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Mirror Symmetry for K3 Surfaces with Non-symplectic Automorphism

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ABSTRACT

Mirror Symmetry for Algebraic K3 Surfaces with Non-Symplectic Automorphism

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Mirror symmetry is the phenomenon, originally discovered by physicists, that Calabi-Yau manifolds come in dual pairs, with each member of the pair producing the same physics. Mathematicians studying enumerative geometry became interested in mirror symmetry around 1990, and since then, mirror symmetry has become a major research topic in pure mathematics. One important problem in mirror symmetry is that there may be several ways to construct a mirror dual for a Calabi-Yau manifold. Hence it is a natural question to ask: when two different mirror symmetry constructions apply, do they agree?

We specifically consider two mirror symmetry constructions for K3 surfaces known as BHK and LPK3 mirror symmetry. BHK mirror symmetry was inspired by the Landau-Ginzburg/Calabi-Yau correspondence, while LPK3 mirror symmetry is more classical. In particular, for algebraic K3 surfaces with a purely non-symplectic automorphism of order n, we ask if these two constructions agree. Results of Artebani-Boissière-Sarti [1] originally showed that they agree when $n = 2$, and more recently Comparin-Lyon-Priddis-Suggs [8] showed that they agree when n is prime. However, the n being composite case required more sophisticated methods. Whenever n is not divisible by four (or $n = 16$), this problem was solved by Comparin and Priddis by studying the associated lattice theory more carefully [9]. In this thesis, we complete the remaining case of the problem when n is divisible by four by finding new isomorphisms and deformations of the K3 surfaces in question, develop new computational methods, and use these results to complete the investigation, thereby showing that the BHK and LPK3 mirror symmetry constructions also agree when n is composite.

Keywords: Mirror Symmetry, Algebraic Geometry, Mathematical Physics, String Theory

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Contents

List of Tables

List of Figures

Chapter 1. Introduction

1.1 String Theory

When it comes to describing the known universe, physicists heavily rely on two, seemingly contradictory models: general relativity and quantum mechanics. In both cases, these theories boil down to describing the fundamental forces of nature. While general relativity quantifies gravitational force, quantum mechanics accounts for electromagnetic, as well as the weak and strong nuclear forces. In other words, general relativity is really good at handling the physics of "large" systems like planets and the universe as a whole, but quantum mechanics excels in the "small" atomic physics realm. The problem is that these models are formulated completely differently and while indispensable in their own spheres, no one has been able to unify them under a single, experimentally validated framework. In fact, such a "theory of everything" is sometimes referred to as "the holy grail of physics."

Perhaps the most popular candidate for a grand, unifying theory of physics is string theory, which models the point-like particles of particle physics with one-dimensional strings. There are a few equivalent formulations of string theory, but in each case the strings are topologically equivalent to either line segments or loops. Theoretically, these strings are Plank length (around 10^{-35} meters), and so from any larger scale appear like ordinary particles, with properties like mass and charge determined by the vibrational state of the string. Under this model, one of the vibrational states corresponds to the graviton, a quantum mechanical particle that carries gravitational force, allowing string theory to bring general relativity and quantum mechanics together under a single mathematical framework.

Although physicists are currently unable to perform experiments at the level of Plank length, string theory has had a remarkable impact on pure mathematics. In particular, one interesting and important feature of string theory is that it requires adding six extra dimensions of spacetime to the four dimensions (three spatial and one time) that we experience every day. A common analogy is that from far away, a garden hose appears to be one dimensional, but locally an ant on the surface of the hose can move in two directions, revealing that the hose has more dimensions than meet the eye from a distance. Similarly, it is theoretically possible that the known universe is more than four-dimensional. Now more specifically, the six extra dimensions of string theory are required to be compactified spaces called Calabi-Yau manifolds, which have been studied by mathematicians since at least the 1950s.

A **Calabi-Yau manifold** of (complex) dimension n is a compact, Kähler n-manifold with trivial canonical bundle. Note that the Calabi-Yau manifolds in string theory are of dimension three and are called threefolds, modeling the extra six (real) dimensions of spacetime. In the definition above, by Kähler we mean that Calabi-Yau manifolds have three mutually compatible, geometric structures: a complex structure, a Riemannian structure, and a symplectic structure. Further, the correction term in the famous Riemann-Roch theorem from algebraic geometry involves the dimension of the canonical bundle, so Riemann-Roch for Calabi-Yau manifolds is extremely nice since that term vanishes. In other words, mathematicians and string theorists alike are interested in Calabi-Yau manifolds because of their rich structure and the fact that they simplify important computations.

1.2 Mirror Symmetry

In the late 1980s, physicists Lance Dixon, Wolfgang Lerche, Cumrun Vafa, and Nick Warner noticed that given a compactification in string theory, it was impossible to uniquely construct a corresponding Calabi-Yau manifold [11], [19]. Instead, Calabi-Yau manifolds came in pairs: two versions of string theory could be compactified, each on a completely different Calabi-Yau manifold, and yet give rise to the same physics. This phenomenon of Calabi-Yau manifolds coming in dual (or "mirror") pairs was coined **mirror symmetry**. In other words, mirror symmetry states that there are two mathematically distinct ways of describing the same physical phenomenon.

Although discovered by physicists, mirror symmetry surprisingly led to solutions of old, unsolved problems in enumerative geometry, which studies combinatorial questions in algebraic geometry. For an example of a well known enumerative geometry result, nineteenth-century mathematicians Arthur Cayley and George Salmon proved that given any cubic surface, one can find precisely 27 straight lines on the surface [7], [21]. Generalizing this problem in the nineteenth century, German mathematician Hermann Schubert showed that every quintic Calabi-Yau threefold (degree five, three dimensional hypersurface in $\mathbb{P}^n(\mathbb{C})$) has exactly 2,875 lines. Further, in 1986 Sheldon Katz proved that the number of (projective) curves of degree two on the quintic Calabi-Yau threefold is 609,250, but attempts to further generalize this result to curves of higher degree became computationally infeasible for mathematicians [26].

In 1991, physicists Philip Candelas, Xenia de la Ossa, Paul Green, and Linda Parks showed that mirror symmetry could be used to translate difficult mathematical questions about one Calabi–Yau manifold into easier questions about its mirror [6]. In particular, they used mirror symmetry to find that the number of curves of degree three on the quintic Calabi-Yau threefold is 317,206,375. Actually, some mathematicians had published a contradicting number for degree three, but it turned out there was an error in their computer code and the physicists were actually right. In fact, the mirror symmetry technique was quickly used to find the number of curves of degree d on a quintic threefold for $d \leq 9$. Ultimately mathematicians caught on to the usefulness of physical intuition, and since then mirror symmetry has grown to be an active research area in pure mathematics. In fact, progress in mirror symmetry has led to the Fields medals of geometer Maxim Kontsevich and physicist Edward Witten, the only physicist ever to win the award.

1.3 Big Questions

Since mirror symmetry is the phenomenon that Calabi-Yau manifolds come in dual pairs, there are a couple of big questions that are natural to ask:

(i) Given a Calabi-Yau manifold, can we find or construct its dual?

(ii) If there are different mirror symmetry constructions for finding a dual, can we show that the duality theories are equivalent?

As it turns out, the answer to (i) is yes, but (ii) is not so easy. Although we can always find a mirror for a given Calabi-Yau manifold, in certain situations there are many mirror symmetry constructions that have been developed by mathematicians, which is a fruitful ground for research.

This thesis gives the answer to (ii) in the affirmative for two specific mirror symmetry constructions on a class of Calabi-Yau manifolds.

In **Chapter 2**, we define this class of Calabi-Yau manifolds, called algebraic K3 surfaces with purely non-symplectic automorphism, and discuss previous results. K3 surfaces have a rich theory and are the closest Calabi-Yau manifolds (only 1 dimension less) to threefolds (the main space of interest in string theory). Thus, understanding their mirror symmetry will be a major stepping stone in understanding that of threefolds.

In **Chapter 3**, we define the two mirror symmetry constructions we use in this thesis. The first, Berglund-Hübsh-Krawitz (BHK) Mirror Symmetry, applies to any quasismooth hypersurface in weighted projective space. BHK mirror symmetry was inspired by the Landau-Ginzburg/Calabi-Yau correspondence and is more general, applying to a large class of Calabi-Yau manifolds in every dimension. In contrast, Dolgachev's lattice polarized K3 (LPK3) Mirror Symmetry applies only to lattice polarized K3 surfaces.

In **Chapter 4**, we introduce new techniques for producing particular isomorphisms and deformations for certain K3 surfaces. We prove that these techniques preserve the structure we care about and outline our proof of our main conjecture.

In **Chapter 5**, we provide the details of our computations, completing the proof that the BHK and LPK3 mirror symmetry constructions coincide for algebraic K3 surfaces with purely non-symplectic automorphism.

Chapter 2. Algebraic K3 Surfaces with Purely Non-Symplectic Automorphism

2.1 K3 Surfaces

In Chapter 1, we defined general Calabi-Yau manifolds and discussed how in string theory, three-dimensional Calabi-Yau manifolds (threefolds) play a prominent role in describing the nature of the universe. However, mirror symmetry applies to Calabi-Yau manifolds of all dimensions, each dimension bringing with it rich geometry and therefore interesting mathematical questions. In particular, every Calabi-Yau manifold of dimension two is either an abelian surface or a K3 surface, the latter being the object of study in this thesis. To distinguish between the two cases, abelian surfaces are not simply connected while K3 surfaces are, so that condition is often used to distinguish them (although there are several other equivalent definitions). Putting everything together a **K3 Surface** is defined as a compact, simply connected, Kähler surface with trivial canonical bundle.

Just like Calabi-Yau threefolds, K3 surfaces are ubiquitous in string theory, due to the fact that string duality and compactification in these spaces is nontrivial, yet simple enough to analyze properties in detail. At the same time; however, K3 surfaces have been studied extensively and classically by mathematicians due to their rich, complicated structure. For example, Kunihiko Kodaira showed that all K3 surfaces are diffeomorphic, completing their classification in the context of differential geometry [16]. Additionally, their second cohomology groups are particularly nice invariants: given a K3 surface X, $H^2(X, \mathbb{Z})$ is free of rank 22, the Hodge numbers of X are $h^{2,0}(X) = h^{0,2}(X) = 1$, $h^{1,1}(X) = 20$, and $h^{1,0}(X) = h^{0,1}(X) = 0$ (note that all interesting behavior occurs in the second cohomology group). Further, the Euler characteristic of X is 24, the Picard group of X coincides with the Néron-Severi group, and both are torsion-free.

André Weil was the first to study K3 surfaces (in 1957), and named them "in honor of Kähler, Kummer, Kodaira and the beautiful Mount K2 in Kashmir", since they too are hard to master [25].

2.2 Some Algebraic Geometry

We now define algebraic K3 surfaces, which will be our main object of study in this thesis. In contrast to general K3 surfaces, which as complex manifolds are hard to describe and understand, algebraic K3 surfaces are much easier to work with, due to the fact that they are also algebro-geometric structures. Although most K3 surfaces are not algebraic, this subclass is particularly important in string theory, and is required for our subsequent mirror symmetry constructions. All of the K3 surfaces in this thesis will come from hypersurfaces in weighted projective space. We review that construction now, starting with ordinary projective space.

Complex projective n-space, denoted $\mathbb{P}^n$, is the set of lines through the origin (one-dimensional subspaces) in $\mathbb{C}^{n+1}$. More specifically, $\mathbb{P}^n = (\mathbb{C}^{n+1}\backslash\{0\})/\mathbb{C}^*$, where $(x_0, \ldots, x_n) \sim (\lambda x_0, \ldots, \lambda x_n)$ for all $\lambda \in \mathbb{C}^*$.

Projective n-space is easily given the structure of a compact, complex n-dimensional manifold (in fact it is Kähler). We denote the **homogeneous coordinates** of $\mathbb{P}^n$ by $[x_0, \ldots, x_n]$ (note that there are $n+1$ coordinates since we start indexing at 0, and that there is no origin in projective space). Further, $\mathbb{P}^n$ is important in algebraic geometry as the ambient space for projective varieties, which are the main objects of study in classical algebraic geometry. Note that any nonconstant polynomial $f \in \mathbb{C}[x_0, \ldots, x_n]$ does not give a function from $\mathbb{P}^n$ to $\mathbb{C}$ via evaluation, since $f(a_0, \ldots, a_n) \neq f(\lambda a_0, \ldots, \lambda a_n)$ for all $\lambda \in \mathbb{C}^*$. However, if f is **homogeneous** of degree d, meaning that each monomial has total degree d, the zero locus of f is well-defined, since if $f(a_0, \ldots, a_n) = 0$, $f(\lambda a_0, \ldots, \lambda a_n) = \lambda^d f(a_0, \ldots, a_n) = 0$ as well.

A **homogeneous ideal** is an ideal of $\mathbb{C}[x_0, \ldots, x_n]$ that is generated by homogeneous polynomials. For any homogeneous ideal $I \subseteq \mathbb{C}[x_0, \ldots, x_n]$, define

$$V(I) = \{[a_0, \ldots, a_n] \in \mathbb{P}^n : f([a_0, \ldots, a_n]) = 0$$

for all $f \in I\}$. We say that $S \subseteq \mathbb{P}^n$ is a **projective variety** if $S = V(I)$ for some homogeneous ideal $I \subseteq \mathbb{C}[x_0, \ldots, x_n]$. Additionally, Hilbert's Basis Theorem from commutative algebra gives that $\mathbb{C}[x_0, \ldots, x_n]$ is a Nötherian ring, one consequence being that each of its ideals is finitely generated. Hence, every homogeneous ideal $I \subseteq \mathbb{C}[x_0, \ldots, x_n]$ is of the form $I = (f_1, \ldots, f_k)$ for some homogeneous $f_1, \ldots, f_k \in \mathbb{C}[x_0, \ldots, x_n]$. In other words, every projective variety is of the form $V(f_1, \ldots, f_k)$ for some homogeneous $f_1, \ldots, f_k$, which gives intuition behind why it is often stated that algebraic geometry is the study of solutions to systems of polynomial equations.

In the case of K3 surfaces which are **hypersurfaces** in projective space (the zero locus of a single homogeneous polynomial, i.e. of the form $V(f)$), it is natural to ask if their dual Calabi-Yau manifold (which will always be a K3 surface) can also be embedded as a hypersurface in projective space. Unfortunately, the answer to this question in general is "no". However, the dual will always be embedded in a slightly more general space, called a (complex) weighted projective space. Let $q_0, \ldots, q_n \in \mathbb{Z}^{\geq 1}$. **Weighted projective n-space with weights** $(q_0, \ldots, q_n)$, denoted by $W\mathbb{P}^n(q_0, \ldots, q_n)$, is defined to be $(\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*$, where $(x_0, \ldots, x_n) \sim (\lambda^{q_0} x_0, \ldots, \lambda^{q_n} x_n)$ for all $\lambda \in \mathbb{C}^*$.

Note that weighted projective space is a generalization of projective space, since $\mathbb{P}^n = W\mathbb{P}^n(1, \ldots, 1)$. With this new group action of $\mathbb{C}^*$ (via λ) on $(\mathbb{C}^{n+1} \setminus \{0\})$, to again get zero loci of polynomials to be well-defined, we have to broaden our class of polynomials. This time, instead of homogeneous polynomials, we need quasihomogeneous polynomials to get projective varieties in weighted projective space.

Definition 2.1. A polynomial $f \in \mathbb{C}[x_0, \ldots, x_n]$ is **quasihomogeneous** with weight system $(q_0, \ldots, q_n; d)$ if $\gcd(q_0, \ldots, q_n) = 1$, $q_0 + \ldots + q_n = d$, and $f(\lambda^{q_0} x_0, \ldots, \lambda^{q_n} x_n) = \lambda^d f(x_0, \ldots, x_n)$ for all $\lambda \in \mathbb{C}^*$.

Again, note that a homogeneous polynomial of degree d is a special case of quasihomogeneous polynomial, with weights $(1, \ldots, 1; d)$. We then get projective varieties of the form $V(f_1, \ldots, f_k)$ in weighted projective space, this time where $f_1, \ldots, f_k \in \mathbb{C}[x_0, \ldots, x_n]$ are quasihomogeneous with the same weights, but with possibly different degrees.

We now turn our attention to our algebraic K3 surfaces, which are are resolutions of hypersurfaces of the form $V(f)$. Since the zero locus of a single polynomial will have codimension one, this means that our K3 surface will specifically be embedded in $W\mathbb{P}^3(q_0, \ldots, q_3)$ for some weights $(q_0, \ldots, q_3)$, so in particular f will be a four variable polynomial. We have already mentioned that f is required to be quasihomogeneous, but what other restrictions are necessary for $V(f)$ to be a K3 surface? As it turns out, Reid (in an unpublished work) and Yonemura [27] each independently classified that there 95 distinct weight systems that will give rise to algebraic K3 surfaces that are hypersurfaces. For each of these weight systems, one can then list all polynomials which are quasihomogeneous with respect to that weight system.

2.3 Our Main Conjecture

Given a K3 surface X and an automorphism σ, we get an induced Hodge isometry σ^* which preserves $H^2(X, \mathbb{Z})$, i.e. $\sigma^* \omega_X = \lambda_\sigma \omega_X$ for some $\lambda_\sigma \in \mathbb{C}^*$. We call σ **symplectic** if $\lambda_\sigma = 1$ and **non-symplectic** otherwise. If σ is an automorphism with non-prime order n, we say σ is **purely non-symplectic** if $\lambda_\sigma = \zeta_n$, with ζ_n a primitive nth root of unity.

Among the K3 surfaces resulting from Reid and Yonemura's classification of weight systems, certain hypersurfaces stood out as being of particular interest, due to the fact that they have purely non-symplectic automorphism. Specifically, they are hypersurfaces defined by invertible polynomials f, which are of the form $f = x^n + g(y, z, w)$. For such hypersurfaces, there is a group action of $\mathbb{Z}/n\mathbb{Z}$, viewed as the nth roots of unity under multiplication, on $V(f)$ by $\zeta_n \cdot (x, y, z, w) = (\zeta_n x, y, z, w)$. Note that f is invariant under this group action, revealing a purely non-symplectic automorphism of the K3 surface $V(f)$. K3 surfaces with such automorphisms have even richer geometric structure.

In mirror symmetry specifically, algebraic K3 hypersurfaces of the form $V(f)$ with $f(x, y, z, w) = x^n + g(y, z, w)$ have two very different mirror constructions. We will go into detail about these constructions (called BHK and LPK3 mirror symmetry, respectively) next chapter, but for now it suffices to mention that BHK mirror symmetry applies to Calabi-Yau hypersurfaces in general weighted projective spaces, while LPK3 mirror symmetry applies to lattice polarized K3 surfaces (which our algebraic K3 hypersurfaces are). Hence in this setting, we have the situation that both BHK and LPK3 mirror symmetry constructions can be applied. We then come back to our "big questions" in Chapter 1. When do these two mirror constructions agree? We have the following conjecture.

Conjecture 2.2. *Let $V(f)$ be an algebraic K3 hypersurface with $f(x, y, z, w) = x^n + g(y, z, w)$. Then, the dual K3 surfaces given by the BHK and LPK3 mirror symmetry constructions agree.*

There has already been a lot of progress towards the solution of this conjecture. The first paper in 2011 was by Artebani-Boissière-Sarti [1],and originally showed that the conjecture is true for $n = 2$. Subsequently, Comparin-Lyon-Priddis-Suggs [8] showed that the conjecture holds when n is prime. Finally, Comparin and Priddis have recently shown that the constructions agree for all composite n, excluding the cases when n is 4, 8, or 12 [9]. While these numbers may seem arbitrary, it should be noted that orders 8 and 12 heavily rely on order 4, so $n = 4$ is where much of the difficulty lies.

Chapter 3. Mirror Symmetry Constructions

3.1 Berglund-Hübsch-Krawitz Mirror Symmetry

We now define the BHK mirror symmetry construction. Specifically, this construction applies to Calabi-Yau manifolds coming from hypersurfaces defined by a polynomial satisfying certain conditions (which we will later call an invertible polynomial), and a group of symmetries which acts on the hypersurface. Recall from Chapter 2 that a hypersurface in weighted projective space is of the form $V(f) \subseteq W\mathbb{P}^n(q_0, \ldots, q_n)$ where for the zero locus of f to be well-defined, we require f to be quasihomogeneous with integer weights $(q_0, \ldots, q_n)$. Further, we want these weights to be unique to avoid polynomials with terms like $x_0 x_1 \ldots x_n$ which could have infinitely many weight systems.

Though given by the vanishing of quasihomogeneous polynomials, hypersurfaces are not in general complex manifolds since projective varieties generally have singularities, i.e. points where the dimension of the tangent space may have an inconsistent dimension. Intuitively, we can visualize some of the possibilities for singularities as being cusps or self-intersection points. In order for a hypersurface given by the vanishing set of a quasihomogeneous polynomial to be a complex manifold, we must impose an additional constraint on f called nondegeneracy, guaranteeing that the only possible singularities of $V(f)$ are those in common with the singularities of its ambient space $W\mathbb{P}^n(q_0, \ldots, q_n)$, which in general are singular spaces. Hence hypersurfaces will be called **quasismooth**, and not just "smooth."

Definition 3.1. A polynomial $f \in \mathbb{C}[x_0, \ldots, x_n]$ is **nondegenerate** if $\nabla f = 0$ only has $(0, \ldots, 0)$ as a solution.

Note that normally, $\nabla f = 0$ at the origin would indicate that $V(f)$ has a singularity there. However, recall that weighted projective spaces do not contain an origin: we first omit the origin from $\mathbb{C}^{n+1}$ before making the quotient.

Given a polynomial $f = \sum_{i=0}^{m} \prod_{j=0}^{n} x_j^{a_{ij}} \in \mathbb{C}[x_0, \ldots, x_n]$, we can construct an **exponent matrix** $A = (a_{ij})$ that has the exponents for the monomials of f as rows, with each variable representing a column.

Due to the fact that an exponent matrix encodes all the information we need to construct a polynomial (we will soon describe how coefficients can be omitted for our purposes), for notational convenience, instead of writing f, we may write F_A for a generic polynomial with exponent matrix A.

In order to get a nice mirror symmetry theory, we put further constraints on F_A, requiring the polynomial to have the same number of monomials as variables. This way, A is a square matrix, which will allow us to form a dual Calabi-Yau manifold by finding a "dual polynomial" for F_A. Further, our earlier distinction that we want quasihomogeneous polynomials to have unique weights makes A invertible if A is square (which can be proven from the atomic type classification). With all of these constraints and their motivations in mind, we are ready to define our final classes of multivariate polynomials.

Definition 3.2. A polynomial $F_A \in \mathbb{C}[x_0, \ldots, x_n]$ is **admissible** if it is quasihomogeneous with unique weights, and nondegenerate. An admissible polynomial with square matrix A is called **invertible**, and an admissible polynomial whose exponent matrix A is not square is called **non-invertible**.

There is an extremely useful classification of invertible polynomials that we will also use.

Definition 3.3. The following types of quasihomogeneous polynomials are called **atomic types**:

(i) **Fermat**: $f = x^n$

(ii) **Loop**: $f = x_0 x_1^{a_1} + x_1 x_2^{a_2} + \ldots + x_{n-1} x_n^{a_n} + x_n x_0^{a_0}$ where $a_i \geq 2$ for all i

(iii) **Chain**: $f = x_0^{a_0} + x_0 x_1^{a_1} + x_1 x_2^{a_2} + \ldots + x_{n-1} x_n^{a_n}$ where $a_i \geq 2$ for all i

Proposition 3.4 (Kreuzer-Skarke [18]). *A polynomial $f \in \mathbb{C}[x_0, \ldots, x_n]$ is invertible if and only if it can be written as a finite sum of atomic types in distinct variables.*

For our current purposes, we will require that all of our polynomials F_A are invertible. In algebraic geometry, hypersurfaces are **projectively equivalent** if there is a linear change of coordinates that takes the defining polynomial of one to the other. For hypersurfaces defined by invertible polynomials, there is such a change of variables, allowing us to to scale all coefficients to 1 (as may have been noticed in the atomic types above). We do not distinguish between projectively equivalent hypersurfaces.

Given an invertible polynomial F_A, we can define a "dual" polynomial F_{A^T} by transposing the exponent matrix A to get a new exponent matrix A^T. The dual polynomial is then simply the corresponding polynomial for A^T (monomials corresponding to rows, variables to columns). With the conditions above satisfied, F_{A^T} will also be an invertible polynomial (which again can be proven from the atomic type decomposition).

Finding this dual polynomial construction seems simple, but was a major breakthrough in mirror symmetry from Burglund and Hübsch [4]. Greene and Plesser were the first ones to find a dual Calabi-Yau manifold explicitly, and did so for the polynomial $F_A = x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5$ (the "Fermat quintic"), which was of particular interest in string theory [13]. However, their work could not be generalized because they used F_A as its own dual polynomial, which does not work for most cases.

For a more interesting example, let $F_A = x^3w + y^4 + z^4 + w^4$, which is invertible since it is the sum of the "chain" $x^3w + w^4$ and the two "Fermats" y^4 and z^4. Furthermore, F_A has weights $(1, 1, 1, 1)$ since $F_A(\lambda x, \lambda y, \lambda z, \lambda w) = \lambda^4 F_A(x, y, z, w)$. In particular, $W\mathbb{P}^3(1, 1, 1, 1) = \mathbb{P}^3$ (this is because F_A is homogeneous, not just quasihomogeneous).

Now, the exponent matrix is given by $A = \begin{bmatrix} 3 & 0 & 0 & 1 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$, so $A^T = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 1 & 0 & 0 & 4 \end{bmatrix}$
.

By the definition of an exponent matrix, we then can see from the rows and columns of A^T that $F_{A^T} = x^3 + y^4 + z^4 + xw^4$, which again is invertible since it is the sum of the chain $x^3+xw^4 = w^4x+x^3$ and the two Fermats y^4 and z^4. However, F_{A^T} is not homogeneous, since it has monomials of degree 3,4,4, and 5, respectively. However, it is quasihomogeneous with weights $(4,3,3,2;12)$ since $F_{A^T}(\lambda^4 x, \lambda^3 y, \lambda^3 z, \lambda^2 w) = \lambda^{12} F_{A^T}(x,y,z,w)$. In other words, for BHK mirror symmetry we really do need quasihomogeneous polynomials and weighted projective space, since even the dual of an invertible homogeneous polynomial is not in general homogeneous, but always will be quasihomogeneous.

From now on we call invertible polynomials W instead of F_A for notational convenience. As far as complex manifolds are concerned, weighted projective space does have some problems. Recall that for an invertible polynomial W, the corresponding variety $V(W)$ is a subspace of $W\mathbb{P}^n(q_0, \ldots, q_n)$, which is in general a singular space. Hence, $V(W)$ may be quasismooth due to nondegeneracy, but not smooth in the sense of complex manifolds. To remedy this situation, we use the "blow up" construction from algebraic geometry.

As a motivating example, the cone $V(x^2+y^2-z^2) \in \mathbb{C}^3$ is a variety that is not smooth if viewed in $\mathbb{C}^3$ (instead of $\mathbb{P}^2$), because it has a singularity at the origin (see Figure 3.1. To get a smooth surface, we "blow up" the origin, which means replace it with a copy of $\mathbb{P}^1$, which is homeomorphic to the Riemann sphere. After the blow up, the resulting surface is then homeomorphic (topologically equivalent) to a cylinder, which is a smooth manifold. Now of course the two surfaces are not homeomorphic, which is the first criteria for varieties to be isomorphic (they also have algebraic structures that need to be preserved). However, away from the origin, the two surfaces are isomorphic (they are even identical). In algebraic geometry, isomorphism of varieties (which is already a finer equivalence than projective equivalence) is often too difficult a condition to check, but if two varieties are isomorphic on dense open subsets (in the Zariski topology, these are complements of varieties), there is another equivalence relation between them that is called **birational equivalence** that algebraic geometers use for classification. Hence, after blowing up the origin on the cone, we see that the cone is birationally equivalent to the resulting surface, which in this case is a cylinder.

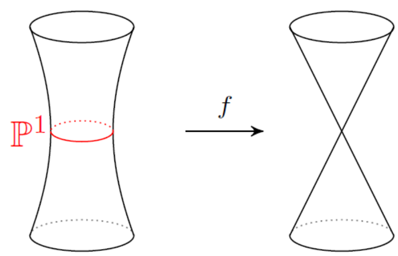

Figure 3.1: Blowing Up Singularities

For an invertible polynomial W satisfying the Calabi-Yau condition, the corresponding variety $V(W) \subseteq W\mathbb{P}^n(q_0, \dots, q_n)$ has singularities to blow up (which it has in common with its ambient weighted projective space). After this blow up procedure, we get a new variety that is birationally equivalent to the original one. We denote this new surface by Z_W. However, unlike $V(W)$, Z_W is a Calabi-Yau manifold embedded in projective space of some large dimension, which is expected since every compact complex manifold can be embedded in projective space.

Now that we have a Calabi-Yau manifold Z_W associated to every invertible polynomial $W = F_A$, we need a dual manifold construction. Earlier we found a dual polynomial $W^T = F_{A^T}$ which will be key in the BHK mirror symmetry construction, but it turns out that Z_{W^T} is not the desired mirror for Z_W (it doesn't produce the same physics or duality on the level of cohomology). Instead, it turns out that we need to act on Z_{W_T} by a group of symmetries and again blow up singularities to get the desired dual. To do so, there are several important groups that we now define that are associated to every invertible polynomial W.

Definition 3.5. Let W be an invertible polynomial with weights $(q_0,\ldots,q_n;d)$ (i.e. $W(\lambda^{q_0}x_0,\ldots,\lambda^{q_n}x_n)=\lambda^d W(x_0,\ldots,x_n)$.)

(i) The **maximal symmetry group** $G_W^{\max}$ for W is defined by

$$G_W^{max}=\{(g_0,\ldots,g_n)\in\mathbb{C}^{n+1}:W(g_0x_0,\ldots,g_nx_n)=W(x_0,\ldots,x_n)\}.$$

(ii) Viewing the elements of $G_W^{\max}$ as diagonal matrices, the **special linear symmetry group** for W is $SL_W=G_W^{\max}\cap SL_{n+1}(\mathbb{C})$, where $SL_{n+1}(\mathbb{C})$ is the special linear group (the $(n+1)\times(n+1)$ invertible matrices with complex entries and determinant 1).

(iii) The **exponential grading operator group**, is denoted $J_W=\langle(e^{2\pi i q_0/d},\ldots,e^{2\pi i q_n/d})\rangle$.

There is much to be said about these symmetry groups. First of all, each of these groups can be represented by diagonal matrices and the usual matrix multiplication, and so are abelian. Next, it can be shown that the entries of these matrices are all roots of unity, so $(g_0,\ldots,g_n)$=$(e^{2\pi i\cdot a_0},\ldots,e^{2\pi i\cdot a_n})$ for some $a_0,\ldots,a_n\in\mathbb{Q}/\mathbb{Z}$. Since multiplication of roots of unity corresponds to addition of exponents, for convenience we often will write these groups additively with elements $(a_0,\ldots,a_n)\in(\mathbb{Q}/\mathbb{Z})^{n+1}$. With this view in mind, and viewing these groups as acting on $\mathbb{C}^{n+1}$, $G_W^{\max}$ represents the tuples that send points of Z_W to Z_W (i.e. the action lifts to an action on Z_W), SL_W is the subgroup with entries that add up to 1, and $J_W=\langle q_0/d,\ldots,q_n/d\rangle$ (again viewed additively).

We now present some nice properties of $G_W^{\max}$ that will be used hereafter.

Proposition 3.6 (Krawitz [17])**.** *Let W be an invertible polynomial with exponent matrix A_W and $G_W^{\max}$ its maximal symmetry group, viewed additively.*

(i) $|G_W^{\max}|=|\det(A_W)|$. *In particular, $G_W^{\max}$ is a finite abelian group.*

(ii) $G_W^{\max}$ is generated by the columns of A_W^{-1}.

Any subgroup of $G_W^{\max}$ can act on Z_W However, for the resulting quotient space to be a Calabi-Yau manifold, the group must satisfy the "Calabi-Yau condition", which concretely means that the subgroup must also be a subgroup of SL_W. Note also that the equivalence relation defining points in weighted projective space is just the action of J_W, so J_W acts trivially on Z_W. Thus we require symmetry groups $\tilde{\mathrm{G}}$ acting on Z_W to be of the form $\tilde{\mathrm{G}}= G/J_W$, $J_W \leq G \leq SL_W$. Elements of $\tilde{\mathrm{G}}$ are the equivalence classes where $(g_0, \ldots, g_n) \sim (h_0, \ldots, h_n)$ if there exists some $j \in J_W$ such that $j \cdot (g_0, \ldots, g_n) = (h_0, \ldots, h_n)$. Note further that if we take $G = J_W$, $\tilde{\mathrm{G}}= \{0\}$, so $Z_W/\{0\} = Z_W$. In the case where $G \neq J_W$, the group action may introduce new singularities. Thus, if we blow up singularities one more time, the final variety, denoted $Z_{W,G}$, is a Calabi-Yau manifold.

Recall that we mentioned how the dual to the Calabi-Yau manifold $Z_W = Z_{W,J_W}$ is not Z_{W^T}, but instead a quotient of it. In other words, we will find a dual for Z_W that is of the form $Z_{W',G'}$ for some polynomial W' and some group $J_{W'} \leq G' \leq SL_{W'}$. As can be expected from our above discussion, W' will be the dual polynomial W^T of W, but we have not yet determined how to compute a dual group. In fact, finding a dual group was a huge breakthrough in mirror symmetry, given in the PhD dissertation of Mark Krawitz [17]. We give the definition and some important properties below.

Definition 3.7. For an invertible polynomial W with exponent matrix A_W and subgroup $G \leq G_W^{\max}$, we define the **dual group** to be $G^T = \{g \in G_{W^T}^{\max} : gA_W h^T \in \mathbb{Z} \text{ for all } h \in G\}$.

While the definition of the dual group may be a bit confusing, it does in fact have some very nice properties:

Proposition 3.8 (Artebani-Boissiére-Sarti [1])**.** *Let W be an invertible polynomial and $G, G_1, G_2 \subseteq G_W^{\max}$.*

(i) $(G^T)^T = G$

(ii) If $G_1 \subseteq G_2, G_2^T \subseteq G_1^T$, and $G_2/G_1 \cong G_1^T/G_2^T$

(iii) $(G_W^{\max})^T = \{0\} \subseteq G_{W^T}^{\max}$

(iv) $\{0\}^T = G_{W^T}^{\max}$

(v) $(J_W)^T = SL_{W^T}$

(vi) $(SL_W)^T = J_{W^T}$

In particular, if $J_W \leq G \leq SL_W$, we have that $J_{W^T} \leq G^T \leq SL_{W^T}$. Hence for the Calabi-Yau manifold of the form $Z_{W,G}$ with $J_W \leq G \leq SL_W$, we can define the **BHK mirror** (dual manifold) to be Z_{W^T,G^T}. In particular, we have that the dual of $Z_W = Z_{W,J_W}$ is Z_{W^T,SL_W}, and even more importantly, we will later be able to construct algebraic K3 surfaces and their BHK duals in this fashion.

3.2 Non-invertible Polynomials and Non-abelian Groups in BHK Mirror Symmetry

We now take a quick detour to discuss some current work in generalizing BHK mirror symmetry, including our resolution of a conjecture in this area. Recall that in the last section, Calabi-Yau manifolds were constructed from an invertible polynomial W and a group of symmetries G. In our previous discussion, we only considered diagonal groups of symmetries, which are abelian, but there was no physical or mathematical reason to do so, except for simplifying the theory and computations. By using faithful representations of non-abelian groups, it is conjectured that the BHK mirror symmetry constructions as given above can be generalized to produce a dual group for certain non-abelian groups.

In addition to BHK mirror symmetry on $Z_{W,G}$, there are two algebraic structures associated to these Calabi-Yau manifolds that are also important in string theory, called the **A-model** and the **B-model**. The A-model and B-model are graded Frobenius algebras (over $\mathbb{C}$) and contain a lot of important geometric and physical information, via the categorical equivalence between graded Frobenius algebras and certain topological quantum field theories. The classical construction of the A-model is closely related to Gromov-Witten Theory, while the B-model is well-known from singularity theory. However, due to the Landau-Ginzburg/Calabi-Yau correspondence, there is another equivalent (and easier to compute) A-model for each Calabi-Yau manifold $Z_{W,G}$.

The construction of this new A-model was another breakthrough in mirror symmetry given by Fan-Jarvis-Ruan, named **FJRW theory** (the W is for Edward Witten) [12]. With this new A-model construction came another mirror symmetry theory for each $Z_{W,G}$ called Landau-Ginzburg mirror symmetry, which states that the A-model associated to each $Z_{W,G}$ should be isomorphic as graded Frobenius algebras to the B-model associated to Z_{W^T,G^T}, and there has been a lot of success in proving this, assuming as in the last section that G is abelian.

Although we can construct $Z_{W,G}$ when G is a non-abelian group of symmetries acting on W, the FJRW theory A-model is much more complicated in this case and deals with gauged linear sigma models. There have also been attempts at defining a B-model in such cases [5], [22], [10]. Ongoing work by Nathan Priddis and some of his other students is also promising in this area.

In addition to generalizing BHK mirror symmetry to include non-abelian groups, efforts have also been made to generalize the situation to allow polynomials W that are non-invertible. Recall that a non-invertible polynomial does not need to have the same number of monomials as variables. Unlike the case for non-abelian groups, there is no current candidate for W^T for non-invertible W (the transpose of the exponent matrix does not usually give an admissible polynomial as in the case for invertible polynomials). However, one possible way around this (until such a dual polynomial construction is discovered) is the breakthrough by Julian Tay in his Master's thesis that the FJRW theory A-model only depends on the symmetry group and the weights of the polynomial, not the polynomial itself. The following proposition is then called the **Group-Weights Theorem**.

Proposition 3.9 (Tay [24])**.** *Let W_1 and W_2 be admissible polynomials (invertible or non-invertible) with some shared fixed group of symmetries G (up to isomorphism). Then, the A-models associated to $Z_{W_1,G}$ and $Z_{W_2,G}$ are isomorphic as graded Frobenius algebras.*

As a consequence of the Group-Weights Theorem, at least at the level of A-models, we can replace a non-invertible polynomial with some group of symmetries with an invertible polynomial, provided that the new polynomial also shares that group of symmetries. In particular, since J_W is a shared symmetry group among all admissible polynomials with the same weights, the theory is especially nice. Hence if $G_W^{\max} = J_W$ for some non-invertible polynomial W, the Group-Weights Theorem states that the invertible theory is enough to classify all A-models (as long as there is an invertible polynomial in the same weight class, which "usually" is the case), leading to our next definition.

Definition 3.10. Let W be a non-invertible polynomial. W is called a **J-singularity** if $G_W^{\max} = J_W$. On the other hand, if $G_W^{\max} \neq J_W$, we call W a **C-singularity**.

Note that the definition applies only to non-invertible polynomials, not invertible ones. While it is well understood that for three or more variables there are many C-singularities, for the two variable case there were no known examples. In fact due to a large number of computations showing that all computed two variable non-invertible polynomials were J-Singularities, the following conjecture (also known as the **C-singularity conjecture**) was given by Tyler Jarvis (personal communication):

Conjecture 3.11. *In two variables, there are no C-Singularities, i.e. all non-invertible W have $G_W^{\max} = J_W$.*

If the C-singularity conjecture were true, again in the two variable case A-models for two variable non-invertible polynomials would be able to be computed with invertible representatives via the Group-Weights Theorem, completing Landau-Ginzburg mirror symmetry for non-invertible polynomials. However, in an unexpected turn of events, the author was able to show that the C-singularity conjecture was in fact false. While computing a list of all

A-models with weights up to a small fractional size in an attempt to classify all two variable A-model isomorphisms, we came across a few examples of non-invertible polynomials that did not have J_W as their maximal symmetry group. It turns out that these C-singularities were particularly rare, which explains why they had remained undetected for so long.

For example, in the weight system $(1, 1; 28)$, among all the non-invertible polynomials with those weights, only 0.002% were C-singularities. Additionally for this weight system, only three A-models (up to isomorphism) came from invertible polynomials, while eight more came from C-singularities, revealing that C-singularities (though unexpected) provide the majority of distinct A-models for two variable weight systems. Once C-singularities were discovered in two variables, we were also able to find a pattern among the ones we found and were able to classify all of them by finding necessary and sufficient conditions.

Definition 3.12. Let $W(x, y) = x^{a_1}y^{b_1} + x^{a_2}y^{b_2} + \ldots + x^{a_n}y^{b_n}$. The **difference GCD** of W, denoted $dg(W) = GCD(a_i - a_j)$ running over all $i \neq j$.

Note here that $dg(W)$ only depends on the exponents for one of the variables. In our definition above we used the exponents of x, but just as well we could have used the exponents of y. For consistency in this thesis we will only use $dg(W)$ in terms of the exponents of x, as defined above. Now that we have $dg(W)$, we come to our main theorem for non-invertible polynomials.

Theorem 3.13. *Let W be a non-invertible polynomial in two variables. Then, W is a C-singularity if and only if $dg(W) \geq 1$.*

This theorem takes some time to justify and is an aside from the main topic of this thesis. It is further original work done jointly with Benjamin Pachev that was never published, so with his permission we have included a proof in full detail in the appendix.

With necessary and sufficient conditions for two variable C-singularities, there is hope for a similar criterion in more variables. Additionally, it is conjectured that C-singularities with the same weights and difference GCDs have isomorphic A-models (this has been verified in several cases), providing another potential source of A-model isomorphisms previously unexpected. Perhaps more importantly, this reveals that there is still more interesting work to be done in BHK mirror symmetry.

3.3 Lattice Polarized K3 Mirror Symmetry

We now depart from BHK mirror symmetry which concerns Calabi-Yau manifolds that are algebraic hypersurfaces, and instead consider mirror symmetry specifically for K3 surfaces. We also impose the condition that our K3 surfaces be lattice polarized, since we will be summarizing Dolgachev's lattice polarized K3 (LPK3) mirror symmetry. In particular, this implies that all the K3 surfaces we will be considering are algebraic. The lattices associated to our K3 surfaces are key for our future discussion, so we take the time here to develop some required lattice theory. We will follow the definitions given in [20].

A **lattice** is defined to be a tuple (L, B), where L is a free abelian group of finite rank (i.e. isomorphic to $\mathbb{Z}^n$ for some finite n), and where B is a non-degenerate symmetric bilinear form $B : L \times L \to \mathbb{Z}$. Here **non-degenerate** means that if $B(x, y) = 0$ for all y, then $x = 0$, **symmetric** means $B(x, y) = B(y, x)$ for all $x, y \in L$, and **bilinear** means that B is $\mathbb{Z}$-linear (i.e. a group homomorphism) in both components, separately. In other words, after fixing a basis for L (which is always possible since L is free), we can represent B as a non-degenerate, symmetric, integer-valued matrix (also denoted B) in that basis with $B(x, y) = x^T By$ for all $x, y \in L$ (viewed as column vectors). Since B is symmetric, from linear algebra we know that there exists a change of basis over $\mathbb{R}$ where B is diagonal with eigenvalues along the diagonal. The number of positive eigenvalues of B is denoted by l_+ and the number of negative eigenvalues l_-. We call the tuple (l_+, l_-) the **signature** of the lattice (L, B). A **sublattice** $(L', B') \subseteq (L, B)$ is a free abelian subgroup $L' \subseteq L$, where B restricted to L' is B'.

From now on, we denote a lattice (L, B) by L for convenience, making B explicit when required. A sublattice $L' \subseteq L$ is **primitive** if L/L' is free, and is called an **overlattice** if L/L' is a finite abelian group. We denote $L^* = \mathrm{Hom}(L, \mathbb{Z})$, and note that we can view $L \subseteq L^*$ via $x(y) = B(x, y)$ for all $x, y \in L$. Hence we can define the **discriminant group** $A_L = L^*/L$, which is always a finite abelian group. In fact if we write B as a matrix, then $|A_L| = |\det(B)|$. If A_L is trivial (i.e. $L = L^*$), we call the lattice L **unimodular**. A lattice is called **even** if $L(x, x) \in 2\mathbb{Z}$ for all $x \in L$.

Given a finite abelian group A, a (finite) **quadratic form** is a map $q : A \to \mathbb{Q}/2\mathbb{Z}$ satisfying

(i) For all $n \in \mathbb{Z}$ and $a \in A$, $q(na) = n^2 q(a)$, and

(ii) There exists a symmetric, bilinear form $b : A \times A \to \mathbb{Q}/\mathbb{Z}$ such that for all $a, a' \in A$, $q(a + a') \equiv q(a) + q(a') + 2b(a, a') \pmod{2\mathbb{Z}}$.

In particular, we can extend the bilinear form B on L to L^* (now taking values in $\mathbb{Q}$), so if L is even, we get an induced quadratic from $q_L : A_L \to \mathbb{Q}/2\mathbb{Z}$ (with $b = B$). Using this quadratic form, we obtain a useful notion of orthogonality for lattices.

We already have the notion of an orthogonal complement from linear algebra: if $L \subseteq S$, $L_S^\perp = \{s \in S : B(s, l) = 0 \text{ for all } l \in L\}$ (Here we consider B to be the form on S. Two lattices L and K are said to be **orthogonal** if there exists an even, unimodular lattice S such that $L \subseteq S$ and $L_S^\perp \cong K$, where isomorphism of lattices, called an **isometry**, is an isomorphism of free abelian groups that preserves the bilinear form. Perhaps even more importantly, we have the useful criterion that L and K are orthogonal if and only if $q_L \cong -q_K$, where isomorphism of quadratic forms is really an isomorphism of A_L and A_K that preserves the maps to $\mathbb{Q}/2\mathbb{Z}$. While orthogonality may not at present seem important, it will be key in our later definition of LPK3 mirror symmetry.

Some lattices of particular interest are the rank 2 hyberbolic lattice whose bilinear form is given by the matrix $U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, the lattices A_l,D_m,and E_n ($l \geq 1$,$m \geq 4$, $6 \leq n \leq 8$) whose matrix representations are the adjacency matrices for the Dynkin diagrams of the classic ADE-singularities, and the $T_{p,q,r}$ lattices which are the adjacency matrices for graphs in the form of a T with p, q, r the respective lengths of the legs (see Figure 3.2 for $T_{4,4,4}$ below). We also have the rank 1 lattices defined by $\langle n\rangle$ for some $n \in \mathbb{Z}$ that act by multiplication, i.e. $B(x, y) = xny$. For notation purposes, $U(2)$ means the lattice whose values are 2 times those for U, etc.

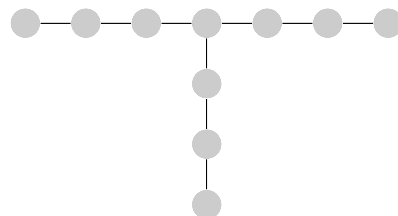

Figure 3.2: $T_{4,4,4}$

The set of all finite quadratic forms forms a semi-group under direct sum, where direct sum of matrices is the corresponding block diagonal matrix. There are three classes of quadratic forms which generate the semi-group of all finite quadratic forms under direct sum, which we present now:

(i) For $p \neq 2$ prime, $k \in \mathbb{Z}^{\geq 1}$, and $\epsilon \in \{-1, 1\}$, let a be the smallest even integer that has ϵ as quadratic residue modulo p. Then we define

$$w^{\epsilon}_{p,k} : \mathbb{Z}/p^k\mathbb{Z} \to \mathbb{Q}/2\mathbb{Z} \text{ by } w^{\epsilon}_{p,k}(1) = \tfrac{a}{p^k}$$

(ii) For $p = 2$, $k \in \mathbb{Z}^{\geq 1}$, and $\epsilon \in \{-1, 1, -5, 5\}$, we define

$$w^{\epsilon}_{2,k} : \mathbb{Z}/2^k\mathbb{Z} \to \mathbb{Q}/2\mathbb{Z} \text{ by } w^{\epsilon}_{2,k}(1) = \tfrac{\epsilon}{2^k}$$

(iii) For $k \in \mathbb{Z}^{\geq 1}$, we define the quadratic forms u_k and v_k on $\mathbb{Z}/2^k\mathbb{Z} \times \mathbb{Z}/2^k\mathbb{Z}$ by

$$u_k = \begin{bmatrix} 0 & \frac{1}{2^k} \\ \frac{1}{2^k} & 0 \end{bmatrix} \text{ and } v_k = \frac{1}{2^k}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

We list relevant lattices, their signatures, and their associated quadratic forms in Table 3.1. These lattices are described in much greater detail in [3].

Note that the lattices (and hence quadratic forms) for the K3 surfaces we will be studying may not just be entries in Table 3.1, but direct sums of them. For example, $U \oplus D_4$ has discriminant form u, while $U(2) \oplus D_4$ has discriminant form $u \oplus v$.

Lattice	Signature	Form
U	(1,1)	trivial
$U(2)$	(1,1)	u
A_1	(0,1)	$w_{2,1}^{-1}$
$A_1(2)$	(0,1)	$w_{2,2}^{-1}$
A_2	(0,2)	$w_{3,1}^{1}$
A_3	(0,3)	$w_{2,2}^{5}$
D_4	(0,4)	v
D_5	(0,5)	$w_{2,2}^{-5}$
D_6	(0,6)	$(w_{2,1}^{1})^2$
D_9	(0,9)	$w_{2,2}^{-1}$
E_6	(0,6)	$w_{3,1}^{-1}$
E_7	(0,7)	$w_{2,1}^{1}$
E_8	(0,8)	trivial
$T_{4,4,4}$	(1,9)	v_2
$\langle 4\rangle$	(1,0)	$w_{2,2}^{1}$
$\langle -4\rangle$	(0,1)	$w_{2,2}^{-1}$
$\langle 8\rangle$	(1,0)	$w_{2,3}^{1}$
$\langle -8\rangle$	(0,1)	$w_{2,3}^{-1}$

Table 3.1: Lattices and Quadratic Forms

One more relevant fact that we should mention is that in what follows we will be identifying lattices via their discriminant quadratic forms and signatures. The following result ensures that we can do this.

Proposition 3.14 (Nikulin [20]). *An even lattice with signature (l_+, l_-) and discriminant quadratic form q exists and is unique if $l_+ \geq 1$,$l_- \geq 1$, and $l_+ + l_- \geq 2 + l(A_q)$, where $l(A_q)$ denotes the minimum number of generators of A_q.*

With all this machinery, we are now ready to describe LPK3 mirror symmetry. Let X be a K3 surface with purely non-symplectic automorphism σ. It is well known that $H^2(X,\mathbb{Z})$ is an even, unimodular lattice of signature $(3,19)$, so isomorphic to the **K3 lattice** $L_{K3} = U^3 \oplus (E_8)^2$. We can then look at some interesting sublattices. $\text{Pic}(X) = H^2(X,\mathbb{Z}) \cap H^{1,1}(X,\mathbb{C})$ be the **picard lattice** of X, and let $S(\sigma) \subseteq H^2(X,\mathbb{Z})$ be the **invariant lattice** of X given by $S(\sigma) = \{x \in H^2(X,\mathbb{Z}) : \sigma^*(x) = x\}$. The signature of $S(\sigma)$ is $(1,t)$ for some $t \leq 19$.

Let M be a lattice of signature $(1,t)$, $t \leq 18$. If there exists some primitive embedding of M $j : M \to \mathrm{Pic}(X)$, we call X an **M-polarized K3 surface**. For an M-polarized K3 surface X, the lattice M naturally embeds into L_{K3}, leading to our final definition.

Definition 3.15. Let M be a primitive sublattice of L_{K3} of signature $(1,t)$ with $t \leq 18$ such that $(M_{L_{K3}})^{\perp} \cong U \oplus M^{\vee}$. We define $M^{\vee}$ (up to isometry) to be the **dual lattice** to M. Given an M-polarized K3 surface X and an M'-polarized K3 surface X' with $M' = M^{\vee}$ (or equivalently $M = (M')^{\vee}$), we call X and X' **LPK3 dual K3 surfaces**.

Note that the dual of X is a whole family of K3 surfaces, each of which is $M^{\vee}$-polarized. As for duality, one can check that $M^{\vee}$ is also primitively embedded in L_{K3}, has signature $(1, 18-t)$, $q_M \cong -q_{M^{\vee}}$, and that $(M^{\vee})^{\perp}_{L_{K3}} \cong U \oplus M$, so $(M^{\vee})^{\vee} = M$. Hence our notion of M-polarized X and $M^{\vee}$-polarized X' being mirror K3 surfaces makes sense.

Now that we have defined both the BHK and LPK3 mirror symmetry constructions, we turn our attention specifically to K3 surfaces of the form $Z_{W,G}$, where $W = x^n + g(y,z,w)$ is invertible from one of the 95 previously mentioned weight systems, and which are polarized by the invariant lattice $S(\sigma_n)$, where $\sigma_n(x,y,z,w) = (\zeta_n x, y, z, w)$ is the non-symplectic automorphism acting by a primitive nth root of unity ζ_n. Now via BHK mirror symmetry, we get the dual K3 surface Z_{W^T,G^T}, which is polarized by the invariant lattice $S(\sigma_n^T)$. If we show that $S(\sigma_n)^{\vee} \cong S(\sigma_n^T)$, we then have that Z_{W^T,G^T} is in the LPK3 mirror family of $Z_{W,G}$ as well. This is what we mean for the BHK and LPK3 mirror symmetry constructions "to agree." Computationally, we verify this by checking the rank and quadratic form of a K3 surface and its mirror.

In the next two chapters, we describe methods and computations showing that the BHK and LPK3 mirror symmetry constructions agree. Besides some listed exceptions, we then prove our main theorem:

Theorem 3.16. *Let W be an invertible polynomial of the form $x^n + g(y,z,w)$ from one of Reid and Yonemura's 95 weight systems, and $J_W \leq G \leq SL_W$. Then the K3 surface $Z_{W,G}$ and its BHK mirror X_{W^T,G^T} are LPK3 mirror pairs.*

There has already been a lot of progress on this problem: $n = 2$ was proven by Artebani-Boissière-Sarti [1], n prime by Comparin-Lyon-Priddis-Suggs [8], and $n \neq 4, 8, 12$ by Comparin-Priddis [9]. Hence for the rest of our discussion, we look specifically at the cases $n = 4, 8, 12$ to complete the problem, where previous methods no longer apply.

Chapter 4. Classification Project

4.1 Isomorphisms of K3 Surfaces

At the end of the last section, we discussed how the key to showing that BHK and LPK3 mirror symmetry agree (when both apply) is to show that the invariant lattice $S(\sigma_n^T)$ of Z_{W^T,G^T} is the same as the LPK3 dual lattice. Here the main problem is that invariant lattices are in general very difficult to compute, and in particular this is the situation for $n = 4, 8, 12$, being the precise reason why these cases have been saved for last. In this section, we develop new tools that will aid in us resolving this situation. Our first technique is a new way to show K3 surfaces are isomorphic, which is useful since isomorphic K3 surfaces have the same invariant lattice. Here by isomorphism, we mean as algebraic varieties, which also shows that they are isomorphic as complex manifolds. Hence, our method of attack for this problem is to use a Theorem of Kelly and Shoemaker to find isomorphism classes within our desired list K3 surfaces and to show that these isomorphisms preserve the non-symplectic automorphism. In many of these classes, we then find a representative that has a computable invariant lattice, and then use this representative to find the invariant lattice for all members of its isomorphism class.

Theorem 4.1 (Kelly and Shoemaker [15] [23]). *If W and W' are invertible polynomials and $J_{W^T} \leq G^T \leq SL_{W^T}$, $J_{(W')^T} \leq (G')^T \leq SL_{(W')^T}$ with $G^T = (G')^T$, then $Z_{W,G}$ and $Z_{W',G'}$ are birationally equivalent.*

Kelly proved Theorem 4.1 using Shioda maps, and we go into more specifics about his arguments in our proof of Lemma 4.3 below. The following theorem is a standard result in the minimal model program in algebraic geometry and the classification of compact complex surfaces in algebraic geometry. A proof can be found in [2], which follows quickly from the fact that by definition, K3 surfaces have trivial canonical bundle.

Theorem 4.2. *Every birational map between K3 surfaces is an isomorphism.*

Together, these two theorems give an original way to find isomorphisms of K3 surfaces, aiding in showing the equivalence of BHK and LPK3 mirror symmetry for $W = x^n + g(y, z, w)$ (up to change of variables) with $n = 4, 8, 12$. Since our K3 surfaces have the additional structure of a purely non-symplectic automorphism, we just need to verify that the isomorphism of Kelly preserves the automorphism σ_n. In other words, if X and X' are two of our K3 surfaces with automorphism of order n σ_n (which may act on them differently), and $\phi : X \to X'$ is Kelly's birational equivalence, then for all $x \in X$, we need $\phi(\sigma_n \cdot x) = \sigma_n \cdot \phi(x)$. A map that preserves σ_n in this way is called σ_n**-equivariant**.

Lemma 4.3. *The birational equivalence given in Theorem 4.1 is σ_n-equivariant.*

Proof. The proof of Theorem 4.1 uses what are called Shioda maps, which are rational. We introduce the following notation. Let W be an invertible polynomial with exponent matrix A, and let $B = dA^{-1}$, where d is any integer such that B has all integer entries. Further, let X_{dI} be the hypersurface in $\mathbb{P}^n$ cut out by the Fermat polynomial $x_1^d + ... + x_n^d$. Then, the Shioda map $\phi_B : X_{dI} \to X_A$ is given by $\phi_B(y_0 : ... : y_n) = (x_0 : ... : x_n)$, where $x_j = \prod_{k=0}^{n} y_k^{b_{jk}}$. We denote by ϕ_{B^T} the Shioda map for W^T. Kelly proves that these Shioda maps push forward any action by a symmetry group G via $\phi_{B_*}(g) = Bg$, and similarly for W^T.

Now, Kelly's proof shows that if $G = G'$, then both Z_{W^T,G^T} and $Z_{W'^T,G'^T}$ are birational to $X_{dd'I}/H$, for some group H. Now, in our specific case this is an isomorphism, and σ_n acts on both Z_{W^T,G^T} and $Z_{W'^T,G'^T}$. This is a purely non-symplectic automorphism $\sigma_{dd'}$ of $X_{dd'}$, which descends to an action on $X_{dd'I}/H$. Notice that $\sigma_{dd'}$ has order n in $X_{dd'I}/H$, and so this gives an equivalent action of σ_n on both $Z_{W,G}$ and Z_{W^T,G^T}. □

Although lattice computations are difficult and there are at present no known methods for carrying out many of them, we had success in using Theorem 4.1 and Theorem 4.2 together to find isomorphism classes of K3 surfaces. Since the invariant lattice is preserved under isomorphism, many computations thus reduce to finding a method that works for just a representative in each isomorphism class. Unfortunately, not every isomorphism class has such a nice representative, so we need another tool. This comes from deformations, and in particular pencils.

4.2 Deformations and Embeddings of K3 Surfaces

In this section we will describe certain deformations which will further facilitate our computations. A **deformation** is a continuous family $\mathcal{X} \to S$ for some base S. We will mostly be interested when each fiber is a K3 surface. When $S = \mathbb{P}^1$, we call such a family a **pencil**. For our purposes, we take invertible polynomials W and W' with the same system where $G = G'$ (they have the same group of symmetries acting on them). We then form the pencil of hypersurfaces in $W\mathbb{P}^n(q_0, \ldots, q_n)$ $sW + tW'$ for $(s,t) \in \mathbb{P}^1$ and then do a blow up for this entire family since each fiber has the same singularities. It is a well known fact that all K3 surfaces are deformations, but not all of these deformations will preserve the invariant lattice if these K3 surfaces have a purely non-symplectic automorphism. However, if we can show that the invariant lattice is also preserved under certain deformations, then deformations combined with isomorphisms creating a larger pool in each class to draw from for developing computational methods.

The following two theorems together show that if one member of the pencil of K3 surfaces we just described is polarized by an invariant lattice, every member of the pencil is polarized by the same invariant lattice. Let L_B denote a lattice generated by a set of curves $B \in \operatorname{Pic}(X)$.

Theorem 4.4. *L_B is primitively embedded in $H^2(X, \mathbb{Z})$ if and only if L_B is primitively embedded in* $\operatorname{Pic}(X)$.

Proof. Note that from previous discussion $\operatorname{Pic}(X)$ is primitively embedded in $H^2(X, \mathbb{Z})$. By the Third Isomorphism Theorem, $(H^2(X, \mathbb{Z})/L_B)/(\operatorname{Pic}(X)/L_B) \cong H^2(X, \mathbb{Z})/\operatorname{Pic}(X)$, which is free. Since we know that $H^2(X, \mathbb{Z})/L_B$ is a finitely generated abelian group, $\operatorname{Pic}(X)/L_B$ cannot be free or else the quotient will have torsion, unless $H^2(X, \mathbb{Z})/L_B$ has no torsion to begin with, i.e. is free itself. For the other direction, we just note that the subgroup of a free group is always free. ☐

Let L_{B_t} be the lattice generated by the curves over $t \in \mathbb{P}^1$.

Theorem 4.5. *If L_{B_0} is primitively embedded in $H^2(X_0, \mathbb{Z})$, then L_{B_t} is primitively embedded in $H^2(X_t, \mathbb{Z})$ for all $t \in \mathbb{P}^1$.*

Proof. This proof actually follows from Proposition 2.10 in Huybrechts, since it is straightforward that the specialization map is a primitive embedding [14]. Using the same idea as Theorem 4.4, we can show this gives the desired primitive embedding. Since his treatment uses the modern language of sheaves and schemes from algebraic geometry, we omit the technical details. ☐

While techniques for computing invariant lattices are scarce, for our specific class of K3 surfaces, there is a lattice L_B that is easy to compute defined as the adjacency matrix of the graph that is the intersection (a set B) of copies of $\mathbb{P}^1$ which arise in the blow up process. Since the Picard lattice $\mathrm{Pic}(X)$ consists of equivalence classes of (finite formal sums of) curves on the K3 surface X, $L_B \subseteq \mathrm{Pic}(X)$. In the case where this is a primitive embedding and the group of symmetries acts trivially on X, Sarah-Marie Belcastro in her PhD dissertation [3] computed $\mathrm{Pic}(X_t)$ for a general member of a family of K3 surfaces for each of the 95 weight systems. Our goal is to find a representative in each class that has a trivial group action and a primitive embedding, and to find deformations that preserve the invariant lattice. Then we can find a primitive embedding using Theorem 4.3 and Belcastro's work. We will rely on the following result.

Theorem 4.6. *If L_B is primitively embedded in $\mathrm{Pic}(X)$, then $L_B = S(\sigma_n)$.*

Proof. The key to this proof is that we can find L_B and $L_{B'}$ such that L_B is primitively embedded in $L_{B'}$. In the next chapter, we complete the proof of this lemma computationally by showing via our MAGMA function "dicompare" (see appendix A) that $L_{B'} \cong \mathrm{Pic}(X_t)$ for

Belcastro's X_t, which is always primitively embedded in $H^2(X,\mathbb{Z})$. Similarly, we always have that $S(\sigma_n)$ is primitively embedded in $H^2(X,\mathbb{Z})$. Further, $L_B \subseteq S(\sigma_n)$ with the same rank, so $S(\sigma_n)/L_B$ is a finite abelian group contained in $H^2(X,\mathbb{Z})/L_B$, which is free. However, the only finite subgroup of a free group is $\{0\}$, so it must be that $L_B = S(\sigma_n)$. $\square$

4.3 Outline of Proof

The proof of our main theorem that the BHK and LPK3 mirror symmetry constructions agree when they overlap in the $n = 4, 8, 12$ (with some listed exceptions) is computational. In the preceding sections we have described some original methods for finding isomorphism classes, deformations, and embeddings of relevant K3 surfaces. Here, we summarize the process of our computations, with more specific details in Chapter 5 and the appendix, outlining our computational proof of the theorem. In the next section, we go through our outline's process for $n = 4$, rank 4 to illustrate these points so they are more clear. The steps for the computations are as follows:

(i) Classify each K3 surface by order of automorphism (which in our case the possibilities are $4, 8, 12$). We list all pairs (W, G), where $W = x^n + g(y, z, w)$ is invertible and $J_W \leq G \leq SL_W$. For some polynomials, there are several choices of G, each giving rise to a different K3 surface.

(ii) Classify each K3 surface's lattice by the rank of its invariant lattice. These ranks are computed using a theorem of Comparin-Priddis [9].

(iii) Find isomorphism classes for our K3 surfaces

  (a) Classify the W^T's by weight system (this is the only way presently known to find possible $G^T = G'^T$).

  (b) For each $W^T = x^n + g(y, z, w)$ (up to change of variables) with $n = 4, 8, 12$, find all possible G^T such that $J_{W^T} \leq G^T \leq SL_{W^T}$.

(c) Collect the $(W^T, G^T), (W'^T, G'^T)$ in each weight system with $G^T = G'^T$.

(d) By Theorem 4.1 and Theorem 4.2, we then have $Z_{W,G} \cong Z_{W',G'}$, separating K3 surfaces into isomorphism classes.

(iv) In a given rank, look for K3 surfaces that have the same weights and group, and find deformations. If there is a deformation between representatives for different isomorphism classes, we have enlarged our pool of candidates in that case.

(v) In each isomorphism/deformation class, find a representative that has a trivial group. If invariant lattice matches Belcastro's work, we're done. If not, find an embedding of L_B into $\mathrm{Pic}(X)$ by showing L_B is primitively embedded in some $L_{B'} \cong \mathrm{Pic}(X)$, and then we're done. At this point we have shown that $S(\sigma_n)$ is the same for each member of a given class.

(vi) Check that $S(\sigma_n)^\vee = S(\sigma_n^T)$ for some $Z_{W,G}$ in each class and its BHK mirror Z_{W^T,G^T} by comparing their ranks and discriminant quadratic forms. If they agree, the BHK mirrors are LPK3 mirrors as well.

(vii) If there is no representative with trivial group in a given equivalence class, we call that class **exceptional**. Additional (perhaps case specific) methods will have to be developed in these cases, but our work has at least made it so these methods only need to apply to a single member of each exceptional class.

4.4 Illustrative Example

We now go over our process for the $n = 4$, rank 4 case. Most of our proof is too computational to show every step, but this example illustrates enough so that our short descriptions in the next chapter are sufficient. In the $n = 4$, rank 4 case, there are three invertible polynomials with symmetry groups that satisfy Reid and Yonemura's conditions (just a small excerpt from our table in Chapter 5):

No	Rk	Dual	Weights	Polynomial	G/J_W	Form
11	4	77	(4,2,1,1;8)	$x^2 + y^4 + z^8 + w^8$	$\mathbb{Z}/2\mathbb{Z}$	$w^1_{2,2} + w^5_{2,2}$
12	4	78	(3,2,2,1;8)	$x^2z + y^4 + z^4 + w^8$	trivial	$w^1_{2,2} + w^5_{2,2}$
13	4	79	(3,2,2,1;8)	$x^2z + y^4 + z^4 + xw^5$	trivial	$w^1_{2,2} + w^5_{2,2}$

Table 4.1: Order 4, Rank 4 Example

Note that the dual polynomial for 11 above is itself, while the dual polynomial for 12 (using the transpose of its exponent matrix) is $x^2 + y^4 + xz^4 + w^8$. Both of these dual polynomials have the same weight system (4,2,1,1;8).

From subgroup lattices (depicted below) where edges correspond to index, and using duality of quotient groups, we see that the dual groups for 11 and 12 both have order 32, and with some order considerations we can see that they are the same group (using the classification of finite abelian groups). Hence by our results on isomorphisms, 11 and 12 are birationally equivalent, and in fact by Theorem 4.2 they are isomorphic, and so their invariant lattices are the same.

$$\begin{tikzcd}
G_W^{\max} \arrow[d, dash] & \{0\} \arrow[d, dash] \\
SL_W \arrow[d, dash] & J_{W^T} \arrow[d, dash] \\
J_W \arrow[d, dash] & SL_{W^T} \arrow[d, dash] \\
\{0\} & G_{W^T}^{\max}
\end{tikzcd}$$

Figure 4.1: Subgroup Lattice for 12 and its Dual

Now that we know that we can relate 11 and 12 via isomorphism, we continue by observing how 12 and 13 have the same weight system with G/J_W the trivial group acting on both of them, which is an indicator that they are in a shared pencil of K3 surfaces. Blowing up the pencil, we get the corresponding adjacency graphs for the intersections of special fibers for 12 and 13.

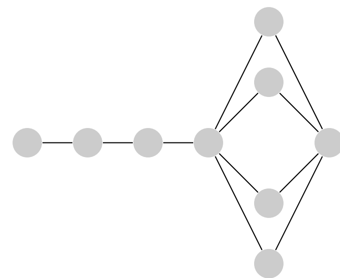

Figure 4.2: Blowup Graph for 12

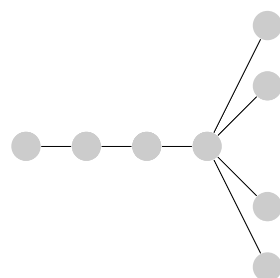

Figure 4.3: Blowup Graph for 13

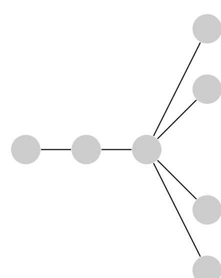

Figure 4.4: Belcastro's Picard Lattice Computation

Looking at the adjacency graphs above for the blow ups corresponding to 12 and 13, we observe that they are almost the same, with the exception that 12 has one additional note missing from 13. Now, Belcastro's computation (see Figure 4.4) for Pic(X) for this weight system indicates that it should have rank 7, which doesn't quite match either 12 or 13 (have 9 and 8 generators, respectively). Removing the node that 12 has, but 13 doesn't, along with one additional node, we get a sublattice $L_{B'}$ which our magma code shows is (potentially after a change of basis) isomorphic to Belcastro's rank 7 Pic(X). Further, it is easy to find a rank 4 $L_B \subseteq L_{B'}$ as our invariant lattice (manipulating the defining equations), so by our results above, 12 and 13 are deformations of one another that preserve the non-symplectic automorphism (since the missing node in 13 wasn't necessary, and that was what dropped out during the deformation).

Hence, 11 and 12 are connected via isomorphism, and 12 and 13 are connected deformation, so all three have the same invariant lattice. We then primitively embedded 13 into the lattice computed by Belcastro (easy to check that it is a sublattice). Looking at the table in Ch. 5, we can verify that the rank and quadratic form of 11, 12, and 13 correspond dually with their duals 77,78, and 79, respectively having done the same computations there, showing that the two mirror symmetry constructions agree for these K3 Surfaces.

Before we move on from this example, we note that 11 did not have $\{0\}$ as its G/J_W like 12 and 13 did, so Belcastro's results do not apply for 11. In other words, if it weren't for the isomorphism of 11 and 12 via our use of Tyler and Kelly's Theorem, 11 would be an exceptional K3 surface.

Chapter 5. Computations

Recall that we construct our K3 surfaces from an invertible polynomial of the form $W = x^n + g(y, z, w)$ and group G with $J_W \leq G \leq SL_W$. These K3 surfaces have a purely non-symplectic automorphism σ_n, and our goal for $n = 4, 8, 12$ is to find $S(\sigma_n)$ and show that $S(\sigma_n)^\vee \cong S(\sigma_n^T)$. This last step is verified by checking the below tables to compare the rank and quadratic form of a K3 surface with that of its dual.

5.1 Order 4 Computations

We begin with our computations of an exhaustive list of invertible polynomials of the form $W = x^4 + g(y, z, w)$ and their symmetry groups G that generate lattice polarized, algebraic K3 surfaces $Z_{W,G}$, along with their weights and the rank of their invariant lattice. Note here that "x" may not be the order four Fermat term, but it is convenient and equivalent to say the polynomials are of the form $x^4 + g(y, z, w)$. Further, weights come from a list derived from Reid and Yonemura's work, and are of the form $(q_0, \ldots, q_3; d)$, where for the corresponding polynomial W, we have $W(\lambda^{q_0}x, \lambda^{q_1}y, \lambda^{q_2}z, \lambda^{q_3}w) = \lambda^d W(x, y, z, w)$ for all $\lambda \in \mathbb{C}^*$. Since rank is a lattice-invariant, if two lattices have different ranks they are not the same. Hence, note that our list and following remarks are ordered by rank. We also number our list to make referencing efficient.

Notice the table is also grouped (notationally by double lines) by isomorphism/deformation classes, the details of which are summarized afterward, separated by rank. In this discussion, we mention that some of the K3 surfaces are exceptional. In the table for those K3 surfaces, we include the conjectured quadratic forms and duals, even though we have not been able to show they are primitively embedded with our current methods. For all other K3 surfaces, we verify Conjecture 2.2 by checking the rank and quadratic forms for each K3 surface and that of their mirror.

No	Rk	Dual	Weights	Polynomial	G/J_W	Form
1	1	85	(1,1,1,1;4)	$x^4+y^3z+z^3w+yw^3$	trivial	$w^1_{2,1}$
2	1	86	(1,1,1,1;4)	$x^4+y^4+z^3w+zw^3$	trivial	$w^1_{2,1}$
3	1	87	(1,1,1,1;4)	$x^4+y^4+z^4+w^4$	trivial	$w^1_{2,1}$
4	1	89	(1,1,1,1;4)	$x^3w+y^4+z^4+w^4$	trivial	$w^1_{2,1}$
5	1	88	(1,1,1,1;4)	$x^3z+y^4+z^3w+w^4$	trivial	$w^1_{2,1}$
6	2	80	(4,2,1,1;8)	$x^2+y^4+z^7w+zw^7$	trivial	u
7	2	83	(4,2,1,1;8)	$x^2+y^4+z^8+w^8$	trivial	u
8	2	81	(4,2,1,1;8)	$x^2+y^4+xz^4+w^8$	trivial	u
9	2	84	(4,2,1,1;8)	$x^2+y^4+z^7w+w^8$	trivial	u
10	2	82	(4,2,1,1;8)	$x^2+y^4+xz^4+zw^7$	trivial	u
11	4	77	(4,2,1,1;8)	$x^2+y^4+z^8+w^8$	$\mathbb{Z}/2\mathbb{Z}$	$w^1_{2,2}+w^5_{2,2}$
12	4	78	(3,2,2,1;8)	$x^2z+y^4+z^4+w^8$	trivial	$w^1_{2,2}+w^5_{2,2}$
13	4	79	(3,2,2,1;8)	$x^2z+y^4+z^4+xw^5$	trivial	$w^1_{2,2}+w^5_{2,2}$
14	5	72	(6,3,2,1;12)	$x^2+y^4+xz^3+w^{12}$	trivial	$u+w^5_{2,2}$
15	5	73	(6,3,2,1;12)	$x^2+y^4+z^6+w^{12}$	trivial	$u+w^5_{2,2}$
16	5	74	(6,3,2,1;12)	$x^2+y^4+z^6+xw^6$	trivial	$u+w^5_{2,2}$
17	5	75	(6,3,2,1;12)	$x^2+y^4+z^6+zw^{10}$	trivial	$u+w^5_{2,2}$
18	5	76	(6,3,2,1;12)	$x^2+y^4+xz^3+zw^{10}$	trivial	$u+w^5_{2,2}$
19	6	60	(1,1,1,1;4)	$x^4+y^4+z^3w+zw^3$	$\mathbb{Z}/2\mathbb{Z}$	$v+v_2$
20	6	63	(1,1,1,1;4)	$x^4+y^4+z^4+w^4$	$\mathbb{Z}/4\mathbb{Z}$	$v+v_2$
21	6	64	(4,3,2,2;12)	$x^3+y^4+z^4+xw^4$	trivial	$v+v_2$
22	6	65	(4,3,2,2;12)	$x^3+y^4+z^4+w^6$	trivial	$v+v_2$
23	6	61	(1,1,1,1;4)	$x^4+y^4+z^4+w^4$	$\mathbb{Z}/2\mathbb{Z}$	$v+v_2$
24	6	62	(1,1,1,1;4)	$x^3w+y^4+z^4+w^4$	$\mathbb{Z}/2\mathbb{Z}$	$v+v_2$
25	6	66	(10,5,4,1;20)	$x^2+y^4+z^5+w^{20}$	trivial	$u+v$
26	6	69	(8,4,3,1;16)	$x^2+y^4+z^5w+w^{16}$	trivial	$u+v$
27	6	67	(10,5,4,1;20)	$x^2+y^4+z^5+xw^{10}$	trivial	$u+v$
28	6	71	(8,4,3,1;16)	$x^2+y^4+z^5w+xw^8$	trivial	$u+v$
29	6	68	(10,5,4,1;20)	$x^2+y^4+z^5+zw^{16}$	trivial	$u+v$
30	6	70	(8,4,3,1;16)	$x^2+y^4+z^5w+zw^{13}$	trivial	$u+v$
31	8	58	(4,2,1,1;8)	$x^2+y^4+z^8+w^8$	$\mathbb{Z}/2\mathbb{Z}$	$v+w^1_{2,2}+w^5_{2,2}$
32	8	59	(4,2,1,1;8)	$x^2+y^4+xz^4+w^8$	$\mathbb{Z}/2\mathbb{Z}$	$v+w^1_{2,2}+w^5_{2,2}$
33	9	54	(6,3,2,1;12)	$x^2+y^4+z^6+w^{12}$	$\mathbb{Z}/2\mathbb{Z}$	$u+v+w^5_{2,2}$
34	9	57	(10,5,3,2;20)	$x^2+y^4+z^6w+w^{10}$	trivial	$u+v+w^5_{2,2}$
35	9	55	(6,3,2,1;12)	$x^2+y^4+z^6+xw^6$	$\mathbb{Z}/2\mathbb{Z}$	$u+v+w^5_{2,2}$
36	9	56	(10,5,3,2;20)	$x^2+y^4+z^6w+xw^5$	trivial	$u+v+w^5_{2,2}$
37	10	self	(1,1,1,1;4)	$x^4+y^4+z^4+w^4$	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$	u^3

Table 5.1: Order 4 Computations pt.1

No	Rk	Dual	Weights	Polynomial	G/J_W	Form
38	10	41	(4,2,1,1;8)	$x^2+y^4+z^7w+zw^7$	$\mathbb{Z}/3\mathbb{Z}$	u^3
39	10	42	(4,2,1,1;8)	$x^2+y^4+z^8+w^8$	$\mathbb{Z}/4\mathbb{Z}$	u^3
40	10	43	(14,7,4,3;28)	$x^2+y^4+z^7+zw^8$	trivial	u^3
41	10	38	(4,2,1,1;8)	$x^2+y^4+z^7w+zw^7$	$\mathbb{Z}/2\mathbb{Z}$	u^3
42	10	39	(4,2,1,1;8)	$x^2+y^4+z^8+w^8$	$\mathbb{Z}/2\mathbb{Z}$	u^3
43	10	40	(4,2,1,1;8)	$x^2+y^4+z^7w+w^8$	$\mathbb{Z}/2\mathbb{Z}$	u^3
44	10	52	(8,7,6,3;24)	$x^3+y^3w+z^4+w^8$	trivial	v_2
45	10	49	(4,4,3,1;12)	$x^3+y^3+z^4+w^{12}$	trivial	v_2
46	10	self	(4,4,3,1;12)	$x^2y+xy^2+z^4+w^{12}$	trivial	v_2
47	10	50	(6,3,2,1;12)	$x^2+y^4+xz^3+w^{12}$	$\mathbb{Z}/2\mathbb{Z}$	v_2
48	10	53	(7,4,3,2;16)	$x^2w+y^4+xz^3+w^8$	trivial	v_2
49	10	45	(4,4,3,1;12)	$x^3+y^3+z^4+w^{12}$	$\mathbb{Z}/3\mathbb{Z}$	v_2
50	10	47	(4,4,3,1;12)	$x^2y+y^3+z^4+w^{12}$	trivial	v_2
51	10	self	(6,3,2,1;12)	$x^2+y^4+z^6+w^{12}$	$\mathbb{Z}/2\mathbb{Z}$	v_2
52	10	44	(4,4,3,1;12)	$x^3+y^3+z^4+yw^8$	trivial	v_2
53	10	48	(4,4,3,1;12)	$x^3+xy^2+z^4+yw^8$	trivial	v_2
54	11	33	(6,3,2,1;12)	$x^2+y^4+z^6+w^{12}$	$\mathbb{Z}/2\mathbb{Z}$	$u+v+w_{2,2}^{-5}$
55	11	35	(5,3,2,2;12)	$x^2w+y^4+z^6+w^6$	trivial	$u+v+w_{2,2}^{-5}$
56	11	36	(5,3,2,2;12)	$x^2w+y^4+z^6+zw^5$	trivial	$u+v+w_{2,2}^{-5}$
57	11	34	(6,3,2,1;12)	$x^2+y^4+z^6+zw^{10}$	$\mathbb{Z}/2\mathbb{Z}$	$u+v+w_{2,2}^{-5}$
58	12	31	(4,2,1,1;8)	$x^2+y^4+z^8+w^8$	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$	$v+w_{2,2}^{-1}+w_{2,2}^{-5}$
59	12	32	(3,2,1,1;8)	$x^2z+y^4+z^4+w^8$	$\mathbb{Z}/2\mathbb{Z}$	$v+w_{2,2}^{-1}+w_{2,2}^{-5}$
60	14	19	(1,1,1,1;4)	$x^4+y^4+z^3w+zw^3$	$\mathbb{Z}/4\mathbb{Z}$	$v+v_2$
61	14	23	(1,1,1,1;4)	$x^4+y^4+z^4+w^4$	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z}$	$v+v_2$
62	14	24	(4,3,2,2;12)	$x^3+y^4+z^4+xw^4$	$\mathbb{Z}/2\mathbb{Z}$	$v+v_2$
63	14	20	(1,1,1,1;4)	$x^4+y^4+z^4+w^4$	$\mathbb{Z}/4\mathbb{Z}$	$v+v_2$
64	14	21	(1,1,1,1;4)	$x^3w+y^4+z^4+w^4$	$\mathbb{Z}/4\mathbb{Z}$	$v+v_2$
65	14	22	(4,3,2,2;12)	$x^3+y^4+z^4+w^6$	$\mathbb{Z}/2\mathbb{Z}$	$v+v_2$
66	14	25	(10,5,4,1;20)	$x^2+y^4+z^5+w^{20}$	$\mathbb{Z}/2\mathbb{Z}$	$u+v$
67	14	27	(9,5,4,2;20)	$x^2w+y^4+z^5+w^{10}$	trivial	$u+v$
68	14	29	(8,4,3,1;16)	$x^2+y^4+z^5w+w^{16}$	$\mathbb{Z}/2\mathbb{Z}$	$u+v$
69	14	26	(10,5,4,1;20)	$x^2+y^4+z^5+zw^{16}$	$\mathbb{Z}/2\mathbb{Z}$	$u+v$
70	14	30	(8,4,3,1;16)	$x^2+y^4+z^5w+zw^{13}$	$\mathbb{Z}/2\mathbb{Z}$	$u+v$
71	14	28	(9,5,4,2;20)	$x^2w+y^4+z^5+zw^8$	trivial	$u+v$
72	15	14	(4,4,3,1;12)	$x^2y+y^3+z^4+w^{12}$	$\mathbb{Z}/2\mathbb{Z}$	$u+w_{2,2}^{-5}$
73	15	15	(6,3,2,1;12)	$x^2+y^4+z^6+w^{12}$	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$	$u+w_{2,2}^{-5}$
74	15	16	(5,3,2,2;12)	$x^2w+y^4+z^6+w^6$	$\mathbb{Z}/2\mathbb{Z}$	$u+w_{2,2}^{-5}$

Table 5.2: Order 4 Computations pt. 2

No	Rk	Dual	Weights	Polynomial	G/J_W	Form
75	15	17	(10,5,3,2;20)	$x^2+y^4+z^6w+w^{10}$	$\mathbb{Z}/2\mathbb{Z}$	$u+w_{2,2}^{-5}$
76	15	18	(7,6,5,2;20)	$x^2y+y^3w+z^4+w^{10}$	trivial	$u+w_{2,2}^{-5}$
77	16	11	(4,2,1,1;8)	$x^2+y^4+z^8+w^8$	$\mathbb{Z}/4\mathbb{Z}$	$w_{2,2}^{-1}+w_{2,2}^{-5}$
78	16	12	(4,2,1,1;8)	$x^2+y^4+xz^4+w^8$	$\mathbb{Z}/4\mathbb{Z}$	$w_{2,2}^{-1}+w_{2,2}^{-5}$
79	16	13	(8,5,4,3;20)	$x^2z+y^4+z^5+xw^4$	trivial	$w_{2,2}^{-1}+w_{2,2}^{-5}$
80	18	6	(4,2,1,1;8)	$x^2+y^4+z^7w+zw^7$	$\mathbb{Z}/6\mathbb{Z}$	u
81	18	8	(3,2,2,1;8)	$x^2z+y^4+z^4+w^8$	$\mathbb{Z}/4\mathbb{Z}$	u
82	18	10	(11,7,6,4;28)	$x^2z+y^4+z^4w+w^7$	trivial	u
83	18	7	(4,2,1,1;8)	$x^2+y^4+z^8+w^8$	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z}$	u
84	18	9	(14,7,4,3;28)	$x^2+y^4+z^7+zw^8$	$\mathbb{Z}/2\mathbb{Z}$	u
85	19	1	(1,1,1,1;4)	$x^4+y^3z+z^3w+yw^3$	$\mathbb{Z}/7\mathbb{Z}$	$w_{2,2}^{-1}$
86	19	2	(1,1,1,1;4)	$x^4+y^4+z^3w+zw^3$	$\mathbb{Z}/8\mathbb{Z}$	$w_{2,2}^{-1}$
87	19	3	(1,1,1,1;4)	$x^4+y^4+z^4+w^4$	$\mathbb{Z}/4\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z}$	$w_{2,2}^{-1}$
88	19	5	(12,9,8,7;36)	$x^3+y^4+xz^3+zw^4$	trivial	$w_{2,2}^{-1}$
89	19	4	(4,3,3,2;12)	$x^3+y^4+z^4+xw^4$	$\mathbb{Z}/4\mathbb{Z}$	$w_{2,2}^{-1}$

Table 5.3: Order 4 Computations pt.3

Separated by rank, we now describe our computations using the methods described in the previous chapter. These computations together constitute a proof of our main conjecture that the BHK and LPK3 mirror symmetry constructions agree for algebraic K3 hypersurfaces with non-symplectic automorphism of order four.

5.1.1 Rank 1 Computations.

Actually for our rank 1 K3 surfaces, there were no overlattices, so there was nothing to show.

5.1.2 Rank 2 Computations.

We showed 6-10 were deformations. Then, we embedded 6.

5.1.3 Rank 4 Computations.

We showed that 11 and 12 were isomorphic, and then that 12 and 13 were deformations. We then embedded 12.

5.1.4 Rank 5 Computations.

We showed 14-18 were deformations. Then, we embedded 16.

5.1.5 Rank 6 Computations.

First, we showed 19-21 were isomorphic. We then showed 21 and 22 were deformations, and that 19, 23, and 24 were deformations, connecting 19-24. Then, we embedded 21.

Next, we showed that 25 and 26, 27 and 28, and 29 and 30 were pairwise isomorphic, respectively. We also showed that 25, 27, and 29 were deformations, connecting 25-30, and embedded 25.

5.1.6 Rank 8 Computations.

We showed that 31 and 32 were deformations, both of which are exceptional cases.

5.1.7 Rank 9 Computations.

We showed that 33 and 34, and 35 and 36, were pairwise isomorphic, respectively. We then showed that 34 and 36 were deformations, connecting 33-36. Finally, we embedded 34.

5.1.8 Rank 10 Computations.

For rank 10, there are 4 groups:

37 is by itself, and is an exceptional case.

We showed 38-40 were isomorphic, and embedded 35 .

We showed 41-43 were deformations, all of which are exceptional cases.

We showed that 44-48 were isomorphic, and that 50-52 were isomorphic. Then 44, 50, and 53 matched Belcastro perfectly. 49 is an exceptional case.

5.1.9 Rank 11 Computations.

We showed that 54 and 55, and 56 and 57 were pairwise isomorphic, respectively. Then 55 and 56 each matched Belcastro perfectly.

5.1.10 Rank 12 Computations.

We showed that 58 and 59 were isomorphic, both of which are exceptional cases.

5.1.11 Rank 14 Computations.

We showed that 60-62 are isomorphic, 60, 63, and 64 are deformations, and that 62 and 65 were deformations. This connects 60-65, each of which are exceptional cases.

We then showed that 66-68 are isomorphic, and that 69-71 were isomorphic. Then, 67 and 71 matched Belcastro perfectly.

5.1.12 Rank 15 Computations.

We showed that 72-76 were isomorphic, and 76 matched Belcastro perfectly.

5.1.13 Rank 16 Computations.

We showed that 77 and 78 are deformations, and that 78 and 79 are isomorphic, connection 77-79. 79 then matched Belcastro perfectly.

5.1.14 Rank 18 Computations.

We showed that 80-84 were isomorphic, and then 82 matched Belcastro perfectly.

5.1.15 Rank 19 Computations.

We showed that 85-89 were isomorphic, and then 88 matched Belcastro perfectly.

5.2 Order 8 Computations

We now give the required computations by rank, finishing the proof of our main conjecture for the order eight case.

5.2.1 Rank 3 Computations.

We showed that 1, 2, 3, 4, and 6 were deformations, and that 4 and 5 were isomorphic, connecting 1-6. 1 then matched Belcastro perfectly.

5.2.2 Rank 6 Computations.

We showed that 7-8, 9-10, and 11-12 were pairwise isomorphic, and then showed that 8, 10, and 12 were deformations, connecting 7-12. We then embedded 8 .

5.2.3 Rank 7 Computations.

We showed that 13-14 and 15-16 were pairwise isomorphic, respectively. 14 and 16 then matched Belcastro perfectly.

No	Rk	Dual	Weights	Polynomial	G/J_W	Form
1	3	36	(4,2,1,1;8)	$x^2+y^4+xz^4+w^8$	trivial	$w_{2,2}^{-1}$
2	3	34	(4,2,1,1;8)	$x^2+y^4+yz^6+w^8$	trivial	$w_{2,2}^{-1}$
3	3	35	(4,2,1,1;8)	$x^2+xy^2+yz^6+w^8$	trivial	$w_{2,2}^{-1}$
4	3	32	(4,2,1,1;8)	$x^2+xy^2+z^8+w^8$	trivial	$w_{2,2}^{-1}$
5	3	37	(4,2,1,1;8)	$x^2+y^4+z^8+w^8$	$\mathbb{Z}/2\mathbb{Z}$	$w_{2,2}^{-1}$
6	3	33	(4,2,1,1;8)	$x^2+y^4+z^8+w^8$	trivial	$w_{2,2}^{-1}$
7	6	29	(12,8,3,1;24)	$x^2+y^3+z^8+xw^12$	trivial	v
8	6	27	(8,5,2,1;16)	$x^2+y^3w+z^8+xw^8$	trivial	v
9	6	30	(12,8,3,1,24)	$x^2+y^3+z^8+yw^16$	trivial	v
10	6	28	(8,5,2,1;16)	$x^2+y^3w+z^8+yw^11$	trivial	v
11	6	31	(12,8,3,1;24)	$x^2+y^3+z^8+w^24$	trivial	v
12	6	26	(8,5,2,1;16)	$x^2+y^3w+z^8+w^16$	trivial	v
13	7	25	(4,2,1,1;8)	$x^2+y^4+yz^6+w^8$	$\mathbb{Z}/2\mathbb{Z}$	$v+w_{2,3}^{-1}$
14	7	23	(3,2,2,1;8)	$x^2z+y^4+yz^3+w^8$	trivial	$v+w_{2,3}^{-1}$
15	7	24	(4,2,1,1;8)	$x^2+y^4+z^8+w^8$	$\mathbb{Z}/2\mathbb{Z}$	$v+w_{2,3}^{-1}$
16	7	22	(3,2,2,1;8)	$x^2z+y^4+z^4+w^8$	trivial	$v+w_{2,3}^{-1}$
17	10	20	(4,2,1,1;8)	$x^2+y^4+xz^4+w^8$	$\mathbb{Z}/2\mathbb{Z}$	$w_{2,2}^{-1}+w_{2,3}^{1}$
18	10	18	(4,2,1,1;8)	$x^2+xy^2+z^8+w^8$	$\mathbb{Z}/2\mathbb{Z}$	$w_{2,2}^{-1}+w_{2,3}^{1}$
19	10	21	(4,2,1,1,;8)	$x^2+y^4+z^8+w^8$	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$	$w_{2,2}^{-1}+w_{2,3}^{1}$
20	10	17	(3,2,2,1;8)	$x^2z+y^4+z^4+w^8$	$\mathbb{Z}/2\mathbb{Z}$	$w_{2,2}^{-1}+w_{2,3}^{1}$
21	10	19	(4,2,1,1;8)	$x^2+y^4+z^8+w^8$	$\mathbb{Z}/2\mathbb{Z}$	$w_{2,2}^{-1}+w_{2,3}^{1}$
22	13	16	(4,2,1,1;8)	$x^2+y^4+xz^4+w^8$	$\mathbb{Z}/4\mathbb{Z}$	$v+w_{2,3}^{1}$
23	13	14	(12,5,4,3;24)	$x^2+y^4z+xz^3+w^8$	trivial	$v+w_{2,3}^{1}$
24	13	15	(4,2,1,1;8)	$x^2+y^4+z^8+w^8$	$\mathbb{Z}/4\mathbb{Z}$	$v+w_{2,3}^{1}$
25	13	13	(12,5,4,3;24)	$x^2+y^4z+z^6+w^8$	trivial	$v+w_{2,3}^{1}$
26	14	12	(12,8,3,1;24)	$x^2+y^3+z^8+yw^16$	$\mathbb{Z}/2\mathbb{Z}$	v
27	14	8	(11,8,3,2;24)	$x^2w+y^3+z^8+yw^8$	trivial	v
28	14	10	(8,5,2,1;16)	$x^2+y^3w+z^8+yw^11$	$\mathbb{Z}/2\mathbb{Z}$	v
29	14	7	(11,8,3,2;24)	$x^2w+y^3+z^8+w^12$	trivial	v
30	14	9	(8,5,2,1;16)	$x^2+y^3w+z^8+w^16$	$\mathbb{Z}/2\mathbb{Z}$	v
31	14	11	(12,8,3,1;24)	$x^2+y^3+z^8+w^24$	$\mathbb{Z}/2\mathbb{Z}$	v
32	17	4	(4,2,1,1;8)	$x^2+xy^2+z^8+w^8$	$\mathbb{Z}/4\mathbb{Z}$	$w_{2,2}^{1}$
33	17	6	(4,2,1,1;8)	$x^2+y^4+z^8+w^8$	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/4\mathbb{Z}$	$w_{2,2}^{1}$
34	17	2	(12,5,4,3;24)	$x^2+y^4z+z^6+w^8$	$\mathbb{Z}/2\mathbb{Z}$	$w_{2,2}^{1}$
35	17	3	(10,7,4,3;24)	$x^2z+xy^2+z^6+w^8$	trivial	$w_{2,2}^{1}$
36	17	1	(3,2,2,1;8)	$x^2z+y^4+z^4+w^8$	$\mathbb{Z}/4\mathbb{Z}$	$w_{2,2}^{1}$
37	17	5	(4,2,1,1;8)	$x^2+y^4+z^8+w^8$	$\mathbb{Z}/4\mathbb{Z}$	$w_{2,2}^{1}$

Table 5.4: Order 8 Computations

5.2.4 Rank 10 Computations.

We showed that 17, 18, and 21 were deformations, and that 18-20 were isomorphic, connecting 17-21. Each of these is an exceptional case.

5.2.5 Rank 13 Computations.

We showed that 22-23 and 24-25 were pairwise isomorphic, respectively. We also showed that 23 and 25 were deformations, connecting 22-25, and embedded 23.

5.2.6 Rank 14 Computations.

We showed that 26-28 were isomorphic, and that 29-31 were isomorphic. 27 and 29 then matched Belcastro perfectly.

5.2.7 Rank 17 Computations.

We showed that 32-36 were isomorphic, and that 32 and 37 were deformations, connecting 32-37. 35 then matched Belcastro perfectly.

5.3 Order 12 Computations

We now give a reference table for our calculations for those of order twelve, i.e. of the form $x^{12} + g(y, z, w)$.

We now give the required computations by rank, finishing the proof of our main conjecture for the order twelve case.

5.3.1 Rank 2 Computations.

No overlattices, so nothing to show.

5.3.2 Rank 7 Computations.

No overlattices, so nothing to show.

5.3.3 Rank 8 Computations.

No overlattices, so nothing to show.

5.3.4 Rank 10 Computations.

$S(\sigma_1 2) = S(\sigma_4)$ in this case, so all computations reduce to those done above in order 4.

5.3.5 Rank 12 Computations.

No overlattices, so nothing to show.

No	Rk	Dual	Weights	Polynomial	G/J_W	Form
1	2	26	(6,4,1,1;12)	$x^2+y^3+xz^6+w^{12}$	trivial	trivial
2	2	27	(6,4,1,1;12)	$x^2+y^3+yz^8+w^{12}$	trivial	trivial
3	2	28	(6,4,1,1;12)	$x^2+y^3+z^{12}+w^{12}$	trivial	trivial
4	7	22	(6,3,2,1;12)	$x^2+y^4+xz^3+w^{12}$	trivial	$w_{2,2}^1+w_{3,1}^{-1}$
5	7	23	(6,3,2,1;12)	$x^2+xy^2+z^6+w^{12}$	trivial	$w_{2,2}^1+w_{3,1}^{-1}$
6	7	24	(6,3,2,1;12)	$x^2+y^4+z^6+w^{12}$	trivial	$w_{2,2}^1+w_{3,1}^{-1}$
7	7	25	(6,3,2,1;12)	$x^2+y^4+z^6+w^{12}$	$\mathbb{Z}/2\mathbb{Z}$	$w_{2,2}^1+w_{3,1}^{-1}$
8	8	20	(5,4,2,1;12)	$x^2z+y^3+z^6+w^{12}$	trivial	$v+w_{3,1}^1$
9	8	19	(5,4,2,1;12)	$x^2z+y^3+yz^4+w^{12}$	trivial	$v+w_{3,1}^1$
10	8	18	(6,4,1,1;12)	$x^2+y^3+yz^8+w^{12}$	$\mathbb{Z}/2\mathbb{Z}$	$v+w_{3,1}^1$
11	8	21	(6,4,1,1;12)	$x^2+y^3+z^{12}+w^{12}$	$\mathbb{Z}/2\mathbb{Z}$	$v+w_{3,1}^1$
12	10	16	(4,4,3,2;12)	$x^2y+y^3+z^4+w^{12}$	trivial	v_2
13	10	14	(4,4,3,2;12)	$x^3+y^3+z^4+w^{12}$	trivial	v_2
14	10	13	(4,4,3,2;12)	$x^3+y^3+z^4+w^{12}$	$\mathbb{Z}/3\mathbb{Z}$	v_2
15	10	self	(4,4,3,2;12)	$x^2y+xy^2+z^4+w^{12}$	trivial	v_2
16	10	12	(6,3,2,1;12)	$x^2+y^4+xz^3+w^{12}$	$\mathbb{Z}/2\mathbb{Z}$	v_2
17	10	self	(6,3,2,1;12)	$x^2+y^4+z^6+w^{12}$	$\mathbb{Z}/2\mathbb{Z}$	v_2
18	12	10	(12,7,3,2;24)	$x^2+y^3z+z^8+w^{12}$	trivial	$v+w_{3,1}^{-1}$
19	12	9	(12,7,3,2;24)	$x^2+y^3z+xz^4+w^{12}$	trivial	$v+w_{3,1}^{-1}$
20	12	8	(6,4,1,1;12)	$x^2+y^3+xz^6+w^{12}$	$\mathbb{Z}/3\mathbb{Z}$	$v+w_{3,1}^{-1}$
21	12	11	(6,4,1,1;12)	$x^2+y^3+z^{12}+w^{12}$	$\mathbb{Z}/3\mathbb{Z}$	$v+w_{3,1}^{-1}$
22	13	4	(4,4,3,1;12)	$x^2y+y^3+z^4+w^{12}$	$\mathbb{Z}/2\mathbb{Z}$	$w_{2,2}^{-1}+w_{3,1}^1$
23	13	5	(6,3,2,1;12)	$x^2+xy^2+z^6+w^{12}$	$\mathbb{Z}/2\mathbb{Z}$	$w_{2,2}^{-1}+w_{3,1}^1$
24	13	6	(6,3,2,1;12)	$x^2+y^4+z^6+w^{12}$	$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$	$w_{2,2}^{-1}+w_{3,1}^1$
25	13	7	(6,3,2,1;12)	$x^2+y^4+z^6+w^{12}$	$\mathbb{Z}/2\mathbb{Z}$	$w_{2,2}^{-1}+w_{3,1}^1$
26	18	1	(5,4,2,1;12)	$x^2z+y^3+z^6+w^{12}$	$\mathbb{Z}/3\mathbb{Z}$	trivial
27	18	2	(12,7,3,2;24)	$x^2+y^3z+z^8+w^{12}$	$\mathbb{Z}/2\mathbb{Z}$	trivial
28	18	3	(6,4,1,1;12)	$x^2+y^3+z^{12}+w^{12}$	$\mathbb{Z}/6\mathbb{Z}$	trivial

Table 5.5: Order 12 Computations

5.3.6 Rank 13 Computations.

No overlattices, so nothing to show.

5.3.7 Rank 18 Computations.

No overlattices, so nothing to show.

Appendix A. C-Singularities

The following results consisted of joint work with a research partner, Benjamin Pachev.

Our first result is about the number of C-singularities in a given weight system. For this appendix only, we do note that our weight system notation is different. Instead of writing quasihomogeneous weights as $(q_0, \ldots, q_n; d)$, we write weight systems in the form $(\frac{q_0}{d}, \ldots, \frac{q_n}{d} \in (\mathbb{Q}/\mathbb{Z})^{n+1}$. Both notations are common in the literature.

Definition A.1. Let $T(k) = \sum_{p|k} 2^{\frac{k}{p}-1}$ - $\sum_{p_1,p_2|k} 2^{\frac{k}{p_1p_2}-1} + 2 \sum_{p_1,p_2,p_3|k} 2^{\frac{k}{p_1p_2p_3}-1} + \cdots + (n-1)(-1)^{n-1} \sum_{p_1,\cdots,p_n|k} 2^{\frac{k}{p_1p_2\cdots p_n}-1}$ where only prime divisors of k are counted in each sum, and n is the number of distinct prime divisors of k.

Theorem A.2. *T(k) counts the number of nonempty tuples of distinct natural numbers $(a_1, a_2, \cdots, a_n)$ satisfying $\gcd(a_1, a_2, \cdots, a_n) \rangle$ 1 and $k = \max(a_1, a_2, \cdots, a_n)$. We call such tuples admissible.*

Proof. Let $d = \gcd(a_1, a_2, \cdots, a_n)$, for any admissible tuple. We must have that $d|k$, by definition of the gcd. Now fix d, and suppose $d|\gcd(a_1, a_2, \cdots, a_n)$, hence $d|k$. Any such admissible tuple must contain k, and the remaining elements must be divisible by d and less than k. Hence they form a subset of $\{d, 2d, \cdots, (\frac{k}{d}-1)d\}$. Moreover, the union of any such subset with k will yield an admissible tuple whose gcd is divisible by d. Hence the number of admissible tuples whose gcd is divisible by d is the number of subsets of a set of $\frac{k}{d}-1$ elements, $2^{\frac{k}{d}-1}$.

Let m denote the number of distinct prime divisors of k, and p_i the ith prime divisor. Let A denote the set of all admissible tuples. A_i denote the set of all admissible tuples $(a_1, a_2, \cdots, a_n)$ with $p_i \mid \gcd(a_1, a_2, \cdots, a_n)$. We have that $A = \bigcup_{i=1}^{i=n} A_i$, since every gcd of an admissible tuple must be divisible by some prime divisor of k.

Thus, the cardinality $|A| = \sum_{i=1}^{i=m} |A_i| + \sum_{j=2}^{j=m} (j-1)(-1)^{j-1} \sum_{i_1,\cdots i_j \in 1\cdots m,\text{distinct}} |\bigcap_{l=1}^{l=j} A_{i_l}|$

Note that $\bigcap_{l=1}^{l=j} A_{i_l}$ corresponds to the set of admissible tuples whose gcd is divisible by each of the primes p_{i_l}, and hence by their product. Thus the number of such tuples is given by $2^{\frac{k}{p_{i_1}\cdots p_{i_j}}-1}$. Substituting into the formula for the cardinality of A gives the desired result. $\square$

Remark. The necessary and sufficient conditions on C-singularities boil down to a correspondence between tuples of the form prescribed in the previous theorem and polynomials that are C-singularities. Hence, for a given weight system and invertible representative, the number of C-singularities can be found straightforwardly from the previous theorem. Formulas for each individual case are given in the table below.

Weight System	$(\frac{1}{d}, \frac{1}{d})$	$(\frac{a}{d}, \frac{1}{d})$	$(\frac{1}{d}, \frac{1}{ad})$
Chain	$T(d-1)$	$T(\frac{d-1}{a})$	$T(d-1)$
Loop	$T(d-2)$	$T(\frac{d-1}{a}-1)$	X
Fermat	$T(d)$	X	$T(d)$

Table A.1: Number of C-Singularities by Weight Type

Definition A.3. Let p be an admissible two-variable polynomial with weights (q_1, q_2). The **difference gcd** $dg(p)$ of p is defined as the greatest common divisor of the differences of the exponents of the variable corresponding to the larger weight.

The general pattern noticed is that if $dg(p) = r$, and d is the denominator of the smaller weight of p, then rd is the maximal denominator any symmetry of p can have. Also, all polynomials with the same dg and the same invertible representative will have the same symmetry group. Hence by the Group-Weights Theorem, the corresponding A-models are isomorphic.

Theorem A.4. *Let p be a C-singularity with a loop invertible representative, and weights $(\frac{1}{d}, \frac{a}{d})$. Let $dg(p) = r \geq 1$. Then $G_p^{\max} = \langle(\frac{1}{rd}, \frac{(r-1)d+a}{rd})\rangle = \langle(g1, g2)\rangle$.*

Proof. By the proof of the necessary and sufficient conditions, (g_1, g_2) is a symmetry, hence the group generated by it is contained within $G_p^{\max}$. By the same proof, all symmetries have maximal denominator rd, or more precisely, all symmetries are of the form $(\frac{k}{rd}, \frac{j}{rd})$.
Let g be any such symmetry. Utilizing the monomial $x^{d-a}y$, we have $\frac{(d-a)k+j}{rd}$ an integer. Hence $j \equiv k*(rd-(d-a)) \pmod{rd}$, and thus $j \equiv k*((r-1)d+a) \pmod{rd}$, hence $g \in \langle(g_1, g_2)\rangle$. □

Theorem A.5. *Let p have a chain invertible representative, with weights* $(\frac{1}{d}, \frac{a}{d})$ *and let* $dg(p) = r \geq 1$. $G_p^{\max} = \langle(\frac{d-\frac{d-1}{ra}}{d}, \frac{1}{rd})\rangle = \langle(g_1, g_2)\rangle$.

Proof. Suppose $g \in G_p^{\max}$ is of the form $(\frac{k}{d}, \frac{j}{rd})$. Because p contains the monomials of a chain, it must contain the monomial $xy^{\frac{d-1}{a}}$. Since g is a symmetry, we have $\frac{k}{d} + \frac{\frac{j(d-1)}{ra}}{d}$ an integer. Hence $k + \frac{j(d-1)}{ra} \equiv 0 \pmod{d}$. Hence $k = j(d - \frac{(d-1)}{ra})$, and g $\in \langle(g_1, g_2)\rangle$.

It remains to show that all symmetries are of the proposed form, and that all group elements of the proposed form are actually symmetries. Note that $r|\frac{d-1}{a}$ because $r = dg(p)$, $\frac{d-1}{a}$ is an exponent of the proper variable, and 0 is also an exponent of the proper variable.

It suffices to show that the generating element (g_1, g_2) of the group is in $G_p^{\max}$. Let $x^{a_i}y^{b_i}$ be any monomial of p. Using the identity $a_i = d - ab_i$, and collecting terms, $a_ig_1 + b_ig_2 = \frac{b_i*(ard+(d-1)+1)}{rd} = \frac{b_i(ar+1)}{r}$, which is an integer since $r|b_i$, so (g_1, g_2) is a symmetry. That all group elements are of the desired form follows from the proof of the necessary and sufficient conditions. □

Theorem A.6. *Let p have a chain invertible representative, with weights* $(\frac{1}{d}, \frac{1}{ad})$ *and let* $dg(p) = r \geq 1$. $G_p^{\max} = \langle(\frac{d-\frac{d-1}{r}}{d}, \frac{1}{rad})\rangle = \langle(g_1, g_2)\rangle$.

Proof. Following the previous proof, assume g is a symmetry of the form $(\frac{k}{d}, \frac{j}{rad})$ Using the monomial $xy^{a(d-1)}$, we obtain k $= \frac{j(d-\frac{d-1}{r})}{adr} \in \langle(g_1, g_2)\rangle$, as desired. Again, from the proof of the necessary and sufficient conditions we have that all symmetries are of the desired form. It now remains to show that the generating group element is a symmetry. Letting $x^{a_i}y^{b_i}$ be any monomial of p, and using the identity $aa_i = ad - b_i$, we have that $a_ig_1 + b_ig_2 = 1 + \frac{b_i(d-rd)}{adr}$

$= 1 + \frac{b_i(1-r)}{ar} = 1 + \frac{(d-a_i)(1-r)}{r}$, which is an integer since p contains the monomial x^d, and hence by definition $r = dg(p)$ divides $d - a_i$. □

Theorem A.7. *Let p have a Fermat invertible representative, with weights $(\frac{1}{d}, \frac{1}{ad})$ and let $dg(p) = r \geq 1$. $G_p^{\max} = \langle(\frac{1}{r}, 0), (\frac{1}{d}, \frac{1}{ad})\rangle = \langle(g_1, g_2)\rangle$.*

Proof. Let $g = (\frac{k}{d}, \frac{j}{ad})$. Because p contains the monomials x^d and y^{ad}, all symmetries of p must be of this form. For every monomial $x^{a_i}y^{b_i}$ of p, we have, after some algebra, $kaa_i + jb_i \equiv 0 \pmod{ad}$. Using the identity $aa_i + b_i = ad$, we have $a(k-j)a_i \equiv 0 \pmod{ad}$, hence $(k-j)a_i \equiv 0 \pmod{d}$, which holds for every i if and only if $(k-j)r \equiv 0 \pmod{d}$. Thus, $k \equiv j \pmod{\frac{d}{r}}$, and $g \in \langle(g_1, g_2)\rangle$. □

The explicit forms for the groups are summarized below:

Weight System	$(\frac{1}{d}, \frac{1}{d})$	$(\frac{a}{d}, \frac{1}{d})$	$(\frac{1}{d}, \frac{1}{ad})$
Chain	$\langle(\frac{d-\frac{d-1}{r}}{d}, \frac{1}{rd})\rangle$	$\langle(\frac{d-\frac{d-1}{ra}}{d}, \frac{1}{rd})\rangle$	$\langle(\frac{d-\frac{d-1}{r}}{d}, \frac{1}{rad})\rangle$
Loop	$\langle(\frac{1}{rd}, \frac{(r-1)d+1}{rd})\rangle$	$\langle(\frac{1}{rd}, \frac{(r-1)d+a}{rd})\rangle$	X
Fermat	$\langle(\frac{1}{r}, 0), (\frac{1}{d}, \frac{1}{d})\rangle$	X	$\langle(\frac{1}{r}, 0), (\frac{1}{d}, \frac{1}{ad})\rangle$

Table A.2: C-Singularity Groups by Weight Type

Theorem A.8. *Given an admissible polynomial $p(x,y) = \sum_{i=1}^{n} x^{a_i}y^{b_i}$ with weights $(\frac{1}{d}, \frac{1}{ad})$, where d and a are nonzero integers, if $\gcd(a_1, a_2, \cdots, a_n) \geq 1$ or $\gcd(b_1, b_2, \cdots, b_n) \geq a$, then p is a C-singularity.*

Proof. Case 1. $\gcd(a_1, a_2, \cdots, a_n) = k \geq 1$. Then consider $g_1 = (\frac{1}{k}, 0)$, in additive notation. For all i, $\frac{a_i}{k}$ is an integer, hence $g_1 \in G_p^{\max}$. Then suppose by way of contradiction that $g_1 \in \langle(\frac{1}{d}, \frac{1}{ad})\rangle$. Thus for some j, $g_1 = (\frac{j}{d}, \frac{j}{ad})$. Then $0 = \frac{j}{ad}$ in additive notation, so $ad|j$. But then, $d|j$, hence $\frac{j}{d} = 0$. Then $\frac{1}{k} = 0$. But this is a contradiction because $k \geq 1$. Thus g_1 is not in $\langle J\rangle$. Thus, $G_p^{\max} \neq \langle J\rangle$.

Case 2. $\gcd(b_1, b_2, \cdots, b_n) = k \geq a$. Then define $g_l = (0, \frac{l}{k})$, for $0 \leq l \leq k$. For all i, $\frac{lb_i}{k}$ is an integer, hence $g_l \in G_p^{\max}$ for all l. Then suppose by way of contradiction that for all l, $g_1 \in \langle(\frac{1}{d}, \frac{1}{ad})\rangle = \langle J\rangle$. Now consider all $(x, y) \in \langle J\rangle$ with $x = 0$. We have that $(x, y) = (\frac{j}{d}, \frac{j}{ad})$ so $d|j$. So we have a distinct possibilities for $y = \frac{j}{ad}$, hence there are at most a elements of $\langle J\rangle$ whose first entry is zero. However, the first element of each g_l is zero, and each distinct, and each is in $\langle J\rangle$ which implies there must be at least k elements of $\langle J\rangle$ whose first entry is zero. However, $k \geq a$, so we have a contradiction. Hence $g_l \notin \langle J\rangle$ for at least one $0 \leq l \leq k$, hence $G_p^{\max} \neq \langle J\rangle$. □

Theorem A.9. *Given an admissible polynomial* $p(x, y) = \sum_{i=1}^{n} x^{a_i} y^{b_i}$ *with weights* $(\frac{1}{d}, \frac{a}{d})$, *where d and a are nonzero integers, if* $\gcd(b_1, b_2, \cdots, b_n) \geq 1$ *or* $\gcd(a_1, a_2, \cdots, a_n) \geq \gcd(a, d)$, *then p is a C-singularity.*

Proof. Case 1. $\gcd(b_1, b_2, \cdots, b_n) = k \geq 1$. This is similar to the first case of the previous theorem. As before, if $g_1 = (0, \frac{1}{k})$, we have $g_1 \in G_p^{\max}$. Then assume by way of contradiction that $g_1 \in \langle J\rangle$. Hence for some j, $0 = \frac{j}{d}$ and $\frac{1}{k} = \frac{aj}{d}$. Thus $d|j$, hence $d|aj$ and $0 = \frac{aj}{d} = \frac{1}{k}$, a contradiction because $k \geq 1$. Hence $g_1 \notin \langle J\rangle$ and thus $G_p^{\max} \neq \langle J\rangle$

Case 2. $\gcd(a_1, a_2, \cdots, a_n) = k \geq \gcd(a, d)$. First consider all $(x, y) = (\frac{j}{d}, \frac{aj}{d}) \in \langle J\rangle$ with $y = 0$. But $y = 0$ implies $d|aj$, $0 \leq j \leq d$. Thus $\frac{d}{gcd(a,d)} | \frac{aj}{\gcd(a,d)}$. But $\frac{d}{gcd(a,d)}$ is coprime to $\frac{aj}{\gcd(a,d)}$, so we have $\frac{d}{\gcd(a,d)} | j$. This holds for exactly $\frac{d}{\frac{d}{\gcd(a,d)}} = \gcd(a, d)$ different $0 \leq j \leq d$. Now note that $(l/k, 0) \in G_p^{\max}$ for all $0 \leq l \leq k$. Thus, the first entry of at least k elements of $G_p^{\max}$ is zero, but at most $\gcd(a, d) \leq k$ elements of $\langle J\rangle$ have their first entry equal to zero. Thus, $G_p^{\max} \neq \langle J\rangle$. □

Theorem A.10. *Given $p(x, y)$ as in the hypotheses of Theorem 1,* $\gcd(b_1, b_2, \cdots, b_n) \geq a$ *if and only if* $\gcd(d - a_1, d - a_2, \cdots, d - a_n) \geq 1$. *Thus, to check if Theorem 1 applies to a given polynomial, one need only examine the exponents of the first variable.*

Proof. For all $1 \le i \le n$, $\frac{a_i}{d} + \frac{b_i}{ad} = 1$. Hence, $b_i = ad - aa_i$. Thus, $\gcd(b_1, b_2, \cdots, b_n) = \gcd(ad - aa_1, ad - aa_2, \cdots, ad - aa_n) = a\gcd(d - a_1, d - a_2, \cdots, d - a_n)$, and the result follows immediately. □

It turns out in some cases that the above sufficient conditions for a C-singularity are necessary as well. We begin with a few lemmas.

Lemma A.11. *Given the natural numbers $a_1, a_2, \cdots, a_n$, d, j, and k, if $j \not\equiv k \pmod d$, and for all i, $a_i j \equiv a_i k \pmod d$, then $\gcd(a_1, a_2, \cdots, a_n) \ge 1$.*

Proof. Assume by way of contradiction that $\gcd(a_1, a_2, \cdots, a_n) = 1$. Then some linear combination $\sum_{i=1}^{n} c_i a_i = 1$. Because modular equivalence is linear, $j = \sum_{i=1}^{n} c_i a_i j \equiv \pmod d$ $\sum_{i=1}^{n} c_i a_i k = k$. Thus $j \equiv k \pmod d$, a contradiction. □

Lemma A.12. *If p is as in the hypotheses of Theorem 1, sans the condition on the exponents, the following hold.*

(i) $a|b_i$ for all i

(ii) $ad = b_i + aa_i$

(iii) $-a_i \equiv \frac{b_i}{a} \pmod d$

(iv) If $g = (g1, g2) = (\frac{j}{d}, \frac{k}{ad})$, $g \in \langle J \rangle$ if and only if $j \equiv k \pmod d$

(v) If $g = (g1, g2) = (\frac{j}{d}, \frac{k}{ad}) \in G_p^{\max}$, then either $g \in \langle J \rangle$ or $\gcd(a_1, a_2, \cdots, a_n) \ge 1$

Proof. The verification of the first three properties is straightforward from the definition of weights.

(iv) If $g \in \langle J \rangle$, the result holds trivially. If $j \equiv k \pmod d$, then $g = (\frac{j}{d}, \frac{k}{ad}) = (\frac{k}{d}, \frac{k}{ad}) \in \langle J \rangle$.

(v) Because (g_1, g_2) is a symmetry of p, we have that, for all $1 \le i \le n$, $\frac{a_i j}{d} + \frac{b_i k}{ad} = z \in \mathbb{Z}$. Hence, $aa_i j + b_i k = adz$. After dividing through by a and rearranging, we see that $a_i j \equiv \frac{-b_i k}{a} \pmod d$, so applying (iii), $a_i j \equiv a_i k \pmod d$ for all k. Thus, by Lemma 4, either $\gcd(a_1, a_2, \cdots, a_n) \ge 1$ or $j \equiv k \pmod d$, in which case by Lemma (iv) $(g_1, g_2) \in \langle J \rangle$. □

Theorem A.13. *Given a non-invertible, admissible polynomial* $p(x, y) = \sum_{i=1}^{n} x^{a_i} y^{b_i}$ *that can be written as the sum of a Fermat and other monomials, if* p *has the weights* $(\frac{1}{d}, \frac{1}{ad})$*, and is also a C-Singularity, then* $\gcd(a_1, a_2, \cdots, a_n) \geq 1$*. This is one case of the converse to Theorem 1.*

Proof. Since, p is a C-singularity, there exists $(g_1, g_2) \in G_p^{\max}$ but not in $\langle J \rangle$. Since p is the sum of a Fermat and other monomials, it contains the monomials x^d and y^{ad}. Hence dg_1 and adg_2 must be integers, and $(g_1, g_2) = (\frac{j}{d}, \frac{k}{ad})$ for some integers j and k. Then, by Lemma 5 (v), either $\gcd(a_1, a_2, \cdots, a_n) \geq 1$ or $g \in \langle J \rangle$, a contradiction. □

Theorem A.14. *Let* p *be as in the previous theorem, except a sum of a chain and other monomials as opposed to a Fermat. Either* $\gcd(a_1, a_2, \cdots, a_n) \geq 1$ *or* $\gcd(b_1, b_2, \cdots, b_n) \geq a$*. This is the second case of the converse to Theorem 1.*

Proof. Since, p is a C-singularity, there exists $(g_1, g_2) \in G_p^{\max}$ but not in $\langle J \rangle$. Since p is the sum of a chain and other monomials, it contains the monomial x^d. Hence $g_1 = \frac{j}{d}$ for some j. Let $g_2 = \frac{k}{r}$ for some k and r, and assume without loss of generality that $\gcd(k, r) = 1$. Then for all $1 \leq i \leq n$, $\frac{a_i j}{d} + \frac{b_i k}{r} \in \mathbb{Z}$. Simplifying, we obtain $\frac{db_i k}{r} \in \mathbb{Z}$. Hence, $r | db_i k$. But k and r are coprime, so $r | db_i$. By Lemma 5 (i), $\gcd(b_1, b_2, \cdots, b_n) \geq$ a. If the inequality is strict, the result holds. If not, then $\gcd(b_1, b_2, \cdots, b_n) = a$. Hence some linear combination $\sum_{i=1}^{n} c_i b_i = a$. We have $r | \sum_{i=1}^{n} c_i b_i d = ad$. Thus, $ad = qr$ for some $q \in \mathbb{Z}$, and $g_2 = \frac{qk}{ad}$. The result now follows from Lemma 5 (v). □

To deal with the weight system $(\frac{1}{d}, \frac{a}{d})$ we need another lemma.

Lemma A.15. *If* p *is admissible with weights* $(\frac{1}{d}, \frac{a}{d})$*, the following hold.*

(i) $a_i = d$ - ab_i

(ii) If $(\frac{j}{q}, \frac{k}{l}) \in G_p^{\max}$*, and is a loop added to other monomials, then* $l = q$*.*

(iii) If $(\frac{j}{d}, \frac{k}{d}) \in G_p^{\max}$*, then it is in* $\langle J \rangle$*.*

Proof. (i) By the definition of weights, we have $\frac{a_i}{d} + \frac{ab_i}{d} = 1$. The result follows trivially.

(ii) Assume that both fractions are in reduced form. We have that $\frac{ja_i}{q} + \frac{kb_i}{l} \in \mathbb{Z}$ for all i, and in particular when $a_i = 1$, and when $b_i = 1$. Since p contains the monomials of a loop, this is guaranteed to occur. Respectively, these yield $\frac{jl}{q}, \frac{kq}{l} \in \mathbb{Z}$. Since $\gcd(j, q) = \gcd(k, l) = 1$, we have $q|l$ and $l|q$. Hence $q = l$.

(iii) We have $\frac{ja_i}{d} + \frac{kb_i}{d} \in \mathbb{Z}$ for all i, and in particular for $b_i = 1$, $a_i = d - a$. Hence $d|(k - ja)$, and $k \equiv ja \pmod{d}$. Hence $(\frac{j}{d}, \frac{k}{d}) = (\frac{j}{d}, \frac{ja}{d}) \in \langle J \rangle$. $\square$

Theorem A.16. *Let p be an admissible polynomial with weights $(\frac{1}{d}, \frac{a}{d})$ that can be written as the sum of a loop and other monomials. We have that p is a C-singularity if and only if the greatest common divisor of the differences $b_i - b_j$ of the exponents of the variable corresponding to the weight $\frac{a}{d}$ is greater than one.*

Proof. $\Longrightarrow$ Since p is a C-singularity, it has at least one symmetry, g_1 not in $\langle J \rangle$. By combining (ii) and (iii) of the previous lemma, we have $g_1 = (\frac{j}{q}, \frac{k}{q})$ where $q \neq d$ and j and k are coprime to q. Using the definition of a symmetry, we have $q|(a_i j + b_i k)$ for all i. By applying (i) of Lemma 8 in two different ways, we have, for all i, $q|dj + b_{i_1}(k - aj)$ and $q|aa_{i_2}j + ab_{i_2}k = aa_{i_2}j + dk - a_{i_2}k$. Hence, we have $q|d(j + k) + (a_{i_2} - b_{i_1})(aj - k)$ for all i_2 and i_1. In particular, setting $a_{i_2} = b_{i_1} = 1$, we have $q|d(j + k)$. Combining this with the previous line, we have $q|(a_{i_2} - b_{i_1})(aj - k)$ for all i_2 and i_1. If the $\gcd(a_{i_2} - b_{i_1})$ over all i_1 and i_2 is 1, we have $q|aj - k$. Applying this result to the first line, we have $q|dj$. But $\gcd(q, j) = 1$, hence $q|d$. But this is a contradiction, because then g_1 can be re-written so that the denominators of both fractions equal d, and then by (iii) of Lemma 8 we have $g_1 \in \langle J \rangle$, a contradiction. Hence $\gcd(a_{i_2} - b_{i_1}) \geq 1$. From this we easily obtain the weaker condition $\gcd(b_{i_1} - b_{i_2}) \geq 1$.

$\Longleftarrow$ If $\gcd(b_{i_1} - b_{i_2}) = r \geq 1$, let $g = (\frac{1}{rd}, \frac{(r-1)d+a}{rd}) = (g_1, g_2)$. To show that g is a symmetry, it suffices to show that $a_i g_1 + b_i g_2$ is an integer for all i. We have $a_i g_1 + b_i g_2 = \frac{a_i + ab_i + (r-1)db_i}{rd} = \frac{1+(r-1)b_i}{r} = b_i + \frac{1-b_i}{r}$. However, $b_j = 1$ for some j because p contains the monomials of a loop, hence, $r|1 - b_i$, so $\frac{1-b_i}{r}$ is an integer, and g is a symmetry. However, $r \geq 1$, so $rd \geq d$, and hence $g_1 = \frac{1}{rd}$ cannot be a multiple of $\frac{1}{d}$, so $g \notin \langle J \rangle$, and p is a C-singularity. $\square$

Theorem A.17. *If p is as in the previous theorem, but is the sum of a chain and other monomials as opposed to a loop, then if P is a C-singularity,* $\gcd(b_1, b_2, \cdots, b_n) \geq 1$.

Proof. Let $g = (g_1, g_2)$ be a singularity of P not contained in $\langle J \rangle$. Because p contains the monomial x^d, g_1 must have denominator d. Then, $g = (\frac{j}{d}, \frac{k}{r})$, where j, r, and k are some integers, with $\gcd(r, k) = 1$. Because g is a symmetry, for all a_i, b_i we must have $\frac{ja_i}{d} + \frac{kb_i}{r} \in \mathbb{Z}$. Hence $r | dkb_i$, and since r is coprime to k, $r | db_i$. If $\gcd(b_1, b_2, \cdots, b_n) = 1$, we have $r | d$, so $g_2 = \frac{k}{r}$ can be written with denominator d, and then by Lemma 8 (iii) $g \in \langle J \rangle$, a contradiction. Thus, $\gcd(b_1, b_2, \cdots, b_n) \geq 1$. $\square$

Appendix B. Magma Code

In chapter 5, we remarked on our computations showing isomorphisms of invariant lattices. In this appendix, we give our code for the corresponding deformations and embeddings.

```
    Main Functions


disc:=function(M)
 S,A,B:=SmithForm(M);
 l:=[[S[i,i],i]: i in [1..NumberOfColumns(S)]| S[i,i] notin {0,1}];
 sA:=Matrix(Rationals(),ColumnSubmatrixRange(B,l[1][2],l[#l][2]));
 for i in [1..#l] do
  MultiplyColumn(~sA,1/l[i][1],i);
 end for;
 Q:=Transpose(sA)*Matrix(Rationals(),M)*sA;
 for i,j in [1..NumberOfColumns(Q)] do
  if i ne j then
   Q[i,j]:=Q[i,j]-Floor(Q[i,j]);
  else
   Q[i,j]:=Q[i,j]-Floor(Q[i,j])+ (Floor(Q[i,j]) mod 2);
  end if;
 end for;
 return [l[i][1]: i in [1..#l]], Q;
end function;


mod2:=function(Q);
 for i,j in [1..Nrows(Q)] do
  if i ne j then Q[i,j]:=Q[i,j]-Floor(Q[i,j]);
```

```
  else Q[i,j]:=Q[i,j]-2*Floor(Q[i,j]/2);
  end if;
 end for;
 return Q;
end function;

isot:=function(M,n)
 v,U:=disc(M);
 Q:=Rationals();
 A:=AbelianGroup(v);
 return [Eltseq(a) : a in A |
 mod2(Matrix(Q,1,#v,Eltseq(a))*U*Matrix(Q,#v,1,Eltseq(a)))[1,1] eq n];
end function;

dicompare:=function(M,Q)
 v,U:=disc(M);
 w,D:=disc(Q);
 if v ne w then return false; end if;
 A:=AbelianGroup(v);
 Aut:=AutomorphismGroup(A);
 f,G:=PermutationRepresentation(Aut);
 h:=Inverse(f);
 ll:=[Matrix(Rationals(),[Eltseq(Image(h(g),A.i)) :
 i in [1..Ngens(A)]]) : g in G];
 dd:=[mod2(a*U*Transpose(a)) : a in ll];
 return D in dd;
end function;
```

Order 4

Rank 2

Belcastro Equivalence and Deformation

```
M2:=Matrix(Integers(), [[-2,0,1,1],
                        [0,-2,1,1],
                        [1,1,0,0],
                        [1,1,0,0]]);
```

```
n:=4;
pp:=[SetToSequence({1..n} diff {i}): i in [1..n]];
[IsConsistent(Matrix(M2[pp[i]]), Matrix(M2[i])): i in [1..n]];
```

(Got ‘‘true" for last row, last row omitted gives Belcastro’s exactly)

Rank 4

Belcastro Equivalence and Deformation

```
M4:=Matrix(Integers(),[[-2,1,0,4,1,1,1],
                        [1,-2,1,0,0,0,0],
                        [0,1,-2,0,0,0,0],
                        [4,0,0,-8,-2,-2,-2],
                        [1,0,0,-2,-2,0,0],
```

```
                        [1,0,0,-2,0,-2,0],
                        [1,0,0,-2,0,0,-2]]);

MB4:=Matrix(Integers(),[[-2,1,1,1,1,1,0],
                        [1,-2,0,0,0,0,0],
                        [1,0,-2,0,0,0,0],
                        [1,0,0,-2,0,0,0],
                        [1,0,0,0,-2,0,0],
                        [1,0,0,0,0,-2,1],
                        [0,0,0,0,0,1,-2]]);

dicompare(M4,MB4); (Got ‘‘true")

Rank 5

Belcastro Equivalence and Deformation

M5:=Matrix(Integers(),[[-2,0,0,0,0,0,1],
                        [0,-2,0,0,0,0,1],
                        [0,0,-2,-2,1,1,0],
                        [0,0,-2,-4,1,2,0],
                        [0,0,1,1,-2,-2,1],
                        [0,0,1,2,-2,-4,2],
                        [1,1,0,0,1,2,-2]]);

MB5:=Matrix(Integers(),[[-2,1,1,1,0,1,0],
                        [1,-2,0,0,0,0,0],
```

```
                    [1,0,-2,0,0,0,0],
                    [1,0,0,-2,1,0,0],
                    [0,0,0,1,-2,0,0],
                    [1,0,0,0,0,-2,1],
                    [0,0,0,0,0,1,-2]]);

dicompare(M5,MB5); (Got ‘‘true")

Rank 6

Deformation of Class 1a and 1b

M6:=DiagonalMatrix(Integers(),12,[-2,-2,-2,-2,-2,-2,-2,-2,0,0,0,0]);
M6[1,9]:=1; M6[9,1]:=1;
M6[1,10]:=1; M6[10,1]:=1;
M6[2,9]:=1; M6[9,2]:=1;
M6[2,10]:=1; M6[10,2]:=1;
M6[3,9]:=1; M6[9,3]:=1;
M6[3,10]:=1; M6[10,3]:=1;
M6[4,9]:=1; M6[9,4]:=1;
M6[4,10]:=1; M6[10,4]:=1;
M6[5,11]:=1; M6[11,5]:=1;
M6[5,12]:=1; M6[12,5]:=1;
M6[6,11]:=1; M6[11,6]:=1;
M6[6,12]:=1; M6[12,6]:=1;
M6[7,11]:=1; M6[11,7]:=1;
M6[7,12]:=1; M6[12,7]:=1;
```

```
M6[8,11]:=1; M6[11,8]:=1;
M6[8,12]:=1; M6[12,8]:=1;


n:=12;
pp:=[SetToSequence({1..n} diff {i}): i in [1..n]];
[IsConsistent(Matrix(M6[pp[i]]), Matrix(M6[i])): i in [1..n]];
(Got ‘‘true" for last row, so deformations)
```

Belcastro Equivalence and Deformation for Class 1 (v+v_2) (Using 1c)

```
M6:=DiagonalMatrix(Integers(),15,[-2,-2,-2,-2,-2,
-2,-2,-2,-2,-2,-2,-2,0,0,0]);
M6[1,5]:=1; M6[5,1]:=1;
M6[1,13]:=1; M6[13,1]:=1;
M6[2,6]:=1; M6[6,2]:=1;
M6[2,13]:=1; M6[13,2]:=1;
M6[3,7]:=1; M6[7,3]:=1;
M6[3,13]:=1; M6[13,3]:=1;
M6[4,8]:=1; M6[8,4]:=1;
M6[4,13]:=1; M6[13,4]:=1;
M6[5,12]:=1; M6[12,5]:=1;
M6[6,12]:=1; M6[12,6]:=1;
M6[7,12]:=1; M6[12,7]:=1;
M6[8,12]:=1; M6[12,8]:=1;
M6[9,14]:=1; M6[14,9]:=1;
M6[9,15]:=1; M6[15,9]:=1;
M6[10,14]:=1; M6[14,10]:=1;
```

```
M6[10,15]:=1; M6[15,10]:=1;
M6[11,14]:=1; M6[14,11]:=1;
M6[11,15]:=1; M6[15,11]:=1;
M6[12,14]:=1; M6[14,12]:=1;
M6[12,15]:=1; M6[15,12]:=1;
M6[13,14]:=2; M6[14,13]:=2;
M6[13,15]:=2; M6[15,13]:=2;

n:=15;
pp:=[SetToSequence({1..n} diff {i}): i in [1..n]];
[IsConsistent(Matrix(M6[pp[i]]), Matrix(M6[i])): i in [1..n]];

M61:=Submatrix(M6,[1,2,3,4,5,6,7,8,9,10,11,12,13,15],
[1,2,3,4,5,6,7,8,9,10,11,12,13,15]);

n:=14;
pp:=[SetToSequence({1..n} diff {i}): i in [1..n]];
[IsConsistent(Matrix(M61[pp[i]]), Matrix(M61[i])): i in [1..n]];

M62:=Submatrix(M6,[1,2,3,4,5,6,7,8,9,10,11,12,15],
[1,2,3,4,5,6,7,8,9,10,11,12,15]);

n:=13;
pp:=[SetToSequence({1..n} diff {i}): i in [1..n]];
[IsConsistent(Matrix(M62[pp[i]]), Matrix(M62[i])): i in [1..n]];

M63:=Submatrix(M6,[2,3,4,5,6,7,8,9,10,11,12,15],
```

```
[2,3,4,5,6,7,8,9,10,11,12,15]);
```

```
n:=12;
pp:=[SetToSequence({1..n} diff {i}): i in [1..n]];
[IsConsistent(Matrix(M63[pp[i]]), Matrix(M63[i])): i in [1..n]];
```

```
E6:=Matrix(Integers(),[[-2,1,0,0,0,0],
                        [1,-2,1,0,0,0],
                        [0,1,-2,1,1,0],
                        [0,0,1,-2,0,0],
                        [0,0,1,0,-2,1],
                        [0,0,0,0,1,-2]]);
```

```
D4:=Matrix(Integers(),[[-2,1,0,0],
                        [1,-2,1,1],
                        [0,1,-2,0],
                        [0,1,0,-2]]);
```

```
U3:=Matrix(Integers(),[[0,3],[3,0]]);
```

```
M6B:=DiagonalJoin(<E6,D4,U3>);
```

```
dicompare(M63,M6B); (Got ``true")
```

Belcastro Equivalence and Deformation for Class 2 (u+v)

```
M6:=DiagonalMatrix(Integers(),10,[-2,-2,-2,-2,-2,-2,-2,-2,-2,-2]);
```

```
M6[1,3]:=1; M6[3,1]:=1;

M6[1,10]:=1; M6[10,1]:=1;

M6[2,4]:=1; M6[4,2]:=1;

M6[2,10]:=1; M6[10,2]:=1;

M6[3,5]:=1; M6[5,3]:=1;

M6[4,6]:=1; M6[6,4]:=1;

M6[5,7]:=1; M6[7,5]:=1;

M6[6,8]:=1; M6[8,6]:=1;

M6[9,10]:=1; M6[10,9]:=1;


T:=DiagonalMatrix(Integers(),10,[-2,-2,-2,-2,-2,-2,-2,-2,-2,-2]);

T[1,2]:=1; T[2,1]:=1;

T[1,3]:=1; T[3,1]:=1;

T[1,7]:=1; T[7,1]:=1;

T[3,4]:=1; T[4,3]:=1;

T[4,5]:=1; T[5,4]:=1;

T[5,6]:=1; T[6,5]:=1;

T[7,8]:=1; T[8,7]:=1;

T[8,9]:=1; T[9,8]:=1;

T[9,10]:=1; T[10,9]:=1;


dicompare(M6,T); (Got ‘‘true")
```

Rank 8

Deformation

```
M8:=Matrix(Integers(), [[-2,0,1,0,0,0,0,0,0,0,1,0,0],
                        [0,-2,0,1,0,0,0,0,0,0,1,0,0],
                        [1,0,-2,0,1,0,0,0,0,0,0,0,0],
                        [0,1,0,-2,0,1,0,0,0,0,0,0,0],
                        [0,0,1,0,-2,0,0,0,0,0,0,1,0],
                        [0,0,0,1,0,-2,0,0,0,0,0,1,0],
                        [0,0,0,0,0,0,-2,0,0,0,1,0,1],
                        [0,0,0,0,0,0,0,-2,0,0,1,0,1],
                        [0,0,0,0,0,0,0,0,-2,0,0,1,1],
                        [0,0,0,0,0,0,0,0,0,-2,0,1,1],
                        [1,1,0,0,0,0,1,1,0,0,-2,0,0],
                        [0,0,0,0,1,1,0,0,1,1,0,-2,0],
                        [0,0,0,0,0,0,1,1,1,1,0,0,0]]);

print M8;

n:=13;
pp:=[SetToSequence({1..n} diff {i}): i in [1..n]];
[IsConsistent(Matrix(M8[pp[i]]), Matrix(M8[i])): i in [1..n]];

(Got ‘‘true" for last row, so can remove.
Hence, they are deformations.)

Rank 9

M9:=DiagonalMatrix(Integers(),14,[-2,-2,-2,-2,-2,
-2,-2,-2,-2,-2,-2,-2,0,-2]);
```

```
M9[1,2]:=1; M9[2,1]:=1;
M9[1,13]:=1; M9[13,1]:=1;
M9[3,13]:=1; M9[13,3]:=1;
M9[3,14]:=1; M9[14,3]:=1;
M9[4,13]:=1; M9[13,4]:=1;
M9[4,14]:=1; M9[14,4]:=1;
M9[5,7]:=1; M9[7,5]:=1;
M9[5,14]:=1; M9[14,5]:=1;
M9[6,8]:=1; M9[8,6]:=1;
M9[6,14]:=1; M9[14,6]:=1;
M9[7,9]:=1; M9[9,7]:=1;
M9[8,10]:=1; M9[10,8]:=1;
M9[9,11]:=1; M9[11,9]:=1;
M9[10,12]:=1; M9[12,10]:=1;

n:=14;
pp:=[SetToSequence({1..n} diff {i}): i in [1..n]];
[IsConsistent(Matrix(M9[pp[i]]), Matrix(M9[i])): i in [1..n]];

M91:=Submatrix(M9,[1,2,4,5,6,7,8,9,10,11,12,13,14],
[1,2,4,5,6,7,8,9,10,11,12,13,14]);

n:=13;
pp:=[SetToSequence({1..n} diff {i}): i in [1..n]];
[IsConsistent(Matrix(M91[pp[i]]), Matrix(M91[i])): i in [1..n]];

T:=Matrix(Integers(), [[-2,1,1,0,0,0,1,0,0,0],
```

```
                        [1,-2,0,0,0,0,0,0,0,0],
                        [1,0,-2,1,0,0,0,0,0,0],
                        [0,0,1,-2,1,0,0,0,0,0],
                        [0,0,0,1,-2,1,0,0,0,0],
                        [0,0,0,0,1,-2,0,0,0,0],
                        [1,0,0,0,0,0,-2,1,0,0],
                        [0,0,0,0,0,0,1,-2,1,0],
                        [0,0,0,0,0,0,0,1,-2,1],
                        [0,0,0,0,0,0,0,0,1,-2]]);

A3:=Matrix(Integers(), [[-2,1,0],
                        [1,-2,1],
                        [0,1,-2]]);

M9B:=DiagonalJoin(<T,A3>);

dicompare(M91,M9B); (Got ‘‘true")
```

Rank 10

Belcastro Equivalence for Class 2 (u^3)

```
M0:=DiagonalMatrix(Integers(),17,[-2,-2,-2,-2,-2,-2,
-2,-2,-2,-2,-2,-2,-2,-2,-2,-2,0]);
M0[1,16]:=1; M0[16,1]:=1;
M0[1,17]:=1; M0[17,1]:=1;
M0[2,3]:=1; M0[3,2]:=1;
```

```
M0[2,17]:=1; M0[17,2]:=1;
M0[4,6]:=1; M0[6,4]:=1;
M0[4,16]:=1; M0[16,4]:=1;
M0[5,7]:=1; M0[7,5]:=1;
M0[5,16]:=1; M0[16,5]:=1;
M0[6,8]:=1; M0[8,6]:=1;
M0[7,9]:=1; M0[9,7]:=1;
M0[8,10]:=1; M0[10,8]:=1;
M0[9,11]:=1; M0[11,9]:=1;
M0[10,12]:=1; M0[12,10]:=1;
M0[11,13]:=1; M0[13,11]:=1;
M0[12,14]:=1; M0[14,12]:=1;
M0[13,15]:=1; M0[15,13]:=1;

n:=17;
pp:=[SetToSequence({1..n} diff {i}): i in [1..n]];
[IsConsistent(Matrix(M0[pp[i]]), Matrix(M0[i])): i in [1..n]];

M01:=Submatrix(M0,[1,2,4,5,6,7,8,9,10,11,12,13,14,15,16,17],
[1,2,4,5,6,7,8,9,10,11,12,13,14,15,16,17]);

n:=16;
pp:=[SetToSequence({1..n} diff {i}): i in [1..n]];
[IsConsistent(Matrix(M01[pp[i]]), Matrix(M01[i])): i in [1..n]];

E8:=Matrix(Integers(), [[-2,1,0,0,0,0,0,0],
                        [1,-2,1,0,0,0,0,0],
```

```
                        [0,1,-2,1,1,0,0,0],
                        [0,0,1,-2,0,0,0,0],
                        [0,0,1,0,-2,1,0,0],
                        [0,0,0,0,1,-2,1,0],
                        [0,0,0,0,0,1,-2,1],
                        [0,0,0,0,0,0,1,-2]]);

A6:=Matrix(Integers(), [[-2,1,0,0,0,0],
                        [1,-2,1,0,0,0],
                        [0,1,-2,1,0,0],
                        [0,0,1,-2,1,0],
                        [0,0,0,1,-2,1],
                        [0,0,0,0,1,-2]]);

U:=Matrix(Integers(), [[0,1],
                        [1,0]]);

MOB:=DiagonalJoin(<E8,A6,U>);

dicompare(M01,MOB); (Got ‘‘true")

Deformation for Class 3 (u^3)

M0:=DiagonalMatrix(Integers(),13,[-2,-2,-2,-2,-2,-2,
-2,-2,-2,4,-2,0,0]);
M0[1,12]:=1; M0[12,1]:=1;
M0[1,13]:=1; M0[13,1]:=1;
```

```
M0[2,10]:=1; M0[10,2]:=1;
M0[2,11]:=1; M0[11,2]:=1;
M0[3,10]:=1; M0[10,3]:=1;
M0[3,11]:=1; M0[11,3]:=1;
M0[4,10]:=1; M0[10,4]:=1;
M0[4,11]:=1; M0[11,4]:=1;
M0[5,10]:=1; M0[10,5]:=1;
M0[5,11]:=1; M0[11,5]:=1;
M0[6,10]:=1; M0[10,6]:=1;
M0[6,11]:=1; M0[11,6]:=1;
M0[7,10]:=1; M0[10,7]:=1;
M0[7,11]:=1; M0[11,7]:=1;
M0[8,10]:=1; M0[10,8]:=1;
M0[8,11]:=1; M0[11,8]:=1;
M0[9,10]:=1; M0[10,9]:=1;
M0[9,11]:=1; M0[11,9]:=1;

n:=13;
pp:=[SetToSequence({1..n} diff {i}): i in [1..n]];
[IsConsistent(Matrix(M0[pp[i]]), Matrix(M0[i])): i in [1..n]];
```

(last row got ‘‘true", so can remove. Hence, they are deformations)

Rank 14

Deformation for 1a (Order 16)

```
M:=DiagonalMatrix(Integers(),18,[-2,-2,-2,-2,-2,-2,
-2,-2,-2,-2,-2,-2,-2,-2,-2,-2,0,0]);
M[1,5]:=1; M[5,1]:=1;
M[1,15]:=1; M[15,1]:=1;
M[2,6]:=1; M[6,2]:=1;
M[2,15]:=1; M[15,2]:=1;
M[3,7]:=1; M[7,3]:=1;
M[3,15]:=1; M[15,3]:=1;
M[4,8]:=1; M[8,4]:=1;
M[4,15]:=1; M[15,4]:=1;
M[5,9]:=1; M[9,5]:=1;
M[6,10]:=1; M[10,6]:=1;
M[7,11]:=1; M[11,7]:=1;
M[8,12]:=1; M[12,8]:=1;
M[9,16]:=1; M[16,9]:=1;
M[10,16]:=1; M[16,10]:=1;
M[11,16]:=1; M[16,11]:=1;
M[12,16]:=1; M[16,12]:=1;
M[13,17]:=1; M[17,13]:=1;
M[13,18]:=1; M[18,13]:=1;
M[14,17]:=1; M[17,14]:=1;
M[14,18]:=1; M[18,14]:=1;

n:=18;
pp:=[SetToSequence({1..n} diff {i}): i in [1..n]];
[IsConsistent(Matrix(M[pp[i]]), Matrix(M[i])): i in [1..n]];
```

(Last row was ‘‘true", so can omit. Thus, they are deformations)

Deformation for 1b (Order 24)

```
M:=DiagonalMatrix(Integers(),19,[-2,-2,-2,-2,-2,
-2,-2,-2,-2,-2,-2,-2,-2,-2,-2,-2,0,-2,-2]);
M[1,4]:=1; M[4,1]:=1;
M[1,19]:=1; M[19,1]:=1;
M[2,5]:=1; M[5,2]:=1;
M[2,19]:=1; M[19,2]:=1;
M[3,6]:=1; M[6,3]:=1;
M[3,19]:=1; M[19,3]:=1;
M[4,7]:=1; M[7,4]:=1;
M[5,8]:=1; M[8,5]:=1;
M[6,9]:=1; M[9,6]:=1;
M[7,18]:=1; M[18,7]:=1;
M[8,18]:=1; M[18,8]:=1;
M[9,18]:=1; M[18,9]:=1;
M[10,16]:=1; M[16,10]:=1;
M[10,19]:=1; M[19,10]:=1;
M[11,16]:=1; M[16,11]:=1;
M[11,18]:=1; M[18,11]:=1;
M[12,16]:=1; M[16,12]:=1;
M[12,14]:=1; M[14,12]:=1;
M[13,15]:=1; M[15,13]:=1;
M[13,16]:=1; M[16,13]:=1;
M[14,17]:=1; M[17,14]:=1;
```

```
M[15,17]:=1; M[17,15]:=1;
M[17,18]:=1; M[18,17]:=1;
M[17,19]:=1; M[19,17]:=1;

n:=19;
pp:=[SetToSequence({1..n} diff {i}): i in [1..n]];
[IsConsistent(Matrix(M[pp[i]]), Matrix(M[i])): i in [1..n]];
```

(Last row was ‘‘true", so can omit. Hence, they are deformations)

Rank 16

Deformation

(Were identical so nothing was removed in the transformation)

Order 8

Rank 6

Belcastro Equivalence

```
M6:=DiagonalMatrix(Integers(),8,[-2,-2,-2,-2,-2,-2,4,0]);
M6[1,2]:=1; M6[2,1]:=1;
M6[2,3]:=1; M6[3,2]:=1;
M6[3,4]:=1; M6[4,3]:=1;
```

```
M6[4,8]:=1; M6[8,4]:=1;

M6[5,7]:=1; M6[7,5]:=1;

M6[6,7]:=1; M6[7,6]:=1;

M6[7,8]:=2; M6[8,7]:=2;


n:=8;

pp:=[SetToSequence({1..n} diff {i}): i in [1..n]];

[IsConsistent(Matrix(M6[pp[i]]), Matrix(M6[i])): i in [1..n]];


M61:=Submatrix(M6,[2..8],[2..8]);


n:=7;

pp:=[SetToSequence({1..n} diff {i}): i in [1..n]];

[IsConsistent(Matrix(M61[pp[i]]), Matrix(M61[i])): i in [1..n]];


D5:=Matrix(Integers(),[[-2,1,0,0,0],
                        [1,-2,1,0,0],
                        [0,1,-2,1,1],
                        [0,0,1,-2,0],
                        [0,0,1,0,-2]]);


U:=Matrix(Integers(),[[0,1],[1,0]]);


M6B:=DiagonalJoin(<D5,U>);


dicompare(M61,M6B); (Got ‘‘true")
```

Rank 10

Deformation

Actually all identical, so nothing to show...

Rank 13

Belcastro Equivalence

```
M3:=DiagonalMatrix(Integers(),16,[-2,-2,-2,-2,-2,
-2,-2,-2,-2,-2,-2,-2,-2,-2,-2,-2]);
M3[1,2]:=1; M3[2,1]:=1;
M3[1,16]:=1; M3[16,1]:=1;
M3[2,3]:=1; M3[3,2]:=1;
M3[3,4]:=1; M3[4,3]:=1;
M3[5,7]:=1; M3[7,5]:=1;
M3[5,16]:=1; M3[16,5]:=1;
M3[6,8]:=1; M3[8,6]:=1;
M3[6,16]:=1; M3[16,6]:=1;
M3[7,9]:=1; M3[9,7]:=1;
M3[8,10]:=1; M3[10,8]:=1;
M3[9,15]:=1; M3[15,9]:=1;
M3[10,15]:=1; M3[15,10]:=1;
M3[11,13]:=1; M3[13,11]:=1;
M3[11,15]:=1; M3[15,11]:=1;
M3[12,14]:=1; M3[14,12]:=1;
```

```
M3[12,15]:=1; M3[15,12]:=1;


n:=16;
pp:=[SetToSequence({1..n} diff {i}): i in [1..n]];
[IsConsistent(Matrix(M3[pp[i]]), Matrix(M3[i])): i in [1..n]];


M31:=Submatrix(M3,[1,2,3,5,6,7,8,9,10,11,12,13,14,15,16],
[1,2,3,5,6,7,8,9,10,11,12,13,14,15,16]);


n:=15;
pp:=[SetToSequence({1..n} diff {i}): i in [1..n]];
[IsConsistent(Matrix(M31[pp[i]]), Matrix(M31[i])): i in [1..n]];


E6:=Matrix(Integers(),[[-2,1,0,0,0,0],
                        [1,-2,1,0,0,0],
                        [0,1,-2,1,1,0],
                        [0,0,1,-2,0,0],
                        [0,0,1,0,-2,1],
                        [0,0,0,0,1,-2]]);


A7:=Matrix(Integers(),[[-2,1,0,0,0,0,0],
                        [1,-2,1,0,0,0,0],
                        [0,1,-2,1,0,0,0],
                        [0,0,1,-2,1,0,0],
                        [0,0,0,1,-2,1,0],
                        [0,0,0,0,1,-2,1],
                        [0,0,0,0,0,1,-2]]);
```

```
U:=Matrix(Integers(),[[0,1],[1,0]]);

M3B:=DiagonalJoin(<E6,A7,U>);

dicompare(M31,M3B); (Got ‘‘true")
```

Rank 17

Deformation

Identical, so nothing to show...

Bibliography

[1] Michela Artebani, Samuel Boissière, and Alessandra Sarti, *The Berglund-Hübsch-Chiodo-Ruan mirror symmetry for K3 surfaces*, J. Math. Pures Appl. (9) **102** (2014), no. 4, 758–781.

[2] Wolf P. Barth, Klaus Hulek, Chris A. M. Peters, and Antonius Van de Ven, *Compact complex surfaces*, Second, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, vol. 4, Springer-Verlag, Berlin, 2004.

[3] Sarah-Marie Belcastro, *Picard Lattices of Families of K3 Surfaces*, Ph.D. Thesis, 1997.

[4] Per Berglund and Tristan Hübsch, *A generalized construction of mirror manifolds*, Nuclear Phys. B **393** (1993), no. 1-2, 377–391.

[5] Matthew Brown, *Construction and isomorphism of landauginzburg b-model frobenius algebras*, 2016.

[6] Philip Candelas, Xenia C. de la Ossa, Paul S. Green, and Linda Parkes, *A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory*, Essays on mirror manifolds, 1992, pp. 31–95. MR1191420

[7] Arthur Cayley, *On the triple tangent planes to a surface of the third order*, The Cambridge and Dublin Mathematical Journal **4** (1849), 445–456.

[8] Paola Comparin, Christopher Lyons, Nathan Priddis, and Rachel Suggs, *The mirror symmetry of K3 surfaces with non-symplectic automorphisms of prime order*, Adv. Theor. Math. Phys. **18** (2014), no. 6, 1335–1368.

[9] Paola Comparin and Nathan Priddis, *Bhk mirror symmetry for k3 surfaces with non-symplectic automorphism*, 2018.

[10] Nathan Cordner, *Isomorphisms of landau-ginzburg b-models*, 2016.

[11] Lance J. Dixon, *Some world-sheet properties of superstring compactifications, on orbifolds and otherwise*, Superstrings, unified theories and cosmology 1987 (Trieste, 1987), 1988, pp. 67–126. MR1104035

[12] Huijun Fan, Tyler Jarvis, and Yongbin Ruan, *The Witten equation, mirror symmetry, and quantum singularity theory*, Ann. of Math. (2) **178** (2013), no. 1, 1–106. MR3043578

[13] B. R. Greene and M. R. Plesser, *Duality in Calabi-Yau moduli space*, Nuclear Phys. B **338** (1990), no. 1, 15–37. MR1059831

[14] Daniel Huybrechts, *Lectures on K3 surfaces*, Cambridge Studies in Advanced Mathematics, vol. 158, Cambridge University Press, Cambridge, 2016. MR3586372

[15] Tyler Kelly, *Berglund-Hübsch-Krawitz mirrors via Shioda maps*, Adv. Theor. Math. Phys. **17** (2013), 1425–1449.

[16] K. Kodaira, L. Nirenberg, and D. C. Spencer, *On the existence of deformations of complex analytic structures*, Ann. of Math. (2) **68** (1958), 450–459. MR0112157

[17] Marc Krawitz, *FJRW rings and Landau-Ginzburg Mirror Symmetry*, Ph.D. Thesis, 2010.

[18] Maximilian Kreuzer and Harald Skarke, *On the classification of quasihomogeneous functions*, Comm. Math. Phys. **150** (1992), no. 1, 137–147. MR1188500

[19] Wolfgang Lerche, Cumrun Vafa, and Nicholas P. Warner, *Chiral rings in $N = 2$ superconformal theories*, Nuclear Phys. B **324** (1989), no. 2, 427–474. MR1025424

[20] V. V. Nikulin, *Integer symmetric bilinear forms and some of their geometric applications*, Izv. Akad. Nauk SSSR Ser. Mat. **43** (1979), no. 1, 111–177, 238.

[21] George Salmon, *On the triple tangent planes to a surface of the third order*, The Cambridge and Dublin Mathematical Journal **4** (1849), 252–260.

[22] Ryan Sandberg, *A nonabelian landau-ginzburg b-model construction*, 2015.

[23] Mark Shoemaker, *Birationality of Berglund-Hübsch-Krawitz mirrors*, Comm. Math. Phys. **331** (2014), no. 2, 417–429. MR3238520

[24] Julian Tay, *Poincaré polynomial of fjrw rings and the group-weights conjecture*, 2007.

[25] André Weil, *Final report on contract*, Scientific works. collected papers, ii, 1958, pp. 390–395,545–547. MR0537935

[26] Shing-Tung Yau and Steve Nadis, *The shape of inner space*, Basic Books, New York, 2010. String theory and the geometry of the universe's hidden dimensions. MR2722198

[27] Takashi Yonemura, *Hypersurface simple K3 singularities*, Tohoku Math. J. (2) **42** (1990), no. 3, 351–380.