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Existence of a Periodic Brake Orbit in the Fully SymmetricPlanar Four Body Problem

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ABSTRACT

Existence of a Periodic Brake Orbit in the Fully Symmetric
Planar Four Body Problem

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We investigate the existence of a symmetric singular periodic brake orbit in the equal mass, fully symmetric planar four body problem. Using regularized coordinates, we remove the singularity of binary collision for each symmetric pair. We use topological and symmetry tools in our investigation.

Keywords: four body problem, brake orbits, binary collision, topological tools

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Chapter 1

Introduction

1.1 The Four Body Problem

In the classical four-body problem we study the motion of four bodies moving according to Newton's law of gravitation. In the symmetric planar equal mass four body problem, the masses of the four bodies are all equal, and they lie on the same plane with their positions symmetric to each other along the horizontal and vertical direction. Chen [1] has proved existence of periodic orbits in the planar equal mass four body problem with parallelogram configuration. Bakker, Ouyang, Yan and Simmons [2] have proved the existence and determined stability of symmetric planar periodic orbits with simultaneous binary collision in the symmetric four-body problem.

Our main focus is on periodic brakes orbits in four-body problem. A brake orbit is a solution to the four-body problem for which, at some instant, all four bodies have zero velocity. Weinstein [3], Rabinowitz [4] and Chen [5] have studied periodic brake orbit under different set ups. Here we are looking for periodic brake orbit in the symmetric planar equal mass four-body problem.

Using numerical stimulations, we identified a potential candidate for a periodic brake orbit. Figure 1.1 is the periodic brake orbit that we are going to study, and we give it a name periodic brake orbit 121 because it hits the x_1 axis and the x_2 axis then the x_1 axis again in that order. We will give a rigorous definition of the periodic brake orbit 121 in the Main Results in Chapter 2.

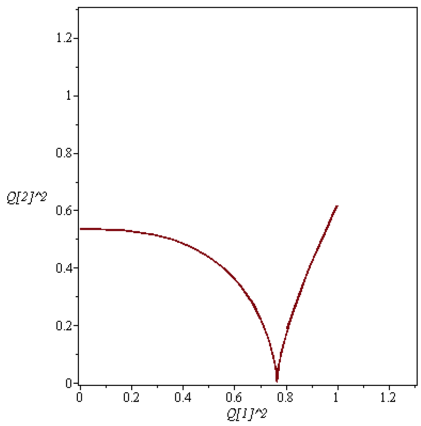

FIGURE 1.1: Periodic brake orbit 121

In the periodic brake orbits that we study simultaneous binary collision (SBC) is involved, meaning that four bodies would collide with each other (with infinite speed and infinite acceleration at the collision), and bounce back to the opposite direction. The singularity at collision creates problems while trying to analyze the orbits. When trying to prove the existence of particular solutions to an n-body problem, the method of minimization of action (a variational technique) is often used. However the orbit we are trying to prove existence has a singularity at collision between our boundary conditions. Marchal [6] showed that for the method of minimization of action to be used, singularities can only occur at initial or final times but not intermediate times. Therefore the method of minimization of action cannot be used to prove existence of our target orbit. As a result we turn to topological techniques. We first need to regularize the simultaneous binary collisions. While binary collisions are well known to be regularizable, Simó[7] showed that simultaneous binary collision can be regularized too. After regularization we prove the existence of our target orbit using the shooting method. Topological techniques and regularization has been used in several instances to prove existence of solutions with regularizable collisions. Moeckel [8] used topological techniques to prove existence of a solution with regularizable collisions in the collinear three body problem. Duokui Yan [9] [10] used topological techniques to prove existence of a solution with regularizable collisions in a planar three body problem and a planar equal mass four body problem. Regina Martínez [11] gives topological proof of solutions with regularizable collisions in several even number n-body problems. Ouyang, Yan, and Simmons [12] used topological techniques to prove existences of periodic solutions with regularizable collisions in several even number n-body problems.

Besides using regularization, to apply the shooting method we used several differential inequalities to use get estimates of time intervals and information about the orbit between collisions. Symmetries are used to generate the full orbit from a part of the orbit. One part of the results has not been proven analytically so we provide only numerical results for that part. At the end we will discuss other SBC periodic brake orbits orbits that we have found numerically. Techniques that we used here can likely be applied as we try proving the existence of those orbits too.

Chapter 2

Main Results

Our goal is to prove the existence of a symmetric periodic brake orbit with simultaneous binary collision (SBC) in the equal mass, fully symmetric planar four body problem. Section 2.1 lays out the set up for the fully symmetric planar four body problem and the resulting system of differential equations. In section 2.2 we transform the original coordinates into regularized coordinates and gives its corresponding system of differential equations. Section 2.3 describes the SBC periodic brake orbit we are trying to prove existence for. Section 2.4 describes the way of generating a full periodic orbit from part of it using symmetry. Section 2.5 shows how we can scale any solution to a particular energy level of the same sign. Section 2.6 and 2.7 proves the existence of an orbit going up and an orbit going down at x_2 axis collision. In section 2.8 we prove the existence of the periodic solution by Intermediate Value Theorem, using the results in all previous sections.

2.1 The Equations

We start with the equations of the equal mass planar four-body problem. Let $m \in \mathbb{R}^+$ and $\pi_k \in \mathbb{R}^2, k = \{1, 2, 3, 4\}$, be the masses and positions of three particles. The motions of the four particles are governed by the differential equations

$$m\frac{d^2\pi_k}{dt^2} = \frac{\partial U}{\partial \pi_k},\ \ k = 1, 2, 3, 4,$$

where

$$U(\pi_1,\ \pi_2,\ \pi_3,\ \pi_4) = \sum_{i<j} \frac{m^2}{|\pi_i - \pi_j|}\ \ i, j \in 1, 2, 3, 4.$$

Dividing these equations through by m, and carrying out the partial derivatives gives

$$\frac{d^2\pi_k}{dt^2} = \sum_{j<k} \frac{m(\pi_j - \pi_k)}{|\pi_j - \pi_k|^3},\ \ j, k = 1, 2, 3, 4.$$

Writing these out we get

$$\begin{aligned}
\frac{d^2\pi_1}{dt^2} &= \frac{m(\pi_2 - \pi_1)}{|\pi_2 - \pi_1|^3} + \frac{m(\pi_3 - \pi_1)}{|\pi_3 - \pi_1|^3} + \frac{m(\pi_4 - \pi_1)}{|\pi_4 - \pi_1|^3} \\
\frac{d^2\pi_2}{dt^2} &= \frac{m(\pi_1 - \pi_2)}{|\pi_1 - \pi_2|^3} + \frac{m(\pi_3 - \pi_2)}{|\pi_3 - \pi_2|^3} + \frac{m(\pi_4 - \pi_2)}{|\pi_4 - \pi_2|^3} \\
\frac{d^2\pi_3}{dt^2} &= \frac{m(\pi_1 - \pi_3)}{|\pi_1 - \pi_3|^3} + \frac{m(\pi_2 - \pi_3)}{|\pi_2 - \pi_3|^3} + \frac{m(\pi_4 - \pi_3)}{|\pi_4 - \pi_3|^3} \\
\frac{d^2\pi_4}{dt^2} &= \frac{m(\pi_1 - \pi_4)}{|\pi_1 - \pi_4|^3} + \frac{m(\pi_2 - \pi_4)}{|\pi_2 - \pi_4|^3} + \frac{m(\pi_3 - \pi_4)}{|\pi_3 - \pi_4|^3}.
\end{aligned}$$

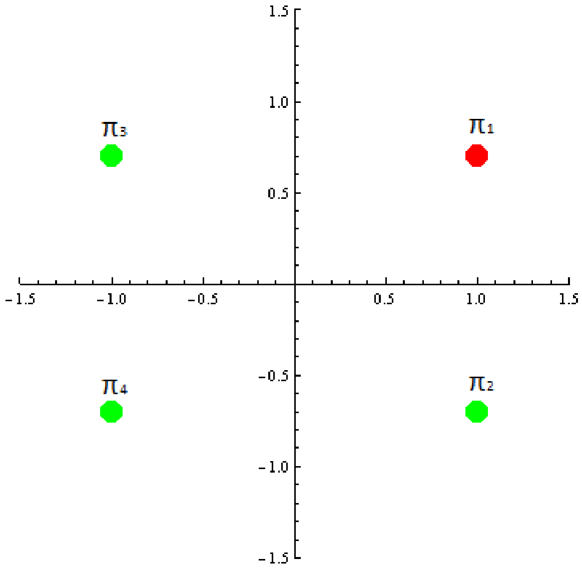

FIGURE 2.1: Position of the four particles

We let $\pi_1 = (x_1, x_2)$. Through an Ansätz we impose a symmetry constraint, and set the coordinates of the four vectors to be respectively:

$$\pi_1 = (x_1, x_2)$$
$$\pi_2 = (-x_1, x_2)$$
$$\pi_3 = (-x_1, -x_2)$$
$$\pi_4 = (x_1, -x_2)$$

for $x_1 \geq 0$ and $x_2 \geq 0$.

So π_1 is in the 1st quadrant, π_2 is in the 2nd quadrant, π_3 is in the 3rd quadrant, π_4 is in the 4th quadrant. (See Figure 2.1)

Now we can simplify the equations once we plug in the coordinates of each π_i

$$\begin{aligned}
\frac{d^2\pi_1}{dt^2} &= \frac{m(-2x_1,0)}{|(-2x_1,0)|^3} + \frac{m(-2x_1,-2x_2)}{|(-2x_1,-2x_2)|^3} + \frac{m(0,-2x_2)}{|(0,-2x_2)|^3}\\
\frac{d^2\pi_2}{dt^2} &= \frac{m(2x_1,0)}{|(2x_1,0)|^3} + \frac{m(0,-2x_2)}{|(0,-2x_2)|^3} + \frac{m(2x_1,-2x_2)}{|(2x_1,-2x_2)|^3}\\
\frac{d^2\pi_3}{dt^2} &= \frac{m(2x_1,2x_2)}{|(2x_1,2x_2)|^3} + \frac{m(0,2x_2)}{|(0,2x_2)|^3} + \frac{m(2x_1,0)}{|(2x_1,0)|^3}\\
\frac{d^2\pi_4}{dt^2} &= \frac{m(0,2x_2)}{|(0,2x_2)|^3} + \frac{m((-2x_1,2x_2))}{|(-2x_1,2x_2)|^3} + \frac{m(2x_1,0)}{|(2x_1,0)|^3}.
\end{aligned}$$

If we let $\dfrac{d^2\pi_1}{dt^2} = (\ddot{x}_1, \ddot{x}_2)$ then

$$\ddot{x}_1 = m\left(-\frac{1}{4x_1^2} - \frac{x_1}{4(x_1^2+x_2^2)^{3/2}}\right) \tag{2.1}$$

$$\ddot{x}_2 = m\left(-\frac{1}{4x_2^2} - \frac{x_2}{4(x_1^2+x_2^2)^{3/2}}\right) \tag{2.2}$$

and we would have

$$\begin{aligned}
\frac{d^2\pi_1}{dt^2} &= (\ddot{x}_1, \ddot{x}_2)\\
\frac{d^2\pi_2}{dt^2} &= (-\ddot{x}_1, \ddot{x}_2)\\
\frac{d^2\pi_3}{dt^2} &= (-\ddot{x}_1, -\ddot{x}_2)\\
\frac{d^2\pi_4}{dt^2} &= (\ddot{x}_1, -\ddot{x}_2).
\end{aligned}$$

Reduce to Studying One Particle Only

Since particles π_2, π_3, π_4 have their position (and thus acceleration) strictly determined by π_1, it is sufficient to study the behavior of particle π_1 alone.

Theorem 2.1. *The mass m in differential equations 2.1 and 2.2 can be scaled to one under appropriate time change.*

Proof. Begin with differetial equations 2.1 and 2.2:

$$\ddot{x}_1(t) = m\Big(- \frac{1}{4x_1^2(t)} - \frac{x_1(t)}{4(x_1^2(t)+x_2^2(t))^{3/2}}\Big) \tag{2.3}$$

$$\ddot{x}_2(t) = m\Big(- \frac{1}{4x_2^2(t)} - \frac{x_2(t)}{4(x_1^2(t)+x_2^2(t))^{3/2}}\Big), \tag{2.4}$$

introduce a time change $\sqrt{m}\tau = t$,then the differential equations become:

$$\frac{\partial^2 x_1(\sqrt{m}\tau)}{\partial t^2} = m\Big(- \frac{1}{4x_1^2(\sqrt{m}\tau)} - \frac{x_1(\sqrt{m}\tau)}{4(x_1^2(\sqrt{m}\tau)+x_2^2(\sqrt{m}\tau))^{3/2}}\Big)$$

$$\frac{\partial^2 x_2(\sqrt{m}\tau)}{\partial t^2} = m\Big(- \frac{1}{4x_2^2(\sqrt{m}\tau)} - \frac{x_2(\sqrt{m}\tau)}{4(x_1^2(\sqrt{m}\tau)+x_2^2(\sqrt{m}\tau))^{3/2}}\Big),$$

use chain rule to change the partical derivative from t to τ

$$m\frac{\partial^2 x_1(\sqrt{m}\tau)}{\partial \tau^2} = m\Big(- \frac{1}{4x_1^2(\sqrt{m}\tau)} - \frac{x_1(\sqrt{m}\tau)}{4(x_1^2(\sqrt{m}\tau)+x_2^2(\sqrt{m}\tau))^{3/2}}\Big)$$

$$m\frac{\partial^2 x_2(\sqrt{m}\tau)}{\partial \tau^2} = m\Big(- \frac{1}{4x_2^2(\sqrt{m}\tau)} - \frac{x_2(\sqrt{m}\tau)}{4(x_1^2(\sqrt{m}\tau)+x_2^2(\sqrt{m}\tau))^{3/2}}\Big),$$

therefore we have

$$\frac{\partial^2 x_1(\sqrt{m}\tau)}{\partial \tau^2} = - \frac{1}{4x_1^2(\sqrt{m}\tau)} - \frac{x_1(\sqrt{m}\tau)}{4(x_1^2(\sqrt{m}\tau)+x_2^2(\sqrt{m}\tau))^{3/2}}$$

$$\frac{\partial^2 x_2(\sqrt{m}\tau)}{\partial \tau^2} = - \frac{1}{4x_2^2(\sqrt{m}\tau)} - \frac{x_2(\sqrt{m}\tau)}{4(x_1^2(\sqrt{m}\tau)+x_2^2(\sqrt{m}\tau))^{3/2}}$$

sytem of differential equations with the mass scaled to one.

□

Since mass can be scaled away, we can study the differential equations assuming mass $m = 1$.

Our system of differential equations become:

$$\ddot{x}_1 = -\frac{1}{4x_1^2} - \frac{x_1}{4(x_1^2 + x_2^2)^{3/2}} \tag{2.5}$$

$$\ddot{x}_2 = -\frac{1}{4x_2^2} - \frac{x_2}{4(x_1^2 + x_2^2)^{3/2}}. \tag{2.6}$$

on the open first quadrant $x_1 > 0, x_2 > 0$.

From the system of differential equations above we can obtain a Hamiltonian

$$\begin{aligned} H &= \frac{1}{2}(p_1^2 + p_2^2) + (x_1\ddot{x}_1 + x_2\ddot{x}_2) \\ &= \frac{1}{2}(p_1^2 + p_2^2) - \frac{1}{4x_1} - \frac{1}{4x_2} - \frac{1}{4(x_1^2 + x_2^2)^{1/2}}. \end{aligned} \tag{2.7}$$

where $p_1 = \dot{x}_1$ and $p_2 = \dot{x}_2$ (velocity of the particle in the first quadrant).

Checking that H is a Hamiltonian:

$$
\begin{aligned}
\frac{dH}{dp_1} &= p_1 = \dot{x}_1 \\
\frac{dH}{dp_2} &= p_2 = \dot{x}_2 \\
\frac{dH}{dx_1} &= -\frac{1}{4x_1^2} - \frac{x_1}{4(x_1^2 + x_2^2)^{3/2}} = \dot{p}_1 \\
\frac{dH}{dx_2} &= -\frac{1}{4x_2^2} - \frac{x_2}{4(x_1^2 + x_2^2)^{3/2}} = \dot{p}_2.
\end{aligned}
$$

Notice here that as $x_1 \to 0$ or $x_2 \to 0$ (near collision), both $\ddot{x}_1$, $\ddot{x}_2$ and our Hamiltonian H would blow up. In order to study the behavior of the particles near collision, we need a change of coordinates.

2.2 Introducing Regularized Coordinate

We introduce new coordinates Q_1, Q_2, P_1 and P_2, from which we apply a symplectic transformation which preserves the form of the Hamiltonian equations. As the first step of the symplectic transformation, we define a generating function

$$F = p_1 Q_1^2 + p_2 Q_2^2.$$

Through our generating function we define

$$
\begin{aligned}
x_1 &= \frac{dF}{dp_1} = Q_1^2 \\
x_2 &= \frac{dF}{dp_2} = Q_2^2 \\
P_1 &= \frac{dF}{dQ_1} = 2p_1 Q_1 \\
P_2 &= \frac{dF}{dQ_2} = 2p_2 Q_2.
\end{aligned}
$$

Therefore the relationship between the old coordinate and the regularized coordinate is:

$$
\begin{aligned}
x_1 &= Q_1^2 \\
x_2 &= Q_2^2 \\
p_1 &= \frac{P_1}{2Q_1} \\
p_2 &= \frac{P_2}{2Q_2}.
\end{aligned} \tag{2.8}
$$

Now replace the old coordinates by the regularized coordinates into our Hamiltonian H (equation 2.7) to obtain $\hat{H}$

$$\hat{H} = \frac{1}{2}\left(\frac{P_1^2}{4Q_1^2} + \frac{P_2^2}{4Q_2^2}\right) - \frac{1}{4}\left(\frac{1}{Q_1^2} + \frac{1}{Q_2^2} + \frac{1}{(Q_1^4 + Q_2^4)^{1/2}}\right) \tag{2.9}$$

In order to get rid of the singularity when $Q_1 \to 0$ or $Q_2 \to 0$, we introduce a time change from t to s. As a particle approaches binary collision, we would observe that $p_i \to \infty$ as $Q_i \to 0$. Defining s by $\frac{dt}{ds} = Q_1^2 Q_2^2$ means that as $Q_i \to 0$, a small change in t would correspond to a big change in s, resulting in a time dilation in s near collision. Through which the 'velocity' in s, P_i is made finite at collision.

Multiply $\hat{H}$ by $Q_1^2 Q_2^2$ to get

$$\frac{dt}{ds}\hat{H} = \frac{1}{8}\left(P_1^2 Q_2^2 + P_2^2 Q_1^2\right) - \frac{1}{4}\left(Q_2^2 + Q_1^2 + \frac{Q_1^2 Q_2^2}{(Q_1^4 + Q_2^4)^{1/2}}\right).$$

Define a Hamiltonian Γ to be

$$\Gamma = \frac{dt}{ds}(\hat{H} - E)$$

and we get

$$\Gamma = \frac{1}{8}\left(P_1^2 Q_2^2 + P_2^2 Q_1^2\right) - \frac{1}{4}\left(Q_2^2 + Q_1^2 + \frac{Q_1^2 Q_2^2}{(Q_1^4 + Q_2^4)^{1/2}}\right) - EQ_1^2 Q_2^2, \tag{2.10}$$

a Hamiltonian defined in the extended phase space.

Proposition 2.2. *Γ is C^1 in the whole plane.*

Proof. Looking at equation 2.10

$$\Gamma = \frac{1}{8}\Big(P_1^2 Q_2^2 + P_2^2 Q_1^2\Big) - \frac{1}{4}\Big(Q_2^2 + Q_1^2 + \frac{Q_1^2 Q_2^2}{(Q_1^4 + Q_2^4)^{1/2}}\Big) - E Q_1^2 Q_2^2,$$

we readily see that Γ as a functions of P_1, P_2, Q_1 and Q_2 has no discontinuity except possible when $Q_1 \to 0$ and $Q_2 \to 0$.

Consider $\left|\frac{Q_1^2 Q_2^2}{(Q_1^4 + Q_2^4)^{1/2}}\right|$ when $Q_1 \to 0$ and $Q_2 \to 0$. Because

$$\begin{aligned}\left|\frac{Q_1^2 Q_2^2}{(Q_1^4 + Q_2^4)^{1/2}}\right| &\leq \left|\frac{Q_1^2 Q_2^2}{(Q_1^4)^{1/2}}\right| \\ &\leq |Q_2^2| \to 0,\end{aligned}$$

there is no singularity for Γ when $Q_1 \to 0$ and $Q_2 \to 0$ or at any point in the whole plane. Therefore Γ is C^1 in the whole plane. □

We are assuming a system with no energy lost (perfectly elastic collision). Therefore E does not vary with time. Under this set up, the level curve given by $\Gamma = 0$ would give us a system of differential equations in the regularized coordinates.

$\Gamma = 0$ is the new Hamiltonian system given by

$$\begin{aligned}-\dot{P}_1 &= \frac{\partial \Gamma}{\partial Q_1} \\ -\dot{P}_2 &= \frac{\partial \Gamma}{\partial Q_2} \\ \dot{Q}_1 &= \frac{\partial \Gamma}{\partial P_1} \\ \dot{Q}_2 &= \frac{\partial \Gamma}{\partial P_2}.\end{aligned}$$

Therefore

$$
\begin{aligned}
\dot{Q}_1 &= \frac{1}{4}P_1Q_2^2 \\
\dot{Q}_2 &= \frac{1}{4}P_2Q_1^2 \\
\dot{P}_1 &= -\frac{1}{4}P_2^2Q_1 + \frac{1}{2}Q_1 + \frac{1}{2}\frac{Q_1Q_2^6}{(Q_1^4+Q_2^4)^{3/2}} + 2EQ_1Q_2^2 \\
\dot{P}_2 &= -\frac{1}{4}P_1^2Q_2 + \frac{1}{2}Q_2 + \frac{1}{2}\frac{Q_2Q_1^6}{(Q_1^4+Q_2^4)^{3/2}} + 2EQ_2Q_1^2.
\end{aligned}
\tag{2.12}
$$

It is important to point out that in the regularized coordinate system we have a C^1 system of differential equations away from the origin, even at simlutaneous binary collision.

Theorem 2.3. *Under regularized coordinate, the differential equation is real analytic except possibly the origin, even through simultaneous binary collision.*

Proof. As long as Q_1 and Q_2 are not zero at the same time (which covers away from collision and binary collision), there is no singularity in our differential equation. Therefore our differential equation is continuous away from the origin. □

Proposition 2.4. *$P_1 = \pm\sqrt{2}$ when $Q_1 = 0$ and $Q_2 \neq 0$; P_2 equals to $\pm\sqrt{2}$ when $Q_2 = 0$ and $Q_1 \neq 0$.*

Proof. We begin with Hamiltonian Γ (equation 2.10) and set $\Gamma = 0$.

$$
0 = \frac{1}{8}\Big(P_1^2Q_2^2 + P_2^2Q_1^2\Big) - \frac{1}{4}\Big(Q_2^2 + Q_1^2 + \frac{Q_1^2Q_2^2}{(Q_1^4+Q_2^4)^{1/2}}\Big) - EQ_1^2Q_2^2.
$$

Set $Q_1 = 0$ and we get

$$
0 = \frac{1}{8}P_1^2Q_2^2 - \frac{1}{4}Q_2^2.
$$

Since $Q_2 \neq 0$,

$$
\begin{aligned}
P_1^2 &= 2 \\
P_1 &= \pm\sqrt{2}.
\end{aligned}
$$

The result for P_2 equals to $\pm\sqrt{2}$ when $Q_2 = 0$ and $Q_1 \neq 0$ can be obtained similarly.

$\square$

Note that P_2 equals to $\pm\sqrt{2}$ for multiple solution that has binary collision at the same point does not violates the uniqueness of solution. Two different solution can have all four coordinates the same in the extended regularized plane because $Q_1 = 0$ forces $P_1 = \pm\sqrt{2}$ or $Q_2 = 0$ forces $P_2 = \pm\sqrt{2}$. However this does not violates the uniqueness of solution because the energy H is still different for the two different solutions.

Proposition 2.5. *For each $(P_1, P_2) \in \mathbb{R}^2$, the point $(Q_1 = 0, Q_2 = 0, P_1, P_2)$ is an equilibrium in the regularized coordinate of the system 2.12.*

Proof. Consider $\left|\frac{Q_1 Q_2^6}{(Q_1^4+Q_2^4)^{3/2}}\right|$ and $\left|\frac{Q_2 Q_1^6}{(Q_1^4+Q_2^4)^{3/2}}\right|$ when $Q_1 \to 0$ and $Q_2 \to 0$. Now

$$\begin{aligned}
\left|\frac{Q_2 Q_1^6}{(Q_1^4+Q_2^4)^{3/2}}\right| &\leq \left|\frac{Q_2 Q_1^6}{(Q_1^4)^{3/2}}\right| \\
&\leq |Q_2| \to 0 \\
\left|\frac{Q_1 Q_2^6}{(Q_1^4+Q_2^4)^{3/2}}\right| &\leq \left|\frac{Q_1 Q_2^6}{(Q_2^4)^{3/2}}\right| \\
&\leq |Q_1| \to 0.
\end{aligned}$$

As a result for an fixed P_1 and P_2, when $Q_1 \to 0$ and $Q_2 \to 0$ we have

$$\begin{aligned}
\dot{Q}_1 &= \frac{1}{4} P_1 Q_2^2 \to 0 \\
\dot{Q}_2 &= \frac{1}{4} P_2 Q_1^2 \to 0 \\
\dot{P}_1 &= -\frac{1}{4} P_2^2 Q_1 + \frac{1}{2} Q_1 + \frac{1}{2} \frac{Q_1 Q_2^6}{(Q_1^4+Q_2^4)^{3/2}} + 2E Q_1 Q_2^2 \to 0 \\
\dot{P}_2 &= -\frac{1}{4} P_1^2 Q_2 + \frac{1}{2} Q_2 + \frac{1}{2} \frac{Q_2 Q_1^6}{(Q_1^4+Q_2^4)^{3/2}} + 2E Q_2 Q_1^2 \to 0.
\end{aligned}$$

Therefore for any fixed $(P_1, P_2) \in \mathbb{R}^2$, the point $(Q_1 = 0, Q_2 = 0, P_1, P_2)$ is an equilibrium in the regularized coordinate system. $\square$

2.3 Periodic Brake Orbit 121

Our main goal is to prove the existence of a SBC periodic brake orbit. Using numerical stimulations, we have generated a solution that is close to satisfying the the SBC brake orbit criteria. The two plots below are the numerical stimulated solution in the original coordinates and the regularized coordinates.

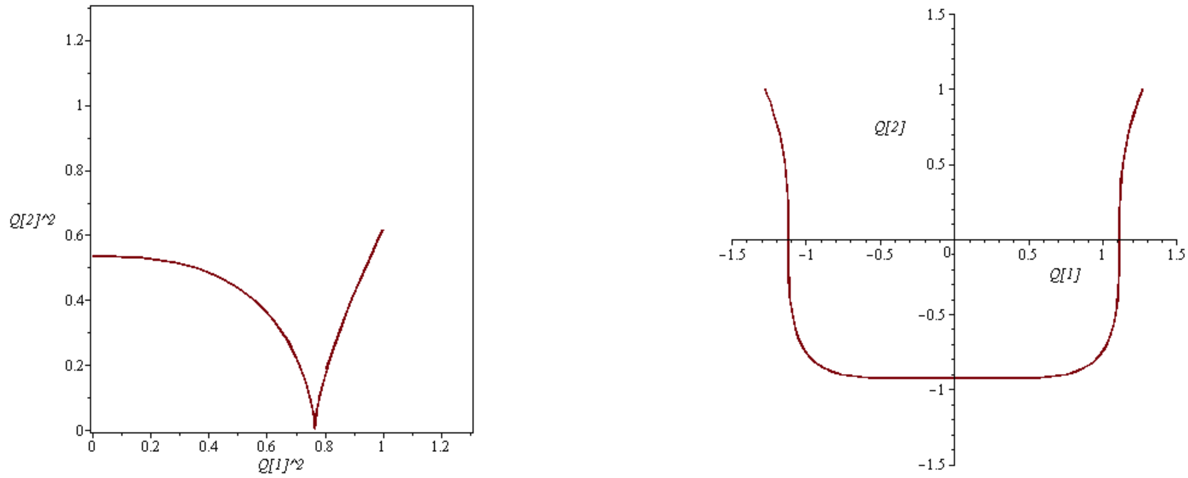

FIGURE 2.2: Periodic brake orbit 121

Figure 2.2 plotted the SBC periodic brake orbit we are looking for on both x_1-x_2 plane and Q_1-Q_2 plane. Looking that the left plot (x_1-x_2 plane), the particle starts at a brake, first collides on the x_1 axis, then hits the x_2 axis and bounces back under the exact same track, hits the x_1 axis at the same spot, and finally reach a brake again at the starting position. To explain the relationship between x_i and Q_i, notice that as $x_i \to 0$, $Q_i \to 0$ at the same time. While in the x_1-x_2 plane the particle bounces back from the axis after collision, in the Q_1-Q_2 plane the particle would pass through the axis in a C^1 manner.

From now on we are going to refer to this particular SBC periodic brake orbit as periodic brake orbit 121 (121 tells us the order of the binary collisions against x_1 and x_2 axis between two brakes).

In Chapter 3 Future Works we will talk about other possible SBC periodic brake orbit with a different set of collisions, and show the numerical stimulations for those potential SBC periodic brake orbits.

2.4 Generating the Full Periodic Orbit from Part of it Using Symmetry

In the regularized plane, periodic brake orbit 121 can be divided into four parts that are symmetric to each other. By symmetry and a proper time change we can generate the full periodic orbit from one part of the orbit.

Consider periodic brake orbit 121 as illustrated in Figure 2.3.

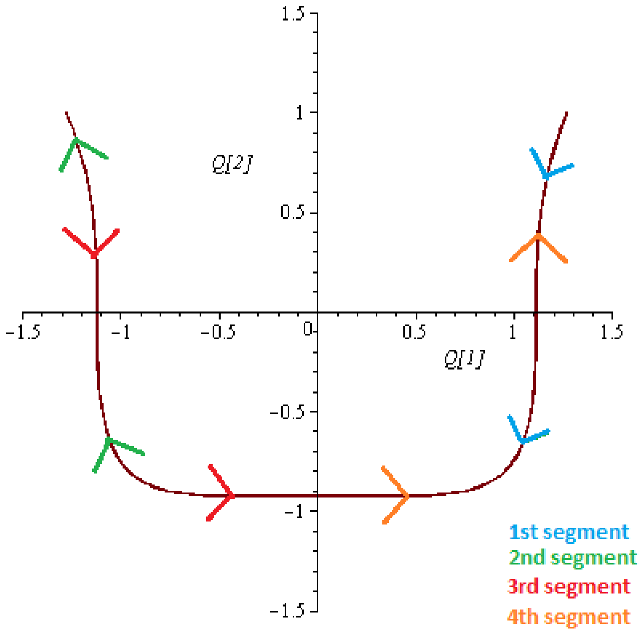

FIGURE 2.3: Periodic brake orbit 121 in four segments

We can describe periodic brake orbit 121 in terms of quadrants in the Q_1-Q_2 plane by the flow chart below: (I for 1st quadrant, II for 2nd quadrant, etc.)

$$\text{brake at } I \rightarrow IV \rightarrow III \rightarrow \text{brake at } II \rightarrow III \rightarrow IV \rightarrow \text{brake at } I$$

From the brake in the I quadrant to the brake in the II quadrant, we refer to the orbit in I and IV quadrants as the 1st segment, the orbit in III and II quadrants as the 2nd segment. From the brake in the II quadrant to the brake in the I quadrant, we refer to the orbit in III and II quadrants to as 3rd segment, the orbit in I and IV quadrants as the 4th segment.

Let T denote the full period of the orbit.

In order for the 1st segment to be able to generate the other three segment in a continuous way such that the other three segments satisfy the differential equations 2.12 at the same time, we need to impose the restriction that $P_2 = 0$ when $Q_1 = 0$. (In other words, to use the argument below, our 1st segment orbit must satisfy the condition that $P_2 = 0$ when $Q_1 = 0$)

Suppose the 1st segment of the periodic orbit is given by

$$(Q_1, Q_2, P_1, P_2)(s) \qquad 0 \leq s \leq \frac{T}{4}$$

then we define the others segments in the following way:

2nd segment:

$$(-Q_1, Q_2, P_1, -P_2)(\frac{T}{2} - s) \qquad \frac{T}{4} \leq s \leq \frac{T}{2}$$

3rd segment:

$$(-Q_1, Q_2, -P_1, P_2)(s - \frac{T}{2}) \qquad \frac{T}{2} \leq s \leq \frac{3T}{4}$$

4th segment:

$$(Q_1, Q_2, -P_1, -P_2)(T - s) \qquad \frac{3T}{4} \leq s \leq T$$

Our claim is that the 2nd, 3rd and 4th segment defined in this way does combine with the 1st segment to form a periodic orbit.

First, we have to check that the head and tails of these segments do join together. Note that we have the following constraints:

$$Q_1(t) = 0 \quad \text{and} \quad Q_2(t) = 0 \qquad \text{when } t = \frac{T}{4} \text{ or } \frac{3T}{4}$$

$$P_1(t) = 0 \quad \text{and} \quad P_2(t) = 0 \qquad \text{when } t = 0 \text{ or } \frac{T}{2} \text{ or } T$$

End of 1st segment:

$$(Q_1, Q_2, P_1, P_2)(\frac{T}{4}) = (0, Q_2(\frac{T}{4}), P_1(\frac{T}{4}), 0)$$

Beginning of 2nd segment:

$$(-Q_1, Q_2, P_1, -P_2)(\frac{T}{4}) = (0, Q_2(\frac{T}{4}), P_1(\frac{T}{4}), 0)$$

End of 2nd segment:

$$(-Q_1, Q_2, P_1, -P_2)(\frac{T}{2}) = (-Q_1(\frac{T}{2}), Q_2(\frac{T}{2}), 0, 0)$$

Beginning of 3rd segment:

$$(-Q_1, Q_2, -P_1, P_2)(\frac{T}{2}) = (-Q_1(\frac{T}{2}), Q_2(\frac{T}{2}), 0, 0)$$

End of 3rd segment:

$$(-Q_1, Q_2, -P_1, P_2)(\frac{3T}{4}) = (0, Q_2(\frac{3T}{4}), P_1(\frac{3T}{4}), 0)$$

Beginning of 4th segment:

$$(Q_1, Q_2, -P_1, -P_2)(\frac{3T}{4}) = (0, Q_2(\frac{3T}{4}), P_1(\frac{3T}{4}), 0)$$

End of 4th segment:

$$(Q_1, Q_2, -P_1, -P_2)(T) = (Q_1(0), Q_2(0), 0, 0)$$

Beginning of 1st segment:

$$(Q_1, Q_2, P_1, P_2)(0) = (Q_1(0), Q_2(0), 0, 0)$$

So we see that the heads and tails of the segments do join together.

Now we have to check that, suppose segment 1 satisfies the system of differential equations 2.12, then segment 2, 3 and 4 also satisfy the same system of differential equations.

Suppose segment 1 satisfies the system of differential equations 2.12

Segment 2:

Use $\hat{Q}_1, \hat{Q}_2, \hat{P}_1, \hat{P}_2$ to represent the the 4 coordinates of segment 2. Let

$$\begin{aligned}
\hat{Q}_1 &= -Q_1(\frac{T}{2} - s)\\
\hat{Q}_2 &= Q_2(\frac{T}{2} - s)\\
\hat{P}_1 &= P_1(\frac{T}{2} - s)\\
\hat{P}_2 &= -P_2(\frac{T}{2} - s).
\end{aligned}$$

Then

$$\begin{aligned}
\frac{d\hat{Q}_1(s)}{ds} &= \frac{d\big(-Q_1(\frac{T}{2} - s)\big)}{ds}\\
&= \dot{Q}_1(\frac{T}{2} - s)\\
&= \frac{1}{4}P_1(\frac{T}{2} - s)Q_2^2(\frac{T}{2} - s)\\
&= \frac{1}{4}\hat{P}_1(s)\hat{Q}_2^2(s)
\end{aligned}$$

and

$$\begin{aligned}
\frac{d\hat{Q}_2(s)}{ds} &= \frac{d\big(Q_2(\frac{T}{2} - s)\big)}{ds}\\
&= -\dot{Q}_2(\frac{T}{2} - s)\\
&= -\frac{1}{4}P_2(\frac{T}{2} - s)Q_1^2(\frac{T}{2} - s)\\
&= \frac{1}{4}\hat{P}_2(s)\hat{Q}_1^2(s)
\end{aligned}$$

and

$$
\begin{aligned}
\frac{d\hat{P}_1(s)}{ds} &= \frac{d\big(P_1(\frac{T}{2}-s)\big)}{ds} \\
&= -\dot{P}_1(\frac{T}{2}-s) \\
&= \frac{1}{4}P_2^2(\frac{T}{2}-s)Q_1(\frac{T}{2}-s) - \frac{1}{2}Q_1(\frac{T}{2}-s) - \frac{1}{2}\frac{Q_1(\frac{T}{2}-s)Q_2^6(\frac{T}{2}-s)}{\big(Q_1^4(\frac{T}{2}-s)+Q_2^4(\frac{T}{2}-s)\big)^{\frac{3}{2}}} \\
&\qquad - 2EQ_1(\frac{T}{2}-s)Q_2^2(\frac{T}{2}-s) \\
&= -\frac{1}{4}\hat{P}_2^2(s)\hat{Q}_1(s) + \frac{1}{2}\hat{Q}_1(s) + \frac{1}{2}\frac{\hat{Q}_1(s)\hat{Q}_2^6(s)}{\big(\hat{Q}_1^4(s)+\hat{Q}_2^4(s)\big)^{\frac{3}{2}}} + 2E\hat{Q}_1(s)\hat{Q}_2^2(s)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{d\hat{P}_2(s)}{ds} &= \frac{d\big(-P_2(\frac{T}{2}-s)\big)}{ds} \\
&= \dot{P}_2(\frac{T}{2}-s) \\
&= -\frac{1}{4}P_1^2(\frac{T}{2}-s)Q_2(\frac{T}{2}-s) + \frac{1}{2}Q_2(\frac{T}{2}-s) + \frac{1}{2}\frac{Q_2(\frac{T}{2}-s)Q_1^6(\frac{T}{2}-s)}{\big(Q_1^4(\frac{T}{2}-s)+Q_2^4(\frac{T}{2}-s)\big)^{\frac{3}{2}}} \\
&\qquad + 2EQ_2(\frac{T}{2}-s)Q_1^2(\frac{T}{2}-s) \\
&= -\frac{1}{4}\hat{P}_1^2(s)\hat{Q}_2(s) + \frac{1}{2}\hat{Q}_2(s) + \frac{1}{2}\frac{\hat{Q}_2(s)\hat{Q}_1^6(s)}{\big(\hat{Q}_1^4(s)+\hat{Q}_2^4(s)\big)^{\frac{3}{2}}} + 2E\hat{Q}_2(s)\hat{Q}_1^2(s).
\end{aligned}
$$

Therefore segment 2 also satisfies the system of differential equations 2.12.

Segment 3:

Use $\check{Q}_1, \check{Q}_2, \check{P}_1, \check{P}_2$ to represent the the 4 coordinates of segment 3. Let

$$\begin{aligned}
\check{Q}_1 &= -Q_1(s - \frac{T}{2}) \\
\check{Q}_2 &= Q_2(s - \frac{T}{2}) \\
\check{P}_1 &= -P_1(s - \frac{T}{2}) \\
\check{P}_2 &= P_2(s - \frac{T}{2}).
\end{aligned}$$

Then

$$\begin{aligned}
\frac{d\check{Q}_1(s)}{ds} &= \frac{d\big(-Q_1(s - \frac{T}{2})\big)}{ds} \\
&= -\dot{Q}_1(s - \frac{T}{2}) \\
&= -\frac{1}{4}P_1(s - \frac{T}{2})Q_2^2(s - \frac{T}{2}) \\
&= \frac{1}{4}\check{P}_1(s)\check{Q}_2^2(s)
\end{aligned}$$

and

$$\begin{aligned}
\frac{d\check{Q}_2(s)}{ds} &= \frac{d\big(Q_2(s - \frac{T}{2})\big)}{ds} \\
&= \dot{Q}_2(s - \frac{T}{2}) \\
&= \frac{1}{4}P_2(s - \frac{T}{2})Q_1^2(s - \frac{T}{2}) \\
&= \frac{1}{4}\check{P}_2(s)\check{Q}_1^2(s)
\end{aligned}$$

and

$$\begin{aligned}
\frac{d\check{P}_1(s)}{ds} &= \frac{d\big(-P_1(s-\frac{T}{2})\big)}{ds} \\
&= -\dot{P}_1(s-\frac{T}{2}) \\
&= \frac{1}{4}P_2^2(s-\frac{T}{2})Q_1(s-\frac{T}{2}) - \frac{1}{2}Q_1(s-\frac{T}{2}) - \frac{1}{2}\frac{Q_1(s-\frac{T}{2})Q_2^6(s-\frac{T}{2})}{\big(Q_1^4(s-\frac{T}{2}) - Q_2^4(s-\frac{T}{2})\big)^{\frac{3}{2}}} \\
&\qquad - 2EQ_1(s-\frac{T}{2})Q_2^2(s-\frac{T}{2}) \\
&= -\frac{1}{4}\check{P}_2^2(s)\check{Q}_1(s) + \frac{1}{2}\check{Q}_1(s) + \frac{1}{2}\frac{\check{Q}_1(s)\check{Q}_2^6(s)}{\big(\check{Q}_1^4(s) + \check{Q}_2^4(s)\big)^{\frac{3}{2}}} + 2E\check{Q}_1(s)\check{Q}_2^2(s)
\end{aligned}$$

and

$$\begin{aligned}
\frac{d\check{P}_2(s)}{ds} &= \frac{d\big(P_2(s-\frac{T}{2})\big)}{ds} \\
&= \dot{P}_2(s-\frac{T}{2}) \\
&= -\frac{1}{4}P_1^2(s-\frac{T}{2})Q_2(s-\frac{T}{2}) + \frac{1}{2}Q_2(s-\frac{T}{2}) + \frac{1}{2}\frac{Q_2(s-\frac{T}{2})Q_1^6(s-\frac{T}{2})}{\big(Q_1^4(s-\frac{T}{2}) + Q_2^4(s-\frac{T}{2})\big)^{\frac{3}{2}}} \\
&\qquad + 2EQ_2(s-\frac{T}{2})Q_1^2(s-\frac{T}{2}) \\
&= -\frac{1}{4}\check{P}_1^2(s)\check{Q}_2(s) + \frac{1}{2}\check{Q}_2(s) + \frac{1}{2}\frac{\check{Q}_2(s)\check{Q}_1^6(s)}{\big(\check{Q}_1^4(s) + \check{Q}_2^4(s)\big)^{\frac{3}{2}}} + 2E\check{Q}_2(s)\check{Q}_1^2(s).
\end{aligned}$$

Therefore segment 3 also satisfies the system of differential equations 2.12.

Segment 4:

Use $\widetilde{Q}_1, \widetilde{Q}_2, \widetilde{P}_1, \widetilde{P}_2$ above to represent the the 4 coordinates of segment 4.

Let

$$\begin{aligned}\widetilde{Q}_1 &= Q_1(T-s)\\ \widetilde{Q}_2 &= Q_2(T-s)\\ \widetilde{P}_1 &= -P_1(T-s)\\ \widetilde{P}_2 &= -P_2(T-s).\end{aligned}$$

Then

$$\begin{aligned}\frac{d\widetilde{Q}_1(s)}{ds} &= \frac{d\big(Q_1(T-s)\big)}{ds}\\ &= -\dot{Q}_1(T-s)\\ &= -\frac{1}{4}P_1(T-s)Q_2^2(T-s)\\ &= \frac{1}{4}\widetilde{P}_1(s)\widetilde{Q}_2^2(s)\end{aligned}$$

and

$$\begin{aligned}\frac{d\widetilde{Q}_2(s)}{ds} &= \frac{d\big(Q_2(T-s)\big)}{ds}\\ &= -\dot{Q}_2(T-s)\\ &= -\frac{1}{4}P_2(T-s)Q_1^2(T-s)\\ &= \frac{1}{4}\widetilde{P}_2(s)\widetilde{Q}_1^2(s)\end{aligned}$$

and

$$\begin{aligned}\frac{d\widetilde{P}_1(s)}{ds} &= \frac{d\big(-P_1(T-s)\big)}{ds}\\ &= \dot{P}_1(T-s)\\ &= -\frac{1}{4}P_2^2(T-s)Q_1(T-s) + \frac{1}{2}Q_1(T-s) + \frac{1}{2}\frac{Q_1(T-s)Q_2^6(T-s)}{\big(Q_1^4(T-s)+Q_2^4(T-s)\big)^{\frac{3}{2}}}\\ &\qquad + 2EQ_1(T-s)Q_2^2(T-s)\\ &= -\frac{1}{4}\widetilde{P}_2^2(s)\widetilde{Q}_1(s) + \frac{1}{2}\widetilde{Q}_1(s) + \frac{1}{2}\frac{\widetilde{Q}_1(s)\widetilde{Q}_2^6(s)}{\big(\widetilde{Q}_1^4(s)+\widetilde{Q}_2^4(s)\big)^{\frac{3}{2}}} + 2E\widetilde{Q}_1(s)\widetilde{Q}_2^2(s)\end{aligned}$$

and

$$\begin{aligned}
\frac{d\widetilde{P}_2(s)}{ds} &= \frac{d\big(-P_2(T-s)\big)}{ds} \\
&= \dot{P}_2(T-s) \\
&= -\frac{1}{4}P_1^2(T-s)Q_2(T-s) + \frac{1}{2}Q_2(T-s) + \frac{1}{2}\frac{Q_2(T-s)Q_1^6(T-s)}{\big(Q_1^4(T-s)+Q_2^4(T-s)\big)^{\frac{3}{2}}} \\
&\qquad + 2EQ_2(T-s)Q_1^2(T-s) \\
&= -\frac{1}{4}\widetilde{P}_1^2(s)\widetilde{Q}_2(s) + \frac{1}{2}\widetilde{Q}_2(s) + \frac{1}{2}\frac{\widetilde{Q}_2(s)\widetilde{Q}_1^6(s)}{\big(\widetilde{Q}_1^4(s)+\widetilde{Q}_2^4(s)\big)^{\frac{3}{2}}} + 2E\widetilde{Q}_2(s)\widetilde{Q}_1^2(s).
\end{aligned}$$

Therefore segment 4 also satisfies the system of differential equations 2.12.

Now since all 4 segments' head and tail do join together, and they all satisfy differential equations 2.12, by uniqueness of solutions to the initial value problem, they must form a periodic orbit.

2.5 Zero Momentum Curve and Scaling Energy

In the coming sections we will consider solutions with different energy. It turns out that finding one of those solutions would mean that we can find it with all energy of the same sign. First we talk about what a zero momentum curve is. Then we show that any orbit can be scaled to a chosen energy level without changing the shape of the orbit, so we can scale one zero momentum curve to another.

While looking for a brake orbit, it is very difficult to specific the right initial position and velocity so that the orbit would momentarily reach a brake at some point. As a result we pick a family of orbits that starts stationary, and see whether a correctly chosen initial position would result in an orbit that returns back to the initial position at some time. Since energy is conserved, if the orbit returns to its initial stationary position it must reach a brake at that point (Having a non-zero velocity would means it has extra kinetic energy that is not present at the initial position, contradicting energy conservation).

Now given any negative energy, we can identify a zero momentum curve for which under zero initial velocity, all the initial positions along the curve has the same energy. Figure 2.4 showed the zero momentum curves for a few different energy levels.

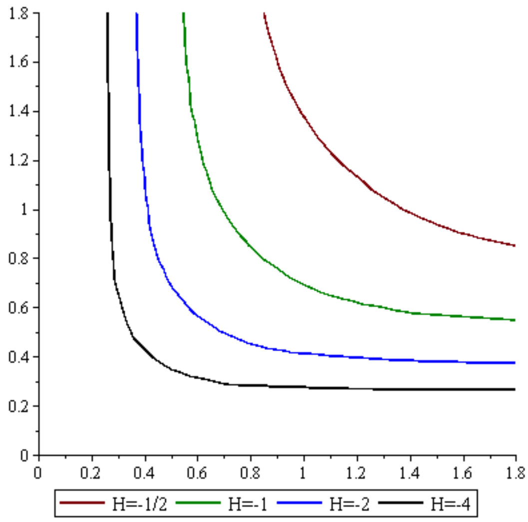

FIGURE 2.4: Zero momentum curves

Now given energy $H_0 < 0$, the zero momentum curve divides the 1st quadrant into two regions, a permissible region and a impermissible region as illustrated in Figure 2.5. An orbit with energy H_0 has no way of reaching the impermissible region, and always stays within the permissible region (except possibly being stationary at the boundary of the permissible region).

Theorem 2.6. *Any solution of our system of differential equations 2.5 and 2.6 can be scaled to any particular energy of the same sign.*

Proof. Given $x_1(t)$, $x_2(t)$ a solution to our system of differential equations 2.5 and 2.6 with energy $H < 0$, for $a > 0$, we define a new position variable

$$s_1(t) = ax_1(a^{-\frac{3}{2}}t) > 0$$
$$s_2(t) = ax_2(a^{-\frac{3}{2}}t) > 0.$$

Now $s_1(t)$ and $s_2(t)$ satisfy the exact same differential equation because

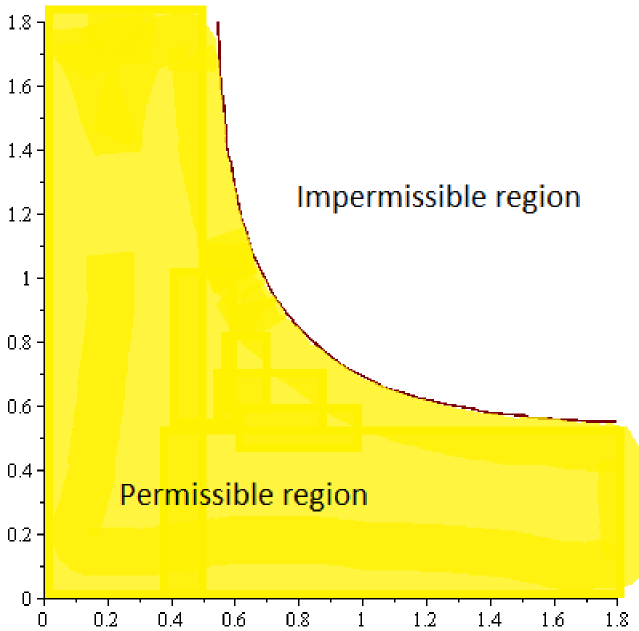

FIGURE 2.5: Permissible and impermissible region

$$\begin{aligned}
\ddot{s}_1(t) &= \frac{\ddot{x}_1(a^{-\frac{3}{2}}t)}{a^2} \\
&= \frac{1}{a^2}\left(-\frac{1}{4x_1^2(a^{-\frac{3}{2}}t)} - \frac{x_1(a^{-\frac{3}{2}}t)}{4\big(x_1^2(a^{-\frac{3}{2}}t) + x_2^2(a^{-\frac{3}{2}}t)\big)^{3/2}}\right) \\
&= -\frac{1}{4a^2x_1^2(a^{-\frac{3}{2}}t)} - \frac{ax_1(a^{-\frac{3}{2}}t)}{4\big(a^2x_1^2(a^{-\frac{3}{2}}t) + a^2x_2^2(a^{-\frac{3}{2}}t)\big)^{3/2}} \\
&= -\frac{1}{4s_1^2(t)} - \frac{s_1(t)}{4\big(s_1^2(t) + s_2^2(t)\big)^{3/2}}, \\
\ddot{s}_2(t) &= \frac{\ddot{x}_2(a^{-\frac{3}{2}}t)}{a^2} \\
&= \frac{1}{a^2}\left(-\frac{1}{4x_2^2(a^{-\frac{3}{2}}t)} - \frac{x_2(a^{-\frac{3}{2}}t)}{4\big(x_1^2(a^{-\frac{3}{2}}t) + x_2^2(a^{-\frac{3}{2}}t)\big)^{3/2}}\right) \\
&= -\frac{1}{4a^2x_2^2(a^{-\frac{3}{2}}t)} - \frac{ax_2(a^{-\frac{3}{2}}t)}{4\big(a^2x_1^2(a^{-\frac{3}{2}}t) + a^2x_2^2(a^{-\frac{3}{2}}t)\big)^{3/2}} \\
&= -\frac{1}{4s_2^2(t)} - \frac{s_2(t)}{4\big(s_1^2(t) + s_2^2(t)\big)^{3/2}},
\end{aligned}$$

which matches our system of differential equations 2.5 and 2.6.

Substitute s_1 and s_2 into equation 2.7, The energy H_a for s_1 and s_2 is given by

$$\begin{aligned}
H_a &= \frac{1}{2}(\dot{s}_1^2(t) + \dot{s}_2^2(t)) - \frac{1}{4s_1(t)} - \frac{1}{4s_2(t)} - \frac{1}{4(s_1^2(t) + s_2^2(t))^{1/2}} \\
&= \frac{1}{2a}(\dot{x}_1^2(a^{-\frac{3}{2}}t) + \dot{x}_2^2(a^{-\frac{3}{2}}t)) - \frac{1}{4ax_1(a^{-\frac{3}{2}}t)} - \frac{1}{4ax_2(a^{-\frac{3}{2}}t)} - \frac{1}{4a(x_1^2(a^{-\frac{3}{2}}t) + x_2^2(a^{-\frac{3}{2}}t))^{1/2}} \\
&= \frac{H}{a}.
\end{aligned}$$

By going from x_1,x_2 to s_1,s_2 under parameter a, we can scale the energy of our orbit by $\frac{1}{a}$. Thus when a particular orbit exists for one energy, it exists for all energy of the same sign. $\square$

2.6 Orbit Going Up

Conjecture 2.7. *An orbit that starts at a brake, hits the x_1 axis, then collides on the x_2 axis with $\dot{x}_2 > 0$ exists.*

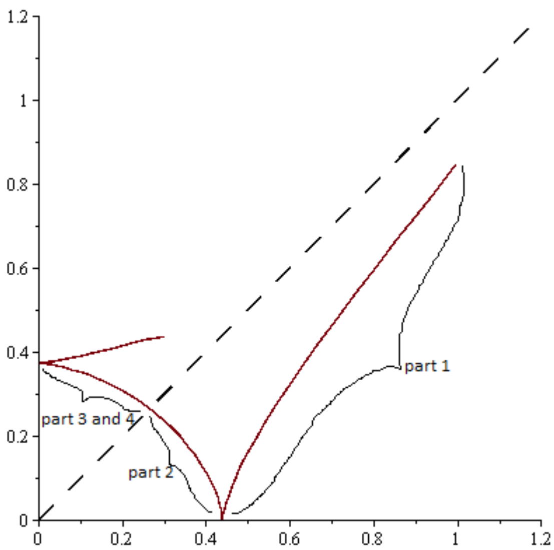

FIGURE 2.6: Four parts of showing orbit going up exist

In this section we want to prove Conjecture 2.7: the existence of an orbit that starts at a brake, hits the x_1 axis, then hits the x_2 axis with $\dot{x}_2 > 0$. We will simply call it an orbit going up. Proving the existence of an orbit going up involves 4 parts, as shown in Figure 2.6. In part 1 (section 2.6.1) we have to show that an orbit that starts at a brake on the right side of the $x_1 = x_2$ line , then hits the x_1 axis exists. In part 2 (section 2.6.2) we have to show that there exist an orbit that extends from part 1 and cross the $x_1 = x_2$ line with $\dot{x}_2 > 0$. In part 3 (section 2.6.3-2.6.7) we estimate the time it takes from the orbit crossing $x_1 = x_2$ to $x_2(t)$ going to zero. In part 4 (section 2.6.8) we estimate the time it takes for the orbit from crossing $x_1 = x_2$ to hitting x_2 axis. We have obtained analytical proof for part 1, 3 and 4; We rely on numerical results for part 2.

In the following sections the existence of orbits are proved under a certain energy, but again by Theorem 2.6 we know that we can scale the orbit to achieve all energy of the same sign.

2.6.1 Orbit that Starts at a Brake and Hits the x_1 Axis

In this section we want to show that an orbit that starts at a brake on the right side of the $x_1 = x_2$ line , then hits the x_1 axis exists. We begin with proving a lemma.

Lemma 2.8. *Under differential equations 2.5 and 2.6,*

$$x_1 > x_2 \iff \ddot{x}_2 < \ddot{x}_1$$

$$x_1 < x_2 \iff \ddot{x}_2 > \ddot{x}_1.$$

Proof. First we show that

$$x_1 = x_2 \iff \ddot{x}_1 = \ddot{x}_2$$

$(\Rightarrow)$

If $x_1 = x_2$, then

$$\frac{1}{4x_1^2} = \frac{1}{4x_x^2}$$

and

$$\frac{x_1}{4(x_1^2 + x_2^2)^{3/2}} = \frac{x_2}{4(x_1^2 + x_2^2)^{3/2}}$$

so obviously $\ddot{x}_1 = \ddot{x}_2$.

($\Leftarrow$)

Now suppose $\ddot{x}_1 = \ddot{x}_2$. Then

$$\begin{aligned}
-\frac{1}{4x_1^2} - \frac{x_1}{4(x_1^2+x_2^2)^{3/2}} &= -\frac{1}{4x_2^2} - \frac{x_2}{4(x_1^2+x_2^2)^{3/2}} \\
\frac{(x_1^2+x_2^2)^{3/2}+x_1^3}{x_1^2(x_1^2+x_2^2)^{\frac{3}{2}}} &= \frac{(x_1^2+x_2^2)^{3/2}+x_2^3}{x_2^2(x_1^2+x_2^2)^{\frac{3}{2}}} \\
x_2^2[(x_1^2+x_2^2)^{3/2}+x_1^3] &= x_1^2[(x_1^2+x_2^2)^{3/2}+x_2^3] \\
x_2^2(x_1^2+x_2^2)^{\frac{3}{2}} + x_1^3x_2^2 &= x_1^2(x_1^2+x_2^2)^{\frac{3}{2}} + x_2^3x_1^2 \\
(x_2^2-x_1^2)(x_1^2+x_2^2)^{\frac{3}{2}} &= x_1^3x_2^2 - x_2^3x_1^2 \\
(x_2-x_1)(x_2+x_1)(x_1^2+x_2^2)^{\frac{3}{2}} &= x_1^2x_2^2(x_2-x_1) \\
0 &= (x_2-x_1)[(x_2+x_1)(x_1^2+x_2^2)^{\frac{3}{2}} - x_1^2x_2^2].
\end{aligned}$$

Now we want to show that

$$f(x_1,x_2) = (x_2+x_1)(x_1^2+x_2^2)^{\frac{3}{2}} - x_1^2x_2^2$$

is positive when $x_1 > 0$ and $x_2 > 0$

Assuming $x_1 > 0$ and $x_2 > 0$ we can use arithmetic and geometric mean inequality, to get the following two relationships:

$$\begin{aligned}
\frac{x_1+x_2}{2} &> \sqrt{x_1x_2} \\
\frac{x_1^2+x_2^2}{2} &> x_1x_2.
\end{aligned}$$

Combine the two inequalities to get

$$(x_1+x_2)(x_1^2+x_2^2)^{\frac{3}{2}} > 2^{\frac{5}{2}}x_1^2x_2^2 > x_1^2x_2^2,$$

therefore

$$f(x_1,x_2) = (x_2+x_1)(x_1^2+x_2^2)^{\frac{3}{2}} - x_1^2x_2^2 > 0$$

when $x_1 > 0$ and $x_2 > 0$. Thus

$$\ddot{x}_1 = \ddot{x}_2 \implies x_1 = x_2$$

Now for $x_1 >> x_2$ we have

$$\ddot{x}_1 \approx 0, \quad \ddot{x}_2 \approx -\frac{1}{4x_2^2} < 0,$$

and for $x_1 << x_2$ we have

$$\ddot{x}_2 \approx 0, \quad \ddot{x}_1 \approx -\frac{1}{4x_1^2} < 0.$$

Thus

$$x_1 > x_2 \iff \ddot{x}_1 < \ddot{x}_2$$

$$x_1 < x_2 \iff \ddot{x}_1 > \ddot{x}_2.$$

□

Theorem 2.9. *For solution $(x_1(t), x_2(t))$ with $x_1(0) > x_2(0)$ and $\dot{x}_1(0) = 0, \dot{x}_2(0) = 0$, there exist $\beta > 0$ and $t^* \in (0, \beta)$ such that $x_1(t) < x_2(t)$ for all $t \in [0, \beta)$ and $x_2(t^*) = 0$ with $x_1(0) - x_2(0) \leq x_1(t^*) < x_1(0)$. Furthermore $\dot{x}_1(t) < 0$ for all $t \in (0, \beta)$.*

Proof. Let $(x_1(t), x_2(t))$ be the solution satisfying

$$x_1(0) > x_2(0)$$

$$\dot{x}_1(0) = 0, \ \dot{x}_2(0) = 0$$

By continuity of the continuously extended solution (regularization) with respect to its intial conditions, there exist a maximal and energy perserving $\beta > 0$ such that $x_1(t) > x_2(t)$ for all $t \in [0, \beta)$. This implies that

$$\lim_{t \to \beta} x_1(t) = \lim_{t \to \beta} x_2(t),$$

otherwise it contradicts β being maximal. This implies that $x_1(t) > 0$ for all $t \in [0, \beta$. From our initial coniditions we know that $x_2(0) > 0$. Let $t^* \in (0, \beta]$ be the largest possible time such that $x_2(t) > 0$ for all $t \in [0, t^*)$.

Claim: $t^* < \beta$ and $x_1(0) - x_2(0) \leq x_1(t^*) < x_1(0)$.

Since $x_1(t) > 0$ and $x_2(t) > 0$ for all $t \in [0, t^*)$, we know that $\dot{x}_1(t)$ and $\dot{x}_2(t)$ are strictly decreasing on $[0, t^*)$, i.e.

$$\dot{x}_1(t) < \dot{x}_1(0) = 0$$

$$\dot{x}_2(t) < \dot{x}_2(0) = 0$$

for all $t \in (0, t^*)$.

Since $x_1(t) > x_2(t)$ for all $t \in [0, t^*)$, by Lemma 2.8 we know that

$$\ddot{x}_2(t) < \ddot{x}_1(t)$$

for all $t \in [0, t^*)$. Intergration of $\ddot{x}_2(t) < \ddot{x}_1(t)$ gives

$$\dot{x}_2(t) - \dot{x}_2(0) < \dot{x}_1(t) - \dot{x}_1(0)$$

Since $\dot{x}_1(0) = 0$ and $\dot{x}_2(0) = 0$ we get

$$\dot{x}_2(t) < \dot{x}_1(t) < 0$$

for all $t \in [0, t^*)$.

From $\dot{x}_1(0) = 0$ for all $t \in (0, t^*)$, continuity of $\dot{x}_1$ and integration gives us

$$x_1(t^*) < x_1(0).$$

Now integration of $\dot{x}_2(t) < \dot{x}_1(t) < 0$ for $t \in [0, t^*)$ gives,

$$x_2(t) - x_2(0) < x_1(t) - x_1(0)$$

for all $t \in [0, t^*)$. So

$$x_1(t) \geq x_2(t) + x_1(0) - x_2(0) > x_2(t) + k$$

because $x_1(0) - x_2(0) = k > 0$. From the equation above we know that

$$x_1(0) > x_1(t^*) \geq x_1(0) - x_2(0)$$

If $t^* = \beta$, then we have

$$x_1(\beta) > x_2(\beta) + k$$

for $k > 0$, contradicting $\lim_{t\to\beta} x_1(t) = \lim_{t\to\beta} x_2(t)$.

Therefore it must be that $t^* < \beta$. (Done with proof of our claim) $\square$

Theorem 2.9 essentially means that an orbit that starts at a brake on the right side of the $x_1 = x_2$ line , then hits the x_1 axis exists.

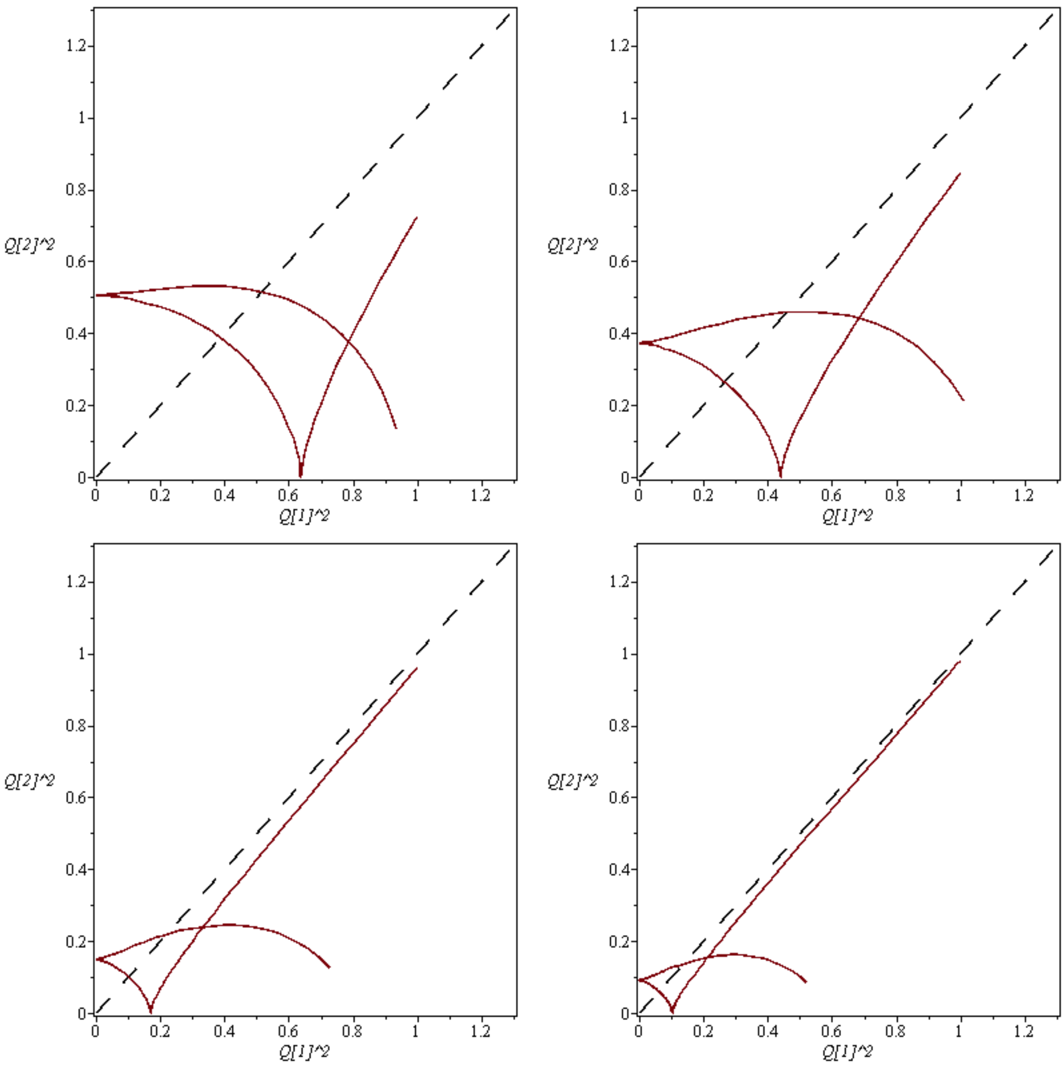

FIGURE 2.7: Numerical Results for Conjecture 2.10

2.6.2 Orbit that Cross the $x_1 = x_2$ Line with $\dot{x}_2 > 0$

Conjecture 2.10. *An orbit that starts at a brake, hits the x_1 axis, then crosses the $x_1 = x_2$ line with $\dot{x}_2 > 0$ exists.*

We provide numerical results for Conjecture 2.10 in this section. Conjecture 2.10 is essentially part 2 of the orbit going up that we are trying to prove. Analytical results for this part have not been obtained yet. As shown in Figure 2.7, all four orbits here crosses the $x_1 = x_2$ line with $\dot{x}_2 > 0$. Numerical simulations suggest that orbits that starts close enough to the $x_1 = x_2$ line will cross the $x_1 = x_2$ line with $\dot{x}_2 > 0$.

In the Future Work section we will mention approaches to proving Conjecture 2.10 analytically.

2.6.3 Set up for proving part 3 and 4 of the orbit going up

Assuming Conjecture 2.10 is true, we now have an orbit that cross the $x_1 = x_2$ line with $\dot{x}_2 > 0$. Scale this orbit until the particle would cross the $x_1 = x_2$ line at $x_1 = x_2 = 1$, corresponding to energy $\hat{H}_0 > H_0$.

For any energy level higher than $\hat{H}_0$, we can always find a corresponding initial x_1 and x_2 such that the orbit would start at a brake, collide at x_1 axis then cross the $x_1 = x_2$ line at $x_1 = x_2 = 1$. The closer the negative energy is to 0, the closer the initial position approaches the $x_1 = x_2$ line.

Consider the case when energy level is negative and close to 0. Let $E = -\epsilon$ for some $\epsilon > 0$. We now define a set of times we are going to look at while proving part 3 and 4 of the orbit going up (refer to Figure 2.8).

Let 0 denote the starting time,

t_1 denote the time when $x_1 = x_2 = 1$,

$t_{\frac{1}{2}}$ denote the time when $x_1 = \frac{1}{2}$,

t_0 denote the time when $x_1 = 0$ (collision at x_2 axis),

t_m denote the time when x_2 reaches a local maximum ($\dot{x}_2 = 0$).

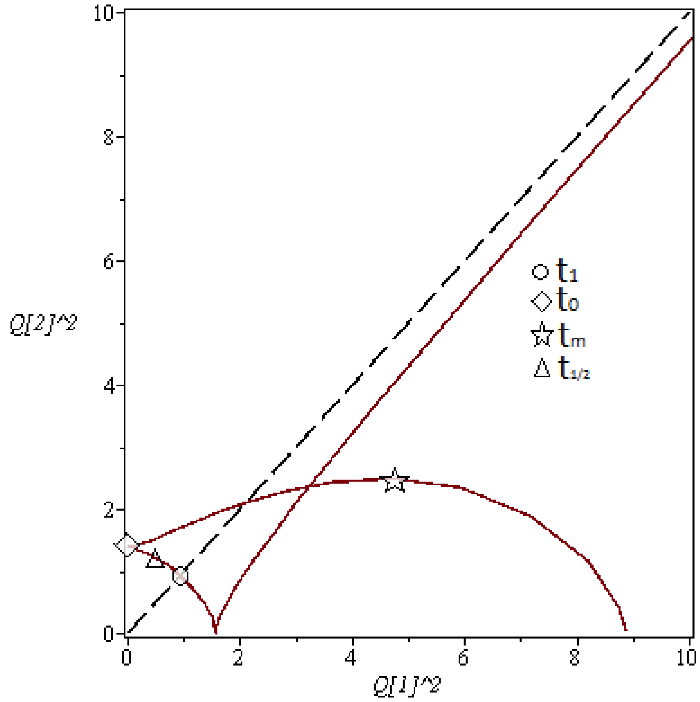

FIGURE 2.8: Times labeled

2.6.4 Lower Bound on $x_1(0)$ and $x_2(0)$

We need to first obtain a lower bound on $x_1(0)$ and $x_2(0)$ using the energy constraint. Begin with the energy equation 2.7:

$$H = \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2) - \left(\frac{1}{4x_1} + \frac{1}{4x_2} + \frac{1}{4(x_1^2 + x_2^2)^{\frac{1}{2}}}\right).$$

Under $H = -\epsilon$, at the initial time when $\dot{x}_1(0) = 0$ and $\dot{x}_2(0) = 0$ we have,

$$\begin{aligned}
-\epsilon &= \frac{1}{2}(\dot{x}_1^2(0) + \dot{x}_2^2(0)) - \left(\frac{1}{4x_1(0)} + \frac{1}{4x_2(0)} + \frac{1}{4(x_1^2(0) + x_2^2(0))^{\frac{1}{2}}}\right) \\
-\epsilon &= -\left(\frac{1}{4x_1(0)} + \frac{1}{4x_2(0)} + \frac{1}{4(x_1^2(0) + x_2^2(0))^{\frac{1}{2}}}\right) \\
\epsilon &= \frac{1}{4x_1(0)} + \frac{1}{4x_2(0)} + \frac{1}{4(x_1^2(0) + x_2^2(0))^{\frac{1}{2}}}.
\end{aligned}$$

All there terms on the right are positive. We get the two inequalities,

$$\epsilon \geq \frac{1}{4x_1(0)} \quad , \quad \epsilon \geq \frac{1}{4x_2(0)}.$$

Rearrange the terms to get

$$x_1(0) \geq \frac{1}{4\epsilon} \quad , \quad x_2(0) \geq \frac{1}{4\epsilon}. \tag{2.13}$$

2.6.5 Lower and Upper bound for $\dot{x}_1^2(t_1)$

Before crossing $x_1 = x_2$ line, we can derive the following inequalities using $x_2 \leq x_1$:

$$\begin{gathered} x_1^2 + x_2^2 \leq 2x_1^2 \\ -\frac{x_1\dot{x}_1}{4(x_1^2+x_2^2)^{\frac{3}{2}}} \geq -\frac{x_1\dot{x}_1}{4(2x_1^2)^{\frac{3}{2}}}. \end{gathered} \tag{2.14}$$

(The inequality sign flipped three times, the third time because $\dot{x}_1 \leq 0$)

First we get a lower bound for $\dot{x}_1(t_1)^2$ by integrating $\ddot{x}_1$ (equation 2.5):

$$\begin{aligned} \ddot{x}_1 &= -\frac{1}{4x_1^2} - \frac{x_1}{4(x_1^2+x_2^2)^{3/2}} \\ \dot{x}_1\ddot{x}_1 &= -\frac{\dot{x}_1}{4x_1^2} - \frac{\dot{x}_1 x_1}{4(x_1^2+x_2^2)^{3/2}}, \end{aligned}$$

using inequality 2.14, we get

$$\begin{aligned} \dot{x}_1\ddot{x}_1 &\geq -\frac{\dot{x}_1}{4x_1^2} - \frac{x_1\dot{x}_1}{4(2x_1^2)^{\frac{3}{2}}} \\ \dot{x}_1\ddot{x}_1 &\geq -\left(\frac{1}{4} + \frac{1}{8\sqrt{2}}\right)\frac{\dot{x}_1}{x_1^2} \\ \frac{1}{2}\dot{x}_1^2\Big|_0^{t_1} &\geq \left(\frac{1}{4} + \frac{1}{8\sqrt{2}}\right)\frac{1}{x_1}\Big|_0^{t_1}, \end{aligned}$$

since $\frac{1}{x_1(0)} \leq 4\epsilon$ and $x_1(t_1) = 1$ and $\dot{x}_1(0) = 0$,

$$\begin{aligned}
\frac{1}{2}\dot{x}_1(t_1)^2 &\geq \left(\frac{1}{4} + \frac{1}{8\sqrt{2}}\right)\left(1 - 4\epsilon\right) \\
\dot{x}_1(t_1)^2 &\geq \frac{1}{2} + \frac{1}{4\sqrt{2}} - 2\epsilon - \frac{\epsilon}{\sqrt{2}}. \qquad (2.15)
\end{aligned}$$

Now we get a upper bound for $\dot{x}_1(t_1)^2$:

$$\begin{aligned}
\ddot{x}_1 &= -\frac{1}{4x_1^2} - \frac{x_1}{4(x_1^2 + x_2^2)^{3/2}} \\
\dot{x}_1\ddot{x}_1 &= -\frac{\dot{x}_1}{4x_1^2} - \frac{\dot{x}_1 x_1}{4(x_1^2 + x_2^2)^{3/2}} \\
\dot{x}_1\ddot{x}_1 &\leq -\frac{\dot{x}_1}{4x_1^2} - \frac{x_1\dot{x}_1}{4(x_1^2)^{\frac{3}{2}}} \qquad \text{using } x_1^2 + x_2^2 \geq x_1^2 \\
\dot{x}_1\ddot{x}_1 &\leq -\frac{1}{2}\frac{\dot{x}_1}{x_1^2} \\
\frac{1}{2}\dot{x}_1^2\Big|_0^{t_1} &\leq \frac{1}{2}\frac{1}{x_1}\Big|_0^{t_1}.
\end{aligned}$$

Since $x_1(t_1) = 1$ and $\dot{x}_1(0) = 0$,

$$\begin{aligned}
\dot{x}_1^2(t_1) &\leq 1 - \frac{1}{x_1(0)} \\
\dot{x}_1^2(t_1) &\leq 1. \qquad (2.16)
\end{aligned}$$

Combining with equation 2.15 to get

$$\frac{1}{2} + \frac{1}{4\sqrt{2}} - 2\epsilon - \frac{\epsilon}{\sqrt{2}} \leq \dot{x}_1^2(t_1) \leq 1. \qquad (2.17)$$

2.6.6 Lower and Upper Bound for $\dot{x}_2^2(t)$

We go back to the energy constraint (Equation 2.7) and consider the time t_1 (the time when $x_1 = x_2 = 1$). Since $H = -\epsilon$ and $x_1(t_1) = x_2(t_1) = 1$, we have

$$\begin{aligned} -\epsilon &= \frac{1}{2}(\dot{x}_1^2(t_1) + \dot{x}_2^2(t_1)) - \left(\frac{1}{4} + \frac{1}{4} + \frac{1}{4(1+1)^{\frac{1}{2}}}\right) \\ \frac{1}{2} + \frac{1}{4\sqrt{2}} - \epsilon &= \frac{1}{2}(\dot{x}_1^2(t_1) + \dot{x}_2^2(t_1)) \\ 1 + \frac{1}{2\sqrt{2}} - 2\epsilon &= \dot{x}_1^2(t_1) + \dot{x}_2^2(t_1). \end{aligned} \tag{2.18}$$

Combine this with the bounds on $\dot{x}_1(t_1)^2$ (equation 2.17) to get

$$\frac{1}{2\sqrt{2}} - 2\epsilon \leq \dot{x}_2(t_1)^2 \leq \frac{1}{2} + \frac{1}{4\sqrt{2}} + \frac{\epsilon}{\sqrt{2}}. \tag{2.19}$$

To summarize, we now have bounds for $\dot{x}_1^2(t_1)$ and $\dot{x}_2^2(t_1)$ from equation 2.17 and 2.19:

$$\begin{aligned} \frac{1}{2} + \frac{1}{4\sqrt{2}} - 2\epsilon - \frac{\epsilon}{\sqrt{2}} \leq \dot{x}_1(t_1)^2 \leq 1 \\ \frac{1}{2\sqrt{2}} - 2\epsilon \leq \dot{x}_2(t_1)^2 \leq \frac{1}{2} + \frac{1}{4\sqrt{2}} + \frac{\epsilon}{\sqrt{2}}. \end{aligned}$$

Using above inequalities we can get a upper bound on time elapsed from crossing $x_1 = x_2$ to hitting x_2 axis $(t_0 - t_1)$, and also get a lower bound on the time needed for $\dot{x}_2(t_1)$ to go to zero $(t_m - t_1)$.

2.6.7 Estimating Time Needed for $\dot{x}_2(t)$ to Go to Zero

Consider $t_1 \leq t \leq t_m$ (after crossing $x_1 = x_2$, before $\dot{x}_2(t)$ goes to zero). Since $\dot{x}_2(t)$ is positive in this time interval, we know that $x_2(t) \geq 1$.
Time needed for $\dot{x}_2(t)$ to start from t_1 and go to zero is given by $t_m - t_1$.

Begin by estimating $\ddot{x}_2$ (equation 2.6):

$$\begin{aligned}
\ddot{x}_2(t) &= -\frac{1}{4x_2^2(t)} - \frac{x_2}{4(x_2^2(t)+x_1^2(t))^{3/2}} \\
\ddot{x}_2(t) &\geq -\frac{1}{4x_2^2(t)} - \frac{x_2}{4(x_2^2(t))^{3/2}} && \text{using } x_1^2 + x_2^2 \geq x_1^2 \\
\ddot{x}_2(t) &\geq -\frac{1}{2x_2^2(t)},
\end{aligned}$$

because $x_2(t) \geq 1$,

$$\begin{aligned}
\ddot{x}_2(t) &\geq -\frac{1}{2} \\
\dot{x}_2(t)\Big|_{t_1}^{t_m} &\geq -\frac{1}{2}(t_m - t_1) \\
\dot{x}_2(t_m) - \dot{x}_2(t_1) &\geq -\frac{1}{2}(t_m - t_1) \\
\frac{1}{2}(t_m - t_1) + \dot{x}_2(t_m) \geq \dot{x}_2(t_1) &\geq \sqrt{\frac{1}{2\sqrt{2}} - 2\epsilon} && \text{by equation (2.19).}
\end{aligned}$$

Since $\dot{x}_2(t_m) = 0$, we have

$$t_m - t_1 \geq 2\sqrt{\frac{1}{2\sqrt{2}} - 2\epsilon}. \tag{2.20}$$

In the case of $\epsilon = 0.01$ we would get

$$t_m - t_1 \geq 1.155. \tag{2.21}$$

This estimate gets bigger as $\epsilon \to 0$.

2.6.8 Estimating Time Elapsed from Crossing $x_1 = x_2$ to Hitting x_2 Axis

Consider $t_1 \leq t \leq t_0$ (after crossing $x_1 = x_2$, before hitting x_2 axis).
Begin with estimating $\ddot{x}_1$ (equation 2.5)

$$\begin{aligned}
\ddot{x}_1 &= -\frac{1}{4x_1^2} - \frac{x_1}{4(x_1^2+x_2^2)^{3/2}} \\
\ddot{x}_1 &\leq -\frac{1}{4x_1^2} \\
\ddot{x}_1 &\leq -\frac{1}{4}, && \text{because } x_1(t) \leq 1
\end{aligned}$$

now take $t_1 \leq t^* \leq t_{\frac{1}{2}}$

$$\begin{aligned}
\dot{x}_1(t)\Big|_{t_1}^{t^*} &\leq -\frac{1}{4}t\Big|_{t_1}^{t^*} \\
\dot{x}_1(t^*) - \dot{x}_1(t_1) &\leq -\frac{1}{4}(t^* - t_1) \\
\dot{x}_1(t^*) &\leq \dot{x}_1(t_1) - \frac{1}{4}(t^* - t_1) \\
\dot{x}_1(t^*) &\leq -\sqrt{\frac{1}{2} + \frac{1}{4\sqrt{2}} - 2\epsilon - \frac{\epsilon}{\sqrt{2}}} - \frac{1}{4}(t^* - t_1) \\
x_1(t^*)\Big|_{t_1}^{t_{\frac{1}{2}}} &\leq -t^*\sqrt{\frac{1}{2} + \frac{1}{4\sqrt{2}} - 2\epsilon - \frac{\epsilon}{\sqrt{2}}} - \frac{1}{4}\frac{(t^* - t_1)^2}{2}\Big|_{t_1}^{t_{\frac{1}{2}}} \\
x_1(t_{\frac{1}{2}}) - x_1(t_1) &\leq -(t_{\frac{1}{2}} - t_1)\sqrt{\frac{1}{2} + \frac{1}{4\sqrt{2}} - 2\epsilon - \frac{\epsilon}{\sqrt{2}}} - \frac{1}{8}(t_{\frac{1}{2}} - t_1)^2 \\
(\frac{1}{2} - 1) &\leq -(t_{\frac{1}{2}} - t_1)\sqrt{\frac{1}{2} + \frac{1}{4\sqrt{2}} - 2\epsilon - \frac{\epsilon}{\sqrt{2}}} - \frac{1}{8}(t_{\frac{1}{2}} - t_1)^2 \\
0 &\geq \frac{1}{8}(t_{\frac{1}{2}} - t_1)^2 + (t_{\frac{1}{2}} - t_1)\sqrt{\frac{1}{2} + \frac{1}{4\sqrt{2}} - 2\epsilon - \frac{\epsilon}{\sqrt{2}}} - \frac{1}{2}. \qquad (2.22)
\end{aligned}$$

Solving the quadratic inequality with $\epsilon = 0.01$, we get

$$-7.01826 \leq t_{\frac{1}{2}} - t_1 \leq 0.570. \qquad (2.23)$$

Since $\dot{x}_1$ is going more and more negative as $x_1 \to 0$, the time needed to get from $x_1 = 1$ to $x_1 = \frac{1}{2}$ is going to be more than the time needed to get from $x_1 = \frac{1}{2}$ to $x_1 = 0$. Double the value of $(t_{\frac{1}{2}} - t_1)$ to get an upper bound for $(t_0 - t_1)$,

$$t_0 - t_1 \leq 2 \cdot 0.570 = 1.14.$$

Under $H = -0.01$, equation 2.21 gives us $t_m - t_1 \geq 1.155$, which inturn implies that $t_0 - t_1 \leq t_m - t_1$, (time it takes for the particle to hit x_2 axis is less than time needed for $\dot{x}_2$ to become negative) so the $\dot{x}_2$ is positive at collision.

A energy level closer to zero (less negative) would lead to tighter bound for both $t_m - t_1$ and $t_0 - t_1$ (a bigger $t_m - t_1$ and a smaller $t_0 - t_1$). The inequality $t_0 - t_1 \leq t_m - t_1$ holds for all negative energy $H > -0.01$

2.6.9 Summary of Section 2.6

The goal of section 2.6 is to prove Conjecture 2.7: existence of an orbit that starts at a brake, hits the x_1 axis, then hits the x_2 axis with $\dot{x}_2 > 0$. The orbit is divided into four parts. Theorem 2.9 gives us part 1 of the orbit. We provided numerical results for Conjecture 2.10, which would give us part 2 of the orbit if assumed to be true. After several time estimation, we showed that the time it takes from the orbit crossing $x_1 = x_2$ to $\dot{x}_2(t)$ going to zero (part 3) is greater than the time it takes for the orbit from crossing $x_1 = x_2$ to hitting x_2 axis (part 4). Which means when the orbit hits the x_2 axis it must have $\dot{x}_2 > 0$.

With all four parts combined, assuming Conjecture 2.10 is true, then we know that Conjecture 2.7 is true; or that an orbit going up exist.

2.7 Orbit Going Down

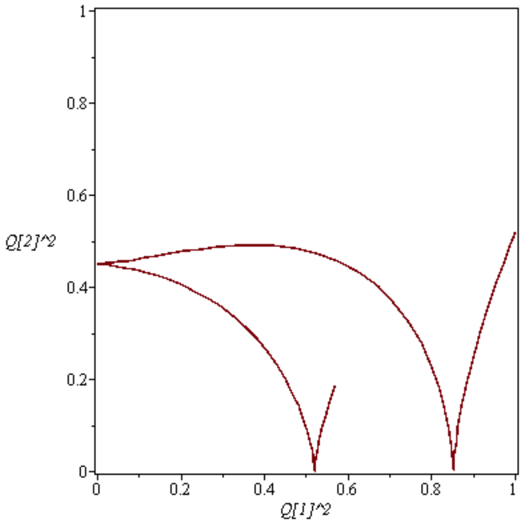

FIGURE 2.9: Orbit going down

We call an orbit that starts at a brake, hits the x_1 axis, then hits the x_2 axis with $\dot{x}_2 < 0$ an orbit going down. To prove the existence of an orbit going down, we are going to first prove the existence of an orbit that bounces off x_1 axis once, then ends in total collision. After that we apply a perturbation of the initial condition of the total collision solution to obtain an orbit going down.

Lemma 2.11. *An orbit that bounces off the x_1 axis once, then ends in total collision exists.*

Proof. We first show that we can begin with an orbit that bounces off x_1 axis many times before reaching x_2 axis.

Recall our energy equation 2.7 :

$$H = \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2) - \left(\frac{1}{4x_1} + \frac{1}{4x_2} + \frac{1}{4(x_1^2 + x_2^2)^{\frac{1}{2}}}\right)$$

Given an energy level $H = H_0$. Set $\dot{x}_1(0) = \dot{x}_2(0) = 0$, we get

$$H_0 = -\left(\frac{1}{4x_1(0)} + \frac{1}{4x_2(0)} + \frac{1}{4(x_1^2(0) + x_2^2(0))^{\frac{1}{2}}}\right)$$

for any large enough $x_1(0)$ there is a corresponding $x_2(0)$ that would give our energy H_0 (We can show this by a simple implicit function theorem argument).

As $x_1(0) \to \infty$, we have $x_2(0) \to -\dfrac{1}{4H_0}$, a finite value.

Looking at $\ddot{x}_1$ and $\ddot{x}_2$,

$$\begin{aligned}
\ddot{x}_1 &= \frac{-1}{4x_1^2} + \frac{-x_1}{4(x_1^2+x_2^2)^{3/2}} \\
\lim_{x_1\to\infty} \ddot{x}_1 &= \lim_{x_1\to\infty} \frac{-1}{4x_1^2} + \frac{-x_1}{4(x_1^2+x_2^2)^{3/2}} \\
&= 0 \\
\ddot{x}_2 &= \frac{-1}{4x_2^2} + \frac{-x_2}{4(x_2^2+x_1^2)^{3/2}} \\
\lim_{x_1\to\infty} \ddot{x}_2 &= \lim_{x_1\to\infty} \frac{-1}{4x_2^2} + \frac{-x_2}{4(x_2^2+x_1^2)^{3/2}} \\
&= \lim_{x_1\to\infty} \frac{-1}{4x_2^2} \\
&= -4H_0^2 \text{ which is bounded away from zero.}
\end{aligned}$$

Pick a large enough $x_1(0)$, the acceleration along the x_2 direction would dominate the acceleration along the x_1 direction. Thus we can ensure that the particle would bounce off the x_1 axis at least twice before getting close to x_2 axis. Let $x_1 = c_2$ denote the x_1 coordinate of the second collision. Notice that since $\ddot{x}_2$ is strictly negative, $x_2(t)$ has a unique maximum between the first and the second collision at x_1 axis.

Now move the initial position to the left; That is, fix $x_2(0) = x_2^*$, and reduce $x_1(0)$. Notice that the energy of the system would change correspondingly. We want the show that the whole orbit before the second collision at x_1 axis would shift to the left.

Observe that magnitude of acceleration along the x_1 direction gets stronger as $x_1(t)$ gets smaller. We obtain this result by simply taking derivative of $\ddot{x}_1$ with respect to x_1.

$$\begin{aligned}
\frac{\partial \ddot{x}_1}{\partial x_1} &= \frac{3{x_1}^2}{4\left({x_1}^2 + {x_2}^2\right)^{5/2}} - \frac{1}{4\left({x_1}^2 + {x_2}^2\right)^{3/2}} + \frac{1}{2{x_1}^3} \\
&\geq \frac{3x_1^2}{4\left({x_1}^2 + {x_2}^2\right)^{5/2}} - \frac{1}{4\left({x_1}^2 + {x_2}^2\right)^{3/2}} + \frac{1}{2\left({x_1}^2 + {x_2}^2\right)^{3/2}} \\
&= \frac{3x_1^2}{4\left({x_1}^2 + {x_2}^2\right)^{5/2}} + \frac{1}{4\left({x_1}^2 + {x_2}^2\right)^{3/2}} \\
&> 0.
\end{aligned}$$

Note that $\ddot{x}_1$ is negative while x_1 is positive, so as x_1 decreases $\ddot{x}_1$ also decreases and becomes more negative, therefore the magnitude of acceleration would increase as $x_1(t)$ gets smaller.

Therefore as we move the initial position to the left, the acceleration of the whole orbit before the second collision would get stronger, leading to a smaller value of c_2. We keep moving the initial position to the left until c_2 exactly reaches zero. That gives us our orbit that bounces off x_1 axis once, then ends in total collision. $\square$

Theorem 2.12. *An orbit that starts at a brake, hits the x_1 axis, then collides on the x_2 axis with $\dot{x}_2 < 0$ exists.*

Proof. By Lemma 2.11 there exists an orbit that bounces off x_1 axis once, then ends in total collision. A perturbation of the orbit's initial position to the left causes the second collision to happen at the x_2 axis. Continuity of $\dot{x}_2$ gives us that the particle would cross the $x_1 = x_2$ line with $\dot{x}_2 < 0$. Because $\ddot{x}_2$ is strictly negative and continuous until it reaches the second collision at the x_1 axis, we can guarantee that $\dot{x}_2$ is negative at collision.

Thus we have proved the existence of the orbit that goes down after collision at x_2 axis. $\square$

2.8 Using Intermediate Value Theorem to Prove the Existence of the Periodic Solution

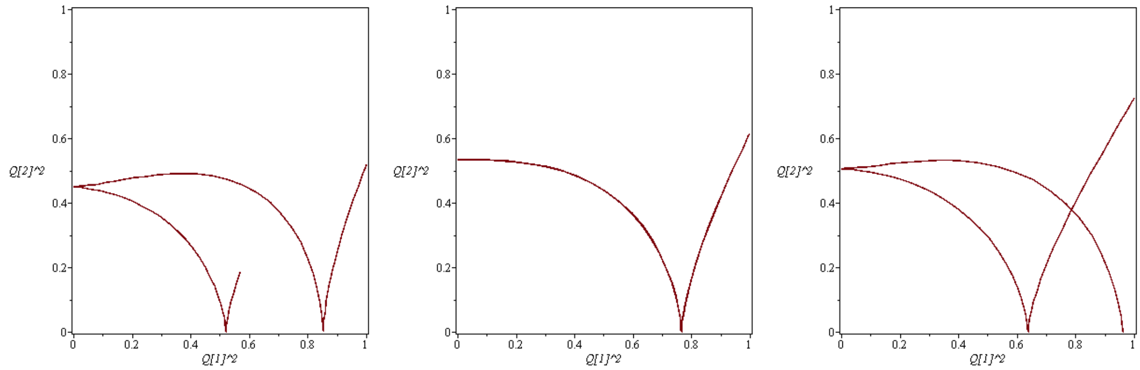

FIGURE 2.10: Using intermediate value theorem

Theorem 2.13. *The periodic brake orbit 121 exist. (An orbit that starts at a brake, first collides on the x_1 axis, then hits the x_2 axis and bounces back under the exact same track, hits the x_1 axis at the same spot, and finally reach a brake again at the starting position.)*

Proof. Assuming Conjecture 2.7 is true, then we have the existence of an orbit that starts at a brake, hits the x_1 axis, then collides on the x_2 axis with $\dot{x}_2 > 0$. Theorem 2.12 gives us the existence of an orbit that starts at a brake, hits the x_1 axis, then collides on the x_2 axis with $\dot{x}_2 < 0$. By Theorem 2.6, we can fix an energy level and scale those two orbit to match the same energy level.

Now the initial position of these two orbits both lies on the zero momentum curve of the chosen energy. Let $u(t)$ be the orbit going up and $d(t)$ be the orbit going down. Since we have a C^1 system of differential equations under the regularized coordinate, under continuous dependence on initial conditions, as we slide along the zero momentum curve to get from $u(0)$ to $d(0)$, at some point $\dot{x}_2 = 0$ at the collision on the x_2 axis (Intermediate Value Theorem). Let this solution be $(Q_1^*, Q_2^*, P_1^*, P_2^*)$ in the regularized coordinate. Looking at equation 2.8, we see that at the x_2 axis collision $\dot{x}_2 = 0$ implies that $P_2^* = 0$. Thus the segment of the solution $(Q_1^*, Q_2^*, P_1^*, P_2^*)$ from the brake to the x_2 axis collision satisfies the condition required to generate a full periodic orbit from on part, as talked about in section 2.4 . Using the result of section 2.4 we can obtain a full periodic orbit, that starts at a brake, collides on the x_1 axis, then hits the x_2 axis and bounces back under the exact same track, hits the x_1 axis at the same spot, and finally reach a brake again at the starting position.

□

2.9 Summary of Results

Using regularized coordinate we obtained a C^1 system of differential equations through binary collision. We obtained estimates on moving direction of the orbit under stationary initial conditions and through Intermediate Value Theorem we showed existence of the desired orbit in the 1st quadrant. Extending the orbit through symmetry to the whole plane, and we have given a analytic existence of a symmetric periodic brake orbit with simultaneous binary collision(SBC) in the equal mass, fully symmetric planar four body problem.

Chapter 3

Future Work

In Conjecture 2.5, we proposed that an orbit that starts at a brake, hits the x_1 axis, then collides on the x_2 axis with $\dot{x}_2 > 0$ exists. Although numerical simulations seems imply the statement is true, we are still working on finding an analytical prove of the result.

From the function $I = x_1^2 + x_2^2$, we can obtain that $\ddot{I} = T + h$ (the Lagrange-Jacobi identity, see Meyer, Hall and Offing [13]). From that we know $\ddot{I}$ is positive near the origin. If we can prove that $\dot{I} = 0$ at someone point before I reaches zero, then conjecture 2.5 can be proved to be true. Proving this requires more analysis on the function I and its derivatives.

The next step to studying periodic orbit 121 is to determine its stability. Techniques that Bakker, Ouyang, Yan and Simmons[2] has used in studying stability of symmetric planar periodic orbits with simultaneous binary collision in symmetric four-body problem might be useful.

Along with periodic orbit 121, we have also numerically found other plausible periodic brake orbits as illustrated in Figure 3.1. The orbit on the left starts stationary, hit the x_1 axis, then the x_2 axis, when it crosses the $x_1 = x_2$ line the second time, the remaining orbit is symmetric to the previous orbit. The orbit on the right starts stationary, hit the x_1 axis, then the x_2 axis, then the x_1 axis, and x_2 axis twice. After that it traces back its path until reaches a brake at the original position.

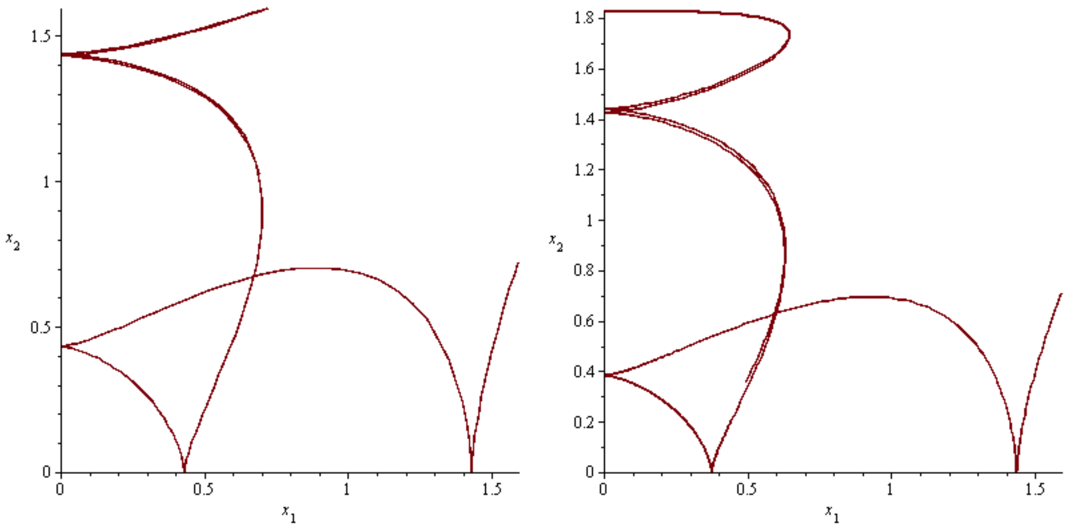

FIGURE 3.1: Two other periodic brake orbit

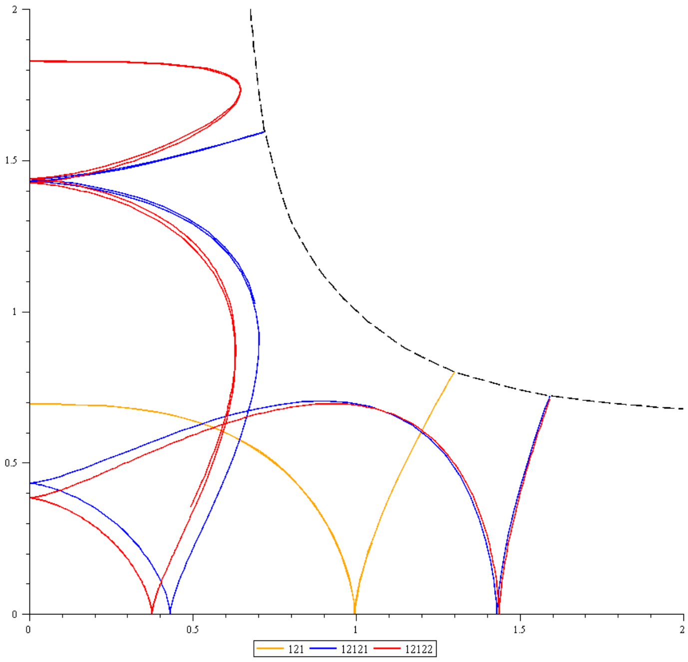

FIGURE 3.2: Three periodic brake orbits on the same plot

In Figure 3.2 we plotted all three periodic brake orbits the same plane. The black dotted line represent the constant energy level curve. It seems that as we move the initial position further to the right under the same energy level, we could discover more periodic brake orbits.while there are more potential periodic brake orbits undiscovered yet, numerical stimulation seems to suggest that a 1221 periodic brake orbit does not exist.

The two other orbits that we have not prove existence for has similarity to periodic brake orbit 121. First, there's regularizable simultaneous binary collision, which direct us to using topological techniques. Second, there is useful symmetry embedded in the orbit. The shooting method and differential inequalities we used to prove existence of periodic brake orbit 121 will likely be useful as we explore how to prove existence of the remaining two orbits. In future research we can try to prove the existence of the remaining two orbits.

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