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# Construction and Isomorphism of Landau-Ginzburg B-Model Frobenius Algebras

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ABSTRACT

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Landau-Ginzburg Mirror Symmetry provides for the construction of two algebraic objects, called the A- and B-models. Special cases of these models–constructed using invertible polynomials and abelian symmetry groups–are well understood. In this thesis, we consider generalizations of the B-model, and specifically address the associativity of the multiplication in these models. We also prove an explicit B-model isomorphism for a class of polynomials in three variables.

Keywords: Mirror Symmetry, Frobenius Algebras, Algebraic Geometry

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# Chapter 1. Background

Part of string theory includes the formation to two algebraic objects, called the A- and B-models. These objects, created from polynomials, have many layers of structure, including that of graded Frobenius algebra. The methods of construction differ significantly, and we want to understand both models. In some cases, even the method of construction is incomplete. Descriptions of the B-model can be found in [2] and [3], and details of their construction follow in [9], [8], and [10]. The construction of the A-model, often called an FJRW model, was formalized more recently in [4].

In order to understand these structures, we must first define several objects and properties required in the construction.

## 1.1 Definitions

**Definition 1.1.** A polynomial, $W(x_1, \cdots, x_n)$, is called *quasihomogeneous*, or *weighted homogeneous*, if there exists a vector of rational numbers $q = (q_1, \cdots, q_n)$ such that $W(c^{q_1}x_1, \cdots, c^{q_n}x_n) = cW(x_1, \cdots, x_n)$ for all $c \in \mathbb{C}$. Such a vector $q$ is called a *weight system*, or even just the *weights of* $W$.

**Definition 1.2.** A polynomial is said to be *nondegenerate* if it has a unique critical point at the origin.

**Definition 1.3.** A polynomial is called *admissible* if it is nondegenerate and quasihomogeneous with a unique weight system. If an admissible polynomial has the same number of monomials as variables, it is called *invertible*.

**Definition 1.4.** A polynomial $W = W_1 + W_2 + \cdots + W_m$ is called a *disjoint sum* if no two $W_i$s have any variables in common.

*Remark.* It was proven in [7] that all invertible polynomials consist of disjoint sums of three atomic types of polynomials. These are as follows.

$$\begin{aligned}
&Fermat : x_1^{a_1};\\
&Chain : x_1^{a_1} + x_1x_2^{a_2} + \cdots + x_{n-1}x_n^{a_n};\\
&Loop : x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_n^{a_n}x_1,
\end{aligned}$$

where $a_i \geq 2$ for all $i$.

**Definition 1.5.** Let $W = w_1 + w_2 + \cdots w_n$ be a polynomial, where each $w_i$ is a monomial. Denote the degree of $w_j$ as a polynomial of $x_i$ by $deg_{x_i}(w_j)$. Then the *exponent matrix* of a polynomial, denoted $A_W$, is defined to be $A_W = (deg_{x_i}(w_j))$. For example, consider the polynomial $W = x^2y + y^3$. Its exponent matrix is

$$A_W = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}.$$

The *transpose polynomial* is determined by the transpose of the exponent matrix, and is denoted $W^T$. For the above polynomial, the transpose matrix is

$$A_{W^T} = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix},$$

so the transpose polynomial is $W^T = x^2 + xy^3$.

**Definition 1.6.** The *maximal diagonal symmetry group*, $G_W^{max}$, of a polynomial, $W(x_1, \cdots, x_n)$, is defined to be the set of all diagonal matrices $g \in GL(n, \mathbb{C})$ such that $W \circ g = W$. As these maps are diagonal, we often write them as the vector of their diagonal entries. It is shown in [12] that these entries are always roots of unity, and thus we can write them as a vector of the complex arguments of the diagonal entries.

$$\begin{pmatrix} e^{2\pi i\theta_1} & & \\ & \ddots & \\ & & e^{2\pi i\theta_n} \end{pmatrix} \longleftrightarrow (\theta_1, \cdots, \theta_n).$$

Note that this change in notation changes the group operation from matrix multiplication to vector addition. If we consider the weights of the polynomial as a vector, $J = (q_1, q_2, \cdots, q_n)$, then $J$ is always an element of $G_W^{max}$.

**Definition 1.7.** The *transpose* of a group $G \leq G_W^{max}$ is defined to be

$$G^T = \left\{ g \in G_{W^T}^{max} \mid gA_W h^T \in \mathbb{Z} \text{ for all } h \in G \right\},$$

where $G$ and $G^T$ are written in additive notation as subgroups of $(\mathbb{Q}/\mathbb{Z})^n$. Several nice properties of this transpose are proven in [1], such as:

$$(G_W^{max})^T = \{0\};$$
$$\left(G^T\right)^T = G;$$
$$G_1 \leq G_2 \implies G_2^T \leq G_1^T.$$

With an admissible polynomial and a subgroup of its maximal symmetry group that contains $J$, we can construct an A-model. Using the transpose polynomial and transpose group, which will always be contained in $SL(n, \mathbb{C})$, we can also construct a B-model.

## 1.2 Landau-Ginzburg Mirror Symmetry

The Landau-Ginzburg Mirror Symmetry Conjecture says that for any admissible polynomial, $W$, and corresponding symmetry group, $G$:

$$\mathcal{A}_{W,G} \cong \mathcal{B}_{W^T,G^T}. \tag{1.1}$$

This is an isomorphism of graded Frobenius algebras. This means that these models are isomorphic as vector spaces, and that the isomorphism respects the grading, the pairing and the muliplication that we define on the model. In [10], Krawitz proves that the A- and B-models for an invertible polynomial and its transpose are isomorphic on the level of graded vector spaces. He also proves that if the B-model is constructed with the trivial group, and if all of the weights are strictly less than $\frac{1}{2}$, then the isomorphism holds on the level of Frobenius algebras.

More cases of this isomorphism were proven in [6]. Let $W = W_1 + W_2 + \cdots + W_m$ be a disjoint sum of invertible polynomials with symmetry group $G$. Suppose that for each $g \in G$, $g$ fixes every variable of $W_i$ or no variable of $W_i$ for every $i \leq m$. Similarly, suppose that each $h \in G^T$ fixes every variable of $W_i^T$ or no variable of $W_i^T$ for every $i \leq m$. Then Theorem 3.0.3 in [6] gives that (1.1) holds on the level of Frobenius algebras. This theorem covers all B-models for loops and fermats, but only some chains. Work continues to prove the remaining cases.

## 1.3 Generalizing B-Models

The construction of the B-model is well understood for invertible polynomials, but not in other cases. It is natural to generalize the B-model in two ways. First, it has not yet been proven that the multiplication of B-models for admissible but noninvertible polynomials is associative. In his dissertation, [10], Krawitz proved that the multiplication of a B-model constructed with an inveritble polynomial is always associative. Unfortunately, Krawitz's proof of associativity in the invertible case depends heavily on something that does not apply to noninvertible polynomials, so his method of proof cannot be used again. We will look at one special case in Chapter 4.

Another way to generalize is to allow for more complicated symmetry groups. In the traditional construction, we restrict our group representations to only include diagonal matrices. Many polynomials admit symmetry groups with nondiagonal representations. Such groups, which may no longer be abelian, are also of interest. In Chapter 2, we will detail the new construction process, and look at a multiplication that can be defined for the generalized B-model. We will also include several examples.

Fan, Jarvis, and Ruan showed that A-models can be constructed using arbitrary symmetry groups [5]. If mirror symmetry were to hold for these new A-models, it is reasonable to assume that the transpose groups will also be nonabelian. Defining B-models with such groups will then provide candidates for the mirror model.

This generalized construction is difficult from the first few steps. Finding nonabelian symmetry groups of an admissible polynomial often requires solving large systems of nonlinear equations. As this quickly becomes untractable, we must find other ways of finding symmetry groups. One approach is to start with a specific group representation, and find all of the admissible polynomials for which it is a symmetry group. In Chapter 3, we will demonstrate a technique for doing exactly that.

## 1.4 Group-Weights

In 2013, Julian Tay proved a conjecture concerning the A-models in [11]. His theorem, called the Group-Weights Theorem says that if two admissible polynomials have the same weights, and some group $G$ is a symmetry group for both polynomials, then the A-models for these polynomials with this common group are always isomorphic. In Chapter 5, we will look at what this can tell us about specific B-models.

# Chapter 2. Nonabelian Construction and Examples

The B-model is well understood in the case of invertible polynomials and diagonal symmetry groups. Many cases of noninvertible polynomials have also been described, but it remains to show in general that the multiplication of such models is associative. In the case of nondiagonal groups, the B-model construction is not well understood. This case is the main focus of this chapter, and is still a topic of current research.

Below, we explain the construction of an B-model, up to the level of Frobenius algebra, using both diagonal and nonabelian symmetry groups. More details and examples of this generalization can be found in Ryan Sandberg's thesis, [13]. We highlight the differences between the original and the nonabelian constructions. We also include several complete examples.

In order to create a B-model, we start by picking an admissible polynomial and a group of symmetries. In the abelian case, the polynomial must be quasihomogenous, but in our generalization, we require the stronger condition of begin homogeneous. In both cases, we require that the symmetry group be a subgroup of $SL(n, \mathbb{C})$. We will see that this condition is necessary for the existence of a multiplicative identity.

## 2.1 State Space

**2.1.1 Abelian State Space.** The first step is to define a basis for a vector space. Throughout this thesis, we will refer to vector spaces as state spaces. In the abelian case, we start by computing the restriction of a polynomial $W(x_1, \cdots, x_n)$ to the fixed locus of each group. With the group elements written in additive notation, $g = (\theta_1, \cdots, \theta_n)$, this amounts to setting $x_i = 0$ if $\theta_i \neq 0$. The restricted polynomial is denoted $W|_{\text{fix}(g)}$.

Next, we construct the Milnor ring of each of these restricted polynomials. The Milnor ring $W$ is defined to be

$$\mathcal{Q}_W = \frac{\mathbb{C}[x_1, \cdots, x_n]}{\left(\frac{\partial W}{\partial x_1}, \cdots, \frac{\partial W}{\partial x_n}\right)}.$$

Each of these rings, $\mathcal{Q}_{W|_{\text{fix}(g)}}$, is called a *sector*. Each sector forms a finite-dimensional vector space, and the unprojected state space, $\mathcal{B}_{W,\{0\}}$, is defined to be the direct sum of all of these sectors.

$$\mathcal{B}_{W,\{0\}} = \left(\bigoplus_{g\in G} \mathcal{Q}_{W|_{\text{fix}(g)}}\right).$$

When we need to distinguish between sectors, we call $\mathcal{Q}_{W|_{\text{fix}(g)}}$ the $g$-sector.

**Abelian Example.** Consider $V = x^3 + y^3$ with symmetry group $H =< h_1 >$, where $h_1 = (\frac{1}{3}, \frac{2}{3})$. We can directly compute the following rings.

$\mathcal{Q}_V = \text{span}\{1, x, y, xy\}$;

$\mathcal{Q}_{V|_{\text{fix}(h_1)}} = \mathcal{Q}_{W|_{fix(h_1^2)}} = \mathbb{C}$.

In the abelian case, we denote a state space element as $\lfloor m; g \rceil$, where $m$ is a monomial in $\mathcal{Q}_{W|_{\text{fix}(g)}}$.

### 2.1.2 Nonabelian State Space.

In the nonableian case, restricting a polynomial, $W$, to the fixed locus of a group element, $g$, is not as straightforward. In the diagonal case, a nonzero polynomial was fixed by $g$ exactly when each of the monomials were fixed. This is not true in the nonabelian case. As an example, it is possible for a group element $g$ to fix $x + y$, and fix neither $x$ nor $y$. Because of this, we cannot construct $\mathcal{Q}_{W|_{\text{fix}(g)}}$ in the same way. Instead, we define a similar object, $H_g$.

**Computing $H_g$.** There are a few different ways of defining $H_g$. Here, we give one of the more intuitive definitions.

$$H_g = \frac{\mathbb{C}[x_1, x_2, \cdots x_n]}{(\mathcal{J}_W, x_i - gx_i)} d\mathbf{x},$$

where $\mathcal{J}_W$ is the Jacobian ideal of $W$ (this is generated by all of the first partial derivatives of $W$), and $d\mathbf{x}$ is a volume form (this will be explained later on).

**Nonabelian Example.** Let $W = x^3 + y^3 + z^3 + w^3$, and let $G \cong A_4$, where $G$ acts by permuting the variables of $W$. Let $g_1$ be the identity element of $G$. Let $g_2$ be the element corresponding to (12)(34). Then

$$\begin{aligned}
H_{g_1} &\cong \frac{\mathbb{C}[x,y,z,w]}{(3x^3, 3y^2, 3z^2, 3w^2, x-x, y-y, z-z, w-w)} dx \wedge dy \wedge dz \wedge dw \\
&\cong \frac{\mathbb{C}[x,y,z,w]}{(x^2, y^2, z^2, w^2)} dx \wedge dy \wedge dz \wedge dw \\
&\cong \operatorname{span}\{1, x, y, z, w, xy, xz, xw, yz, yw, zw, xyz, xyw, xzw, yzw, xyzw\} dx \wedge dy \wedge dz \wedge dw,
\end{aligned}$$

and

$$\begin{aligned}
H_{g_2} &\cong \frac{\mathbb{C}[x,y,z,w]}{(3x^3, 3y^2, 3z^2, 3w^2, x-y, y-x, z-w, w-z)} dx \wedge dy \wedge dz \wedge dw \\
&\cong \frac{\mathbb{C}[x_{(12)(34)}, z_{(12)(34)}]}{(x^2_{(12)(34)}, z^2_{(12)(34)})} dx_{(12)(34)} \wedge dz_{(12)(34)} \\
&\cong \operatorname{span}\{1, x_{(12)(34)}, z_{(12)(34)}, x_{(12)(34)} z_{(12)(34)}\} dx_{(12)(34)} \wedge dz_{(12)(34)}.
\end{aligned}$$

Here, $x_{(12)(34)}$ is the image of $x$ and $y$ in the quotient, and $z_{(12)(34)}$ is the image of $z$ and $w$. This is then repeated for every element of $G$. As in the abelian case, we then define the unprojected state space to be the direct sum of these sectors:

$$\mathcal{B}_{W,\{0\}} = \left( \bigoplus_{g \in G} H_g \right).$$

**Another way of finding $H_g$.** There is another way to do this construction using more linear algebra. In the previous construction, we restricted to the fixed locus of $g$ by equating each variable with its image under $g$ in a quotient. Alternatively, we can compute $W|_{\mathrm{fix}(g)}$ directly using a change of basis.

Let $k$ be the geometric multiplicity of 1 as an eigenvalue of $g$. Let $[T]$ be an eigenvector matrix of $g$, where the first $k$ columns are eigenvectors with eigenvalue 1. Let the new variables be labeled $s_1, \cdots, s_n$. Then

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = [T] \begin{pmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{pmatrix}.$$

With this new basis, restricting $W$ to the fix locus of $g$ only requires setting $s_i = 0$ for $i > k$. This gives $W|_{\text{fix}(g)}$, and $H_g$ is computed by finding the Milnor ring using the same methods as in the abelian case. This method is useful because it gives an explicit way of writing $W|_{\text{fix}(g)}$, which is required for computing Hessians later in the construction.

**Nonabelian Example.** Let $W$, $G$, $g_1$, $g_2$ be defined as in the previous example. For $g_1$, note that 1 is an eigenvalue of geometric multiplicity 4, so $T$ will be a square matrix. However, we can pick our eigenvectors to be

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

In this case, $T$ is the identity, so $\tilde{W} = W$. Hence $H_{g_1}$ is just the Milnor ring of $W$. This is equivalent to what we found using the other method.

For $g_2$, note that

$$g_2 - I_4 = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

We can then choose the eigenvectors to be

$$\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

Now, let

$$T = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

Since the geometric multiplicity of 1 is 2, we need to introduce two new indeterminants. Call them $x_{(12)(34)}$ and $z_{(12)(34)}$. Then

$$T \begin{pmatrix} x_{(12)(34)} \\ z_{(12)(34)} \end{pmatrix} = \begin{pmatrix} x_{(12)(34)} & x_{(12)(34)} & z_{(12)(34)} & z_{(12)(34)} \end{pmatrix}^T.$$

Then $\tilde{W} = W(x_{(12)(34)}, x_{(12)(34)}, z_{(12)(34)}, z_{(12)(34)}) = 2x^3_{(12)(34)} + 2z^3_{(12)(34)}$.

The Milnor ring of $\tilde{W}$ can then be computed as

$$\begin{aligned} H_g &\cong \frac{\mathbb{C}[x_{(12)(34)}, z_{(12)(34)}]}{(6x^2_{(12)(34)}, 6z^2_{(12)(34)})} dx_{(12)(34)} \wedge dz_{(12)(34)} \\ &\cong \frac{\mathbb{C}[x_{(12)(34)}, z_{(12)(34)}]}{(x^2_{(12)(34)}, z^2_{(12)(34)})} dx_{(12)(34)} \wedge dz_{(12)(34)} \\ &\cong \text{span}\{1, x_{(12)(34)}, z_{(12)(34)}, x_{(12)(34)}z_{(12)(34)}\} dx_{(12)(34)} \wedge dz_{(12)(34)} \end{aligned}$$

We can see that this is also equivalent to the other method of construction. As in the other case, this process would need to be repeated for every element in the group $G$.

The direct sum of the following spaces can be shown to form the entire unprojected state space.

$H_{(1)} = \{1, x, y, z, w, xy, xz, xw, yz, yw, zw, xyz, xyw, xzw, yzw, xyzw\} dx \wedge dy \wedge dz \wedge dw;$

$H_{(12)(34)} = \{1, x_{(12)(34)}, z_{(12)(34)}, x_{(12)(34)} z_{(12)(34)}\} dx_{(12)(34)} \wedge dz_{(12)(34)};$

$H_{(13)(24)} = \{1, x_{(13)(24)}, y_{(13)(24)}, x_{(13)(24)} y_{(13)(24)}\} dx_{(13)(24)} \wedge dy_{(13)(24)};$

$H_{(14)(23)} = \{1, x_{(14)(23)}, y_{(14)(23)}, x_{(14)(23)} y_{(14)(23)}\} dx_{(14)(23)} \wedge dy_{(14)(23)};$

$H_{(123)} = \{1, x_{(123)}, w_{(123)}, x_{(123)} w_{(123)}\} dx_{(123)} \wedge dx_{(123)};$

$H_{(132)} = \{1, x_{(132)}, w_{(132)}, x_{(132)} w_{(132)}\} dx_{(132)} \wedge dx_{(123)};$

$H_{(124)} = \{1, x_{(124)}, z_{(124)}, x_{(124)} z_{(124)}\} dx_{(124)} \wedge dz_{(124)};$

$H_{(142)} = \{1, x_{(142)}, z_{(142)}, x_{(142)} z_{(142)}\} dx_{(142)} \wedge dz_{(142)};$

$H_{(134)} = \{1, x_{(134)}, y_{(134)}, x_{(134)} y_{(134)}\} dx_{(134)} \wedge dy_{(134)};$

$H_{(143)} = \{1, x_{(143)}, y_{(143)}, x_{(143)} y_{(143)}\} dx_{(143)} \wedge dy_{(143)};$

$H_{(234)} = \{1, y_{(234)}, x_{(234)}, y_{(234)} x_{(234)}\} dy_{(234)} \wedge dx_{(234)};$

$H_{(243)} = \{1, y_{(243)}, x_{(243)}, y_{(243)} x_{(243)}\} dy_{(243)} \wedge dx_{(243)}.$

Many of these spaces are isomorphic, as the fixed locus of $g$ is equal to the fixed locus of $g^{-1}$. It is important to remember that each of these is a different sector, even if they are trivially isomorphic. In order to to simplify notation, we will denote the volume form for each sector as $e_g$.

**Taking Invariants.** After all of the $H_g$'s are computed, we take a direct sum of these spaces. This is what we call the unprojected state space. Next, we want to take $G$-invariants of this whole space, which will result in the projected state space. To do this, we make use of the map $\pi$, which is defined below. This map is called the Reynolds operator, and is a surjection from the unprojected state space onto its $G$-invariant subspace. This operator is also discussed in Chapter 3.

$$\pi : \left( \bigoplus_{g \in G} H_g \right) \to \left( \bigoplus_{g \in G} H_g \right)^G$$
$$\mathbf{x} \mapsto \frac{1}{|G|} \sum_{g \in G} \mathbf{x} \cdot g$$

The image of a basis for the unprojected state space will span the $G$-invariant space, but it may not be linearly independent. In fact, many of the basis elements may be zero under this map. An appropriate subset of the image of the basis of the unprojected state space can be chosen to use as a basis for the projected state space.

One more thing to consider is the right group action of $G$ on the unprojected state space. If we let $X$ represent our space, and $X^g = \{x | xg = x\}$ be the $g$-invariants of this space, note the following.

$$\begin{aligned}
(X^g)h &= \{xh \mid xg = x\} \\
&= \{x \mid xh^{-1}g = xh^{-1}\} \\
&= \{x \mid xh^{-1}gh = x\} \\
&= X^{h^{-1}gh}.
\end{aligned}$$

So, if we act on something that is $g$-invariant on the right by $h$, we get something that is $h^{-1}gh$ invariant. This will be important in many computations when using the $\pi$ map.

**Nonabelian Example.** We will use the same polynomial and group as above. Some of the basis elements of the unprojected state space can be shown to be $1e_1$, $xe_1$, $1e_{(12)(34)}$, and $ue_{(12)(34)}$. We will now project each of these using the $\pi$ map.

$1e_1$**:**

$$\pi(1e_1) = \frac{1}{12} \sum_{g \in G} g(1e_1).$$

Note that this basis element came from the sector $H_{(1)}$. So when we act on this by an element $g$, we should land in the sector corresponding to $g^{-1}(1)g = (1)$. Also, 1 is fixed by every element of $G$, so we really only need to consider how each $g$ acts on the volume form (the wedge product). As an example, consider the group element (12)(34).

$$\begin{aligned}
(12)(34)(e_1) &= (12)(34)(dx \wedge dy \wedge dz \wedge dw) \\
&= dy \wedge dx \wedge dw \wedge dz \\
&= -dx \wedge dy \wedge dw \wedge dz \\
&= dx \wedge dy \wedge dz \wedge dw \\
&= e_1.
\end{aligned}$$

The third and fourth steps follow from the anticommutativity of wedge products. This shows that (12)(34) fixes this volume form. In fact, all of the elements of $G$ fix this volume form, so

$$\begin{aligned}
\pi(1e_1) &= \frac{1}{12}\sum_{g\in G} g(1dx \wedge dy \wedge dz \wedge dw) \\
&= \frac{1}{12}\sum_{g\in G} 1dx \wedge dy \wedge dz \wedge dw \\
&= 1dx \wedge dy \wedge dz \wedge dw \\
&= e_1.
\end{aligned}$$

So $1e_1$ can be taken as a basis element of the projected state space.

$xe_1$**:** Next, we consider the element $xe_1$. We have already seen that every element of $G$ fixes the volume form, but we need to check how they act on $x$. We can directly compute each of these actions.

| $g$ | (1) | (12)(34) | (13)(24) | (14)(23) | (123) | (132) |
|---|---|---|---|---|---|---|
| $xg$ | $x$ | $y$ | $z$ | $w$ | $y$ | $z$ |

| $g$ | (124) | (142) | (134) | (143) | (234) | (243) |
|---|---|---|---|---|---|---|
| $xg$ | $y$ | $w$ | $z$ | $w$ | $x$ | $x$ |

.

This gives us that

$$\begin{aligned}\pi(xe_1) &= \frac{1}{12}\sum_{g\in G}(xe_1)g \\ &= \frac{1}{12}e_1(3x+3y+3z+3w) \\ &= \frac{1}{4}(x+y+z+w)e_1.\end{aligned}$$

Since this is linearly independent of the previously calculated basis element, we can choose $(x+y+z+w)e_1$ as another basis element of our projected state space.

$1e_{(12)(34)}$: Doing this computation is more difficult, as the group element is no longer central. In order to see how group elements act on variables that we introduced during the construction, it is easiest to think about them in terms of the eigenvectors. When we constructed a matrix, $T$, from the eigenvalues, we fixed an ordering. The variables that we introduced, $u_1, \cdots, u_r$, can then be equated to these eigenvectors in our state space. So, if we want to see how $h$ acts on $u_i$, we multiply the $i$th column of $T$ on the left by $h$. This will give us a vector in the eigenspace of $hgh^{-1}$. This vector can similarly be equated with some linear combination of basis elements in $H_{hgh^{-1}}$.

For example, consider how (13)(24) and (132) act on $x_{(12)(34)}$. By the above, we can equate $u$ with the corresponding eigenvector, which was $\begin{pmatrix} 1 & 1 & 0 & 0 \end{pmatrix}$. Then

$$x_{(12)(34)}(13)(24) = \begin{pmatrix} 1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 1 \end{pmatrix} = z_{(12)(34)},$$

$$x_{(12)(34)}(132) = \begin{pmatrix} 1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 & 0 \end{pmatrix} = y_{(14)(23)}.$$

Doing the rest of these calculations, we get the following table.

| $h$ | $h^{-1}(12)(34)h$ | $(x_{(12)(34)})h$ | $(z_{(12)(34)})h$ | $(dx_{(12)(34)} \wedge dz_{(12)(34)})h$ | $(e_{(12)(34)})h$ |
|---|---|---|---|---|---|
| $(1)$ | $(12)(34)$ | $x_{(12)(34)}$ | $z_{(12)(34)}$ | $dx_{(12)(34)} \wedge dz_{(12)(34)}$ | $e_{(12)(34)}$ |
| $(12)(34)$ | $(12)(34)$ | $x_{(12)(34)}$ | $z_{(12)(34)}$ | $dx_{(12)(34)} \wedge dz_{(12)(34)}$ | $e_{(12)(34)}$ |
| $(13)(24)$ | $(12)(34)$ | $z_{(12)(34)}$ | $x_{(12)(34)}$ | $-dx_{(12)(34)} \wedge dz_{(12)(34)}$ | $-e_{(12)(34)}$ |
| $(14)(23)$ | $(12)(34)$ | $z_{(12)(34)}$ | $x_{(12)(34)}$ | $-dx_{(12)(34)} \wedge dz_{(12)(34)}$ | $-e_{(12)(34)}$ |
| $(123)$ | $(13)(24)$ | $y_{(13)(24)}$ | $x_{(13)(24)}$ | $-dx_{(13)(24)} \wedge dy_{(13)(24)}$ | $-e_{(14)(23)}$ |
| $(132)$ | $(14)(23)$ | $x_{(14)(23)}$ | $y_{(14)(23)}$ | $dx_{(14)(23)} \wedge dy_{(14)(23)}$ | $e_{(13)(24)}$ |
| $(124)$ | $(14)(23)$ | $y_{(14)(23)}$ | $x_{(14)(23)}$ | $-dx_{(14)(23)} \wedge dy_{(14)(23)}$ | $-e_{(13)(24)}$ |
| $(142)$ | $(13)(24)$ | $x_{(13)(24)}$ | $y_{(13)(24)}$ | $dx_{(13)(24)} \wedge dy_{(13)(24)}$ | $e_{(14)(23)}$ |
| $(134)$ | $(13)(24)$ | $y_{(13)(24)}$ | $x_{(13)(24)}$ | $-dx_{(13)(24)} \wedge dy_{(13)(24)}$ | $-e_{(14)(23)}$ |
| $(143)$ | $(14)(23)$ | $y_{(14)(23)}$ | $x_{(14)(23)}$ | $-dx_{(14)(23)} \wedge dy_{(14)(23)}$ | $-e_{(13)(24)}$ |
| $(234)$ | $(14)(23)$ | $x_{(14)(23)}$ | $y_{(14)(23)}$ | $dx_{(14)(23)} \wedge dy_{(14)(23)}$ | $e_{(13)(24)}$ |
| $(243)$ | $(13)(24)$ | $x_{(13)(24)}$ | $y_{(13)(24)}$ | $dx_{(13)(24)} \wedge dy_{(13)(24)}$ | $e_{(14)(23)}$ |

So projecting $1e_{(12)(34)}$ gives

$$\pi(1e_{(12)(34)}) = \frac{1}{12}\sum_{g\in G}(1dx_{(12)(34)} \wedge dz_{(12)(34)})g$$
$$= \frac{1}{12}\ (1+1-1-1)(dx_{(12)(34)} \wedge dz_{(12)(34)} + dx_{(13)(24)} \wedge dy_{(13)(24)} + dx_{(14)(23)} \wedge dy_{(14)(23)})$$
$$= 0$$

Hence this element projects to 0, and provides no new elements in the projected state space basis.

$ue_{(12)(34)}$**:** We have already found all of the information we need for this computation.

$$\pi(x_{(12)(34)}e_{(12)(34)}) = \frac{1}{12}\sum_{g\in G}(x_{(12)(34)}e_{(12)(34)})g$$
$$= \frac{1}{12}[(2x_{(12)(34)} - 2z_{(12)(34)})e_{(12)(34)} + (2x_{(13)(24)} - 2y_{(13)(24)})e_{(13)(24)}$$
$$+ (2x_{(14)(23)} - 2y_{(14)(23)})e_{(14)(23)}]$$
$$= \frac{1}{6}[(x_{(12)(34)} - z_{(12)(34)})e_{(12)(34)} + (x_{(13)(24)} - y_{(13)(24)})e_{(13)(24)}$$
$$+ (x_{(14)(23)} - y_{(14)(23)})e_{(14)(23)}].$$

Because this is independent of the other basis elements (it lives in different sectors), we can take it to be an element of our basis.

This process is then repeated for every element in the basis of our unprojected state space. The following elements can then be chosen as a basis for the projected state space.

$$\begin{aligned}
A_1 =& 1e_1;\\
A_2 =& (x+y+z+w)e_1;\\
A_3 =& (xy+xz+xw+yz+yw+zw)e_1;\\
A_4 =& (xyz+xyw+xzw+yzw)e_1;\\
A_5 =& (xyzw)e_1;\\
A_6 =& e_{123}+e_{243}+e_{142}+e_{134};\\
A_7 =& x_{(123)}e_{123}+y_{(234)}e_{243}+x_{(124)}e_{142}+x_{(134)}e_{134};\\
A_8 =& w_{(123)}e_{123}+x_{(234)}e_{243}+z_{(124)}e_{142}+y_{(134)}e_{134};\\
A_9 =& x_{(123)}w_{(123)}e_{123}+y_{(234)}x_{(234)}e_{243}+x_{(124)}z_{(124)}e_{142}+x_{(134)}y_{(134)}e_{134};\\
A_{10} =& e_{132}+e_{234}+e_{124}+e_{143};\\
A_{11} =& x_{(123)}e_{132}+y_{(234)}e_{234}+x_{(124)}e_{124}+x_{(134)}e_{143};\\
A_{12} =& w_{(123)}e_{132}+x_{(234)}e_{234}+z_{(124)}e_{124}+y_{(134)}e_{143};\\
A_{13} =& x_{(123)}w_{(123)}e_{132}+y_{(234)}x_{(234)}e_{234}+x_{(124)}z_{(124)}e_{124}+x_{(134)}y_{(134)}e_{143};\\
A_{14} =& (x_{(12)(34)}-z_{(12)(34)})e_{(12)(34)}+(x_{(13)(24)}-y_{(13)(24)})e_{(13)(24)}+(x_{(14)(23)}-y_{(14)(23)})e_{(14)(23)}.
\end{aligned}$$

**Abelian Example.** In the abelian case, we can use the same projection technique. However, with an abelian group, the group action is much simpler. Consider projecting the elements of the unprojected state space previously found for $V$ and $H$. For example,

$$\begin{aligned}
\pi(\lfloor x;(0,0)\rceil) &= \frac{1}{3}\sum_{h\in H}(\lfloor x;(0,0)\rceil)h\\
&= \frac{1}{3}(\lfloor x;(0,0)\rceil + \lfloor \omega x;(0,0)\rceil + \lfloor \omega^2 x;(0,0)\rceil)\\
&= 0.
\end{aligned}$$

Note that when the monomial lives in the Milnor ring of a restricted polynomial, we omit the components of $k$ that correspond to the variables that were set to zero.

Doing this, we can pick a basis of the projected state space to be

$B_1 = \lfloor 1; 0 \rceil$;

$B_2 = \lfloor xy; 0 \rceil$;

$B_3 = \lfloor 1; h_1 \rceil$;

$B_4 = \lfloor 1; h_1^2 \rceil$.

It can be shown that a monomial $m$ will be $H$-invariant if and only if

$$\det(h) h \circ m = m, \text{ for all } h \in H.$$

Because we require that $G \leq SL(n, \mathbb{C})$, $\lfloor 1; 0 \rceil$ will always be invariant. This element is the multiplicative identity in every B-model.

#### 2.1.3 An Alternative Construction.

Using some additional group theory, it is possible to do a simpler set of calculations and get an isomorphic vector space. The steps are as follows:

- Pick conjugacy class representatives in $G$, $g_1, g_2, \cdots, g_s$;
- Construct $H_{g_1}, H_{g_2}, \cdots, H_{g_s}$ in the same way;
- Find the $C_G(g_i)$-invariants of $H_{g_i}$ for each $i$;
- Take the direct sum of all of these spaces.

After following these steps, our new formula for the projected state space is

$$\mathcal{B}_{W,G} = \bigoplus_{(g) \subset G} (H_g)^{C_G(g)}.$$

Note that this computation really is easier to do. We are taking invariants with a smaller group and we are summing over fewer spaces. Also note that each element of the centralizer of $g$ conjugates $g$ to itself. This means that when we implement the $\pi$ map to find the centralizer invariants, an element from the sector $H_g$ will map to another element in that same sector. The only reason that we do not always use this simpler method is that it is less intuitive than the other formulation. If nothing else, this means that it is unclear how to define the pairing and multiplication on this space.

Also note that the difference in these constructions is not apparent in the abelian case. In an abelian group, all elements form their own conjugacy classes.

## 2.2 Grading

Now that we have our state space constructed, the next step toward creating a graded Frobenius algebra is defining the grading. This amounts to assigning a degree to each basis element. While the state space is constructed in the same way for both of the A- and B-models, the gradings are defined differently. We include only the definition for the B-model grading below.

**2.2.1 B-Model Grading.** The formula for the grading on the B-model is:

$$deg(me_g) = 2p + \sum_{\theta_i \notin \mathbb{Z}} 1 - 2q_i.$$

In this formula, the $q_i$ are the quasihomogeneous weights of the variables. It is important to note that in the nonabelian case we have only been working with disjoint sums of homogeneous polynomials. Because of this, the group representations are direct sums, and there is a natural way to relate non-integer phases with weights. This means that this definition is well-defined.

The $\theta$'s are called the phases of $g$. These can be calculated using the eigenvalues of $g$. If $\lambda_i$ is an eigenvalue, then $log(\lambda_i)/2\pi i$ is a phase of $g$. Special care should be taken to choose a branch cut so that this value satisfies $0 \leq \theta_i < 1$.

The value of $p$ is determined by looking at the total weighted degree of $m$. When using a nonabelian group, it is possible that $m$ is no longer a monomial. However, the total weighted degree of each term in $m$ will be constant, as degree is invariant under the Reynolds operator.

**Abelian Example.** Note that in the abelian case, the phases are just the entries of the group elements in additive notation. We can also directly connect $\theta_i$ to $q_i$, so it is not necessary to use homogeneous polynomials. We compute the grading of each element in our basis for $\mathcal{B}_{V,H}$.

| Element | $B_1$ | $B_2$ | $B_3$ | $B_4$ |
|---|---|---|---|---|
| Degree | 0 | 4/3 | 2/3 | 2/3 |

**Nonabelian Example.** For this example, we will compute the degree of the element $A_{14}$. To do this, we compute the degree of one part of this element: $x_{(12)(34)} - z_{(12)(34)}$.

We have already computed the eigenspace of $(12)(34)$ corresponding to the eigenvalue 1. This space had dimension 2. We assumed that $x_{(12)(34)}$ and $z_{(12)(34)}$ have the same weights as the original variables of $W$. In this case, these are all $\frac{1}{3}$.

Next, we consider the phases, $\theta_i$. We first need to compute the eigenvalues of $(12)(34)$. These are $\{1, 1, -1, -1\}$. Taking the natural log of each of these gives $\{0, 0, \pi i, \pi i\}$. Dividing by $2\pi i$ gives $\{0, 0, \frac{1}{2}, \frac{1}{2}\}$. Because these values are in the interval $[0, 1)$, we know that they are acceptable values. If they were not, we would need to define a different branch cut of the log function.

Last, we need to find the total weighted degree of $x_{(12)(34)} - z_{(12)(34)}$. We define the degree of these new variable to be the same as the degree of the variables in the original polynomial. This means that $x_{(12)(34)} - z_{(12)(34)}$ has a total weighted degree of $\frac{1}{3}$. Hence

$$\begin{aligned} deg((x_{(12)(34)} - z_{(12)(34)})e_{(12)(34)}) &= 2p + \sum_{\theta_i \notin \mathbb{Z}} 1 - 2q_i \\ &= \frac{2}{3} + \sum_{i=1}^{2} \frac{1}{3} \\ &= \frac{4}{3}. \end{aligned}$$

Let us also consider the element $1e_1$. In the abelian construction of a Landau-Ginzburg B-model, this element is always the multiplicative identity. We should check that it's degree is still 0 in this construction, especially since we want it to still be a multiplicative identity.

Note that $p = 0$, as no variables are present. Also, all of the eigenvalues of (1) are 1. This gives that all of the phases are 0, so the sum in the formula is empty. Hence

$$deg(1e_1) = 2 \cdot 0 + 0 = 0.$$

The rest of the degrees can be computed as

| Element | $A_1$ | $A_2$ | $A_3$ | $A_4$ | $A_5$ | $A_6$ | $A_7$ |
|---|---|---|---|---|---|---|---|
| Degree | 0 | 2/3 | 4/3 | 2 | 8/3 | 2/3 | 4/3 |

| Element | $A_8$ | $A_9$ | $A_{10}$ | $A_{11}$ | $A_{12}$ | $A_{13}$ | $A_{14}$ |
|---|---|---|---|---|---|---|---|
| Degree | 4/3 | 2 | 2/3 | 4/3 | 4/3 | 2 | 4/3 |

.

## 2.3 Pairing

We next define a pairing function:

$$< \cdot, \cdot >: \mathcal{B} \times \mathcal{B} \to \mathbb{C}.$$

on the basis of the state space. This pairing must satisfy properties of symmetry, linearity, and nondegeneracy. By nondegeneracy, we mean that for every $A \in \mathcal{B}$, there exists $B \in \mathcal{B}$ such that $< A, B > \neq 0$. In the abelian case, the pairing is defined by

$$\langle me_g, ne_h \rangle = \begin{cases} \langle m, n \rangle & g = h^{-1} \\ 0 & otherwise \end{cases}$$

Here, the pairing of $m$ and $n$ is found in the same manner as in the original construction, details of which can be found in [10]. This is by solving for the pairing in the formula

$$mn = \frac{\langle m, n \rangle}{\mu_g} Hess(W|_{\text{fix}(g)}) + \text{lower order terms},$$

where $\mu_g$ is the dimension of $H_g$ as a vector space, and $Hess(W)$ is the determinant of the Hessian matrix of $W$.

Once this is done for all of the basis elements, we extend the definition linearly for the whole space. This makes our definition respect the symmetric and linear properties of pairings. In the case of an invertible polynomial and diagonal group representation, this pairing has also been proven to be nondegenerate. It remains to be proven in the case of noninvertible polynomials and in the case of nonabelian groups. We will verify that in pairing is nondegenerate in each of the B-models that we construct.

**Abelian Example.** The condition that the group elements must be inverses gives us that only $< B_1, B_2 >$ and $< B_3, B_4 >$ may be nonzero. We compute these using the formula above.

$$1xy = \frac{< B_1, B_2 >}{\mu} Hess(W) + \text{lower order terms}$$

$$1xy =< B_1, B_2 > 9xy + \text{lower order terms}$$

$$\frac{1}{9} =< B_1, B_2 >$$

$$1 = \frac{< B_3, B_4 >}{\mu|_{\text{fix}\, h_1}} Hess(W|_{\text{fix}\, h_1})$$

$$1 =< B_3, B_4 > 1$$

$$1 =< B_3, B_4 >$$

We include the pairing matrix for reference. In a pairing matrix $\eta_{i,j} =< B_i, B_j >$.

$$\eta = \begin{bmatrix} 0 & \frac{1}{9} & 0 & 0 \\ \frac{1}{9} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Table 2.1: Pairing Matrix for $\mathcal{B}_{x^3+y^3,C_3}$

**Nonabelian Example.** In the nonabelian case, we use the same definition of the pairing. This computation is much more involved, as our basis elements often include several terms. For example, let us take the element $A_{14}$ and pair it with itself.

First, we will use bilinearity to simplify.

$$\begin{aligned} \langle A_{14}, A_{14} \rangle = & \langle (x_{(12)(34)} - z_{(12)(34)}) e_{(12)(34)}, (x_{(12)(34)} - z_{(12)(34)}) e_{(12)(34)} \rangle + \\ & 2\langle (x_{(12)(34)} - z_{(12)(34)}) e_{(12)(34)}, (x_{(13)(24)} - y_{(13)(24)}) e_{(13)(24)} \rangle + \\ & 2\langle (x_{(12)(34)} - z_{(12)(34)}) e_{(12)(34)}, (x_{(14)(23)} - y_{(14)(23)}) e_{(14)(23)} \rangle + \\ & \langle (x_{(13)(24)} - y_{(13)(24)}) e_{(13)(24)}, (x_{(13)(24)} - y_{(13)(24)}) e_{(13)(24)} \rangle + \\ & 2\langle (x_{(13)(24)} - y_{(13)(24)}) e_{(13)(24)}, (x_{(14)(23)} - y_{(14)(23)}) e_{(14)(23)} \rangle + \\ & \langle (x_{(14)(23)} - y_{(14)(23)}) e_{(14)(23)}, (x_{(14)(23)} - y_{(14)(23)}) e_{(14)(23)} \rangle. \end{aligned}$$

Next, note that many of these correspond to pairs of group elements that are not inverses of each other. In each of this cases, the pairing is zero. So this simplifies further to

$$\begin{aligned}\langle A_{14}, A_{14}\rangle = &\langle (x_{(12)(34)} - z_{(12)(34)})e_{(12)(34)}, (x_{(12)(34)} - z_{(12)(34)})e_{(12)(34)}\rangle + \\ &\langle (x_{(13)(24)} - y_{(13)(24)})e_{(13)(24)}, (x_{(13)(24)} - y_{(13)(24)})e_{(13)(24)}\rangle + \\ &\langle (x_{(14)(23)} - y_{(14)(23)})e_{(14)(23)}, (x_{(14)(23)} - y_{(14)(23)})e_{(14)(23)}\rangle.\end{aligned}$$

We have to compute each of these terms directly. We first must calculate the Hessians. To do this, we compose the original variables with an appropriate eigenvector before computing $W$. For example $W|_{\text{fix}(123)}$ would be $3x_{(123)}^3 + w_{(123)}$. Using this method, it can be shown that the three Hessians that we need are $144x_{(12)(34)}z_{(12)(34)}, 144x_{(13)(24)}y_{(13)(24)}, 144x_{(14)(23)}y_{(14)(23)}$ respectively. Also, we have already computed $H_g$ for each $g$, and we know that each has dimension 4. This implies

$$\begin{aligned}(x_{(12)(34)} - z_{(12)(34)})^2 = &\frac{\langle (x_{(12)(34)} - z_{(12)(34)}), (x_{(12)(34)} - z_{(12)(34)})\rangle}{4} 144x_{(12)(34)}z_{(12)(34)} \\ &+ \text{lower order terms.}\end{aligned}$$

We simplify to

$$\begin{aligned}&x_{(12)(24)}^2 - 2x_{(12)(24)}z_{(12)(24)} + z_{(12)(24)}^2 = \\ &\langle (x_{(12)(24)} - z_{(12)(24)}), (x_{(12)(24)} - z_{(12)(24)})\rangle 36x_{(12)(24)}z_{(12)(24)} + \text{lower order terms.}\end{aligned}$$

Here, we have to use the quotient relations defined during the construction to solve for the pairing. Recall that $x_{(12)(34)}^2 = z_{(12)(34)}^2 = 0$ in this ring. So

$$\langle (x_{(12)(24)} - z_{(12)(24)}), (x_{(12)(24)} - z_{(12)(24)})\rangle = -\frac{1}{18},$$

by equating coefficients. Note that identical computations can be done for the other components that we started with. Hence the pairing of $A_{14}$ with itself is $3\langle (x_{(12)(24)} - z_{(12)(24)}), (x_{(12)(24)} -$

$z_{(12)(24)})\rangle = -\frac{1}{6}$.

Similar computations can be carried out for each pair of elements in the basis. We represent these results in a matrix. The $i, j$ coordinate is given by $\langle A_i, A_j\rangle$. Note that this pairing is still nondegenerate as required.

$$\eta = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{1}{81} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{4}{81} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{27} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{4}{81} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{81} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{4}{27} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{4}{27} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{4}{27} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{4}{27} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{4}{27} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{4}{27} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{4}{27} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{4}{27} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{6} \end{bmatrix}$$

Table 2.2: Pairing Matrix for $\mathcal{B}_{x^3+y^3+z^3+w^3, A_4}$

## 2.4 B-Model Multiplication

**2.4.1 Multiplication in Abelian Construction.** In the abelian case, multiplication is defined as follows.

$$\lfloor m; g\rceil \star \lfloor n; h\rceil = \epsilon\lfloor \gamma mn; gh\rceil,$$

where

$$\gamma = \frac{Hess(W|_{\mathrm{fix}(gh)})\mu|_{\mathrm{fix}(g)\cap\mathrm{fix}(h)}}{Hess(W|_{\mathrm{fix}(g)\cap\mathrm{fix}(h)})\mu|_{\mathrm{fix}(gh)}},$$

and $\epsilon = 1$ if every variable is fixed by at least one of $g$, $h$, and $gh$, and $\epsilon = 0$ otherwise. Note that we still reduce in the appropriate quotient rings after multiplying.

**Abelian Example.** Consider $B_1 \star B_3$. Because everything is fixed by the identity group element, $\epsilon = 1$. Also,

$$\begin{aligned}\gamma &= \frac{Hess(W|_{\text{fix}(gh)})\mu|_{\text{fix}(g)\cap\text{fix}(h)}}{Hess(W|_{\text{fix}(g)\cap\text{fix}(h)})\mu|_{\text{fix}(gh)}} \\ &= \frac{1\cdot 1}{1\cdot 1} \\ &= 1.\end{aligned}$$

So,

$$\begin{aligned}B_1 \star B_3 &= \lfloor 1; 0 \rceil \star \lfloor 1; h_1 \rceil \\ &= \lfloor 1; h_1 \rceil \\ &= B_3.\end{aligned}$$

Consider $B_3 \star B_4$. Neither group element fixes any variables, but the product is the identity, so $\epsilon = 1$.

$$\begin{aligned}\gamma &= \frac{Hess(W|_{\text{fix}(gh)})\mu|_{\text{fix}(g)\cap\text{fix}(h)}}{Hess(W|_{\text{fix}(g)\cap\text{fix}(h)})\mu|_{\text{fix}(gh)}} \\ &= \frac{36xy\cdot 1}{1\cdot 4} \\ &= 9xy.\end{aligned}$$

So,

$$\begin{aligned}B_3 \star B_4 &= \lfloor 1; h_1 \rceil \star \lfloor 1; h_1^2 \rceil \\ &= \lfloor 9xy; 0 \rceil \\ &= 9B_2.\end{aligned}$$

We include the multiplication table for the rest of the products.

| $\star$ | $B_1$ | $B_2$ | $B_3$ | $B_4$ |
|---|---|---|---|---|
| $B_1$ | $B_1$ | $B_2$ | $B_3$ | $B_4$ |
| $B_2$ | $B_2$ | $0$ | $0$ | $0$ |
| $B_3$ | $B_3$ | $0$ | $0$ | $9B_2$ |
| $B_4$ | $B_4$ | $0$ | $9B_2$ | $0$ |

Table 2.3: Multiplication Table for $\mathcal{B}_{x^3+y^3,C_3}$

### 2.4.2 Multiplication in Nonabelian Construction.

Generalizing this multiplication to state spaces constructed with nonabelian groups is not a simple process. Now that we have a pairing, it is at least possible to define a multiplication that is consitent with the abelian definition. So far, there is no easy way to compute all of the products. It is also unclear as to what conditions are necessary to make this multiplication associative. The current method of multiplication makes use of many maps. These maps are shown in Figure 2.1.

$$\begin{tikzcd}
H_g \arrow[dr, two heads] & & & \\
& H_{g,h} \arrow[dd, "\flat"] & H_{(gh)^{-1}} \arrow[l, two heads, "f"'] \arrow[r] & H_{gh} \\
H_h \arrow[ur, two heads] & & & \\
& H^*_{g,h} \arrow[r, "f^*"] & H^*_{(gh)^{-1}} \arrow[uu, "\sharp"'] &
\end{tikzcd}$$

Figure 2.1: Multiplication Diagram

The idea here is that we want to multiply an element of $H_g$ with an element of $H_h$. This should result in an element of $H_{gh}$. There is not a natural map into this space. However, all three spaces can be projected into $H_{g,h}$ (restricting the the fixed loci of both $g$ and $h$). On the diagram, we label the map from $H_{(gh)^{-1}}$ to $H_{g,h}$ with $f$. What we need to find, however, is a map between these spaces in the other direction. To do this, we make use of musical isomorphisms.

It is possible to define a map from $H_{g,h}$ to $H^*_{g,h}$ (the asterisk denotes the dual space) using the pairing function. If $m$ is an element of $H_{g,h}$ then this map, which we denote $\flat$, acts by:

$$\flat : m \mapsto \langle m, \cdot \rangle.$$

Next, we use the adjoint of $f$, which is labeled $f^*$, to map from $H^*_{g,h}$ to $H^*_{(gh)^{-1}}$. This map works by:

$$f^* : \langle m, \cdot \rangle \mapsto \langle m, f(\cdot) \rangle.$$

Now, suppose that $H_{(gh)^{-1}}$ has a n-dimensional basis with basis $b_1, \cdots, b_n$. Then we define a row vector

$$[\langle m, f(b_1) \rangle, \cdots, \langle m, f(b_n) \rangle].$$

There exists an element $\lambda$ in $H_{(gh)^{-1}}$ such that the vector

$$[\langle \lambda, b_1 \rangle, \cdots, \langle \lambda, b_n \rangle].$$

is equal to our previous vector. The $\sharp$ map sends our element in $H^*_{(gh)^{-1}}$ to this $\lambda$.

Note that $H_{(gh)^{-1}} \cong H_{gh}$ in an almost trivial manner. This means that the last map on the diagram is easy to compute.

It is important to note that in the diagonal case, these $\flat$ and $\sharp$ maps provide the $\gamma$ present in the multiplication formula:

$$\gamma = \frac{Hess(W|_{\mathrm{fix}(gh)})\mu|_{\mathrm{fix}(g)\cap\mathrm{fix}(h)}}{Hess(W|_{\mathrm{fix}(g)\cap\mathrm{fix}(h)})\mu|_{\mathrm{fix}(gh)}}.$$

These factors appear when taking the pairings in the respective spaces.

**2.4.3 Projecting into $H_{g,h}$.** When projecting from $H_g$ and $H_h$ into $H_{g,h}$ in this multiplication, we must multiply by a factor of $\epsilon$. In the diagonal case, this $\epsilon$ is always either 0 or 1. If $G$ is diagonal, then $\epsilon = 1$ if and only if every variable is fixed by at least one of $g$, $h$, and $gh$. This definition is insufficient in the nonabelian case. In the example below, we suppress the epsilons thoughout the computation. We will scale the final products by choosing specific values for the different epsilons that would arise.

While insufficient, we still have a condition that forces $\epsilon = 0$. If $dim(R) \neq 0$, with $R$ defined as below, then $\epsilon = 0$.

$$R = \ominus X \oplus X^{\vec{m}} \oplus \sum_{i=1}^{3} S_{m_i}$$

We now define each piece of this equation.

$\vec{m} = (g, h, (gh)^{-1})$.

$X = \mathbb{C}^n$, where $n$ is the number of variables in $W$.

$X^{\vec{m}} = \bigcap_{i=1}^{3} X|_{\text{fix}(m_i)}$.

$S_i = \sum_{k=1}^{r-1} \frac{k}{r} E_{m_i,k}$.

$r = |m_i|$, the order of the group element.

$E_{m_i,k}$ is the eigenspace of $m_i$ corresponding to the eigenvalue $e^{2\pi ik/r}$.

When computing the dimension of $R$, we do not worry about how fractions of different spaces are added together, we simply sum up the degrees of all of the pieces.

**Example of $\epsilon$ Condition.** Consider $e_{123} \star e_{123}$.

$$\vec{m} = ((123),(123),(123))$$
$$X = \mathbb{C}^4$$
$$X^{\vec{m}} = \operatorname{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$
$$S_{123} = \frac{1}{3} \operatorname{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 1 \\ \omega \\ \omega^2 \\ 0 \end{pmatrix} \right\} + \frac{2}{3} \operatorname{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 1 \\ \omega^2 \\ \omega \\ 0 \end{pmatrix} \right\}.$$

Hence

$$dim(R) = -4 + 2 + 3 = 1 \neq 0.$$

Thus $e_{(123)} \star e_{(123)} = 0$.

**2.4.4 Nonabelian Example.** We will compute the product of $A_6$ and $A_{10}$. First, note that the definitions give that this multiplication is distributive. We first consider the case where $e_g \neq e_{g^{-1}}$. We can show that our condtion for $\epsilon = 0$ is not satisfied, so we must use the multiplication diagram. If we multiply $e_{123}$ and $e_{234}$, it can be shown that we eventually get a scalar multiple of $)x_{(12)(34)} + z_{12}34)e_{(12)(34)}$. While this does respect the grading, it is not $G$-invariant. When we add in all of the other products, we will still have something not $G$-invariant. Because of this, we set these terms to zero.

So $A_6 \star A_{10}$ reduces quickly to $e_{123} \star e_{132} + e_{243} \star e_{234} + e_{142} \star e_{124} + e_{134} \star e_{143}$. We will show how to compute $e_{123} \star e_{132}$, and the other three products are very similar.

Note that $(123)(132) = 1$, so $H_{gh} = H_1$. Also, note that $H_{123}$ and $H_{132}$ have the same fixed locus, so the projections into $H_{123 \cap 132}$ are inclusion maps. After projecting from both spaces, we take the product in that space. This gives us $1 * 1 = 1$.

Next, we implement the $\flat$ map, and send 1 to $\langle 1, \cdot \rangle$. The adjoint of $f$ then maps this element to $\langle 1, f(\cdot) \rangle$. This is still the pairing in $H_{123 \cap 132}$, even though it is a function on $H_1$. We now evaluate this function on each of the basis elements of $H_1$ (Remember that this is the unprojected Milnor ring. Don't use only the invariant parts). We will use the same ordering as when this Milnor ring was computed and write the results in a vector.

$$\left[0, 0, 0, 0, 0, 0, 0, \frac{1}{27}, 0, \frac{1}{27}, \frac{1}{27}, 0, 0, 0, 0, 0\right].$$

Now, we must consider the pairing matrix of $H_1$. Again, this is not the same as the pairing matrix that we found earlier as we have a different basis. Computing this pairing matrix can be done using the exact same method from the abelian case. It happens to have $\frac{1}{81}$ on the main anti-diagonal and 0 in all other places. This means that we need to find an element that, when paired with each of $1, x, y, z, w, xy, xz, yz, xyz, xyw, xzw, yzw, xyzw$ is 0 and when paired with each of $xw, yw, zw$ is $\frac{1}{27}$. Because our pairing matrix is nicely anti-diagonal, this is easily computed to have the solution $3yz + 3xz + 3xy$.

Notice that this is not an element of our invariant space. That is because we have only computed $e_{123} \star e_{132}$. When we finish out the multiplication of the other terms, we get $6(xy + xz + xw + yz + yw + zw)$, which is $6A_3$.

If we leave all values of $\epsilon$ as 0 or 1, it can be shown that the multiplication is not associative. To rectify this, we replace $\frac{27}{4}$ with 3. This is effectively setting $\epsilon = \frac{4}{9}$ in several cases. The complete multiplication table is given as Table 2.4.4. The lines separate the different conjugacy classes corresponding to the sectors. Blank entries are zero. Sage confirms that the multiplication as presented is associative. However, this multiplication does not respect the pairing. Namely, it does not satisfy the Frobenius property:

$$< A_i \star A_j, A_k > = < A_i, A_j \star A_k > . \tag{2.1}$$

| $\star$ | $A_1$ | $A_2$ | $A_3$ | $A_4$ | $A_5$ | $A_6$ | $A_7$ | $A_8$ | $A_9$ | $A_{10}$ | $A_{11}$ | $A_{12}$ | $A_{13}$ | $A_{14}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $A_1$ | $A_1$ | $A_2$ | $A_3$ | $A_4$ | $A_5$ | $A_6$ | $A_7$ | $A_8$ | $A_9$ | $A_{10}$ | $A_{11}$ | $A_{12}$ | $A_{13}$ | $A_{14}$ |
| $A_2$ | $A_2$ | $2A_3$ | $3A_4$ | $4A_5$ | | $3A_7 + A_8$ | $A_9$ | $3A_9$ | | $3A_{11} + A_{12}$ | $A_{13}$ | $3A_{13}$ | | |
| $A_3$ | $A_3$ | $3A_4$ | $6A_5$ | | | $3A_9$ | | | | $3A_{13}$ | | | | |
| $A_4$ | $A_4$ | $4A_5$ | | | | | | | | | | | | |
| $A_5$ | $A_5$ | | | | | | | | | | | | | |
| $A_6$ | $A_6$ | $3A_7 + A_8$ | $3A_9$ | | | $3(A_{11} + A_{12})$ | $3A_{13}$ | $3A_{13}$ | | $6A_3$ | $3A_4$ | $9A_4$ | $12A_5$ | |
| $A_7$ | $A_7$ | $A_9$ | | | | $3A_{13}$ | | | | $3A_4$ | | $12A_5$ | | |
| $A_8$ | $A_8$ | $3A_9$ | | | | $3A_{13}$ | | | | $9A_4$ | $12A_5$ | | | |
| $A_9$ | $A_9$ | | | | | | | | | $12A_5$ | | | | |
| $A_{10}$ | $A_{10}$ | $3A_{11} + A_{12}$ | $3A_{13}$ | | | $6A_3$ | $3A_4$ | $9A_4$ | $12A_5$ | $3(A_7 + A_8)$ | $3A_9$ | $3A_9$ | | |
| $A_{11}$ | $A_{11}$ | $A_{13}$ | | | | $3A_4$ | | $12A_5$ | | $3A_9$ | | | | |
| $A_{12}$ | $A_{12}$ | $3A_{13}$ | | | | $9A_4$ | $12A_5$ | | | $3A_9$ | | | | |
| $A_{13}$ | $A_{13}$ | | | | | $12A_5$ | | | | | | | | |
| $A_{14}$ | $A_{14}$ | | | | | | | | | | | | | $-6A_5$ |

Table 2.4: Multiplication Table for $\mathcal{B}_{x^3+y^3+z^3+w^3,A_4}$

## 2.5 Another Nonabelian Example

In order to demonstrate that in some cases, we can still get an associative multiplication by picking $\epsilon$ to be either 0 or 1, we work another example. This example will also show another problem that can arise in defining the multiplication. Let $W = x^2y + xy^2 + z^2$. We use the $G \cong S_3$, where

$$(1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, (123) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} (132) = \begin{pmatrix} -1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(12) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, (13) = \begin{pmatrix} 1 & 0 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} (23) = \begin{pmatrix} -1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

We next compute the 1-eigenspaces. For (1), everything is fixed. All other group elements have one-dimensional fixed loci, which we list bases for below.

$$(123), (132) : \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}; (12) : \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}; (13) : \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}; (23) : \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}.$$

The unprojected state spaces for these are as follows.

$H_1 = \{1, x, y, xy\} dx \wedge dy$;

$H_{123} = \{1\} e_{123}$;

$H_{132} = \{1\} e_{132}$;

$H_{12} = \{1, x_{12}\} e_{12}$;

$H_{13} = \{1, x_{13}\} e_{13}$;

$H_{23} = \{1, x_{23}\} e_{23}$.

Using the Reynolds operator, we next project these twelve elements in order to compute the basis of the projected state space. This gives us the following basis:

$A_1 = e_1$;

$A_2 = xye_1$;

$A_3 = e_{123} - e_{132}$;

$A_4 = e_{12} + e_{13} + e_{23}$;

$A_5 = x_{12}e_{12} + x_{13}e_{13} + x_{23}e_{23}$.

Next, we compute the B-model degrees of these basis elements.

| Element | $A_1$ | $A_2$ | $A_3$ | $A_4$ | $A_5$ |
|---|---|---|---|---|---|
| Degree | 0 | 4/3 | 2/3 | 1/3 | 1 |

We follow the same steps to define the multiplication of this model. Again, we suppress $\epsilon$ throughout. These steps successfully compute the multiplication of all pairs of basis elements except for $A_4 \star A_4$.

As in the previous example, all off the non-square terms must be zero because the products are not $G$-invariant.

The square terms must also be zero, but for a different reason. The degree of $A_4$ is 1/3, so the degree of $A_4 \star A_4$ should be 2/3. The product of any of the terms of $A_4$ and itself lives in $H_1$, but this has nothing of weight 2/3.

Setting these products to zero is justified by saying that $\epsilon$ must equal zero in this case. Note that if we were to let $\epsilon$ be a linear combination of $x$ and $y$, we would indeed get something $G$-invariant of the right degree. There is no indication as to why this should done, and at this point, setting $\epsilon = 0$ is preferrable.

The multiplication table, assuming $A_4 \star A_4 = 0$, is given as Table 2.5, and its multiplicative associativity can easily be verified. Note however, that

$$< A_1 \star A_4, A_5 >= 1 \neq \frac{1}{2} =< A_1, A_4 \star A_5 >,$$

| $\star$ | $A_1$ | $A_2$ | $A_3$ | $A_4$ | $A_5$ |
|---|---|---|---|---|---|
| $A_1$ | $A_1$ | $A_2$ | $A_3$ | $A_4$ | $A_5$ |
| $A_2$ | $A_2$ | 0 | 0 | 0 | 0 |
| $A_3$ | $A_3$ | 0 | $-6A_2$ | 0 | 0 |
| $A_4$ | $A_4$ | 0 | 0 | 0 | $3A_2$ |
| $A_5$ | $A_5$ | 0 | 0 | $3A_2$ | 0 |

Table 2.5: Multiplication Table for $\mathcal{B}_{x^2y+xy^2+z^2,S_3}$

so this algebra does not satisfy the Frobenius property. If we scale the product $A_5 \star A_5$ by a factor of 2, the Frobenius property will be satisfied and the multiplication will still be associative.

**2.5.1 Remarks.** We set some of the products above to zero because they were not $G$-invariant. In most of these cases, the products looked similar to something that was $G$-invariant. For example, we would get something that looked like $x + z$, which was not invariant while $x - z$ was. We could force the product to be invariant by redefining the map $f$ from $H_{(gh)^{-1}}$ to $H_{g,h}$ in another way. Similarly, we could make the product invariant by allowing for multiple $\epsilon$ values in the same product of basis elements. This is less natural than simply setting such products to zero, which is what we have done in the examples. If we simply wanted to define an associative multiplication, we could also define all products to be zero, but this is not enlightening.

## 2.6 All Narrow Sectors

Even though much needs to be proven about this multiplication in general, there are things we can say about special cases. We consider B-models with no non-identity broad sectors.

**Definition 2.1.** A sector is said to be narrow if it has trivial fixed locus, or equivalently if 1 is not an eigenvalue of the matrix representation of the group element. If a sector is not narrow, it is called broad.

**Lemma 2.2.** *All basis elements from narrow sectors in the same B-model have the same*

*B-model degree.*

*Proof* : Recall that the B-model degree is given as

$$deg(me_g) = 2p + \sum_{\theta_i \notin \mathbb{Z}} 1 - 2q_i \tag{2.2}$$

If $A_j$ and $A_k$ are two basis elements from narrow sectors, then $p = 0$ for both. This is because all of the variables of the original polynomial are zero when restricted to the fixed locus of each group element.

Also, the summand in (2.2) will be equal in both cases, as $\theta_i \notin \mathbb{Z}$ for all $i$. Since every part of the formula is equal, the total degrees of $A_j$ and $A_k$ are equal.

Note that if at least one weight is strictly less than $\frac{1}{2}$, this common degree is strictly positive. This is, however, also required in the construction for other reasons.

**Proposition 2.3.** *Let $W$ be an admissible polynomial, and $G \subset SL(n, \mathbb{C})$ a corresponding symmetry group. If all non-identity sectors are narrow, then the B-model constructed from $W$ and $G$ is associative, and requires no additional scaling, meaning $\epsilon$ can be chosen to be* 0 *or* 1 *in every case, in defining the multiplication.*

*Proof* : By construction, basis elements consist of sums of elements from sectors whose group elements are conjugate in $G$. Because the identity is central, no basis element can contain terms from the identity sector and any other sector. Using this, we will show that the following equation holds for all choices of basis elements.

$$A_i \star (A_j \star A_k) = (A_i \star A_j) \star A_k \tag{2.3}$$

If any of the three terms in a product is the identity, associativity follows trivially, so suppose all of them are not the identity element. Similarly, if all of $A_i$, $A_j$, and $A_k$ are in the identity sector, then associativity is inherited from the Milnor ring of $W$. Hence, it suffices to show that if any of $A_i$, $A_j$, and $A_k$ are in narrow sectors, then the products on both sides of (2.3) are zero. We start by proving existence of basis elements in narrow sectors.

We can compute the basis elements from narrow sectors in general. If $e_g$ is the volume form corresponding to the narrow sector $H_g$, then the Reynolds operator applied to $e_g$ is

$$\begin{aligned}\pi(e_g) &= \sum_{h\in G} e_g \cdot h \\ &= \sum_{h\in G} e_{h^{-1}gh} \\ &= |C_G(g)| \sum_{h\in(g)} e_h\end{aligned}$$

Here, $(g)$ means the conjugacy class of $g$. This last equality follows from the Orbit-Stabilizer Theorem. Note that if $h$ is conjugate to $g$, then $\pi(e_h) = \pi(e_g)$. As the Reynolds operator preserves conjugacy classes, this is sufficient to show that each narrow sector will have exactly one basis element. This proof of existence means that we must consider a few cases.

If exactly two of $A_i$, $A_j$, and $A_k$ are in the identity sector, then the final product is necessarily zero, so associativity holds. This follows from that fact that either the first or second multiplication computed in each side of (2.3) will include both a nonconstant polynomial and exactly one nontrivial group element. Since every sector except the identity is narrow, the restricted Milnor ring in each nonidentity sector is just $\mathbb{C}$, so a nonconstant polynomial becomes zero in the quotient.

We now consider the case where exactly one of $A_i$, $A_j$, and $A_k$ is in the identity sector. If this element is $A_j$, then the above argument gives that both sides of (2.3) are zero, so assume without loss of generality that $A_i$ is in the identity sector. The right hand side of (2.3) is still zero, but we must justify why the left hand side is also zero.

By Lemma 2.2, all elements of nonidentity sectors are of the same degree, so if the product $A_j \star A_k$ is nonzero, it must belong to the identity sector. If this were not the case, the multiplication would not respect the grading. Hence $\lfloor 1; g \rceil \star \lfloor 1; h \rceil = 0$ if $g$ and $h$ are both nontrivial and $h \neq g^{-1}$.

Consider the B-model multiplication of $\lfloor 1; g \rceil$ and $\lfloor 1; g^{-1} \rceil$. According to the generalized multiplication as defined by Sandberg, this product is a scalar multiple of $Hess(W)$ in the identity sector. This gives that the product of $A_j$ and $A_k$ is a sum of scalar multiples of the identity and some number of zeros, and is thus a scalar (possibly zero) multiple of $Hess(W)$. Because $A_i$ is not the identity, it has strictly positive degree. Since the hessian has maximal degree, we get that the left hand side of (2.3) is zero as desired.

If none of $A_i$, $A_j$, and $A_k$ are in the identity sector, by the above reasoning, the first product computed on each side will be a scalar multiple of the identity. The second product will then include both a nonconstant polynomial and exactly one nontrivial group element. We have already shown that any such product is zero. Therefore (2.3) holds in this case, too.

By exhaustion of cases, this proves the proposition.

**2.6.1 Example.** Let $W = x^4 + y^4$, and let $G$ by $Q_8$ with representation

$$i = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \; j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \; k = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}.$$

Note that all non-identity sectors are narrow, so the B-model that is constructed using this polynomial and group is associative, and requires no additional scaling during the multiplication.

We can pick our basis to be the following:

$$A_1 = e_1;$$

$$A_2 = e_{-1};$$

$$A_3 = e_i + e_{-i};$$

$$A_4 = e_j + e_{-j};$$

$$A_5 = e_k + e_{-k};$$

$$A_6 = x^2y^2e_1.$$

Then, using the definition of the multiplication without scaling, we get the multiplication table below. Using Table 2.6, it is easy to verify that this B-model is in fact associative.

| $\star$ | $A_1$ | $A_2$ | $A_3$ | $A_4$ | $A_5$ | $A_6$ |
|---|---|---|---|---|---|---|
| $A_1$ | $A_1$ | $A_2$ | $A_3$ | $A_4$ | $A_5$ | $A_6$ |
| $A_2$ | $A_2$ | $16A_6$ | 0 | 0 | 0 | 0 |
| $A_3$ | $A_3$ | 0 | $32A_6$ | 0 | 0 | 0 |
| $A_4$ | $A_4$ | 0 | 0 | $32A_6$ | 0 | 0 |
| $A_5$ | $A_5$ | 0 | 0 | 0 | $32A_6$ | 0 |
| $A_6$ | $A_6$ | 0 | 0 | 0 | 0 | 0 |

Table 2.6: Multiplication Table for $\mathcal{B}_{x^4+y^4,Q_8}$

By direct computation, we can see that the pairing, as defined in Chapter 2, satisfies the Frobenius property. Hence this is actually the first example given of a Frobenius algebra constructed using a nonabelian group.

# Chapter 3. Starting from the Group

Given an admissible polynomial, it is difficult to find nonabelian symmetry groups. This often requires solving large systems on nonlinear equations. It becomes even more difficult if we impose restrictions such as a maximum order, requiring a real representation, or requiring that the representation be contained in $SL(n, \mathbb{C})$. To avoid this problem, we attempt to go the other direction.

## 3.1 Invariant Rings

We next look for ways to simplify the above construction using invariant theory found in [16]. Rather than pick a polynomial and look for nonabelian symmetries, we can pick a representation of a nonabelian group, and then pick an admissible polynomial from its ring of invariants. If the representation for a group $G$ is of degree $n$, then we define the ring of invariants as:

$$\mathbb{C}[x_1, \cdots, x_n]^G = \{f \in \mathbb{C}[x_1, \cdots, x_n] \mid f(g \cdot \mathbf{x}) = f(\mathbf{x}),\ \forall g \in G\}.$$

WRecall the Reynolds operator on a polynomial ring:

$$\pi(\mathbf{x}) = \frac{1}{|G|} \sum_{g \in G} \mathbf{x} \cdot g.$$

If $\mathbf{x} = x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}$ is a monomial in $\mathbb{C}[x_1, \cdots, x_n]$, define $\beta(\mathbf{x}) = b_1 + b_2 + \cdots + b_n$.
A theorem by Emmy Noether gives a bound on how many generators are needed to form the whole invariant ring. As presented in [16],

$$\mathbb{C}[x_1, \cdots, x_n]^G = \mathbb{C}[\pi(\mathbf{x}) \mid \mathbf{x} \in \mathbb{C}[x_1, \cdots, x_n],\ \beta(\mathbf{x}) \leq |G|].$$

Hence we only have to consider the image under the Reynolds operator of the monomials of total degree less than or equal to the order of the group. We can improve this bound even further by Molien's Theorem, given in [17], which tells us exactly how many generators we need of each total degree. The Molien series is defined to be:

$$\Phi_G(t) = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(I_n - tg)},$$

where $I_n$ is the $n \times n$ identity matrix. Molien's Theorem states that the coefficient of $t^k$ in this power series is the dimension of the subspace of $\mathbb{C}[x_1, \cdots, x_n]^G$ consiting of all of the homogeneous polynomials of degree $k$.

## 3.2 An example with $Q_8$

Consider the group $Q_8$ with representation:

$$i = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \; j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \; k = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}.$$

The aforementioned theorem gives that we only need to look for generators with degree at most 8. We compute the Molien series of this representation up to the $t^8$ term.

$$\Phi_G(t) = 1 + 2 * t^4 + 1 * t^6 + 3 * t^8 + \cdots$$

This gives that we only need to compute two polynomials of degree 4, one of degree 6, and three of degree 8. All other invariant polynomials can be formed using this eight polynomials. We compute these six polynomials using the Reynolds operator. After scaling, we get the following polynomials:

(i) $x^4 + y^4$;

(ii) $x^2y^2$;

(iii) $x^5y - xy^5$;

(iv) $x^8 + y^8$;

(v) $x^6y^2 + x^2y^6$;

(vi) $x^4y^4$.

Note that of these polynomials, (i), (iii), and (iv) are admissible. In fact, because $G \subset SL(2, \mathbb{C})$, we can construct a B-model using any of these three polynomials with this group. However, the only group element that has 1 as an eigenvalue in this representation is the identity. Because all of the non-identity sectors are narrow, we know from Section 2.6 that the B-model will be associative.

# Chapter 4. $P_8$ B-Model Associativity

As we attempt to define an associative multiplication on these generalized models, it is helpful to also consider the multiplication of B-models created from noninvertible polynomials. Krawitz's proof in [10] that the multiplication is associative for invertible polynomials is done by considering each atomic type separately, and then showing that sums of atomic types also lead to associative multiplication. As no such decomposition of noninvertible polynomials exist, this method of proof cannot work in general. Below, we explicitly show that any B-model constructed with $W = x^3 + y^3 + z^3 + \alpha xyz$ $P_8$, where $\alpha \neq \sqrt[3]{-27}$, is associative. This polynomial, classified as $P_8$ by Vladimir Arnold, is the canonical example of a noninvertible polynomial.

## 4.1 B-Model Construction

Let $W = x^3 + y^3 + z^3 + \alpha xyz$, where $\alpha \neq 0, \sqrt[3]{-27}$. If $\alpha = 0$, then $W$ is invertible, and thus is included in Krawitz's proof. If $\alpha = \sqrt[3]{-27}$, then $W$ is not nondegenerate.

We consider two different groups: $G = \langle g \rangle = \langle (\frac{1}{3}, \frac{2}{3}, 0) \rangle$ and $G_J = \langle j \rangle = \langle (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \rangle$. Note that symmetry gives us that these two cases suffice to show that the B-model created with $P_8$ is associative, regardless of which nontrivial subgroup of $G_W^{max} \cap SL(3, \mathbb{C})$ is used. We exclude the trivial group, as the B-model is then just the Milnor ring, which is clearly associative.

The Jacobian ideal of $W$ is $\mathcal{J} = (3x^2 + \alpha yz, 3y^2 + \alpha xz, 3z^2 + \alpha xy)$.

A basis for the Milnor ring of $W$ is $\{1, x, y, z, xy, xz, yz, xyz\}$. Denote the Milnor ring by $\mathcal{Q}_W$, and the dimension of this Milnor ring by $\mu = 8$.

Next, we compute the determinant of the Hessian matrix of $W$.

$$Hess(W) = \det \begin{vmatrix} 6x & \alpha z & \alpha y \\ \alpha z & 6y & \alpha x \\ \alpha y & \alpha x & 6z \end{vmatrix}$$

$$= \alpha^3 xyz - 6\alpha^2 y^3 + +\alpha^3 xyz - 6\alpha^2 z^3 - 6\alpha^2 x^3 + 216xyz$$

$$= 2\alpha^3 xyz - 2\alpha^2(3x^3 + 3y^3 + 3z^3) + 216xyz$$

$$= (2\alpha^3 + 216)xyz - 2\alpha^2(-3\alpha xyz)$$

$$= (8\alpha^3 + 216)xyz.$$

Note that the assumption that $\alpha \neq \sqrt[3]{-27}$ gives that $Hess(W) \neq 0$.

**4.1.1 Finding $G_J$-Invariants.** Projecting our state space, we can pick a basis for $\mathcal{B}_{W,G_J}$ to be

$B_1 = \lfloor 1; 0 \rceil$;

$B_2 = \lfloor 1; j \rceil$;

$B_3 = \lfloor 1; j^2 \rceil$;

$B_4 = \lfloor xyz; 0 \rceil$;

To demonstrate that the pairing is nondegenerate in this case, we include the pairing matrix. We next compute all of the products. Note that $B_1$ is the multiplicative identity. Note that

$$\eta = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{\alpha^3+27} \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{\alpha^3+27} & 0 & 0 & 0 \end{bmatrix}$$

Table 4.1: Pairing Matrix for $\mathcal{B}_{P_8,G_J}$

by considering the group components of these basis elements,

$$B_2 \star B_2 = B_3 \star B_3 = 0.$$

Also, $x = 0$ in $\mathcal{Q}_W$ when restricted to the fixed locus of $j$ or $j^2$. This gives that $B_2 \star B_4 = B_3 \star B_4 = 0$.

Now, we compute $B_2 \star B_3$. Note that $j \cdot j^2 = 0$, and that $Hess(W|_{\mathrm{fix}(j)}) = \mu|_{\mathrm{fix}(j)} = 1$. So

$$\begin{aligned} B_2 \star B_3 &= \left\lfloor \frac{Hess(W)\mu|_{\mathrm{fix}(j)}}{Hess(W|_{\mathrm{fix}(j)})\mu}; 0 \right\rceil \\ &= \left\lfloor \frac{(8\alpha^3 + 216)xyz}{8}; 0 \right\rceil \\ &= (\alpha^3 + 27)B_4 \end{aligned}$$

Collectively, this gives the multiplication table below. Note that any nonzero product of

| $\star$ | $B_1$ | $B_2$ | $B_3$ | $B_4$ |
|---|---|---|---|---|
| $B_1$ | $B_1$ | $B_2$ | $B_3$ | $B_4$ |
| $B_2$ | $B_2$ | $0$ | $(\alpha^3 + 27)B_2$ | $0$ |
| $B_3$ | $B_3$ | $(\alpha^3 + 27)B_2$ | $0$ | $0$ |
| $B_4$ | $B_4$ | $0$ | $0$ | $0$ |

Table 4.2: Multiplication Table for $\mathcal{B}_{P_8,G_J}$

three basis elements necessarily includes the identity. This means that associativity follows from commutativity.

**4.1.2 Finding the $G$-Invariants.** Projecting our state space, we can pick a basis for $\mathcal{B}_{W,G}$ to be:

$B_1 = \lfloor 1; 0 \rceil$;

$B_2 = \lfloor z; 0 \rceil$;

$B_3 = \lfloor 1; g \rceil$;

$B_4 = \lfloor 1; g^2 \rceil$;

$B_5 = \lfloor xy; 0 \rceil$;

$B_6 = \lfloor z; g \rceil$;

$B_7 = \lfloor z; g^2 \rceil$;

$B_8 = \lfloor xyz; 0 \rceil$.

To demonstrate that the pairing is nondegenerate in this case, we include the pairing matrix. We next consider the multiplication. Note that $B_1$ is still the multiplicative identity

$$\eta = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\alpha^3+27} \\ 0 & 0 & 0 & 0 & \frac{1}{\alpha^3+27} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{\alpha^3+27} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{\alpha^3+27} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Table 4.3: Pairing Matrix for $\mathcal{B}_{P_8,G}$

element.

Also, note that $W|_{\text{fix}(g)} = z^3$, so $Hess(W|_{\text{fix}(g)}) = 6z$, and $\mu|_{\text{fix}(g)} = 2$.

The following products are zero by considering the sectors' fixed loci: $B_3 \star B_3$, $B_3 \star B_6$, $B_6 \star B_6$, $B_4 \star B_4$, $B_4 \star B_7$, $B_7 \star B_7$.

The following products are zero because of the quotient relations in $\mathcal{Q}_W$: $B_5 \star B_5$, $B_8 \star B_8$, $B_2 \star B_8$, $B_5 \star B_8$, $B_6 \star B_7$.

The following products are zero because of the quotient relations in $\mathcal{Q}_W|_{\text{fix}(g)}$: $B_2 \star B_6$, $B_2 \star B_7$, $B_3 \star B_5$, $B_3 \star B_8$, $B_4 \star B_8$, $B_5 \star B_6$, $B_5 \star B_7$, $B_6 \star B_8$, $B_7 \star B_8$.

The rest of the products are computed directly. We will summarize the results in the table below. Also, let $r = \frac{\alpha^3+27}{3}$.

Consider $(B_i \star B_j) \star B_k \neq 0$, with $1 \leq i \leq j \leq k \leq 8$. Note that we can impose this ordering because multiplication is commutative. If $i \neq 1$, then $(i, j, k)$ is either $(2, 2, 2)$ or $(2, 3, 4)$. If one of $i = 1$, then associativity follows immediately because $B_1$ is the multiplicative identity. If $i = j = k = 2$, then associativity follows from commutativity. The last case can be checked

| $\star$ | $B_1$ | $B_2$ | $B_3$ | $B_4$ | $B_5$ | $B_6$ | $B_7$ | $B_8$ |
|---|---|---|---|---|---|---|---|---|
| $B_1$ | $B_1$ | $B_2$ | $B_3$ | $B_4$ | $B_5$ | $B_6$ | $B_7$ | $B_8$ |
| $B_2$ | $B_2$ | $-\frac{\alpha}{3}B_5$ | $B_6$ | $B_7$ | $B_8$ | 0 | 0 | 0 |
| $B_3$ | $B_3$ | $B_6$ | 0 | $rB_5$ | 0 | 0 | $rB_8$ | 0 |
| $B_4$ | $B_4$ | $B_7$ | $rB_5$ | 0 | 0 | $rB_8$ | 0 | 0 |
| $B_5$ | $B_5$ | $B_8$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $B_6$ | $B_6$ | 0 | 0 | $rB_8$ | 0 | 0 | 0 | 0 |
| $B_7$ | $B_7$ | 0 | $rB_8$ | 0 | 0 | 0 | 0 | 0 |
| $B_8$ | $B_8$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table 4.4: Multiplication Table for $\mathcal{B}_{P_8,G}$

directly. Note that

$$
\begin{aligned}
(B_2 \star B_3) \star B_4 &= B_6 \star B_4 \\
&= rB_8
\end{aligned}
$$

This gives that

$$
\begin{aligned}
B_2 \star (B_3 \star B_4) &= B_2 \star rB_5 \\
&= rB_8 \\
&= (B_2 \star B_3) \star B_4
\end{aligned}
$$

This associativity was also verified using Sage's FiniteDimensionalAlgebra module's is_associative() attribute. As this was the last case, we therefore have that every B-model constructed from $P_8$ is associative, regardless of the admissible diagonal symmetry group used.

# Chapter 5. Using Group-Weights in Several Variables

The Group-Weights Theorem has provided a new way to approach some problems. This theorem gives isomorphisms between many pairs of A-models. By the Landau-Ginzburg Mirror Symmetry Conjecture, each of these A-models should be isomorphic to a B-model. By transitivity, these B-models should also be isomorphic. We depict this in Figure 5.1.

$$\begin{tikzcd}
\mathcal{A}_{W_1,G} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \mathcal{B}_{W_1^T,G_1^T} \arrow[d, leftrightarrow] \\
\mathcal{A}_{W_2,G} \arrow[r, leftrightarrow] & \mathcal{B}_{W_2^T,G_2^T}
\end{tikzcd}$$

Figure 5.1: Isomorphism Diagram

We are particularly interested in cases where one of the B-models in this isomorphism has a trivial group. The higher structure of such models is well understood, and isomorphisms to B-models with more complicated groups will help us understand the B-model in the future. In order to have one of the transpose groups be trivial, we require that the common group used on the A-side be the maximal diagonal symmetry group for one of the polynomials.

In his thesis, [14], Nathan Cordner proves all relevant cases for polynomials of two variables. Here, we will consider only polynomials in three or more variables.

To begin, we list and prove several properties of invertible polynomials and their symmetry groups.

**Property 5.1.** *$G_W^{max}$, as an additive group, is generated by the columns of $A_W^{-1}$*

A proof of this is given in [15].

**Property 5.2.** *If $W = W_1 + W_2 + \cdots + W_m$ is a disjoint sum, then $A_W$ is the block diagonal matrix:*

$$A_W = \begin{bmatrix} A_{W_1} & & & \\ & A_{W_2} & & \\ & & \ddots & \\ & & & A_{W_m} \end{bmatrix}.$$

This follows immediately from the definition of the exponent matrix and from the fact that the $W_i$ are in distinct variables.

**Property 5.3.** *If $W = W_1 + W_2 + \cdots + W_m$ is a disjoint sum, then $G_W^{max} = G_{W_1}^{max} \times G_{W_2}^{max} \times \cdots \times G_{W_m}^{max}$*

By Property 5.1, this group is generated by the columns of $A_W^{-1}$. By Property 5.2, $A_W$ is a direct sum of the $A_{W_i}$. Because $A_W$ is block diagonal, $A_W^{-1}$ is also block diagonal with blocks $A_{W_i}^{-1}$. The columns of $A_W^{-1}$ thus generate a group that is the direct product of the $G_{W_i}^{max}$. $\square$

**Property 5.4.** *Let $W$ be a loop with maximal symmetry group, $G_W^{max}$. For any non-identity group element $h = (h_1, h_2, \cdots, h_n)$ in $G_W^{max}$, if all of the rational numbers are reduced so that the numerator and denominator are relatively prime, then all of the $h_i$'s will have the same denominator.*

Let $W = x_1^{a_1} x_2 + x_2^{a_2} x_3 + \cdots + x_n^{a_n} x_1$. Then

$$A_W = \begin{bmatrix} a_1 & 1 & 0 & \cdots & 0 \\ 0 & a_2 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a_{n-1} & 1 \\ 1 & 0 & \cdots & 0 & a_n \end{bmatrix}.$$

Note that $\det A_W = a_1 a_2 \cdots a_n + (-1)^{n+1}$. Call this value $D$. The inverse of this matrix is given as Figure 5.2.

$$A_W^{-1} = \frac{1}{D}\begin{bmatrix} a_2a_3\cdots a_n & -a_3a_4\cdots a_n & a_4\cdots a_n & \cdots & \pm a_n & \mp 1 \\ \mp 1 & a_1a_3\cdots a_n & a_1a_4\cdots a_n & \cdots & \mp a_1a_n & \pm a_1 \\ \pm a_2 & \mp 1 & a_1a_2a_4\cdots a_n & \cdots & \pm a_1a_2a_n & \mp a_1a_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_2a_3\cdots a_{n-1} & -a_3\cdots a_{n-1} & a_4\cdots a_{n-1} & \cdots & \mp 1 & a_1a_2\cdots a_{n-1} \end{bmatrix}$$

Figure 5.2: Inverse Exponent Matrix for a Loop

By Property 5.1, $G_W^{max}$ is generated by the columns of the matrix in Figure 5.2. The last column of this matrix, call it $g = (\frac{g_1}{D}, \cdots, \frac{g_n}{D})$, generates the other columns, so $G_W^{max}$ is generated by this last column. Note that all of the $g_i$ are relatively prime to $D$.

Let $h \in G_W^{max}$, with $h = (h_1, \cdots, h_n)$, where $h$ is not the identity. Then $h = kg$ for some $k \in \mathbb{N}$. Then $h_i = \frac{(k/\gcd(k,D))g_i}{D/\gcd(k,D)}$, where $D/\gcd(k, D)$ is relatively prime to $k/\gcd(k, D)$. So all of the $h_i$ have denominator $D/\gcd(k, D)$. $\square$

**Property 5.5.** *Let $W$ be a loop with maximal symmetry group, $G_W^{max}$. For any group element $h = (h_1, h_2, \cdots, h_n)$ in $G_W^{max}$, if $h_i = 0$ for some $i$, then $h_j = 0$, $\forall j, 0 \le j \le n$.*

Let $h \in G_W^{max}$, with $h = (h_1, \cdots, h_n)$, with $h_i = 0$ for some $i$. Let $g = (\frac{g_1}{D}, \cdots, \frac{g_n}{D}$ be defined as in the proof of Property 5.4. Then $h = kg$ for some $k \in \mathbb{N}$, since $g$ generates $G_W^{max}$. Since $0 = h_i = k\frac{g_i}{D}$, and $\gcd(g_i, D) = 1$, we have that $D \mid k$. Thus $h_j = k\frac{g_j}{D} = 0$ for all $j$. $\square$

**Property 5.6.** *Let $W$ be a chain with maximal symmetry group, $G_W^{max}$. For any generator of this group, $h = (h_1, h_2, \cdots, h_n)$, if the numerator and denominator of each $h_i$ are relatively prime, then the denominator of $h_i$ is strictly less than the denominator of $h_{i+1}$.*

Let $W = x_1^{a_1} + x_1x_2^{a_2} + \cdots + x_{n-1}x_n^{a_n}$. Then

$$A_W = \begin{bmatrix} a_1 & 0 & \cdots & \cdots & 0 \\ 1 & a_2 & 0 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & a_{n-1} & 0 \\ 0 & \cdots & 0 & 1 & a_n \end{bmatrix}.$$

The determinant of this matrix is $D = \det A_W = a_1a_2\cdots a_n$. The inverse of this matrix is given in Figure 5.3. Denote the first column the matrix in Figure 5.3 by $g$. By Property

$$A_W^{-1} = \begin{bmatrix} \frac{1}{a_1} & 0 & 0 & \cdots & 0 & 0 \\ \frac{-1}{a_1a_2} & \frac{1}{a_2} & 0 & & & 0 \\ \frac{1}{a_1a_2a_3} & \frac{-1}{a_2a_3} & \frac{1}{a_3} & \ddots & & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ \frac{\pm 1}{a_1\cdots a_{n-1}} & \frac{\mp 1}{a_2\cdots a_{n-1}} & \frac{\pm 1}{a_3\cdots a_{n-1}} & \cdots & \frac{1}{a_{n-1}} & 0 \\ \frac{\mp 1}{a_1\cdots a_n} & \frac{\pm 1}{a_2\cdots a_n} & \frac{\mp 1}{a_3\cdots a_n} & \cdots & \frac{-1}{a_{n-1}a_n} & \frac{1}{a_n} \end{bmatrix}$$

Figure 5.3: Inverse Exponent Matrix for a Chain

5.1, the columns of this matrix generate $G_W^{max}$, but all of the columns are generated by $g$, so $G_W^{max}$ is generated by $g$. As $G_W^{max}$ is cyclic of order $D$, if $h = (h_1, \cdots, h_n)$ is a generator of $G_W^{max}$, then $h = kg$ for some $k \in \mathbb{N}$ such that $\gcd(k, D) = 1$. The denominators of the entries of $h$ are $a_1, a_1a_2, \cdots a_1a_2\cdots a_n$, in that order, and $k$ is relatively prime with each of these. As each $a_i \geq 2$, the denominator of $h_i$ is strictly less than the denominator of $h_{i+1}$. $\square$

**Property 5.7.** *Let $W$ be a chain with maximal symmetry group, $G_W^{max}$. For any group element $h = (h_1, h_2, \cdots, h_n)$ in $G_W^{max}$, if $h_i = 0$ for some $i$, then $h_j = 0$, $\forall j < i$.*

From the proof of Property 5.6, we know tha $g = \left(\frac{1}{a_1}, \frac{-1}{a_1a_2}, \cdots, \frac{\pm 1}{a_1\cdots a_n}\right)$ is a generator of $G_W^{max}$. Let $h = (h_1, h_2, \cdots, h_n)$ be an element of $G_W^{max}$ with $h_i = 0$. There exists $k \in \mathbb{N}$ such that $h = kg$. Since $h_i = k\frac{\pm 1}{a_1a_2\cdots a_i} = 0$, we know that $a_1a_2\cdots a_i \mid k$. Therefore $a_1a_2\cdots a_j \mid k$ for all $j < i$. Hence $h_j = k\frac{\pm 1}{a_1a_2\cdots a_j} = 0$ for all $j < i$. $\square$

**Property 5.8.** *Let $W = x_1^{a_1} + x_1x_2^{a_2} + \cdots + x_{n-1}x_n^{a_n}$ be a chain with weights $q = (q_1, q_2, \cdots, q_n)$. Any weight, $q_i$, $i > 1$, can be determined solely from $q_{i-1}$ and $a_i$.*

Since $W$ is invertible, there exists a unique weights vector $q = (q_1, \cdots, q_n)$. By definition $W(c^{q_1}x_1, \cdots, c^{q_n}x_n) = cW(x_1, \cdots, x_n)$ for all $c \in \mathbb{C}$. This gives that

$$c(x_1^{a_1} + x_1x_2^{a_2} + \cdots + x_{n-1}x_n^{a_n}) = c^{q_1{}^{a_1}}x_1^{a_1} + c^{q_1}x_1^{a_1}c^{q_2{}^{a_2}}x_2 \cdots c^{q_{n-1}}x_{n-1}c^{q_n{}^{a_n}}x_n^{a_n}.$$

Equating coefficients, we get that if $i > 1$, then $c^{q_{i+1}}c^{q_i{}^{a_i}} = c$, so $q_{i+1}q_i^{a_i} = 1$. Hence $q_i$ can be determined solely from $q_{i-1}$ and $a_i$. $\square$

**Property 5.9.** *If $W$ is an invertible polynomial with exponent matrix $A_W$, then there is an entry in every row and in every column of $A_W^{-1}$ of the form $\frac{\pm 1}{k}$, where $k \in \mathbb{Z}, k > 1$.*

By Property 5.2, $A_W$ is a block diagonal matrix, where each block is the exponent matrix of an atomic type polynomial. Hence it is sufficient to show that this holds for each atomic type.

If $W = x^a$ is a fermat, then $A_W^{-1} = [\frac{1}{a}]$, and the result holds. If $W$ is a loop, then we can see that the result holds from Figure 5.2. Similarly, if $W$ is a chain, the result holds from Figure 5.3. $\square$

**Property 5.10.** *Given a system of weights, there is at most one distinct polynomial of each atomic type that has that weight system.*

We consider the three cases. If $W_1 = x^a$ and $W_2 = x^b$ are both fermat polynomials with the same weight $q$, then $qa = 1 = qb$, so $a = b$ and $W_1 = W_2$.

Next, let $W_1 = x_1^{a_1} + x_1x_2^{a_2} + \cdots + x_{n-1}x_n^{a_n}$ and $W_2 = x_1^{b_1} + x_1x_2^{b_2} + \cdots + x_{n-1}x_n^{b_n}$ be chains with weights $q = (q_1, q_2, \cdots, q_n)$. By the same argument as the fermat case, $a_1 = b_1$. From the proof of Property 5.8, if $i > 1$ then $q_{i+1}q_i^{a_i} = 1 = q_{i+1}q_i^{b_i}$. This implies that $a_i = b_i$ for all $i > 1$. Hence $W_1 = W_2$.

Now, let $W_1 = x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_n^{a_n}x_1$ and $W_2 = x_1^{b_1}x_2 + x_2^{b_2}x_3 + \cdots + x_n^{b_n}x_1$ be loops with weights $q = (q_1, q_2, \cdots, q_n)$. Then by again equating coefficients, we get that

$q_i^{a_i} q_{i+1} = 1 = q_i^{b_i} q_{i+1}$ for $1 \le i < n$ and $q_n^{a_n} q_1 = 1 = q_n^{b_n} q_1$. Simplifying, we get that $a_i = b_i$ for all $i$. Hence $W_1 = W_2$. $\square$

With these properties, we now show when it is possible for $W_1$ and $W_2$ to be invertible polynomials with the same weights satisfying $G_{W_1}^{max} \subset G_{W_2}^{max}$. We do this by considering several cases below.

**Lemma 5.1.** *Let $W_1$ and $W_2$ be invertible polynomials. If If $G_{W_1}^{max} \subset G_{W_2}^{max}$, then $A_{W_2} A_{W_1}^{-1}$ is an integer matrix.*

Recal from Property 5.1 that the columns of $A_{W_1}^{-1}$ generate $G_{W_1}^{max}$ and that the columns of $A_{W_2}^{-1}$ generate $G_{W_2}^{max}$. Since $G_{W_1}^{max} \subset G_{W_2}^{max}$, the columns of $A_{W_1}^{-1}$ must be sums of the columns of $A_{W_2}^{-1}$. Hence there exists an integer matrix $B$ such that $A_{W_1}^{-1} = A_{W_2}^{-1} B$. So $A_{W_2} A_{W_1}^{-1} = A_{W_2} A_{W_2}^{-1} B = B$ is an integer matrix. $\square$

**Lemma 5.2.** *If $W_1$ is a loop in three or more variables, and $W_1$ and $W_2$ have the same weights, and $G_{W_1}^{max} \subset G_{W_2}^{max}$, then $W_2$ does not contain a fermat or a chain.*

By way of contradiction, suppose that $W_2$ contains a chain or a fermat. Then $x_i^{b_i}$ is a monomial of $W_2$ for some $i$. Thus by Lemma 5.1,

$$(0, \cdots, 0, b_i, 0, \cdots, 0) A_{W_1}^{-1} \in \mathbb{Z}^n.$$

Since $W_1$ is invertible, by Property 5.9, there is a column of $A_{W_1}^{-1}$ for which the $i$th element is $\pm 1/\det(A_{W_1})$. Thus $b_i = k \det(A_{W_1})$, for some $k \in \mathbb{Z}$. Equating the weights of $W_1$ and $W_2$, we get

$$\frac{1}{k \det(A_{W_1})} = \frac{a_{i+1} a_{i+2} \cdots a_{i+n} - a_{i+2} a_{i+3} \cdots a_{i+n} + a_{i+3} a_{i+4} \cdots a_{i+n} \cdots \mp a_{i+n} \pm 1}{\det(A_{W_1})}.$$

with the indices taken modulo $n$. This implies that

$$1 = k(a_{i+1} a_{i+2} \cdots a_{i+n} - a_{i+2} a_{i+3} \cdots a_{i+n} + a_{i+3} a_{i+4} \cdots a_{i+n} \cdots \mp a_{i+n} \pm 1).$$

Some simple algebra shows that this can only hold if $n = 2$. This is a contradiction, since $W_1$ has at least three variables. $\square$

**Lemma 5.3.** *If $W_1$ is a chain in three or more variables, $W_1$ and $W_2$ have the same weights, and $G_{W_1}^{max} \subset G_{W_2}^{max}$, then $W_2$ has no monomial of the form $x_i^{b_i}$ for $i > 2$.*

By way of contradiction, suppose $x_i^{b_i}$ with $i > 2$ is a monomial of $W_2$. Since $G_{W_1}^{max} \subset G_{W_2}^{max}$, $A_{W_2}A_{W_1}^{-1}$ is an integer matrix by Lemma 5.1. Hence

$$(0, \cdots, 0, b_i, 0, \cdots, 0)A_{W_1}^{-1} \in \mathbb{Z}^n.$$

This implies that for some integer $k$, $b_i = ka_1a_2 \cdots a_i$

Equating the weights of $x_i$ in $W_1$ and $W_2$, we get

$$\frac{a_1a_2 \cdots a_{i-1} - a_1a_2 \cdots a_{i-2} + \cdots \pm 1}{a_1a_2 \cdots a_i} = \frac{1}{ka_1a_2 \cdots a_i}.$$

This simplifies to

$$1 = k(a_1a_2 \cdots a_{i-1} - a_1a_2 \cdots a_{i-2} + \cdots \pm 1).$$

Algebraically, this can only hold for $i \leq 2$. Since $i > 2$, this is a contradiction. $\square$

**Lemma 5.4.** *If $W_1$ is the sum of two atomic types, and $W_2$ consists of a single atomic type, then $G_{W_1}^{max} \not\subset G_{W_2}^{max}$.*

If $W_2$ is a fermat, this follows immediately from the fact that chains and loops each have a minimum of two variables. Suppose that $W_2$ is either a chain or a loop. If $W_1$ is the sum of two atomic types, it is clear from Property 5.3 that there exists an element of $\mathbf{g} = (g_1, g_2, \cdots, g_n)$ in $G_{W_1}^{max}$ for which $g_i = 0$ and for some $j > i$, $g_j \neq 0$. This violates Properties 5.5 and 5.7. Therefore, $G_{W_1}^{max} \not\subset G_{W_2}^{max}$. $\square$

**Lemma 5.5.** *If $G_{W_1}^{max} \subset G_{W_2}^{max}$, and $W_2$ consists of the sum of atomic types, no two variables that are part of the same atomic type in $W_2$ form parts of distinct loops or chains in $W_1$.*

Suppose $G_{W_1}^{max} \subset G_{W_2}^{max}$. By way of contradiction, suppose that there exist variables that form part of the same atomic type in $W_2$, that form parts of distinct chains or loops in $W_1$. Reorder the variables so that if $x_i$ and $x_j$ form part of $W_1$, then so does $x_k$ for $i < k < j$. Then there exists $i$ such that the monomial $x_i^{a_i}x_j$ is a monomial of $W_2$, but $x_i$ and $x_j$ do not form part of the same atomic type in $W_1$. The inverse of $A_{W_1}$ is block matrix, where each block is the inverse of an exponent matrix of an atomic type polynomial. Thus there exist at least $m$ columns of $A_{W_1}^{-1}$ such that the $i$th entry is zero, where $m$ is the number of monomials in the atomic type to which $x_j$ belongs in $W_1$. From Property 5.9, for one of these columns, the $j$th entry is of the form $\frac{\pm 1}{k}$ for $k \in \mathbb{Z}, k > 1$. Call this column $v$. We need $A_{W_2}A_{W_1}^{-1}$ to be an integer matrix. However, we now have a row of $A_{W_2}$–the row corresponding to the monomial, $x_i^{a_i}x_j$–and a column of $A_{W_1}^{-1}$, $v$, that have a product of $\frac{\pm 1}{k}$, which is not an integer. This is a contradiction. □

**Lemma 5.6.** *If we have two pairs of polynomials in distinct variables, $W_1$, $W_2$ and $W_3$, $W_4$ satisfying the conditions that both polynomials in each pair have the same weights and $G^{max}$ of the first is contained in the second, then the polynomials $W_1 + W_3$ and $W_2 + W_4$ have the same weights and $G^{max}$ of the first is contained in $G^{max}$ of the second.*

This follows almost directly from Property 5.3. Note that strict containment of the symmetry groups in at least one of the initial pairs is required to guarantee strict containment for the symmetry groups of the sums. □

**Proposition 5.7.** *If $W_1$ and $W_2$ are invertible polynomials with the same weights satisfying $G_{W_1}^{max} \subset G_{W_2}^{max}$, then $W_1$ is a single atomic type.*

Lemmas 5.5 and 5.6 show that if $W_1$ and $W_2$ have the same weights and $G_{W_1}^{max} \subset G_{W_2}^{max}$, then for some atomic type in $W_1$, call it $w_1$, there is a subset of the monomials of $W_2$, call it $w_2$, for which, when treated as polynomials in the appropriate number of variables, $w_1$ and $w_2$ have the same weights and the maximal symmetry group of $w_1$ is properly contained in that of $w_2$. Therefore, we can consider each of the atomic types in $W_1$ independently. □

**Lemma 5.8.** *Let $W_1$ and $W_2$ be invertible polynomials with the same weights satisfying $G_{W_1}^{max} \subset G_{W_2}^{max}$. If $W_1$ is a chain, then the only non-trivial $W_2$ is a fermat plus a chain.*

From Properties 5.4 and 5.6, it is clear that if $G_{W_1}^{max} \subset G_{W_2}^{max}$, then $W_2$ cannot contain a loop. From Property 5.10, we do not need to consider the case of another chain. Therefore, we need only consider the cases of fermats, and fermats plus chains.

By Lemma 5.3, at least one variable of any chain will have a weight equal to $\frac{\pm 1}{k}$, for some $k > 1$. Similarly, all fermats have weights of this form. Hence, $W_2$ cannot contain more then one fermat, and that fermat must be of either $x_1$ or $x_2$.

First, we consider the case where the fermat is in $x_1$. Let $W_1 = x_1^{a_1} + x_1 x_2^{a_2} + \cdots + x_{n-1} x_n^{a_n}$ and $W_2 = x_1^{b_1} + x_2^{b_2} + x_2 x_3^{b_3} + \cdots + x_{n-1} x_n^{b_n}$. From Property 5.6, we are allowed to assume that the relative order of the variables must remain consistent. Equating the weights of $W_1$ and $W_2$, we see that:

$$\begin{aligned} a_1 &= b_1 \\ \frac{a_1 a_2}{a_1 - 1} &= b_2 \end{aligned}$$

In order for $G_{W_1}^{max} \subset G_{W_2}^{max}$, we need $C = A_{W_2} A_{W_1}^{-1}$ to be an integer matrix. In this matrix, $c_{22} = \frac{b_2}{a_2}$. Therefore, $b_2 = k a_2$ for some integer $k$. Substituting this into the weights, we see that:

$$\begin{aligned} a_2 \frac{a_1}{a_1 - 1} &= k a_2 \\ \frac{a_1}{a_1 - 1} &= k \end{aligned}$$

This holds iff $a_1 = 2$. Substituting this into the weights for $x_2$, we see that $b_2 = 2a_2$. Since the weights of $x_1$ and $x_2$ match, Property 5.8 gives us that the rest of the weights also match. Also, all of the other monomials of $W_2$ are fixed by $G_{W_1}^{max}$, as they are also monomials of $W_1$.

Now, consider the case where the fermat is of $x_2$. Let $W_1 = x_1^{a_1} + x_1 x_2^{a_2} + \cdots + x_{n-1} x_n^{a_n}$

and $W_2 = x_1^{b_1} + x_2^{b_2} + x_1 x_3^{b_3} + \cdots + x_{n-1} x_n^{b_n}$. Equating weights, we get

$$a_1 = b_1$$
$$\frac{a_1 a_2}{a_1 - 1} = b_2$$
$$\frac{a_1 a_2 - a_1 + 1}{a_1 a_2 a_3} = \frac{b_1 - 1}{b_1 b_3}$$

Computing $C = A_{W_2} A_{W_1}^{-1}$, we find that $c_{21} = \frac{-b_2}{a_1 a_2}$ must be an integer. Therefore, $b_2 = k a_1 a_2$ for some integer $k$. Substituting this into the weights, we see that

$$a_1 a_2 = k a_1 a_2 (a_1 - 1)$$
$$1 = k(a_1 - 1)$$

This is true if and only if $k = 1$ and $a_1 = 2$. Therefore, $b_2 = 2a_2$. Substituting these equalities into the weights for $x_3$, we get

$$\frac{a_1 a_2 - a_1 + 1}{a_1 a_2 a_3} = \frac{b_1 - 1}{b_1 b_3}$$
$$\frac{2a_2 - 1}{2a_2 a_3} = \frac{1}{2b_3}$$
$$\frac{a_2 a_3}{2a_2 - 1} = b_3$$

However, $c_{32} = \frac{-b_3}{a_2 a_3}$. Hence $b_3 = k a_2 a_3$ for some integer $k$. Making this substitution, we get

$$k = \frac{1}{2a_2 - 1}.$$

This is never true, hence the fermat in $W_2$ cannot be in $x_2$.

Now consider the case where $W_2$ consists of the sum of two chains. From Lemma 5.3, if $W_1 = x_1^{a_1} + x_1 x_2^{a_2} + \cdots + x_{n-1} x_n^{a_n}$, then there exist monomials $x_1^{b_1}$ and $x_2^{b_2}$ in $W_2$. Since $W_2$ is comprised of two chains, this requires that there exist a monomial $x_1 x_i^{b_i}$. Equating the weights of $x_i$, we get:

$$\frac{1-q_{i-1}}{b_i} = \frac{1-q_1}{a_i}.$$

From the proofs of the other sections of this Lemma, we know that $a_1 = 2$, and thus $q_1 = \frac{1}{2}$. Hence,

$$2b_i(1-q_{i-1}) = a_i.$$

Computing $C = A_{W_1}^{-1}$, we find that $c_{12} = 0$ and $c_{i2} = \frac{\pm 1}{a_2 a_3 \cdots a_i}$. Thus we require $b_i = k a_2 \cdots a_i$ for some integer $k$.

Using $q_{i-1}$ from $W_1$ and substituting in this new value for $b_i$, we get that

$$\begin{aligned}
&a_i = k a_1 \cdots a_i (1-q_{i-1}) \\
&a_i = k(a_1 \cdots a_i - a_1 \cdots a_{i-2} + a_1 \cdots a_{i-3} + \cdots \pm 1) \\
&a_i = k(a_1 \cdots a_{i-2} a_i + a_1 \cdots a_{i-2}(a_{i-1}-1)a_i - a_1 \cdots a_{i-2} + a_1 \cdots a_{i-3} + \cdots \pm 1) \\
&a_i > k a_i
\end{aligned}$$

This is a contradiction, hence $W_2$ cannot consist of two chains. $\square$

**Lemma 5.9.** *If $W_1$ is a single loop in three or more variables, then $W_2$ does not consist of a sums of loops.*

By way of contradiction, let $W_2$ be the sum of loops such that $W_1$ and $W_2$ have the same weights, and $G_{W_1}^{max} \subset G_{W_2}^{max}$. Reorder the variables so that if $x_i$ and $x_j$ are a part of the same loop, $x_k$ also forms part of that loop for all $k$ such that $i < k < j$. This makes $A_{W_2}^{-1}$ a diagonal block matrix.

Let $g$ be a column of $G_{W_1}^{max}$. Then for some integer vector $b$,

$$g = A_{W_2}^{-1} b.$$

By Property 5.9, for some $m$, the $m$th component of $g$ is of the form $\frac{\pm 1}{l}$. Since $W_1$ is a loop, in this case $l = \det(A_{W_1})$. Now consider the $m$th component of $A_{W_2}^{-1}b$. Because the numerators of the components of $A_{W_2}^{-1}$ are integers, and $b$ is an integer vector, the numerators of their product is also an integer.

If we do not allow the fraction to reduce, the denominator is equal to $\det(A_{W_m})$, where $W_m$ is the loop in $W_2$ that contains $x_m$. This follows from the fact that $A_{W_2}^{-1}$ is also a block matrix. Since $g = A_{W_2}^{-1}b$, their $m$th entries must also be equal, hence for some integer $l$,

$$\frac{\pm 1}{\det(A_{W_1})} = \frac{l}{\det(A_{W_m})}.$$

This implies that

$$\det(A_{W_1}) \mid \det(A_{W_m}) \tag{5.1}$$

Since $W_1$ is a loop, the determinants of its exponent matrix is of the form

$$\det(A_{W_1}) = \prod_{i \in V} \left( \frac{1 - q_i}{q_i} \right) \pm 1,$$

where $V$ is the set of indices for which $x_i$ is a part of the loop. Similarly,

$$\det(A_{W_m}) = \prod_{i \in V_m} \left( \frac{1 - q_i}{q_i} \right) \pm 1.$$

Because $W_1$ is a loop, the weights of $W_1$ are strictly less than $\frac{1}{2}$. Also, if $\frac{1-q_i}{q_i} = k$, then $q_i = \frac{1}{k+1}$. Therefore, $k > 1$.

Since the weights of $W_m$ are a strict subset of the weights of $W_1$,

$$\prod_{i \in V_{W_m}} \left( \frac{1 - q_i}{q_i} \right) < \prod_{j \in V_{W_1}} \left( \frac{1 - q_j}{q_j} \right).$$

But then by (5.1), $\det(A_{W_1}) \mid \det(A_{W_m}) < \det(A_{W_1})$ which is a contradiction. $\square$

Collectively, these lemmas and propositions show that there is only one interesting case in three or more variables: $W_1$ is a single chain, and $W_2$ is a fermat plus a chain. In Section 5.1, we provide a proof that the B-model isomorphism suggested by the Group-Weights Theorem holds.

## 5.1 Explicit Isomorphism

Lemma 5.8 and the Group-Weights Theorem give an A-model isomorphism using the polynomials $W_1 = x_1^2 + x_1x_2^{a_2} + \cdots + x_{n-1}x_n^{a_n}$ and $W_2 = x_1^2 + x_2^{2a_2} + x_2x_3^{a_3} + \cdots + x_{n-1}x_n^{a_n}$. Here, we explicitly prove the B-model isomorphism found by transposing from A- to B-models in the three variable case.

Let $W = x^2 + y^{2a}z + z^b$, $V = x^2y + y^az + z^b$, $G =< (1/2, 1/2, 0) >$ and $H = \{0\}$. We prove that $\mathcal{B}_{W,G} \cong \mathcal{B}_{V,H}$ by constructing an isomorphism. In order to do this, we will create both models, compute all of the possible products, and then show that a system of nonlinear equations has a solution. We follow the construction and multiplication methods described at length in Chapter 2. We start by finding everything needed to construct $\mathcal{B}_{W,G}$.

$$q_W = \left(\frac{1}{2}, \frac{b-1}{2ab}, \frac{1}{b}\right);$$

$$\nabla W = (2x, 2ay^{2a-1}z, y^{2a} + bz^{b-1});$$

$$Hess(W) = 4ab(2ab - b + 1)y^{2a-2}z^{b-1};$$

$$\mathcal{Q}_W \text{ basis: } \left(\{1, y, \cdots, y^{2a-2}\} \times \{1, z, \cdots, z^{b-1}\}\right) \cup \{y^{2a-1}\};$$

$$(\mathcal{Q}_W)^G \text{ basis: } \{1, y^2, \cdots y^{2a-2}\} \times \{1, z, \cdots, z^{b-1}\};$$

$$\left(\mathcal{Q}_W|_{\text{fix}(g)}\right)^G = \{1, z, \cdots z^{b-2}\};$$

$$\mathcal{B}_{W,G} = (\mathcal{Q}_W)^G \oplus \left(\mathcal{Q}_W|_{\text{fix}(g)}\right)^G;$$

$$dim\left(\mathcal{B}_{W,G}\right) = 2ab - b + 1.$$

So, for the appropriate exponents, a basis can be chosen as follows.

| Basis Element | Degree |
|---|---|
| $\lfloor 1\,;(0,0,0)\rceil$ | $0$ |
| $\lfloor y^{2r_1}\,;(0,0,0)\rceil$ | $\frac{2r_1(b-1)}{ab}$ |
| $\lfloor z^{r_2}\,;(0,0,0)\rceil$ | $\frac{2r_2}{b}$ |
| $\lfloor y^{2r_1}z^{r_2}\,;(0,0,0)\rceil$ | $\frac{2r_2}{b}$ |
| $\lfloor z^{r_2}\,;(1/2,1/2,0)\rceil$ | $\frac{2ar_2+ab-b+1}{ab}$ |

Now, we compute everything that we will need about $\mathcal{B}_{V,H}$.

$$q_V = \left(\frac{ab-b+1}{2ab}, \frac{b-1}{ab}, \frac{1}{b}\right);$$

$$\nabla V = (2xy, x^2 + ay^{a-1}z, y^a + bz^{b-1});$$

$$Hess(V) = 2ab(ab+b-1)y^{a-1}z^{b-1};$$

$$\mathcal{Q}_V \text{ basis: } \{x, xz, \cdots, xz^{b-2}\} \cup \left(\{1, y, \cdots, y^{a-1}\} \times \{1, z, \cdots, z^{b-1}\}\right);$$

$$dim\,(\mathcal{Q}_V) = 2ab - b + 1;$$

$$\mathcal{B}_{V,H} = \mathcal{Q}_V.$$

Let $c_x$, $c_y$, $c_z$ be nonzero constants in $\mathbb{C}$. They will be described explicitly later on. We can then choose a basis of $\mathcal{B}_{V,H}$ to be:

| Basis Element | Degree |
|---|---|
| $\lfloor 1\,;(0,0,0)\rceil$ | $0$ |
| $c_y^{r_1}\lfloor y^{r_1}\,;(0,0,0)\rceil$ | $\frac{2r_1(b-1)}{ab}$ |
| $c_z^{r_2}\lfloor z^{r_2}\,;(0,0,0)\rceil$ | $\frac{2r_2}{b}$ |
| $c_y^{r_1}c_z^{r_2}\lfloor y^{r_1}z^{r_2}\,;(0,0,0)\rceil$ | $\frac{2r_2}{b}$ |
| $c_x c_z^{r_2}\lfloor xz^{r_2}\,;(0,0,0)\rceil$ | $\frac{2ar_2+ab-b+1}{ab}$ |

Define $\varphi$ such that

$$\begin{aligned}
&\lfloor 1\,;(0,0,0)\rceil_W \mapsto \lfloor 1\,;(0,0,0)\rceil_V\,;\\
&\left\lfloor y^{2r_1}\,;(0,0,0)\right\rceil_W \mapsto c_y^{r_1}\lfloor y^{r_1}\,;(0,0,0)\rceil_V\,;\\
&\lfloor z^{r_2}\,;(0,0,0)\rceil_W \mapsto c_z^{r_2}\lfloor z^{r_2}\,;(0,0,0)\rceil_V\,;\\
&\left\lfloor y^{2r_1}z^{r_2}\,;(0,0,0)\right\rceil_W \mapsto c_y^{r_1}c_z^{r_2}\lfloor y^{r_1}z^{r_2}\,;(0,0,0)\rceil_V\,;\\
&\lfloor z^{r_2}\,;(1/2,1/2,0)\rceil_W \mapsto c_x c_z^{r_2}\lfloor xz^{r_2}\,;(0,0,0)\rceil_V\,.
\end{aligned}$$

We extend linearly to all of $\mathcal{B}_{W,G}$. Note that $\lfloor 1\,;(0,0,0)\rceil_W$ and $\lfloor 1\,;(0,0,0)\rceil_V$ are the multiplicative identities, and they are mapped to each under $\varphi$.

Consider the multiplication on the basis for $\mathcal{B}_{W,G}$:

$$\lfloor y^{2r_1}\,;(0,0,0)\rceil\star\lfloor y^{2s_1}\,;(0,0,0)\rceil=\begin{cases}\lfloor y^{2r_1+2s_1}\,;(0,0,0)\rceil & r_1+s_1<a\\ -b\left\lfloor y^{2r_1+2s_1-2a}z^{b-1}\,;(0,0,0)\right\rceil & \text{otherwise}\end{cases}$$

$$\lfloor y^{2r_1}\,;(0,0,0)\rceil\star\lfloor z^{s_2}\,;(0,0,0)\rceil=\lfloor y^{2r_1}z^{s_2}\,;(0,0,0)\rceil$$

$$\lfloor y^{2r_1}\,;(0,0,0)\rceil\star\lfloor y^{2s_1}z^{s_2}\,;(0,0,0)\rceil=\begin{cases}\lfloor y^{2r_1+2s_1}z^{s_2}\,;(0,0,0)\rceil & r_1+s_1<a\\ 0 & \text{otherwise}\end{cases}$$

$$\lfloor y^{2r_1}\,;(0,0,0)\rceil\star\lfloor z^{s_2}\,;(1/2,1/2,0)\rceil=0$$

$$\lfloor z^{r_2}\,;(0,0,0)\rceil\star\lfloor z^{s_2}\,;(0,0,0)\rceil=\begin{cases}\lfloor z^{r_2+s_2}\,;(0,0,0)\rceil & r_2+s_2<b\\ 0 & \text{otherwise}\end{cases}$$

$$\lfloor z^{r_2}\,;(0,0,0)\rceil\star\lfloor y^{2s_1}z^{s_2}\,;(0,0,0)\rceil=\begin{cases}\lfloor y^{2s_1}z^{r_2+s_2}\,;(0,0,0)\rceil & r_2+s_2<b\\ 0 & \text{otherwise}\end{cases}$$

$$\lfloor z^{r_2}\,;(0,0,0)\rceil\star\lfloor z^{s_2}\,;(1/2,1/2,0)\rceil=\begin{cases}\lfloor z^{r_2+s_2}\,;(1/2,1/2,0)\rceil & r_2+s_2<b-1\\ 0 & \text{otherwise}\end{cases}$$

$$\lfloor y^{2r_1} z^{r_2} \, ; (0,0,0) \rceil \star \lfloor y^{2s_1} z^{s_2} \, ; (0,0,0) \rceil = \begin{cases} \lfloor y^{2r_1+2s_1} z^{r_2+s_2} \, ; (0,0,0) \rceil & r_1 + s_1 < a \\ & \text{and } r_2 + s_2 < b \\ 0 & \text{otherwise} \end{cases}$$

$$\lfloor y^{2r_1} z^{r_2} \, ; (0,0,0) \rceil \star \lfloor z^{s_2} \, ; (1/2,1/2,0) \rceil = 0$$

$$\lfloor z^{r_2} \, ; (1/2,1/2,0) \rceil \star \lfloor z^{s_2} \, ; (1/2,1/2,0) \rceil = \begin{cases} 4a \lfloor y^{2a-2} z^{1+r_2+s_2} \, ; (0,0,0) \rceil & r_2 + s_2 < b - 1 \\ 0 & \text{otherwise} \end{cases}$$

The multiplication on our scaled basis of $\mathcal{B}_{V,H}$ is computed below.

$$c_y^{r_1} \lfloor y^{r_1} \, ; (0,0,0) \rceil \star c_y^{s_1} \lfloor y^{s_1} \, ; (0,0,0) \rceil = \begin{cases} c_y^{r_1+r_2} \lfloor y^{r_1+s_1} \, ; (0,0,0) \rceil & r_1 + s_1 < a \\ -b c_y^{r_1+s_1} \left\lfloor y^{r_1+s_1-a} z^{b-1} \, ; (0,0,0) \right\rceil & \text{otherwise} \end{cases}$$

$$c_y^{r_1} \lfloor y^{r_1} \, ; (0,0,0) \rceil \star c_z^{s_2} \lfloor z^{s_2} \, ; (0,0,0) \rceil = c_y^{r_1} c_z^{s_2} \lfloor y^{r_1} z^{s_2} \, ; (0,0,0) \rceil$$

$$c_y^{r_1} \lfloor y^{r_1} \, ; (0,0,0) \rceil \star c_y^{s_1} c_z^{s_2} \lfloor y^{s_1} z^{s_2} \, ; (0,0,0) \rceil$$

$$= \begin{cases} c_y^{r_1+s_1} c_z^{s_2} \lfloor y^{r_1+s_1} z^{s_2} \, ; (0,0,0) \rceil & r_1 + s_1 < a \\ 0 & \text{otherwise} \end{cases}$$

$$c_y^{r_1} \lfloor y^{r_1} \, ; (0,0,0) \rceil \star c_x c_z^{s_2} \lfloor x z^{s_2} \, ; (0,0,0) \rceil = 0$$

$$c_z^{r_2} \lfloor z^{r_2} \, ; (0,0,0) \rceil \star c_z^{s_2} \lfloor z^{s_2} \, ; (0,0,0) \rceil = \begin{cases} c_z^{r_2+s_2} \lfloor z^{r_2+s_2} \, ; (0,0,0) \rceil & r_2 + s_2 < b \\ 0 & \text{otherwise} \end{cases}$$

$$c_z^{r_2} \lfloor z^{r_2} \, ; (0,0,0) \rceil \star c_y^{s_1} e^{s_2} \lfloor y^{s_1} z^{s_2} \, ; (0,0,0) \rceil$$

$$= \begin{cases} c_y^{s_1} c_z^{r_2+s_2} \lfloor y^{s_1} z^{r_2+s_2} \, ; (0,0,0) \rceil & r_2 + s_2 < b \\ 0 & \text{otherwise} \end{cases}$$

$$c_z^{r_2} \lfloor z^{r_2} \, ; (0,0,0) \rceil \star c_x c_z^{s_2} \lfloor x z^{s_2} \, ; (0,0,0) \rceil = \begin{cases} c_x c_z^{r_2+s_2} \lfloor x z^{r_2+s_2} \, ; (0,0,0) \rceil & r_2 + s_2 < b - 1 \\ 0 & \text{otherwise} \end{cases}$$

$$c_y^{r_1} c_z^{r_2} \lfloor y^{r_1} z^{r_2} \, ; (0,0,0) \rceil \star c_y^{s_1} c_z^{s_2} \lfloor y^{s_1} z^{s_2} \, ; (0,0,0) \rceil$$

$$= \begin{cases} c_y^{r_1+s_1} c_z^{r_2+s_2} \lfloor y^{r_1+s_1} z^{r_2+s_2} ; (0,0,0) \rceil & r_1+s_1 < a \\ & \text{and } r_2+s_2 < b \\ 0 & \text{otherwise} \end{cases}$$

$$c_y^{r_1} c_z^{r_2} \lfloor y^{r_1} z^{r_2} ; (0,0,0) \rceil \star c_x c_z^{s_2} \lfloor x z^{s_2} ; (0,0,0) \rceil = 0$$

$$c_x c_z^{r_2} \lfloor x z^{r_2} ; (0,0,0) \rceil \star c_x c_z^{s_2} \lfloor x z^{s_2} ; (0,0,0) \rceil$$

$$= \begin{cases} -a c_x^2 c_z^{r_2+s_2} \lfloor y^{a-1} z^{1+r_2+s_2} ; (0,0,0) \rceil & r_2+s_2 < b-1 \\ 0 & \text{otherwise} \end{cases}$$

Next, we look at the pairings on the basis of $\mathcal{B}_{W,G}$. Allowing for exponents of zero, we get the following.

$$< \lfloor y^{2r_1} z^{r_2} ; (0,0,0) \rceil , \lfloor y^{2s_1} z^{s_2} ; (0,0,0) \rceil >= \begin{cases} \frac{1}{4ab} & r_1+s_1 = a-1 \\ & \text{and } r_2+s_2 = b-1 \\ 0 & \text{otherwise} \end{cases}$$

$$< \lfloor z^{r_2} ; (1/2,1/2,0) \rceil , \lfloor z^{s_2} ; (1/2,1/2,0) \rceil >= \begin{cases} \frac{1}{b} & r_2+s_2 = b-2 \\ 0 & \text{otherwise} \end{cases}$$

All other pairings are zero. Now, consider the pairings on the basis of $\mathcal{B}_{V,H}$:

$$< c_y^{r_1} c_z^{r_2} \lfloor y^{r_1} z^{r_2} ; (0,0,0) \rceil , c_y^{s_1} c_z^{s_2} \lfloor y^{s_1} z^{s_2} ; (0,0,0) \rceil >$$

$$= \begin{cases} c_y^{a-1} c_z^{b-1} (\frac{1}{2ab}) & r_1+s_1 = a-1 \\ & \text{and } r_2+s_2 = b-1 \\ 0 & \text{otherwise} \end{cases}$$

$$< c_x c_z^{r_2} \lfloor x z^{r_2} ; (0,0,0) \rceil , c_x c_z^{s_2} \lfloor x z^{s_2} ; (0,0,0) \rceil >$$

$$= \begin{cases} c_x^2 c_z^{b-2} (\frac{-1}{2b}) & r_2+s_2 = b-2 \\ 0 & \text{otherwise} \end{cases}$$

Thus, in order for $\varphi$ to be an isomorphism on the graded Frobenius algebra level, we have four conditions, two from the multiplication, two from the pairings. They are:

$$4c_y^{a-1}c_z = -c_x^2 \tag{5.2}$$

$$c_z^{b-1} = c_y^a \tag{5.3}$$

$$c_x^2c_z^{b-2} = -2 \tag{5.4}$$

$$c_y^{a-1}c_z^{b-1} = 1/2 \tag{5.5}$$

Combining (5.3) and (5.5), we see that $c_y^{2a-1} = 1/2$, so $c_y = 2^{\frac{1}{1-2a}}$. Again from (5.3), we see that $c_z = c_y^{\frac{a}{b-1}} = 2^{\frac{a}{(1-2a)(b-1)}}$. Using this value of $c_z$, we can solve for $c_x$ in (5.4). This gives that $c_x = -2^{\frac{1}{2}+\frac{(2-b)a}{2(1-2a)(b-1)}} = -2^{\frac{(1-2a)(b-1)+a(2-b)}{2(1-2a)(b-1)}} = -2^{\frac{4a-3ab+b-1}{2(1-2a)(b-1)}}$. These numbers were constructed so as to be solutions to (5.3), (5.4), and (5.5). Therefore, we only need to check that they are also solutions to (5.2). Note that:

$$-c_x^2 = 2^{\frac{4a-3ab+b-1}{(1-2a)(b-1)}}$$

and

$$\begin{aligned}
4c_y^{a-1}c_z &= 2^2 * 2^{\frac{a-1}{1-2a}} * 2^{\frac{a}{(1-2a)(b-1)}} \\
&= 2^{\frac{(2b+4a-4ab-2)+(ab-a-b+1)+a}{(1-2a)(b-1)}} \\
&= 2^{\frac{4a-3ab+b-1}{(1-2a)(b-1)}} \\
&= -c_x^2
\end{aligned}$$

Therefore, if we let $c_x$,$c_y$,$c_z$ be defined as follows, then $\varphi$, as defined above, is a ring homomorphism that respects the pairing.

$$c_x = -2^{\frac{4a-3ab+b-1}{2(1-2a)(b-1)}}$$

$$c_y = 2^{\frac{1}{1-2a}}$$

$$c_z = 2^{\frac{a}{(1-2a)(b-1)}}$$

Since $\varphi$ is a diagonal map, with all non-zero entries on the diagonal, it is also a vector space isomorphism. Thus, $\varphi$ is an isomorphism on the graded Frobenius algebra level. We conclude that $\mathcal{B}_{W,G} \cong \mathcal{B}_{V,H}$. $\square$

# Chapter 6. Conclusion

We have described the construction of a generalized Landau-Ginzburg B-model. In this construction, we have allowed nondiagonal group representations to be used. We have demonstrated all of the changes and allowances that must be made in this case, and have worked on defining a multiplication. We have shown through explicit computation that this multiplication can be made to be associative in several cases. We have proven that such a multiplication is also possible in the case of a certain type of group representation.

To avoid some of the difficulties apparent in the nonabelian construction, we have also shown that invariant theory can be used to create less unwieldy examples. Picking a nice group representation and then finding an admissible polynomial will facilitate future examples. These examples may be useful in determining necessary and sufficient conditions for an associative multiplication, and reveal patterns in the requisite $\epsilon$ values.

We have also looked at another generalization of B-models, using noninvertible polynomials. While we still do not have a proof for associativity in general, we have shown that any B-model constructed using $P_8$ has an associative multiplication, regardless of which abelian group is used.

To further understand the B-model, we have also found two sets of B-models–one using a trivial group, one using a nontrivial abelian group–that are isomorphic. We defined a map and proved explicitly that it is an isomorphism. As the higher structure of B-models created with nontrivial groups is not well understood, this particular case may be of use in the future.

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