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The Minimum Rank Problem for Outerplanar Graphs

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ABSTRACT

The Minimum Rank Problem for Outerplanar Graphs

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Given a simple graph G with vertex set $V(G) = \{1, 2, \ldots, n\}$ define $\mathcal{S}(G)$ to be the set of all real symmetric matrices A such that for all $i \neq j$, $a_{ij} \neq 0$ if and only if $ij \in E(G)$. The range of the ranks of matrices in $\mathcal{S}(G)$ is of interest and can be determined by finding the minimum rank. The *minimum rank of a graph*, denoted $\mathrm{mr}(G)$, is the minimum rank achieved by a matrix in $\mathcal{S}(G)$. The *maximum nullity of a graph*, denoted $M(G)$, is the maximum nullity achieved by a matrix in $\mathcal{S}(G)$. Note that $\mathrm{mr}(G) + M(G) = |V(G)|$ and so in finding the maximum nullity of a graph, the minimum rank of a graph is also determined. The minimum rank problem for a graph G asks us to determine $\mathrm{mr}(G)$ which in general is very difficult. A simple graph is *planar* if there exists a drawing of G in the plane such that any two line segments representing edges of G intersect only at a point which represents a vertex of G. A planar drawing partitions the rest of the plane into open regions called *faces*. A graph is *outerplanar* if there exists a planar drawing of G such that every vertex lies on the outer face. We consider the class of outerplanar graphs and summarize some of the recent results concerning the minimum rank problem for this class.

The *path cover number* of a graph, denoted $P(G)$, is the minimum number of vertex-disjoint paths needed to cover all the vertices of G. We show that for all outerplanar graphs G, $P(G) \geq M(G)$. We identify a subclass of outerplanar graphs, called *partial 2-paths*, for which $P(G) = M(G)$. We give a different characterization for another subset of outerplanar graphs, unicyclic graphs, which determines whether $M(G) = P(G)$ or $M(G) = P(G) - 1$. We give an example of a 2-connected outerplanar graph for which $P(G) > M(G)$.

A *cover* of a graph G is a collection of subgraphs $G_1, \ldots, G_k$ of G such that $\cup E(G_i) = E(G)$. The *rank-sum* of a cover $\mathcal{C} = \{G_1, \ldots, G_k\}$ is denoted $\mathrm{rs}(\mathcal{C})$ and is equal to $\sum \mathrm{mr}(G_i)$. We show that for an outerplanar graph G, there exists an edge-disjoint cover of G consisting of cliques, stars, cycles, and double cycles such that the rank-sum of the cover is equal to the minimum rank of G. Using the fact that such a cover exists allows us to show that the minimum rank of a weighted outerplanar graph is equal to the minimum rank of its underlying simple graph.

Keywords: outerplanar graph, minimum rank, maximum nullity, path cover number, partial 2-path, edge-disjoint cover, weighted graph

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Chapter 1. Introduction

1.1 Introduction

The minimum rank problem for a graph is introduced fully in Chapter 2. A graph is given which determines the zero/nonzero structure of a real symmetric matrix. The set of all matrices with the specified structure form an uncountable class of matrices. What is the smallest rank attained by a matrix in the class? Equivalently, what is the largest dimension of the null space of a matrix in the class? It is also equivalent to determining the maximum multiplicity of any eigenvalue of a matrix in the class.

The first paper, [1], describing the problem as the minimum rank problem for a graph was published in 1996. Since then interest has steadily grown and a survey [2] was published in 2007. The minimum rank problem for acyclic graphs and unicyclic graphs were considered in [3] and [4], respectively. In both papers the path cover number (see Chapter 4) played an important role. Both acyclic graphs and unicyclic graphs are subsets of the larger class of graphs known as outerplanar graphs. Outerplanar graphs are defined and discussed in Chapter 3.

The first result specifically mentioning outerplanar graphs is found in [5] and establishes the path cover number as an upperbound for the maximum nullity of an outerplanar graph. The main results from [5] are found in Sections 4.2 and 4.3. These results sparked interest in outerplanar graphs as noted by the publication of [6]. The main result of Section 4.2, Theorem 4.16, was the main tool used in [7]. Outerplanar graphs and covers are the subject of [8] and a modified proof of the main result from that paper is given in Section 5.2.

The minimum rank problem for a graph is difficult in general, but if the graph is outerplanar there are many tools available to calculate the minimum rank.

1.2 Basic Matrix Theory and Graph Theory

In most cases notation and definitions follow those found in [9]. A matrix A is *symmetric* if $A^T = A$. The $n \times n$ matrix with a one in every entry will be denoted as J_n. The $n \times n$ identity matrix will be denoted as I_n.

Given an $m \times n$ matrix, $\alpha \subseteq \{1, 2, \ldots, m\}$, and $\beta \subseteq \{1, 2, \ldots, n\}$, the *submatrix* of A that lies in the rows of A indexed by α and the columns indexed by β is denoted by $A[\alpha, \beta]$. In the case that A is square and $\alpha = \beta$, $A[\alpha] := A[\alpha, \alpha]$ is a *principal* submatrix of A. The notation $A(k)$ will be used to represent the principal submatrix of A obtained by deleting the kth row and column.

Fact 1. Let B be a submatrix of A. Then $\operatorname{rank} A \geq \operatorname{rank} B$.

The following is a useful fact (see [9] page 16).

Fact 2. Let A and B be matrices such that $A + B$ is defined. Then

$$\operatorname{rank}(A + B) \leq \operatorname{rank} A + \operatorname{rank} B.$$

Let $G = (V, E)$ be a simple graph with vertex set V and edge set E. The vertex set V is usually the set of natural numbers from 1 to n, while the edge set E consists of 2-element subsets of V such as $\{1, 2\}$ or $\{3, 4\}$. An edge $\{x, y\}$ will usually be written simply as xy unless this notation creates some abiguity.

While our main focus is on simple graphs, it will be necessary to consider a larger class of graphs which contains all simple graphs. By extending the definition of E and allowing it to be a multiset of 2-element subsets of V, *multiple edges* or *parallel edges* may be present between a pair of vertices. Such graphs have been called *multigraphs* and *graphs of parallel edges*. For example if $G = (V, E)$ where $V = \{1, 2, 3, 4\}$ and $E = \{12, 12, 23, 24\}$, the resulting graph has a pair of edges between vertices 1 and 2, and single edges between vertices 2 and 3 and between vertices 2 and 4. Since 23 is in the edge set we say that 2 is *adjacent* to 3 and edge 23 is *incident* to vertices 2 and 3. When v and w are adjacent it is sometimes convenient to write $v \sim w$.

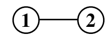
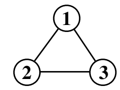
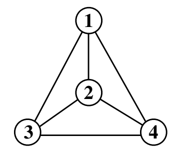
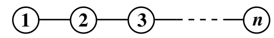
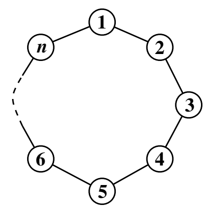
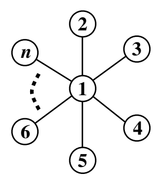
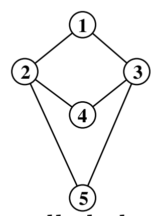
The *order* of a graph G is the number of vertices in $V(G)$, and will be denoted as $|G|$. The *degree* of a vertex v of G is equal to the number of edges incident to v. A vertex v of G is a *dominating vertex* if v is adjacent to every other vertex in $V(G)$. A *pendant* vertex is a vertex of degree 1.

Given graphs $G = (V, E)$ and $G' = (V', E')$, if $V' \subseteq V$ and $E' \subseteq E$ we say G' is a *subgraph* of G and G is a *supergraph* of G'. Continuing with the assumption that $V' \subseteq V$, the subgraph of G *induced* by V' is the graph $H = (V', E')$ where $e = v_1v_2 \in E'$ if and only if $v_1, v_2 \in V'$ and $v_1v_2 \in E(G)$. The *complement* of a graph $G = (V, E)$, denoted $\overline{G}$, is the graph on the same vertex set as G and edge set E' where $e \in E'$ if and only if $e \notin E$.

Given $w \in V(G)$, the graph $G-w$ is obtained from G by deleting w and all edges incident to w. In other words $G-w$ is the subgraph of G induced by $V \setminus w$. Similarly given $e \in E(G)$, the graph $G - e$ is the graph obtained from G by deleting the edge e. Given graphs G and G', $G \cup G'$ is defined to be $(V \cup V', E \cup E')$. Unless otherwise indicated the union of two graphs is a disjoint union, i.e. $V \cap V' = \emptyset$.

A *path* P in a graph G consists of distinct vertices $v_1, v_2, \ldots, v_k$ such that $v_i \sim v_{i+1}$ for all $i < k$. At times it will convenient to simply write $P = v_1v_2 \ldots v_k$ and say P is a path from v_1 to v_k. The vertices v_1 and v_k are the *ends* of the path and will at times be referred to as the *pendant vertices* of P. The *length* of a path is its number of edges. The *path graph* of length $k-1$ has k vertices and will be denoted as P_k. A graph G is *connected* if for every pair of distinct vertices v and w, there exists a path from v to w. Otherwise, G is disconnected.

A *cycle* C can be defined as a path $P = v_1v_2 \ldots v_k$ with the edge v_1v_k added to it. We write $C = v_1v_2 \ldots v_kv_1$ and the *length* of C is its number of edges. The *cycle graph* of length k has k vertices and will be denoted as C_k. A graph for which all vertices are pairwise adjacent is called *complete*. The complete graph on n vertices is denoted as K_n. The *complete r-partite* graph, denoted $K_{n_1,n_2,\ldots,n_r}$, is defined to be the graph $\overline{(K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_r})}$. When $r = 2$, the graph is a *complete bipartite* graph. A *tree* is a connected acyclic graph. The unique tree on $n \geq 3$ vertices with $n-1$ pendant vertices is called a *star* and is denoted S_n. Also S_n is the complete bipartite graph $K_{1,n-1}$. Notice that $K_1 = P_1$, $K_2 = P_2$, $C_3 = K_3$, and $P_3 = S_3$.

K_2	The complete graph on 2 vertices	
K_3	The complete graph on 3 vertices	
K_4	The complete graph on 4 vertices	
P_n	The path on $n \geq 1$ vertices	
C_n	The cycle on $n \geq 3$ vertices	
S_n	The star on $n \geq 3$ vertices	
$K_{2,3}$	A complete bipartite graph	

The maximal connected subgraphs of G are called the *components* of G. A graph $G = (V, E)$ is *k-connected* if $|G| > k$ and $G - X$ is connected for every set $X \subseteq V$ with $|X| < k$. A more intuitive definition, see [10], is that a graph is k-connected if any two vertices can be joined by k independent paths (paths with vertex-disjoint interiors). All connected graphs on 2 or more vertices are 1-connected. C_n is both 1-connected and 2-connected, but not 3-connected. The greatest integer k such that $G \neq K_n$ is k-connected is the *connectivity* $\kappa(G)$ of G and we define $\kappa(K_n) = n - 1$ for $n \geq 1$. For example $\kappa(C_n) = 2$ and $\kappa(T) = 1$ for every non-trivial tree.

The *neighborhood* of a vertex v is the set of vertices which are adjacent to v and is denoted $N(v)$. The *closed neighborhood* of a vertex v is $N(v) \cup v$ and is denoted $N[v]$. A *clique* is a subset of the vertex set which induces a complete graph. A *simplicial vertex* is a vertex whose neighborhood is a clique.

Let $G = (V, E)$ be a simple graph on n vertices. The *adjacency matrix* of G, denoted $A(G)$, is the $n \times n$, (0,1)-matrix where $a_{ij} = 1$ if and only if $ij \in E(G)$. The *Laplacian matrix* of G, denoted $L(G)$, is the $n \times n$ matrix $D - A(G)$, where $D = diag(d_1, \ldots, d_n)$ and d_i is the degree of vertex i.

Other graph theory terminology will be defined as the need arises.

Chapter 2. The Minimum Rank Problem

The minimum rank problem for a graph was first defined for simple graphs. Given a simple graph G with vertices labeled from 1 to n, $\mathcal{S}(G)$ is defined to be the set of all $n \times n$ real symmetric matrices A whose off-diagonal entries a_{ij} are zero if $i \nsim j$ and nonzero if $i \sim j$. Note that a diagonal entry may be any real number. The minimum rank of a simple graph is defined as the smallest attainable rank of a matrix in $\mathcal{S}(G)$. It is denoted $\mathrm{mr}(G)$ and can be expressed as

$$\min\{\operatorname{rank} A : A \in \mathcal{S}(G)\}.$$

Similarly, the maximum nullity of a simple graph is the largest attainable nullity of a matrix in $\mathcal{S}(G)$ and is denoted $M(G)$. The following observation is a simple consequence of the Rank-Nullity Theorem.

Observation 2.1. *If G is a graph, then* $\mathrm{mr}(G) + M(G) = |G|$.

Given a graph on n vertices and a matrix $A \in \mathcal{S}(G)$ such that $\operatorname{rank} A = \mathrm{mr}(G)$, there exists a constant k such that $A + kI$ is nonsingular. Letting A_1 be formed from A by adding k to a_{11}, and inductively A_t be formed from A_{t-1} by adding k to a_{tt}, $A_1, A_2, \ldots, A_n$ is a sequence of matrices in $\mathcal{S}(G)$ such that $A_n = A + kI$. By Fact 2, adding a rank 1 matrix increases the rank by at most 1. Since $\operatorname{rank} A_1 = \operatorname{mr}(G)$ and $\operatorname{rank} A_n = n$, every rank from $\operatorname{mr}(G)$ to n must be achieved by some matrix in the sequence. Thus $\operatorname{mr}(G)$ determines all possible ranks for matrices in $\mathcal{S}(G)$.

Since the rank of a symmetric matrix is the number of nonzero eigenvalues, minimizing the rank or equivalently maximizing the nullity, is the same as maximizing the multiplicity of zero as an eigenvalue. Note that if $A \in \mathcal{S}(G)$ for some graph G, $A + kI$ is also in $\mathcal{S}(G)$. Since $A + kI$, shifts all eigenvalues of A by k, finding $M(G)$ is equivalent to finding the maximum multiplicity of any eigenvalue for any matrix in $\mathcal{S}(G)$.

Finally, the minimum rank problem is a relaxation of the inverse inertia problem for a graph and the inverse eigenvalue problem for a graph.

2.1 Characterizations of Graphs with Extremal Minimum Rank

In this section we list results which characterize the graphs G with minimum rank equal to 1, 2, and $|G| - 1$. Given a graph G, a matrix $A \in \mathcal{S}(G)$, and an eigenvalue λ of A, we have that $A - \lambda I \in \mathcal{S}(G)$. Since $A - \lambda I$ is symmetric, $\operatorname{rank}(A - \lambda I)$ is equal to the number of nonzero eigenvalues. Since zero is an eigenvalue of $A - \lambda I$ with multiplicity at least 1, $\operatorname{rank}(A - \lambda I) \leq |G| - 1$. Therefore for all graphs G we have that $\operatorname{mr}(G) \leq |G| - 1$.

Example 2.2. Consider the complete graph on $n \geq 2$ vertices, K_n. Note that the all-ones matrix J_n is in $\mathcal{S}(K_n)$ and that $\operatorname{rank} J_n$=1. Since only the zero matrix has rank equal to 0, $\operatorname{mr}(K_n) = 1$ for all $n \geq 2$.

In fact it is not too difficult to show that the only connected graphs with minimum rank equal to 1 are the complete graphs on 2 or more vertices. The following observations are restatements of Observations 1 and 3 in [11].

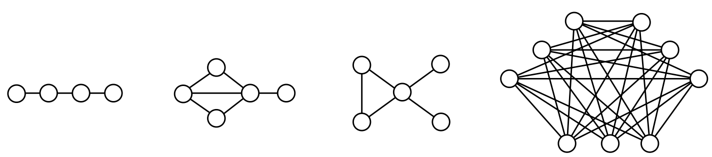
Observation 2.3. *Let G be a connected graph. Then $\operatorname{mr}(G) = 1$ if and only if $G = K_n$ for some $n \geq 2$.*

Observation 2.4. *Let $K_{m,n}$ be the complete bipartite graph with $m, n \geq 1$ and $m + n \geq 3$. Then* $\mathrm{mr}(K_{m,n}) = 2$.

Example 2.5. Consider S_n, the star graph on $n \geq 3$ vertices where the dominating vertex is labeled 1. The adjacency matrix $A(S_n) = \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix}$ has two distinct columns. Thus $2 \geq \operatorname{rank} A(S_n) \geq \mathrm{mr}(S_n)$. By Observation 2.3, $\mathrm{mr}(S_n) \neq 1$. Thus $\mathrm{mr}(S_n) = 2$

In 2004, Barrett, van der Holst, and Loewy, [11], characterized the graphs with minimum rank 2 using forbidden subgraphs. The following is Theorem 9 in [11] and gives a characterization for connected graphs with minimum rank less than or equal to 2.

Theorem 2.6. *Let G be a connected graph. Then* $\mathrm{mr}(G) \leq 2$ *if and only if*



are not induced subgraphs of G.

In 1969 Fiedler [12] proved the following theorem which has implications to the minimum rank problem.

Theorem 2.7. *Let A be an $n \times n$ real symmetric matrix. Then* $\operatorname{rank}(A + D) \geq n - 1$, *for every $n \times n$ real diagonal matrix D, if and only if A is permutation similar to an irreducible tridiagonal matrix.*

The graph corresponding to an $n \times n$ symmetric irreducible tridiagonal matrix is P_n. Being permutation similar in graph theoretical terms is just a renumbering of the vertices. Thus in the language of minimum rank we have the following theorem.

Theorem 2.8. *Let G be a connected graph on n vertices. Then* $\mathrm{mr}(G) = n - 1$ *if and only if $G = P_n$ for some $n \geq 2$.*

At this point it seems appropriate to mention that Johnson, Loewy, and Smith, [13], characterized all the graphs on n vertices for which the minimum rank is equal to $n-2$. Their result will be easier to describe in a later section. One of the graphs which has minimum rank equal to $n-2$ is C_n.

Proposition 2.9. *Let C_n be the cycle on n vertices. Then* $\mathrm{mr}(C_n) = n-2$.

Proof. Since C_n is not a path, by Theorem 2.8 $M(C_n) \neq 1$. Thus $M(C_n) \geq 2$ and by Observation 2.1, $\mathrm{mr}(C_n) \leq n-2$. Since deleting a vertex of C_n yields P_{n-1}, by Proposition 2.11, $\mathrm{mr}(C_n) \geq \mathrm{mr}(P_{n-1})$. By Theorem 2.8, $\mathrm{mr}(P_{n-1}) = n-2$. Thus $\mathrm{mr}(C_n) = n-2$. $\square$

2.2 Formulas for the Minimum Rank of Graphs with Low Connectivity

In this section some basic, but useful early results concerning minimum rank are cited. Formulas for determining the minimum rank which can be used on graphs with low connectivity are also cited.

In 1996, Nylen [1] authored one of the first papers using the terminology minimum rank. The following three propositions are parts of Proposition 1.2 in [1] and give bounds on the minimum rank for some common subgraphs of a graph G. The corresponding bounds in terms of the maximum nullity are also given.

Proposition 2.10. *Let $G_1, \ldots, G_k$ be the components of G. Then*

$$\mathrm{mr}(G) = \sum_{i=1}^{k} \mathrm{mr}(G_i).$$

Proposition 2.11. *Let G be a graph and v a vertex of G. Then*

- $\mathrm{mr}(G) \geq \mathrm{mr}(G-v) \geq \mathrm{mr}(G) - 2$
- $M(G-v) + 1 \geq M(G) \geq M(G-v) - 1.$

Proposition 2.12. *Let G be a graph and e an edge of G. Then*

- $\mathrm{mr}(G) + 1 \geq \mathrm{mr}(G - e) \geq \mathrm{mr}(G) - 1$

- $M(G) + 1 \geq M(G - e) \geq M(G) - 1$.

In [10], a *separation* is an unordered pair $\{A, B\}$ such that $A \cup B = V$ and G has no edge between $A \setminus B$ and $B \setminus A$. As previously defined by van der Holst in [14] and [15] a separation is a pair of subgraphs (G_1, G_2) on the vertex sets A and B, respectively. More precisely, a *separation* (G_1, G_2) of a graph $G = (V, E)$ is a pair of subgraphs $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ such that $V_1 \cup V_2 = V$, $E_1 \cup E_2 = E$, and $E_1 \cap E_2 = \emptyset$. The *order* of a separation is $|V_1 \cap V_2|$. A *k-separation* is a separation of order k. If (G_1, G_2) is a 1-separation of G with $V(G_1) \cap V(G_2) = \{v\}$, we say that G is the *vertex-sum* at v of G_1 and G_2.

If G is a connected graph with $\kappa(G) = k$, there exists a set of k vertices R of G such that $G - R$ has more than one component. Such a set is called a *separator* and when it consists of a single vertex v, v is a *cutvertex* of G. Note that if G has a separator of size k, then G has a k-separation.

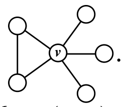
The following theorem was proven independently by Hsieh in her PhD dissertation [16] and by Barioli, Fallat, and Hogben in [17]. In [16] it is Theorem 16 and in [17] it is a special case of Theorem 2.3. It gives a formula for determining the minimum rank of a graph with a cutvertex v in terms of the graphs in its 1-separation and the components of $G - v$. At times it is useful to consider the maximum nullity instead of the minimum rank. So we give both versions and note that it is a simple exercise using Observation 2.1 to arrive at one formula using the other.

Theorem 2.13. *Let G be the vertex-sum at v of G_1 and G_2. Then*

- $M(G) = \max\{M(G_1) + M(G_2) - 1, M(G_1 - v) + M(G_2 - v) - 1\}$,

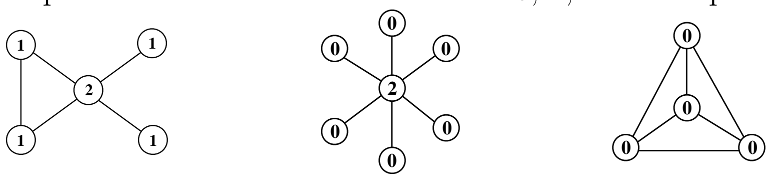
- $\mathrm{mr}(G) = \min\{\mathrm{mr}(G_1) + \mathrm{mr}(G_2), \mathrm{mr}(G_1 - v) + \mathrm{mr}(G_2 - v) + 2\}$.

Example 2.14. Let $G_1 = K_3$ and $G_2 = S_4$. Label a vertex of K_3 and the dominating vertex of S_4 as v. Then the vertex-sum at v of G_1 and G_2 is

$G =$ . Notice that $G_1 - v = K_2$ and $G_2 - v = 3K_1$. Applying Theorem 2.13, $\mathrm{mr}(G) = \min\{\mathrm{mr}(K_3) + \mathrm{mr}(S_4), \mathrm{mr}(K_2) + \mathrm{mr}(3K_1) + 2\}$. By Observation 2.3, $\mathrm{mr}(K_3) = \mathrm{mr}(K_2) = 1$ and from Example 2.5, $\mathrm{mr}(S_4) = 2$. Thus $\mathrm{mr}(G) = \min\{1 + 2, 1 + 0 + 2\} = 3$.

In [17] the *rank-spread* of a vertex v of G, denoted $r_v(G)$, is equal to the difference between $\mathrm{mr}(G)$ and $\mathrm{mr}(G - v)$.

Example 2.15. The following graphs have the vertices labeled with their corresponding rank-spreads. Their minimum ranks are 3, 2, and 1 respectively.



One interesting open question is:

Question 2.16. Does there exist a graph for which each vertex has rank-spread two?

Using the notation for rank-spread, Proposition 2.11 becomes the following proposition.

Proposition 2.17. *Let v be a vertex of a graph G. Then $0 \leq r_v(G) \leq 2$.*

Furthermore, in [17], Theorem 2.13 appears in the following form:

Theorem 2.18. *Let G be the vertex-sum at v of $G_1, G_2, \ldots, G_k$. Then*

$$\mathrm{mr}(G) = \sum_{i=1}^{k} \mathrm{mr}(G_i - v) + r_v(G) \ \textit{where}\ r_v(G) = \min\left\{\sum_{i=1}^{k} r_v(G_i), 2\right\}.$$

In an effort to learn more about the minimum rank problem for simple graphs, the family of graphs under consideration was expanded to include graphs with parallel edges. In doing so it was necessary to modify the definition of $\mathcal{S}(G)$. The following extension of $\mathcal{S}(G)$ was given by Hein van der Holst in [14].

Definition 2.19. *Let G be a graph of parallel edges with vertices $1, \ldots, n$. Let $\mathcal{S}(G)$ be the set of all $n \times n$ real symmetric matrices $A = [a_{ij}]$ such that*

(i) $a_{ij} = 0$ if $i \neq j$ and $i \nsim j$,

(ii) $a_{ij} \neq 0$ *if* $i \neq j$ *and there is exactly one edge joining* i *and* j*,*

Example 2.20. *Let* $G = (V, E)$ *be the graph with* $V = \{1, 2, 3, 4\}$ *and* $E = \{12, 12, 23, 34, 34, 14, 24\}$*.*

A matrix $A \in \mathcal{S}(G)$ *is of the form* $\begin{bmatrix} d_1 & x & 0 & a \\ x & d_2 & c & b \\ 0 & c & d_3 & y \\ a & b & y & d_4 \end{bmatrix}$ *where* $d_i, x, y \in \mathbb{R}$ *and the product* $abc \neq 0$*.*

A *simple realization* H of G is a simple subgraph of G in which each set of multiple edges between a pair of vertices is replaced either by exactly one edge or deleted completely. Thus the minimum rank of a graph G with parallel edges is just the minimum of the minimum ranks of the simple realizations of G. In other words, if G is a graph with parallel edges,

$$\mathrm{mr}(G) = \min\{\mathrm{mr}(H) : H \text{ is a simple realization of } G\}.$$

Given a multigraph $G(V, E)$ and two vertices v_1,v_2 of G, define $G/v_1v_2 = (V', E')$ as the graph obtained from G by *identifying* v_1 and v_2 (see [18] page 55). Specifically, V' consists of the vertices in V with the exception that v_1 and v_2 are replaced by a single vertex v. The edge set E' consists of the edges of E with the exception that every edge of the form v_1x or v_2x where x is any vertex distinct from v_1 or v_2 is replaced with vx, and any edge of the form v_1v_2 is deleted.

With this new notation in place, the 2-separation formula for calculating the minimum rank of a graph can be stated. The following theorem of van der Holst is Theorem 14 and Corollary 15 in [14].

Theorem 2.21. *Let* (G_1, G_2) *be a 2-separation of* G *with* $R = \{r_1, r_2\} = V(G_1) \cap V(G_2)$*. Let* H_1 *and* H_2 *be obtained from* G_1 *and* G_2*, respectively, by inserting an edge between* r_1 *and* r_2*.*

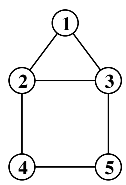
Then $\mathrm{mr}(G) = \min\{\mathrm{mr}(G_1) + \mathrm{mr}(G_2),$

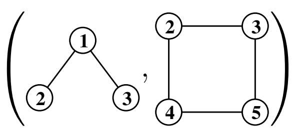
$$\mathrm{mr}(H_1) + \mathrm{mr}(H_2),$$

$$\begin{aligned}
&\operatorname{mr}(G_1/r_1r_2) + \operatorname{mr}(G_2/r_1r_2) + 2,\\
&\operatorname{mr}(G_1 - r_1) + \operatorname{mr}(G_2 - r_1) + 2,\\
&\operatorname{mr}(G_1 - r_2) + \operatorname{mr}(G_2 - r_2) + 2,\\
&\operatorname{mr}(G_1 - R) + \operatorname{mr}(G_2 - R) + 4\}.
\end{aligned}$$

and $M(G) = \max\{M(G_1) + M(G_2) - 2,$

$$\begin{aligned}
&M(H_1) + M(H_2) - 2,\\
&M(G_1/r_1r_2) + M(G_2/r_1r_2) - 2,\\
&M(G_1 - r_1) + M(G_2 - r_1) - 2,\\
&M(G_1 - r_2) + M(G_2 - r_2) - 2,\\
&M(G_1 - R) + M(G_2 - R) - 2\}.
\end{aligned}$$

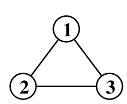
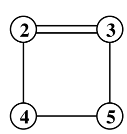
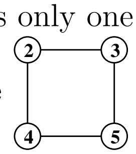
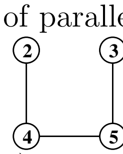
Example 2.22. Let G be the graph  and consider the 2-separation

$(G_1, G_2) =$ . Applying Theorem 2.21, $\operatorname{mr}(G)$ is the minimum of the 6 terms given.

- $\operatorname{mr}(G_1) + \operatorname{mr}(G_2)$

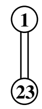
  Note that $G_1 = P_3$ and $G_2 = C_4$. By Theorem 2.8 and Proposition 2.9, $\operatorname{mr}(P_3) = 2$ and $\operatorname{mr}(C_4) = 2$. Thus $\operatorname{mr}(G_1) + \operatorname{mr}(G_2) = \operatorname{mr}(P_3) + \operatorname{mr}(C_4) = 2 + 2 = 4$.

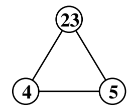
- $\operatorname{mr}(H_1) + \operatorname{mr}(H_2)$

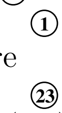
  The graphs H_1 and H_2 are  and , respectively. Notice that $H_1 = K_3$ and so by Observation 2.3, $\operatorname{mr}(K_3) = 1$. The graph H_2 is a graph with parallel edges. Thus to calculate the minimum rank we must find the minimum of the minimum ranks of all simple realizations of H_2. Since there is only one set of parallel edges, there are only two simple realizations of H_2. They are  and . Using the same theorems as in the previous case, $\operatorname{mr}(C_4) = 2$ and $\operatorname{mr}(P_4) = 3$. Thus $\operatorname{mr}(H_2) =$

$\min\{\mathrm{mr}(C_4), \mathrm{mr}(P_4)\} = \min\{2,3\} = 2$. Thus $\mathrm{mr}(H_1) + \mathrm{mr}(H_2) = \mathrm{mr}(K_3) + \mathrm{mr}(C_4) = 1 + 2 = 3$.

- $\mathrm{mr}(G_1/23) + \mathrm{mr}(G_2/23) + 2$

  The graphs $G_1/23$ and $G_2/23$ are 

 and 

. Now $G_1/23$ is a graph with parallel edges. The simple realizations of $G_1/23$ are 

 and 

 which are $2K_1$ and K_2, respectively. So $\mathrm{mr}(G_1/23) = \min\{\mathrm{mr}(2K_1), \mathrm{mr}(K_2)\} = \min\{0,1\} = 0$. Note that $G_2/23 = K_3$, so $\mathrm{mr}(G_2/23) = 1$. Thus $\mathrm{mr}(G_1/23) + \mathrm{mr}(G_2/23) + 2 = \mathrm{mr}(2K_1) + \mathrm{mr}(K_3) + 2 = 0 + 1 + 2 = 3$.

- $\mathrm{mr}(G_1 - 2) + \mathrm{mr}(G_2 - 2) + 2$

  The graphs $G_1 - 2$ and $G_2 - 2$ are K_2 and P_3, respectively. Thus $\mathrm{mr}(G_1 - 2) + \mathrm{mr}(G_2 - 2) + 2 = \mathrm{mr}(K_2) + \mathrm{mr}(P_3) + 2 = 1 + 2 + 2 = 5$.

- $\mathrm{mr}(G_1 - 3) + \mathrm{mr}(G_2 - 3) + 2$.

  The graphs $G_1 - 3$ and $G_2 - 3$ are K_2 and P_3, respectively. Thus $\mathrm{mr}(G_1 - 3) + \mathrm{mr}(G_2 - 3) + 2 = \mathrm{mr}(K_2) + \mathrm{mr}(P_3) + 2 = 1 + 2 + 2 = 5$.

- $\mathrm{mr}(G_1 - \{2,3\}) + \mathrm{mr}(G_2 - \{2,3\}) + 4$

  The graphs $G_1 - \{2,3\}$ and $G_2 - \{2,3\}$ are K_1 and K_2, respectively. Thus $\mathrm{mr}(G_1 - \{2,3\}) + \mathrm{mr}(G_2 - \{2,3\}) + 4 = \mathrm{mr}(K_1) + \mathrm{mr}(K_2) + 4 = 0 + 1 + 4 = 5$.

Thus the minimum of the 6 terms is 3, and $\mathrm{mr}(G) = 3$.

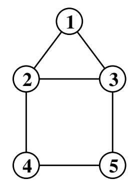
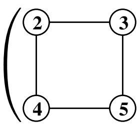
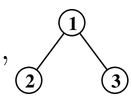
The following lemma is a generalization of Theorem 17 and Corollary 18 in [14] and is a useful special case of Theorem 2.21. If $G_2 = P_k$ for some $k \geq 3$, it is only necessary to check 2 of the 6 terms.

Lemma 2.23. *Let (G_1, P_k), $k \geq 3$ be a 2-separation of a graph G with $V(G_1) \cap V(G_2) = \{r_1, r_2\}$. Then*

- $\mathrm{mr}(G) = \min\{\mathrm{mr}(H_1) + k - 2, \mathrm{mr}(G/r_1r_2) + k - 1\}$
- $M(G) = \max\{M(H_1), M(G_1/r_1r_2)\}$

where H_1 is obtained from G_1 by inserting an edge between r_1 and r_2.

It should also be pointed out that the formula for the maximum nullity does not depend on the length of the path.

Example 2.24. Let G be the same graph as in Example 2.22, , and consider the 2-separation $(G_1, P_3) = \Bigg($ , , $\Bigg)$. Applying Lemma 2.23, $\mathrm{mr}(G) = \min\{\mathrm{mr}(H_1)+1, \mathrm{mr}(G_1/23) + 2\}$. Using the information derived in Example 2.22, $\mathrm{mr}(H_1) = \mathrm{mr}(C_4) = 2$ and $\mathrm{mr}(G_1/23) = \mathrm{mr}(K_3) = 1$. Thus as was seen in the previous example, $\mathrm{mr}(G) = 3$.

The following lemma relates $\mathrm{mr}(G)$ to $\mathrm{mr}(G/v_1v_2)$ and the second inequality appears in van der Holst [14] as Lemma 10. Note that v_1 and v_2 are any two vertices of G.

Lemma 2.25. *Let G be a non-trivial graph with labeled vertices v_1 and v_2. Then*

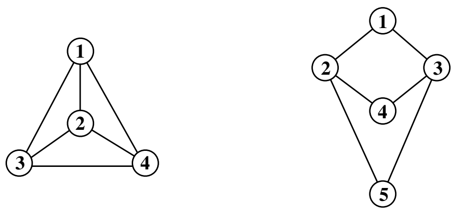
$$\mathrm{mr}(G) \geq \mathrm{mr}(G/v_1v_2) \geq \mathrm{mr}(G) - 2.$$

Chapter 3. Outerplanar Graphs

3.1 Outerplanar Graphs

For the most part the following terminology is taken from [19], [10], and [20]. In a *drawing* of a graph G, vertices are represented by small circles and edges are represented by line segments (or curves if necessary). A *crossing* is a point of a graph drawing where two edges intersect. A *planar drawing* is a drawing in which no two edges cross and is referred to as a *plane graph.* A graph G is *planar* if there exists a planar drawing of G. A planar drawing partitions the rest of the plane into open sets called *faces*. Each plane graph has exactly one unbounded face, called the *outer face*. A graph G is *outerplanar* if there exists a planar drawing of G such that every vertex is incident with the outer face. Such a drawing is at times referred to as an *outerplane graph* or an *outerplanar drawing*.

Example 3.1. Here are the smallest two graphs which are not outerplanar. In other words there does not exist a planar drawing of either K_4 or $K_{2,3}$ such that all the vertices are incident to the outer face and no two edges intersect other than at their endpoints.



Example 3.2. Of the graphs which have been introduced to this point, any graph on 3 or fewer vertices is outerplanar. All forests and cycles are examples of outerplanar graphs.

Before continuing it is necessary to define a few terms. Given a graph G and an edge e of G, to *subdivide* e is to delete e, add a new vertex v, and join v to the ends of e. In other words the edge e is replaced by a path of length 2. The graph resulting from subdividing e in G is denoted G_e. Any graph created from a graph G by a sequence of edge subdivisions is called a *subdivision* of G. Two graphs are defined to be *homeomorphic* if both can be obtained from the same graph by a sequence of edge subdivisions.

The following theorem appears in [19] as Theorem 11.10 and is due to G. Chartrand and F. Harary in 1967. Let e be an edge of K_4. The *diamond* graph is $K_4 - e$.

Theorem 3.3. *A graph which is not the diamond is outerplanar if and only if it has no subgraph homeomorphic to K_4 or $K_{2,3}$.*

A *cut-edge* is an edge which upon deletion increases the number of components. A *bridge* is the K_2 subgraph induced by the vertices of a cut-edge. A *block* is a maximal connected subgraph of G which does not contain a cutvertex. Thus a block is either a maximal 2-connected subgraph, a bridge, or an isolated vertex.

Given a plane graph its *dual*, denoted G^*, is constructed by placing a vertex in each face of G and if two faces have an edge e in common joining their corresponding vertices by an edge e^* crossing only e. The *weak dual* of a plane graph G, denoted G^w, is obtained from G^* by deleting the vertex corresponding to the outer face. A planar graph may have many weak duals each depending on a distinct planar drawing. In 1932 Whitney proved that a 3-connected planar graph has essentially one planar drawing (see [10] page 90 or [20] page 628). In fact a 2-connected outerplanar graph has essentially one outerplanar drawing(see [21]). Given an outerplanar drawing of a graph G, the weak dual is a disjoint union of the weak duals of the blocks of G ([22]). In Figure 3.1, G has solid lines for edges and larger circles for vertices while G^* has dashed lines for edges and smaller circles for vertices. The subgraph with the alternating dashes and dots as edges is the weak dual G^w.

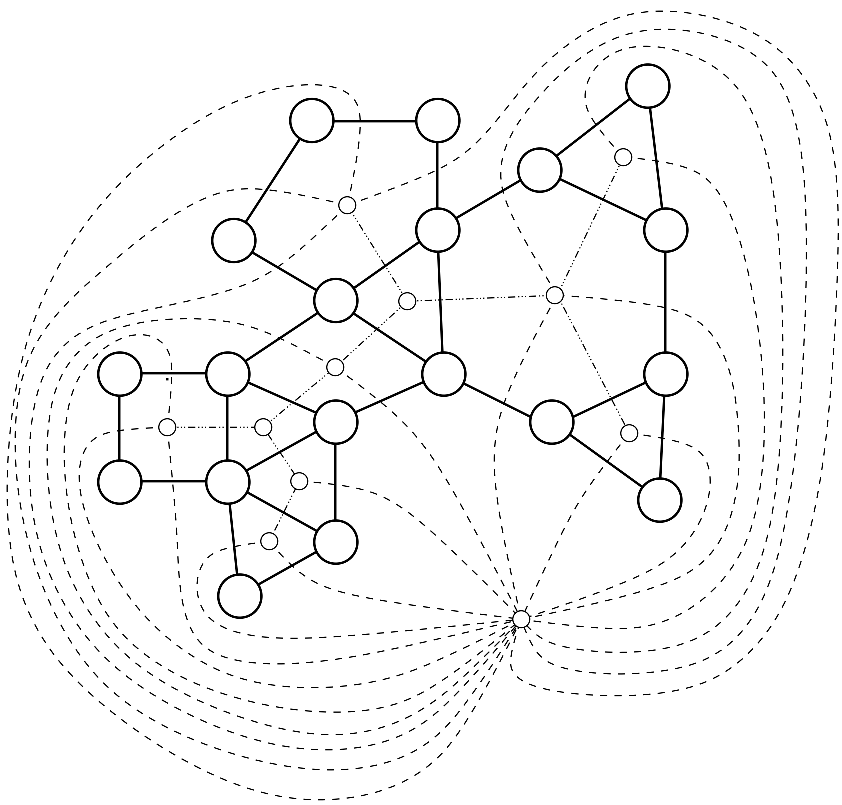


Figure 3.1: A 2-connected outerplanar graph and its dual

The following theorem is found in [22].

Theorem 3.4. *A graph G is outerplanar if and only if it has a weak dual G^w which is a forest.*

Since a 2-connected graph has only one block, Theorem 3.4 implies the following useful lemma.

Lemma 3.5. *If G is a 2-connected outerplanar graph, then it has a weak dual G^w which is a tree.*

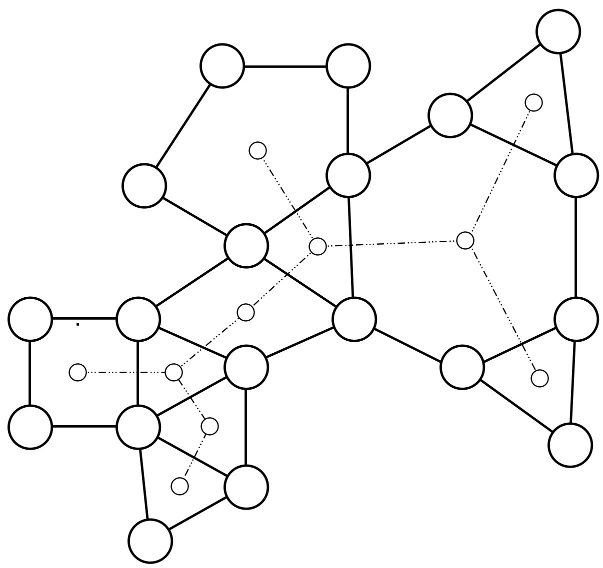


Figure 3.2: A 2-connected outerplanar graph and its weak dual

Every subgraph of G can be obtained from G by a sequence of edge and vertex deletions. A vertex v which is incident to the outer face in some outerplanar drawing of G, will still be adjacent to the outer face after any sequence of edge and vertex deletions. Thus we have the following observation:

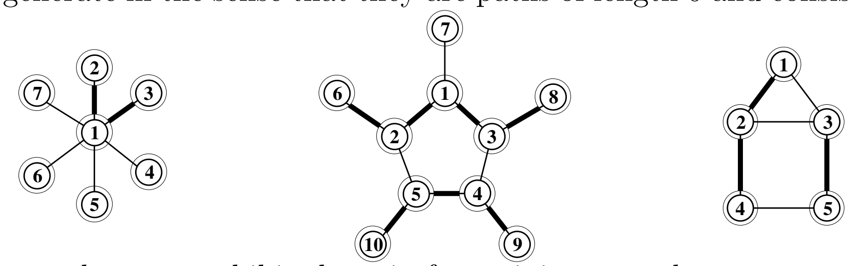
Observation 3.6. *Every subgraph of an outerplanar graph is outerplanar.*

Chapter 4. The Path Cover Number

4.1 The Path Cover Number

A *path cover* for a simple graph G is a collection of vertex-disjoint induced paths which cover all the vertices of G. The *path cover number* of a graph G, denoted $P(G)$, is the minimum number of paths required in a path cover for G.

Example 4.1. In the following graphs the paths in the path cover have thicker edges and vertices of larger diameter. The graphics are meant to give the sense that the paths are physically covering the vertices and edges of the graph. Notice that some of the paths are degenerate in the sense that they are paths of length 0 and consist of only an isolated vertex.



The path covers exhibited are in fact minimum path covers.

In the case of S_n, since the paths must be vertex-disjoint, any path containing the dominating vertex, must isolate the remaining vertices not in that path. Since every vertex besides the dominating vertex is pendant, the maximum number of vertices in a path containing the dominant vertex, is 3. Thus $P(S_n) = n - 2$, and in particular $P(S_7) = 5$.

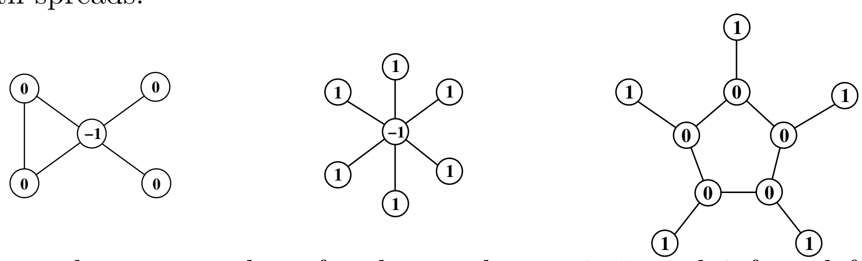
The graph in the middle is the 5-sun. Since the 5-sun has 5 pendant vertices, and at most 2 pendant vertices can belong to any one path, $P(\text{5-sun}) \geq 3$. But a path cover with 3 paths has been given and so $P(\text{5-sun}) = 3$.

The last graph is known as the house. Notice that the house is not a path, and so $P(\mathit{house}) \geq 2$. Since a path cover for the house has been given consisting of 2-paths, $P(\mathit{house}) = 2$.

In [4], Barioli, Fallat, and Hogben, define the analog of rank-spread for the path cover number as well as some additional terms related to path covers. The *path-spread* of a vertex

v of G, denoted $p_v(G)$, is the difference of $P(G)$ and $P(G-v)$. A vertex v of G is *doubly terminal* if there exists a minimum path cover of G in which v is a degenerate path. A vertex v of G is *simply terminal* if v is not doubly terminal and is a pendant vertex of a path in a minimum path cover for G.

Example 4.2. The following graphs have the vertices labeled with their corresponding path-spreads.



The path cover numbers for the graphs are 2, 5, and 3 from left to right. Note that the graph on the left has a unique path cover and that all the path-spread 0 vertices are simply terminal. However the path-spread 0 vertices of the 5-sun, are not simply terminal. Also all vertices with path-spread 1 are doubly terminal.

The following proposition is Lemma 2.1 in [4] and in the interest of self-containment we include a proof.

Proposition 4.3. *Let G be a graph and v a vertex of G. Then*

(a) $P(G) - P(G-v) \geq -1$,

(b) $1 \geq P(G) - P(G-v)$,

(c) $1 \geq p_v(G) \geq -1$,

(d) *v is doubly terminal if and only if $p_v(G) = 1$,*

(e) *if v is simply terminal, then $p_v(G) = 0$.*

Proof. (a) Let v be a vertex of G. Let R be a minimum path cover of G and P the path in R containing v. There are three cases to consider. If P is a degenerate path consisting of only v, then $R \backslash P$ is a path cover for G with $|R|-1$ paths. If v is a pendant vertex of

P, then $R \cup \{P - v\}$ is a path cover for $G - v$ with $|R|$ paths. If v is not a pendant vertex of P, then $P - v$ consists of two paths P_1 and P_2. Thus $(R \setminus P) \cup \{P_1, P_2\}$ is a path cover for $G - v$ with $|R| + 1$ paths. In all three cases $P(G - v) \leq |R| + 1 = P(G) + 1$. Therefore $P(G) - P(G - v) \geq -1$.

(b) On the other hand let R' be a minimum path cover for $G - v$. Let v be covered by a degenerate path P. Then $R' \cup P$ is a path cover for G and $1 = |R' \cup P| - |R'| \geq P(G) - P(G - v)$. as desired.

(c) From parts (a) and (b), $p_v(G) \geq -1$ and $p_v(G) \leq 1$.

(d) If v is doubly terminal, then there exists a minimum path cover R of G such that v is a degenerate path. Then $R \setminus P$ is a path cover for $G - v$ with $P(G) - 1$ paths. Thus $P(G - v) \leq P(G) - 1$. By part b, $P(G - v) \geq P(G) - 1$. Thus $p_v(G) = P(G) - P(G - v) = 1$.

Let R' be a minimum path cover for $G - v$ and P a degenerate path for v. Then $R = R' \bigcup P$ is a path cover for G, with $P(G - v) + 1$ paths. Since $p_v(G) = 1$, R is a minimum path cover for G. Since P is a degenerate path covering v in a minimum path cover of G, v is doubly terminal.

(e) Let v be simply terminal. Then v is not doubly terminal and there exists a minimum path cover R of G and nontrivial path $P \in R$ such that v is a pendant vertex of P. Thus $R' = (R \setminus P) \cup (P - v)$ is a path cover for $G - v$, and $P(G) - P(G - v) \geq |R| - |R'| = 0$. Thus by part (c), $p_v(G) = 0$ or $p_v(G) = 1$. Since v is not doubly terminal, part (d) implies that $p_v(G) \neq 1$. Thus $p_v(G) = 0$.

□

In order to get a lower bound for the path cover number of a graph it is convenient to know which vertices must be either simply terminal or doubly terminal.

Lemma 4.4. *Let v be a simplicial vertex of G. Then exactly one of the following is true:*

- *$P(G - v) < P(G)$ and v is doubly terminal in G*
- *$P(G - v) = P(G)$ and v is simply terminal in G*

Proof. Since v is simplicial the neighborhood of v is a clique. Since a path cover consists of induced paths in G, any path containing v can contain at most one neighbor of v. Thus v is either simply terminal or doubly terminal in G. If v is simply terminal, then by Proposition 4.3 part (e), $P(G) - P(G - v) = p_v(G) = 0$. Thus $P(G) = P(G - v)$. If v is doubly terminal, then by Proposition 4.3 part (d), $P(G) - P(G - v) = p_v(G) = 1$. Thus $P(G) > P(G - v)$. □

Thus Lemma 4.4 concludes that every simplicial vertex is either simply terminal or doubly terminal. Since each path in a path cover may cover at most 2 simplicial vertices, we have the following corollary.

Corollary 4.5. *Let G have k simplicial vertices. Then $P(G) \geq \left\lceil \frac{k}{2} \right\rceil$.*

Proposition 4.6. *Let G be the vertex-sum at v of G_1 and G_2. Then*

(*a*) $P(G) \geq P(G_1) + P(G_2) - 1$

(*b*) $P(G) \geq P(G_1 - v) + P(G_2 - v) - 1$.

Proof. Let R be a minimum path cover of G and P the path in R containing v. For $i = 1, 2$, define P_i to be the path induced by the vertices of P which lie in G_i. Note that v is in both P_1 and P_2. Let R_i contain all the paths of $(R \setminus P) \cup \{P_i\}$ which lie in G_i. Then R_i is a path cover for G_i and

$$P(G_1) + P(G_2) \leq |R_1| + |R_2| = |(R \setminus P) \cup \{P_1, P_2\}| = |R| + 1 = P(G) + 1.$$

This proves the first part of the conclusion.

By Proposition 4.3 part (a), $P(G - v) - 1 \leq P(G)$. Since v is a cutvertex, $G - v$ is isomorphic to the union of $G_1 - v$ and $G_2 - v$. Thus

$$P(G_1 - v) + P(G_2 - v) - 1 = P(G - v) - 1 \leq P(G),$$

proving the second part of the conclusion. $\square$

The following lemmas will yield a formula for finding the path-spread of a cutvertex. A generalized version of Lemma 4.8 can be found in [4] as Proposition 2.2.

Lemma 4.7. *Let G be the vertex-sum at v of G_1 and G_2 and let $p_v(G_1) = p_v(G_2) = 0$. Then v is simply terminal in each G_i if and only if $p_v(G) = -1$.*

Proof. Let G be the vertex-sum at v of G_1 and G_2 and let $p_v(G_1) = p_v(G_2) = 0$.

Case 1. v is simply terminal in at most one G_i.

Let R be a minimum path cover of G and P the path in R covering v.

Subcase 1. P lies completely in some G_i

Without loss of generality, renaming if necessary, let P lie completely in G_1. Let R_i be the set of paths in R which lie completely in G_i. Since R is a minimum path cover for G, R_1 is a minimum path cover for G_1 and R_2 is a minimum path cover for $G_2 - v$ (if they weren't, a path cover smaller than $|R|$ could be constructed for G). Thus $P(G) = |R| = |R_1| + |R_2| = P(G_1) + P(G_2 - v)$. Since $p_v(G_2) = 0$, $P(G_2 - v) = P(G_2)$. Thus $P(G) = P(G_1) + P(G_2)$ and

$$\begin{aligned} p_v(G) &= P(G) - P(G - v) \\ &= P(G_1) + P(G_2) - (P(G_1 - v) + P(G_2 - v)) \\ &= p_v(G_1) + p_v(G_2) = 0. \end{aligned}$$

Subcase 2. P contains vertices of $G_1 - v$ and $G_2 - v$.

If P has vertices which lie in both $G_1 - v$ and $G_2 - v$, then we split P at v into two (non-degenerate) paths P_1 and P_2 which lie in G_1 and G_2, respectively. Define R_i as the paths in $(R \setminus P) \cup \{P_1, P_2\}$ which lie completely in G_i. Then R_i is a path cover for G_i for each i. Note v is a pendant vertex of both P_1 and P_2. Since v is simply terminal in at most one G_i, R_i is not a minimum path cover of G_i for some i. Thus $P(G) = |R| = |R_1| + |R_2| - 1 > P(G_1) + P(G_2) - 1$. So

$$\begin{aligned} p_v(G) &= P(G) - P(G - v) \\ &> P(G_1) + P(G_2) - 1 - (P(G_1 - v) + P(G_2 - v)) \\ &= p_v(G_1) + p_v(G_2) - 1 = -1. \end{aligned}$$

Thus $p_v(G) \neq -1$.

Case 2. v is simply terminal in both G_i.

For $i = 1, 2$, let R_i be a path cover for G_i such that v is a pendant vertex of a path P_i in R_i. Let P be the path in G created by the union of P_1 and P_2. Since P_1 and P_2 are induced paths in G_1 and G_2, respectively, P is an induced path in G. So $R = (R_1 \setminus P_1) \cup (R_2 \setminus P_2) \cup P$ is a path cover for G, and $|R| = |R_1 \setminus P_1| + |R_2 \setminus P_2| + 1 = P(G_1) - 1 + P(G_2) - 1 + 1 = P(G_1) + P(G_2) - 1$. Thus

$$\begin{aligned} p_v(G) &= P(G) - P(G - v) \\ &\leq P(G_1) + P(G_2) - 1 - (P(G_1 - v) - P(G_2 - v)) \\ &= p_v(G_1) + p_v(G_2) - 1 = -1. \end{aligned}$$

By Proposition 4.3 part (c), $p_v(G) \geq -1$. Therefore $p_v(G) = -1$.

$\square$

Lemma 4.8. *Let G be the vertex-sum at v of G_1 and G_2. Then $p_v(G) = \min\{p_v(G_1), p_v(G_2)\}$ unless v is simply terminal in each G_i in which case $p_v(G) = -1$.*

Proof. Without loss of generality, renaming if necessary, let $p_v(G_1) \leq p_v(G_2)$. Let R_1 be a minimum path cover for G_1 and R_2 be a minimum path cover for $G_2 - v$. Notice that $R_1 \cup R_2$ is a path cover for G.

$$\begin{aligned}
p_v(G) &= P(G) - P(G - v) && (4.1)\\
&\leq |R_1| + |R_2| - P(G - v) && (4.2)\\
&= P(G_1) + P(G_2 - v) - P(G - v) && (4.3)\\
&= P(G_1) + P(G_2 - v) - (P(G_1 - v) - P(G_2 - v)) && (4.4)\\
&= p_v(G_1) && (4.5)\\
&= \min\{p_v(G_1), p_v(G_2)\}. && (4.6)
\end{aligned}$$

Thus $p_v(G) \leq \min\{p_v(G_1), p_v(G_2)\}$.

Case 1. $p_v(G_2) = 1$

Since $p_v(G_2) = 1$, $P(G_2 - v) = P(G_2) - 1$. Substituting this into line (4.3) for $P(G_2 - v)$, we see that $P(G) \leq P(G_1) + P(G_2) - 1$. On the other hand Proposition 4.6 part (a) says that $P(G) \geq P(G_1) + P(G_2) - 1$. Thus in (4.2) we have equality and there is equality throughout the equation. Therefore $p_v(G) = \min\{p_v(G_1), p_v(G_2)\}$

Case 2. $p_v(G_1) = -1$

Since $p_v(G_1) = -1$, $p_v(G) \leq -1$. By Proposition 4.3 part (c), $p_v(G) \geq -1$. Thus $p_v(G) = -1 = \min\{p_v(G_1), p_v(G_2)\}$

Case 3. $p_v(G_1) \neq -1$ and $p_v(G_2) \neq 1$

By Proposition 4.3 part (c), $-1 \leq p_v(G_i) \leq 1$. Since $p_v(G_1) \neq -1$, $p_v(G_1) \geq 0$. Since $p_v(G_2) \neq 1$, $p_v(G_2) \leq 0$. Finally, since $p_v(G_1) \leq p_v(G_2)$, $p_v(G_1) = p_v(G_2) = 0$. Since $p_v(G) \leq \min\{p_v(G_1), p_v(G_2)\}$, $p_v(G) \leq 0$. By Lemma 4.7, $p_v(G) = -1$ if and only if v is simply terminal in both G_1 and G_2. Thus $p_v(G) = 0 = \min\{p_v(G_1), p_v(G_2)\}$ unless v is simply terminal in both G_i in which case $p_v(G) = -1$.

$\square$

4.2 $P(G) \geq M(G)$ for Outerplanar Graphs G

In Examples 2.5 and 4.1 it is shown that $\text{mr}(S_n) = 2$ and that $P(S_n) = n - 2$. Thus we see that $P(S_n) = M(S_n)$. In 1999, Johnson and Duarte [3], proved the following theorem.

Theorem 4.9. *Let T be a tree. Then $M(T) = P(T)$.*

Of course Theorem 4.9 has an equivalent statement in terms of minimum rank, $\text{mr}(T) = |T| - P(T)$. Some natural questions which arose as a result are as follows.

Question 4.10. For what graphs G, does $P(G) = M(G)$?

Question 4.11. Is $P(G)$ an upper bound or lower bound for $M(G)$?

Question 4.12. How large can $|P(G) - M(G)|$ be?

In [17], Barioli, Fallat, and Hogben investigate these questions. It turns out that $P(G)$ is neither an upper bound or lower bound for $M(G)$ and that $|P(G) - M(G)|$ can be arbitrarily large. The following examples demonstrate these facts.

Example 4.13. Consider the complete graph K_n, $n \geq 2$. By Observation 2.3, $\text{mr}(K_n) = 1$, and thus $M(K_n) = n - 1$. Since every set of 3 vertices induces a triangle, the largest length of an induced path is 1. Thus $P(K_n) = \left\lceil \frac{n}{2} \right\rceil$. So $|P(K_n) - M(K_n)| = n - 1 - \left\lceil \frac{n}{2} \right\rceil = \left\lfloor \frac{n}{2} \right\rfloor - 1$.

Example 4.14. In [17] it is shown that $M(\text{5-sun}) = 2$ while $P(\text{5-sun}) = 3$. In Example 4.1 it was shown that $P(\text{5-sun}) = 3$, and Theorem 4.37 implies that $M(\text{5-sun}) = 2$.

Thus Examples 4.13 and 4.14 illustrate that there exist graphs such that $P(G) > M(G)$ and graphs such that $P(G) < M(G)$.

In this section we give the necessary results to show that $P(G) \geq M(G)$ for all outerplanar graphs.

Lemma 4.15. *Let G be the vertex-sum at v of G_1 and G_2. If $M(G_i) \leq P(G_i)$ and $M(G_i - v) \leq P(G_i - v)$ for all i, then $M(G) \leq P(G)$.*

Proof. By Theorem 2.13,

$$M(G) = \max\{M(G_1) + M(G_2) - 1, M(G_1 - v) + M(G_2 - v) - 1\}.$$

Thus there are two cases to consider.

Case 1: $M(G) = M(G_1) + M(G_2) - 1$.

Using the case, the hypothesis, and part (a) of Proposition 4.6,

$$M(G) = M(G_1) + M(G_2) - 1 \leq P(G_1) + P(G_2) - 1 \leq P(G).$$

Case 2: $M(G) = M(G_1 - v) + M(G_2 - v) - 1$.

Using the case, the hypothesis and part (b) of Proposition 4.6,

$$M(G) = M(G_1 - v) + M(G_2 - v) - 1 \leq P(G_1 - v) + P(G_2 - v) - 1 \leq P(G).$$

In both cases we see that $M(G) \leq P(G)$. □

Theorem 4.16. *If G is an outerplanar graph, $P(G) \geq M(G)$.*

Proof. Suppose by way of contradiction that there exists an outerplanar graph whose path cover number is strictly less the maximum nullity of the graph. Let $G = (V, E)$ be an outerplanar graph such that $P(G) < M(G)$ and G is the smallest such graph with respect to the sum $|V(G)| + |E(G)|$. In other words, every outerplanar graph whose total number of vertices and edges is less than $|V(G)| + |E(G)|$, must satisfy the conclusion of the theorem. It is easily checked that all graphs with 3 or fewer vertices satisfy $P(G) = M(G)$. Thus $|V(G)| > 3$.

Since both the maximum nullity of a graph and the path cover number of a graph are additive on components, the minimality of G implies that G is connected.

In the case that G has a cutvertex, we label it v. Thus there exist proper induced subgraphs of G, G_1 and G_2, such that G is the vertex-sum at v of G_1 and G_2. By Observation 3.6, G_i and $G_i - v$ are outerplanar for all i. By the minimality of G, $M(G_i) \leq P(G_i)$ and $M(G_i - v) \leq P(G_i - v)$ for all i. Thus by Lemma 4.15, $M(G) \leq P(G)$, a contradiction.

Thus G has no cutvertices and is 2-connected. By Lemma 3.5, the weak dual of G, G^w, is a tree. Since $M(C_n) = 2 = P(C_n)$, $G \neq C_n$ and G has at least two induced cycles. Thus G^w has at least two vertices and at least one pendant vertex. Each pendant vertex of G^w corresponds to a pendant cycle C in G. Since C is a pendant cycle, there are exactly two adjacent vertices u and w of C which have degree greater than 2 in G. Thus G has a 2-separation (G_1, P_k) where $k \geq 3$, $V(G_1) \cap V(P_k) = \{u, w\}$, u and w are adjacent in G_1, and u, w are the pendant vertices of P_k. By Lemma 2.23,

$$M(G) = \max\{M(H_1), M(G_1/uw)\} \tag{4.7}$$

where H_1 is created from G_1 by adding an additional edge between between u and w.

Note that equation (4.7) implies that $M(G)$ depends on the graph G_1 and not on the length of the path P_k. In the case that $k \geq 4$, the path in a minimum path cover for G which covers a degree two vertex of P_k may be shortened or eliminated all together, to create a path cover of equal or smaller size for the graph corresponding to $k = 3$. Thus by the minimality of G we may assume that $k = 3$.

Let v be the vertex of degree 2 of the subgraph P_3. Note this implies that $G - v = G_1$. By the minimality of G, $M(G - v) \leq P(G - v)$. Since v is simplicial Lemma 4.4 implies $P(G - v) \leq P(G)$. By Proposition 2.11, $M(G) - 1 \leq M(G - v)$. Thus

$$M(G) - 1 \leq M(G - v) \leq P(G - v) \leq P(G) < M(G)$$

which implies

$$M(G - v) = M(G) - 1 \qquad \text{and} \qquad P(G - v) = P(G). \tag{4.8}$$

Since $P(G-v) = P(G)$, Lemma 4.4 implies that v is simply terminal in G.

Using equation (4.7), there are two cases to consider.

Case 1: $M(G) = M(H_1)$.

Since there are two edges between u and w, $M(H_1) = \max\{M(G_1), M(G_1 - uw)\}$. Thus using equation (4.8),

$$M(G) - 1 = M(G-v) = M(G_1) \leq M(H_1) = M(G).$$

Thus $M(H_1) > M(G_1)$ and it must be the case that $M(H_1) = M(G_1 - uw)$.

Since v is simply terminal in G there exists a minimum path cover R of G which does not use edge uw. Modifying R by shortening the path which covers v we have a path cover for $G_1 - uw$. Thus $P(G_1 - uw) \leq P(G)$.

Summarizing the case and using the minimality of G,

$$M(G) = M(H_1) = M(G_1 - uw) \leq P(G_1 - uw) \leq P(G),$$

a contradiction.

Case 2: $M(G) = M(G_1/uw)$.

Consider the graph $G - uw$ which has a 2-separation $(G_1 - uw, P_3)$. Since the edge uw is not present, Lemma 2.23 implies that $M(G - uw) = \max\{M(G_1), M(G_1/uw)\}$. Thus $M(G - uw) \geq M(G_1/uw) = M(G)$ where the equality is due to the case.

Since v is simply terminal in G, there exists a minimum path cover R of G which does not use the edge uw. Thus R is a path cover for $G - uw$ and $P(G - uw) \leq P(G)$.

Summarizing the case,

$$P(G - uw) \leq P(G) < M(G) \leq M(G - uw),$$

which contradicts the minimality of G.

Therefore there does not exist an outerplanar graph G such that $M(G) > P(G)$ and for every outerplanar graph G, $M(G) \leq P(G)$. □

A construction was given in [4] which demonstrates that for every natural number k there exists an outerplanar graph G for which $P(G) - M(G) > k$. Given k, take k copies of the 5-sun $G_1, G_2, \ldots, G_k$ and create a string of 5-suns by vertex-summing G_1 and G_2 at pendant vertices, then choosing a different pendant vertex from G_2 and summing it to G_3 at a pendant vertex and so forth. One possibility for the resulting graph is Figure 4.1. We

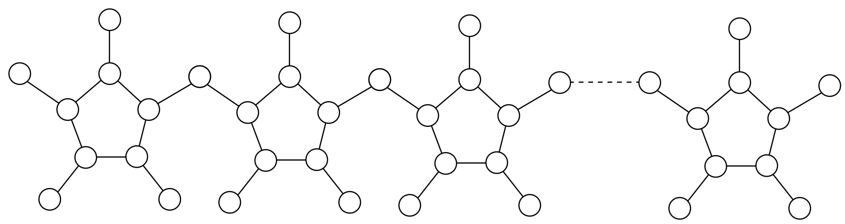


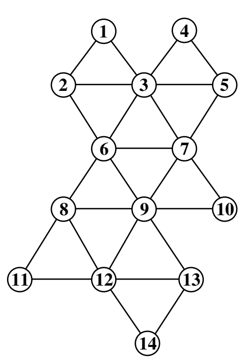
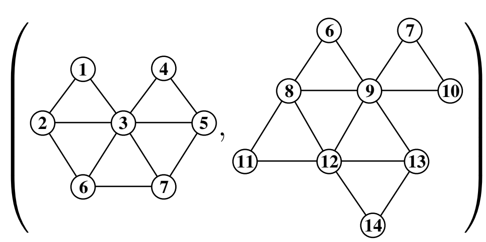
Figure 4.1: A construction to make $P(G) - M(G)$ arbitrarily large.

do not give a proof of the claim concerning this construction and refer the reader to [4] for a proof. This construction is mentioned to give context to the following question which was asked in [5].

Question 4.17. Does there exists a 2-connected outerplanar graph such that $P(G) > M(G)$.

The following example answers this question in the affirmative.

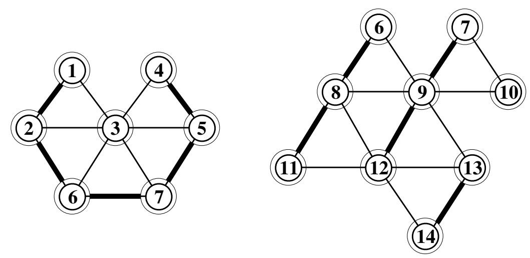
Example 4.18. Let G be the following 2-tree on 14 vertices with the given 2-separation (G_1, G_2).

$G =$  $(G_1, G_2) =$ 

First we show that $M(G) \leq 4$ using Theorem 2.21. A path cover for each graph is demonstrated in its drawing. If the graph has parallel edges, a path cover is given that does not cover the parallel edges. By doing so, we get an upper bound on the path cover number for each simple realization. Using Theorem 4.16 we get an upper bound for the maximum nullity of each graph and thus for each term in Theorem 2.21.

- $M(G_1) + M(G_2) - 2$

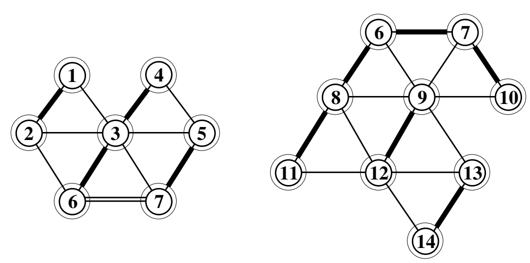
  The graphs G_1 and G_2 are as follows.

  Thus $P(G_1) \leq 2$ and $P(G_2) \leq 4$. By Theorem 4.16, $M(G_1) + M(G_2) - 2 \leq P(G_1) + P(G_2) - 2 \leq 2 + 4 - 2 = 4$.

- $M(H_1) + M(H_2) - 2$

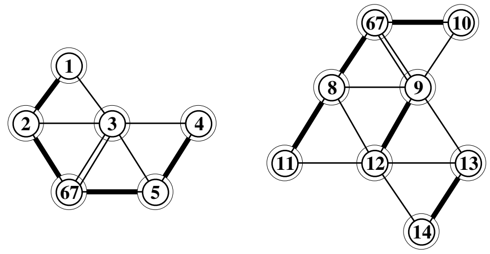
  The graphs H_1 and H_2 are as follows.

  Thus $P(H_1) \leq 3$ and $P(H_2) \leq 3$. By Theorem 4.16, $M(H_1) + M(H_2) - 2 \leq P(H_1) + P(H_2) - 2 \leq 3 + 3 - 2 = 4$.

- $M(G_1/67) + M(G_2/67) - 2$

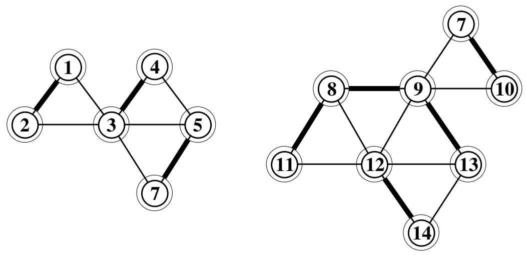
  The graphs $G_1/67$ and $G_2/67$ are as follows.



Thus $P(G_1/67) \leq 2$ and $P(G_2/67) \leq 3$. By Theorem 4.16, $M(G_1/67) + M(G_2/67) - 2 \leq P(G_1/67) + P(G_2/67) - 2 \leq 2 + 3 - 2 = 3$.

- $M(G_1 - 6) + M(G_2 - 6) - 2$

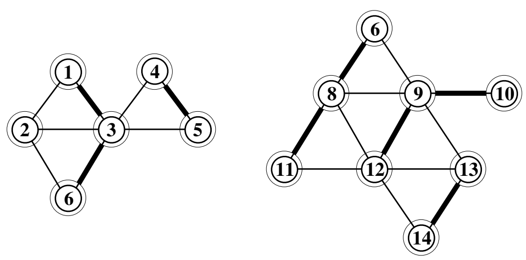
  The graphs $G_1 - 6$ and $G_2 - 6$ are as follows.



  Thus $P(G_1 - 6) \leq 3$ and $P(G_2 - 6) \leq 3$. By Theorem 4.16, $M(G_1 - 6) + M(G_2 - 6) - 2 \leq P(G_1 - 6) + P(G_2 - 6) - 2 \leq 3 + 3 - 2 = 4$.

- $M(G_1 - 7) + M(G_2 - 7) - 2$

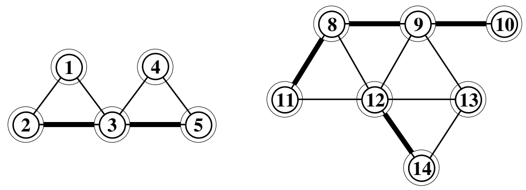
  The graphs $G_1 - 7$ and $G_2 - 7$ are as follows.



  Thus $P(G_1 - 7) \leq 3$ and $P(G_2 - 7) \leq 3$. By Theorem 4.16, $M(G_1 - 7) + M(G_2 - 7) - 2 \leq P(G_1 - 7) + P(G_2 - 7) - 2 \leq 3 + 3 - 2 = 4$.

- $M(G_1 - \{6, 7\}) + M(G_2 - \{6, 7\}) - 2$

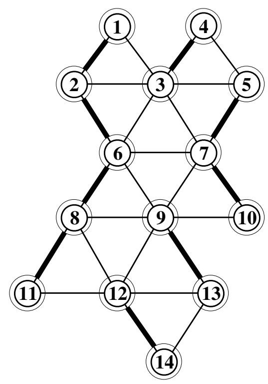
  The graphs $G_1 - \{6, 7\}$ and $G_2 - \{6, 7\}$ are as follows.



Thus $P(G_1 - \{6,7\}) \le 3$ and $P(G_2 - \{6,7\}) \le 3$. By Theorem 4.16, $M(G_1 - \{6,7\}) + M(G_2 - \{6,7\}) - 2 \le P(G_1 - \{6,7\}) + P(G_2 - \{6,7\}) - 2 \le 3 + 3 - 2 = 4$.

Since each term in Theorem 2.21 is less than or equal to 4, $M(G) \le 4$.

The following path cover of G demonstrates that $P(G) \le 5$.



We now show that $P(G) \ge 5$. We begin with an observation about minimum path covers: If R is a minimum path cover for G which does not cover an edge e, then R is a path cover for $G - e$ implying $P(G - e) \le P(G)$.

Suppose by way of contradiction that $P(G) < 5$. There exists a path cover R of G such that $|R| < 5$. Label the edges $\{7, 10\}$ and $\{9, 10\}$ of G as e_1 and e_2, respectively. Since vertices 7, 9, and 10 form a clique, at least one of the e_i is not covered by paths in R. By the observation above, $P(G - e_i) < 5$ for some i. We claim that $M(G - e_i) \ge 5$ for all i. If so, then by Theorem 4.16, $P(G - e_i) \ge 5$ for all i, a contradiction. It remains to show that $M(G - e_i) \ge 5$ for all i. In Example 5.3 it is shown using covers that $M(G - e_i) \ge 5$ for all i, completing the proof.

Thus $M(G) = 4 < 5 = P(G)$.

4.3 Partial 2-paths

In this section we give an example of a subclass of outerplanar graphs for which the path cover number is equal to the maximum nullity. A *k-tree* is a graph that can be built up from a k-clique by adding one vertex at a time adjacent to exactly the vertices in an existing k-clique. A *k-path* is a k-tree with either at most $k+1$ vertices or exactly two vertices of degree k. A *partial k-path* is a subgraph of a k-path.

All 2-paths and consequently partial 2-paths are outerplanar graphs. The following are some useful lemmas regarding partial 2-paths.

Lemma 4.19. *If G is a 2-path and $v \in V(G)$, then $G - v$ can be completed to a 2-path on $V(G - v)$.*

Proof. Let v be a vertex of degree k in the 2-path G. If $k = 2$, then $G - v$ is still a 2-path. If $k \geq 3$, then the graph induced by v and its neighbors is a 2-path with v being a dominating vertex. Thus the graph induced by only the neighbors of v is a path. Label the vertices of the path consecutively $v_1, v_2, \ldots, v_k$, see Figure 4.2. In $G - v$ add the new edges $v_1v_3, v_1v_4, \ldots, v_1v_k$. to $E(G-v)$, see Figure 4.3. The graph induced by $v_1, \ldots, v_k$ is a 2-path and thus the graph induced by $V(G - v)$ is a 2-path. □

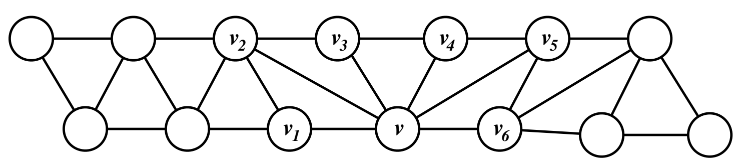


Figure 4.2: A labelled 2-path G

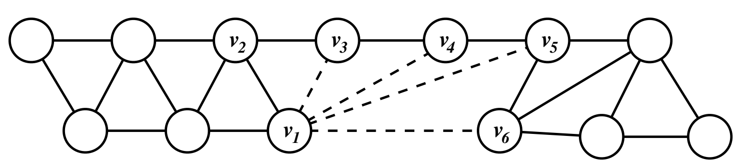


Figure 4.3: The completion of $G - v$ to a 2-path

Lemma 4.20. *If G is a partial 2-path, then G may be completed to a 2-path on $V(G)$.*

Proof. Let G be a partial 2-path. By definition G may be completed to a 2-path H. If $|V(H)| \neq |V(G)|$ then by Lemma 4.19 we may delete any $v \notin V(G)$ and still complete $H - v$ to a 2-path. Thus after repeated applications if necessary we obtain a completion of G to a 2-path on $V(G)$. □

In [13] a graph G is defined to be C_2 if it is connected and has no pendant vertices. A graph of *two parallel paths* has a specific structure which is defined in [13]. The relevant property is that the path cover number of such a graph is 2. A *linear singly edge articulated cycle* or LSEAC graph is basically a "path" of cycles where neighboring cycles share exactly one edge. A 2-path is an LSEAC graph where all the cycles are triangles. The following is Theorem 4.9 in [13] and shows that for a 2-path G, $P(G) = M(G) = 2$.

Theorem 4.21. *If G is a C_2 graph, then the following three statements are equivalent:*

- $M(G) = 2$
- *G is a graph of two parallel paths (i.e., $P(G) = 2$), and*
- *G is an LSEAC graph.*

In an outerplanar drawing of a graph the edges which are not adjacent to the outer face will be called *interior* edges, while those adjacent to the outer face are *exterior* edges. In an LSEAC graph which has more than one cycle there are two unique cycles which have only one neighboring cycle. These cycles are referred to as *pendant cycles.*

Lemma 4.22. *A graph G is an LSEAC graph if and only if it is a 2-connected partial 2-path.*

Proof. Let G be an LSEAC graph. Then G is 2-connected by construction. To show that G is a partial 2-path it is sufficient to show that a cycle of length 4 or more may be completed to an appropriate 2-path. In each such cycle of G, except the pendant cycles, there are two edges of articulation. Using induction it can be shown that any cycle may be completed to a 2-path H in such a way that these edges of articulation correspond to different pendant cycles of H.

Let G be a 2-connected partial 2-path. Then by Lemma 4.20, G can be completed to a 2-path H on $V(G)$. Since deletion of an exterior edge creates a cutvertex, only interior edges can be deleted from H to create G. Thus the same path cover that works for H will be a path cover for G. By Theorem 4.21, $P(H) = 2$. Thus $P(G) \leq 2$. By Theorem 4.16, $P(G) \geq M(G)$. Thus $M(G) \leq 2$. From Theorem 2.8, $M(G) = 1$ if and only if G is a path. Thus $M(G) = 2$. Since G is 2-connected it is C_2, and by Theorem 4.21, G is an LSEAC graph. □

Combining Theorem 4.21 and Lemma 4.22 gives the following corollary.

Corollary 4.23. *If G is a 2-connected partial 2-path, then $P(G) = M(G) = 2$.*

In Figure 4.4 an illustration of a 2-connected partial 2-path G is given as well as a possible path cover. The cycles $A = 1234$ and $B = 567$ are the pendant cycles of G.

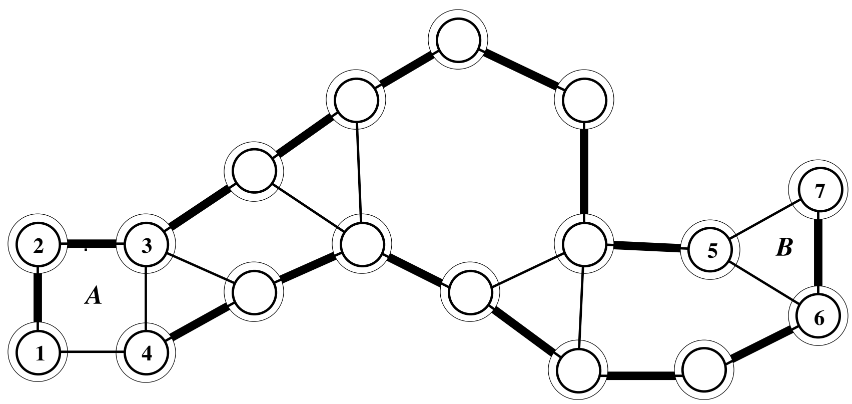


Figure 4.4: A 2-connected partial 2-path

Given any subset of two edges of C_n, there are two vertex-disjoint paths beginning at the vertices of one edge and ending at the vertices of the other edge. We summarize this as:

Observation 4.24. *Given any two edges of C_n, there exists a minimum path cover whose paths begin at the vertices of one edge and end at the vertices of another.*

This idea may be extended to 2-connected partial 2-paths. Since the two paths in a minimum path cover start at one of the pendant cycles, there is an exterior edge whose vertices are the starting points for the paths. These paths can be shortened and lengthened respectively so that the paths can start at any exterior edge of the pendant cycle. For example consider the minimum path cover in Figure 4.4 for a 2-connected partial 2-path. Note that the path P_1 begins at vertex 1 and ends at vertex 5, while P_2 begins at vertex 4 and ends at vertex 7. It is easily observed that P_1 may be shortened to begin at vertex 3 and end at vertex 5, while P_2 may be lengthened to start at vertex 2 and end at vertex 7. This leads to the following observation.

Observation 4.25. *Let G be a 2-connected partial 2-path with at least 2 induced cycles. Given two exterior edges $e_1 = v_1v_2$ and $e_2 = w_1w_2$ from each pendant cycle of G, there exists a minimum path cover of G such that the paths begin at vertices v_1 and v_2 and end at vertices w_1 and w_2.*

The object of this section is to show that for any partial 2-path G, $P(G) = M(G)$. By Corollary 4.23 we know that $M(G) = P(G) = 2$ for 2-connected partial 2-paths. Thus we will be considering partial 2-paths that have at least one cutvertex. We will now proceed with a few more lemmas necessary for the proof of the main result of this section.

Lemma 4.26. *Let G be a graph such that $P(G) > M(G)$ and for each proper induced subgraph K of G, $P(K) = M(K)$. Then*

(i) $r_v(G) + p_v(G) > 1$ for every vertex v of G, and

(ii) for every proper induced subgraph K of G and every vertex v of K, $r_v(K)+p_v(K) = 1$.

Proof. Since $G - v$ is a proper induced subgraph of G, $P(G - v) = M(G - v)$ and $\mathrm{mr}(G - v) + P(G - v) = \mathrm{mr}(G - v) + M(G - v) = |G - v|$. Now

$$\begin{aligned} r_v(G) + p_v(G) &= \mathrm{mr}(G) - \mathrm{mr}(G - v) + P(G) - P(G - v) \\ &= \mathrm{mr}(G) + P(G) - (\mathrm{mr}(G - v) + P(G - v)) \\ &> |G| - |G - v| = 1. \end{aligned}$$

Let K be a proper induced subgraph of G. Then $\text{mr}(K)+P(K) = \text{mr}(K)+M(K) = |K|$. Since $K - v$ is also a proper induced subgraph of G,

$$\begin{aligned} r_v(K) + p_v(K) &= \text{mr}(K) - \text{mr}(K - v) + P(K) - P(K - v) \\ &= \text{mr}(K) + P(K) - (\text{mr}(K - v) + P(K - v)) \\ &= |K| - |K - v| = 1. \end{aligned}$$

□

Lemma 4.27. *Let G be a graph such that $P(G) > M(G)$ and for each proper induced subgraph K of G, $P(K) = M(K)$. Then for every cutvertex v of G, $G - v$ has exactly two components one of which is an isolated vertex.*

Proof. Let G be as stated in the hypothesis. Let G be the vertex-sum at v of $G_1, G_2, \ldots, G_k$ where $k \geq 2$ and each G_i is connected. By Proposition 2.17, $0 \leq r_v(G) \leq 2$. We first consider four of five possible cases and show that they cannot occur because they contradict Lemma 4.26 part (i).

Case 1 $r_v(G) = 0$.

By Proposition 4.3 part (c), $p_v(G) \leq 1$. Thus $r_v(G) + p_v(G) \leq 1$.

Case 2 $r_v(G) = 1$.

By Theorem 2.18, there is exactly one i, $1 \leq i \leq k$ such that $r_v(G_i) = 1$. Thus G can be expressed as the vertex-sum at v of H_1 and H_2 where $r_v(H_1) = 1$ and $r_v(H_2) = 0$. By Lemma 4.26, $p_v(H_1) = 0$ and $p_v(H_2) = 1$. Since $p_v(H_2) = 1$, by Proposition 4.3 part (d), v is doubly terminal in H_2. Thus v can only possibly be simply terminal in H_1. By Lemma 4.8, $p_v(G) = \min_i p_v(H_i) = \min\{0, 1\} = 0$.

Thus $r_v(G) + p_v(G) = 1$.

Case 3 $r_v(G) = 2$ and $k > 2$

By Theorem 2.18 and Proposition 2.17, either $r_v(G_i) = 2$ for some i or $r_v(G_i) = 1$ for two distinct values of i. Thus G can be expressed as the vertex-sum at v of H_1 and H_2 where $r_v(H_1) = 2$. We can ensure the existence of such a vertex-sum since $k > 2$. By Lemma 4.26, $p_v(H_1) = -1$. Thus by Lemma 4.8, $p_v(G) = -1$, and $r_v(G) + p_v(G) = 1$.

Case 4 $r_v(G) = 2$ and $k = 2$ and $r_v(G_i) = 2$ for at least one i

Without loss of generality $r_v(G_1) = 2$. By Lemma 4.26, $p_v(G_1) = -1$. Thus by Lemma 4.8, $p_v(G) = -1$, and $r_v(G) + p_v(G) = 1$.

In all of the above cases $p_v(G) + r_v(G) \leq 1$. This contradicts Lemma 4.26.

The only remaining case is when $r_v(G) = 2$ and $k = 2$ and $r_v(G_i) = 1$ for $i = 1, 2$.

So G is the vertex-sum at v of G_1 and G_2 where $r_v(G_i) = 1$ for $i = 1, 2$. Since each G_i is a proper induced subgraph of G, by Lemma 4.26, $p_v(G_i) = 0$ for $i = 1, 2$.

If v were simply terminal in both G_1 and G_2, then by Lemma 4.8, $p_v(G) = -1$. Further $r_v(G) + p_v(G) = 1$ which would contradict Lemma 4.26.

Therefore v is simply terminal in at most one of G_1 and G_2. Without loss of generality let v be not simply terminal in G_1.

Let X be the vertex-sum at v of G_1 and K_2 the complete graph on two vertices. Note that $r_v(K_2) = 1$ and $p_v(K_2) = 0$. By Theorem 2.18, $r_v(X) = 2$. By Lemma 4.8, $p_v(X) = 0$. Now X is an induced subgraph of G and $r_v(X) + p_v(X) = 2$. By Lemma 4.26, X is not a proper induced subgraph of G. Thus $X = G$ and further $G_2 = K_2$ the complete graph on two vertices.

Since v was an arbitrary cutvertex, for any cutvertex v, $G - v$ has exactly two components one of which is an isolated vertex. □

Lemma 4.28. *Let G be a partial 2-path such that $P(G) > M(G)$ and for each proper induced subgraph K of G, $P(K) = M(K)$. Then $G - S$, where S is the set of all pendant vertices of G, is a 2-connected partial 2-path.*

Proof. Let G be as described in the hypothesis. By the minimality of G it must be connected. If G is 2-connected, then by Corollary 4.23, $M(G) = 2 = P(G)$. Thus it must be the case that G has a cutvertex.

By Lemma 4.27, for every cutvertex v of G, $G - v$ has exactly two components one of which is an isolated vertex. Thus every cutvertex in G is adjacent to a pendant vertex. By deleting the set S of pendant vertices of G, the unique neighbor of each pendant vertex is no longer a cutvertex. Note that any cutvertex in $G - S$, is a cutvertex in G. Thus $G - S$ has no cut vertices. In light of Theorem 4.9, G is not a tree and consequently neither is $G - S$. Thus $|G - S| \geq 3$ and $G - S$ is a 2-connected partial 2-path. □

Lemma 4.29. *Let G be a partial 2-path such that $P(G) > M(G)$ and for each proper induced subgraph K of G, $P(K) = M(K)$. Then $P(G) \leq 2$.*

Proof. Let G be as described in the hypothesis. By Lemma 4.28, G is a partial 2-path with a pendant vertex set S and $G - S$ is a 2-connected partial 2-path (LSEAC graph). By Lemma 4.20, G can be completed to a 2-path G' on $V(G)$. Consider the collection $\mathcal{C}$ of 2-connected subgraphs of G' which contain G. Let H be a graph in $\mathcal{C}$ with the smallest number of edges. Thus H is 2-connected, $|H| = |G|$, and G is a subgraph of H. Let W be the set of edges of H such that $H - W = G$. By the minimality of H any edge in W is adjacent to at least one vertex in S and thus deleting S deletes all the edges of W. Thus $H - S = G - S$ and consequently $H - S$ is an LSEAC graph. Since $H - S$ is 2-connected, each vertex of S is a degree 2 vertex of a pendant cycle of H. Thus all the edges in W belong to a pendant cycle of H and to no other cycle.

Now there can be at most one edge from W in each pendant cycle of H otherwise upon deletion the graph would become disconnected. By Observation 4.25, there exists a minimal path cover for H which does not use the edges in W. By Theorem 4.21, this path cover is of size two and is also a valid path cover for G. Thus $P(G) \leq 2$. □

Theorem 4.30. *If G is a partial 2-path, then $M(G) = P(G)$.*

Proof. Let G be a partial 2-path. By Theorem 4.16, $P(G) \geq M(G)$. Suppose by way of contradiction that there exists a partial 2-path G such that $P(G) > M(G)$. We take G such that $P(H) = M(H)$ for each proper induced subgraph H of G. By Lemma 4.29, $P(G) \leq 2$. Certainly $P(G) \neq 1$, otherwise $M(G) < 1$. Thus $P(G) = 2$ and $M(G) = 1$. By Theorem 2.8, $M(G) = 1$ if and only if G is a path. Since G is a path, $P(G) = 1 \neq 2$ which is a contradiction. Thus $P(G) = M(G)$. □

4.4 Unicyclic Graphs

A *unicyclic* graph G is a graph which has exactly one cycle. The *girth* of a graph is the length of its shortest cycle. In the case of a unicyclic graph, its girth is the length of the only cycle. Barioli, Fallat, and Hogben [4] show that for all unicyclic graphs G, either $M(G) = P(G)$ or $M(G) = P(G) - 1$. Since unicyclic graph are outerplanar, the previous sections will provide a different approach to proving a similar result concerning the maximum nullity of unicyclic graphs. We begin by slightly modifying Lemma 4.26.

Lemma 4.31. *Let G be the vertex-sum at v of G_1 and G_2 such that $M(G_i) = P(G_i)$ and $M(G_i - v) = P(G_i - v)$ for all $i = 1, 2$. If $P(G) > M(G)$, then*

(*a*) $r_v(G) + p_v(G) > 1$, *and*

(*b*) $r_v(G_i) + p_v(G_i) = 1$ *for* $i = 1, 2$.

Proof. (*a*) Note that $P(G - v) = P(G_1 - v) + P(G_2 - v) = M(G_1 - v) + M(G_2 - v) = M(G - v)$ and $\operatorname{mr}(G - v) + P(G - v) = \operatorname{mr}(G - v) + M(G - v) = |G - v|$. Using Observation 2.1 and the hypothesis that $P(G) > M(G)$ we have

$$\begin{aligned} r_v(G) + p_v(G) &= \operatorname{mr}(G) - \operatorname{mr}(G - v) + P(G) - P(G - v) \\ &= \operatorname{mr}(G) + P(G) - (\operatorname{mr}(G - v) + P(G - v)) \\ &> \operatorname{mr}(G) + M(G) - (\operatorname{mr}(G - v) + M(G - v)) \\ &= |G| - |G - v| = 1. \end{aligned}$$

(b) By similar reasoning, for each i,

$$\begin{aligned} r_v(G_i) + p_v(G_i) &= \mathrm{mr}(G_i) - \mathrm{mr}(G_i - v) + P(G_i) - P(G_i - v) \\ &= \mathrm{mr}(G_i) + P(G_i) - (\mathrm{mr}(G_i - v) + P(G_i - v)) \\ &= \mathrm{mr}(G_i) + M(G_i) - (\mathrm{mr}(G_i - v) + M(G_i - v)) \\ &= |G_i| - |G_i - v| = 1. \end{aligned}$$

□

Lemma 4.32. *Let G be a graph and v a vertex of G such that $M(G-v) = P(G-v)$. Then $P(G) = M(G)$ if and only if $r_v(G) + p_v(G) = 1$.*

Proof. Let v be a vertex of a graph G such that $M(G-v) = P(G-v)$.

Assume that $P(G) = M(G)$. Then

$$\begin{aligned} r_v(G) + p_v(G) &= \mathrm{mr}(G) - \mathrm{mr}(G - v) + P(G) - P(G - v) \\ &= \mathrm{mr}(G) + P(G) - (\mathrm{mr}(G - v) + P(G - v)) \\ &= \mathrm{mr}(G) + M(G) - (\mathrm{mr}(G - v) + M(G - v)) \\ &= |G| - |G - v| = 1. \end{aligned}$$

Assume that $r_v(G) + p_v(G) = 1$. Then

$$\begin{aligned} \mathrm{mr}(G) + P(G) - (\mathrm{mr}(G - v) + P(G - v)) &= \mathrm{mr}(G) - \mathrm{mr}(G - v) + P(G) - P(G - v) \\ &= r_v(G) + p_v(G) = 1 = |G| - |G - v| \\ &= \mathrm{mr}(G) + M(G) - (\mathrm{mr}(G - v) + M(G - v)) \end{aligned}$$

Since $P(G-v) = M(G-v)$, we have $P(G) = M(G)$. □

The following lemma will help to show that for unicyclic graphs either $M(G) = P(G)$ or $M(G) = P(G) - 1$.

Lemma 4.33. *Let G be the vertex-sum at v of G_1 and G_2 such that $M(G_i) = P(G_i)$ and $M(G_i - v) = P(G_i - v)$ for all $i = 1, 2$. Then the following are equivalent*

(a) $P(G) > M(G)$

(b) $r_v(G_i) = 1$ *and* $p_v(G_i) = 0$ *for* $i = 1, 2$ *and* v *is simply terminal in at most one* G_i

(c) $P(G) = M(G) + 1$.

Proof. Let G be the vertex-sum at v of G_1 and G_2 such that $M(G_i) = P(G_i)$ and $M(G_i-v) = P(G_i - v)$ for all $i = 1, 2$.

Assume $P(G) > M(G)$. By Proposition 2.17, $0 \leq r_v(G_i) \leq 2$ for all $i = 1, 2$. By Proposition 4.3 part (c), $-1 \leq p_v(G_i) \leq 1$ for all $i = 1, 2$. By Lemma 4.31 part (b), $r_v(G_i) + p_v(G_i) = 1$. Since there are three different values for $p_v(G_1)$ as well as for $p_v(G_2)$, there are 9 different cases to consider. In addition, by Lemma 4.8, $p_v(G) = \min\{p_v(G_1), p_v(G_2)\}$ unless v is simply terminal in both G_i, in which case $p_v(G) = -1$. By Proposition 4.3 part (e), if v is simply terminal in G_i, then $p_v(G_i) = 0$. Thus there are two cases in which $p_v(G_1) = p_v(G_2) = 0$; one in which v is simply terminal in both graphs, and one in which v is simply terminal in at most one graph. By Theorem 2.18, $r_v(G) = \min\{r_v(G_1) + r_v(G_2), 2\}$. The different possibilities are considered in Table 4.1.

Table 4.1: Summary of Possible Cases

$r_v(G_1)$	$p_v(G_1)$	$r_v(G_2)$	$p_v(G_2)$	$r_v(G)$	$p_v(G)$	$p_v(G) + r_v(G)$
2	-1	2	-1	2	-1	1
2	-1	1	0	2	-1	1
2	-1	0	1	2	-1	1
1	0	2	-1	2	-1	1
1	0	1	0	2	0	2
1	0	1	0	2	-1	1
1	0	0	1	1	0	1
0	1	2	-1	2	-1	1
0	1	1	0	1	0	1
0	1	0	1	0	1	1

By Lemma 4.31 part (a), $r_v(G) + p_v(G) > 1$. This occurs only when $p_v(G_i) = 0$ and $r_v(G_i) = 1$ for $i = 1, 2$ and v is simply terminal in at most one G_i.

Assume $r_v(G_i) = 1$ and $p_v(G_i) = 0$ for $i = 1, 2$ and v is simply terminal in at most one G_i. By Theorem 2.18, $r_v(G) = 2$. By Lemma 4.8, $p_v(G) = 0$. Thus using Theorem 4.16,

$$
\begin{aligned}
0 = P(G) - P(G-v) &\geq M(G) - (P(G_1 - v) + P(G_2 - v)) && (4.9)\\
&= M(G) - (M(G_1 - v) + M(G_2 - v)) = M(G) - M(G-v) && (4.10)\\
&= |G| - \mathrm{mr}(G) - (|G-v| - \mathrm{mr}(G-v)) = 1 - r_v(G) = -1 && (4.11)\\
& && (4.12)
\end{aligned}
$$

Thus the inequality must be strict and $P(G) = M(G) + 1$

Thus we have shown that (a) imples (b) implies (c). Certainly (c) implies (a) so the proof is complete. □

A unicyclic graph G with girth n is the union of a cycle C_n with vertices $v_1, \ldots, v_n$ and n possibly degenerate trees $T_1, \ldots, T_n$ with a vertex v_i labeled in each T_i. This is illustracted in Figure 4.5. The trees $T_1, \ldots, T_n$ will be called the *branches* of G. In Example 4.1, the

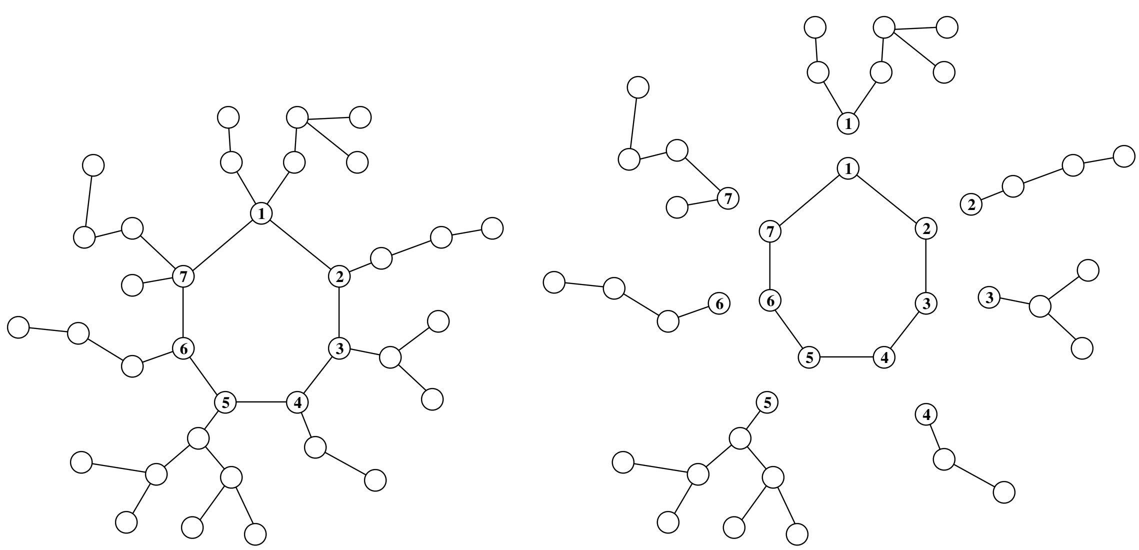


Figure 4.5: A unicyclic graph and its decomposition into a cycle and branches.

5-sun was introduced. In general an n-sun is a unicyclic graph where the cycle is C_n and each branch is a K_2. As in [17] and [4], H_n will denote the n-sun. A *partial n-sun*, is an n-sun with one or more pendant vertices deleted.

Theorem 4.34. *Let G be a unicyclic graph with girth n such that H_n is not induced. Then $M(G) = P(G)$.*

Proof. Suppose by way of contradiction that there exists a unicyclic graph with girth n where H_n is not induced such that the maximum nullity of the graph is not equal to the path cover of the graph. Let G be the smallest such graph with respect to the size of the vertex set. Since all unicyclic graphs are outerplanar, G is outerplanar. Since $P(G) \neq M(G)$, we have by Theorem 4.16 that $P(G) > M(G)$. Note that every proper induced subgraph of G is either a tree or a unicyclic graph on a smaller vertex set. Using Theorem 4.9 and the minimality of G, all proper induced subgraphs K of G have $P(K) = M(K)$. By Lemma 4.27 for every cutvertex v of G, $G - v$ has exactly two components one of which is an isolated vertex. Let C be the induced cycle of G of length n. Since every vertex of C which is not of degree 2 is a cutvertex of G, and H_n is not induced, G is a partial n-sun.

Since G is a partial n-sun, there is at least one vertex of degree 2 in G. Since $M(C_n) = P(C_n)$, $G \neq C_n$ and G has at least one pendant vertex. Since every vertex of C is either adjacent to a pendant vertex or is a vertex of degree 2, there exists a vertex of degree 2, v_2, adjacent to a vertex of degree 3, v_1, with pendant neighbor u. Label the remaining vertices of C in order $v_3, \ldots, v_n$. Thus G is the vertex-sum at v_1 of $H = G - u$ and K_2. Since H and $H - v_1$ are proper induced subgraphs, $P(H) = M(H)$ and $P(H - v_1) = M(H - v_1)$. Since $P(G) > M(G)$, Lemma 4.33 part (b) implies that $p_{v_1}(H) = 0$ and v_1 is simply terminal in at most one of H and K_2. Note that v_1 is simply terminal in K_2, and so v_1 is not simply terminal in H.

Since $p_{v_1}(H) = 0$, $P(H) = P(H - v_1)$. Since H is a unicyclic graph, $P(H) \geq 2$ and $P(H - v_1) \geq 2$. We construct a minimum path cover R for H by modifying a minimum path cover of $H - v_1$ without increasing the number of paths. Let Q be a minimal path cover for $H - v_1$. Let P_2 be the path in Q which covers v_2. Both v_1 and v_2 are vertices of degree 2 in H. Since $v_1 \sim v_2$ in H, v_2 is a pendant vertex in $H - v_1$. Let P_1 be the path formed by extending P_2 to cover v_1 and let $R = (Q \setminus P_2) \cup P_1$.

If P_1 is an induced path in H, then R is a minimum path cover of H in which v_1 is a pendant vertex of P_1.

On the other hand if P_1 is not an induced path, then P_1 induces the cycle C. Since $P(H-v_1) \geq 2$, $|Q| \geq 2$. Since $|Q| = |R|$, $|R| \geq 2$. Since P_1 covers all the vertices of C, there exists a degenerate path of R which covers a pendant vertex p of H. Let v_i be adjacent to p and note that i is not equal to 1, 2, or n. Let $P' = v_1 v_2 \ldots v_i p$ and let P'' start at v_{i+1} and end at the last vertex of P_1 which is either v_n or its pendant neighbor if it has one. Let $R' = (R \setminus \{P_1, \{p\}\}) \cup \{P', P''\}$. Thus $|R'| = |R|$ and R' is a minimum path cover for H. Further v_1 is a pendant vertex of P'.

Since $p_{v_1}(H) = 0$, Proposition 4.3 part (d) implies that v_1 is not doubly terminal. Since in either case above v_1 is a pendant vertex of a path in a minimum path cover for H, v_1 is simply terminal in H, a contradiction. □

Theorem 4.35. *If G is a unicyclic graph with even girth, then $P(G) = M(G)$.*

Proof. Suppose by way of contradiction that there exists a unicyclic graph with even girth such that the path cover number of the graph is not equal to the maximum nullity of the graph. Let G be the smallest such graph with respect to the size of the vertex set. Since all unicyclic graphs are outerplanar, G is outerplanar. Since $P(G) \neq M(G)$, we have by Theorem 4.16 that $P(G) > M(G)$. Note that every proper induced subgraph of G is either a tree or a unicyclic graph on a smaller vertex set. Using Theorem 4.9 and the minimality of G, all proper induced subgraphs K of G have $P(K) = M(K)$. By Lemma 4.27, for every cutvertex v of G, $G - v$ has exactly two components one of which is an isolated vertex. Let C be the induced cycle of G with length n. Since every vertex of C which is not of degree 2, is a cutvertex of G, G is either an n-sun or a partial n-sun. By Theorem 4.34, G is not a partial n-sun. Thus G is an n-sun where n is even. There is a path cover R for G in which every path is of length 3 and covers exactly 2 pendant vertices. Thus $P(G) \leq \dfrac{n}{2}$. By Corollary 4.5, $P(G) \geq \left\lceil \dfrac{n}{2} \right\rceil$, so $P(G) = \dfrac{n}{2}$. Thus R is a minimum path cover for G.

Label a vertex of degree 3 as v. Now G is the vertex-sum at v of G_1 and K_2. Since G_1 has

$n-1$ pendant vertices and $n-1$ is odd, Corollary 4.5 implies that $P(G_1) \geq \left\lceil \frac{n-1}{2} \right\rceil = \frac{n}{2}$. Since R can be easily modified by shortening the path that covers v to a path cover for G_1, $P(G_1) = \frac{n}{2}$. Thus this modification of R is a minimum path cover in which v is a pendant vertex of a path.

Note that G satisfies the hypothesis of Lemma 4.33. Since $P(G) > M(G)$, Lemma 4.33 implies that $p_v(G_1) = 0$ and v is simply terminal in at most one of G_1 and K_2. Since v is simply terminal in K_2, v is not simply terminal in G_1.

Since $p_v(G_1) = 0$, Proposition 4.3 part (d) implies that v is not doubly terminal in G_1. Since v is a pendant vertex of a path in a minimum path cover for G_1, v is simply terminal in G_1, a contradiction. □

Before proving the main theorem of this section, we will need the following lemma.

Lemma 4.36. *Let T be a tree with a vertex v. Then $p_v(T) = 0$ if and only if v is simply terminal in T.*

Proof. The reverse implication follows from Proposition 4.3 part (e).

Assume $p_v(T) = 0$. By Proposition 4.3 part (d), v is not doubly terminal in T. Thus it remains to show that v is a pendant vertex of a path in a minimum path cover for T.

If v is not a cutvertex, it is a pendant vertex and thus simplicial. By Lemma 4.4, v is simply terminal. Thus we may assume that v is a cutvertex.

Let T be the vertex-sum at v of $T_1, \ldots, T_k$. Since $T - v$ is either a forest or a tree and $T_i - v$ are trees for each i, Theorem 4.9 implies that $P(T) = M(T)$, $P(T - v) = M(T - v)$ and $P(T_i - v) = M(T_i - v)$ for each i. Since $p_v(T) = 0$, Lemma 4.32 implies $r_v(T) = 1$. By Theorem 2.18, $r_v(T) = \min\left\{\sum_{i=1}^{k} r_v(T_i), 2\right\}$. Since $r_v(T) = 1$, there exists j such that $r_v(T_j) = 1$ and $r_v(T_i) = 0$ for all $i \neq j$. Renaming if necessary, let $r_v(T_1) = 1$. Let T_q be the vertex-sum at v of $T_2, \ldots, T_k$. By Theorem 2.18, $r_v(T_q) = 0$. Now T is the vertex-sum at v of T_1 and T_q. Since T_q is a tree and $T_q - v$ is a forest, Theorem 4.9 implies that $M(T_q) = P(T_q)$ and $M(T_q - v) = P(T_q - v)$. Thus by Lemma 4.32, $p_v(T_q) = 1$. By Proposition 4.3 part (d),

v is doubly terminal in T_q. Since $r_v(T_1) = 1$, Lemma 4.32 implies that $p_v(T_1) = 0$. Since v is a pendant vertex of T_1 it is simplicial and by Lemma 4.4, v is simply terminal in T_1.

Let R_1 be a minimum path cover for T_1 in which v is a pendant vertex of some path $P_1 \in R_1$. Let R_2 be a minimum path cover for T_q in which v is a degenerate path. Then $R = R_1 \cup (R_2 - \{v\})$ is a path cover for T with $|R_1| + |R_2| - 1$ paths. By Proposition 4.6 part (a), $P(T) \geq P(T_1) + P(T_q) - 1 = |R_1| + |R_2| - 1$. Thus R is a minimum path cover for T in which v is the pendant vertex of some path in R. So v is simply terminal in T. □

Theorem 4.37. *Let H_n be the n-sun with $n > 3$ odd. Then $P(H_n) > M(H_n)$.*

Proof. Note that H_n has n pendant vertices. By Corollary 4.5, $P(H_n) \geq \left\lceil \frac{n}{2} \right\rceil$. Since n is odd this implies that $P(H_n) \geq \frac{n+1}{2}$. There is a path cover R of H_n using paths of length 3 and a path of length 1, where each path of length 3 covers two pendant vertices of H_n and the path of length 1 covers the remaining pendant vertex. Thus $|R| = \frac{n-1}{2} + 1 = \frac{n+1}{2}$. Therefore $P(H_n) \leq |R| = \frac{n+1}{2}$ and it must be that $P(H_n) = \frac{n+1}{2}$.

Let v be a vertex of degree 3 in H_n. Thus H_n is the vertex-sum at v of G_1 and K_2. It is clear that $p_v(K_2) = 0$ and that v is simply terminal in K_2. The claim is that v is not simply terminal in G_1 and that $p_v(G_1) = 0$. Since $n > 3$, G_1 has $n - 1 > 2$ pendant vertices. By Corollary 4.5, $P(G_1) \geq \left\lceil \frac{n-1}{2} \right\rceil$. Since n is odd, $n - 1$ is even and $P(G_1) \geq \frac{n-1}{2}$. Since $n > 3$ is odd, there exists a path cover for G_1 consisting of at least 1 path of length 3 each of which cover 2 pendant vertices of G_1 and one path of length 4 which covers v and 2 pendant vertices of G_1. This path cover has exactly $\frac{n-1}{2}$ paths, and thus $P(G_1) = \frac{n-1}{2}$.

At this point it is convenient to show that v is not simply terminal in G_1. Suppose there exists a minimum path cover R_1 for G_1 where v is a pendant vertex of some path in R_1. Since all the pendant vertices of G_1 are necessarily pendant vertices of paths in R_1 or are themselves degenerate paths, $|R_1| \geq \left\lceil \frac{n}{2} \right\rceil = \frac{n+1}{2}$. Since R_1 is minimal, this contradicts that $P(G_1) = \frac{n-1}{2}$.

Note that $G_1 - v$ is a tree with $\frac{n-1}{2}$ pendant vertices. By Corollary 4.5, $P(G_1 - v) \geq \left\lceil \frac{n-1}{2} \right\rceil$. Since $n - 1$ is even, we have that $P(G_1 - v) \geq \frac{n-1}{2}$. There exists a path cover for $G_1 - v$ consisting of paths of length 3 each of which cover exactly 2 pendant vertices of $G_1 - v$. Thus $P(G_1 - v) \leq \dfrac{n-1}{2}$ and it follows that $P(G_1 - v) = \dfrac{n-1}{2}$. Thus $p_v(G_1) = 0$.

Now G_1 is a partial n-sun and so by Theorem 4.34, $P(G_1) = M(G_1)$. Since $G_1 - v$ is a tree, Theorem 4.9 implies that $P(G_1 - v) = M(G_1 - v)$. Thus by Lemma 4.32, $r_v(G_1) + p_v(G_1) = 1$. Thus $r_v(G_1) = 1$.

So $r_v(G_1) = 1$, $p_v(G_1) = 0$, and v is not simply terminal in G_1. Also $r_v(K_2) = 1$ and $p_v(K_2) = 0$. Thus by Lemma 4.33 we have that $P(H_n) > M(H_n)$. □

Theorem 4.38. *Let G be a unicyclic graph of girth n with the vertices of the cycle labeled as $v_1, \ldots, v_n$. Then $P(G) = M(G) + 1$ if and only if for each branch T_i, $p_{v_i}(T_i) = 0$ and $n > 3$ is odd.*

Proof. Let G be a unicyclic graph with the vertices of the cycle labeled $v_1, \ldots, v_n$.

Assume that $P(G) = M(G) + 1$. By Theorem 4.35, n is odd. Let $T_1, \ldots, T_n$ be the branches of G. By Theorem 4.34, none of the T_i are degenerate. Further, G is the vertex-sum at v_1 of a unicyclic graph G_1 and T_1. By Theorem 4.34, $M(G_1) = P(G_1)$. By Theorem 4.9, $M(G_1 - v_1) = P(G_1 - v_1)$, $M(T_1) = P(T_1)$, and $M(T_1 - v_1) = P(T_1 - v_1)$. Thus by Lemma 4.33, $p_{v_1}(T_1) = 0$. Similarly, $p_{v_i}(T_i) = 0$ for all $i = 1, \ldots, n$.

All that remains for the forward direction is to show that $n > 3$. Suppose by way of contradiction that G has girth 3. By Theorem 4.34, $P(G_1) = M(G_1)$. Since $G_1 - v_1$ and T_1 are trees and $T_1 - v_1$ is either a forest or a tree, Theorem 4.9 implies that $P(G_1 - v_1) = M(G_1 - v_1)$, $P(T_1) = M(T_1)$, and $P(T_1 - v_1) = M(T_1 - v_1)$. Thus by Lemma 4.33, $p_{v_1}(G_1) = 0$, $p_{v_1}(T_1) = 0$ and v_1 is simply terminal in at most one of T_1 and G_1. By Lemma 4.36, v_1 is simply terminal in T_1. Since $n = 3$, v_1 is a simplicial vertex of G_1. By Lemma 4.4, v_1 is simply terminal in G_1. So v_1 is simply terminal in both T_1 and G_1, a contradiction.

Assume that for each branch T_i of G, $p_{v_i}(T_i) = 0$ and $n > 3$ is odd. Suppose by way of contradiction that there exists a unicyclic graph with the given properties such that $M(G) = P(G)$. Let G be a the smallest such graph with respect to the number of vertices. Since $G - v_i$ is a forest for all i, Theorem 4.9 implies that $P(G - v_i) = M(G - v_i)$ for all i.

Thus by Lemma 4.32, $r_{v_i}(G) + p_{v_i}(G) = 1$.

Consider a vertex v_i of G on the cycle. Since $p_{v_i}(T_i) = 0$, Lemma 4.36 implies v_i is simply terminal in T_i. Since T_i is a tree and $T_i - v_i$ is either a tree or a forest, Theorem 4.9 implies that $M(T_i) = P(T_i)$ and $M(T_i - v_i) = P(T_i - v_i)$. By Lemma 4.32, $r_{v_i}(T_i) = 1$. Let H be the graph obtained from G by replacing branch T_i with K_2. Note that $p_{v_i}(K_2) = 0$, v_i is simply terminal in K_2, and $r_{v_i}(K_2) = 1$. Since T_i and K_2 have the exact same path-spread and rank-spread, using Theorem 2.18 and Lemma 4.8, we have $r_{v_i}(H) + p_{v_i}(H) = 1$. By Lemma 4.32, we have $M(H) = P(H)$. All the branches of H still have the property that the path-spread is equal to 0, and n is still odd. Since each branch of G can be replaced by K_2 and still keep the desired characteristics, the minimality of G implies G is an n-sun for $n > 3$ odd. By Theorem 4.37, $P(G) > M(G)$, a contradiction. $\square$

Theorem 4.39. *Let G be a unicyclic graph of girth n with the vertices of the cycle labeled as $v_1, \ldots, v_n$.*

- $M(G) = P(G) - 1$ *if $n > 3$ is odd and $p_{v_i}(T_i) = 0$ for every branch T_i*
- $M(G) = P(G)$ *otherwise.*

Proof. Let G be as described in the statement of the theorem. Since G is outerplanar, Theorem 4.16 implies $P(G) \geq M(G)$. If $T_i = K_1$ for all i , then $G = C_n$ and $M(G) = P(G)$. Thus we may assume that at least one T_i is not degenerate. Renaming the vertices of the cycle if necessary, let G be the vertex-sum at v_1 of G_1 and T_1. By Theorem 4.34, $M(G_1) = P(G_1)$. By Theorem 4.9, $P(G_1 - v_1) = M(G_1 - v_1)$, $P(T_1) = M(T_1)$, and $P(T_1 - v_1) = M(T_1 - v_1)$. By Lemma 4.33, $P(G) > M(G)$ if and only if $P(G) = M(G) + 1$. By Theorem 4.38, $P(G) = M(G) + 1$ if and only if $p_{v_i}(T_i) = 0$ for all $i \in \{1, \ldots n\}$ and $n > 3$ is odd. $\square$

Corollary 4.40. *Let G be a unicyclic graph of girth n with the vertices of the cycle labeled as $v_1, \ldots, v_n$.*

- $M(G) = P(G) - 1$ *if $n > 3$ is odd and $r_{v_i}(T_i) = 1$ for every branch T_i*

- $M(G) = P(G)$ *otherwise.*

Proof. By Theorem 4.39, it is sufficient to show that $p_{v_i}(T_i) = 0$ if and only if $r_{v_i}(T_i) = 1$. By Theorem 4.9, $P(T_i) = M(T_i)$ and $P(T_i - v_i) = M(T_i - v_i)$ for all $i \in \{1, \ldots, n\}$. Thus the result follows by Lemma 4.32. $\square$

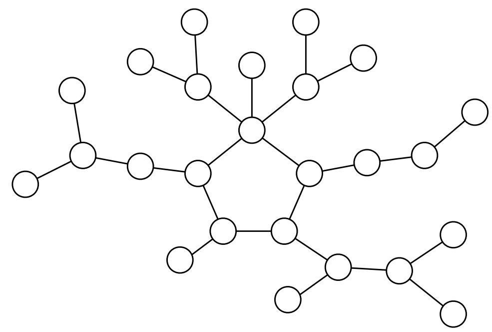


Figure 4.6: A unicyclic graph G such that $P(G) > M(G)$

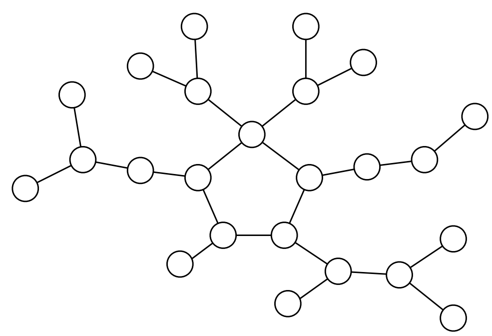


Figure 4.7: A unicyclic graph G such that $P(G) = M(G)$

Using Theorem 4.39 the unicyclic graph in Figure 4.6 has $P(G) = M(G) + 1$ and the unicyclic graph in Figure 4.7 has $P(G) = M(G)$. Notice that the graphs differ by a single vertex.

Chapter 5. Covers and Minimum Rank of a Graph

The main result of this chapter is Theorem 5.8. It states that every outerplanar graph can be covered by a cliques, stars, cycles, and double cycles in such a way that the sum of the minimum ranks of the graphs in the cover equals the minimum rank of the graph. The first proof of this result is found in [8]. The proof given here is slightly easier to follow and somewhat shorter. There are many consequences of Theorem 5.8 and most are found in [8]. One of the nicer implications found in [8] is that the minimum rank of an outerplanar graph is the same whether the matrix has real entries or entries from an arbitrary field. In particular the minimum rank of an outerplanar graph is equal to the minimum rank of the graph when considered over the field of two elements. In Section 5.3 we give another consequence of Theorem 5.8 to weighted graphs. Theorem 5.16 shows that for outerplanar graphs, the zero/nonzero pattern of the off-diagonal entries in the matrix determines the minimum rank and not the value of the nonzero entries. Theorem 5.16 can also be applied to signed outerplanar graphs.

5.1 Covers and Minimum Rank of a Graph

A *cover* for a graph G is a finite collection $\mathcal{C} = \{G_1, G_2, \ldots, G_k\}$ of subgraphs of G such that $\bigcup_{i=1}^{k} E(G_i) = E(G)$. In other words every edge of G is in at least one graph of $\mathcal{C}$. If every edge of G is in exactly one G_i, then the cover is *edge-disjoint*. Given a graph G and a cover $\mathcal{C}$ of G, we say a vertex (edge) of G is *covered by* an element of the cover $H \in \mathcal{C}$ if the vertex (edge) is in the vertex (edge) set of H. The *rank-sum* of a cover $\mathcal{C} = \{G_1, G_2, \ldots, G_k\}$, denoted $\mathrm{rs}(\mathcal{C})$, is equal to $\sum_{i=1}^{k} \mathrm{mr}(G_i)$.

Before proving the next lemma, we introduce some useful notation. Let $\mathcal{C} = \{G_1, \ldots, G_k\}$ be a cover for a graph G. It will be convenient to sum matrices corresponding to the subgraphs G_i. Since $|V(G_i)|$ will vary, and thus the size of the matrices as well, define $\widetilde{G_i}$ as the graph with vertex set $V(G)$ and edge set $E(G_i)$. In effect, $\widetilde{G_i}$ is the union of G_i and a finite set of isolated vertices. Since $\mathrm{mr}(K_1) = 0$, $\mathrm{mr}(\widetilde{G_i}) = \mathrm{mr}(G_i)$.

Lemma 5.1. *Let* $\mathcal{C} = \{G_1, \ldots, G_k\}$ *be a cover for a graph* G. *Then* $\mathrm{rs}(\mathcal{C}) = \sum_{i=1}^{k} \mathrm{mr}(G_i) \geq \mathrm{mr}(G)$.

Proof. Let $\mathcal{C} = \{G_1, G_2, \ldots, G_k\}$. Let A_i be a minimum rank matrix for $\widetilde{G_i}$ for $i = \{1, \ldots, k\}$, in other words $\operatorname{rank} A_i = \operatorname{mr}(\widetilde{G_i})$. There exists nonzero real numbers $c_1, \ldots, c_k$ so that $A = \sum_{i=1}^{k} c_i A_i$ is in $\mathcal{S}(G)$. Then using Fact 2,

$$\operatorname{rs}(\mathcal{C}) = \sum_{i=1}^{k} \operatorname{mr}(G_i) = \sum_{i=1}^{k} \operatorname{mr}(\widetilde{G_i}) = \sum_{i=1}^{k} \operatorname{rank} c_i A_i \geq \operatorname{rank} \sum_{i=1}^{k} c_i A_i = \operatorname{rank} A \geq \operatorname{mr}(G).$$

□

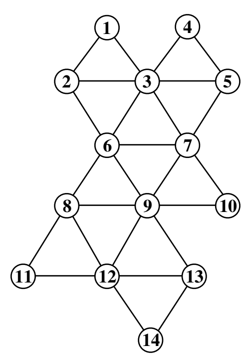
Observation 5.2. *Let G be a disconnected graph with components $G_1, \ldots, G_k$. If $\mathcal{C}_1, \ldots, \mathcal{C}_k$ are covers for $G_1, \ldots, G_k$ respectively, such that $\operatorname{rs}(\mathcal{C}_i) = \operatorname{mr}(G_i)$ for all i, then $\operatorname{rs}(\cup \mathcal{C}_i) = \operatorname{mr}(G)$.*

A *minimum rank cover* of a graph G is a cover $\mathcal{C}$ of G such that $\operatorname{rs}(\mathcal{C}) = \operatorname{mr}(G)$.

The *clique cover number* of a graph G, denoted $cc(G)$, is the minimum number of complete subgraphs of G required to cover the edges of G. Since $\operatorname{mr}(K_n) = 1$ for all $n \geq 2$,

$$cc(G) = \min\{\operatorname{rs}(\mathcal{C}) : \mathcal{C} \text{ consists of complete graphs}\}.$$

Example 5.3. Let G be as in Example 4.18.

$G =$ 

It was shown that $M(G) \leq 4$. We will use a clique cover to get an upper bound on $\operatorname{mr}(G)$. Let $\mathcal{C}$ consist of the all the triangles of G except those induced by the vertex sets $\{3, 6, 7\}$ and $\{8, 9, 12\}$. Then $\operatorname{rs}(\mathcal{C}) = 10$ and by Lemma 5.1, $\operatorname{mr}(G) \leq 10$. Since $\operatorname{mr}(G) \leq 10$, we see that $M(G) \geq |G| - 10 = 4$. Thus $M(G) = 4$.

We now finish Example 4.18 by showing that $M(G - e_i) \geq 5$ for $i = \{1, 2\}$ where $e_1 = \{7, 10\}$ and $e_2 = \{9, 10\}$. To do so, for each i we find a cover of $G - e_i$ whose rank-sum is less than or equal to 9.

Let $\mathcal{C}_1$ consist of the triangles induced by $\{1, 2, 3\}$, $\{3, 4, 5\}$, $\{3, 5, 7\}$, $\{8, 11, 12\}$, and $\{12, 13, 14\}$ as well as the star subgraph with vertex 6 as the dominant vertex and the star subgraph with vertex 9 as the dominant vertex. Then $\mathcal{C}_1$ is a cover for $G - e_1$ and $\operatorname{rs}(\mathcal{C}_1) = 5 \operatorname{mr}(K_3) + 2 \operatorname{mr}(S_6) = 5 + 4 = 9$.

Let $\mathcal{C}_2$ consist of the triangles induced by $\{1, 2, 3\}$, $\{3, 4, 5\}$, $\{2, 3, 6\}$, $\{6, 8, 9\}$, $\{8, 11, 12\}$, $\{9, 12, 13\}$, and $\{12, 13, 14\}$ as well as the star subgraph with vertex 7 as the dominant vertex. Then $\mathcal{C}_2$ is a cover for $G - e_2$ and $\operatorname{rs}(\mathcal{C}_2) = 7 \operatorname{mr}(K_3) + \operatorname{mr}(S_6) = 7 + 2 = 9$.

Since $\operatorname{rs}(\mathcal{C}_i) = 9$ for all i, by Lemma 5.1 $\operatorname{mr}(G - e_i) \leq 9$. Thus $M(G - e_i) \geq |G - e_i| - 9 = 14 - 9 = 5$ for $i = \{1, 2\}$.

5.2 Minimum Rank for Outerplanar Graphs

The major difficulty in proving Theorem 5.8 is determining to which 2-separation Theorem 2.21 should be applied. To find and describe this 2-separation we make use of the weak dual. The following proposition will be applied to the weak dual of a 2-connected outerplanar graph.

Proposition 5.4. *Let T be a tree which is not P_n. Then there exists a vertex v of T of degree $k \geq 3$ such that T is the vertex-sum at v of $T_1, \ldots, T_k$ where at most one T_i is not a path.*

Proof. Proceed by induction on the number of vertices in T. The only tree on four vertices which is not a path is S_4. Since S_4 is the vertex-sum of 3 copies of K_2, its dominating vertex satisfies the conclusion of the proposition. Let T be a tree on more than 4 vertices which is not a path. Let P be a diametrical path in T with pendant vertex w. Since P is a diametrical path the neighbor u of w is adjacent to at most one vertex which is not pendant in T. If u has degree 3 or greater, then it satisfies the conclusion. If u has degree 2, consider the graph $T - w$. Since u has degree 2 and T is not a path, $T - w$ is not a path. By the induction hypothesis there exists a vertex v of $T - w$ of degree $k \geq 3$ such that $T - w$ is the vertex-sum at v of $T_1, \ldots, T_k$ where at most one T_i is not a path. Since w was a pendant vertex of T adjacent to a degree 2 vertex, v is a vertex of T which satisfies the conclusion. $\square$

The graph G described in the next lemma will be one of two graphs in the desired 2-separation of a 2-connected outerplanar graph in Case 2 of the proof of Theorem 5.8. An example of such a graph and how it relates to a 2-connected outerplanar graph can be seen in Figure 5.1.

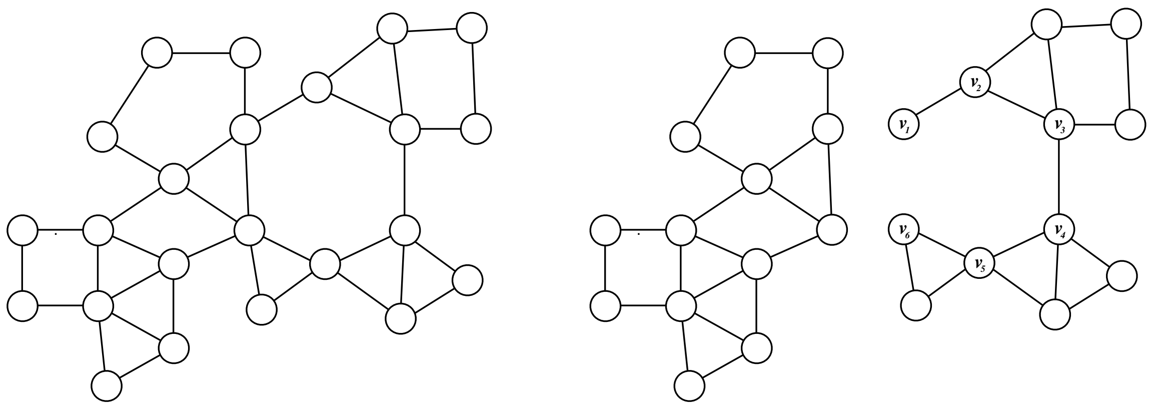


Figure 5.1: A 2-connected outerplanar graph with a particular 2-separation

Lemma 5.5. *For $k \geq 3$, let each of $G_1, \ldots, G_k$ be either a 2-connected partial 2-paths or be isomorphic to K_2. If G_i is a 2-connected partial 2-path choose an exterior edge from a pendant cycle and label its vertices v_i and v_{i+1}. If G_i is isomorphic to K_2 label the only vertices of G_i as v_i and v_{i+1}. Let G be created by first vertex-summing G_1 and G_2 at v_2, then vertex-summing the resulting graph with G_3 at v_3, and so on until the resulting graph is vertex-summed with G_k at v_k. Then $\operatorname{mr}(G) = \sum_{i=1}^{k} \operatorname{mr}(G_i)$ and $M(G) = P(G)$.*

Proof. Let G be constructed as described in the lemma. Since the v_i are chosen to be on an exterior edge of a pendant cycle of G_i or in the case that G_i is K_2, are the only vertices in

G_i, by Observations 4.24 and 4.25, there is a path cover for G with $\sum_{i=1}^{k} P(G_i) - (k-1)$ paths. Thus using Theorem 4.16, Observation 2.1, and Lemma 5.1 we have

$$\sum_{i=1}^{k} P(G_i) - (k-1) \geq P(G) \geq M(G) = |G| - \mathrm{mr}(G) = \sum_{i=1}^{k} |G_i| - (k-1) - \mathrm{mr}(G)$$

$$\geq \sum_{i=1}^{k} |G_i| - (k-1) - \sum_{i=1}^{k} \mathrm{mr}(G_i) = \sum_{i=1}^{k} M(G_i) - (k-1).$$

Since $P(G_i) = M(G_i)$ for all G_i, there is equality throughout. Thus $\mathrm{mr}(G) = \displaystyle\sum_{i=1}^{k} \mathrm{mr}(G_i)$ and $M(G) = P(G)$. □

In the following lemma we will be working with the graphs G/v_1v_k and H which are formed from G and occur in the use of the 2-separation formula found in Theorem 2.21. In Figure 5.2 we give an example of what G/v_1v_6 and H would be for the graph G.

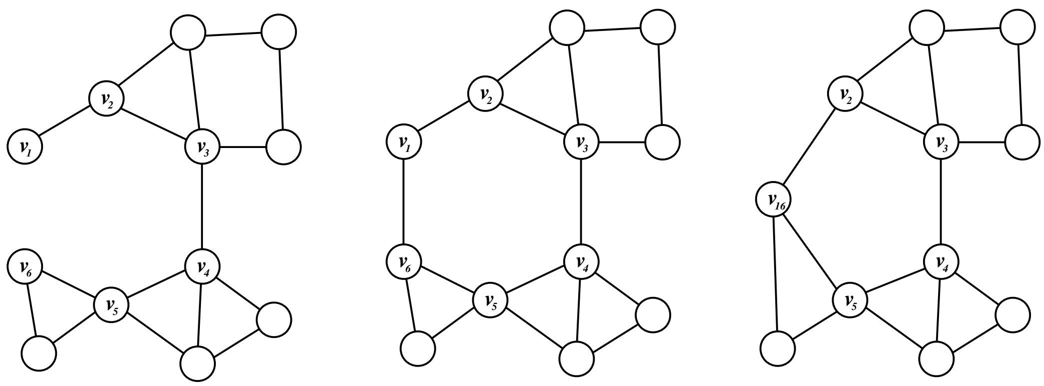


Figure 5.2: The graphs G, H, and G/v_1v_6

Lemma 5.6. *Let G be constructed and labeled as in Lemma 5.5 and let H be the graph obtained from G by adding an edge between v_1 and v_{k+1}. If at least two of the G_i are 2-connected partial 2-paths, then* $\mathrm{mr}(G/v_1v_{k+1}) = \mathrm{mr}(G)$ *and* $\mathrm{mr}(H) = \mathrm{mr}(G) + 1$.

Proof. Note that since G is outerplanar, all graphs considered are outerplanar as well. Construct a minimum path cover for G (as in the proof of Lemma 5.5) where v_1 and v_{k+1} are pendant vertices in their respective paths. In the graph G/v_1v_{k+1}, v_1 and v_{k+1} are identified together and in the graph H, v_1 and v_{k+1} are adjacent. Thus there is the possibility that the two paths which covered v_1 and v_{k+1} may become one path. It must be shown that this new path does not induce the cycle consisting of the v_i which was created by identifying v_1 and v_{k+1} or adding the edge v_1v_{k+1} in their respective graphs. Since there are at least two G_i which are 2-connected partial 2-paths, there exists a vertex v_j which is covered by a path different from the paths covering v_1 and v_{k+1} in G. Thus the path created from joining up the paths covering v_1 and v_{k+1} is an induced path and there is a path cover for G/v_1v_{k+1} and H consisting of $P(G)-1$ paths. Thus $P(G/v_1v_{k+1}) \leq P(G)-1$ and $P(H) \leq P(G)-1$.

Then using Theorem 4.16, Observation 2.1, Lemma 5.1, and Lemma 5.5,

$$P(G)-1 \geq P(G/v_1v_{k+1}) \geq M(G/v_1v_{k+1}) = |G/v_1v_{k+1}| - \mathrm{mr}(G/v_1v_{k+1})$$

$$= |G|-1-\mathrm{mr}(G/v_1v_{k+1}) \geq |G|-1-\sum_{i=1}^{k}\mathrm{mr}(G_i) = |G|-1-\mathrm{mr}(G) = M(G)-1.$$

By Lemma 5.5, $P(G) = M(G)$ and so there is equality throughout. Therefore $\mathrm{mr}(G/v_1v_{k+1}) = \mathrm{mr}(G)$.

The edge between v_1 and v_{k+1} can be covered with K_2. Using Theorem 4.16, Observation 2.1, Lemma 5.1, and Lemma 5.5,

$$P(G)-1 \geq P(H) \geq M(H) = |H| - \mathrm{mr}(H) = |G| - \mathrm{mr}(H) \geq |G| - \sum_{i=1}^{k}\mathrm{mr}(G_i) - \mathrm{mr}(K_2)$$

$$= |G| - \mathrm{mr}(G) - 1 = M(G) - 1.$$

By Lemma 5.5, $P(G) = M(G)$ and so there is equality throughout. Therefore $\mathrm{mr}(H) = \mathrm{mr}(G)+1$. $\square$

Recall from the beginning of Section 4.3, that partial 2-paths are outerplanar graphs. A *double cycle* is a 2-connected partial 2-path consisting of exactly 2 induced cycles. In other words, a double cycle is a cycle with exactly one chord. By Corollary 4.23, the path cover number and maximum nullity of a double cycle is 2.

Theorem 5.7. *If G is a 2-connected partial 2-path, then there exists an edge-disjoint cover $\mathcal{C}$ of G consisting of cliques, cycles, and double cycles such that the rank-sum of $\mathcal{C}$ is equal to $mr(G)$.*

Proof. Let G be a 2-connected partial 2-path. Proceed by induction on the number of induced cycles in G. Since cycles and double cycles are part of the covering class, the base cases are clearly true. Assume that G has at least 3 induced cycles. Let C_r be a pendant cycle of G and C_s its neighboring cycle. Let H be the 2-connected partial 2-path obtained from G by deleting the vertices of C_r and C_s which do not belong to any other cycle of G. By the inductive hypothesis there exists an edge-disjoint cover $\mathcal{C}'$ of H consisting of cliques, cycles, and double cycles such that $\mathrm{rs}(\mathcal{C}') = \mathrm{mr}(H)$. Using Observation 2.1 and Corollary 4.23,

$$|H| - \mathrm{mr}(H) = M(H) = 2 = M(G) = |G| - \mathrm{mr}(G) = |H| + |C_r| + |C_s| - 4 - \mathrm{mr}(G)$$

$$= |H| + M(C_r) + \mathrm{mr}(C_r) + M(C_s) + \mathrm{mr}(C_s) - 4 - \mathrm{mr}(G)$$

$$= |H| + \mathrm{mr}(C_r) + \mathrm{mr}(C_s) - \mathrm{mr}(G)$$

Thus

$$\mathrm{mr}(G) = \mathrm{mr}(H) + \mathrm{mr}(C_r) + \mathrm{mr}(C_s).$$

Let $\mathcal{C} = \mathcal{C}' \cup \{C_r\} \cup \{(s-2)K_2\}$. The cover $\mathcal{C}'$ will cover all the edges of H, C_r will cover the edges of the pendant cycle C_r, and the $s-2$ copies of K_2 will cover the remaining edges of C_s. Thus $\mathcal{C}$ is an edge-disjoint cover for G and

$$\mathrm{rs}(\mathcal{C}) = \mathrm{rs}(\mathcal{C}') + \mathrm{mr}(C_r) + s - 2 = \mathrm{mr}(H) + \mathrm{mr}(C_r) + \mathrm{mr}(C_s) = \mathrm{mr}(G).$$

□

In the proof of Theorem 5.7 it may not be apparent why the double cycles were necessary. A cover consisting of the two induced cycles of a double cycle, is a cover whose rank-sum is equal to the minimum rank of the double cycle. The problem is that such a cover is not edge-disjoint. There does not exist an edge-disjoint cover of a double cycle consisting of cliques, stars, and cycles whose rank-sum is the minimum rank of the double cycle.

Theorem 5.8. *If G is an outerplanar graph, then there exists an edge-disjoint cover $\mathcal{C}$ of G consisting of cliques, stars, cycles, and double cycles, such that the rank-sum of $\mathcal{C}$ is equal to* $\mathrm{mr}(G)$.

Proof. Let G be an outerplanar graph. Proceed by induction on the number of vertices of G. The base cases K_1, K_2, $2K_1$, K_3, P_3, $K_1 \cup K_2$, and $3K_1$ are trivial. If $|G| > 3$, then consider the connectivity of G. In the case that G is disconnected the inductive hypothesis yields edge-disjoint covers for each component. By Observation 5.2 the union of such covers will be an edge-disjoint cover for G with the correct rank-sum. So assume now that G is connected.

Case 1 G has a cutvertex v.

By Theorem 2.13, $\mathrm{mr}(G) = \min\{\mathrm{mr}(G_1) + \mathrm{mr}(G_2), \mathrm{mr}(G_1 - v) + \mathrm{mr}(G_2 - v) + 2\}$. By Observation 3.6, G_1, G_2, $G_1 - v$, and $G_2 - v$ are all outerplanar graphs with less vertices than G. By the inductive hypothesis there exist edge-disjoint covers for these graphs consisting of cliques, stars, cycles, and doubles cycles, such that the rank-sum of the covers is equal to the minimum rank of the graphs. Let $\mathcal{C}_i$ be such a cover for G_i and $\mathcal{C}'_i$ be such a cover for $G_i - v$, for $i = \{1, 2\}$.

Subcase 1 $\mathrm{mr}(G) = \mathrm{mr}(G_1) + \mathrm{mr}(G_2)$.

Let $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$. Then $\mathcal{C}$ is an edge-disjoint cover for G and

$$\mathrm{rs}(\mathcal{C}) = \mathrm{rs}(\mathcal{C}_1) + \mathrm{rs}(\mathcal{C}_2) = \mathrm{mr}(G_1) + \mathrm{mr}(G_2) = \mathrm{mr}(G).$$

Subcase 2 $\operatorname{mr}(G) = \operatorname{mr}(G_1 - v) + \operatorname{mr}(G_2 - v) + 2$.

Let v have degree k in G and let $\mathcal{C} = \mathcal{C}'_1 \cup \mathcal{C}'_2 \cup \{S_{k+1}\}$. The star S_{k+1} will cover vertex v and all the edges incident to v, while $\mathcal{C}'_i$ covers $G_i - v$. Thus $\mathcal{C}$ is an edge-disjoint cover for G and

$$\operatorname{rs}(\mathcal{C}) = \operatorname{rs}(\mathcal{C}'_1) + \operatorname{rs}(\mathcal{C}'_2) + \operatorname{mr}(S_{k+1}) = \operatorname{mr}(G_1 - v) + \operatorname{mr}(G_2 - v) + 2 = \operatorname{mr}(G).$$

Case 2 G is 2-connected.

Then G is a 2-connected outerplanar graph and by Lemma 3.5 the weak dual G^w of G is a tree. If G^w is a path, then G is a 2-connected partial 2-path and by Theorem 5.7 the conclusion follows. If G^w is not a path, then by Proposition 5.4 there exists a vertex v of G^w with degree $k \geq 3$ such that G^w is the vertex-sum at v of $T_1, T_2, \ldots, T_k$ where at most one of the T_i is not a path. The vertex v of G^w corresponds to an induced cycle in G with at least 3 neighboring cycles. Further, the T_i which are paths correspond to 2-connected partial 2-paths. Thus G has a 2-separation (G_1, G_2) such that G_2 is constructed as in Lemma 5.5 and $V(G_1) \cap V(G_2) = \{v_1, v_{k+1}\}$. By Theorem 2.21, $\operatorname{mr}(G)$, is the minimum of 6 terms. It will now be shown that two of the six terms are unnecessary for this particular 2-separation.

Consider the term $\operatorname{mr}(G_1/v_1v_{k+1}) + \operatorname{mr}(G_2/v_1v_{k+1}) + 2$. By Lemma 2.25 and Lemma 5.6,

$$\operatorname{mr}(G_1/v_1v_{k+1}) + \operatorname{mr}(G_2/v_1v_{k+1}) + 2 \geq \operatorname{mr}(G_1) - 2 + \operatorname{mr}(G_2) + 2 = \operatorname{mr}(G_1) + \operatorname{mr}(G_2).$$

Consider the term $\operatorname{mr}(H_1) + \operatorname{mr}(H_2)$. Since the edge v_1v_{k+1} is already present in G_1, H_1 has two edges between v_1 and v_{k+1}. Thus

$$\operatorname{mr}(H_1) = \min\{\operatorname{mr}(G_1), \operatorname{mr}(G_1 - v_1v_{k+1})\}.$$

By Proposition 2.12, $\text{mr}(G_1 - v_1 v_{k+1}) \geq \text{mr}(G_1) - 1$. Thus $\text{mr}(H_1) \geq \text{mr}(G_1) - 1$. Using this fact and Lemma 5.6,

$$\text{mr}(H_1) + \text{mr}(H_2) \geq \text{mr}(G_1) - 1 + \text{mr}(G_2) + 1 = \text{mr}(G_1) + \text{mr}(G_2).$$

Thus one of the four terms $\text{mr}(G_1) + \text{mr}(G_2), \text{mr}(G_1 - v_1) + \text{mr}(G_2 - v_1) + 2, \text{mr}(G_1 - v_{k+1}) + \text{mr}(G_2 - v_{k+1}) + 2$, or $\text{mr}(G_1 - R) + \text{mr}(G_2 - R) + 4$ is equal to $\text{mr}(G)$. By Observation 3.6, all the graphs in the four terms are outerplanar. Thus by the inductive hypothesis, every graph has an edge-disjoint cover consisting of cliques, stars, cycles, and double cycles whose rank-sum is equal its minimum rank. Let $\mathcal{C}_i$ be such a cover for G_i, $\mathcal{C}'_i$ be such a cover for $G_i - v_1$, and $\mathcal{C}''_i$ be such a cover for $G_i - R$.

Subcase 1 $\text{mr}(G) = \text{mr}(G_1) + \text{mr}(G_2)$.

Let $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$. Then $\mathcal{C}$ is an edge-disjoint cover for G and

$$\text{rs}(\mathcal{C}) = \text{rs}(\mathcal{C}_1) + \text{rs}(\mathcal{C}_2) = \text{mr}(G_1) + \text{mr}(G_2) = \text{mr}(G).$$

Subcase 2 $\text{mr}(G) = \text{mr}(G_1 - v_1) + \text{mr}(G_2 - v_1) + 2$.

Let v_1 have degree p and let $\mathcal{C} = \mathcal{C}'_1 \cup \mathcal{C}'_2 \cup \{S_{p+1}\}$. In this case S_{p+1} will cover the vertex v_1 and the edges incident to it. Then $\mathcal{C}$ is an edge-disjoint cover for G and

$$\text{rs}(\mathcal{C}) = \text{rs}(\mathcal{C}'_1) + \text{rs}(\mathcal{C}'_2) + \text{mr}(S_{p+1}) = \text{mr}(G_1 - v_1) + \text{mr}(G_2 - v_2) + 2 = \text{mr}(G).$$

Subcase 3 $\text{mr}(G) = \text{mr}(G_1 - v_{k+1}) + \text{mr}(G_2 - v_{k+1}) + 2$.

This case is almost identical to Subcase 2, with the only change being that the star will cover vertex v_{k+1} and its incident edges.

Subcase 4 $\text{mr}(G) = \text{mr}(G_1 - R) + \text{mr}(G_2 - R) + 4$.

Let v_1 have degree p and let v_{k+1} have degree q. Since v_1 and v_{k+1} are adjacent, let S_{p+1} and S_q be the stars needed to cover v_1, v_{k+1}, and their incident edges. So $\mathcal{C} = \mathcal{C}_1'' \cup \mathcal{C}_2'' \cup \{S_{p+1}, S_q\}$. Then $\mathcal{C}$ is an edge-disjoint cover for G and

$$\mathrm{rs}(\mathcal{C}) = \mathrm{rs}(\mathcal{C}_1'')+\mathrm{rs}(\mathcal{C}_2'')+\mathrm{mr}(S_{p+1})+\mathrm{mr}(S_q) = \mathrm{mr}(G_1-R)+\mathrm{mr}(G_2-R)+4 = \mathrm{mr}(G).$$

Therefore in all cases an edge-disjoint cover of G consisting of cliques, stars, cycles, and double cycles has been found whose rank-sum is equal to the minimum rank of G. □

5.3 Weighted Graphs

A *weighted graph* G_w is a pair (G, w) where G is a simple graph and w is a function from $E(G)$ to $\mathbb{R} \setminus \{0\}$. In other words each edge of G receives a nonzero real number as a label. In the minimum rank problem for a simple graph the value of each nonzero off-diagonal entry of a matrix in $\mathcal{S}(G)$ is not specified. There are many papers whose subject is weighted graphs. However, usually the diagonal entries are assumed to be zero. In this sense our definition of a weighted graph is more general. It seems logical that by specifying the value of each nonzero off-diagonal entry the range of attainable ranks for this subset of $\mathcal{S}(G)$ would decrease. However we will see that this is not the case for outerplanar graphs.

Given a weighted graph G_w let $\mathcal{S}(G_w)$ be the set of all symmetric matrices $A = [a_{ij}]$ such that $a_{ij} = w(ij)$ if $ij \in E(G)$, $a_{ij} = 0$ if $i \neq j$ and $ij \notin E(G)$, and $a_{ij} \in \mathbb{R}$ if $i = j$. The minimum rank of a weighted graph G_w is $\min\{\operatorname{rank} A : A \in \mathcal{S}(G_w)\}$.

Observation 5.9. *Let G_w be a weighted graph. Then* $\mathrm{mr}(G) \leq \mathrm{mr}(G_w)$.

The following is an example where $\mathrm{mr}(G) < \mathrm{mr}(G_w)$.

Example 5.10. Let G_w be K_4 where all the weights on the edges are 1 except one edge weighted 2. Let edge 14 be weighted 2. Then all matrices in $\mathcal{S}(G_w)$ have the following form:

$$\begin{bmatrix} a & 1 & 1 & 2 \\ 1 & b & 1 & 1 \\ 1 & 1 & c & 1 \\ 2 & 1 & 1 & d \end{bmatrix}.$$

There is a 2×2 submatrix with rank 2, and so by Fact 1 any matrix in $\mathcal{S}(G_w)$ has rank at least 2. Letting $a = d = 2$ and $b = c = 1$ yields a rank 2 matrix in $\mathcal{S}(G_w)$. Thus $\operatorname{mr}(G_w) = 2$, while $\operatorname{mr}(G) = \operatorname{mr}(K_4) = 1$.

Proposition 5.11. *Let G_w be a weighted graph with $G = K_2$. Then* $\operatorname{mr}(G_w) = 1$.

Proof. Given a weighted graph G_w with $G = K_2$ every matrix in $\mathcal{S}(G_w)$ has the form $\begin{bmatrix} d_1 & a \\ a & d_2 \end{bmatrix}$ where $a = w(12) \neq 0$. Since $\operatorname{mr}(K_2) = 1$, Observation 5.9 implies $\operatorname{mr}(G_w) \geq 1$. Let $A \in \mathcal{S}(G_w)$ with $d_1 = d_2 = a$. Then $\operatorname{rank} A = 1$ and so $\operatorname{mr}(G_w) = 1$. □

Proposition 5.12. *Let G_w be a weighted graph with $G = K_3$. Then* $\operatorname{mr}(G_w) = 1$.

Proof. Given a weighted graph G_w with $G = K_3$ every matrix in $\mathcal{S}(G_w)$ has the form $\begin{bmatrix} d_1 & a & b \\ a & d_2 & c \\ b & c & d_3 \end{bmatrix}$ where $a = w(12)$, $b = w(13)$, and $c = w(23)$ are all nonzero. Since $\operatorname{mr}(K_3) = 1$, Observation 5.9 implies $\operatorname{mr}(G_w) \geq 1$. Let $A \in \mathcal{S}(G_w)$ with $d_1 = \dfrac{ab}{c}$, $d_2 = \dfrac{ac}{b}$, and $d_3 = \dfrac{bc}{a}$. Then $\operatorname{rank} A = 1$ and so $\operatorname{mr}(G_w) = 1$. □

Proposition 5.13. *Let G_w be a weighted graph with $G = S_n$. Then* $\operatorname{mr}(G_w) = 2$.

Proof. Given a weighted graph G_w with $G = S_n$ every matrix in $\mathcal{S}(G_w)$ has the form

$$\begin{bmatrix} d_1 & a_2 & a_3 & \dots & a_n \\ a_2 & d_2 & 0 & \dots & 0 \\ a_3 & 0 & d_3 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ a_n & 0 & \dots & 0 & d_n \end{bmatrix}$$ where $a_i = w(1i) \neq 0$ for $i \in \{2, \dots, n\}$. Since $\mathrm{mr}(S_n) = 2$, Observation 5.9 implies $\mathrm{mr}(G_w) \geq 2$. Let $A \in \mathcal{S}(G_w)$ with $d_i = 0$ for all $i \in \{2, \dots, n\}$. Then $\operatorname{rank} A = 2$ and so $\mathrm{mr}(G_w) = 2$. □

Lemma 5.14. *Let G_w be a weighted graph with $G = C_n$. Then* $\mathrm{mr}(G_w) = n - 2$.

Proof. We proceed by induction on the number of vertices in the cycle. By Proposition 5.12 the base case $n = 3$ is true. Assume that if G_w is a weighted graph with $G = C_{n-1}$, we have $\mathrm{mr}(G_w) = n - 3$. Let G_w be a weighted graph with $G = C_n$. Then $A \in \mathcal{S}(G_w)$ has the form

$$\begin{bmatrix} d_1 & a_1 & 0 & 0 & \dots & 0 & a_n \\ a_1 & d_2 & a_2 & 0 & \dots & 0 & 0 \\ 0 & a_2 & d_3 & a_3 & \ddots & \vdots & \vdots \\ 0 & 0 & a_3 & d_4 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & \ddots & d_{n-1} & a_{n-1} \\ a_n & 0 & \dots & 0 & 0 & a_{n-1} & d_n \end{bmatrix}.$$

Let $B = \begin{bmatrix} q_1 & a_1 & 0 & 0 & \dots & 0 & a_n \\ a_1 & q_2 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & 0 & \ddots & \vdots & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & \ddots & 0 & 0 \\ a_n & 1 & 0 & \dots & 0 & 0 & q_n \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & r_2 & a_2 & 0 & \dots & 0 & -1 \\ 0 & a_2 & r_3 & a_3 & \ddots & \vdots & 0 \\ 0 & 0 & a_3 & r_4 & \ddots & 0 & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & \ddots & r_{n-1} & a_{n-1} \\ 0 & -1 & 0 & \dots & 0 & a_{n-1} & r_n \end{bmatrix}$.

By Proposition 5.12 there exist diagonal entries q_1, q_2, and q_n such that $\operatorname{rank} B = 1$. By the inductive hypothesis there exist diagonal entries $r_2, r_3, \ldots, r_n$ such that $\operatorname{rank} C = n-3$. By Fact 2, $\operatorname{rank}(B+C) \leq \operatorname{rank} B + \operatorname{rank} C = 1+n-3 = n-2$. Let $A = B+C$. Note that $A \in \mathcal{S}(G_w)$. Thus $\operatorname{mr}(G_w) \leq n-2$. Since $\operatorname{mr}(C_n) = n-2$, Observation 5.9 implies that $\operatorname{mr}(G_w) \geq n-2$. Therefore $\operatorname{mr}(G_w) = n-2$. □

Lemma 5.15. *Let G_w be a weighted graph with G a double cycle on n vertices. Then* $\operatorname{mr}(G_w) = n-2$.

Proof. Let G_w be a weighted graph with G a double cycle on n vertices. Let C_r and C_s be the two induced cycles of G and let e be the common edge. Let $a = w(e)$. By Lemma 5.14, there exist matrices B and C corresponding to C_r and C_s such that $\operatorname{rank} B = r-2$ and $\operatorname{rank} C = s-2$ and the off-diagonal entries correspond to the weights given by w except that the weight for the common edge e is $a/2$. Appropriately embedding B and C so as to match the labeling and size of G, their sum A is in $\mathcal{S}(G_w)$. Further by Fact 2, $\operatorname{rank} A \leq r-2+s-2 = n-2$. Since a double cycle has maximum nullity 2, Observation 5.9 implies that $\operatorname{mr}(G_w) \geq n-2$. Therefore $\operatorname{mr}(G_w) = n-2$. □

Theorem 5.16. *Let G_w be a weighted graph with G outerplanar. Then* $\operatorname{mr}(G_w) = \operatorname{mr}(G)$.

Proof. Let G_w be a weighted graph with G outerplanar. By Theorem 5.8 there exists an edge-disjoint cover $\mathcal{C}$ of G consisting of cliques, stars, cycles and double cycles such that $\operatorname{rs}(\mathcal{C}) = \operatorname{mr}(G)$. Using Propositions 5.11, 5.12, 5.13 and Lemmas 5.14, 5.15, for each graph in the cover there exists a minimum rank matrix which has the appropriate off-diagonal entries given by w. Appropriately embedding each matrix to match the size and labeling of G, and noting that the cover is edge-disjoint, their sum A is in $\mathcal{S}(G_w)$. Since each matrix in the sum is a minimum rank matrix, using Fact 2 we have $\operatorname{mr}(G_w) \leq \operatorname{rank} A \leq \operatorname{rs}(\mathcal{C}) = \operatorname{mr}(G)$. By Observation 5.9, $\operatorname{mr}(G) \leq \operatorname{mr}(G_w)$. Therefore $\operatorname{mr}(G_w) = \operatorname{mr}(G)$. □

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