2014-12-01

Pro-Covering Fibrations of the Hawaiian Earring

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A thesis submitted to the faculty of
Brigham Young University
in partial fulfillment of the requirements for the degree of

Master of Science

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December 2014

ABSTRACT

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Let $\mathbf{H}$ be the Hawaiian Earring, and let $\mathcal{H}$ denote its fundamental group. Assume (B_i) is an inverse system of bouquets of circles whose inverse limit is $\mathbf{H}$. We give an explicit bijection between finite normal covering spaces of $\mathbf{H}$ and finite normal covering spaces of B_i. This bijection induces a correspondence between a certain family of inverse sequences of these covering spaces. The correspondence preserves the inverse limit of these sequences, thus offering two methods of constructing the same limit. Finally, we characterize all spaces that can be obtained in this fashion as a particular type of fibrations of $\mathbf{H}$.

Keywords: fibration, Hawaiian Earring, pro-cover, covering space, inverse limit, bouquet of circles

Acknowledgments

Completing this thesis would have been impossible without the help and support of so many in my life. First and foremost, I am grateful to my God for my understanding and any talent I possess, as well as the constant inspiration and sustaining influence of His Spirit.

I am also particularly grateful to my advisor, Greg Conner, and the many professors from whom I have learned so much. I am especially indebted to Dr. Conner and Petar Pavešić for the countless hours they have spent with me in counsel. I also acknowledge Lawrence Fearnley, David Wright and Denise Daniels for their roles in introducing me to the beauty of mathematics.

My family, friends and students were also indispensable. I will forever be indebted to my parents for the time, love and money they put into raising me and supporting me in all my ambitions. I am especially grateful for their prayers on my behalf. My siblings and extended family were not a whit behind them in their support of me as well.

I also thank Andrew Logan for his dutiful proof-reading, cheerful encouragement and inimitable friendship. I am also grateful to those who helped me by taking up my other responsibilities so that I could spend more time on my thesis. Of special note are Dominique Fitch and Krisnet Hernandez, who were always there for me whether I needed help to focus or a distraction; my flatmates Ryan Hintze, Daniel Anderson and David Seletos, who handled cleaning checks and were great morale boosters; and my officemates Michael Andersen, Tyler Hills and Ben Schoonmaker, who substituted classes for me, helped grade exams and served as sounding boards for my ideas.

I truly cannot express the love and gratitude I feel for these individuals and the many others who have not been named. Thank you.

Contents

List of Figures

Chapter 1. Introduction

1.1 Introduction

Cannon and Conner, in [1], [2] and [3], define a generalization of the fundamental group and show that in this new theory the Hawaiian Earring serves an analagous role to that of bouquets of circles for regular fundamental groups. This inspired the question of whether the Hawaiian Earring serves as a generalization of bouquets of circles in other settings as well. For instance, there is a bijection between covering spaces of bouquets of circles and subgroups of the fundamental group of bouquets of circles. Does a similar bijection exist between fibrations of the Hawaiian Earring and subgroups of the fundamental group of the Hawaiian Earring.

Our primary result is that a particular family of sequences of covering spaces of the Hawaiian Earring yields fibrations with several of the usual properties one may desire in a fibration, such as compactness and unique path lifting. Furthermore, any fibration with these properties is obtainable as the inverse limit of some member of this family.

We also show that a bijection exists between the same family of sequences and a certain family of subgroups of the Hawaiian Earring Group. Thus we obtain also a relation between fibrations of the Hawaiian Earring and this family of subgroups.

There is in fact a canonical bijection, in the sense that inverse limits are preserved, between this family of sequences of covering spaces of the Hawaiian Earring and a similar family of sequences of covering spaces of bouquets of circles. The method of proof is constructive, so that we obtain an algorithm for converting between these spaces.

We suspect that these results extend to settings over spaces other than the Hawaiian Earring, specifically compact Peano Continua. The construction used for the general case would almost certainly require the Axiom of Choice, though, so the Hawaiian Earring serves as an important transition point between the intuition of bouquets of circles and the abstract reasoning of general spaces.

1.2 Outline

In Chapter 2 , we will give an overview of the concepts we wish to generalize as well as some standard topological methods which will be of use to us. These facts and methods are quite standard and so we will generally refer the reader to certain introductory topology texts for proofs and examples.

In Chapter 3, we will define finite normal covering sequences and finite approximate covering sequences, which are the families of sequences involved in our main theorem. We will then state and prove the first half of the main theorem: these sequences yield fibrations with certain desirable properties.

In Chapter 4, we will provide the algorithm for converting between these families of sequences in the case that our base space is the Hawaiian Earring.

In Chapter 5, we prove the second half of the main theorem: any fibration of the Hawaiian Earring with certain intrinsic properties is obtainable as the inverse limit of a finite approximate covering sequence, or equivalently a finite normal covering sequence, of the Hawaiian Earring.

Chapter 2. Background Information

We assume the reader has a basic knowledge of introductory topology. For more intermediate level topics, however, we will endeavor to provide the pertinent definitions and results before venturing into new ideas. To not detract from the current research, we will usually express old definitions and results in a way that gives us the most utility later on, rather than the most generality. Each section will include references for the reader to consult if desired.

When we refer to a *space* X we will mean that X is a compact metric space, unless specifically stated otherwise. Given a point $x \in X$, a *neighborhood* of x will mean an open subset of X containing the point x. Whenever we write $f : X \longrightarrow Y$, we mean that f is a continuous function from X to Y if X and Y are topological spaces or a homomorphism if X and Y are algebraic groups. We may also say that f is a *map* from X to Y. Similarly, $f : (X, x) \longrightarrow (Y, y)$ means that f is a map from X to Y and $f(x) = y$.

2.1 Fundamental Group

We will begin with the same concept that motivated this research, namely the fundamental group. We will refer the reader to [4, Chapter 1.1] and [5, Sections 51 and 52] for a more detailed discussion on the fundamental group and its properties in general.

For the following section, X and Y are metric spaces, though not necessarily compact. I is the standard unit interval.

Definition 2.1. Suppose $\alpha : I \longrightarrow X$ such that $\alpha(0) = x_0$ and $\alpha(1) = x_1$. We say that α is a *path* in X from x_0 to x_1. Furthermore, if $x_0 = x_1$, we say α is a *loop* in X based at x_0.

Definition 2.2. If for every pair of points $x, y \in X$ there exists a path in X from x_0 to x_1, we say that X is *path-connected*.

Definition 2.3. If for every $x \in X$ and every neighborhood U of x there exists a path-connected neighborhood of x contained in U, we say X is *locally path-connected*.

Definition 2.4. If α and α' are two paths in X from x_0 to x_1, we say α and α' are *path homotopic* if there is exists $F : I \times I \longrightarrow X$ such that

$$F(s,0) = \alpha(s) \qquad \text{and} \qquad F(s,1) = \alpha'(s),$$
$$F(0,t) = x_0 \qquad \text{and} \qquad F(1,t) = x_1,$$

for each $s \in I$ and each $t \in I$. F is called a *path homotopy* between α and α' and we write $\alpha \simeq \alpha'$.

Lemma 2.5. *The relation $\simeq$ is an equivalence relation.*

Definition 2.6. If α is a path, we denote its $\simeq$ *equivalence class* by $[\alpha]$.

Definition 2.7. If α is a path in X from x_0 to x_1 and β is a path in X from x_1 to x_2, we define $\alpha * \beta$ to be the path γ from x_0 to x_2 given by

$$\gamma(t) = \begin{cases} \alpha(2t) & \text{for } t \in [0, \frac{1}{2}], \\ \beta(2t-1) & \text{for } t \in [\frac{1}{2}, 1]. \end{cases}$$

Lemma 2.8. *The operation $[\alpha] * [\beta] = [\alpha * \beta]$ is a well-defined operation on path-homotopy classes.*

Theorem 2.9. *Suppose $x_0 \in X$. The set of homotopy classes of loops based at x_0 with the product $*$ forms a group.*

Definition 2.10. The group in Theorem 2.9 is called the *fundamental group* of X relative to the base point x_0. We will denote this by $\pi_1(X, x_0)$.

Theorem 2.11. *Given $f : I \longrightarrow X$, we define $\hat{f} : \pi_1(X, f(0)) \longrightarrow \pi_1(X, f(1))$ by*

$$\hat{f}([\alpha]) = [\bar{f}] * [\alpha] * [f],$$

where $\bar{f}(t) = f(1-t)$. Furthermore, $\hat{f}$ is an isomorphism.

Corollary 2.12. *If X is path-connected, then $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X, x_1)$ for all choices of $x_0, x_1 \in X$.*

Lemma 2.13. *Given $h : (X, x_0) \longrightarrow (Y, y_0)$, the map $[\alpha] \mapsto [h \circ \alpha]$ is a homomorphism from $\pi_1(X, x_0)$ to $\pi_1(Y, y_0)$.*

Definition 2.14. The map defined in Lemma 2.13 is call the *induced homomorphism* of h. We denote it by h_*.

Definition 2.15. If X is path-connected and $\pi_1(X, x_0)$ is the trivial group for some $x_0 \in X$, we say that X is *simply-connected*.

Theorem 2.16. *Given $h : (X, x_0) \longrightarrow (Y, y_0)$ and $k : (Y, y_0) \longrightarrow (Z, z_0)$, then*

$$(k \circ h)_* = k_* \circ h_*.$$

Furthermore, if h is a homeomorphism, then h_* is an isomorphism.

Corollary 2.17. *Given $A \subset X$, let $\imath : A \longrightarrow X$ denote the inclusion map. Suppose there exists $r : X \longrightarrow A$ so that $r \circ \imath$ is the identity, then*

- *r_* is surjective and*
- *$\imath_*$ is injective.*

2.2 Inverse Limit

We next discuss the concept of an inverse system, which is the tool that enables us to construct the fibrations we desire. Though this concept may be defined quite generally, we will discuss here only those types of systems which will be pertinent in the following discussion. For a more complete discussion of the topic, including proofs of the theorems below, the reader is referred to [6, Appendix 2, Section 2].

Definition 2.18. Let (X_i) be a sequence of spaces. Suppose that for each $i \geq 1$ there exists a map $f_i : X_{i+1} \longrightarrow X_i$. Then the sequence of pairs (X_i, f_i) is called an *inverse sequence*.

Whenever we have an inverse sequence (X_i, f_i) we define the following maps for $i > j$.

$$f_{i,i} : X_i \longrightarrow X_i, \quad \text{given by} \quad f_{i,i}(x) = x;$$
$$f_{i,j} : X_i \longrightarrow X_j, \quad \text{given by} \quad f_{i,j} = f_j \circ f_{j+1} \circ \cdots \circ f_{i-1}.$$

Thus our simplified definition is merely a restriction of [6, Definition 2.1 on p. 427].

Definition 2.19. Let (X_i, f_i) be an inverse sequence. Let

$$\rho_i : \prod_{i=1}^{\infty} X_i \longrightarrow X_i$$

be the natural projection onto the ith factor. The *inverse limit* of (X_i, f_i) is

$$X = \{x \in \prod_{i=1}^{\infty} X_i \big| \rho_i(x) = f_i \circ \rho_{i+1}(x)\}.$$

We denote this as $(X, (F_i)) = \lim\limits_{\leftarrow f_i} X_i$, where $F_i = \rho_i\big|_X$.

$$\begin{tikzcd}
\lim\limits_{\leftarrow f_i} X_i \arrow[dr, "F_{i+1}"'] \arrow[drr, "F_i"] & & \\
& X_{i+1} \arrow[r, "f_i"] & X_i
\end{tikzcd}$$

Figure 2.1: Representing $\lim\limits_{\leftarrow f_i} X_i$

The points of $\lim\limits_{\leftarrow f_i} X_i$ are 'coherent' sequences of points, in the sense that $F_i = f_i \circ F_{i+1}$. In other words, Figure 2.1 is a commutative diagram. As with normal limits, inverse limits do not depend on the first terms of the inverse sequence. In fact, we will show that, up to homeomorphism, inverse limits of inverse sequences may be determined by any infinite subsequence.

Definition 2.20. Suppose (X_i, f_i) and (Y_j, g_j) are inverse sequences. We say that (X_i, f_i) is a *cofinal subsequence* of (Y_j, g_j) if for each i there exists an integer m_i such that

$$m_{i+1} > m_i, \quad X_i = Y_{m_i}, \quad f_i = g_{m_{i+1}, m_i}.$$

Clearly $m_1 \geq 1$ and thus $m_i > i$ for all i. Therefore $\lim_{i \to \infty} m_i = \infty$, so this agrees with [6, p. 424]. Therefore we obtain the following result relating different inverse subsequences.

Theorem 2.21. *Suppose (X_i, f_i) and (Y_i, g_i) are inverse subsequences. If either of the following is true,*

(a) (X_i, f_i) is a cofinal subsequence of (Y_j, g_j), or

(b) There exist homeomorphisms $h_i : X_i \longrightarrow Y_i$, for all i, such that $h_i \circ f_i = g_i \circ h_{i+1}$, for all i,

then $\varprojlim_{f_i} X_i \cong \varprojlim_{g_i} Y_i$.

Thus we will define an equivalence relation on inverse sequences as follows.

Definition 2.22. $(X_i, f_i) \sim (Y_i, g_i)$ if there are cofinal subsequences (X_{i_k}, f_{i_k}) and (Y_{j_k}, g_{j_k}) that satisfy condition (b) of Theorem 2.21.

Corollary 2.23. *If $(X_i, f_i) \sim (Y_i, g_i)$, then $\varprojlim_{f_i} X_i \cong \varprojlim_{g_i} Y_i$.*

Now that we have some tools to help compute inverse limits, we will present some of the properties that will be of use to us.

Theorem 2.24. *Suppose (X_i, f_i) is an inverse sequence, where each X_i is compact and nonempty. Then $\varprojlim_{f_i} X_i$ is compact and nonempty.*

Theorem 2.25. *Suppose there exist maps $g_i : Y \longrightarrow X_i$ so that $g_i = f_i \circ g_{i+1}$ for all i. Then there exists a unique map $G : Y \longrightarrow \varprojlim_{f_i} X_i$ such that*

$$F_i \circ G = g_i \textit{ for all } i,$$

where $F_i : \varprojlim_{f_i} X_i \longrightarrow X_i$ is the standard projection map.

In other words, there is a map G so that Figure 2.2 commutes for all i.

$$\begin{tikzcd}
\varprojlim_{f_i} X_i \arrow[dr, "F_{i+1}"] \arrow[drr, "F_i"] \arrow[dd, dashed, "G"'] & & \\
& X_{i+1} \arrow[r, "f_i"] & X_i \\
Y \arrow[ur, "g_{i+1}"] \arrow[urr, "g_i"'] & &
\end{tikzcd}$$

Figure 2.2: The universal mapping property of inverse limits

2.3 Covering Space

We will now define what a covering space is as well as some related properties which we will desire these spaces to have. For a complete treatment of this topic, the reader may wish to see [4, Chapter 1.3] or [5, Section 53].

Definition 2.26. Suppose that $p : \tilde{X} \longrightarrow X$ is a continuous, surjective map. We say that $U \subset X$ is *evenly covered by p* if $p^{-1}(U)$ is a disjoint union of sets in $\tilde{X}$, each of which is mapped homeomorphically onto U by p.

Figure 2.3 illustrates the idea of $p^{-1}(U)$ as the classic stack of disjoint copies of U.

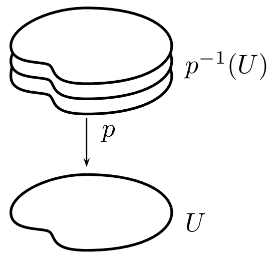


Figure 2.3: Evenly covered neighborhood

Definition 2.27. Suppose $p : \tilde{X} \longrightarrow X$ is a continuous, surjective map. If there exists an open cover $\{U_\alpha\}$ of X such that each U_α is evenly covered by p for all α, we say that p is a *covering map of* X. If we wish to emphasize $\tilde{X}$, we say that $(\tilde{X}, p)$ is a *covering space of* X.

Though not at all obvious from the definition, the primary source of interest in covering spaces is that they interact very nicely with respect to 'lifting' maps, as outlined in the following definitions and propositions.

Definition 2.28. Suppose $p : \tilde{X} \longrightarrow X$ and $f : Y \longrightarrow X$. We say that a function $\tilde{f} : Y \longrightarrow \tilde{X}$ is a *lift* of f if $p \circ \tilde{f} = f$.

Definition 2.29. We say that $p : \tilde{X} \longrightarrow X$ has the *homotopy lifting property* if for every homotopy $f_t : Y \longrightarrow X$ and lift of f_0, $\tilde{f}_0 : Y \longrightarrow \tilde{X}$, there exists a homotopy $\tilde{f}_t : Y \longrightarrow \tilde{X}$ of $\tilde{f}_0$ that lifts f_t. If $\tilde{f}_t$ is unique, we say p has the *unique homotopy lifting property*.

See Figure 2.4 for a visualization of the homotopy lifting property.

Definition 2.30. We say $p : \tilde{X} \longrightarrow X$ has the *(unique) path lifting property* if for every path $\alpha : I \longrightarrow X$ and lift $\tilde{x}_0$ of $\alpha(0)$, there exists a (unique) path $\tilde{\alpha} : I \longrightarrow \tilde{X}$ that is a lift of α and $\tilde{\alpha}(0) = \tilde{x}_0$.

Theorem 2.31. *Every covering space $(\tilde{X}, p)$ of a space X has the unique homotopy lifting property and p_* is injective.*

Of course, as one always does in mathematics after defining an object, we now define when two such objects should be considered equivalent. Thus we define an isomorphism

$$
\begin{tikzcd}
Y \arrow[r, "\tilde{f}_0"] \arrow[d, "y \times 0"'] & \tilde{X} \arrow[d, "p"] \\
Y \times I \arrow[r, "f_t"'] \arrow[ur, dashed, "\tilde{f}_t"] & X
\end{tikzcd}
$$

Figure 2.4: Homotopy lifting property

between covering spaces and state a basic result that demonstrates why this equivalence is something of interest.

Definition 2.32. Suppose (X_1, p_1) and (X_2, p_2) are covering spaces of X. We say that these covering spaces are *isomorphic* if there exists a homeomorphism $f : X_1 \longrightarrow X_2$ so that $p_1 = p_2 \circ f$.

Theorem 2.33. *Let X be a path-connected, locally path-connected space and let $x_0 \in X$. Suppose (X_1, p_1) and (X_2, p_2) are covering spaces of X. These covering spaces are isomorphic for some homeomorphism $f : X_1 \longrightarrow X_2$ if and only if*

$$p_{1*}(\pi_1(X_1, \tilde{x}_1)) = p_{2*}(\pi_1(X_2, \tilde{x}_2)), \quad \textit{for some } \tilde{x}_1 \in p_1^{-1}(x_0), \tilde{x}_2 \in p_2^{-1}(x_0).$$

Thus we see the beginnings of a correspondence between subgroups of $\pi_1(X, x_0)$ and covering spaces, up to isomorphism. Though our current interests will lead us in a different direction, we state a particular instance of this correspondence below.

Theorem 2.34. *Let X be a path-connected, locally path-connected, and semilocally simply-connected space. There is a bijection between isomorphism classes of path-connected covering spaces of X and conjugacy classes of subgroups of $\pi_1(X, x_0)$, for any choice of $x_0 \in X$.*

In fact, the main theorem of this paper may be thought of as a generalization of this fact in many ways. On one hand, we will work with a generalized version of path-connected covering spaces, but we will also be dealing with a space that is not semilocally simply-connected.

We close this section with a few results on restrictions of covering maps.

Lemma 2.35. *Suppose $(\tilde{X}, p)$ is a covering space of X. If U is evenly covered by p, then every open subset of U is also evenly covered by p.*

Lemma 2.36. *Suppose $(\tilde{X}, p)$ is a covering space of X. For a subspace $A \subset X$, let $\tilde{A} = p^{-1}(A)$. Then the restriction $p\big|_{\tilde{A}} : \tilde{A} \longrightarrow A$ is a covering space of A.*

2.4 Finite Covering Space

Having defined general covering spaces, we must now define the class of covering spaces that will be used in the constructions described in Chapters 3 and 4. As with the previous section on covering spaces, we refer the reader to [4, Chapter 1] for additional understanding, though we will state the relevant facts here.

Lemma 2.37. *Suppose $(\tilde{X}, p)$ is a covering space of X. If X and $\tilde{X}$ are path-connected, then the cardinality of $p^{-1}(x)$ is constant for all $x \in X$ and is equal to the index of $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ in $\pi_1(X, x_0)$, for any $x_0 \in X$ and $\tilde{x}_0 \in p^{-1}(x_0)$.*

Definition 2.38. Suppose that $(\tilde{X}, p)$ is a covering space of X. Further suppose that X and $\tilde{X}$ are path-connected. Then we say that $(\tilde{X}, p)$ is a *finite covering space* of X if $|p^{-1}(x)|$ is finite for all x.

Clearly if $(\tilde{X}, p)$ is a finite covering space of X and both X and $\tilde{X}$ are path-connected, then for some integer k, $|p^{-1}(x)| = k$ for all $x \in X$. In this case, we say that $(\tilde{X}, p)$ is a *k-cover* of X.

Lemma 2.39. *Suppose that $(\tilde{X}, p)$ is a k-cover of X and $(\tilde{\tilde{X}}, \tilde{p})$ is an ℓ-cover of $\tilde{X}$. Then $(\tilde{\tilde{X}}, p \circ \tilde{p})$ is a $k\ell$-cover of X.*

Proof. Let $x \in X$. By definition, there exists $U \subset X$ such that $x \in U$ and U is evenly covered by p. Let $\{V_\alpha | \alpha \in A\}$, for some indexing set A, be the collection of disjoint open sets in the definition of evenly covered. Since $p\big|_{V_\alpha}$ is a homeomorphism for each $\alpha \in A$, then there is exactly one point of $p^{-1}(x)$ in each V_α. Thus $|A| = |p^{-1}(x)| = k$, so we may assume $A = \{1, 2, \ldots, k\}$.

Let $y_i \in p^{-1}(x) \cap V_i \subset \tilde{X}$. Then since $\tilde{p}$ is an ℓ-cover of $\tilde{X}$, we have an evenly covered neighborhood V_i' in $\tilde{X}$ such that $y_i \in V_i'$. By Lemma 2.35, $V_i' \cap V_i \subset V_i'$ is also evenly covered, so we may assume $V_i' \subset V_i$. Now let

$$U' = \bigcap_{y \in p^{-1}(x)} p(V_i').$$

Since $V_i' \subset V_i$ and p is a homeomorphism on V_i, we see that $p(V_i')$ is an open neighborhood of x in X, so U' is also an open neighborhood of x. Since $(p \circ \tilde{p})^{-1} = \tilde{p}^{-1} \circ p^{-1}$, we see that $p^{-1}(U')$ consists of k disjoint open neighborhoods W_i such that $p\big|_{W_i}$ is a homeomorphism. But $W_i \subset V_i'$, so $\tilde{p}^{-1}(W_i)$ consists of ℓ disjoint open neighborhoods W_j' such that $\tilde{p}\big|_{W_j'}$ maps W_j' homeomorphically onto W_i.

Therefore $(p \circ \tilde{p})\big|_{W_j'}$ maps W_j' homeomorphically onto U', so U' is evenly covered by $p \circ \tilde{p}$. We also see that $(p \circ \tilde{p})^{-1}(U)'$ consists of kl disjoint copies of U', so $(\tilde{\tilde{X}}, p \circ \tilde{p})$ is a $k\ell$-cover of X. $\square$

Definition 2.40. Suppose $G \leq \pi_1(X, x_0)$. We say that G is a *coverable subgroup* of $\pi_1(X, x_0)$ if there exists a covering space (C, p) of X so that $p_*(\pi_1(C, c_0)) = G$ for some $c_0 \in p^{-1}(x_0)$.

Definition 2.41. Suppose $\mathcal{U}$ is an open covering of X. For $x_0 \in X$, we define $\pi_1(\mathcal{U}, x_0)$ to be the subgroup of $\pi_1(X, x_0)$ generated by elements of the form $[\bar{\omega} * \alpha * \omega]$, where $\mathrm{im}(\alpha)$ is contained in some element of $\mathcal{U}$, and ω is a path from x_0 to $\alpha(0)$.

Theorem 2.42. *Suppose $p : \tilde{X} \longrightarrow X$ has the homotopy lifting property and the unique path lifting property. Further suppose that X and $\tilde{X}$ are connected, locally path-connected spaces. Then (X, p) is a covering space of X if and only if there exists an open covering $\mathcal{U}$ of X and a point $\tilde{x}_0 \in \tilde{X}$ such that $\pi_1(\mathcal{U}, p(\tilde{x}_0)) \subset p_*(\pi_1(\tilde{X}, \tilde{x}_0))$.*

For a proof of Theorem 2.42 see [7, p. 81].

2.5 Automorphism Group

Because most results for covering spaces are 'up to isomorphism,' we wish to understand those isomorphisms which map from one space to itself, called automorphisms. The concept we will discuss is a generalization of the deck transformation group described in [4, Chapter 1.3].

Definition 2.43. Suppose $p : X \longrightarrow Y$. We define the *automorphism group* of p to be

$$\text{Aut}(p) = \{h : \tilde{X} \longrightarrow \tilde{X} | p \circ h = p, h \text{ is a homeomorphism}\}.$$

Proposition 2.44. Aut(p) *forms a group under composition of functions.*

Proof. Suppose $p : X \longrightarrow Y$ and $f, g \in \text{Aut}(p)$. Clearly $f \circ g : X \longrightarrow X$ is a homeomorphism. Thus $f \circ g \in \text{Aut}(p)$ since $p \circ (f \circ g) = (p \circ f) \circ g = p \circ g = p$.

Let $I : X \longrightarrow X$ be the identity map. Clearly I is a homeomorphism and $p \circ I = p$, so $I \in \text{Aut}(p)$. In fact, $f \circ I = I \circ f = f$, so I is the group identity.

Since f is a homeomorphism, f^{-1} is also a homeomorphism. Thus $f^{-1} \in \text{Aut}(p)$ since $p \circ f^{-1} = (p \circ f) \circ f^{-1} = p \circ I = p$. In fact, $f \circ f^{-1} = f^{-1} \circ f = I$, so f^{-1} is the inverse of f.

Composition of functions is associative, so Aut(p) is a group under composition. □

Definition 2.45. We will say that a map $p : X \longrightarrow Y$ is *normal* if for each $y \in Y$ and each pair of points $x, x' \in p^{-1}(y)$, there exists $f \in \text{Aut}(p)$ such that $f(x) = x'$.

Theorem 2.46. *Suppose $p : (\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$ is a covering map, $\tilde{X}$ is path-connected, and X is path-connected and locally path-connected. Let $H = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$. Then:*

- *p is normal if and only if H is a normal subgroup of $\pi_1(X, x_0)$,*
- Aut(p) *is isomorphic to $N(H)/H$, where $N(H)$ is the normalizer of H in $\pi_1(X, x_0)$.*

In particular, if p is normal, then Aut(p) *is isomorphic to the quotient $\pi_1(X, x_0)/H$.*

For the proof of this theorem, see [4, p. 71]. It should be noted that this proof does not require the full strength of the covering map hypothesis. In fact, it is sufficient to assume that p has the unique path lifting property, rather than that p be a covering map.

2.6 Fibration

We now define a fibration, which is exactly the generalization of a covering space we wanted in the last result of Section 2.5.

Definition 2.47. Suppose $p : \tilde{X} \longrightarrow X$. We say that $(\tilde{X}, p)$ is a *fibration* over X if it has the homotopy lifting property, as defined in Section 2.3. We say $(\tilde{X}, p)$ is a *unique path-lifting fibration* over X if it also satisfies the unique path lifting property.

Theorem 2.48. *Suppose $(\tilde{X}, p)$ is a unique path-lifting fibration over a path-connected space X. Then for all $x, y \in X$, $p^{-1}(x)$ is homeomorphic to $p^{-1}(y)$.*

Proof. Let $\alpha : I \longrightarrow X$ be a path from x to y. Then for each $\tilde{x} \in p^{-1}(x)$, there is a unique path $\tilde{\alpha}_{\tilde{x}} : I \longrightarrow \tilde{X}$ that lifts α and begins at $\tilde{x}$.

Since $p \circ \tilde{\alpha}_{\tilde{x}} = \alpha$, $\tilde{\alpha}_{\tilde{x}}(1) \in p^{-1}(y)$. Thus we have a continuous map $f : p^{-1}(x) \longrightarrow p^{-1}(y)$. Conversely, $\bar{\alpha}(t)$ goes from y to x, and so there is a unique path $\tilde{\bar{\alpha}}_{\tilde{x}} : I \longrightarrow \tilde{X}$ that lifts $\bar{\alpha}$ and begins at $\tilde{\alpha}_{\tilde{x}}(1)$. Thus we have a continuous map $g : p^{-1}(y) \longrightarrow p^{-1}(x)$.

$\bar{\alpha} * \alpha$ is homotopic to the constant path at x, so $\widetilde{(\bar{\alpha} * \alpha)}_{\tilde{x}}$ is also homotopic to a constant path at $\tilde{x}$. Thus $\tilde{\gamma}_{\tilde{x}}(1) = \tilde{x}$ so $g \circ f$ is the identity. Therefore f is a homeomorphism. □

Definition 2.49. The space $p^{-1}(x)$ in the previous theorem is called the *fibre* of p.

As a consequence of Theorem 2.31, every covering space is a fibration. Of course, not all fibrations will be covering spaces.

Theorem 2.50. *Suppose (Z, q) is a fibration over Y and (Y, p) is a fibration over X. Then $(Z, p \circ q)$ is a fibration over X.*

Proof. Suppose $f_t : W \longrightarrow X$ is a homotopy and $\tilde{\tilde{f}}_0 : W \longrightarrow Z$ is a lift of f_0 to Z. Note that $p(q \circ \tilde{\tilde{f}}_0) = f_0$, so $\tilde{f}_0 = q \circ \tilde{\tilde{f}}_0 : W \longrightarrow Y$ is a lift of f_0 to Y.

Since (Y, p) is a fibration over X, we have a homotopy $\tilde{f}_t : W \longrightarrow Y$ that lifts f_t. However, $q \circ \tilde{\tilde{f}}_0 = \tilde{f}_0$, so $\tilde{\tilde{f}}_0$ is a lift of $\tilde{f}_0$ to Z.

Since (Z, q) is a fibration over Y, we have a homotopy $\tilde{\tilde{f}}_t : W \longrightarrow Z$ that lifts $\tilde{f}_t$. Thus $\tilde{\tilde{f}}_t$ is a lift of f_t since $p \circ q \circ \tilde{\tilde{f}}_t = p \circ \tilde{f}_t = f_t$. Therefore $(Z, p \circ q)$ has the homotopy lifting property and hence is a fibration over X. □

The next theorem extends this result to infinitely many nested fibrations.

Theorem 2.51. *Suppose (X_i, f_i) is an inverse sequence such that $f_i : X_{i+1} \longrightarrow X_i$ is a fibration for all i and let $(X, F_i) = \lim\limits_{\leftarrow f_i} X_i$. Then for each i, $F_i : X \longrightarrow X_i$ is a fibration.*

Proof. Fix i and for each $j > i$ let $f_{i,j} = f_i \circ f_{i+1} \circ \cdots \circ f_{j-1}$. By the previous theorem, if $|j - i| = 1$, then $f_{i,j} = f_i \circ f_{i+1}$ is a fibration, since it is the composition of two fibrations. Clearly by induction $f_{i,j}$ is a fibration for all $j > i$.

Suppose $h_t : Y \longrightarrow X_i$ is a homotopy and $\tilde{h}_0 : Y \longrightarrow X$ is a lift of h_0. By definition, for $j > i$ $F_j \circ \tilde{h}_0 : Y \longrightarrow X_j$ will be a lift of h_0 to X_j. Now $(X_j, f_{i,j})$ is a fibration over X_i, so there exists a lift $(\tilde{h}_t)_j : Y \longrightarrow X_j$ of h_t to X_j.

Now we have maps from Y to each X_j for $j \geq i$, so by the universal mapping property of inverse limits, we have a map $\tilde{H}_t : Y \longrightarrow X$. Since $F_i \circ \tilde{H}_t = h_t$, we see that (X, F_i) is a fibration over X_i. □

Definition 2.52. Suppose $(\tilde{X}, p)$ is a fibration over X, with fibre F. We say this is a *minimal fibration* if there exists a dense path-connected subset of $\tilde{X}$.

2.7 Topological Groups

We will need relatively little about the concept of topological groups, but the reader is invited to see [8] for more details. We will include proofs of the necessary lemmas, however. This is in part because we will use a slightly different definition, but also because we require some additional results.

Definition 2.53. Suppose G is a group, together with a topology τ. We say that G is a *topological group* if $(x, y) \mapsto xy^{-1}$ from $G \times G$ to G is continuous.

Lemma 2.54. *Suppose G is a topological group. Then the following are all continuous:*

(a) $y \mapsto y^{-1}$;

(b) $(x, y) \mapsto xy$;

(c) $x \mapsto xy$ *for fixed* y;

(d) $y \mapsto xy$ *for fixed* x.

Proof. Let 1 denote the identity of G. Let $T : G \times G \longrightarrow G$ denote the map $(x, y) \mapsto xy^{-1}$.

(a) Since $\{1\} \times G$ is a closed subset of $G \times G$, then $f(y) = T(1, y) = y^{-1}$ is continuous.

(b) Since $P(x, y) = T(x, f(y)) = xy$ is the composition of continuous maps, $P(x, y)$ is continuous.

(c) For fixed y, $G \times \{y\}$ is a closed subset of $G \times G$, so $x \mapsto P(x, y) = xy$ is continuous.

(d) For fixed x, $\{x\} \times G$ is a closed subset of $G \times G$, so $y \mapsto P(x, y) = xy$ is continuous. □

Definition 2.55. Suppose G is a metric topological group. We say that G is *uniformly equibicontinuous* if for every $\epsilon > 0$, there exists $\delta > 0$ so that

$$d(gh, g'h') < \epsilon, \quad \text{whenever} \quad d(g, g') < \delta \text{ and } d(h, h') < \delta.$$

Lemma 2.56. *If G is a compact metric topological group, then G is uniformly equibicontinuous.*

Proof. Suppose that G is not uniformly equibicontinuous. Then for some $\epsilon > 0$, for all $n \in \mathbb{N}$, there exist $g_n, g'_n, h_n, h'_n \in G$ so that

$$d(g_n, g'_n) < \delta, \quad d(h_n, h'_n) < \delta, \quad d(g_n h_n, g'_n h'_n) \geq \epsilon.$$

Since G is a topological group, the maps $x \mapsto gx$ and $x \mapsto xg$ are continuous for each $g \in G$. Since G is compact, we may assume that each of these sequences converges by passing to a convergent subsequence. Let g, g', h and h' denote the limits of $(g_n), (g'_n), (h_n)$ and (h'_n), respectively. Furthermore, since $d(g_n, g'_n)$ and $d(h_n, h'_n)$ both go to 0, $g = g'$ and $h = h'$.

Since $x \mapsto gx$ is continuous for all $g \in G$, for sufficiently large n, $d(g_n h_n, g_n h)$ and $d(g_n h'_n, g_n h)$ are both less than $\frac{1}{4}\epsilon$. Therefore, for sufficiently large n, we have

$$d(g_n h_n, g_n h'_n) \leq d(g_n h_n, g_n h) + d(g_n h, g_n h'_n) < \frac{1}{2}\epsilon.$$

Similarly, $x \mapsto xh$ is continuous for all $h \in G$, so for sufficiently large n, $d(g_n h'_n, g h'_n)$ and $d(g h'_n, g'_n h'_n)$ are both less than $\frac{1}{4}\epsilon$. Therefore, for sufficiently large n, we have

$$d(g_n h'_n, g'_n h'_n) \leq d(g_n h'_n, g h'_n) + d(g h'_n, g'_n h'_n) < \frac{1}{2}\epsilon.$$

Therefore, for sufficiently large n, we have

$$d(g_n h_n, g'_n h'_n) \leq d(g_n h_n, g_n h'_n) + d(g_n h'_n, g'_n h'_n) < \epsilon.$$

This contradicts our assumption, though, so G is indeed uniformly equibicontinuous. $\square$

Definition 2.57. Let $p : E \longrightarrow X$ be a fibration. We say that a subgroup of $\mathrm{Aut}(p)$, T, *acts topologically* if T is a topological group, under the uniform topology, and $f \mapsto f(\tilde{x}_0)$ is a homeomorphism from T to $p^{-1}(\tilde{x}_0)$ for each $\tilde{x}_0 \in E$, where the topology of $p^{-1}(\tilde{x}_0)$ is the subspace topology induced from E.

2.8 Wedge Product

Definition 2.58. Let X and Y be topological spaces. For $x \in X$ and $y \in Y$, we define the (x, y) *wedge* of X and Y to be

$$X \;{}_x\!\vee_y Y = (X \sqcup Y)/(x = y).$$

Similarly, for $A = \{x_1, x_2, \ldots, x_k\} \subset X$ and $y \in Y$, we define the (A, y) *wedge* of X and Y to be

$$X\ {}_{A}\!\vee_{y} Y = \left(\cdots\left(\left(X\ {}_{x_1}\!\vee_{y} Y\right)\ {}_{x_2}\!\vee_{y} Y\right)\cdots\right)\ {}_{x_k}\!\vee_{y} Y.$$

Theorem 2.59. *Suppose $p : C \longrightarrow X$ is a finite covering space of a simplicial complex X. Let Y be a topological space. For $x_0 \in X$, let $A = p^{-1}(x_0) \subset C$. Then for any $y_0 \in Y$ there exists a unique covering map $\tilde{p} : C\ {}_{A}\!\vee_{y_0} Y \longrightarrow X\ {}_{x_0}\!\vee_{y_0} Y$ such that $\tilde{p}(c) = p(c)$ for all $c \in C$ and $\tilde{p}$ is the identity on each copy of Y.*

Proof. Let $U = X\ {}_{x_0}\!\vee_{y_0} Y$, $W = C\ {}_{A}\!\vee_{y_0} Y$. We will think of X and Y as being contained in U and C as being contained in W. Furthermore, we will label the copies of Y in W as $Y_1, \ldots, Y_k$.

Since p is a finite covering space, A is a discrete set of points in W and $|A| = k$. Thus $Y_1, \ldots, Y_k$ are pairwise disjoint closed sets in U. Let $\tilde{x}_i$ be the point in $Y_i \cap A$.

By construction of the wedge product, $Y_i \cap Y_j = \emptyset$ if $i \neq j$. Since A is a discrete set of points, for each $\tilde{x}_i$ there exists an open neighborhood U_i so that $U_i \cap U_j = \emptyset$ for $i \neq j$. Thus $(Y_1 \cup U_1), \ldots, (Y_k \cup U_k)$ is a collection of pairwise disjoint open subsets of W.

Let $I_i : Y_i \longrightarrow Y$ be the map that identifies Y_i with Y. Define $\tilde{p}$ by

$$\tilde{p}(x) = \begin{cases} p(x) & \text{if } x \in C, \\ I_i(x) & \text{if } x \in Y_i. \end{cases} \tag{2.1}$$

Note that if $\tilde{x} \in C \cap Y_i$ for some i, then $\tilde{x} \in A$, so $p(\tilde{x}) = x_0 = I_i(\tilde{x})$. Thus $\tilde{p}$ is well-defined. Also observe that $\tilde{p}(C) = p(C) = X$ and $\tilde{p}(Z) = \cup_I (I_i(Y_i)) = Y$.

Now consider $x \in U$ and $\tilde{p}^{-1}(x)$. We will consider three cases.

Case 1: If $x \in X - Y$, then there exists an open neighborhood in U that contains x and is contained in $X - Y$. Since p is a covering map, there exists an open neighborhood in X (but not necessarily in U) that contains x and is evenly covered by p. Let $N(x)$ be the intersection of these two neighborhoods. Note that $N(x)$ is an open neighborhood in U, $x \in N(x)$ and

$N(x) \subset X - Y$. Furthermore, $N(x)$ is contained in an evenly covered neighborhood, so $N(x)$ is evenly covered by p. Since $\operatorname{im} I_i = Y$ for all i, we see that $\tilde{p}^{-1}(N(x)) = p^{-1}(N(x))$, so $\tilde{p}$ also evenly covers $N(x)$.

Case 2: If $x \in Y - X$, then there exists an open neighborhood $M(x)$ of U that contains x and is contained in $Y - X$. Since $\operatorname{im} p = X$, we see that $\tilde{p}^{-1}(M(x)) = \cup_i I_i^{-1}(M(x))$. Since $I_i^{-1}(M(x)) \subset Y_i$ and I_i is the identity on Y_i, we see that $\cup_i I_i^{-1}(M(x))$ is the disjoint union of k open sets $I_i^{-1}(M(x))$ each of which is mapped identically by $\tilde{p}$ onto $M(x)$. Thus $\tilde{p}$ evenly covers $M(x)$.

Case 3: If $x \in Y \cap X$, then $x = x_0 = y_0$. Let $N(x)$ be an open neighborhood of x in X that is evenly covered by p and let $M(x)$ be an open neighborhood of x in Y. Since $X \cup Y = U$ and $X \cap Y = x$, $N(x) \cup M(x)$ is an open neighborhood of x in U. Furthermore, we see that

$$\tilde{p}^{-1}(N(x) \cup M(x)) = \tilde{p}^{-1}(N(x)) \cup \tilde{p}^{-1}(M(x)) = p^{-1}(N(x)) \cup_i I_i^{-1}(M(x)). \tag{2.2}$$

Since p evenly covers $N(x)$, $p^{-1}(N(x))$ is the union of k disjoint open sets in C, which we shall call $\tilde{N}_1(x), \ldots, \tilde{N}_k(x)$, each of which is mapped homeomorphically onto $N(x)$. Each $\tilde{x}_i$ is contained in exactly one of these sets, so we may assume $\tilde{x}_i \in \tilde{N}_i(x)$. Likewise, $\tilde{x}_i \in I_i^{-1}(M(x))$, which is an open set in Y_i.

Let $\tilde{L}(x) = \tilde{N}_i(x) \cup I_i^{-1}(x)$. Since $C \cup_i Y_i = W$ and the only common points of C and the Y_i are in A, we see that $\tilde{L}(x)$ is open in W. Furthermore, $\tilde{p}$ is clearly a homeomorphism on each $\tilde{L}(x)$ onto $N(x) \cup M(x)$, so $N(x) \cup M(x)$ is evenly covered by $\tilde{p}$.

Therefore $\tilde{p}$ is a covering map. In fact, we see that it is a k-covering map, where $k = |A|$, and hence finite. Furthermore, if $\tilde{q}$ is another covering map that agrees with p on C and is the identity on each Y_i, then by our definition of $\tilde{p}$, $\tilde{q} = \tilde{p}$, so this map is unique. $\square$

2.9 Hawaiian Earring

We begin our discussion of the Hawaiian Earring by defining it as a particular compact subset of the plane. The arguments in this paper will not rely on this embedding, but it will enable us to write several maps explicitly. In turn, this enable us to give explicit descriptions in later constructions. The description we give will differ slightly from the more standard one given in [1]. However, we will show that it is equivalent.

Definition 2.60. Fix $\chi \in (0,1)$. For each $n \in \mathbb{N}$, define A_n to be the following subset of the plane.

$$A_n = \{(x,y) \in \mathbb{R}^2 \big| (x-\chi^n)^2 + y^2 = \chi^{2n}\}.$$

Each A_n is clearly a circle in the plane of radius χ^n centered at $(\chi^n, 0)$. As a result, $A_i \cap A_j = \{(0,0)\}$ for all $i \neq j$. Since $(0,0)$ will usually be our basepoint for any discussion of $\mathbf{H}$, we will denote it by $\mathbf{0}$. We now define the Hawaiian Earring as follows.

Definition 2.61. The *Hawaiian Earring*, denoted $\mathbf{H}$, is defined to be

$$\mathbf{H} = \bigcup_n A_n.$$

We also define the *Nth inner Hawaiian Earring* to be

$$\mathbf{H}_N = \bigcup_{n>N} A_n.$$

Figure 2.5 depicts the Hawaiian Earring at a $\chi = \frac{2}{3}$ scale.

Theorem 2.62. *The Hawaiian Earring in Definition 2.61 is homeomorphic to the one defined in [1]:*

$$\mathbf{H} = \bigcup_n \left\{(x,y) \in \mathbb{R}^2 \middle| x^2 + \left(y + \frac{1}{n}\right)^2 = \frac{1}{n^2}\right\}.$$

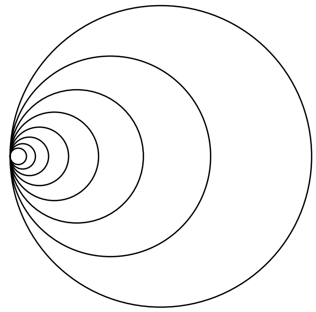


Figure 2.5: The Hawaiian Earring, $\mathbf{H}$

Proof. For each i, let $S_i = \{(x, y) \in \mathbb{R}^2 \big| x^2 + \left(y + \frac{1}{i}\right)^2 = \frac{1}{i^2}\}$. Define $f_i : A_i \longrightarrow \mathbb{R}^2$ by $f_i(x, y) = \left(\frac{1}{i\chi^i} y, \frac{-1}{i\chi^i} x\right)$. Now define $f : \mathbf{H} \longrightarrow \mathbb{R}^2$ by

$$f(u) = \begin{cases} f_1(u) & \text{if } u \in A_1, \\ f_2(u) & \text{if } u \in A_2, \\ \vdots & \quad \vdots \end{cases}$$

Note that $A_i \cap A_j = \{\mathbf{0}\}$ if $i \neq j$. Since each f_i is clearly continuous and well-defined on its domain, $\mathbf{0}$ is therefore the only point at which f may not be well-defined or discontinuous. However, $f_i(\mathbf{0}) = \mathbf{0}$ for all i, so f is indeed well-defined and continuous.

Furthermore, $f_i(A_i) \subset S_i$ for all i since

$$\left(\frac{1}{i\chi^i} y\right)^2 + \left(\frac{-1}{i\chi^i} x + \frac{1}{i}\right)^2 = \left(\frac{1}{i\chi^i}\right)^2 (y^2 + (x - \chi^i)^2) = \left(\frac{1}{i\chi^i}\right)^2 (\chi^{2i}) = \left(\frac{1}{i}\right)^2.$$

Therefore $f(\mathbf{H}) = \cup_i f_i(A_i) = \cup_i S_i$. Furthermore, if $u, v \in \mathbf{H}$, then for some i and j, $u \in A_i$ and $v \in A_j$. There are only two cases to consider.

Case 1: If $i = j$, then $f(u) = f_i(u)$ and $f(v) = f_i(v)$. Since f_i is injective on A_i, $f_i(u) \neq f_i(v)$, so $f(u) \neq f(v)$.

Case 2: If $i \neq j$, then $f(u) = f_i(u) \in S_i$ and $f(v) = f_j(v) \in S_j$. Thus $f(u) = f(v)$ implies $f_i(u) = f_j(v) \in S_i \cap S_j = \{\mathbf{0}\}$. Then $u \in f_i^{-1}(\mathbf{0}) = \{\mathbf{0}\}$ and $v \in f_j^{-1}(\mathbf{0}) = \{\mathbf{0}\}$. Therefore $f(u) = f(v)$ implies $u = v$.

In both cases, f is injective. It now remains to show that $f^{-1} : \cup_i S_i \longrightarrow \mathbf{H}$ is also continuous and well-defined. From the work above, we see that f^{-1} is given by

$$f^{-1}(u) = \begin{cases} f_1^{-1}(u) & \text{if } u \in S_1, \\ f_2^{-1}(u) & \text{if } u \in S_2, \\ \vdots & \vdots \end{cases}$$

Of course, $f_i^{-1}(x, y) = (-(i\chi^i)y, (i\chi^i)x)$, which is continuous. Thus the only point at which f^{-1} may not be well-defined or discontinuous is $\mathbf{0}$. However, it is clear that $f_i^{-1}(\mathbf{0}) = \mathbf{0}$ for all i, so f^{-1} is indeed well-defined and continuous. Thus f is a homeomorphism. $\square$

It will be useful to also present $\mathbf{H}$ as the inverse limit of finite unions of A_i. Thus we have the following definition and theorem.

Definition 2.63. For $i, j \in \mathbb{Z}$ and $0 \leq i < j$, define the following:

$${}_iB_j = \cup_{n=i+1}^{j} A_n, \quad \text{and} \quad B_j = {}_0B_j.$$

Also define $\theta_i : B_{i+1} \longrightarrow B_i$ by

$$\theta_i(u) = \begin{cases} u & \text{if } u \in A_k \text{ for some } k \leq i, \\ \mathbf{0} & \text{if } u \in A_k \text{ for some } k > i. \end{cases}$$

Theorem 2.64. *For all $i < j$, $\theta_{j,i}$ is a retraction. Furthermore, (B_i, θ_i) is an inverse sequence and $\varprojlim_{\theta_i} B_i \cong (\mathbf{H}, \Theta_i)$, where $\Theta_i : \mathbf{H} \longrightarrow B_i$ is the map*

$$\Theta_i(u) = \begin{cases} u & \textit{if } u \in A_k \textit{ for some } k \leq i, \\ \mathbf{0} & \textit{otherwise.} \end{cases}$$

Proof. First note that by definition $B_i \subset B_{i+1}$ for all i. Then clearly θ_i is the identity on $B_i = \cup_{k \leq i} A_k$. Since $A_k \cap A_\ell = \{\mathbf{0}\}$ for $k \neq \ell$, we see that θ_i is well-defined.

Since both $u \mapsto u$ and $u \mapsto \mathbf{0}$ are continuous, θ_i is continuous. Thus θ_i is a retraction.

Therefore (B_i, θ_i) is clearly an inverse sequence. Let $(X, F_i) = \varprojlim_{\theta_i} B_i$. We then define a map $f : X \longrightarrow \mathbf{H}$ as follows. If $(x_i) = (\mathbf{0})$ let $f((x_i)) = \mathbf{0}$, otherwise $f((x_i)) = x_k$, where k is the smallest positive integer for which $x_k \neq \mathbf{0}$. Now we verify that f is a homeomorphism.

- Since any subset of the positive integers has a least element, f is well-defined.

- Suppose that $f((x_i)) = f((y_i))$. If $f((x_i)) = f((y_i)) = \mathbf{0}$, then by definition $x_i = \mathbf{0}$ and $y_i = \mathbf{0}$ for all i and thus $(x_i) = (y_i)$. Thus we may assume $f((x_i)) \neq \mathbf{0}$, so for some positive integers k and ℓ, we have $x_k = y_\ell \neq \mathbf{0}$ and $x_i = y_j = \mathbf{0}$ for all $i < k$ and $j < \ell$. However, this means that $\theta_{k-1}(x_k) = \mathbf{0}$, so $x_k \in A_k$. Likewise $y_\ell \in A_\ell$.

  Since $A_k \cap A_\ell = \{\mathbf{0}\}$ for $k \neq \ell$, we must have $k = \ell$. Furthermore, if $u \in B_i - \{\mathbf{0}\}$, then $\theta_i^{-1}(u) = \{u\}$. Thus $x_{k+1} \in \{x_k\}$ and $y_{k+1} \in \{y_k\}$, so $x_{k+1} = x_k = y_k = y_{k+1}$. If $x_i = y_i \neq \mathbf{0}$, then $x_{i+1} \in \{x_i\}$ and $y_{i+1} \in \{y_i\}$, so $x_{i+1} = x_i = y_i = y_{i+1}$. Thus by induction $x_i = y_i$ for all $i \geq k$. By choice of k we have $x_i = y_i = \mathbf{0}$ for all $i < k$, so in fact $(x_i) = (y_i)$. Therefore f is injective.

- Suppose $x \in \mathbf{H}$. If $x = \mathbf{0}$, then $f(\mathbf{0}) = x$, so we may assume that $x \neq \mathbf{0}$. Since $x \neq \mathbf{0}$, there is a smallest positive integer k such that $x \in B_k$. For $i \geq k$, let $x_i = x_k$ and for $i < k$, let $x_i = \mathbf{0}$. Clearly $\theta_i(x_{i+1}) = x_i$, so $(x_i) \in X$ and by design $f((x_i)) = x_k$. Therefore f is surjective.

- Since B_i is compact for all i, by Lemma 2.24, X is compact. $\mathbf{H}$ is a subset of the plane, so it is Hausdorff. Therefore f is a continuous bijection from a compact space to a Hausdorff space, which implies that f is a homeomorphism.

Finally, note that $\Theta_i \circ F = F_i$, so $(\mathbf{H}, \Theta_i) \cong (X, F_i)$. $\square$

Thus we have three different ways in which to view the Hawaiian Earring: two different compact subsets of the plane or as an inverse limits of bouquets of circles. We will usually maintain the inverse limit perspective except when giving explicit functions, in which case we will use the subset perspective with which we began this section.

Now that we have some understanding of what the Hawaiian Earring is, we wish to state, and prove, several elementary facts about it.

Theorem 2.65. *For all $N > 0$, there is a homeomorphism $h_N : \mathbf{H} \longrightarrow \mathbf{H}_N$.*

Proof. First, define $f_N : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ by $f_N(x, y) = \left(\chi^{N+1}x, \chi^{N+1}y\right)$, where $\chi = \frac{2}{3}$. This is clearly a homeomorphism of the plane since it is merely multiplication by a constant.

We verify, however, that $f_N(A_m) = A_{m+N+1}$ for all $m \geq 1$. Suppose that $(x, y) \in A_m$, then

$$\begin{aligned}
\left(\left(\chi^{N+1}x\right) - \chi^{m+N+1}\right)^2 + \left(\chi^{N+1}y\right)^2 &= \chi^{2N+2}\left((x - \chi^m)^2 + y^2\right) \\
&= \chi^{2N+2}\left(\chi^{2m}\right) \\
&= \left(\chi^{m+N+1}\right)^2
\end{aligned}$$

Thus $f_N(\mathbf{H}) = \mathbf{H}_N$, so $h_N = f_N\big|_{\mathbf{H}}$ is the desired homeomorphism. □

This theorem justifies our decision to call $\mathbf{H}_N$ the Nth inner Hawaiian Earring, since it is in fact a Hawaiian Earring. The next theorem highlights the importance of $\mathbf{H}_N$ in understanding covering spaces of $\mathbf{H}$.

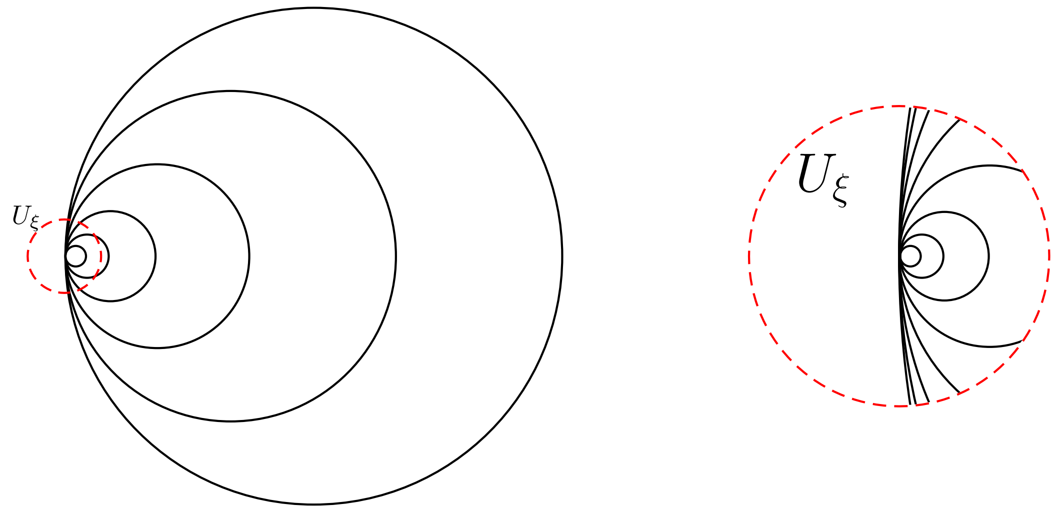
Theorem 2.66. *Suppose (C, p) is a connected, finite covering space of $\mathbf{H}$. Then for some $N > 0$, $\mathbf{H}_N$ is evenly covered by p.*

Proof. For $\varepsilon > 0$, $U_\varepsilon = \{(x, y) \in \mathbf{H} : x^2 + y^2 < \varepsilon\}$ is open in $\mathbf{H}$ and $\mathbf{0} \in U_\varepsilon$. Since p is a covering map, U_ξ is evenly covered by p for some $\xi > 0$. We may assume that $\xi \leq 2$ since otherwise $U_\xi = \mathbf{H}$ so p is a homeomorphism and the result follows immediately.

Since $0 < \xi \leq 2$, $1 \leq \frac{\ln 3 - \ln(\xi)}{\ln(3) - \ln(2)}$, so there exists an integer $N > 1$ such that $\frac{\ln 3 - \ln(\xi)}{\ln(3) - \ln(2)} < N$. For all $k > N$, if $x \in {}_N B_k$, there exists an integer m, with $N + 1 \leq m \leq k$ and $x \in A_m$, so

$$d(\mathbf{0}, x) \leq d(\mathbf{0}, (\chi^m, 0)) + d((\chi^m, 0), x) = \chi^m + \chi^m = 2\chi^m \leq 2\chi^{N+1} < \xi.$$

Thus for all $k > N$, ${}_N B_k \subset U_\xi$. Thus $\mathbf{H}_N = \cup_{k>N} {}_N B_k \subset U_\xi$. Figure 2.6 illustrates a particular open set U_ξ and shows a close-up view of the set.



(a) $\mathbf{H}$, normal (b) U_ξ, 4x zoom

Figure 2.6: Evenly covered neighborhood of the Hawaiian Earring

Now $p^{-1}(\mathbf{H}_N)$ is contained in the $p^{-1}(U_\xi)$, which is the disjoint union of sets $W_1, \ldots, W_k$, each of which is homeomorphic to U_ξ. Let $V_i = p^{-1}(\mathbf{H}_N) \cap W_i$. Since $W_1, \ldots, W_k$ are pairwise disjoint, $V_1, \ldots, V_k$ must also be pairwise disjoint.

Furthermore, $V_i \subset W_i$ and $p\big|_{W_i}$ is a homeomorphism, so $p\big|_{V_i}$ is also a homeomorphism, for each i. Thus $\mathbf{H}_N$ is evenly covered by p. $\square$

Chapter 3. New Ideas

We will now define the families of sequences of covering spaces in which we shall be interested. We will also state and prove a generalization of the first half of our main theorem. For this chapter, we will suspend our convention that spaces are compact. However, we will continue to assume that all spaces are metric and nonempty.

3.1 Finite Normal Covering Sequence

Definition 3.1. Suppose X is a path-connected, locally path-connected space. A *finite normal covering sequence* of X is a sequence (E_i, p_i, ϕ_i) so that for all i

(i) E_i is path-connected and locally path-connected space,

(ii) $p_i : E_i \longrightarrow X$ is a finite normal covering map,

(iii) $\phi_i : E_{i+1} \longrightarrow E_i$ is a covering map,

(iv) $p_{i+1} = p_i \circ \phi_i$.

If X is understood from context, we will usually just say (E_i, p_i, ϕ_i) is a *F.N.C.S.*.

Figure 3.1 depicts a F.N.C.S. as a commutative diagram, where all maps involved are finite covering maps.

$$
\begin{tikzcd}
\cdots & E_{i+1} \arrow[r, "\phi_i"] \arrow[ddl, "p_{i+1}"'] & E_i \arrow[ddll, "p_i"] \\
\cdots & & \\
X & &
\end{tikzcd}
$$

Figure 3.1: Finite normal covering sequence

Observe that since p_i is a finite normal covering map, $(p_i)_*(\pi_1(E_i, \tilde{x}_i))$, which we will denote by G_i, is a finite index normal subgroup of $\pi_1(X, x_0)$ for some $\tilde{x}_i \in p_i^{-1}(x_0)$. But $(p_{i+1})_* = (p_i \circ \phi_i)_*$, so $G_{i+1} \leq G_i$. Hence G_{i+1} is a finite index normal subgroup of G_i.

On the other hand, ϕ_i is also a covering map, so $(\phi_i)_*(\pi_1(E_{i+1}, \tilde{x}_{i+1})) = (p_i)_*^{-1}(G_{i+1})$ is a finite index normal subgroup of $\pi_1(E_i, \tilde{x}_i) = (p_i)_*^{-1}(G_i)$. Hence ϕ_i is a finite normal covering map as well.

Theorem 3.2. *Let X be a path-connected, locally path-connected space. If (E_i, p_i, ϕ_i) is a F.N.C.S., then (E_i, ϕ_i) is an inverse sequence.*

Furthermore, if $E = \lim\limits_{\leftarrow \phi_i} E_i$, then there exists a unique map $p : E \longrightarrow X$ such that $p = \Phi_i \circ p_i$ for all i, where $\Phi_i : E \longrightarrow E_i$ is the natural projection from E to E_i. $p : E \longrightarrow X$ is a fibration.

The commutative diagram in Figure 3.2 illustrates the result of Theorem 3.2:

$$\begin{tikzcd}
E \arrow[dr, "\Phi_{i+1}"] \arrow[drr, "\Phi_i"] \arrow[dd, dashed, "p"] & & \\
& E_{i+1} \arrow[r, "\phi_i"] \arrow[dl, "p_{i+1}"] & E_i \arrow[dll, "p_i"] \\
X & &
\end{tikzcd}$$

Figure 3.2: Inverse limit of F.N.C.S. is a fibration

This is in fact a special case of Theorem 3.4. Though the proof of Theorem 3.2 is simpler than that of Theorem 3.4, we will only give the more general proof as found in Section 3.2.

3.2 Finite Approximate-Covering Sequence

As is often the case in mathematics, computations may be simplified by using approximations of the original object. In this spirit, we define finite approximate-covering sequences.

Definition 3.3. Suppose $X = \lim\limits_{\leftarrow \theta_i} X_i$ for some inverse sequence (X_i, θ_i), where X_i and X are path-connected and locally path-connected spaces. A *finite approximate-covering sequence* of X, relative (X_i, θ_i), is a sequence (C_i, q_i, ψ_i) so that for all i

(i) C_i is a path-connected and locally path-connected space,

(ii) $q_i : C_i \longrightarrow X_i$ is a finite normal covering map,

(iii) (C_i, ψ_i) is an inverse sequence,

(iv) $\theta_i \circ q_{i+1} = q_i \circ \psi_i$.

When X and (X_i, θ_i) are understood, we will usually just say that (C_i, q_i, ψ_i) is a *F.A.C.S.*.

Figure 3.3 depicts a F.A.C.S. as a commutative diagram, where the vertical maps are covering maps.

$$
\begin{tikzcd}
\cdots & C_{i+1} \arrow[r, "\psi_i"] \arrow[d, "q_{i+1}"] & C_i \arrow[d, "q_i"] \\
\cdots & X_{i+1} \arrow[r, "\theta_i"] & X_i \\
\cdots & & \\
X \arrow[uur, "\Theta_{i+1}"] \arrow[uurr, "\Theta_i"'] & &
\end{tikzcd}
$$

Figure 3.3: Finite approximate covering sequence

Since we may always take $X_i = X$ and θ_i to be the identity, we see that every F.N.C.S. is a F.A.C.S relative (X, id). Therefore we see that any facts about F.A.C.S. are automatically true for F.N.C.S. as well, such as Theorem 3.4.

Nevertheless, we have given these separate definitions because it is not yet known whether they are, in general, equivalent in terms of what information can be obtained from each. Of course, one of the main results of this paper is that for $X = \mathbf{H}$, F.N.C.S. and F.A.C.S. are in fact essentially equivalent.

Theorem 3.4. *Let* $(X, \Theta_i) = \varprojlim_{\theta_i} X_i$. *If* (C_i, q_i, ψ_i) *is a F.A.C.S. of* X *and* $(E, \Psi_i) = \varprojlim_{\psi_i} C_i$, *then there is a unique induced map* $p : E \longrightarrow X$ *such that*

$$\Theta_i \circ p = q_i \circ \Psi_i, \qquad \textit{for all } i.$$

Furthermore, (E, p) *is a fibration of* X.

The commutative diagram in Figure 3.4 illustrates the result of Theorem 3.4.

$$\begin{tikzcd}
\varprojlim_{\Psi_i} C_i \arrow[dr, "\Psi_{i+1}"] \arrow[drr, "\Psi_i"] \arrow[ddd, dashed, "p"] & & \\
& C_{i+1} \arrow[r, "\psi_i"] \arrow[d, "q_{i+1}"] & C_i \arrow[d, "q_i"] \\
& X_{i+1} \arrow[r, "\theta_i"] & X_i \\
X \arrow[ur, "\Theta_{i+1}"] \arrow[urr, "\Theta_i"'] & &
\end{tikzcd}$$

Figure 3.4: Inverse limit of F.A.C.S. is a fibration

Proof. We first prove the existence and uniqueness of the map p. Let $(E, \Psi_i) = \varprojlim_{\psi_i} C_i$. Then $q_i \circ \Psi_i$ is a map from E to X_i for each i. Since $X = \varprojlim_{\theta_i} X_i$ by hypothesis, then there exists a unique map $p : E \longrightarrow X$ so that

$$\Theta_i \circ p = q_i \circ \Psi_i, \quad \text{for all } i. \tag{3.1}$$

Thus it remains only to show that (E, p) is a fibration. Suppose that $f_t : Y \longrightarrow X$ is a homotopy and $\tilde{f}_0 : Y \longrightarrow E$ is a partial lift of f_0. Since Θ_i is continuous, $\Theta_i \circ f_t : Y \longrightarrow X_i$ is also a homotopy. Furthermore, $\Psi_i \circ \tilde{f}_0$ is a lift of $\Theta_i \circ f_0$ since

$$q_i \circ \Psi_i \circ \tilde{f}_0(y) = \Theta_i \circ p \circ \tilde{f}_0(y) = \Theta_i \circ f_0(y). \tag{3.2}$$

Since (C_i, q_i) is a finite covering space of X_i, there exists a unique map $\tilde{f}_{t,i} : Y \longrightarrow C_i$ such that

$$q_i \circ \tilde{f}_{t,i} = \Theta_i \circ f_t, \quad \text{and} \quad \tilde{f}_{0,i} = \Psi_i \circ \tilde{f}_0. \tag{3.3}$$

Now $\tilde{f}_{t,i}$ is a continuous map from $Y \times I$ to C_i for all i, so by Theorem 2.25 there exists a unique map $\tilde{f} : Y \times I \longrightarrow E$ so that

$$\Psi_i \circ \tilde{f}(y, t) = \tilde{f}_{t,i}(y). \tag{3.4}$$

Therefore $\Psi_i \circ \tilde{f}(y, 0) = \tilde{f}_{0,i}(y) = \Psi_i \circ \tilde{f}_0(y)$ for all i. However, $\tilde{f}(y, 0)$ is the unique map that accomplishes this, so in fact $\tilde{f}(y, 0) = \tilde{f}_0(y)$. Therefore we may set $\tilde{f}_t(y) = \tilde{f}(y, t)$.

$X = \lim\limits_{\leftarrow \theta_i} X_i$, so we may think of points of X as sequences of points in X_i. Thus it suffices to show that for all i, $\Theta_i \circ f_t = \Theta_i \circ (p \circ \tilde{f}_t)$. In fact, we have

$$\begin{aligned}
\Theta_i \circ f_t &= q_i \circ \tilde{f}_{t,i}, && \text{by equation 3.3,} \\
&= q_i \circ \Psi_i \circ \tilde{f}_t, && \text{by equation 3.4,} \\
&= \Theta_i \circ p \circ \tilde{f}_t, && \text{by equation 3.1.}
\end{aligned}$$

Since Y, f_t and $\tilde{f}_0$ were arbitrary, this shows that (E, p) has the homotopy lifting property and is therefore a fibration. In fact, we have shown that (E, p) has the unique homotopy lifting property, since $\tilde{f}_t$ was uniquely determined by the $\tilde{f}_{t,i}$, which were in turn uniquely determined by $\tilde{f}_0$ and f_t. $\square$

3.3 Pro-Cover

Definition 3.5. Suppose $p : E \longrightarrow X$ is a fibration, where $X = \lim_{\leftarrow \theta_i} X_i$ for some inverse sequence (X_i, θ_i). (E, p) is a *pro-cover* of X relative (X_i, θ_i) if there exists a F.A.C.S. (C_i, q_i, ψ_i) such that

- $(E, (\Psi_i)) = \lim_{\leftarrow \psi_i} C_i$,
- p is the same map obtained from Theorem 3.4.

If the inverse sequence (X_i, θ_i) is the trivial sequence (X, id), then we simply say (E, p) is a pro-cover of X.

We immediately have the following corollaries to Theorem 3.4.

Corollary 3.6. *Every pro-cover of X is a minimal fibration of X.*

Proof. Suppose (C_i, q_i, ψ_i) be a F.A.C.S. where $(E, (\Psi_i)) = \lim_{\leftarrow \psi_i} C_i$ and p is the induced map from E to X. Then let $\tilde{x}, \tilde{y} \in E$ and let $\epsilon > 0$ be given.

Since E has the product topology, there exists $N \in \mathbb{N}$ so that if $\Psi_i(\tilde{z}) = \Psi_i(\tilde{y})$ for all $i < N$ then $d(\tilde{z}, \tilde{y}) < \epsilon$.

Since C_N is path-connected, let γ_N be a null-homotopic path in C_N from $\Psi_N(\tilde{x})$ to $\Psi_N(\tilde{y})$. Clearly $\psi_{N-1} \circ \gamma_N$ is a path in C_{N-1} from $\psi_{N-1} \circ \Psi_N(\tilde{x}) = \Psi_{N-1}(\tilde{x})$ to $\Psi_{N-1}(\tilde{y})$. Similarly we obtain paths γ_i in C_i from $\Psi_i(\tilde{x})$ to $\Psi_i(\tilde{y})$ for all $i \leq N$.

Conversely, C_{N+1} is a covering space of C_N, so there exists a unique lift of γ_N to a path γ_{N+1} based at $\Psi_{N+1}(\tilde{x})$. Inductively we obtain paths γ_i in C_i based at $\Psi_i(\tilde{x})$ that lift γ_N for all $i > N$.

Clearly $\gamma(t) = (\gamma_i(t))$ is a path in E by the homotopy lifting property of inverse limits. Furthermore, $\gamma_i(1) = \Psi_i(\tilde{y})$ for $i \leq N$, so $d(\gamma(1), \tilde{y}) < \epsilon$.

Since $\epsilon > 0$ and $\tilde{y} \in E$ were arbitrary, we see that the path-component of E containing $\tilde{x}$ is dense in E. In other words, E is a minimal fibration. $\square$

Corollary 3.7. *Suppose (E,p) is a pro-cover of X relative (X_i,θ_i), where X and X_i are nonempty metric spaces, not necessarily compact. Let (C_i,q_i,ψ_i) denote a F.A.C.S. that produces (E,p).*

(a) (E,p) is a unique path-lifting fibration;

(b) If C_i is compact for all i, then E is compact; and

(c) If X_i is compact for all i, then E is compact.

Proof. *(a)* This follows immediately from the proof of Theorem 3.4, since we showed (E,p) has the unique homotopy lifting property.

(b) Since (C_i,q_i) is a finite normal covering space of X_i is nonempty. Thus by Theorem 2.24, E is compact and nonempty.

(c) Since (C_i,q_i) is a finite normal covering space of X_i, which is compact and nonempty by hypothesis, then C_i is compact and nonempty. The result then follows from part (c). □

The next result characterizes pro-covers in terms of subgroups of the fundamental group of the base space.

Theorem 3.8. *Let H be a subgroup of $\pi_1(X,x_0)$. There exists a pro-cover (E,p) of X so that $p_*(\pi_1(E,\tilde{x}_0)) = H$, for some $\tilde{x}_0 \in p^{-1}(x_0)$ if and only if there exists a sequence of finite index normal coverable subgroups of $\pi_1(X,x_0)$, H_i, such that $H_i \leq H_{i-1}$ and $H = \cap H_i$.*

Suppose (E,p) is a pro-cover of X, where (C_i,q_i,ψ_i) is the necessary finite approximate-covering sequence. For $x_0 \in X$, fix $\tilde{x}_0 \in E$ and $\tilde{x}_i \in C_i$ so that $p(\tilde{x}_0) = q_i(\tilde{x}_i) = x_0$. Then

$$\pi_1(E,\tilde{x}_0) \cong \cap (q_i)_*(\pi_1(C_i,\tilde{x}_i)).$$

Proof. First, suppose there exists a pro-cover (E,p) of X so that $p_*(\pi_1(E,\tilde{x}_0)) = H$ for some $\tilde{x}_0 \in p^{-1}(x_0)$. Let (C_i,q_i,ψ_i) be a finite approximate-covering sequence such that $(E,(\Psi_i)) = \varprojlim_{\psi_i} C_i$ and p is the induced map from Theorem 3.4.

Let $\tilde{x}_i = \Psi_i(\tilde{x}_0)$, which is contained in $q_i^{-1}(x_0)$, and let $H_i = (q_i)_*(\pi_1(C_i, \tilde{x}_i))$. Then since $p = q_i \circ \Psi_i$ for all i,

$$p_*(\pi_1(E, \tilde{x}_0)) \leq H_i \leq \pi_1(X, x_0). \tag{3.5}$$

Consider $[\alpha] \in \pi_1(E, \tilde{x}_0)$ and suppose $p_*([\alpha]) = 0$. Then $(q_i \circ \Psi_i)_*([\alpha]) = 0$ for all i. However, $q_i : C_i \longrightarrow X$ is a finite covering map, so by Lemma 2.31, this means $(\Psi_i)_*([\alpha]) = 0$ for all i. Then $[\alpha] = 0$ since $(E, (\Psi_i)) = \varprojlim_{\psi_i} C_i$, so p_* is injective.

On the other hand, suppose α_0 is a loop in X based at x_0 such that $[\alpha_0] \in \cap H_i$. Since $[\alpha_0] \in H_i$, there is a loop $\alpha_i \in C_i$ so that $q_i \circ \alpha_i = \alpha_0$ and $\alpha_i(0) = \tilde{x}_i$. Furthermore, by Lemma 2.31, this loop is unique.

Furthermore, $q_i \circ \psi_i \circ \alpha_{i+1} = q_{i+1} \circ \alpha_{i+1} = \alpha_0$. By uniqueness, $\psi_i \circ \alpha_{i+1} = \alpha_i$ for all i, and so (α_i) is a loop in E. Then $p(\alpha_i) = q_i \circ \Psi_i((\alpha_i)) = q_i(\alpha_i) = \alpha_0$, so p_* maps onto $\cap H_i$. Thus

$$H = p_*(\pi_1(E, \tilde{x}_0)) = \cap H_i.$$

Now suppose there exists a sequence of finite index normal coverable subgroups of $\pi_1(X, x_0)$, H_i, such that $H_i \leq H_{i-1}$ and $H = \cap H_i$.

Since H_1 is a finite index normal coverable subgroup of $\pi_1(X, x_0)$, there exists a finite normal covering space (C_1, q_1) of X such that $(q_1)_*(\pi_1(C_1, \tilde{x}_1)) = H_1$ for some $\tilde{x}_1 \in q_1^{-1}(x_0)$.

By way of induction, suppose that (C_i, q_i) is a finite index normal covering space of X such that $(q_i)_*(\pi_1(C_i, \tilde{x}_i)) = H_i$ for some $\tilde{x}_i \in q_i^{-1}(x_0)$ for $i < n$.

Since $H_n \leq H_{n-1} \leq \pi_1(X, x_0)$, H_n is a finite index normal subgroup of H_{n-1}. Furthermore, since H_n was a coverable subgroup of $\pi_1(X, x_0)$, there exists an open cover $\mathcal{U}_n$ of X so that $\pi_1(\mathcal{U}_n, x_0) \leq H_n$. Let $\mathcal{V}_n = \{q_{n-1}^{-1}(U) | U \in \mathcal{U}_n\}$. Then $\pi_1(\mathcal{V}_n, \tilde{x}_{n-1}) \leq (q_{n-1})_*^{-1}(H_n) \leq \pi_1(C_{n-1}, \tilde{x}_{n-1})$, so $(q_{n-1})_*^{-1}(H_n)$ is a coverable subgroup for C_{n-1} as well. Therefore there exists a finite index normal covering space of C_{n-1}, (C_n, ψ_{n-1}) such that $(\psi_{n-1})_*(\pi_1(C_n, \tilde{x}_n)) = (q_{n-1})_*(H_n)$ for some $\tilde{x}_n \in \psi_{n-1}^{-1}(\tilde{x}_{n-1})$.

Let $q_n = q_{n-1} \circ \psi_{n-1}$. Then (C_n, q_n) is a finite index normal covering space of X so that

$$(q_n)_*(\pi_1(C_n, \tilde{x}_n)) = (q_{n-1})_*((\psi_{n-1})_*(\pi_1(C_n, \tilde{x}_n))) = (q_{n-1})_*((q_{n-1})_*^{-1}(H_n)) = H_n.$$

Therefore (C_i, q_i, ψ_i) is a F.N.C.S. for X. By the first part of this proof, if (E, p) is the pro-cover obtained from (C_i, q_i, ψ_i), then

$$p_*(\pi_1(E, \tilde{x}_0)) = \cap (q_i)_*(\pi_1(C_i, \tilde{x}_i)) = \cap H_i = H,$$

where $\tilde{x}_0 = (\tilde{x}_i) \in E$. □

Theorem 3.9. *Suppose (E, p) is a pro-cover of a compact, path-connected, locally-path connected space X, where (E_i, p_i, ϕ_i) is the corresponding F.N.C.S.. For each $\tilde{x} \in E$, we obtain an inverse sequence $(\mathrm{Aut}(p_i), \upsilon_i(\tilde{x}))$. The inverse system obtained by the union of these sequences, denoted L, acts topologically and the fibre is totally disconnected.*

Proof. Let $x_0 \in X$ and $\tilde{x}_0 = (\tilde{x}_i) \in p^{-1}(x_0)$. We will first show that there exist maps $\upsilon_i : \mathrm{Aut}(p_{i+1}) \longrightarrow \mathrm{Aut}(p_i)$ so that $(\mathrm{Aut}(p_i), \upsilon_i)$ is an inverse sequence.

Suppose $f_{i+1} \in \mathrm{Aut}(p_{i+1})$. For $\tilde{y}_i \in E_i$, let $\alpha_{\tilde{y}_i}$ be a path from $\tilde{x}_i$ to $\tilde{y}_i$. Since ϕ_i is a finite normal covering map, there exists a unique path $\tilde{\alpha}_{\tilde{y}_i}$ that lifts $\alpha_{\tilde{y}_i}$ and begins at $\tilde{x}_{i+1}$. Let $\tilde{y}_{i+1} = \tilde{\alpha}_{\tilde{y}_i}(1)$ and define $f_i(\tilde{y}_i) = \phi_i \circ f_{i+1}(\tilde{y}_{i+1})$. Since we used unique path-lifting to obtain $\tilde{y}_{i+1}$, f_i is well-defined and continuous.

Clearly $f_i^{-1}(\tilde{y}_i)$ is obtained by the same process from f_{i+1}^{-1}. Thus $f_i \in \mathrm{Aut}(p_i)$. Therefore $f_{i+1} \mapsto f_i$ is a map from $\mathrm{Aut}(p_{i+1})$ to $\mathrm{Aut}(p_i)$, which we will denote by $\upsilon_i(\tilde{x}_0)$, since it depends on the particular sequence $\tilde{x}_0$. Thus $(f_i(x_i)) \in E$ for all $(x_i) \in E$ since $\phi_i \circ f_{i+1} = f_i \circ \phi_i$. Likewise, $f_i(\tilde{x}_i) = \tilde{x}_i$, since $\alpha_{\tilde{x}_i}$ and $\tilde{\alpha}_{\tilde{x}_i}$ are both the constant path.

Furthermore, each f_i is a homeomorphism, so clearly $(f_i(x_i))$ is a homeomorphism, with $(f_i^{-1}(x_i))$ being its inverse. Thus $\lim\limits_{\leftarrow \upsilon_i(\tilde{x}_0)} \mathrm{Aut}(p_i) \in \mathrm{Aut}(p)$ for each $\lim\limits_{\leftarrow \upsilon_i(\tilde{x}_0)} \mathrm{Aut}(p_i)$. In particular, this means that $L \subset \mathrm{Aut}(p)$.

We now show that L acts topologically.

Define $\Lambda : L \longrightarrow p^{-1}(x_0)$ by $\Lambda(f) = f(\tilde{x}_0) = (f_i(\tilde{x}_i))$, for $f = (f_i) \in L$. Λ is well-defined by definition of $\mathrm{Aut}(p)$, since $p \circ f(\tilde{x}_0) = p(\tilde{x}_0) = x_0$.

For each $\tilde{x}_0 = (\tilde{x}_i) \in p^{-1}(x_0)$ we construct (f_i) so that $(f_i(\tilde{x}_i)) = (\tilde{x}_i) = \tilde{x}_0$, so Λ is onto.

Let d denote the metric on E and let $\bar{d}$ denote the standard bounded metric induced by d. Furthermore, let $\bar{\rho}$ denote the uniform metric on L. Note that $d(\Lambda(f), \Lambda(g)) = d(f(\tilde{x}_0), g(\tilde{x}_0))$, so if $d(\Lambda(f), \Lambda(g)) < 1$, we have

$$d(\Lambda(f), \Lambda(g)) \leq \sup\{\bar{d}(f(x), g(x)) | x \in E\} = \bar{\rho}(f, g).$$

Hence Λ is continuous. Since the inverse limit of compact spaces is compact, and each $\mathrm{Aut}(p_i)$ is finite, and hence compact, L is compact. E is Hausdorff, so any subset of E is Hausdorff. Thus Λ is a homeomorphism.

Finally we show that the fibre is totally disconnected.

Let $f, g \in L \cong p^{-1}(x_0)$. If $f \neq g$, then for some N, $\Phi_N(f) \neq \Phi_N(g) \in p_N^{-1}(x_0)$. However, E_i is a finite covering space of X, so $p_N^{-1}(x_0)$ is a discrete set. Therefore we have nonempty disjoint open sets U and V such that $\Phi_N(f) \in U$ and $\Phi_N(g) \in V$. Thus $\Phi_N^{-1}(U)$ and $\Phi_N^{-1}(V)$ are nonempty disjoint open sets separating f and g. Thus L, and hence the fibre also, is totally disconnected. $\square$

Chapter 4. Covering Sequences of the Hawaiian Earring

Recall from Section 2.9 that $(\mathbf{H}, \Theta_i) = \varprojlim_{\theta_i} B_i$. Unless stated otherwise, for the remainder of this paper any discussion of F.A.C.S. will be relative to that particular inverse sequence. We will show that there is a natural correspondence between F.N.C.S. and F.A.C.S. for the Hawaiian Earring. This correspondence is natural in the sense that it preserves the inverse limit of the system as well as the internal structure of the sequence. In addition, our proof will provide an explicit algorithm to construct each type of sequence from the other.

4.1 Correspondence between F.A.C.S and F.C.S.

Recall the equivalence relation $\sim$ defined in Section 2.2. We define a similar relation on finite approximate-covering sequences (and hence finite normal covering sequences as well).

Definition 4.1. Suppose (C_i, q_i, ψ_i) and (C'_i, q'_i, ψ'_i) are finite approximate-covering sequences of X, relative (X_i, r_i) and (X'_i, r'_i), respectively. Then we write $(C_i, q_i, \psi_i) \simeq (C'_i, q'_i, \psi'_i)$ if, for all i, there exist homeomorphisms $f_i : X_i \longrightarrow X'_i$ and $g_i : C_i \longrightarrow C'_i$ such that

(i) $f_i \circ r_i = r'_i \circ f_{i+1}$,

(ii) $g_i \circ \psi_i = \psi'_i \circ g_{i+1}$, and

(iii) $f_i \circ q_i = q'_i \circ h_i$;

or if there exist cofinal subsequences of each with this property.

Figure 4.1 illustrates the meaning of $(C_i, q_i, \psi_i) \simeq (C'_i, q'_i, \psi'_i)$ as a commutative diagram.

Theorem 4.2. *Let $\mathcal{C}$ be the set of $\sim$-equivalence classes of finite normal covering sequences of $\mathbf{H}$, and let $\mathcal{A}$ be the set of $\sim$-equivalence classes of finite approximate covering sequences of $\mathbf{H}$, relative subsequences of (B_i, r_i). There exists a bijection $T : \mathcal{C} \longrightarrow \mathcal{A}$ so that if $T([(E_i, p_i, \phi_i)]) = [(C_i, q_i, \psi_i)]$ then*

$$
\begin{tikzcd}
 & & & {} \arrow[dl, dotted] & & & & {} \arrow[dl, dotted] \\
 & & C_{i+1} \arrow[rrrr, leftrightarrow] \arrow[ddll, "\psi_i"'] \arrow[dddd, "q_{i+1}"] & & & & C'_{i+1} \arrow[ddll, "\psi'_i"'] \arrow[dddd, "q'_{i+1}"] & \\
 & & & & & & & \\
C_i \arrow[rrrr, leftrightarrow] \arrow[dddd, "q_i"'] & & & & C'_i \arrow[dddd, "q'_i"] & & & \\
 & & & {} \arrow[dl, dotted] & & & & {} \arrow[dl, dotted] \\
 & & X'_{i+1} \arrow[rrrr, leftrightarrow] \arrow[ddll, "r_i"] & & & & X'_{i+1} \arrow[ddll, "r'_i"] & \\
 & & & & & & & \\
X_i \arrow[rrrr, leftrightarrow] & & & & X'_i & & &
\end{tikzcd}
$$

Figure 4.1: $(C_i, q_i, \psi_i) \simeq (C'_i, q'_i, \psi'_i)$

(a) $\varprojlim_{\phi_i} E_i \cong \varprojlim_{\psi_i} C_i$;

(b) $\mathrm{Aut}(p_i) \cong \mathrm{Aut}(q_i)$ *for all* i.

Proof. First, we define T.

Suppose $(E_i, p_i, \phi_i) \in \mathcal{C}$. By Lemma 2.66, there exists $M_i \geq 0$ such that p_i evenly covers $\mathbf{H}_{M_i}$. Furthermore, since $\mathbf{H}_0 = \mathbf{H}$, we may assume that M_i is the least such integer.

Let $N_1 = \max\{M_1, 1\}$ and for $i > 1$, let $N_i = \max\{M_i, M_{i-1}+1\}$. Thus (N_i) is a strictly increasing sequence with $N_i \geq M_i$ for all i. Therefore $\mathbf{H}_{N_i} \subset \mathbf{H}_{M_i}$, so p_i evenly covers $\mathbf{H}_{N_i}$.

Let $C_i = p_i^{-1}(B_{N_i})$. Since B_{N_i} is a closed subset of $\mathbf{H}$ and p_i is a covering map, by Lemma 2.36, we have a covering map $q_i : C_i \longrightarrow B_{N_i}$ such that $q_i(x) = p_i(x)$ for all $x \in C_i \subset E_i$. Since p_i is a normal cover, q_i is also a normal cover.

Furthermore, since $p_i \circ \phi_i = p_{i+1}$, we have $\phi_i(C_{i+1}) \subset C_i$. Let $\psi_i = \phi_i\big|_{C_{i+1}}$. Therefore we obtain a finite approximate covering sequence of $\mathbf{H}$, (C_i, q_i, ψ_i), which is relative (B_{N_i}, θ_{N_i}). We then define $T([(E_i, p_i, \phi_i)]) = [(C_i, q_i, \psi_i)]$.

We now show that T is well-defined.

Suppose $(E_i, p_i, \phi_i) \sim (E'_i, p'_i, \phi'_i)$. If (E'_i, p'_i, ϕ'_i) is a cofinal subsequence of (E_i, p_i, ϕ_i), then clearly (C'_i, q'_i, ψ'_i) is a cofinal subsequence of (C_i, q_i, ψ_i). Thus we may assume there exist homeomorphisms $h_i : E_i \longrightarrow E'_i$ such that $p_i = p'_i \circ h_i$ and $h_i \circ \phi_i = \phi'_i \circ h_{i+1}$ for all i.

Note that $p_i^{-1}(\mathbf{H}_{M_i}) = (p'_i \circ h_i)^{-1}(\mathbf{H}_{M_i})$, so $h_i \circ p_i^{-1}(\mathbf{H}_{M_i}) = p_i'^{-1}(\mathbf{H}_{M_i})$. Thus p'_i also evenly covers $\mathbf{H}_{M_i}$. Thus both (C_i, q_i, ψ_i) and (C'_i, q'_i, ψ'_i) depend on the same sequence of (N_i). But $h_i \circ p_i^{-1} = p_i'^{-1}$, so we have

$$C_i = p_i^{-1}(\mathbf{H}_{N_i}) = h_i^{-1} \circ p_i'^{-1}(\mathbf{H}_{N_i}) = C'_i,$$

$$q_i = p_i\big|_{C_i} = p'_i \circ h_i\big|_{C_i} = q'_i \circ h_i\big|_{C_i},$$

$$h_i \circ \psi_i = h_i \circ \phi_i\big|_{C_i} = \phi'_i \circ h_{i+1}\big|_{C_{i+1}} = \psi'_i \circ h_{i+1}\big|_{C_{i+1}}.$$

Thus $(C_i, q_i, \psi_i) \simeq (C'_i, q'_i, \psi'_i)$, so T is well-defined.

Next we show that T is surjective.

Suppose (C_i, q_i, ψ_i) is a finite approximate covering sequence of $\mathbf{H}$ relative (B_{N_i}, θ_{N_i}), where (B_{N_i}, θ_{N_i}) is a subsequence of the usual (B_i, θ_i). For each i, let $E_i = C_i\,{}_{q_i^{-1}(0,0)}\!\vee_{(0,0)} \mathbf{H}_{N_i}$. Let G_i denote the union of the copies of $\mathbf{H}_{N_i}$ in E_i.

Since C_i is a finite covering space, $q_i^{-1}(0,0) = \{x_1, \ldots, x_k\}$ for some integer k. Therefore G_i is the disjoint union of k copies of $\mathbf{H}_{N_i}$, say $Y_1, \ldots, Y_k$. In particular, this means we have a map $g_i : G_i \longrightarrow \mathbf{H}_{N_i}$ which is the identity on each Y_i. Then we define $p_i : E_i \longrightarrow \mathbf{H}$ as follows.

$$p_i(x) = \begin{cases} q_i(x), & \text{if } x \in C_i; \\ g_i(x), & \text{if } x \in G_i. \end{cases} \tag{4.1}$$

Note that $p_i(C_i) = B_{N_i}$ and $p_i(G_i) = \mathbf{H}_{N_i}$ and $B_{N_i} \cap \mathbf{H}_{N_i} = (0,0)$.

If $x \in B_{N_i}$, then since p_i was a covering space, there exists a neighborhood $U_i(x) \subset B_{N_i}$ that is evenly covered by q_i. If $(0,0) \neq x$, then we may assume $(0,0) \notin U_i(x)$, so $U_i(x)$ is also open in $\mathbf{H}$. Furthermore, $q_i^{-1}(U_i(x)) = p_i^{-1}(U_i(x))$, so p_i evenly covers $U_i(x)$.

Furthermore, for $\tilde{x}_0, \tilde{x}_1 \in p_i^{-1}(x) = q_i^{-1}(x)$, there exists $f_i \in \mathrm{Aut}(q_i)$ so that $f_i(\tilde{x}_0) = \tilde{x}_1$. Then for each $x_r \in q_i^{-1}(0,0)$, $f_i(x_r) = x_{\sigma(r)}$ for some permutation $\sigma \in S_k$. Then we have a map $g_\sigma : G_i \longrightarrow G_i$ that sends Y_r to $Y_{\sigma(r)}$ by the identity. Thus

$$F_i(x) = \begin{cases} f_i(x), & \text{if } x \in C_i; \\ \\ g_\sigma(x), & \text{if } x \in G_i, \end{cases} \tag{4.2}$$

is an automorphism of p_i that sends $\tilde{x}_0$ to $\tilde{x}_1$.

If $x \in \mathbf{H}_{N_i}$, then since g_i is the natural projection, any neighborhood $V_i(x) \subset \mathbf{H}_{N_i}$ is evenly covered by g_i. If $(0,0) \neq x$, then we may assume $(0,0) \notin V_i(x)$, so $V_i(x)$ is also open in $\mathbf{H}$. Furthermore, $g_i^{-1}(V_i(x)) = p_i^{-1}(V_i(x))$, so p_i evenly covers $V_i(x)$.

Again, for $\tilde{x}_0, \tilde{x}_1 \in p_i^{-1}(x) = g_i^{-1}(x)$, $\tilde{x}_0 \in Y_r$ and $\tilde{x}_1 \in Y_s$ for some r, s. Since q_i is normal, there exists $f_i \in \mathrm{Aut}(q_i)$ such that $f_i(x_r) = x_s$. Then for each t, $f_i(x_t) = x_{\sigma(t)}$ for some $\sigma \in S_k$. Then with g_σ as before, we see that

$$F_i(x) = \begin{cases} f_i(x), & \text{if } x \in C_i; \\ \\ g_\sigma(x), & \text{if } x \in G_i, \end{cases} \tag{4.3}$$

is an automorphism of p_i that sends $\tilde{x}_0$ to $\tilde{x}_1$.

If $x = (0,0)$, then $x = B_{N_i} \cap \mathbf{H}_{N_i}$. Let $U_i(x)$ and $V_i(x)$ be as in the preceding paragraphs. Since $x = (0,0)$, $U_i(x) \cup V_i(x)$ is an open set in $\mathbf{H}$. For each $\tilde{x} \in q_i^{-1}(x)$, we have a unique lift of $U_i(x)$ by q_i and a unique lift of $V_i(x)$ by g_i so that $\tilde{x}$ is contained in both. Thus the union of these two lifts is a unique lift by p_i. Therefore p_i evenly covers $U_i(x) \cup V_i(x)$.

Furthermore, the same F_i constructed for $x \in C_i - \{(0,0)\}$ suffices for this case as well. Then since x was arbitrary, this shows that $p_i : E_i \longrightarrow \mathbf{H}$ is a finite normal covering map.

Next, define $\phi_i : E_{i+1} \longrightarrow E_i$ by

$$\phi_i(x) = \begin{cases} \psi_i(x), & \text{if } x \in q_{i+1}^{-1}(B_{N_i}); \\ (\imath_{N_{i+1}} \circ q_{i+1})(x), & \text{if } x \in q_{i+1}^{-1}({}_{N_i}B_{N_{i+1}}); \\ h_{N_{i+1}-N_i}(x), & \text{if } x \in G_{i+1}. \end{cases} \tag{4.4}$$

Note that

$$\phi_i(q_{i+1}^{-1}(B_{N_i})) = \psi_i(C_{i+1}) = C_i; \quad \phi_i(G_{i+1}) = h_{N_{i+1}-N_i}(\mathbf{H}_{N_{i+1}}) = \mathbf{H}_N; \text{ and}$$
$$\phi_i(q_{i+1}^{-1}({}_{N_i}B_{N_{i+1}})) = \imath_{N_{i+1}}({}_{N_i}B_{N_{i+1}}) = {}_{N_i}B_{N_{i+1}} \subset \mathbf{H}.$$

Thus for $x \neq (0,0)$, there exists a neighborhood $U_i(x)$ such that $\phi_i^{-1}(U_i)$ is contained entirely in one of the three closed sets $q_{i+1}^{-1}(B_{N_i})$, $q_{i+1}^{-1}({}_{N_i}B_{N_{i+1}})$ or G_{i+1}.

Suppose $x \in C_i - q_i^{-1}(0,0)$. Then $q_i(x) \in B_{N_i} \subset B_{N_{i+1}}$. Since q_{i+1} is a finite covering map, then there exists an evenly covered neighborhood of $q_i(x)$ in $B_{N_{i+1}}$, say $U_i(x)$. Since $q_i(x) \neq (0,0)$, we may assume $U_i(x) \cap B_{N_i} = U_i(x)$, so $U_i(x)$ is an open neighborhood of $q_i(x)$ in B_{N_i} as well. Since q_i is a finite covering map, there exists an evenly covered neighborhood of $q_i(x)$ in B_{N_i} by q_i as well, say $V_i(x)$. Let $W_i(x) = U_i(x) \cap V_i(x)$. Then $W_i(x)$ is evenly covered by q_i and q_{i+1}. Thus we have a unique neighborhood $\tilde{W}_i(x)$ in C_i such that $q_i\big|_{\tilde{W}_i(x)}$ is a homeomorphism onto $W_i(x)$. Since $W_i(x) \subset B_{N_i}$, we have that θ_i is the identity on $W_i(x)$, so $\theta_i \circ q_{i+1} = q_{i+1} = q_i \circ \psi_i$. Thus ψ_i must evenly cover $\tilde{W}_i(x)$.

Similarly, if $x \in {}_{N_i}B_{N_{i+1}} - q_i^{-1}(0,0) \subset G_i$, then since $\imath_{N_{i+1}}$ is the identity on ${}_{N_i}B_{N_{i+1}}$, we have $\phi_i(x) = q_{i+1}(x)$, so there is some open neighborhood of x that is evenly covered by q_{i+1}. Since $x \neq q_i^{-1}(0,0)$, we may assume that this open neighborhood is contained entirely in ${}_{N_i}B_{N_{i+1}}$, so $q_{i+1} = \phi_i$ on the pre-image of this neighborhood, so ϕ_i also evenly covers it.

Third, if $x \in G_{i+1} - q_i^{-1}(0,0)$, then some open neighborhood of x, $U_i(x)$ lies entirely within G_{i+1}. Since $h_{N_{i+1}-N_i}$ is a homeomorphism, ϕ_i is a homeomorphism on each component of $\phi_i^{-1}(U_i(x))$.

Furthermore, we have as many copies of $\mathbf{H}_{N_i}$ as we have lifts of the origin, so ϕ_i evenly covers $U_i(x)$ with the same number of components as in the previous steps.

Finally, suppose $x = q_i^{-1}(0,0)$. Then we have

$$\phi_i^{-1}(x) = \psi_i^{-1}(x) = (\imath_{N_{i+1}} \circ q_{i+1})^{-1}(x) = h^{-1}_{N_{i+1}-N_i}(x).$$

Therefore since each of these maps evenly covers some relatively open neighborhood of x, ϕ_i evenly covers the union of these neighborhoods, which is open. Thus ϕ_i is indeed a finite covering map, so $(E_i, p_i, \phi_i) \in \mathcal{C}$.

(a) Let $(a_i) \in \lim\limits_{\leftarrow p_i} E_i = E$. Note that for all i,

$$p_{i+1}(a_{i+1}) = p_i \circ \phi_i(a_{i+1}) = p_i(a_i).$$

Thus $x = p_i(a_i)$ is well-defined. Since $x \in \mathbf{H}$, either $x = (0,0)$ or there is a unique M for which $x \in A_M$. In either case, $x \in B_{N_j}$ for some j. Thus for $k > j$, $a_k \in C_k$ and so $\psi_k(a_{k+1}) = \phi_k(a_{k+1}) = a_k$.

However, T is defined on $\simeq$ equivalence classes, so we may assume that $j = 1$. Thus (a_i) is a coherent sequence of (C_i, q_i, ψ_i).

Conversely, suppose $(a_i) \in \lim\limits_{\leftarrow q_i} C_i$. Observe that $C_i \subset E_i$ and $\psi_i(a_{i+1}) = \phi_i(a_{i+1})$, so $(a_i) \in \lim\limits_{\leftarrow p_i} E_i$. Therefore $\lim\limits_{\leftarrow p_i} E_i = \lim\limits_{\leftarrow q_i} C_i$.

(b) Let $p : E \longrightarrow \mathbf{H}$ be the map induced by (E_i, p_i, ϕ_i) and let $q : E \longrightarrow \mathbf{H}$ be the map induced by (C_i, q_i, ψ_i). As in the preceding argument, $p_{i+1}(a_{i+1}) = p_i(a_i)$ and $\theta_i \circ q_{i+1}(a_{i+1}) = q_i(a_i)$. However, $p_i(a_i) \in A_{N_k}$ for some k, so $q_k(a_k) \in B_{N_k}$.

Since we are concerned with $\simeq$ equivalence classes, we may assume that $k = 1$, and thus

$$p((a_i)) = p_1(a_1) = q_1(a_1) = q((a_i)). \tag{4.5}$$

Thus $p = q$, and so $\mathrm{Aut}(p) = \mathrm{Aut}(q)$. $\square$

Chapter 5. Characterization of Pro-Covers of the Hawaiian Earring

5.1 Characterization of Pro-Covers

Definition 5.1. Suppose X is a metric space. For $x, y \in X$ and $\epsilon > 0$, we write $x \sim_\epsilon y$ if there exists a finite set of points $x = x_0, x_1, \dots, x_n = y$ so that $d(x_i, x_{i-1}) < \epsilon$ for $1 \leq i \leq n$. The set of points $x_0, x_1, \dots, x_n$ is called an ϵ-chain from x_0 to x_n.

Lemma 5.2. *$\sim_\epsilon$ is an equivalence relation.*

Proof. $d(x, x) = 0$ for all $x \in X$, so $\sim_\epsilon$ is symmetric.

If $x_0, x_1, \dots, x_n$ is an ϵ-chain from x_0 to x_n, then

$$x_n, x_{n-1}, \dots, x_1, x_0,$$

is an ϵ-chain from x_n to x_0. Thus $\sim_\epsilon$ is reflexive.

If $x_0, \dots, x_n$ is an ϵ-chain from x_0 to x_n and $x_n = y_0, y_1, \dots, y_m$ is an ϵ-chain from x_n to y_m, then clearly

$$x_0, \dots, x_n, y_1, \dots, y_m,$$

is an ϵ-chain from x_0 to y_m. Thus $\sim_\epsilon$ is transitive. □

Definition 5.3. We denote the equivalence class of x under $\sim_\epsilon$ as

$$[x]_\epsilon = \{y \in X | y \sim_\epsilon x\}.$$

If $[x]_\epsilon = X$ for some $x \in X$, we say X is ϵ-connected.

Note that if $\epsilon > \xi > 0$, then $x \sim_\xi y$ implies $x \sim_\epsilon y$, so $[x]_\xi \subset [x]_\epsilon$.

Lemma 5.4. *Let X be a compact metric space. X is connected if and only if it is ϵ-connected for all $\epsilon > 0$.*

Proof. Suppose X is connected. Let $\epsilon > 0$ and $x \in X$ be given.

Let $B_\epsilon(y) = \{z \in X | d(y,z) < \epsilon\}$. For all $z \in B_\epsilon(y)$, $y \sim_\epsilon z$. Therefore $y \in [x]_\epsilon$ implies $B_\epsilon(y) \subset [x]_\epsilon$, so $[x]_\epsilon$ is open. Conversely, if $y \notin [x]_\epsilon$, then $B_\epsilon(y) \subset X - [x]_\epsilon$, so $[x]_\epsilon$ is closed.

Since $[x]_\epsilon$ is both open and closed, and X is connected, then $[x]_\epsilon = X$. Since ϵ was arbitrary, this means X is ϵ-connected for all $\epsilon > 0$.

Now suppose X is ϵ-connected for all $\epsilon > 0$. Further suppose that U and V are nonempty open subsets of X such that $U \cup V = X$ and $U \cap V = \emptyset$. Note that $U^c = V$ and $V^c = U$, so U and V are also closed.

For all $u \in U$ and $v \in V$, $u \sim_\epsilon v$ for all ϵ, by hypothesis. In particular, this means that $d(U,V) < \epsilon$ for all $\epsilon > 0$, so $d(U,V) = 0$.

However, U and V are closed subsets of a compact space, so they are both compact. Therefore $d(U,V) = 0$ implies that $U \cap V \neq \emptyset$, which contradicts our choice of U and V. Thus no such U and V exist, which means X is connected. □

Theorem 5.5. *Let $p : E \longrightarrow \mathbf{H}$ be a minimal unique path-lifting fibration, with fibre F. If E is a compact metric space and some subgroup of* $\mathrm{Aut}(p)$ *acts topologically then (E,p) is a pro-cover of* $\mathbf{H}$.

Proof. Let $\mathbf{0}$ denote the point $(0,0) \in \mathbf{H}$ as before, and fix some point $\tilde{\mathbf{0}} \in p^{-1}(\mathbf{0})$. Furthermore, let T be a subgroup of $\mathrm{Aut}(p)$ that acts topologically and let $\tau : F \longrightarrow T$ denote the homeomorphism from F to T.

By Lemma 5.4, F is not ϵ-connected for some ϵ so it is disconnected.

Since F is a compact topological group, by Lemma 2.56 there exists $\delta > 0$ so that

$$d(e,e') < \delta \Rightarrow d(f \cdot e, f \cdot e') < \epsilon/2, \quad \text{for all } f, e, e' \in F,$$

where $f \cdot e = \tau^{-1}(\tau(f)(\tau(e))))$, i.e. the induced group multiplication on F from T.

Let $V = \{f \in F | d(\tilde{\mathbf{0}}, f) < \delta\}$. For $f, f' \in F$, write $f \sim_V f'$ if there exists a finite set of points $f = f_0, \ldots, f_n = f'$ such that $f_j(V) \cap f_{j-1}(V) \neq \emptyset$ for $1 \leq j \leq n$.

The same argument as in the proof of Lemma 5.2 shows that $\sim_V$ is an equivalence relation and that the equivalence class of a point is both open and closed. We denote the $\sim_V$ equivalence class of f by $[f]_V$.

By definition, if $f(V) \cap f'(V) \neq \emptyset$, then there exists $x, y \in V$ such that $f(x) = f'(y)$. Since $d(x, \tilde{\mathbf{0}}), d(y, \tilde{\mathbf{0}}) < \delta$, this implies $d(f(x), f(\tilde{\mathbf{0}})), d(f'(y), f'(\tilde{\mathbf{0}})) < \epsilon/2$.

Therefore $d(f(\tilde{\mathbf{0}}), f'(\tilde{\mathbf{0}})) = d(f, f') < \epsilon$, so $f \sim_V f'$ implies $f \sim_\epsilon f'$.

Since there are at least two $\sim_\epsilon$ equivalence classes of F this implies there are at least two $\sim_V$ equivalence classes of F. Since these classes are open sets and F is compact, there are finitely many equivalence classes.

Now we define a group action of $\mathcal{H}$ on $\sim_V$-equivalence classes.

Suppose γ is a loop in $\mathbf{H}$ based at $\mathbf{0}$. Let $\tilde{\gamma}$ denote the unique lift of γ such that $\tilde{\gamma}(0) = \tilde{\mathbf{0}}$.

Let $f_\gamma = \tau(\tilde{\gamma}(1))$. If γ and γ' are homotopic, then $\tilde{\gamma}(1) = \tilde{\gamma}'(1)$. Therefore f_γ depends only on the homotopy class of γ.

Suppose $e(V) \cap e'(V) \neq \emptyset$. Then $e(x) = e'(y)$ for some $x, y \in V \subset F$. Since $F \cong T$ is a topological group, for all $[\gamma] \in \mathcal{H}$ we have

$$([\gamma](e))(x) = (f_\gamma(e))(x) = f_\gamma(e(x)) = f_\gamma(e'(y)) = (f_\gamma(e'))(y) = ([\gamma](e'))(y).$$

This clearly defines an action of $\mathcal{H}$ on the $\sim_V$ equivalence classes, namely

$$[\gamma] \cdot [f]_V = [f_\gamma f]_V.$$

Let $G = \{g \in \mathcal{H} | g([f]_V) = [f]_V, \text{ for all } f \in F\}$. G is the stabilizer of the action defined previously, so G is a normal subgroup of $\mathcal{H}$. Furthermore, the orbit of any $\sim_V$ equivalence class is finite, since there are only finitely many equivalence classes. Therefore $[\mathcal{H} : G]$ is finite.

For each $f \in F$, $[f]_\epsilon$ is an open subset of F with the subspace topology, so there exists an open subset U_f of E so that $[f]_\epsilon = F \cap U_f$.

Furthermore, since $[f]_\epsilon$ is also closed in F, and F is closed in E since it is the pre-image of a point, then $[f]_\epsilon$ is closed in E.

Since E is a metric space, for each $[f]_\epsilon$ there exists a pair of disjoint open neighborhoods so that $[f]_\epsilon$ is contained in one of these and $F-[f]_\epsilon$ is contained in the other. By intersecting U_f with the appropriate neighborhood, we may assume that $U_f \cap [f']_\epsilon = \emptyset$ if $f \not\sim_V f'$.

Let $W = \cap_{f\in F} p(U_f)$. Since $F \cap U_f \neq \emptyset$ and $p(F) = \mathbf{0}$, clearly $W \neq \emptyset$. Let $W^c = \mathbf{H} - \{\mathbf{0}\}$. Thus $\{W, W^c\}$ is an open cover of $\mathbf{H}$.

Note that W^c contains no non-trivial loops. On the other hand, suppose α is a loop contained in W with $\alpha(0) = \mathbf{0}$. Then for each $f \in F$, there is a unique lift of α, $\tilde{\alpha}_f$ so that $\tilde{\alpha}_f(0) = f$.

However, since α is contained in W, then $\tilde{\alpha}_f$ is contained in U_f. In particular, this means that $\tilde{\alpha}_f(1) \in [f]_\epsilon$, so $[\alpha]$ fixes $[f]_\epsilon$ for all $f \in F$. Hence $[\alpha] \subset G$.

By Lemma 2.42 this means that G is a coverable subgroup. Thus we have a finite normal covering space (C, q) so that $\pi_1(C, \tilde{x}) \cong \pi_1(\mathbf{H}, \mathbf{0})/G$ for some $\tilde{x} \in q^{-1}(\mathbf{0})$.

By hypothesis, F is disconnected, so by Lemma 5.4 F is not ϵ_1-connected for some ϵ_1.

Since F is totally disconnected, $[\tilde{\mathbf{0}}]_{\epsilon_1}$ is also disconnected. Thus $[\tilde{\mathbf{0}}]_{\epsilon_1}$ is not ϵ_2-connected for some $\epsilon_2 > 0$. If $\epsilon_2 \geq \frac{1}{2}\epsilon_1$, then $[\tilde{\mathbf{0}}]_{\epsilon_1}$ is also not $\frac{1}{2}\epsilon_1$-connected. Therefore we may assume $\epsilon_2 \leq \frac{1}{2}\epsilon_1$.

For $i > 2$, we define ϵ_i similarly, so $[\tilde{\mathbf{0}}]_{\epsilon_{i-1}}$ is not ϵ_i-connected and $\epsilon_i \leq \frac{1}{2}\epsilon_{i-1}$. Then for each ϵ_i we construct a normal finite covering spaces of $\mathbf{H}$, (C_i, q_i) as above.

Furthermore, if G_i represents that finite index, normal coverable subgroup used to construct C_i, then for all $[\alpha] \in G_i$, $[\alpha]$ fixes $[f]_{V_i} \subset [f]_{V_{i-1}}$ for all $f \in F$. Therefore $[\alpha]$ fixes $[f]_{V_{i-1}}$, so $G_i \subset G_{i-1}$. Hence we may assume that (C_i, ψ_i) is a finite normal covering space of C_{i-1} for some ψ_i for each i. Thus (C_i, q_i, ψ_i) is a finite normal covering sequence of $\mathbf{H}$.

We now show that this F.N.C.S. yields a pro-cover homeomorphic to (E, p).

Let $(C, (\Psi_i)) = \varprojlim_{\psi_i} C_i$ and let $q : C \longrightarrow \mathbf{H}$ be the induced fibration over $\mathbf{H}$. Also let $\tilde{\mathbf{0}}' = (\tilde{\mathbf{0}}'_i) \in \varprojlim_{q_i} C_i$, where $\tilde{\mathbf{0}}'_i \in q_i^{-1}(\mathbf{0})$ for each i.

Suppose $\tilde{x} \in E$. Since (E, p) is a minimal fibration, for $\epsilon > 0$ there exists a point $\tilde{x}_\epsilon$ so that $d(\tilde{x}, \tilde{x}_\epsilon) < \epsilon$ and there exists a path ζ_ϵ from $\mathbf{0}$ to $\tilde{x}_\epsilon$.

Note that since (ϵ_i) goes to 0, if ζ_{ϵ_i} is not the constant path, then ζ_{ϵ_i} is not contained in some $[\mathbf{0}]_{\epsilon_i}$. In particular, this means that for sufficiently large i, $p \circ \zeta_{\epsilon_i}$ does not lift to a loop in C_i. Let $\sigma_i(\tilde{x}_{\epsilon_i}) = \tilde{\zeta}_{\epsilon_i}(1)$, where $\tilde{\zeta}_{\epsilon_i}$ is the unique lift of $p \circ \zeta_{\epsilon_i}$ based at $\tilde{\mathbf{0}}_i$.

Since $\tilde{\zeta}_{\epsilon_i}$ is not a loop, $\sigma_i(\tilde{x}_{\epsilon_i}) \neq \tilde{\mathbf{0}}_i$. Furthermore, $\psi_i(\sigma_{i+1}(\tilde{x}_{\epsilon_{i+1}})) = \sigma_i(\tilde{x}_{\epsilon_i})$ since $\psi_i \circ \tilde{\zeta}_{\epsilon_{i+1}}$ is also a lift of $p \circ \zeta_{\epsilon_i}$ based at $\tilde{\mathbf{0}}_i$, and thus equal to $\tilde{\zeta}_{\epsilon_i}$.

Therefore we have a map from a dense leaf of E to each C_i that commutes with ψ_i. Since C_i is compact, we may extend this to a map from E to C_i that commutes with ψ_i. Thus we obtain a map $\sigma : E \longrightarrow C$. Furthermore, if $\tilde{x} \neq \tilde{y} \in E$, then for some n, $\tilde{x}_{\epsilon_i} \neq \tilde{y}_{\epsilon_i}$ for all $i > n$. Hence $\sigma(\tilde{x}) \neq \sigma(\tilde{y})$, so σ is injective.

Conversely, suppose $\tilde{x}' = (\tilde{x}'_i) \in C$. Then since C is minimal, we again have a sequence of points $(\tilde{x}'_{\epsilon_i})$ so that there exists a path ζ'_{ϵ_i} from $\tilde{\mathbf{0}}$ to $\tilde{x}'_{\epsilon_i}$ so that $q \circ \zeta'_{\epsilon_i}$ is a path from $\mathbf{0}$ to $q(\tilde{x}'_{\epsilon_i})$. Furthermore, this path is determined uniquely by the paths $\Psi_i \circ \zeta'_{\epsilon_i}$ in C_i.

Now since (E, p) is a unique path-lifting fibration, we may lift this path to one from $\tilde{\mathbf{0}}$ to some point $\tilde{x}_{\epsilon_i}$. Clearly the process to determine $\sigma(\tilde{x}_{\epsilon_i})$ yields the path $q \circ \zeta'_{\epsilon_i}$ and hence $\sigma(\tilde{x}_{\epsilon_i}) = \tilde{x}'_{\epsilon_i}$.

Now E is compact, so the sequence $(\tilde{x}_{\epsilon_i})$ has a convergent subsequence. Let $\tilde{x}$ be the limit of this subsequence. Since σ is continuous, $\sigma(\tilde{x})$ is also the limit of $\tilde{x}'_{\epsilon_i}$, which is $\tilde{x}'$. Thus σ is onto and hence σ is a homeomorphism since E is compact and C is Hausdorff.

$\square$

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Index