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ABSTRACT

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This dissertation is concerned with the zeros of holomorphic Hecke cusp forms in the space of newforms. We estimate a lower bound for the number of zeros on the imaginary axis and on the vertical line $\mathcal{R}(z) = \frac{1}{2}$ in the upper half plane, both of which are outside the unit circle centered at the origin, and we denote these by δ_1 and δ_2 respectively. Ghosh and Sarnak call those zeros that lie on the rays 'real' including the arc $z = \exp(i\theta)$, $\frac{\pi}{3} \leq \theta \leq \frac{\pi}{2}$, and they showed that a lower bound for the zeros on those geodesic lines is $C \log k$ for all sufficiently large weight k for the level 1 case. We extend their results to the newforms with levels N which are positive integers not divisible by 4 on δ_2, and N which are positive integers on δ_1. On δ_2 we have $C \log k$ zeros if the weight k is sufficiently large and on δ_1 we assume a nonnegativity result on the first negative Hecke eigenvalue and get a conditional result $C \log k$ zeros as the weight k goes to infinity.

The analysis is closely related to the knowledge of Hecke eigenvalues $\lambda_f(n)$. Most importantly it requires Deligne's bound $\lambda_f(n) \ll n^{\epsilon}$ (for every $\epsilon > 0$) with which we look into the proof of Theorem 3.1 in Ghosh and Sarnak [1], and get the same the approximation theorem for any level in Chapter 2. The estimation of zeros on δ_1 also requires a 'good' upper bound for the first negative Hecke eigenvalue for which we investigate an upper bound for central values of Hecke L-functions and a nonnegativity result on those values. Those will be studied in Chapters 3 and 4. In Chapter 5 we estimate lower bounds for the number of zeros on δ_i, $i = 1, 2$.

Keywords: real zeros, newforms

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Contents

Chapter 1. Introduction

In studying a complex valued function it is natural to ask about locations of zeros of the function. In fact one of the most famous unsolved problems in mathematics is the Riemann hypothesis which asserts that all nontrivial zeros of the Riemann zeta function lie on the critical line. Many authors have studied the the locations of zeros of modular functions under various conditions. For the Eisenstein series, it is well known that all zeros of the Eisenstein series $E_k(z)$ in the standard fundamental domain for $\mathrm{SL}_2(\mathbb{Z})$ lie on the lower arc of the fundamental domain; namely, on the unit circle $|z| = 1$. This was first proven for $4 \leq k \leq 26$ in 1960s by Wohlfahrt [2], and Rankin [3] extended the range of values of k for which this holds. Then Rankin and Swinnerton-Dyer [4] proved this result for all weights $k \geq 4$. Rankin [5] obtained the result for certain Poincaré series, which generalize Eisenstein series. Similar results have been proven for various levels by many authors. The general principle of the proof of the results is to approximate the modular forms by an elementary function having the required number of zeros on the arc. Also the zeros of weakly holomorphic modular forms of several levels were studied by Duke [6], Haddock [7], Garthwaite and Jenkins [8] on a natural basis which they constructed.

As a result of the proof of the QUE (Quantum Unique Ergodicity) conjecture by Holowinsky and Soundararajan [9] the zero set $Z(f)$ of a Hecke eigenform f for $\mathrm{SL}_2(\mathbb{Z})$ is equidistributed. That is, for any nice subset S of the fundamental domain $\mathcal{F}$,

$$\frac{Z(f) \cap S}{Z(f)} \to \frac{\mathrm{Area}(S)}{\mathrm{Area}(\mathcal{F})}$$

as the weight $k \to \infty$, where "Area" is the hyperbolic area with $dA = \frac{dx\,dy}{y^2}$.

For these Hecke eigenforms Ghosh and Sarnak [1] estimated the number of zeros on the boundary and the center line of the fundamental domain. We do a similar analysis on the number of zeros of newforms of various levels N. We deal with one case for N not divisible by 4 and the other case for N any positive integer with an additional assumption on the

first nonnegative Hecke eigenvalue. Because the shape of the lower parts of the fundamental domain becomes very complicated as the level gets larger, we are confronted with immediate technical difficulties, which leads to our focus on the two vertical rays rather than the entire boundary.

Let $f \in S_k^{\text{new}}(\Gamma_0(N))$ be a newform with trivial character, i.e., a normalized cusp form which is an eigenform of all the Hecke operators, where k is an even weight. (See §2.1 for precise definitions.) The Fourier expansion of f at infinity takes the form

$$f(z) = \sum_{n=1}^{\infty} a_f(n) e(nz),$$

where $a_f(1) = 1$, and we write $e(x) = \exp(2\pi i x)$. We denote by $\lambda_f(n) n^{\frac{k-1}{2}}$ the Hecke eigenvalues which are related to the Fourier coefficients of $f(z)$:

$$a_f(n) = a_f(1)\lambda_f(n) n^{\frac{k-1}{2}}$$

with $a_f(1) = 1$ and with $\lambda_f(n)$ multiplicative and real, satisfying the well-known Hecke relations

$$\lambda_f(m)\lambda_f(n) = \sum_{\substack{d|(m,n) \\ (d,N)=1}} \lambda_f\left(\frac{mn}{d^2}\right).$$

Deligne's bound, which is well known, states that $|\lambda_f(n)| \leq d(n) \ll n^{\epsilon}$ for all n, where $d(n)$ is the divisor function. (We shall write $f(x) \gg g(x)$ if there exist two constants $M > 0$ and x_0 such that $|f(x)| > M|g(x)|$ for $x \geq x_0$.) It is well known that the field generated by the Fourier coefficients of a newform in the space of cusp forms $S_k(\Gamma_0(N), \chi)$ is real if and only if the Nebentypus character χ is either trivial or quadratic with $\chi(p)a_f(p) = a_f(p)$ for all primes p. In particular, the coefficients of f are all real.

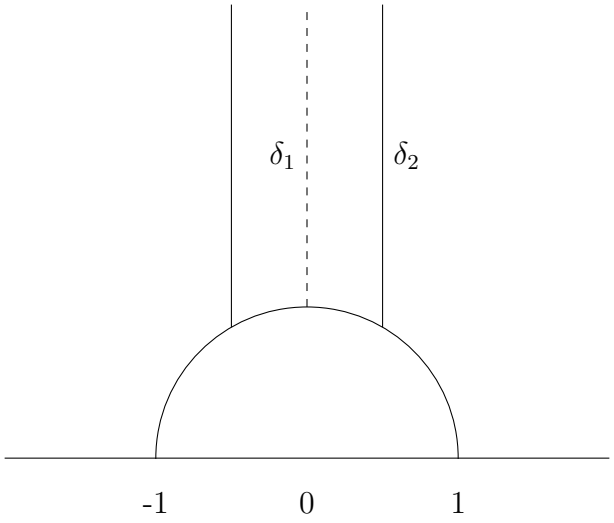

Figure 1.1: Fundamental Domain of $\Gamma_0(1)$

Let δ_1 and δ_2 be two vertical rays on a fundamental domain of $\Gamma_0(N)$ defined by

$$\delta_1 = \{x + yi \mid x = 0, \;\; y \geq 1\},$$
$$\delta_2 = \{x + yi \mid x = \tfrac{1}{2}, \;\; y \geq \tfrac{\sqrt{3}}{2}\}.$$

Then f is a real valued function on the geodesic segments δ_1 and δ_2 since $e(nz)$ are real on δ_1 and δ_2 and the Fourier coefficients are all real. In fact for the full modular group, f is also real on the arc, the lower boundary of the standard fundamental domain, which follows from the relations

$$f(-\overline{z}) = \overline{f(z)}$$

and

$$f(1 - \overline{z}) = \overline{f(z)}.$$

As Ghosh and Sarnak did, we also call the zeros of f on δ_1 and δ_2 the "real" zeros of f. We denote the zero set of f in the fundamental domain of $\Gamma_0(N)$ by $Z(f)$. On these geodesic

lines Ghosh and Sarnak [1] showed that the number of zeros of f goes to infinity as k goes to infinity for the full modular group (level 1 case).

Theorem 1.1. *[1, Theorem 1.4] Let f_k be a Hecke eigenform of even weight k for the full modular group $\mathrm{SL}_2(\mathbb{Z})$. Then the number of zeros on δ_1 and separately δ_2 goes to infinity as $k \to \infty$; we have*

$$|Z(f_k) \cap \delta_j| \gg \log k$$

for $j = 1, 2$.

We extend Theorem 1.1 to newforms with various levels $N \geq 1$. Here we state our main theorems.

Theorem 1.2. *Let k be an even integer and let N be a positive integer. For δ_2 assume $4 \nmid N$. Consider a sequence of newforms $\{f_k\}$ where f_k is in $S_k^{\mathrm{new}}(\Gamma_0(N))$. Then the number of zeros of $f_k(z)$ on δ_2 goes to infinity as k goes to infinity. More quantitatively*

$$|Z(f_k) \cap \delta_2| \gg \log k.$$

For the analysis on δ_1 in [1] Ghosh and Sarnak use subconvexity bounds for Hecke L-functions attached to Hecke eigenforms over the full modular group (see [10] and [11]). They uses the result to prove [1, Proposition 4.4] which is about the first negative Hecke eigenvalue and use it to obtain sign changes on δ_1. For a level $N > 1$ such a strong subconvexity bound is not available yet. Instead we get a conditional result and we have the following theorem on δ_1.

Theorem 1.3. *Let k be an even integer and let N be a positive integer. Consider a sequence of newforms $\{f_k\}$ where f_k is in $S_k^{\mathrm{new}}(\Gamma_0(N))$. Suppose there exist $\epsilon_0 > 0$ and a positive integer N_0 such that if $k \geq N_0$ then there is a power of prime p^r such that $p^r < k^\alpha$, $\alpha < 1/2$ and*

$$\lambda_f(p^r) \leq -\epsilon_0.$$

Then the number of zeros of $f_k(z)$ on δ_2 goes to infinity as k goes to infinity. More quantitatively

$$|Z(f_k) \cap \delta_1| \gg \log k.$$

The proof of the main theorems essentially relies on the approximation theorem in Chapter 2, which basically reduces the problem of counting zeros of the given form to detecting the sign changes of its Hecke eigenvalues. We extend the main approximation theorem by Ghosh and Sarnak [1, Theorem 3.1] to newforms of higher levels in Chapter 2. We can detect sign changes of newforms on δ_2 by considering a natural parity condition on the Fourier coefficients of the form. On the other hand the analysis on δ_1 is intrinsically harder than on δ_2. To get sign changes on δ_1 we need a negative Fourier coefficient that should appear as early as possible. In Chapter 3 we search a first negative Hecke eigenvalue by studying bounds of a sum of Hecke eigenvalues. In Chapter 4 we combine a bound for the cubic moment of central values of Hecke L-functions attached to holomorphic cusp forms and the nonnegativity of those central values to obtain an upper bound for the central values of Hecke L- functions attached to newforms. Lastly, in Chapter 5 we do analysis of counting sign changes of new forms on δ_i, $i = 1,\ 2$.

Chapter 2. Approximation of Hecke Eigenforms in the Space of Newforms

In this chapter we review basic facts from the Atkin-Lehner theory of newforms. After that we derive the main approximation theorem which facilitates counting zeros of a form just by considering normalized Hecke eigenvalues.

2.1 Preliminaries: Revisit Newforms

In this section we recall some important features about the Atkin-Lehner theory of newforms [12] for modular forms with trivial character and Deligne's bound for Fourier coefficients of newforms. For a positive integer N, let $M_k(\Gamma_0(N))$ denote the $\mathbb{C}$-vector space of modular forms of weight k for the congruence group

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \bmod N \right\},$$

and let $S_k(\Gamma_0(N))$ denote the subspace of cusp forms. We denote the complex vector space of modular forms (resp. cusp forms) of weight k with respect to Γ_1 by $M_k(\Gamma_1(N))$ (resp. $S_k(\Gamma_1(N))$) where

$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a \equiv d \equiv 1 \bmod N, \text{ and } c \equiv 0 \bmod N \right\}.$$

Definition 2.1. If χ is a Dirichlet character modulo N, then we say that a form $f(z) \in M_k(\Gamma_1)$ (resp. $S_k(\Gamma_1(N))$) has *Nebentypus character* χ if

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^k f(z)$$

for all $z \in \mathcal{H}$ and all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$. The space of such modular forms (resp. cusp forms) is denoted by $M_k(\Gamma_0(N), \chi)$ (resp. $S_k(\Gamma_0(N), \chi)$). If $\chi = \chi_0$ is trivial, then we denote $M_k(\Gamma_0(N), \chi_0)$ (resp. $S_k(\Gamma_0(N), \chi_0)$) by $M_k(\Gamma_0(N))$ (resp. $S_k(\Gamma_0(N))$).

We recall some important operators on spaces of integer weight modular forms. Recall that

$$\mathrm{GL}_2^+(\mathbb{R}) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R} \text{ and } ad - bc > 0 \right\}$$

acts on functions $f : \mathcal{H} \to \mathbb{C}$ by the operator

$$(f|_k\gamma)(z) = (\det \gamma)^{k/2}(cz + d)^{-k} f(\gamma z),$$

where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{R})$.

Definition 2.2. For a prime divisor p of N with $\mathrm{ord}_p(N) = l$, define the Atkin-Lehner operator $|_k\mathcal{W}_p$ on $M_k(\Gamma_0(N))$ by any matrix

$$\mathcal{W}_p \coloneqq \begin{pmatrix} p^l\alpha & \beta \\ N\gamma & p^l\delta \end{pmatrix} \in \mathrm{M}_2(\mathbb{Z})$$

with determinant p^l, where $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$ (such a choice exists because $(p^l, N/p^l) = 1$ by definition of l, and the operator $\mathcal{W}_p$ are independent of the choices of $\alpha, \beta, \gamma, \delta$ (see [12, Lemma 10])) and the Fricke involution $|_k\mathcal{W}_N$ on $M_k(\Gamma_0(N))$ by the matrix

$$\mathcal{W}_N := \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}.$$

Definition 2.3. Let $q = e^{2\pi iz}$. If d is a positive integer, then the V-operator is defined by

$$\left(\sum_{n=n_0}^{\infty} a(n)q^n \right) \mid V(d) \coloneqq \sum_{n=n_0}^{\infty} a(n)q^{dn}.$$

The U-operator $U(d)$ is defined by

$$\left(\sum_{n=n_0}^{\infty} a(n)q^n\right) \mid U(d) \coloneqq \sum_{n=n_0}^{\infty} a(dn)q^n.$$

The behavior of these operators is described in the following proposition.

Proposition 2.4 (see [28] pg. 28 for a proof)**.** *Suppose that $f(z) \in M_k(\Gamma_0(N), \chi)$ with χ being a Dirichlet character.*

(1) If d is a positive integer, then

$$f(z) \mid V(d) \in M_k(\Gamma_0(dN), \chi).$$

Moreover, if $f(z)$ is a cusp form, then so is $f(z) \mid V(d)$.

(2) If $d \mid N$, then

$$f(z) \mid U(d) \in M_k(\Gamma_0(N), \chi).$$

Moreover, if $f(z)$ is a cusp form, then so is $f(z) \mid U(d)$. And if $d \nmid N$ then simply view $f(z)$ as an element of $M_k(\Gamma_0(dN), \chi)$.

Now suppose that $f(z) \in S_k(\Gamma_0(N))$ and that $d > 1$. Then $f(z) \in S_k(\Gamma_0(dN))$ and $f(dz) = f(z) \mid V(d) \in S_k(\Gamma_0(dN))$. So $f(z)$ sits in $S_k(\Gamma_0(dN))$ in two different ways. One defines "old" forms in $S_k(\Gamma_0(dN))$ by

$$S_k^{\text{old}}(\Gamma_0(N)) \coloneqq \bigoplus_{dM \mid N} S_k(\Gamma_0(M)) \mid V(d),$$

where the sum runs over pairs of positive integers d, M for which $dM \mid N$ and $M \neq N$.

Definition 2.5. Let $f(z)$ and $g(z)$ be cusp forms in $S_k(\Gamma_0(N))$. Their Petersson inner product is defined by

$$\langle f, g \rangle := \frac{1}{[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]} \cdot \int_{\mathcal{F}_N} f(z)\overline{g(z)} y^{k-2}\, dxdy,$$

where $\mathcal{F}_N$ denotes a fundamental domain for the action of $\Gamma_0(N)$ on $\mathcal{H}$ and where $z = x+iy$.

Definition 2.6. Define the space $S_k^{\mathrm{new}}(\Gamma_0(N))$, the subspace of newforms, to be the orthogonal complement of $S_k^{\mathrm{old}}(\Gamma_0(N))$ in $S_k(\Gamma_0(N))$ with respect to the Petersson inner product.

Definition 2.7. A newform in $S_k^{\mathrm{new}}(\Gamma_0(N))$ is a normalized cusp form that is an eigenform of all Hecke operators and all of the Atkin-Lehner involutions $|_k \mathcal{W}_p$, for primes $p \mid N$, and $|_k W(N)$.

The important properties of a newform are captured in the following theorem due to Atkin and Lehner.

Theorem 2.8. *[12, Theorem 3] Suppose that* $f(z) = \sum_{n=1}^{\infty} a_f(n) q^n \in S_k^{\mathrm{new}}(\Gamma_0(N))$ *is a newform. Then the following hold.*

(1) If $p \mid N$*, then there is a* $w_p \in \{\pm 1\}$ *for which*

$$f \mid_k \mathcal{W}_p = w_p f(z).$$

(2) There is an integer $w_N \in \{\pm 1\}$ *for which* $f \mid_k \mathcal{W}_N = w_N f(z)$*. Moreover, we have*

$$w_N = \prod_{p \mid N} w_p.$$

(3) If p *is a prime for which* $p^2 \mid N$*, then* $a_f(p) = 0$*.*

(4) If $p \mid N$ *is a prime for which* $\mathrm{ord}_p(N) = 1$*, then*

$$a_f(p) = -w_p p^{\frac{k-2}{2}}.$$

Note that if $f(z) = \sum_{n=1}^{\infty} a_f(n)q^n \in S_k^{\text{new}}(\Gamma_0(N))$ is a newform, then

$$f \mid U(p) = -p^{\frac{k-2}{2}}(f \mid_k \mathcal{W}_p).$$

The space $S_k^{\text{new}}(\Gamma_0(N))$ has a basis of newforms, and newforms determine distinct Hecke eigenspaces. We add to our basics Deligne's bound for the Fourier coefficients of newforms in the following theorem.

Theorem 2.9. *(Deligne) Let N be any positive integer. If $f(z) = \sum_{n=1}^{\infty} a_f(n)q^n \in S_k^{\text{new}}(\Gamma_0(N))$ is a newform, then*

$$|a_f(n)| \leq d(n)n^{\frac{k-1}{2}}$$

for any $n \geq 1$, where d is the divisor function. Writing $a_f(n) = a_f(1)\lambda_f(n)n^{(k-1)/2}$ with $a_f(n) = 1$, we have

$$|\lambda_f(n)| \leq d(n) \ll n^{\epsilon}$$

for any $\epsilon > 0$.

This was proved by P. Deligne [14] for $k \geq 2$ as a consequence of the Riemann Hypothesis for varieties over finite fields, the Weil conjectures. This bound is essential to extend the approximation theorem from the case of Hecke eigenforms for the full modular group to the case of newforms of arbitrary levels with trivial character.

2.2 Basic Proposition for the Main Approximation Theorem

Following Ghosh and Sarnak [1] let

$$I_s(y) = y^{\frac{s-1}{2}}e^{-y}$$

for $y > 0$ and $s \in \mathbf{C}$. Let $f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{(k-1)/2} e^{2\pi i n z} \in S_k^{\text{new}}(\Gamma_0(N))$ be a newform. Define

$$\Phi_f(s; \alpha, y) = \sum_{n=1}^{\infty} \lambda_f(n) e^{2\pi i n \alpha} I_s(2\pi n y)$$

for any real α. Then we have

$$f(\alpha + iy) = (2\pi y)^{-\frac{k-1}{2}} \Phi_f(k; \alpha, y).$$

We first prove the approximation theorem below.

Theorem 2.10. *Let $f \in S_k^{\text{new}}(\Gamma_0(N))$ be a newform. Let δ be a positive real number. Then there exists a real number N_δ sufficiently large such that for all real $s > N_\delta$, for all y satisfying $\sqrt{s} \ll y < \frac{1}{100} s$, and with $B = \sqrt{\delta s \log s}$, we have*

$$\frac{\Phi_f(s; \alpha, y)}{I_s(s')} = \sum_{\substack{n \\ |2\pi n y - s'| \le B}} \lambda_f(n) e^{2\pi i n \alpha} e^{-|2\pi n y - s'|^2/(2s')} + O(s^{-\delta})$$

where $s' = \frac{s-1}{2}$ and the constant involved only depends on δ.

Ghosh and Sarnak proved the theorem for a Hecke eigenforms for the full modular group $\Gamma_0(1) = SL_2(\mathbb{Z})$, i.e, when the level $N = 1$. Notice that the conclusion of the theorem does not depend on the level N. We follow and examine the proof of Ghosh and Sarnak to verify that the result can be extended to the general case of any congruence group $\Gamma_0(N)$ with N being any positive integer. In the proof the only possible level involvement lies in the estimation of normalized eigenvalues λ_f for which they consistently use Deligne's bound

$$|\lambda_f(n)| \le d(n) \ll n^{\epsilon}.$$

for any $\epsilon > 0$, which is valid for any newform (See 2.9), so that we can avoid the level dependency of the theorem. We will point out whenever the bound appears in the proof.

We study the behavior of the function $I_s(y)$ in the following lemmas.

Lemma 2.11. *[1, Lemma 2.2] For a fixed real $s > 1$, $I_s(y) = y^{\frac{s-1}{2}}e^{-y}$ is strictly increasing for $0 < y < s'$, and strictly decreasing if $y > s'$ where $s' = (s-1)/2$.*

The proof is elementary calculus.

Lemma 2.12. *[1, Lemma 2.3] Let $|h| \ll s^{2/3-\delta}$ for some positive δ sufficiently small. Then*

$$I_s(s'+h) = I_s(s')e^{-\frac{h^2}{2s'}}(1+O(s^{-3\delta})),$$

where $s' = (s-1)/2$.

Proof. We write

$$I_s(s'+h) = e^{-s'-h}(s')^{s'}(1+\frac{h}{s'})^{s'}.$$

Using the Maclaurin series for $\log(1+x)$ we have

$$\log\left(1+\frac{h}{s'}\right)^{s'} = h - \frac{h^2}{2s'} + O\left(\frac{h^3}{s'^2}\right).$$

The result follows after applying the estimation to the last factor of the first equation with $|h| \ll s^{2/3-\delta}$. □

We split the sum Φ_f into three sums using a parameter B which will be determined later to get

$$\Phi_f(s:\alpha,y) = \sum_{i=1}^{3}\Phi_{f,i}(s;\alpha,y)$$

where

$$\Phi_{f,1}(s;\alpha,y) = \sum_{\substack{n\geq 1\\ 2\pi ny-s'<-B}} \lambda_f(n)e^{2\pi in\alpha}I_s(2\pi ny),$$

$$\Phi_{f,2}(s;\alpha,y) = \sum_{\substack{n\geq 1\\ |2\pi ny-s'|\leq B}} \lambda_f(n)e^{2\pi in\alpha}I_s(2\pi ny),$$

$$\Phi_{f,3}(s;\alpha,y) = \sum_{\substack{n\geq 1\\ 2\pi ny-s'>B}} \lambda_f(n)e^{2\pi in\alpha}I_s(2\pi ny).$$

Again Deligne's bound appears in all the estimations of $\Phi_{f,1}$, $\Phi_{f,2}$ and $\Phi_{f,3}$.

Proof of Theorem 2.10. Let $\delta > 0$. We choose $h = 2\pi ny - s'$ in Lemma 2.12, so that $I_s(2\pi ny) = I_s(s'+h)$ and therefore assume that $1 \leq B \ll s^{2/3-\delta}$. We first estimate $\Phi_{f,3}(s;\alpha,y)$. Note that $I_s(t) = t^{(s-s_1)/2}I_{s_1}(t)$, and $|\lambda_f(n)| \ll n^\epsilon$ for any $\epsilon > 0$. Then for sufficiently large s and letting $s_1 = s + 2\epsilon$, we have

$$\begin{aligned}\Phi_{f,3}(s;\alpha,y) &= \sum_{\substack{n\geq 1\\ 2\pi ny-s'>B}} \lambda_f(n)e^{2\pi in\alpha}(2\pi ny)^{(s-(s+2\epsilon))/2}I_{s+2\epsilon}(2\pi ny)\\ &\ll y^{-\epsilon}\sum_{n>\frac{s'+B}{2\pi y}} I_{s+2\epsilon}(2\pi ny).\end{aligned}$$

Since $2\pi ny > s' + B = \frac{s-1}{2} + B$, by Lemma 2.11, $I_{s+2\epsilon}(2\pi ny)$ is decreasing in the sum. Now we approximate the sum as an integral:

$$\begin{aligned}\Phi_{f,3}(s;\alpha,y) &\ll y^{-\epsilon}\left(\int_{\frac{s'+B}{2\pi y}}^{\infty}(2\pi yt)^{s'+\epsilon}e^{-2\pi yt}dt + I_{s+2\epsilon}(s'+B)\right)\\ &\ll y^{-\epsilon}\left(\frac{1}{y}\Gamma(s'+\epsilon+1, s'+B) + I_{s+2\epsilon(s'+B)}\right) \qquad (2.1)\end{aligned}$$

where

$$\Gamma(s,x) = \int_x^\infty t^{s-1}e^{-t}\,dt$$

is the incomplete gamma function.

For the estimation of

$$\frac{\Phi_{f,3}(s;\alpha,y)}{I_s(s')}$$

we first observe that

$$\begin{aligned}\frac{I_{s+2\epsilon(s'+B)}}{I_s(s')} &= e^{-B}(1+\frac{B}{s'})^{s'}(s'+B)^{\epsilon}\\ &\ll e^{-B}(1+\frac{B}{s'})^{s'}(s)^{\epsilon}.\end{aligned}$$

Using

$$\log\left(e^{-B}\left(1+\frac{B}{s'}\right)^{s'}\right) = -\frac{B^2}{2s'}+O\left(\frac{B^3}{s'^2}\right)$$

we get

$$I_{s+2\epsilon}(s'+B) \ll I_s(s')s^{\epsilon}e^{-B^2/(2s')}. \tag{2.2}$$

For the estimation of the incomplete Gamma function in equation (2.1) we use the inequality of Natalini-Palumbo [15]:

Lemma 2.13. *[1, Lemma 2.4] If $a>1, b>1$, and $x>\frac{b}{b-1}(a-1)$, one has*

$$x^{a-1}e^{-x} < |\Gamma(a,x)| < bx^{a-1}e^{-x}.$$

In the lemma with $a=s'+\epsilon+1$ and $b=1+\epsilon+s'/B$ we have

$$\begin{aligned}\Gamma(s'+\epsilon+1,s'+B) &\ll \frac{s'}{B}(s'+B)^{s'+\epsilon}e^{-s'-B}\\ &\ll \frac{s'}{B}\left(1+\frac{B}{s'}\right)e^{-B}I_s(s')s^{\epsilon}.\\ &\ll \frac{s'}{B}e^{-B^2/(2s')}I_s(s')s^{\epsilon}.\end{aligned} \tag{2.3}$$

Combining the estimates of (2.1), (2.2) and (2.3) we estimate

$$\Phi_{f,3}(s;\alpha,y) \ll y^{-\epsilon}\left(\frac{s'}{By}+1\right)e^{-B^2/(2s')}I_s(s')s^{\epsilon}.$$

Now we add the lower bound for B

$$\sqrt{\delta s\log s}\leq B\ll s^{2/3-\delta}$$

and take $y\gg\sqrt{s}$ to derive that

$$\Phi_{f,3}(s;\alpha,y)\ll I_s(s')s^{-\delta/2}. \tag{2.4}$$

For the estimation of $\Phi_{f,1}(s;\alpha,y)$ we can replace $\lambda_f(n)$ with s^{ϵ} using Deligne's bound $\lambda_f(n)\ll n^{\epsilon}$ and the upper bound s for n's in the sum. Note also that $I_s(2\pi ny)$ is strictly increasing in the associated interval, so that we approximate the sum by the integral

$$\Phi_{f,1}(s;\alpha,y)\ll s^{\epsilon}\left(\int_1^{\frac{s'-B}{2\pi y}}(2\pi yt)^{s'}e^{-2\pi yt}\,dt+I_s(2\pi y)+I_s(s'-B)\right). \tag{2.5}$$

Using the definition of $I_s(y)$ we see that

$$\frac{I_s(2\pi y)}{I_s(s')}\ll\left(\frac{2\pi ye}{s'}\right)^{s'}e^{-2\pi y}$$

which is decreasing exponentially in s if $\frac{2\pi ye}{s'}<1$ and it indeed is for $y<s/100$. For the estimation with the last term in (2.5) we use inequality (2.2) and get

$$\frac{I_s(s'-B)}{I_s(s')}\ll e^{-B^2/(2s')}.$$

Now we split the integral part in (2.5) into two pieces with $B_1 > B$ and substituting t by $\frac{s't}{2\pi y}$ to get

$$\begin{aligned}&\int_1^{\frac{s'-B_1}{2\pi y}} (2\pi yt)^{s'} e^{-2\pi yt}\, dt + \int_{\frac{s'-B_1}{2\pi y}}^{\frac{s'-B}{2\pi y}} (2\pi yt)^{s'} e^{-2\pi yt}\, dt \\ &= \frac{s'}{2\pi y} I_s(s') \left(\int_{2\pi y/s'}^{1-B_1/s'} \left[te^{(1-t)}\right]^{s'}\, dt + \int_{1-B_1/s'}^{1-B/s'} \left[te^{(1-t)}\right]^{s'}\, dt \right). \end{aligned} \tag{2.6}$$

The integrand $\left[te^{(1-t)}\right]^{s'}$ is strictly increasing on the involved interval, so the second integral is bounded above by

$$\frac{B_1 - B}{s'} \left(1 - \frac{B}{s'}\right)^{s'} e^B \ll \frac{B_1 - B}{s'} e^{-B^2/(2s')}$$

where the second inequality follows by applying the power series estimation of $\log\left(1 - \frac{B}{s'}\right)^{s'} e^B = s' \log\left(1 - \frac{B}{s'}\right) + B$.

Similarly the first integral is bounded by

$$\begin{aligned}\int_{2\pi y/s'}^{1-B_1/s'} \left[te^{(1-t)}\right]^{s'}\, dt &\ll \left(1 - \frac{B_1}{s'}\right)^{s'} e^{B_1} \\ &= \frac{I_s(s' - B_1)}{I_s(s')} \\ &\ll e^{-B_1^2/(2s')}\end{aligned}$$

provided that $B_1 \ll s^{2/3-\epsilon}$ by Lemma 2.12. By combining these estimations of integrals in (2.6), the integral in (2.5) is

$$\ll \frac{s'}{y} I_s(s') \left(e^{-B_1^2/(2s')} + \frac{B_1 - B}{s'} e^{-B^2/(2s')} \right). \tag{2.7}$$

Now given δ sufficiently small we choose

$$B = \sqrt{\delta s \log s},\ B_1 = \sqrt{\frac{1}{\delta} s \log s}$$

Hence (2.7) is

$$\begin{aligned}
&\ll \frac{s'}{y} I_s(s') \left(e^{-\frac{s\log s}{\delta(s-1)}} + \frac{\sqrt{s\log s}}{s'} e^{-\frac{\delta s\log s}{s-1}} \right) \\
&\ll \frac{s'}{y} I_s(s') \left(s^{-1/\delta} + \frac{\sqrt{s\log s}}{s'} s^{-\delta} \right) \\
&\ll I_s(s') \frac{\sqrt{s\log s}}{y} s^{-\delta} \\
&\ll I_s(s') s^{-\frac{1}{2}\delta}
\end{aligned}$$

where the last inequality follows if $y > \sqrt{s}$. Applying all these estimates to (2.5) we get

$$\frac{\Phi_{f,1}(s;\alpha,y)}{I_s(s')} \ll s^{\epsilon} \left(\left(\frac{2\pi y e}{s'} \right)^{s'} e^{-2\pi y} + e^{-B^2/(2s')} + s^{-\frac{1}{2}\delta} \right) \ll s^{-\frac{1}{3}\delta} \tag{2.8}$$

where $\sqrt{s} \ll y < \frac{s}{100}$.

Finally, we take $h = 2\pi ny - s$ in Lemma 2.12 and apply it to $\Phi_{f,2}(s;\alpha,y)$ to get

$$\begin{aligned}
\Phi_{f,2}(s;\alpha,y) &= \sum_{\substack{n\geq 1 \\ |2\pi ny - s'| \leq B}} \lambda_f(n) e^{2\pi i n\alpha} I_s(2\pi ny) \\
&= \sum_{\substack{n\geq 1 \\ |2\pi ny - s'| \leq B}} \lambda_f(n) e^{2\pi i n\alpha} I_s(s') e^{-|2\pi ny - s'|^2/(2s')} \left(1 + O(s^{-3\delta})\right). \qquad (2.9)
\end{aligned}$$

Again using Deligne's bound we replace $\lambda_f(n)$ with s^{ϵ}, so the error term involved in the sum is estimated as follows:

$$\begin{aligned}
&\sum_{\substack{n\geq 1 \\ |2\pi ny - s'| \leq B}} \lambda_f(n) e^{2\pi i n\alpha} I_s(s') e^{-|2\pi ny - s'|^2/(2s')} O(s^{-3\delta}) \\
&\ll \sum_{\substack{n\geq 1 \\ |2\pi ny - s'| \leq B}} s^{\epsilon} I_s(s') s^{-3\delta}
\end{aligned}$$

$$\ll s^{\epsilon} s^{-3\delta} I_s(s') \sum_{\substack{n\geq 1 \\ |2\pi ny - s'|\leq B}} 1$$
$$\ll I_s(s') \left(1 + \frac{B}{y}\right) s^{-2\delta}$$
$$\ll I_s(s') s^{-\delta}. \tag{2.10}$$

In summary, from (2.4), (2.8), (2.9) and (2.10) we have

$$\Phi_{f,3}(s;\alpha,y) \ll I_s(s') s^{-\delta/2} \qquad \text{if } B = \sqrt{\delta s \log s} \text{ and } \sqrt{s} \ll y,$$
$$\Phi_{f,1}(s;\alpha,y) \ll I_s(s') s^{-\frac{1}{3}\delta} \qquad \text{if } \sqrt{s} < y, \text{ and}$$
$$\Phi_{f,2}(s;\alpha,y) = \sum_{\substack{n\geq 1 \\ |2\pi ny - s'|\leq B}} \lambda_f(n) e^{2\pi i n\alpha} I_s(s') e^{-|2\pi ny - s'|^2/(2s')} + O(I_s(s') s^{-\frac{1}{2}\delta}).$$

With all these approximations combined, the proof is complete. Note that the main term in the statement of Theorem 2.10 comes from the main term of $\Phi_{f,2}(s;\alpha,y)$. □

2.3 Main Approximation Theorem

Now we state and prove the main approximation theorem.

Theorem 2.14. *Let $f \in S_k^{\text{new}}(\Gamma_0(N))$ be a newform. Given $\delta > 0$ there exist positive numbers r_1 and r_2 such that for all integers l satisfying $r_1 < l < r_2\sqrt{\frac{s}{\log s}}$, and for all s sufficiently large, the numbers $y_{l,s} = \frac{s-1}{4\pi l}$ satisfy*

$$\frac{\Phi_f(s;\alpha,y_{l,s})}{I_s\left(\frac{s-1}{2}\right)} = \lambda_f(l) e^{2\pi i\alpha l} + O(s^{-\delta}) \tag{2.11}$$

uniformly for any real number α, where the involved constant only depends on δ.

Again Theorem 2.14 does not depend on the level N. The proof is exactly the same as [1, Theorem 3.1]. We added a detail here.

Proof. Let $y = \frac{s'}{2\pi l}, s' = \frac{s-1}{2}$ with l being a positive integer. Then the bound for n in the sum of Theorem 2.10 is

$$|2\pi ny - s'| \leq B \Leftrightarrow |n - l| \leq B\frac{l}{s'}$$

so that we must have $n = l$ if $l < \frac{s'}{B}$. In this case notice that $2\pi ny - s' = 2\pi ly - s' = 0$. Now observe the condition

$$l < \frac{s'}{B} = \frac{s-1}{2}\frac{1}{\sqrt{\delta s \log s}} < \frac{1}{\sqrt{\delta}}\sqrt{s/\log s}.$$

The condition that $\sqrt{s} \ll y < s/100$ in Theorem 2.10 is translated in terms of l to

$$\left(\frac{100}{4\pi}\right)\frac{s-1}{s} < l \ll \sqrt{s}.$$

Therefore given $\delta > 0$ we can take $r_1 = \frac{100}{2\cdot 4\pi}$ and $r_2 = \frac{1}{\sqrt{\delta}}$. Then for all sufficiently large s such that $s > N_\delta$ in Theorem 2.10, s satisfies

$$l < \frac{1}{\sqrt{\delta}}\sqrt{s/\log s} \ll \sqrt{s}.$$

Hence we necessarily have

$$\frac{\phi_f(s;\alpha,y_{l,s})}{I_s(s')} = \lambda_f(l)e^{2\pi i\alpha l} + O(s^{-\delta})$$

uniformly for any real number α. □

Remark. In Theorem 2.14 if we take $s = k$ an integer, $\alpha = 0$ or $\alpha = 1/2$ then as a function of y, Φ_f is also a real valued function by definition and assuming that the condition of the theorem is satisfied, the main term in 2.11 is

$$\pm\lambda_f(l)$$

since $e^{2\pi i\alpha l} = \pm 1$. Hence the sign of Φ_f and therefore the sign of $f(\alpha + iy)$ is essentially determined by the sign of $\lambda_f(l)$ if its absolute values are bigger than the error term in 2.11. In fact we can make the error term negligible by taking sufficiently large k.

Chapter 3. The First Negative Eigenvalues of Hecke Eigenforms in the Space of Newforms

Again we suppose that $f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{(k-1)/2} e^{2\pi i n z} \in S_k^{\text{new}}(\Gamma_0(N))$ is a newform. The goal of this chapter is to find an upper bound for the first sign change of Hecke eigenvalues. We first state two previous results. Ghosh and Sarnak [1] proved the following theorem.

Theorem 3.1. *[1, Propostion 4.4] Let f be a Hecke eigenform in $S_k^{\text{new}}(\Gamma_0(1))$, where k is an even weight. Then there exists $\epsilon_0 > 0$ such that for all sufficiently large k there is a positive integer n such that $n < k^{0.4963}$ and*

$$\lambda_f(n) \leq -\epsilon_0.$$

Let n_f denote the smallest positive integer such that $\lambda_f(n_f) < 0$ and n_f is relatively prime to N. Matomäki [16] bounds the size of n_f in terms of the analytic conductor $Q = k^2 N$ in the following theorem.

Theorem 3.2. *[16, Theorem 1] Let $k \geq 2$ be an even integer and let N be any positive integer. Then for all newforms in $S_k^{\text{new}}(\Gamma_0(N))$, one has*

$$n_f \ll (k^2 N)^{3/8} = k^{3/4} N^{3/8}$$

where the implied constant is absolute.

Remark. In Theorem 3.1, $n_f < k^{0.4963}$ if k is sufficiently large. In the aspect of upper bound for n_f in terms of weight k the result of Ghosh and Sarnak is better than that of Matomäki and stronger in the sense that the first negative eigenvalue is smaller than some fixed negative number for all sufficiently large weight k.

We combine and modify the proofs of Theorem 3.1 and Theorem 3.2 to prove Theorem 3.6.

3.1 A Lower Bound for a Sum of Normalized Hecke Eigenvalues (Proof of Proposition 3.3)

For a newform $f \in S_k^{\mathrm{new}}(\Gamma_0)$, we begin with adopting the same setting as Matomäki [16]. Let

$$S(f,x) = \sum_{\substack{n\le x\\(n,N)=1}}^{\flat} \lambda_f(n)$$

where the symbol $\flat$ indicates that the sum is over the square-free integers. Let y represent a positive real number.

Proposition 3.3. *There is $\tilde{\epsilon} > 0$ such that for any $\epsilon > 0$ with $\tilde{\epsilon} > \epsilon > 0$, if $\lambda_f(n) \ge -\epsilon$ for all $n \le y$ then*

$$S(f, y^\kappa) \gg y^\kappa$$

for all y, where $\kappa = 1.343461$ and the implied constant depends only on ϵ.

We follow Ghosh and Sarnak for the construction of an auxiliary multiplicative function and modified the proof of [1, Proposition 4] and do estimation on it using Matomäki's proof of [16, Theorem 1].

Here the number $\kappa = 1.343461$ will play an important role later. We first prove several lemmas toward proving Proposition 3.3. Let ϵ, y be positive numbers and fix $K = 100$ and assume that $\lambda(p^k) \ge -\epsilon$ for all prime powers $p^k \le y$. Then by Deligne's bound there exists an angle θ_p such that

$$\lambda_f(p) = \cos\theta_p, \quad \lambda_f(p^m) = \frac{\sin(m+1)\theta_p}{\sin\theta_p} \ge -\epsilon$$

for all $m \le K$ (See the proof of Lemma 5.2). This implies that $\theta_p \le \pi/(m+1)$ we have

$$\lambda_f(p) \ge 2\cos\left(\frac{\pi}{m+1}\right) - C\epsilon$$

for $p \le y^{1/m}$ with $p \nmid N$, where C is a constant depending only on K. We now define

a multiplicative function $h_y(n)$ supported on square-free positive integers with $h_y(p) = \gamma(\log p/\log y)$ on prime numbers, where

$$\gamma(t) = \begin{cases} -2 & \text{if } t \geq 1, \\ 2\cos\left(\frac{\pi}{m+1}\right) - C\epsilon & \text{if } \frac{1}{m+1} \leq t < \frac{1}{m},\ 1 \leq m < M, \\ 2\cos\left(\frac{\pi}{M+1}\right) - C\epsilon & \text{if } t < \frac{1}{M}. \end{cases} \tag{3.1}$$

The required lower bound for $S(f, y^\kappa)$ can be obtained by evaluating the mean value of $h_y(n)$. Matomäki [16] studied the sum

$$\sum_{\substack{n \leq x \\ (n,N)=1}} h(n)$$

with $h(p) = \chi(\log p/\log x)$, where χ is a function on nonnegative numbers into the set of all real numbers such that

$$\chi(t) = \chi_k \quad \text{if} \quad x_k \leq t < x_{k+1}$$

where $0 = x_0 < x_1 < \cdots < x_{K+1} = \infty$, $\chi_k \in \mathbb{R}$ for $k = 0, 1, ..., K$. Let Γ be the Euler gamma function and $\Pi_{q,\kappa}$ be such that

$$\Pi_{q,\kappa} = \left(\frac{\phi(q)}{q}\right)^\kappa \prod_{p \nmid q} \left(1 - \frac{1}{q}\right)^\kappa \left(1 + \frac{\kappa}{p}\right) \tag{3.2}$$

where ϕ is the Euler totient function.

Lemma 3.4. *[16, Lemma 6] Let $U \geq 1$ and let $h(n)$ and $\chi(t)$ be as above with $\chi_0 > 0$. Let further $q \leq x^U$ be a positive integer. Then*

$$\sum_{\substack{n \leq x^u \\ (n,q)=1}} h(n) = (\sigma(u) + o_{\chi,U}(1)) \frac{\Pi_{q,\chi_0}}{\Gamma(\chi_0)} (\log x)^{\chi_0 - 1} x^u$$

uniformly for $u \in [1/U, U]$, where

$$\sigma(u) = u^{\chi_0 - 1} + \sum_{j=1}^{\infty} \frac{(-1)^j}{j!} I_j(u)$$

with

$$I_j(u) = \int_{\Delta_j} (u - t_1 - \cdots - t_j)^{\chi_0 - 1} \prod_{i=1}^{j} (\chi_0 - \chi(t_j)) \frac{dt_1 \cdots dt_j}{t_1 \cdots t_j},$$

integration ranging over the set

$$\Delta_j = \{(t_1, ..., t_j) \in [0, \infty)^j \mid t_1 + \cdots + t_j \leq u\}.$$

The next lemma will allow us to estimate $\sigma(u)$ in the previous lemma effectively. It is in fact a solution of an integral equation.

Lemma 3.5. *[16, Lemma 8] The function $\sigma(u)$ in Lemma 3.4 is the unique solution of the integral equation*

$$u\sigma(u) = \int_0^u \sigma(t)\chi(u - t)dt$$

with the initial condition $\sigma(u) = u^{\chi_0 - 1}$ for $u \in (0, x_1]$.

It is also the unique continuous solution of the differential-difference equation (with the same initial condition)

$$(u^{1-\chi_0}\sigma(u))' = -\frac{1}{u^{\chi_0}} \sum_{k=1}^{K'} \sigma(u - x_k)(\chi_{k-1} - \chi_k) \ \ \textit{when} \ \ u \notin \{x_2, x_3, \cdots, x_K\} \tag{3.3}$$

where $K' \leq K$ is such that $x_{K'} < u$ but $x_{K'+1} > u$.

We integrate both sides of equation (3.3) from $u - \delta$ to u and keep track of the derivatives on the intervals (x_i, x_{i+1}), $i = 1, 2, ..., K$. Then we get

$$u^{1-\chi_0}\sigma(u) = (u - \delta)^{1-\chi_0}\sigma(u - \delta) - \sum_{k=1}^{K'} (\chi_{k-1} - \chi_k) \int_{\max\{u-\delta, x_k\}}^{u} t^{-\chi_0}\sigma(t - x_k)dt \tag{3.4}$$

for any $u > x_1 + \delta, \ \delta > 0$. We write $\alpha(u) = u^{1-\chi_0}\sigma(u)$. If u_0 is the first zero of $\alpha(u)$ then

the function α is decreasing by (3.3) so that we may apply rectangular approximation to the integral in (3.4) and get

$$\alpha(u-\delta)-\sum_{k=1}^{K'}\delta_k(\chi_{k-1}-\chi_k)(u-\delta_k)^{-\chi_0}(u-x_k)^{\chi_0-1}\alpha(u-\delta_k-x_k)$$
$$\leq \alpha(u)\leq \alpha(u-\delta)-\sum_{k=1}^{K'}\delta_k(\chi_{k-1}-\chi_k)(u)^{-\chi_0}(u-\delta_k-x_k)^{\chi_0-1}\alpha(u-x_k),$$

where $\delta_k=\min\{\delta,u-x_k\}$. Starting with $\alpha(x_1)=1$ and $\delta>0$ sufficiently small we can estimate $\alpha(x_1+l\delta)$ recursively for $l=1,2,....$ Now we take $K=M=100$, $\delta=1/(10000\sqrt{73})$, $\chi_0=2\cos(\pi/(K+1))$,

$$x_k=\frac{1}{K-k+1}\ \text{ and }\ \chi_k=2\cos\left(\frac{\pi}{K-k+1}\right)$$

for $k=1,2,...,K$. Using a Sage program [17] (see appendix) we verified that $\alpha(1.343461)>0$. Hence we obtain $\sigma(1.343461)>0$, which is also true for the case $\chi=\gamma$ in (3.1) with the same choices $K=100$ and $\delta=1/(10000\sqrt{73})$ as above by taking sufficiently small $\epsilon>0$ because the recursive estimation just takes finitely many steps. (Notice that γ contains the term $C\epsilon$.) Fix $\tilde{\epsilon}>0$ small enough to guarantee $\sigma(1.343461)>0$ for all $0<\epsilon\leq\tilde{\epsilon}$. Letting $M=100$ in (3.1) we have

$$h_y(p)=\begin{cases}-2 & \text{if } p\geq y,\\ 2\cos\left(\frac{\pi}{m+1}\right)-C\epsilon & \text{if } y^{\frac{1}{m+1}}\leq p<y^{\frac{1}{m}},\ 1\leq m<100,\\ 2\cos\left(\frac{\pi}{101}\right)-C\epsilon & \text{if } p<y^{\frac{1}{100}}.\end{cases}\tag{3.5}$$

We define g_y to be the multiplicative function given by the Dirichlet convolution

$$\lambda_f=g_y*h_y.$$

Then $g_y(n) \geq 0$ for all square free integers $n \geq 1$ such that $(n, N) = 1$, since

$$g_y(p) = \lambda_f(p) - h_y(p) \geq 0$$

by construction for all $p \nmid N$ and the case $p \geq y$ follows from Deligne's inequality $|\lambda_f(n)| \leq d(n)$. (In particular, $|\lambda_f(p)| \leq 2$ for primes p.) Notice that in (3.5) if $m = 1$ and $y^{1/2} \leq p < y$ then $h_y(p) = -C\epsilon < 0$. But since $2\cos(\pi/(m+1)) \geq 1$ for $m \geq 2$, we can take sufficiently smaller $\epsilon > 0$ than the previous one if necessary so that $h_y(l) \geq 0$ if $l \leq y^{1/3}$. Then $\sum_{l \leq z}^{\flat} h_y(l)$ is nonnegative if $z \leq y^{1/3}$. On the other hand for $z > y^{1/3}$, we can get the same result by taking $U = 3$ in Lemma 3.4 since for all sufficiently large y

$$\sum_{l \leq y^u}^{\flat} h_y(l) = (\sigma(u) + o_{\chi,U}(1)) \frac{\Pi_{q,\chi_0}}{\Gamma(\chi_0)} (\log y)^{\chi_0 - 1} y^u$$

uniformly for $u \in [1/3, 3]$ (see (3.2) for Π_{q,χ_0}), and in particular since $\sigma(\kappa) > 0$ ($\kappa = 1.343461$), if y is sufficiently large

$$\sum_{l \leq y^u}^{\flat} h_y(l) \geq cy^u$$

for all $u \in [1/3, \kappa]$ for some positive constant c. Hence

$$S(f, y^\kappa) = \sum_{\substack{n \leq y^\kappa \\ (n,N)=1}}^{\flat} \lambda_f(n) = \sum_{d \leq y^\kappa}^{\flat} \left(\sum_{l \leq y^\kappa/d}^{\flat} h_y(l) \right) g_y(d) \geq \sum_{l \leq y^\kappa}^{\flat} h_y(l) \geq cy^\kappa$$

since $g_y(d) \geq 0$ for every d and $g_y(1) = 1$.

3.2 An Upper Bound for a Sum of Normalized Hecke Eigenvalues and the First Negative Hecke Eigenvalue

Now we prove Theorem 3.6 about the first negative Hecke eigenvalue.

Theorem 3.6. *Let f be a newform in $S_k^{\text{new}}(\Gamma_0(N))$, where k is an even weight and N is a positive integer. Then there exist $\epsilon_0 > 0$ and a positive integer N_0 such that if $k \geq N_0$ then there is a power of prime p^r such that $p^r < k^{0.7444} N^{0.3722}$ and*

$$\lambda_f(p^r) \leq -\epsilon_0.$$

Proof. By Proposition 3.3 if $\lambda_f(n) \geq -\epsilon$ for all $n < y$ then

$$S(f, y^\kappa) \gg y^\kappa$$

where $\kappa = 1.343461$ and the implied constant only depends on ϵ. On the other hand an upper bound can be achieved using the convexity bound for Hecke L-functions and the Perron formula (see [18, Section 2] and [19, Theorem 2.3] respectively) and we have

$$S(f, x) \ll (k^2 N)^{1/4+\epsilon} x^{1/2+\epsilon}$$

for all $x \geq 1$, where the implied constant only depends on ϵ. In particular, for $x = y^\kappa$ we obtain

$$S(f, y^\kappa) \ll (k^2 N)^{1/4+\epsilon} y^{\kappa/2+\epsilon}.$$

Putting together the lower bound and the upper bound for $S(f, y^\kappa)$ we have

$$y^\kappa \ll S(f, y^\kappa) \ll (k^2 N)^{1/4+\epsilon} y^{\kappa/2+\epsilon}. \tag{3.6}$$

Observe that the lower bound grows faster than the upper bound as y goes to ∞. So we fix $\epsilon = \epsilon_0 > 0$ smaller than $\tilde{\epsilon}$ in Proposition 3.3 and take k sufficiently large so that inequalities (3.6) hold first and if necessary take k even larger so that

$$y^\kappa \leq S(f, y^\kappa) \leq (k^2 N)^{1/4+\epsilon} y^{\kappa/2+\epsilon}$$

and then get a contradiction by letting $y \to \infty$. Therefore a contradiction arises when

$$y^{\kappa} \geq (k^2 N)^{1/4+\epsilon} y^{\kappa/2+\epsilon},$$

or equivalently when

$$y \geq (k^2 N)^{\frac{1+4\epsilon}{2\kappa-4\epsilon}}$$

where $1/2\kappa \approx 0.37217$. Thus if there were no negative eigenvalues up to y with $y > k^{0.7444} N^{0.3722}$, this is a contradiction. Recall that for the estimation of the lower bound for $S(f, y^{\kappa})$ we just need to assume that $\lambda_f(p^k) \geq 0$ for all $p^k \leq y$, where p is any prime number with $p \nmid N$. For the upper bound estimates there is no restriction. Therefore for sufficiently large weight k there exists a prime power $p^r < k^{0.7444} N^{0.3722}$ such that $\lambda_f(p^r) \leq \epsilon_0$. $\square$

Chapter 4. An Upper Bound for the Central Values of Hecke L-functions Attached to Newforms in $S_k^{\text{new}}(\Gamma_0)$ and Their Nonnegativity

The goal of this chapter is to prove the nonnegativity of the central values Hecke L-functions associated with the newforms (even integer weight, odd square free level) and to find an upper bound for the values using an upper bound for the cubic moment of the central values.

4.1 An Upper Bound for the Cubic Moment of Central Values

Peng [10] shows an upper bound for the cubic moment of the central value of the L-function associated to Hecke eigenforms for the full modular group and remarked that it is true for arbitrary level. We look into the proof and check the level dependency of the upper bound for higher levels.

Let $\mathcal{F}$ be an orthonormal basis for $S_k^{\text{new}}(\Gamma_0(N))$ consisting of Hecke eigenforms. Then associated to $f \in \mathcal{F}$ is an L-function

$$L_f(s) = \sum_{n=1}^{\infty} \lambda_f(n) n^{-s}.$$

Before the normalization $a_f(n) = a_f(1)\lambda_f(n)n^{\frac{k-1}{2}}$ (we assume $a_f(1) = 1$), one can define a twisted L-function of the form

$$L_f(D, s) = \sum_{n=1}^{\infty} \left(\frac{D}{n}\right) a_f(n) n^{-s}$$

where D is a fundamental discriminant and $\left(\frac{\cdot}{\cdot}\right)$ is the Kronecker symbol.

In [10] Peng proved that for $\mathrm{SL}_2(Z)$, the full modular group with even weight $k \geq 12$,

$$\sum_{f \in \mathcal{F}} L_f^3(1/2) \ll k^{1+\epsilon}$$

for any $\epsilon > 0$ with the implied constant being dependent only on ϵ. Using the nonegativity of the central value of the L-function by Waldspurger [20] or Kohnen-Zagier [21] Peng estimated that an upper bound for individual central values is

$$L_f(1/2) \ll k^{\frac{1}{3}+\epsilon},$$

where the constant depends only on ϵ. He remarked that the result can be extended to holomorphic cusp forms in $S_k(\Gamma_0(N))$ with an additional dependency on N. But the nonnegativity of the central value for $S_k(\Gamma_0(N))$ was not achieved at that time. We are able to ensure the nonnegativity using a later version of a Waldspurger type formula.

Here we begin with Peng's Theorem 3.1.1 [10] and outline the proof to check a level dependency. The result of the theorem can be extended to newforms of level N (See Remark under Corollary of Theorem 3.1.1 [10]).

Theorem 4.1. *[10, Theorem 3.11] Let k be an even number ≥ 12. Then*

$$\sum_{f \in \mathcal{F}} L_f^3(1/2) \ll k^{1+\epsilon} \tag{4.1}$$

for any $\epsilon > 0$, where $\mathcal{F}$ is an orthonormal basis of $S_k(\Gamma_0(1))$ consisting of Hecke eigenforms and the implied constant depends only on ϵ.

Proof. (Outline)

The central value of the Hecke $L-$function can be written

$$L_f(1/2) = 2\sum_{n=1}^{\infty} \frac{\lambda_f(n)}{\sqrt{n}} V(n)$$

where $V(y)$ is the inverse Mellin transform of $(2\pi)^{-s}\Gamma(s+\frac{k}{2})G(s)s^{-1}$,

$$V(y) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \Gamma(s+k/2)G(s)(2\pi y)^{-s}s^{-1}\, ds,$$

and $G(s)$ is holomorphic in $|\Re s| \leq A$, satisfying

$$\begin{aligned} G(s) &= G(-s), \\ \Gamma(k/2)G(0) &= 1, \\ \Gamma(s+k/2)G(s) &\ll (|s|+1)^{-2A}. \end{aligned}$$

Write the third power of $L_f(1/2)$ as

$$L_f^3(1/2) = L_f(1/2)L_f^2(1/2). \tag{4.2}$$

Using the multiplicative property of Hecke eigenvalues the square of the central value can be written as

$$L_f^2(1/2) = 4\sum_d d^{-1}\sum_{n_1}\sum_{n_2}\frac{\lambda_f(n_1n_2)}{\sqrt{n_1n_2}}V(dn_1)V(dn_2). \tag{4.3}$$

To analyze the cubic moment of the central value, consider $\mathcal{C}(k)$, the spectrally normalized cubic moment

$$\mathcal{C}(k) = \sum_{f\in\mathcal{F}}\omega_f L_f^3(1/2)$$

where $\mathcal{F}$ is a Hecke orthonormal basis of $S_k(\Gamma_0(1))$, and

$$\omega_f = \frac{12}{k-1}\left(\sum_{l=1}^{\infty}\frac{\lambda_f(l^2)}{l}\right)^{-1}$$

The coefficient ω_f has bounds

$$k^{-1-\epsilon} \ll \omega_f \ll k^{1+\epsilon} \tag{4.4}$$

for any $\epsilon > 0$. The lower bound can be derived using elementary means and the upper bound is a result of Hoffstein and Lockhart [22]. We only need the lower bound. Now we can combine (4.2) and (4.3) to get

$$\mathcal{C}(k) = 8 \sum_{f \in \mathcal{F}} \omega_f \sum_n \sum_{n_1} \sum_{n_2} \frac{\lambda_f(n)\lambda_f(n_1 n_2)}{\sqrt{n n_1 n_2}} V(n, n_1, n_2), \tag{4.5}$$

where

$$V(y, y_1, y_2) = V(y) \sum_d d^{-1} V(dy_1) V(dy_2).$$

Expressing $V = \int_0^\infty e^{-xy} f(x)\, dx$ as a Laplace transform, (4.5) can be rephrased as

$$\begin{aligned} \mathcal{C}(k) = \sum_d \frac{1}{d} \sum_{n, n_1, n_2} \iiint_{\mathcal{V}} & \frac{exp(-nx - dn_1 x_1 - dn_2 x_2)}{\sqrt{n n_1 n_2}} f(x) f(x_1) f(x_2)\, dx\, dx_1 dx_2 \\ & \times \sum_{f \in \mathcal{F}} \omega_f \lambda_f(n) \lambda_f(n_1 n_2). \end{aligned} \tag{4.6}$$

where $\mathcal{V} = (0, \infty)^3$. In fact, $\mathcal{V}$ is viewed as the union of 27 different products of the intervals

$$\begin{aligned} A &: \left(0, \frac{1}{k^{1+\epsilon}}\right] \\ B &: \left(\frac{1}{k^{1+\epsilon}}, k^\epsilon\right] \\ C &: \left(k^\epsilon, \infty\right). \end{aligned}$$

We let a "word" ABC represent the product $A \times B \times C \subset \mathcal{V}$. The sum (4.6) can be treated by Petersson's formula and written as

$$\sum_{f \in \mathcal{F}} \omega_f \lambda_f(n)\lambda_f(n_1 n_2) = \delta(m,n) + 2\pi i^k \sum_c c^{-1} S(m,n,c) J_{k-1}\left(\frac{4\pi}{c}\sqrt{mn}\right) \tag{4.7}$$

where $\delta(m,n)$ is the diagonal symbol, J_{k-1} is the Bessel function of the first kind of order $k-1$ and $S(m,n,c)$ is the Kloosterman sum

$$S(m,n,c) = \sum_{\substack{x \,(\mathrm{mod})\, c \\ x \text{ coprime to } c}} e\left(\frac{mx + n\bar{x}}{c}\right)$$

for integers m,n, $c \geq 1$, where $\bar{x}$ is the multiplicative inverse of x modulo c. Then $\mathcal{C}(k)$ can be estimated from three different cases. The contribution involving the interval A is measured by

$$\iiint f(x)f(x_1)f(x_2) \sum_{n,n_1,n_2} \frac{\exp(-nx - n_1x_1 - n_2x_2)}{\sqrt{nn_1n_2}} \times \sum_c \frac{1}{c} S(n, n_1n_2, c) J_{k-1}\left(\frac{4\pi\sqrt{nn_1n_2}}{c}\, dx dx_1 dx_2\right).$$

Recall Weil's bound for Kloosterman sums (see Corollary 11.12 in [23] for a proof)

$$|S(m,n;c)| \leq (m,n,c)^{\frac{1}{2}} c^{\frac{1}{2}} \tau(c),$$

where $\tau(c)$ is the number of positive divisor of c. Applying this Weil's bound and bounding the Bessel function (See Lemma A.2 in [10]), the triple integral over the words containing the letter A has the contribution bounded by

$$O(k^{-(A-\frac{3}{4})\epsilon + C_0}).$$

On any "word" that contains C, the contribution is bounded by $O(k^{C_1}e^{-k\epsilon})$ (see [10, Lemma 3.53]), so that $\mathcal{C}(k)$ is further reduced to

$$\mathcal{C}(k) = \iiint_{BBB} f(x)f(x_1)f(x_2) \sum_d \frac{1}{d}\mathcal{C}(x, dx_1, dx_2)\, dxdx_1dx_2 + O(k^{-(A-\frac{3}{4})\epsilon+C_0}) + O(k^{C_1}e^{-k\epsilon}) \tag{4.8}$$

where

$$\mathcal{C}(x, x_1, x_2) = \sum_{n,n_1,n_2} \frac{\exp(-nx - n_1x_1 - n_2x_2)}{\sqrt{nn_1n_2}} \sum_{f\in\mathcal{F}} \omega_f\lambda_f(n)\lambda_f(n_1n_2).$$

Finally this is estimated as

$$\begin{aligned}\mathcal{C}(x, x_1, x_2) &\ll (xx_1x_2)^{-\epsilon} + \sum_c \frac{1}{c^2} \sum_l \sum_{l_1} \sum_{l_2} \exp\left(\frac{ll_1l_2}{c}\right) W(l, l_1, l_2 : c) \\ &\ll (xx_1x_2)^{-\epsilon} + (xx_1x_2)^{-\epsilon}\end{aligned}$$

and

$$\mathcal{C}(x, dx_1, dx_2) \ll d^{-\epsilon}\mathcal{C}(x, x_1, x_2)$$

(See section 3.6 in [10]). Then the summation in d in (4.8) does not affect the final bound. Integration with respect to x, x_1, and x_2 over the "middle range" then gives

$$\mathcal{C}(k) = \sum_{f\in\mathcal{F}} \omega_f L_f^3(1/2) \ll k^{\epsilon}, \tag{4.9}$$

where the implied constant depends only on ϵ. Using the lower bound for ω_f in (4.4) we conclude

$$\sum_{f\in\mathcal{F}} L_f^3(1/2) \ll k^{1+\epsilon}.$$

□

Remark. (1) As Peng stated in the remark under the corollary of [10, Theorem 3.1.1], because of the generality of method of the proof the conclusion holds for Hecke cusp forms of arbitrary level as well, but it will have additional dependency on level N. If nonnegativity of each individual central L-value is secured, we get the same upper bound for central values for level N with the implied constant being dependent only on ϵ and N.

(2) Changing to work with newforms in $S_k^{\text{new}}(\Gamma_0(N))$, Petersson's formula (4.7) will change to

$$\sum_{f\in\mathcal{F}} \omega_f^* \lambda_f(n)\lambda_f(n_1 n_2) = \delta(m,n) + 2\pi i^k \sum_c c^{-1} S(m,n,c) J_{k-1}\left(\frac{4\pi}{c}\sqrt{mn}\right)$$

where

$$w_f^* = \frac{12}{(k-1)N}\left(\sum_{(l,N)=1} \lambda_f(l^2) l^{-1}\right)^{-1} \gg (kN)^{-1-\epsilon}.$$

This is the point where level N is embedded. (See [24] for a related topic of cubic moment of central values.)

4.2 Nonnegativity of the Central Values and an Upper Bound

We now proceed for a nonnegativity result on the central values of Hecke L-functions. Shimura's theory of forms of half-integral weight of modular forms [25] gives a correspondence between Hecke eigenforms $f(z)$ of even integral weight k and half integral weight Hecke eigenforms $g(z)$ of weight $\frac{k}{2}+\frac{1}{2}$. In [20] Waldspurger showed that under the Shimura correspondence there is a relation between the twisted central values $L_f(D, k/2)$ and the Fourier coefficients of $g(z)$ in terms of the language of representation theory. There are many later Waldspurger type theorems. See, for instance, Skoruppa [26] for the connection between Jacobi forms and modular forms of half integral weight and a formula interpreting the associated L-function as the square of a Fourier coefficients of a modular form of weight 3/2. For our purpose we use the result of Baruch and Mao [27] who generalized the Kohnen-Zagier

formula [21] by removing the restriction on the fundamental discriminant D. Before we state their theorem we briefly recall basic facts about half-integral weight forms, Shimura's theory and Kohnen's theory.

Following Shimura [25] we first define $(\frac{c}{d})$ and ϵ_d. If d is an odd prime, then let $(\frac{c}{d})$ be the usual Legendre symbol. For positive odd integers d, define $(\frac{c}{d})$ by multiplicativity. For negative odd d, we define

$$\left(\frac{c}{d}\right) := \begin{cases} \left(\frac{c}{|d|}\right) & \text{if } d < 0 \text{ and } c > 0, \\ -\left(\frac{c}{|d|}\right) & \text{if } d < 0 \text{ and } c < 0. \end{cases}$$

Also let $\left(\frac{0}{\pm 1}\right) = 1$.

Definition 4.2. Let k be a nonnegative integer and N be a positive integer. Let χ be a Dirichlet character modulo $4N$. A meromorphic function $g(z)$ on $\mathcal{H}$ is called a meromorphic half-integral weight modular form with Nebentypus χ and weight $k + \frac{1}{2}$ if it is meromorphic at the cusps of Γ, and if

$$g\left(\frac{az+b}{cz+d}\right) = \chi(d)\left(\frac{c}{d}\right)^{2k+1} \epsilon_d^{-1-2k}(cz+d)^{k+\frac{1}{2}} g(z)$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4N)$. If $g(z)$ is holomorphic on $\mathcal{H}$ and at the cusps of $\Gamma_0(4N)$, then $g(z)$ is called a holomorphic half-integral weight modular form. If $g(z)$ is a holomorphic modular form which vanishes at the cusps of $\Gamma_0(4N)$, then $g(z)$ is called a cusp form. If $g(z)$ is a meromorphic form whose poles are supported at the cusps of $\Gamma_0(4N)$, then $g(z)$ is called a weakly holomorphic modular form.

These forms constitute $\mathbb{C}$-vector spaces. We denote the $\mathbb{C}$-vector space of weight $k + \frac{1}{2}$ modular forms on $\Gamma_0(4N)$ with Nebentypus χ by $M_{k+\frac{1}{2}}(\Gamma_0(4N), \chi)$ and cusp forms by $S_{k+\frac{1}{2}}(\Gamma_0(4N), \chi)$. As in the integer weight case, there are Hecke operators which act on spaces of half-integral weight modular forms.

Definition 4.3. Let $g(z)$ be in the space of half-integral weight modular forms $M_{k+\frac{1}{2}}(\Gamma_0(4N), \chi)$. Then for primes p, the half-integral weight Hecke operator $T(p^2, k, \chi)$ is defined by

$$g(z)|T(p^2, k, \chi) = \sum_{n=0}^{\infty} \left(c(p^2 n) + \chi^*(p) \left(\frac{n}{p}\right) p^{k-1} c(n) + \chi^*(p^2) p^{2k-1} c(n/p^2) \right) q^n$$

where $\chi*$ is the Dirichlet character given by $\chi^*(n) := \left(\frac{(-1)^k}{n}\right) \chi(n)$, and $c(n/p^2) := 0$ if $p^2 \nmid n$.

The Shimura correspondence is a family of maps which take L-functions of half-integral weight cusp forms to L-functions of even integer weight modular forms.

Theorem 4.4. *[28, Theorem 3.14] Let $g(z) = \sum_{n=1}^{\infty} c(n) q^n \in S_{k+\frac{1}{2}}(\Gamma_0(4N), \chi)$ be a half-integral weight cusp form with $k \geq 1$. Let τ be a positive square-free integer, and define the Dirichlet character $\psi_\tau(n) = \chi(n) \left(\frac{-1}{n}\right)^k \left(\frac{\tau}{n}\right)$. Let $a_\tau(n)$ be defined by*

$$\sum_{n=1}^{\infty} \frac{a_\tau(n)}{n^s} := L(s - k + 1, \psi_\tau) \cdot \sum_{n=1}^{\infty} \frac{c(\tau n^2)}{n^s}.$$

Then

$$S_{\tau,k}(g(z)) := \sum_{n=1}^{\infty} a_\tau(n) q^n$$

is a weight $2k$ modular form in $M_{2k}(\Gamma_0(2N), \chi^2)$. If $k \geq 2$, then $S_{\tau,k}$ is a cusp form.

Shimura correspondences commute with the Hecke operators of integral and half-integral weight.

Definition 4.5. Suppose that N is a positive odd square-free integer, and k is a positive integer. The *Kohnen plus space* $S^+_{k+\frac{1}{2}}(\Gamma_0(4N))$ consists of cusp forms whose n-th Fourier coefficients vanish whenever $(-1)^k n \equiv 2, 3 \pmod 4$.

As in the integral weight case $S^+_{k+\frac{1}{2}}(\Gamma_0(4N))$ is the direct sum of two subspaces (See [29, Theorem 2])

$$S^+_{k+\frac{1}{2}}(\Gamma_0(4N)) = S^{\text{new}}_{k+\frac{1}{2}}(\Gamma_0(4N)) \oplus S^{\text{old}}_{k+\frac{1}{2}}(\Gamma_0(4N)).$$

Kohnen showed that the space $S^{\text{new}}_{k+\frac{1}{2}}(\Gamma_0(4N))$ has a basis of cusp forms which are eigenforms of the Hecke operators $T(p^2, \lambda, \chi_0)$ with $p \nmid N$. Such eigenforms are called Kohnen newforms. He established the connection between spaces of half-integral weight and even integer weight. For every positive square free integer τ there is a Shimura correspondence.

$$S_{\tau,k} : S^{\text{new}}_{k+\frac{1}{2}}(\Gamma_0(4N)) \to S_{2k}(\Gamma_0(N))$$

Furthermore Kohnen proves

Theorem 4.6. *[29, Theorem 2] The space of newforms $S^{\text{new}}_{k+\frac{1}{2}}(\Gamma_0(4N))$ of half-integral weight is isomorphic to the space of newforms $S^{\text{new}}_{2k}(\Gamma_0(N))$. In particular, the image of a half-integral weight Kohnen newform in $S^{\text{new}}_{k+\frac{1}{2}}(\Gamma_0(4N))$ is a newform in $S^{\text{new}}_{2k}(\Gamma_0(N))$ with the same system of Hecke eigenvalues.*

Now we state Theorem 10.1 [27] of Baruch and Mao. Recall that D is said to be a fundamental discriminant if it is 1 or the discriminant of a quadratic field. We define $\text{sgn}(D) = \frac{D}{|D|}$. Let $f(z)$ be a cusp form with odd square free level N and weight k. Let S_N be the subset of primes $p \mid N$. Let S be a (possibly empty) subset of S_N. Define $\mathcal{D}_S$ to be the set of fundamental discriminants D such that $(\frac{D}{p}) = -w_p$ if $p \in S$ and $(\frac{D}{p}) \neq -w_p$ if $p \in S_n - S$, where w_p is the eigenvalue of the Atkin-Lehner involution acting on $f(z)$. Then the set of fundamental discriminants is the disjoint union $\cup_{S \subset S_N} \mathcal{D}_S$.

Proposition 4.7. *[27, Theorem 10.1] Let $N' = \prod_{p \in S} p$ and let $\chi = \prod_{p|2N} \chi_p$ be any Dirichlet character of $(\mathbb{Z}/4NN')^*$ such that $\chi_p \equiv 1$ when $pN' \mid N$, $\chi_p(-1) = -1$ when $p \mid N'$, and $\chi(-1) = 1$. Then there exists a unique (up to scalar multiple) cusp form $g_S(z) = \sum_{n=1}^{\infty} c(n) e^{2\pi i n z}$ in $S_{\frac{k+1}{2}}(4NN', \chi)$ which is a Shimura lift of $f(z)$ and lies in the Kohnen space i.e. $c(n) = 0$ when $(1)^{s+k/2} n \equiv 2, 3 \bmod 4$, and $c(|D|) = 0$ if $(-1)^{s+k/2} D$ is a fundamental discriminant that is not in $\mathcal{D}_S$. Moreover for this $g_S(z)$ and for $D \in \mathcal{D}_S$, if*

$(-1)^{s+k/2} \neq \operatorname{sgn}(D)$, *then* $L_f(D, k/2) = 0$*; if* $(-1)^{s+k/2} = \operatorname{sgn}(D)$, *then*

$$\frac{|c(|D|)|^2}{\langle g_S, g_S \rangle} = \frac{L_f(D, k/2)}{\langle f, f \rangle} |D|^{\frac{k-1}{2}} \frac{(k/2-1)!}{\pi^{k/2}} 2^{\nu(N)-t} \prod_{p \in S} \frac{p}{p+1}, \tag{4.10}$$

where s *is the size of* S*,* t *is the number of primes dividing both* D *and* N*, and* $\nu(N)$ *is the number of prime divisors of* N*.*

Following the notations and the setting in Baruch and Mao [27], $f(z)$ is a newform of even weight k, square free odd level N, and of trivial character, and $g(z)$ is the associated newform under the Shimura correspondence. This theorem enables us to get the non-negativity of the central values of newforms.

Corollary 4.8. *Let* $f(z) \in S_k^{\text{new}}(\Gamma_0(N))$ *be a newform of even weight* k*, square free and odd level* N*. Then the central value of the* L*-function satisfies*

$$L_f\left(\frac{1}{2}\right) \geq 0.$$

Proof. Notice that for $D = 1$, $L_f(D, k/2) = L_f(1, k/2) = L_f(\frac{1}{2})$. Choosing $D = 1$ in Proposition 4.7, if $(-1)^{s+k/2} = \operatorname{sgn}(D) = 1$, then $L_f(1, k/2) = 0$; otherwise all the factors involved in equation (4.10) are positive so that $L_f(1, k/2) > 0$. □

We now combine Corollary 4.2 with Peng's estimation (4.1) on the sum of cubic moment of central values to get the following theorem.

Theorem 4.9. *If* f *is a newform in the space* $S_k^{\text{new}}(\Gamma_0(N))$ *with* k *an even integer and* N *a square free odd integer then*

$$0 \leq L_f(1/2) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{\sqrt{n}} \ll k^{\frac{1}{3}+\epsilon}.$$

Chapter 5. Real Zeros of Hecke Eigenforms of Higher Levels

In the main approximation theorem 2.14 above we take $s = k$, the weight of the eigenform. Note that $\Phi_f(k; \alpha, y)$ is real valued when $\alpha = 0$ or $\alpha = \frac{1}{2}$, where we let $z = \alpha + iy$ lying on δ_1 or δ_2 respectively. So the sign changes of $\Phi_f(k; \alpha, y)$ depend on the product $\lambda_f(l)e(\alpha l)$ and the second big O term (See also Remark 2.3). Now the idea of counting zeros of f is to detect the sign changes of $\lambda_f(l)e(\alpha l)$ ensuring that $\lambda_f(l)$ is not too small so that the big O term $O(k^{-\delta})$ does not affect the sign changes once we take weights k large enough. Then we use the bounds on the integers l and the relation between y and l in the assumption of the theorem to get a lower bound for the zeros of $f(z)$.

We first note that $\Phi_f(k; \alpha, y)$ is real valued when $\alpha = 0$ or $= \frac{1}{2}$, where we let $z = \alpha + iy$ lying on δ_1 or δ_2 respectively. To get a lower bound for the number of zeros of $f(z)$ on δ_j, it is enough to get sign-changes of $\Phi_f(k; \alpha, y)$ by detecting sign-changes of $\lambda_f(l)e(\alpha l)$. To this end we need to ensure that $\lambda_f(l)$ is not too small, which can be proven by applying Hecke relations.

5.1 Sign-changes of $f(z)$ on δ_2: Lower Bounds.

We begin with the following lemmas in Ghosh and Sarnak [1].

Lemma 5.1. *[1, See eqn. 20] Let $f \in S_k^{\text{new}}(\Gamma_0(N))$ be a newform and assume p does not divide N. Then either $|\lambda_f(p)| \geq \beta$ or $|\lambda_f(p^2)| \geq \beta$, where $\beta = \frac{\sqrt{5}-1}{2}$.*

Proof. Since $p \nmid N$ we have the Hecke relations for such p's

$$\lambda_f(p)^2 = \lambda_f(p^2) + 1$$

If $\lambda_f(p) = \lambda_f(p^2)$ then the equation gives $|\lambda_f(p)| = \frac{\sqrt{5}\pm 1}{2}$. If $|\lambda_f(p)| \leq \frac{\sqrt{5}-1}{2}$ then $\lambda_f(p^2) = \lambda_f(p)^2 - 1 \leq \frac{6-2\sqrt{5}}{4} - 1 = \frac{1-\sqrt{5}}{2}$. Therefore $|\lambda_f(p^2)| \geq \frac{\sqrt{5}-1}{2}$. □

The first part of the next lemma is attributed to [1, Lemma 4.1] and the second part of the lemma comes from Atkin-Lehner theory of newforms (see Theorem 2.8(4)).

Lemma 5.2. *Let $f \in S_k^{\text{new}}(\Gamma_0(N))$ be a newform, and let p be a prime number. If $p \nmid N$, then for a fixed number $J \geq 1$, there is a constant B depending only on J and a number $b = b(f,p)$, with $1 \leq b \leq B$ such that if $a = p^b$ we have*

$$\lambda_f(a^j) \geq \frac{1}{10}$$

for all $1 \leq j \leq J$. If $p \mid N$ and $p^2 \nmid N$ then

$$\lambda_f(p^2) = \frac{1}{p} > 0.$$

Proof. Suppose $p \nmid N$. Since $|\lambda_f(p)| \leq d(p) = 2$, there exists a number $\theta_p = \theta(f,p)$ with $0 \leq \theta_p \leq \pi$ such that $\lambda_f(p) = 2\cos\theta_p$. Since $p \nmid N$, from the Hecke relation we have $\lambda_f(p^{n+1}) = \lambda_f(p^n)\lambda_f(p) - \lambda_f(p^{n-1})$. Using the relation and by induction on n we get

$$\lambda_f(p^n) = \frac{\sin((n+1)\theta_p)}{\sin\theta_p}$$

for all non-negative integers n. If $\theta_p = 0$ or π, then we may take $b = 2$ since $\lambda_f(p^{2j}) \geq 3$ for all $j \geq 1$. By continuity, we see that $b = 2$ still suffices for θ_p near 0 or π. In other words, there is a number $\theta_0 > 0$ depending only on J such that the conclusion of the lemma holds unless $\theta_0 < \theta_p < \pi - \theta_0$, which we now assume. By Dirichlet's approximation theorem, for any integer $B \geq 1$, there are integers $1 \leq b \leq B$ and b' such that $|b\frac{\theta_p}{2\pi} - b'| \leq \frac{1}{B+1}$. We let $\eta = b\theta_p - 2\pi b'$; then $|\eta| \leq \frac{2\pi}{B+1}$ and

$$\lambda(p^{bj}) = \frac{\sin(bj+1)\theta_p}{\sin\theta_p} = \frac{\sin(\eta j + \theta_p)}{\sin\theta_p} = \cos(j\eta) + \sin(j\eta)\cot\theta_p.$$

We shall choose B sufficiently large so that η is small enough to satisfy $0 < \eta j + \theta_p < \pi$ for

all $1 \leq j \leq J$. Using the Maclaurin series for the cosine function, we conclude that

$$\lambda_f(p^{bj}) \geq 1 - \frac{(j\eta)^2}{2} + \sin(j\eta)\cot\theta_p \geq \frac{1}{10}$$

by choosing B sufficiently large, if necessary.

Now let $p \mid N$ and $p^2 \nmid N$. Then it is well known that

$$\lambda_f(p) = -w_p \frac{1}{\sqrt{p}},$$

where $w_p \in \{1, -1\}$ is the eigenvalue for the Atkin-Lehner involution $\mathcal{W}_p$ attached to p (See Theorem 2.8). The Hecke relation with $p \mid N$ is

$$\lambda_f(p^n) = (\lambda_f(p))^n$$

for all positive integers n. Thus we have, in particular,

$$\lambda_f(p^2) = \frac{1}{p} > 0.$$

□

Now we estimate the number of zeros of newforms on δ_2. The essential idea of the proof is the same as that of Ghosh and Sarnak (See [1, Theorem 4.2]).

Theorem 5.3. *Let $f \in S_k^{\text{new}}(\Gamma_0(N))$ be a newform, where N is a positive integer such $4 \nmid N$. Then there is a constant $C > 0$ such that $f(z)$ has at least $C \log k$ zeros on the line δ_2 with $z = \frac{1}{2} + iy$ and $y \geq \sqrt{k \log k}$ for k sufficiently large.*

Proof. Take $s = k$, with the weight of f sufficiently large so that $L = \left(\beta_2 \sqrt{\frac{k}{\log k}}\right)^{\frac{1}{2}}$ is in the interval $\left(\beta_1, \beta_2 \sqrt{\frac{k}{\log k}}\right)$ in Theorem 2.14. We write $[1, N] \cup [N, L] \supseteq \bigcup_{i=0}^{R} [(2a)^i, (2a)^{i+1}]$ with $R \gg \log k$, $(2)^{R/2} > N$ and $a \geq 1$ some integer. For the choice of a we use Lemma

5.2: if $2|N$ we take $p = 2$ and choose $a = p^2 = 4$, otherwise we take $p = 2$ and $J = 2$ and choose $a = 2^b$. Then $\lambda_f(a) \geq \frac{1}{10}$, $\lambda_f(a^2) \geq \frac{1}{16}$ in either case. Then we only consider the subintervals with $\frac{R}{2} \leq i \leq R$. Each such subinterval we denote by $I = [m, 2am]$ and by I^2 the corresponding subinterval $[m^2, (2am)^2]$. For each $m > N$ the interval $[m, 2m]$ contains an odd prime number $q > N$ since $m \geq (2)^{R/2} > N$, so that both q and aq lie in I. Since we have avoided the case $q|N$, Lemma 5.1 gives either $|\lambda_f(q)| \geq \beta$ or $|\lambda_f(q^2)| \geq \beta$.
Suppose $|\lambda_f(q)| \geq \beta$. Then the product of $\lambda_f(q)e(\frac{1}{2}q)\lambda_f(aq)e(\frac{1}{2}aq)$ with $l = q$ and $l = aq$ on the right hand side of the equation (2.11) is

$$\lambda_f(q)(-1)^q\lambda_f(aq)(-1)^{aq} = -\lambda_f(q)^2\lambda_f(a) < -\frac{\beta^2}{10}.$$

Taking k sufficiently large if necessary we can ignore the big O term, which is possible since the absolute values of the normalized eigenvalues $|\lambda_f(a)|, |\lambda_f(a^2)|$, and for primes q, $|\lambda_f(q)|$ and $|\lambda_f(q^2)|$ are bounded by the number of divisors of a, a^2, q and q^2 respectively. Therefore by Theorem 2.14 we use that $\Phi_f(k, \frac{1}{2}, y)$ has a sign-change between $\frac{k-1}{4\pi q}$ and $\frac{k-1}{4\pi aq}$.
Now suppose $|\lambda_f(q^2)| \geq \beta$. In this case, both q^2 and a^2q^2 lie in I^2 and

$$\lambda_f(q^2)e(\frac{1}{2}q^2)\lambda_f(a^2q^2)e(\frac{1}{2}a^2q^2) = (-1)^{q^2}(-1)^{a^2q^2}\lambda_f(q^2)^2\lambda_f(a^2) < -\frac{\beta^2}{16}.$$

This time for sufficiently large k, we have a sign-change between $\frac{k-1}{4\pi q^2}$ and $\frac{k-1}{4\pi a^2q^2}$.
By considering only subintervals with $\frac{R}{2} \leq i \leq R$, we can ensure that all our subintervals of the type I and I^2 are disjoint, so that there are at least $\frac{R}{2}$ zeros $\gg \log k$ of $f(z)$ with $z = \frac{1}{2} + iy$ and $y \gg \frac{k}{L^2} \gg \sqrt{k \log k}$. □

5.2 Sign-changes of $f(z)$ on δ_1: Lower Bounds.

The analysis on δ_1 is essentially harder than on the case of δ_2 because it requires a construction of negative eigenvalues that must appear soon enough to get desired number of sign changes. To be more specific, in the main approximation theorem, the major term

$\lambda_f(l)$ of the approximation to Φ_f is related to the integer l which is bounded above by $r_2\sqrt{k/\log k}$ (See Theorem 2.14). Therefore we should make sure the existence of a negative Hecke eigenvalue $\lambda_f(n)$ such that $n \ll \sqrt{k/\log k}$. In Theorem 3.6 we prove

$$n_f < k^{0.7444} N^{0.3722}.$$

The power $k^{0.7444}$ is bigger than $\sqrt{k/\log k}$, which is not small enough to apply the method of Ghosh and Sarnak [1, Theorem 4.3]. Instead we assume an upper bound for n_f by $k^\alpha,\ \alpha < 1/2$ and achieve a conditional result.

Let $\pi(x)$ be the number of prime numbers less than equal to x. In 1850 Chebyshev proved that the inequalities

$$c_1 \frac{x}{\log x} \le \pi(x) \le c_2 \frac{x}{\log x} \tag{5.1}$$

hold for all $x \ge 10$, where $c_1 = \log(2^{1/2}3^{1/3}5^{1/5}/30^{1/30}) \approx 0.921292$ and $c_2 = 6c_1/5 \approx 1.1055$. (See [30] for example).

Lemma 5.4. *There are at least* 20 *primes* p *such that*

$$\sqrt{x} \le p \le \sqrt{50x}$$

for every $x \ge 500$.

Proof. By the Chebyshev estimate (5.1),

$$0.9212\ \frac{x}{\log x} \le \pi(x) \le 1.1056\ \frac{x}{\log x}$$

holds for all $x \ge 10$. Then the number of primes between $\sqrt{x}$ and $\sqrt{50x}$ is estimated as

$$\pi(\sqrt{50x}) - \pi(\sqrt{x}) \ge 0.9212\ \frac{\sqrt{50x}}{\log\sqrt{50x}} - 1.1056\ \frac{\sqrt{x}}{\log\sqrt{x}}$$

$$\geq \left(\frac{6.513}{\log\sqrt{50}+\log\sqrt{x}} - \frac{1.1056}{\log\sqrt{x}}\right)\sqrt{x}$$
$$\geq 20$$

for all $x \geq 500$. □

Lemma 5.5. *Given $\xi \geq \max\{1000, N^2\}$ there are six integers m in the interval $(\xi, 50\xi)$ which are relatively prime in pairs and for which*

$$|\lambda_f(m)| \geq \frac{1}{2}.$$

Proof. (We follow Ghosh and Sarnak [1, Theorem 4.2].) Consider the interval $(\sqrt{\xi}, \sqrt{50\xi})$. By Lemma 5.5, it contains at least 18 primes p and since $p > N$, for each one either $\lambda_f(p) \geq \beta$ or $|\lambda_f(p^2)| \geq \beta$ or both by Lemma 5.1. Note that $\beta = (\sqrt{5}-1)/2 \geq 1/2$. If six of these have $|\lambda_f(p^2)| \geq \beta$, then we choose our m's to be these p^2's. Otherwise, we can find twelve distinct primes p_j with $|\lambda_f(p_j)| \geq \beta$. We now take for our m's the six products p_1p_2, p_3p_4, $\cdots$, $p_{11}p_{12}$. Then the six integers will be in the interval $(\xi, 50\xi)$. □

Lemma 5.6. *Given $\xi \geq \max\{1000, N^2\}$ and f, there are relatively prime integers m_1, m_2 in the interval $(\xi, 2500\xi)$ such that*

$$\lambda_f(m_j) \geq \frac{1}{4},\ j = 1, 2.$$

Proof. (See the proof of [1, Lemma 4.6]) Consider the interval $(\sqrt{\xi}, 50\sqrt{\xi})$. By Lemma 5.6, there are integers $n_1, \cdots, n_6$ in the interval $(\xi, 50\xi)$ that are relatively prime in pairs such that $|\lambda_f(n_j)| \geq \frac{1}{2}$. Of the three numbers n_1, n_2 and n_3 at least two have the same sign (we assume the first two) so that $\lambda_f(n_1n_2) \geq \frac{1}{4}$, giving us m_1 and similarly m_2 using the remaining three integers. □

Theorem 5.7. *Let k be an even integer and let N be a positive integer. Consider a sequence of newforms $\{f_k\}$ where f_k is in $S_k^{\text{new}}(\Gamma_0(N))$. Suppose there exist $\epsilon_0 > 0$ and a positive integer N_0 such that if $k \geq N_0$ then there is a power of prime p^r such that $p^r < k^\alpha$, $\alpha < 1/2$ and*

$$\lambda_f(p^r) \leq -\epsilon_0.$$

Then the number of zeros of $f_k(z)$ on δ_2 goes to infinity as k goes to infinity. More quantitatively

$$|Z(f_k) \cap \delta_1| \gg \log k.$$

Proof. Since $\alpha < \frac{1}{2}$, we can take k greater than N_0 satisfying the inequalities

$$\max\{1000, N^2\} \leq k^\alpha < \max\{1000, N^2\}k^\alpha < \frac{1}{2500}\sqrt{\frac{k}{\log k}},$$

where N is the level of the newform space. We denote $\eta = \frac{1}{2500}\sqrt{\frac{k}{\log k}}$. Then by assumption there is a prime power p^r such that $p^r < k^\alpha$ and $\lambda_f(p^r) \leq -\epsilon_0$ for some fixed $\epsilon_0 > 0$ not depending on the weight $k > N_0$.

Note that the interval $I = (\eta, 2500\eta)$ is contained in the interval $(k^\alpha, \sqrt{\frac{k}{\log k}})$, and $\eta \geq \max\{1000, N^2\}$. Thus by Lemma 5.6, there is some $m_1 \in I$ such that $\lambda_f(m_1) \geq \frac{1}{4}$. Because

$$\frac{\eta}{p^r} > \frac{\eta}{k^\alpha} > \max\{1000, N^2\},$$

by applying Lemma 5.6 again to the interval $(\eta/p^r, 2500\eta/p^r)$, we obtain two relatively prime integers v_1 and v_2 such that $\lambda_f(v_j) \geq \frac{1}{4}$ for $j = 1$ and 2. At least one of the v_j's is coprime to p, say v_1. We set $m_2 = p^r v_1$ so that $m_2 \in I$ and $\lambda_f(m_2) \leq -\frac{\epsilon_0}{4}$. Then using Theorem 2.14 we find that $f(iy)$ has a sign-change for a y between $\frac{k-1}{4\pi m_1}$ and $\frac{k-1}{4\pi m_2}$. Since there are $C_1 \log k$ such disjoint subinterval I for some positive constant C_1, we complete the proof. □

Appendix A. Sage Code for Estimation of $\sigma(u)$

This is a Sage code for estimating lower and upper bounds for the values of $\sigma(u)$ (see Lemma 3.5) recursively. The program stops running if it detects the first zero of $\sigma(u)$ and prints a value κ such that $\sigma(\kappa) > 0$.

```
import numpy as np
import math
def x_k(k, K):
if k == 0:
return 0
return 1/(K - k + 1)
def chi_k(k, K):
return 2 * np.cos(np.pi/(K - k + 1))
def findKPrime(u, K):
if u >= 1:
return K
if u < 1/(K):
return 0
return math.floor(-1/u + K + 1)#I think this works.
def estimatePrevGamma(pos, delta, K, gamVals):
if pos == 0:
return [[1,1],[1,1]]
if pos <= 1/K:
return [[1,1],[1,1]]
l = int(math.floor((pos - 1/K)/delta))
if len(gamVals) <= l + 1:
```

```
l = len(gamVals) - 1
return [gamVals[l], gamVals[l]]
return [gamVals[l],gamVals[l+1]]
def findZeroofGamma(delta, K):
gamVals = [[0,0]]
currGamma = [1,1]
gamVals.append(currGamma)
x_kArr = [x_k(i, K) for i in range(K + 1)]
chi_kArr = [chi_k(i, K) for i in range(K + 1)]
sumTerms = [(chi_kArr[i-1] - chi_kArr[i]) for i in range(1, K + 1)]
sumTerms.insert(0,0)
l = 1
while(currGamma[0] >= 0):
l += 1
if  1/K + l*delta <= 1/K:
currGamma = [1,1]
gamVals.append(currGamma)
else:
lower = currGamma[0]
u = 1/K + l*delta
sUpper = 0
kPrime = findKPrime(u, K)
upper = currGamma[1]
sLower = 0
for i in range(1, int(kPrime + 1)):
delK = min(delta, u - x_kArr[i])
prevGamma = estimatePrevGamma(u - delK - x_kArr[i], delta, K, gamVals)
```

```
prevGamma2 = estimatePrevGamma(u - x_kArr[i], delta, K, gamVals)
sLower += delK * sumTerms[i] * (u - delK)**(-chi_kArr[0])
 * (u - x_kArr[i])**(chi_kArr[0] - 1) * max(prevGamma[0][1], prevGamma[1][1])
sUpper += delK * sumTerms[i] * (u)**(-chi_kArr[0]) * (u - delK - x_kArr[i])
**(chi_kArr[0] - 1) * min(prevGamma2[0][0], prevGamma2[1][0])
lower = lower - sLower
upper = upper - sUpper
currGamma = [lower, upper]
gamVals.append(currGamma)
return  (l-1)*delta + 1/K
delta = 1/(np.sqrt(3) * 100)
K = 100
findZeroofGamma(delta, K)
1.2455295760657992
delta = 1/(np.sqrt(3) * 10000)
K = 100
findZeroofGamma(delta, K)
1.3425244212896563
delta = 1/(np.sqrt(31) * 10000)
K = 100
findZeroofGamma(delta, K)
1.3433538411863715
delta = 1/(np.sqrt(73) * 10000)
K = 100
findZeroofGamma(delta, K)
1.3434614941202354
```

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