulate D_0 into triangles which have diameter less than β. If $(v_1, \ldots, v_n)$ and $(u_1, \ldots, u_m)$ are the top and bottom chains of D_0, respectively, then $H|_{D_0^{(0)}}$ is an ε-homotopy between $\big(f(v_1), \ldots, f(v_n)\big)$ and $\big(g(u_1), \ldots, g(u_m)\big)$, so by Lemma 22 we have $\tilde{h}_\varepsilon(f) = \tilde{h}_\varepsilon(g)$. So we can define the map $h_\varepsilon : \pi_1(X, x_0) \to \delta_\varepsilon(X, x_0)$ by $h_\varepsilon([f]) = \tilde{h}_\varepsilon(f)$.

We can choose $t_0 = 0 < t_1 < \ldots < t_n = 1$ so that $\tilde{h}_\varepsilon(f * g) = \big(f(\frac{t_0}{2}, f(\frac{t_1}{2}), \ldots, f(\frac{t_m}{2}), g(2t_{m+1} - 1), \ldots, g(2t_n - 1)\big)$ is equivalent to $\tilde{h}_\varepsilon(f) = \big(f(t_0), \ldots, f(t_m)\big)$ followed by $\tilde{h}_\varepsilon(g) = \big(g(t_{m+1}), \ldots, g(t_n)\big)$ so that $h_\varepsilon([f*g]) = h_\varepsilon([f]) \cdot h_\varepsilon([g])$ and h_ε determines a group homomorphism. If $\varepsilon' < \varepsilon$, then because h_ε is independent of β from the first paragraph, we can choose β small enough so that h_ε maps $\pi_1(X, x_0)$ to both $\delta_{\varepsilon'}(X, x_0)$ and $\delta_\varepsilon(X, x_0)$ simultaneously in the same manner. Furthermore, since the homomorphism $\delta_{\varepsilon'}(X, x_0) \to \delta_\varepsilon(X, x_0)$ is just inclusion on chains, it's clear that $h_{\varepsilon'} = q_{\varepsilon', \varepsilon} \circ h_\varepsilon$. We can summarize this result in the form of the following commutative diagram:



□

Definition 24. *For an open cover C of a space X, we define the cover $2C = \{U \cup V \mid U, V \in C, U \cap V \neq \emptyset\}$.*

The proofs of the next two results use the algebraic fact that if G, H and K are groups, and $f : G \to H$ and $g : G \to K$ are group homomorphisms, then there exists a map $\tilde{f} : K \to H$ satisfying $f = \tilde{f} \circ g$ if and only if $\ker(g) \subseteq \ker(f)$.

Proposition 25. *Let (X, x_0) be a pointed, compact, connected, locally path-connected metric space and $\varepsilon > 0$. If C is an open cover of X, whose mesh is less than $\frac{\varepsilon}{3}$, there exists a group* *homomorphism* $\phi : \pi_1\big(N(C), p(x_0)\big) \to \delta_\varepsilon(X, x_0)$ *such that* $\phi \circ p_* = h_\varepsilon$ *where* h_ε *is the map defined in Proposition 23, and* $p_* : \pi_1(X, x_0) \to \pi_1\big(N(C), p(x_0)\big)$ *is the homomorphism induced by the canonical map* $p : X \to N(C)$*.*

Proof. Let h_ε be as in Proposition 23. We must show $\ker(p_*) \subset \ker(h_\varepsilon)$. Suppose $f : I \to X$ is such that $[f] \in \ker(p_*)$, then there is a nulhomotopy $H : I \times I \to N(C)$ of $p_*([f]) = [p \circ f]$ based at $p(x_0)$.

Consider the cover O of $N(C)$ consisting of the stars of each vertex. Then the cover $C' = \{p^{-1}(U) \mid U \in O\}$ refines the cover C of X, so the mesh of C' is less than the mesh of C, which is less than $\frac{\varepsilon}{3}$. Recall the map $p : X \to N(C)$ is given in barycentric coordinates by the maps from a partition of unity subordinate to C. Since none of these maps given by the partition of unity are identically zero, there exists a point in the star of each vertex which lies in the image of p. Associate to each vertex u in $N(C)$, a point $\tilde{x}_S \in X$ which satisfies $p(\tilde{x}_S)$ is in the star of u.

By uniform continuity, we can find a triangulation D_0 of $I \times I$, so that the image of each triangle, under H, is contained in the star of a vertex of $N(C)$. We define an ε-nulhomotopy $G : D_0^{(0)} \to X$ of the ε-loop $E = \big(G(v_1), \ldots, G(v_n)\big)$ where $(v_1, \ldots, v_n)$ is the top chain of D_0, and E represents $\tilde{h}_\varepsilon(f)$. If v is a vertex of the triangle T in D_0, define $G(v) = \tilde{x}_S$ where S is a star of a vertex containing $H(T)$, with the condition that if there exists a vertex in $N(C)$ whose star contains the full image $H(T)$ and the point $p(x_0)$, then $G(v) = x_0$. The choice of star S containing $H(T)$ doesn't matter because different choices of S give different choices of $\tilde{x}_S$ differing by less than ε. Observe that for each vertex u in the bottom chain of D_0, $G(u) = x_0$, so that G restricted to the bottom chain of D_0 gives an ε-loop F representing the trivial element in $\delta_\varepsilon(X, x_0)$. If $s, t \in D_0^{(0)}$ are adjacent vertices, and $G(s) \in U \in C'$ and $G(t) \in V \in C'$, then there are three cases.

1. $U = V$, so $d\big(G(s), G(t)\big) < \frac{\varepsilon}{3}$.

2. $U \cap V \neq \emptyset$, then $d\big(G(s), G(t)\big) < \frac{2\varepsilon}{3}$.

3. There exists $W \in C'$ satisfying $U \cap W \neq \emptyset$ and $V \cap W \neq \emptyset$. Then, $d\big(G(s), G(t)\big) < \frac{3\varepsilon}{3} = \varepsilon$.

We noted above that G restricted to the top chain of D_0 represents $\tilde{h}_\varepsilon(f)$, and G restricted to the bottom chain of D_0 represents the trivial element of $\delta_\varepsilon(X, x_0)$. Thus $G : D_0^{(0)} \to X$ is an ε-nulhomotopy between E, an ε-loop equivalent to $\tilde{h}_\varepsilon(f)$. By Lemma 22, $\tilde{h}_\varepsilon(f)$ is trivial in $\delta_\varepsilon(X, x_0)$.

Thus $[f] \in \ker(h_\varepsilon)$. This then gives the existence of a map $\phi : \pi_1\big(N(C), p(x_0)\big) \to \delta_\varepsilon(X, x_0)$ yielding the following commutative diagram:

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{h_\varepsilon} & \delta_\varepsilon(X, x_0) \\ \downarrow{\scriptstyle p_*} & \nearrow{\scriptstyle \phi} & \\ \pi_1\big(N(C), p(x_0)\big) & & \end{array}$$

□

Proposition 26. *Let (X, x_0) be a pointed compact, connected, locally path-connected metric space. If C is an open cover of X with nerve $N(C)$, there exists $\varepsilon > 0$ and a map $\psi : \delta_\varepsilon(X, x_0) \to \pi_1\big(N(C), p(x_0)\big)$ such that $p_* = \psi \circ h_\varepsilon$ where h_ε is the map defined in Proposition 23.*

Proof. Suppose $f : I \to X$ is a loop based at x_0 satisfying $[f] \in \ker(h_\varepsilon)$. Let $p : X \to N(C)$ be the map induced by a partition of unity subordinate to C. Let O be the cover of $N(C)$ whose open sets are the stars of each vertex, and consider the open cover $C' = \{p^{-1}(U) \mid U \in O\}$ of X with Lebesgue number L, and choose $\varepsilon = L$.

By uniform continuity, there exists $\beta > 0$ so that whenever $|t_1 - t_0| < \beta$, we have $|f(t_1) - f(t_0)| < L$. We take a partition $\{0 = t_0, t_1 \ldots, t_n = 1\}$ of I with mesh less than β. Then for each i, $0 \leq i \leq n-1$, there exists a $U \in O$ so that the image of $f|_{[t_i, t_{i+1}]}$ is contained in $p^{-1}(U)$. To see this is true, consider x_1, x_2 in the image of $f|_{[t_i, t_{i+1}]}$. Then, there exists $t, t' \in [t_i, t_{i+1}]$ so that $f(t) = x_1$ and $f(t') = x_2$. However, by uniform continuity, $d(x_1, x_2) = d(f(t), f(t')) < L$, so that the diameter of the image of $f|_{[t_i, t_{i+1}]} < L$. Thus, for each i, the image of $p \circ f|_{[t_i, t_{i+1}]}$ is contained in the star of some vertex in $N(C)$.

Since $[f] \in \ker(h_\varepsilon)$, by Lemma 22 there exists an ε-nulhomotopy $H : D_0^{(0)} \to X$ from the vertex set of a triangulation D_0 of $I \times I$ between $E = \big(H(v_1), H(v_1), \ldots, H(v_n)\big)$ - an ε-loop representing $\tilde{h}_\varepsilon(f)$, where $(v_1, v_2, \ldots, v_n)$ is the top chain of D_0 - and the ε-loop $E = \big(H(u_1), H(u_1), \ldots, H(u_n)\big)$ representing the trivial element in $\delta_\varepsilon(X, x_0)$, where $(u_1, \ldots, u_n)$ is the bottom chain of D_0. By property 3) in the definition of ε-homotopy, each triple (s, t, u) of pair-wise adjacent vertices in $D_0^{(0)}$ has the property that the diameter of the set $S = \{H(s), H(t), H(u)\}$ is less than $\varepsilon = L$, and so S is contained in the preimage, under p, of some star of $N(C)$. Thus, the set $\{p \circ H(s), p \circ H(t), p \circ H(u)\}$ is contained in a contractible star U. Therefore, we can continuously map the triangle of D_0 with vertices s, t, and u into U, so that $s \mapsto p \circ H(s)$, $t \mapsto p \circ H(t)$, and $u \mapsto p \circ H(u)$. It's clear that we can continuously map each triangle of the triangulation of D_0 in this way so that two maps on two triangles sharing a common edge agree on the common edge. What's more, for a triangle with an edge J lying on $I \times \{1\}$, the map can be chosen so that, when restricted to J, it agrees with $p \circ f$ restricted to J. For a triangle with an edge K lying on $I \times \{0\}$, the map can be chosen to take all of K to $p(x_0)$. By the pasting lemma, we have a continuous map from $I \times I \to N(C)$ that is a nulhomotopy of $p \circ f$.

Therefore $[p \circ f] = p_*[f]$ is trivial in $\pi_1\big(N(C), p(x_0)\big)$. This concludes the proof, and we have the commutative diagram.

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{h_\varepsilon} & \delta_\varepsilon(X, x_0) \\ \downarrow{\scriptstyle p_*} & \swarrow{\scriptstyle \psi} & \\ \pi_1\big(N(C), p(x_0)\big) & & \end{array}$$

□

Lemma 27. *Let (X, x_0) be a pointed, locally path-connected, metric space, and let (ε_n) be a decreasing sequence of positive numbers converging to zero, so that*
$\Delta_1(X, x_0) = \varprojlim\big(\delta_{\varepsilon_n}(X, x_0), q_{n,n+1}\big)$ with projection maps $q_n : \Delta_1(X, x_0) \to \delta_{\varepsilon_n}(X, x_0)$. Then for any n, we have $q_n\big(\Delta_1(X, x_0)\big) \subseteq h_{\varepsilon_n}\big(\pi_1(X, x_0)\big)$.

Proof. For $n \geq 0$ define $\varepsilon_n = \frac{\varepsilon}{2^n}$ and observe that the sequence (ε_n) decreases to zero, so we can write $\Delta_1(X, x_0) = \varprojlim(\delta_{\varepsilon_n}(X, x_0), q_{n,n+1})$. Further observe that since X is locally path-connected, for all n, there exists $m \geq n$ so that $d(x, y) < \varepsilon_m$ implies there exists a path f from x to y with diameter less than $\frac{\varepsilon_n}{2}$. In particular, this is true for $n = 0$, so that $\varepsilon_n = \varepsilon$.

Choose an arbitrary $C_0 \in \delta_\varepsilon(X, x_0)$ and fix a coherent sequence $\big([C_n]\big) \in \Delta_1(X, x_0)$. Now, let f_m be a loop based at x_0 so that $\big(f_m(\frac{i}{k_m})\big) = C_m$ and the diameter of $f|_{\left[\frac{i}{k_m}, \frac{i+1}{k_m}\right]} < \frac{\varepsilon}{2}$. We claim that $h_{\varepsilon_0}([f_m]) = C_0$.

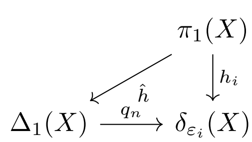
Proof of claim: Notice that if $|s - t| < \frac{1}{k_m}$, then there exists i such that $\frac{i}{k_m} \leq s, t, \leq \frac{i+2}{k_m}$, which implies $d\big(f_m(s), f_m(t)\big) < \varepsilon$. Then, for the partition $0 < \frac{1}{k_m} < \ldots < \frac{k_m - 1}{k_m} < 1$ of I, we have that $\big(f_m(0), f_m(\frac{1}{k_m}), \ldots, f_m(1)\big)$ represents $h_\varepsilon\big([f_m]\big) = C_m$. But, since each ε_m-chain is also an ε-chain, and since $\big([C_n]\big)$ is a coherent sequence, we have $h_\varepsilon\big([f_m]\big) = C_m = C_0$, which concludes the proof. □

We now state and prove the main result of this section.

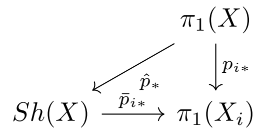
Theorem 28. *Let (X, x_0) be a compact, connected, locally path-connected, pointed, metric space, then $Sh(X, x_0) \cong \Delta_1(X, x_0)$.*

Proof. Throughout this proof all the spaces have a basepoint chosen without ambiguity, so we supress the basepoint from notation. Let $\{O_i\}$ be a refining sequence of finite open covers of X consisting of path-connected, open sets whose mesh goes to 0 with corresponding nerves $X_i = N(O_i)$ and bonding maps $p_{i,i+1}$, so that the shape group of X is $Sh(X, x_0) = \varprojlim(\pi_1(X_i, p_i(x_0)), p_{i,i+1*})$. Further, let $\big(\delta_{\varepsilon_i}(X, x_0)\big)$ be a sequence of discrete homotopy groups with ε_i decreasing monotonically to 0.

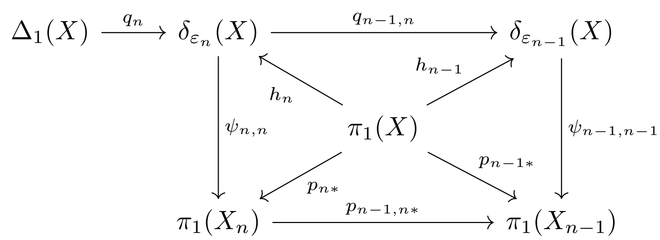
By Proposition 23 and the universal mapping property of inverse limits, we have a map $\hat{h} = \varprojlim h_i$ and commutative diagram



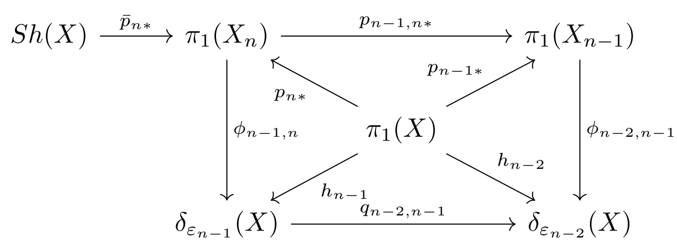
Similarly, we have a map $\hat{p}_* = \varprojlim p_{i*}$



It's clear that we can choose subsequences (X_{i_k}) and (ε_{i_l}) so that, after relabeling, we have the following diagrams:



and



The first diagram commutes by Lemma 27. The second diagram commutes, since because X is locally path-connected, the maps p_{n*} and p_{n-1*} are surjective. Thus, by the universal mapping property of inverse limits, we have a map $F : Sh(X) \to \Delta_1(X)$ defined as follows: if $\big([f_n]\big) \in Sh(X)$, then $F\big(([f_n])\big) = \big(\big[\phi_{n-1,n}([f_n])\big]\big)$. We also have a map $G : \Delta_1(X) \to Sh(X)$ defined as follows: if $\big([C_n]\big) \in \Delta_1(X)$, then $G\big(([C_n])\big) = \big([\psi_{n,n}([C_n])]\big)$. However, by the commutativity of the above diagrams, the maps $F \circ G$ and $G \circ F$ act on coherent sequences in $\Delta_1(X)$ and $Sh(X)$, respectively, by a shift. Thus, $F \circ G$ and $G \circ F$ are the identity maps on $\Delta_1(X)$ and $Sh(X)$, respectively, so F and G are isomorphisms. □

Part II

Further Classifying Sharkovskii Spaces

[19, p.1-3] Sharkovskii's theorem is a well-known result in dynamical systems. It is named after Oleksandr Mikolaiovich Sharkovskii, a prominent Ukrainian mathematician, who submitted a paper titled *Coexistence of cycles of a continuous mapping of a line into itself* to the Ukrainian Mathematical Journal in 1962. The paper was published by the journal in 1964. The paper provided a proof to the following theorem: *If a continuous mapping of the reals into the reals has a point with fundamental period k, and if $k < l$ with respect to the following ordering*

$$\begin{aligned}
&3 < 5 < 7 < 9 < 11 < \dots \\
&< 2(3) < 2(5) < 2(7) < 2(9) < 2(11) < \dots \\
&< 2^2(3) < 2^2(5) < 2^2(7) < 2^2(9) < 2^2(11) < \dots \\
&< 2^3(3) < 2^3(5) < 2^3(7) < 2^3(9) < 2^3(11) < \dots \\
&\vdots \\
&\dots < 2^4 < 2^3 < 2^2 < 2 < 1
\end{aligned}$$

then the mapping also has a point with fundamental period l.

This result was independently published several years later by Tien-Yien Li and James Yorke in a famous paper titled *Period three implies chaos.*

Theorem 29 (3 implies chaos)**.** *Let $J \subset \mathbb{R}$ be an interval, and let $f : J \to J$ be a continuous function. If there exists a point $x \in J$ such that $f^3(x) = x$, and $f^n(x) \neq x$ for $n \in \{1, 2\}$, then for each integer $m \in \mathbb{N}$, there exists a point $x_m \in J$ such that $f^m(x) = x$ and $f^k(x) \neq x$ for all $l \in \{1, 2, \dots, k-1\}$.*

4.1 Generalizing Sharkovskii's Theorem

Since 1975, mathematicians have been seeking other topological spaces on which continuous endomorphisms satisfy the conclusion of Sharkovskii's theorem; we refer to such spaces as *Sharkovskii spaces*. [23] In 1980, Block, Guckenheimer, Misiurewicz, and Young published a paper showing that S^1, the one-dimensional sphere, is a Sharkovskii space. [8, p.164] In 1985, Helga Schirmer first defined a Sharkovskii space as we have done here and proved that an ordered topological space Y is Sharkovskii if and only Y is ordered densely and Y has the least upper bound property for every subset of Y bounded above. [10] In 2012 Grant, Conner, and Meilstrup published a paper with the title *A Sharkovskii Theorem for non-locally Connected Spaces* showing that the following spaces are Sharkovskii: the topologist's sine curve, any n-fold union of topologist sine curves, the Warsaw circle, any n-fold cover of the Warsaw circle, and any line of topologist sine curves.

Even for spaces that are not Sharkovskii, much work has been done in studying periods of orbits of self-maps. For work done on S^1, reference [21, p.221-227] [25, p.351-370] [26, p.5-71]; for n-ods, reference [4, p.249-271] [3, p.475-538] [11, p.84-87]; for trees, reference [2, p.311–341] [5, p.19-31]; for the figure-eight space, reference [12, p.95-106]; for further work on Warsaw circle and k-Warsaw circle, reference [27, p.294–299] [28, p.12–16]; for hereditarily decomposable chainable continua, reference [18, p.549–553].

It is worth noting that all the Sharkovskii spaces mentioned are one-dimensional, since the theorem easily fails for many higher dimensional spaces. For example, rotating a two-dimensional disk by angle $\frac{2\pi}{3}$ is a clear counterexample. Thus, many weaker versions of the theorem have been attempted for higher dimensional spaces, but none have gained the widespread fame as the original theorem.

Many mathematicians are still working to provide a classification of all Sharkovskii spaces.

4.2 Preliminaries

For a set S, we call a function $f : S \to S$ a self function or self map.

Definition 1.1.1: We define a new ordering of the natural numbers called the *Sharkovskii Ordering.*

$$\begin{aligned} & 3 < 5 < 7 < 9 < 11 < \dots \\ & < 2(3) < 2(5) < 2(7) < 2(9) < 2(11) < \dots \\ & < 2^2(3) < 2^2(5) < 2^2(7) < 2^2(9) < 2^2(11) < \dots \\ & < 2^3(3) < 2^3(5) < 2^3(7) < 2^3(9) < 2^3(11) < \dots \\ & \qquad \vdots \\ & \dots < 2^4 < 2^3 < 2^2 < 2 < 1 \end{aligned}$$

Definition 1.1.2: Let f be a continuous function from an interval $I \subseteq \mathbb{R}$ to itself (the interval need not be open or closed). Denote by f^n the nth composition of f with itself. Let $x \in I$. If $f^n(x) = x$ and $f^k(x) \neq x$ for all $k \in \mathbb{N}$, $1 \leq k < n$, we say that x has *orbit* n. If there exists an x with orbit n in the domain of f, we say that f has an n*-orbit.*

Theorem 30 (Sharkovskii)**.** *Let f be a continuous function from an interval $I \subseteq \mathbb{R}$ to itself, where I need not be closed or open. If f has an n-orbit, then f has an m-orbit for all $m \geq n$ with respect to the Sharkovskii Ordering.*

The self map f has the *Sharkovskii property* provided f has period $n \in \mathbb{N}$ whenever it has a period m smaller in the Sharkovskii order. While Sharkovskii originally only studied interval maps of the real line, H. Schirmer ([24]) and e.g. [6] considered connected linear spaces and, more generally, certain 1-dimensional spaces. During the present work we would like to give a space where every self map has the Sharkovskii property a name (as done in [24]).

Definition 31. A topological space X is *Sharkovskii* provided every self map f has the Sharkovskii property. A *Sharkovskii group* is a topological group G whose underlying topological space is a Sharkovskii space.

In [16] it has been shown that every leaf of the dyadic solenoid is Sharkovskii in its induced topology. It can be deduced from e.g. [13, Proposition 6] that the dyadic solenoid itself cannot be Sharkovskii. Every solenoid, and more generally, every connected compact abelian group can be described as the inverse limit of an inverse system $(\mathbb{T}^{n_i}, f_i)$ of tori $\mathbb{T}^{n_i} = \mathbb{R}^{n_i}/\mathbb{Z}^{n_i}$ where $n_i \in \mathbb{N}$, $\mathbb{R}$ is the additive group of reals in its canonical topology, $\mathbb{Z}$ the closed subgroup of integers, and, f_i are continuous group epimorphisms.

Question 32. Classify all Sharkovskii groups.

In this thesis we shall contribute to the question, see Theorem 34. However, for reasons, to be explained below, we shall restrict ourselves to the class of locally compact groups.

Proposition 36 below will show that Sharkovskii groups must be torsion free.

The leaf of the dyadic solenoid is *not* locally compact, yet, by a result of the author (see [16]) it is Sharkovskii. Since one can expect the same result to hold for any p-adic solenoid, there is a wealth of topologies on $\mathbb{R}$ making it Sharkovskii. We therefore will in the sequel restrict ourselves to the task of classifying *locally compact* Sharkovskii groups. An important role will play the particular solenoid just mentioned (see [15, Ch. 10]):

Notation 33. The $\mathbb{Z}$-adic completion of the integers $\mathbb{Z}$ will be denoted by $\hat{\mathbb{Z}}$. Thus every $n \in \mathbb{N}$ allows a $\mathbb{Z}$-adic expansion:

$$n = \sum_{i \geq 0} x_i (i+1)!$$

with $x_0 \in \{0, 1\}$ and $0 \leq x_i \leq i+1$ for $i \geq 1$. Then $\hat{\mathbb{Z}}$ can be viewed the set of all infinite formal expansions and it turns out that $\hat{\mathbb{Z}}$ is a topological ring. The $\mathbb{Z}$-adic *solenoid* is the compact group

$$\Sigma_{\mathbf{a}} := (\mathbb{R} \times \hat{\mathbb{Z}})/\mathbb{Z}$$

with $\mathbf{a} := (2, 3, 4, \ldots)$, where the identification of $\mathbb{Z}$ as a subgroup of $\mathbb{R} \times \hat{\mathbb{Z}}$ is given as z corresponding to the pair $(z, -z)$. As pointed out in [15] this solenoid can also be obtained as the Pontryagin dual of the discrete group $\mathbb{Q}$.

The main result will be

Theorem 34. *Let G be a locally compact Sharkovskii group. If G is not compact it is the additive group of reals $\mathbb{R}$. Otherwise G is the solenoid $\Sigma_{\mathbf{a}}$.*

We have not been able to prove that $\Sigma_{\mathbf{a}}$ is indeed a Sharkovskii group.

4.3 Observations on Sharkovskii Spaces and Groups

Lemma 35. *If a topological group X possesses a self map f, not equal to the identity, satisfying $f^3 = id_X$. Then X is* not *Sharkovskii.*

Proof. Since f is not the identity, it has an orbit of length 3. However, there are no other orbit lengths other than 1 and 3, so f is not Sharkovskii. □

Proposition 36. *If G is a Sharkovskii group then it must be torsion free.*

Proof. Suppose $t \in G$ has order $n > 1$. Define on G a self map f by setting $f(x) = xt$ ("right shift" by t). The possible period lengths of f are all the divisors d of n. Thus f can have only finitely many period lengths and, taking the properties of the Sharkovskii order into account, it follows that n can only be a power of 2. Then, however, f must also possess a fixed point, i.e., there is $x \in G$ with $x = x \cdot t$, leading to the contradiction that t is the identity element of G. Thus G must be torsion free. □

Crucial is the following result about connectedness by H. Schirmer:

Theorem 37 ([24, Theorem 3.3])**.** *Every Sharkovskii space is connected.*

Lemma 38 ([24, Theorem 3.4])**.** *If $r : X \to Y \subseteq X$ is a retraction map and X is a Sharkovskii space then so is Y.*

Proof. Let ϕ be a self map of Y. Then, setting $f := \phi \circ r$ extends ϕ to a self map of X. An easy induction argument shows that $f^k(x) = \phi^k(r(x))$ for all $k \geq 1$. Suppose now that ϕ has a period p. Then there is $y = r(y) \in Y$ with

$$y = r(y) = \phi^p(y) = \phi^p(r(y)) = f^p(y).$$

By the Sharkovskii property of f, for every natural number q bigger than p with respect to the Sharkovskii order there is $x \in X$ with $f^q(x) = x$. Since $q > 1$ one therefore obtains

$$x = \phi^q(r(x)) \in Y$$

and hence $\phi^q(y) = y$ for $y := r(x)$. Thus also ϕ has the Sharkovskii property and therefore the retract Y is itself a Sharkovskii space. □

Corollary 39. *If $G = \prod_i G_i$ is a Sharkovskii group then each factor G_i is also Sharkovskii.*

Proof. Fix i. For $j \neq i$ choose $e_j \in G_j$ to be the identity element, and define $f_j : G_j \to G_j$ by $f_j(x) = e_j$. Now for $j = i$ define $f_j = id : G_j \to G_j$, and observe that the retraction map $\prod_i f_i : G \to G$ shows that the subgroup $\prod_j H_j$, where $H_j = \{e_j\}$ for $j \neq i$ and $H_j = G_j$ for $j = i$, is also Sharkovksii. But, since G_i is isomorphic to $\prod_j H_j$, then G_i is also Sharkovskii. □

Another observation about Cartesian products of topological abelian groups will turn out useful:

Lemma 40. *Suppose an abelian topological group G is the direct product*

$$G = A^{\mathfrak{m}},$$

for a closed subgroup A and some cardinal $\mathfrak{m}$. If G is Sharkovskii then $\mathfrak{m} = 1$.

Proof. If $\mathfrak{m} \geq 2$ then $G = A \times A \times Y$ for some closed (maybe trivial) subgroup Y. Define

$$f : G \to G : \ (a_1, a_2, y) \mapsto (-a_2, a_1 - a_2, y).$$

Then, considering f an element in the endomorphism ring of G, one finds that f satisfies the equation

$$f^2 + f + id_G = 0_G.$$

It follows that f has period 3 and therefore Lemma 35 implies that G cannot be a Sharkovskii space. Hence $\mathfrak{m} = 1$ as claimed. □

Corollary 41. *Let $n \geq 1$. Then $\mathbb{R}^n$ is a Sharkovskii space if, and only if, $n = 1$.*

Proof. Suppose $\mathbb{R}^n$ is a Sharkovskii space. Then Lemma 40 implies that $n = 1$. Conversely, $\mathbb{R}$ is a Sharkovskii space (see [24]). □

4.4 Reduction, up to Homeomorphism, to the Abelian Connected Case

At the end of this section we shall prove Corollary 44, saying that every locally compact Sharkovskii group, up to homeomorphism, must be abelian, connected, and either be compact or homeomorphic to $\mathbb{R} \times K$ with K a compact connected abelian torsion free group.

Proposition 42. *Let G be a compact Sharkovskii group. Then G is homeomorphic to a connected locally compact abelian group.*

Proof. By Theorem 37, G is connected, and by [17, Theorem 9.24] the group G can be presented as a factor group

$$G \cong \frac{Z_0(G) \times \prod_{j \in J} S_j}{\Delta},$$

where $Z_0(G)$ is the connected component of the center of G containing the identity, J is some index set, and, for every $j \in J$ the group S_j is a semisimple Lie group and Δ is a central closed subgroup. If J is empty there is nothing to prove. Else suppose one of the factors S_j is not trivial. Then S_j contains a nontrivial maximal torus $T \cong \mathbb{T}^k$ for some finite $k \geq 1$. In this torus the set of all 3-power elements is dense. Since S_j is the union of its maximal tori it follows that the set X_3 of all 3-power elements of S_j is dense in S_j. Therefore $X_3\Delta/\Delta$ is dense in $S_j\Delta/\Delta$ showing that G contains an element of order 3. Therefore Lemma 35 implies that G cannot be Sharkovskii and thus G must be abelian. □

The following result by J. Cleary and S. Morris will be needed (see [9, Theorem 1]):

Theorem 43. *Let G be any locally compact group. Then the underlying topological space is homeomorphic to*

$$\mathbb{R}^n \times K \times D$$

where n is a natural number, K is compact Hausdorff group, and, D is a discrete space.

We make use of this result for a first reduction:

Corollary 44. *Let G be a locally compact Sharkovskii group. Then there are $0 \leq n \leq 1$ and a compact connected abelian group K such that G is homeomorphic to $\mathbb{R}^n \times K$.*

Proof. (a) By Theorem 43 there are $n \geq 0$ and, compact hausdorff group K and a discrete group D with G homeomorphic to $\mathbb{R}^n \times K \times D$. Since K is a factor then by Corollary 39 it is Sharkovskii; hence K is compact and Sharkovskii, so by Proposition 42, it is abelian. Hence G is abelian, since it is the product of abelian groups. By Theorem 37 G is connected, and so D must be a singleton set. Corollary 39 implies that the factor $\mathbb{R}^n$ is also Sharkovskii, and so by Corollary 41 one obtains $n \leq 1$. □

4.5 Solenoids

Before continuing let us discuss solenoids in more detail. Given an infinite sequence $\mathbf{p} = (p_1, p_2, p_3, \ldots)$ of prime numbers one considers the inverse sequence

$$S^1 \xleftarrow{f_1} S^1 \xleftarrow{f_2} S^1 \xleftarrow{f_3} \ldots$$

for $f_i(z) := z^{p_i}$. The inverse limit $\Sigma_{\mathbf{p}}$ of the system is the $\mathbf{p}$-*adic* solenoid.

Note that the solenoid in Notation 33 corresponds to the sequence of primes

$$\mathbf{p} = (2; 3; 2, 2; 5; 2, 3; 7; 2, 2, 2; 3, 3; \ldots),$$

where the ";" indicate the factorizations of respectively i.

Two sequences $\mathbf{p}$ and $\mathbf{q}$ of primes are *equivalent* if in each sequence one can delete a finite number of elements and after that each prime occurs the same number of times in both sequences. Bing and McCord proved the following:

Theorem 45 ([1, Theorem])**.** *The following statements are equivalent:*

(a) *The sequences* $\mathbf{p}$ *and* $\mathbf{q}$ *are equivalent.*

(b) *The solenoids* $\Sigma_{\mathbf{p}}$ *and* $\Sigma_{\mathbf{q}}$ *are homeomorphic.*

(c) *The solenoids* $\Sigma_{\mathbf{p}}$ *and* $\Sigma_{\mathbf{q}}$ *are algebraically and topologically isomorphic.*

The torsion subgroup of S^1 is (algebraically) isomorphic to $\mathbb{Q}/\mathbb{Z}$ and, for every prime p it has the p-primary subgroup isomorphic to Prüfer's group $\mathbb{Z}(p^\infty) = \mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$.

Lemma 46. *Let* $\mathbf{p}$ *be an infinite sequence of primes. Then* $\Sigma_{\mathbf{p}}$ *has* p*-torsion isomorphic to* $\mathbb{Z}(p^\infty)$ *if, and only if, the prime* p *appears in* $\mathbf{p}$ *only finitely many times.*

Proof. Suppose first that p appears only finitely many times in $\mathbf{p}$. Then Theorem 45 allows us to delete all occurrences of p in $\mathbf{p}$ without changing the solenoid. Now, in each step, the kernel of f_j consists of a finite cyclic group of order $p_j \neq p$. It follows that the p-torsion maps isomorphically under f_j. In particular, in the inverse limit, the p-torsion of S^1 appears embedded, showing that $\operatorname{Tor}(\Sigma_{\mathbf{p}}) = \mathbb{Z}(p^\infty)$ as has been claimed.

Conversely, suppose that $\Sigma_{\mathbf{p}}$ contains a p-torsion element, say x of order p. If infinitely many primes p_{j_k} appear in the sequence $\mathbf{p}$ it means that x maps to 1 under the canonical projection $p_k : \Sigma_{\mathbf{p}} \to S^1$. But then x which is the element $(p_1(x), p_2(x), p_3(x), \ldots)$ would be trivial, a contradiction. Hence p can appear only finitely many times in $\mathbf{p}$. □

One deduces from this the following consequence:

Corollary 47. *Suppose a solenoid* $\Sigma_{\mathbf{p}}$ *is Sharkovskii. Then every prime* p *occurs infinitely often in* $\mathbf{p}$.

Proof. Suppose some prime p appears only finitely many times in the sequence $\mathbf{p}$. By Lemma 46 $\Sigma_{\mathbf{p}}$ contains an element of finite order. Therefore, by Proposition 36, the group $\Sigma_{\mathbf{p}}$ cannot be Sharkovskii. □

4.6 Reduction to Finite Dimension for Compact Sharkovskii Groups

By Proposition 36 we know that every Sharkovskii group is torsion free. In the present section we aim at showing that every compact Sharkovskii group must have finite dimension. In the abelian case our main tool is [15, (25.8) Theorem]:

Theorem 48. *Let G be an abelian compact connected and torsion free group. Then there is a solenoid S and a cardinal $\mathfrak{m}$ with $G \cong S^{\mathfrak{m}}$. Moreover, S is the solenoid from Notation 33.*

The easiest way to describe S is by means of its Pontryagin dual which is nothing but the additive group $\mathbb{Q}$ of rational numbers endowed with the discrete topology. (See [15, (25.4)] where the group in question is denoted by $\Sigma_{\mathbf{a}}$ and $\mathbf{a} = (2, 3, 4, \ldots)$.)

An immediate consequence is:

Corollary 49. *Let G be an compact Sharkovskii group. Then G must be a solenoid of the form $G = \Sigma_{\mathbf{a}}$ for $\mathbf{a} = (2, 3, 4, \ldots)$.*

Proof. Since G is Sharkovskii it is torsion free by Proposition 36 and connected by Theorem 37. However, since G is also compact, then by Proposition 42 it is also abelian. Theorem 48 renders a solenoid S and a cardinal $\mathfrak{m}$ with $G \cong S^{\mathfrak{m}}$ algebraically and topologically. Deduce from Lemma 40 that $\mathfrak{m} = 1$, i.e., $G = S$ is a solenoid for $\mathbf{a} = (2, 3, 4, \ldots)$. □

4.7 Reduction to Dimension 1

We shall prove in this section Theorem 54, saying that every finite dimensional torsion free locally compact Sharkovskii group either is homeomorphic to $\mathbb{R}$ or to a solenoid of the form $\Sigma_{\mathbf{a}}$.

Lemma 50. *Define (polar coordinates) a function on the closed unit disc:*

$$f(re^{i\theta}) := \begin{cases} re^{i(\theta+\frac{2\pi}{3})} & \text{if } 0 \le r < \frac{1}{3} \text{ or if } r = 1 \\ \left(\frac{3}{2}\left(r-\frac{1}{3}\right)^2 + \frac{1}{3}\right) e^{i(\theta+\pi(1-r))} & \text{if } \frac{1}{3} \le r < 1 \end{cases}$$

Then all of the following holds:

(i) *f is a self map of the closed unit disc.*

(ii) *The function $h(r) := |f(re^{i\theta}|$ satisfies*

$$\frac{1}{3} < h(r) < r < 1$$

for all r in the open interval $(\frac{1}{3}, 1)$.

(iii) *f has a period of length 3 but none of length 5.*

(iv) *f is not Sharkovskii.*

Proof. (i) We need to prove that $|f(re^{i\theta}| \le 1$. This clearly holds for $r \in [0, \frac{1}{3}) \cup \{1\}$. For $r \in [\frac{1}{3}, 1]$ the function $r \to r - \frac{1}{3}$ is strictly increasing and therefore

$$|f(re^{i\theta})| = \frac{3}{2}(r-\frac{1}{3})^2 + \frac{1}{3} \le \frac{3}{2}(1-\frac{1}{3})^2 + \frac{1}{3} \le 1,$$

as claimed.

For proving continuity at $r = \frac{1}{3}$ we determine $\lim_{r\uparrow\frac{1}{3}} f(re^{i\theta})$ and find its value to be $\frac{1}{3}e^{i(\theta+\frac{2\pi}{3})}$.

On the other hand, plugging $r = \frac{1}{3}$ into the expression valid for $r \in [\frac{1}{3}, 1)$ yields

$$\frac{1}{3}e^{i(\theta+\pi(1-\frac{1}{3}))} = \frac{1}{3}e^{i(\theta+\frac{2\pi}{3})}.$$

For checking continuity at $r = 1$ we determine $\lim_{r\uparrow 1} f(re^{i\theta})$ and find from the second case its value

$$\left(\frac{3}{2}\left(1-\frac{1}{3}\right)^2+\frac{1}{3}\right)e^{i\theta} = e^{i\theta}.$$

Hence f is continuous and is therefore a self map of the unit disc.

(ii) Since for $\frac{1}{3} < r < 1$ the we have that $0 < r - \frac{1}{3} < 1$ conclude

$$r - h(r) = r - \frac{1}{3} - \frac{3}{2}(r-\frac{1}{3})^2 = (r-\frac{1}{3})(1-\frac{3}{2}(r-\frac{1}{3})) = (r-\frac{1}{3})(1-r) > 0.$$

The estimate in (i) shows $h(r) < 1$.

(iii) Every point $re^{i\theta}$ with $0 < r < \frac{1}{3}$ has period 3 and obviously there is no point with period 5.

Next let $re^{i\theta}$ be a point with $\frac{1}{3} \le r < 1$. As a consequence of (ii) the open annulus $A := \{z \in \mathbb{C} \mid \frac{1}{3} < |z| < 1\}$ is mapped under f into A and for all $z \in A$ we have that

$$|f(z)| < |z|.$$

It follows that neither f nor any iterate can have a fixed point in A and therefore all periodic nontrivial orbits belong to the disc $|z| < 1$.

(iv) is a consequence of (iii). □

Lemma 51. *Let* $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. *Then* $H := \{z \in S^1 : \Im(z) \ge 0\}$ *is a retract of* $\mathbb{T}$.

Proof. On $\mathbb{C}$ define $r(x + iy) := x + i|y|$. Then $\{z \in \mathbb{C} : \Im(z) \ge 0\}$ is a retract of $\mathbb{C}$ with retraction mapping r. Restricting r to S^1 serves our purpose. □

Lemma 52. *Let* $\mathbb{S}$ *be a solenoid. Then* $\mathbb{S}$ *contains a closed subset* C *homeomorphic to* $[0, 1]$ *(i.e., an arc), a retract of* $\mathbb{S}$.

Proof. Consider $p : \mathbb{S} \to S^1$ as a fibration over S^1 and totally disconnected compact fiber homeomorpic to the Čech group of $\mathbb{S}$. Let r be the retraction map of S^1 onto H from Lemma

51. Then H is an arc in S^1 which lifts to an arc $\tilde{H}$ in $\mathbb{S}$. Let $l : H \to \mathbb{S}$ denote the lifting map. The desired retraction map will be $R := i \circ l \circ r \circ p$, where i is the inclusion of $\tilde{H}$ in $\mathbb{S}$.

$$\underbrace{\mathbb{S} \xrightarrow{p} \mathbb{T} \xrightarrow{r} H \xrightarrow{l} \tilde{H} \xrightarrow{i} \mathbb{S}}_{R}$$

Certainly R is continuous. For proving that it restricts to the identity on $\tilde{H}$, pick any $\tilde{h} \in \tilde{H}$. Then, as l is a lifting we have that $l(p(\tilde{h})) = \tilde{h}$. Observing that $p(\tilde{h}) \in H$ and, for every $h \in H$ one has $r(h) = h$, one obtains, setting $h := p(\tilde{h})$, from $i(\tilde{h}) = \tilde{h}$

$$R(\tilde{h}) = i(l(r(p(\tilde{h})))) = i(l(r(h)) = i(l(h)) = i(\tilde{h}) = \tilde{h},$$

as claimed. Thus R is a retraction map. □

Lemma 53. *Let X be a space homeomorphic to $\mathbb{R} \times \mathbb{S}$ for $\mathbb{S}$ some solenoid. Then X is not a Sharkovskii space.*

Proof. Without losing generality we may assume $X = \mathbb{R} \times \mathbb{S}$. Suppose, by way of contradiction, that X is a Sharkovskii space. Let $R : \mathbb{S} \to \mathbb{S}$ be a retract onto an arc inside $\mathbb{S}$ as in Lemma 52. Define a retraction map $\rho : X \to X$ by sending $(r, s) \in \mathbb{R} \times \mathbb{S}$ to $(r, R(s))$. Then ρ is a retraction map from X onto $Y := \mathbb{R} \times im(R)$. Theorem 38 implies that Y must be a Sharkovskii space. Note that Y is homeomorphic to $\mathbb{R} \times [0, 1]$ and thus contains as a retract a closed disk which, again by Theorem 38, must be Sharkovskii. This, however, contradicts Lemma 50. □

Theorem 54. *Let G be a locally compact Sharkovskii group. Then G has dimension 1. In particular, $G \cong \mathbb{R}$ or G is a solenoid.*

Proof. Corollary 44 shows that G can be assumed to be a group

$$G = \mathbb{R}^n \times K$$

for K a compact group. By Corollary 39 both $\mathbb{R}^n$ and K must be Sharkovskii spaces. Corollary 41 shows that $n \leq 1$. If $K \neq \{1\}$ is Sharkovskii we find from Corollary 49 that K is homeomorphic to $\Sigma_{\mathbf{a}}$. Lemma 53 implies that precisely one of the following two cases hold: case 1) $n = 1$ and $K = \{1\}$, or case 2) $K = \Sigma_{\mathbf{a}}$ and $n = 0$.

Up to now we only know that G is homeomorphic either to $\mathbb{R}$ or $\Sigma_{\mathbf{a}}$. However, by what we already proved, it follows that G is a locally compact group of dimension 1. By a theorem of Montgomery (see [22]) such a group is either algebraically and topologically isomorphic to $\mathbb{R}$ or to a solenoid $\mathbb{S}$. Since $\mathbb{S}$ is Sharkovskii it is algebraically and topologically isomorphic to $\Sigma_{\mathbf{a}}$ by Corollary 49. □

Corollary 55. *If G is a nontrivial, Sharkovskii Lie group it must be $\mathbb{R}$.*

Proof. We must exclude that G is compact. If it were, then, being a compact Lie group, it contains a closed subgroup topologically isomorphic to the circle group S^1. This subgroup has an element of order 3, so G has an element of order 3 and is not torsion free, a contradiction. Hence G is not compact, and the result follows. □

Proof of Theorem 34. Let G be a locally compact Sharkovskii group. Therefore Theorem 54 implies that G either equals $\mathbb{R}$ or that G is a solenoid. If G is compact, then Corollary 49 implies that $G \cong \Sigma_{\mathbf{a}}$ for $\mathbf{a} = (2, 3, 4, \ldots)$, whereas if G is not compact, then it must be $\mathbb{R}$. □

Bibliography

[1] J. M. Aarts and R. J. Fokkink. The classification of solenoids. *Proc. Amer. Math. Soc.*, 111(4):1161–1163, 1991.

[2] Alseda, Juher, and Mumbru. Sets of periods for piece-wise monotone tree maps. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, 13:311–341, 2003.

[3] Alseda, Llibre, and Misiurewicz. Periodic orbits of maps of y. *Amer. Math. Soc.*, 313:475–538, 1989.

[4] Baldwin. An extension of sarkovskiı's theorem to the n-od. *Ergodic Theory Systems*, 11:249–271, 1991.

[5] Baldwin. Versions of sharkovskiı's theorem on trees and dendrites. *Topology Proc.*, 18:19–31, 1993.

[6] Stewart Baldwin. Versions of Sharkovskiĭ's theorem on trees and dendrites. *Topology Proc.*, 18:19–31, 1993.

[7] Valera Berestovskii and Conrad Plaut. Uniform universal covers of uniform spaces. *ScienceDirect*, 154:1748–1777, 2007.

[8] Ciesielski and Pogoda. On ordering the natural number, or, the sharkovskii theorem. *The Mathematical Association of America*, 115, 2008.

[9] Joan Cleary and Sidney A. Morris. Topologies on locally compact groups. *Bull. Austral. Math. Soc.*, 38(1):105–111, 1988.

[10] Gregory R. Conner, Christopher Grant, and Meilstrup. A sharkovsky theorem for non-locally connected spaces. *Discrete and Continuous Dynamical Systems*, 32, 2012.

[11] Demin. Coexistence of periodic, almost periodic and recurrent points of transformations of n-od. *Vestnik Moskov Univ. Ser. I Mat. Mekh.*, 51:84–87, 1996.

[12] Gillot and Llibre. Periods for maps of the figure-eight space. *World Sci. Ser. Nonlinear Sci. Ser. B Spec. Theme Issues Proc.*, 8:95–106, 1995.

[13] R. N. Gumerov. On finite-sheeted covering mappings onto solenoids. *Proc. Amer. Math. Soc.*, 133(9):2771–2778, 2005.

[14] Hatcher. *Algebraic Topology.* Cambridge University Press, New York, 2001.

[15] Edwin Hewitt and Kenneth A. Ross. *Abstract harmonic analysis. Vol. I: Structure of topological groups. Integration theory, group representations.* Die Grundlehren der mathematischen Wissenschaften, Bd. 115. Academic Press, Inc., Publishers, New York; Springer-Verlag, Berlin-Göttingen-Heidelberg, 1963.

[16] Tyler W. Hills. The Solenoid and Warsawanoid Are Sharkovskii Spaces. Master's thesis, Brigham Young University, 2015.

[17] Karl H. Hofmann and Sidney A. Morris. *The structure of compact groups*, volume 25 of *De Gruyter Studies in Mathematics*. De Gruyter, Berlin, 2013. A primer for the student—a handbook for the expert, Third edition, revised and augmented.

[18] W. T. Ingram. Periodic points for homeomorphisms of hereditarily decomposable chainable continua. *Proc. Amer. Math. Soc.*, 107:549–553, 1989.

[19] Lawaetz. The sharkovski theorem. *Aktuel Matematik*, 2008.

[20] Sibe Mardesic and Jack Segal. *Shape Theory: An Inverse System Approach*, volume 26 of *III*. North-Holland Publishing, Berlin, 1982.

[21] Misiurewicz. Periodic points of maps of degree one of a circle. *Ergodic Theory Dynamical Systems*, 2:221–227, 1982.

[22] Deane Montgomery. Connected one-dimensional groups. *Ann. of Math. (2)*, 49:110–117, 1948.

[23] Nitecki and Robinson. *Global Theory of Dynamical Systems*. Springer, New York, NY, 2006.

[24] Helga Schirmer. A topologist's view of Sharkovsky's theorem. *Houston J. Math.*, 11(3):385–395, 1985.

[25] Siegberg. Chaotic mappings on s^1, periods one, two, three imply chaos on s^1. *Numerical Solution of Nonlinear Equations (Bremen, 1980), Lecture Notes in Math.*, 878:351–370, 1981.

[26] Alseda Soler. *Periodic points of continuous mappings of the circle*. Publ. Sec. Univ. Autonoma Barcelona, 1981.

[27] Jin Cheng Xiong, Dong Ye Xiang, Zhi Qiang Zhang, and Jun Huang. Some dynamical properties of continuous maps on the warsaw circle. *Acta Math. Sinica (Chin. Ser.)*, 39:294–299, 1996.

[28] Li Zhen Zhou and You Cheng Zhou. Some dynamical properties of continuous self-maps on the k-warsaw circle. *J. Zhejiang Univ. Sci. Ed.*, 29:12–16, 2002.