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Minimum Rank Problems for Cographs

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ABSTRACT

Minimum Rank Problems for Cographs

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Let G be a simple graph on n vertices, and let $\mathcal{S}(G)$ be the class of all real-valued symmetric $n \times n$ matrices whose nonzero off-diagonal entries occur in exactly the positions corresponding to the edges of G. The smallest rank achieved by a matrix in $\mathcal{S}(G)$ is called the minimum rank of G, denoted $\operatorname{mr}(G)$. The maximum nullity achieved by a matrix in $\mathcal{S}(G)$ is denoted $M(G)$. For each graph G, there is an associated minimum rank class, $\mathscr{MR}(G)$ consisting of all matrices $A \in \mathcal{S}(G)$ with $\operatorname{rank} A = \operatorname{mr}(G)$. Although no restrictions are applied to the diagonal entries of matrices in $\mathcal{S}(G)$, sometimes diagonal entries corresponding to specific vertices of G must be zero for all matrices in $\mathscr{MR}(G)$. These vertices are known as nil vertices (see [6]). In this paper I discuss some basic results about nil vertices in general and nil vertices in cographs and prove that cographs with a nil vertex of a particular form contain two other nil vertices symmetric to the first. I discuss several open questions relating to these results and a counterexample. I prove that for all $K_{3,3,3}$-free cographs G, the zero-forcing number $Z(G)$, a graph theoretic parameter, is equal to $M(G)$. In fact this result holds for a slightly larger class of cographs and in particular holds for all threshold graphs. Lastly, I prove that the maximum of the minimum ranks of all cographs on n vertices is $\left\lfloor \frac{2n}{3} \right\rfloor$.

Keywords: cographs, edge subdivision, graph theory, minimum rank, nil vertices, symmetric matrices, threshold graphs, zero forcing

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Chapter 1. Introduction

Cographs have arisen in many diverse areas of mathematics and the sciences. As such, they have many equivalent characterizations and are known by many names. One characterization is that K_1 is a cograph as are all unions and joins of cographs. Equivalently, K_1 is a cograph as are all unions of cographs and the complement of any cograph. This characterization gives rise to the name complement reducible graphs. Cographs are also exactly the graphs that are P_4-free. Other names for cographs include D^*-graphs and Hereditary Dacey graphs. Cographs are perfect graphs, and threshold graphs are a subclass of cographs. There are also various polynomial time algorithms for identifying and computing parameters of cographs, making cographs very computationally tractable [7]. In this paper I prove several results about cographs and about structured matrices corresponding to cographs.

For any cograph, and more generally for any simple graph $G = (V, E)$ with vertex set $V = \{1, 2, \ldots, n\}$, let $\mathcal{S}(G)$ be the set of symmetric $n \times n$ matrices $A = [a_{ij}]$ such that for $i \neq j, a_{ij} \neq 0$ if and only if $ij \in E$. There are no restrictions on the diagonal entries. The minimum rank of G, denoted $\mathrm{mr}(G)$, is the smallest rank achieved by a matrix in $\mathcal{S}(G)$. The minimum rank class of G, denoted $\mathscr{MR}(G)$, is the class of matrices in $\mathcal{S}(G)$ that have rank equal to $\mathrm{mr}(G)$. For small graphs and highly structured graphs it is often straightforward to determine the minimum rank, and for trees minimum rank is computable [12]. A general method for determining the minimum rank of any graph, however, is unknown. See [8] for a survey of minimum rank results from 1960 to 2007. Many other areas of mathematics are related to minimum rank, including the Graham-Pollack Theorem, singular graphs (those whose adjacency matrix is singular), and eigensharp graphs [8]. In 2007, Barioli and Fallat published a result giving the minimum rank of cographs in terms the minimum rank of certain subgraphs [2].

The minimum rank problem asks for the smallest rank achieved by matrices in $\mathcal{S}(G)$. A related question asks for the structure of the matrices in $\mathcal{S}(G)$ that achieve the minimum rank $\mathrm{mr}(G)$. Recall that matrices in $\mathcal{S}(G)$ have unrestricted diagonals. However, sometimes all matrices in $\mathscr{MR}(G)$ have a zero in a particular diagonal entry, or perhaps must have a nonzero entry. If v is a vertex of G such that the corresponding diagonal entry in every matrix in $\mathscr{MR}(G)$ is zero, v is a nil vertex. If instead the corresponding diagonal entry in every matrix in $\mathscr{MR}(G)$ is nonzero, v is a nonzero vertex. Vertices that are neither nil nor nonzero are neutral. Nil, nonzero, and neutral vertices were introduced in [4] and studied extensively in [6]. Another way of studying the structure of matrices in $\mathscr{MR}(G)$ was presented in [3]. The authors gave an alternate proof of a result Hein van der Holst published in [13]. This result gave a formula for the minimum rank of graphs with a 2-separation in terms of the minimum rank of various subgraphs and subgraph derivations. From the alternate method of proof arose another theorem giving the matrices in $\mathscr{MR}(G)$ in terms of matrices in minimum rank classes of subgraphs and subgraph derivations. This gives a way of determining nil vertices from information about nil vertices in subgraphs. I prove that cographs with a particular type of nil vertex in fact have two more nil vertices that are symmetric to the first and form an independent set in an induced $K_{2,3}$. I also present various open questions and a counterexample about nil vertices in cographs and prove that nil vertices are preserved under unions and joins of single vertices and that non-isolated nil vertices in cographs are preserved under joins with K_2.

The zero-forcing number of a graph G, $Z(G)$, is a graph parameter that gives an upper bound on the maximum nullity of G. While $Z(G) = M(G)$ when G is a tree, in general, $Z(G)$ and $M(G)$ are not equal. I show that $Z(G) = M(G)$ when G is a $K_{3,3,3}$-free cograph. As a result, $Z(G) = M(G)$ for the entire class of threshold graphs.

It is well-known and easy to prove that the largest the minimum rank of a graph on n vertices can be is $n - 1$. It follows that a graph on n vertices with k components can have minimum rank at most $n - k$. It is equally well-known, but far less easy to prove, that the class of graphs achieving that upper bound on minimum rank is the class of paths, P_n [9]. I prove a similar result. Instead of finding the structure of the graphs whose minimum ranks achieve a particular bound, I find a bound on the minimum rank of all graphs with a certain structure, namely cographs. I prove that the minimum rank of a cograph on n vertices is at most $\left\lfloor \frac{2n}{3} \right\rfloor$, improving on the preceding bounds by a multiplicative constant.

Chapter 2. Basic Definitions and Preliminary Observations

2.1 Graph Theory

Definition 2.1.

A *graph* $G = (V(G), E(G))$ is a set of vertices $V(G)$ and a set of edges $E(G)$, consisting of 2-element subsets of $V(G)$.

Two vertices u and v in a graph G are said to be *adjacent* if $\{u, v\} \in E(G)$.

The *degree* of a vertex is the number of vertices to which it is adjacent.

A vertex is said to be *isolated* if it is not adjacent to any other vertex.

A degree one vertex is call *pendent*.

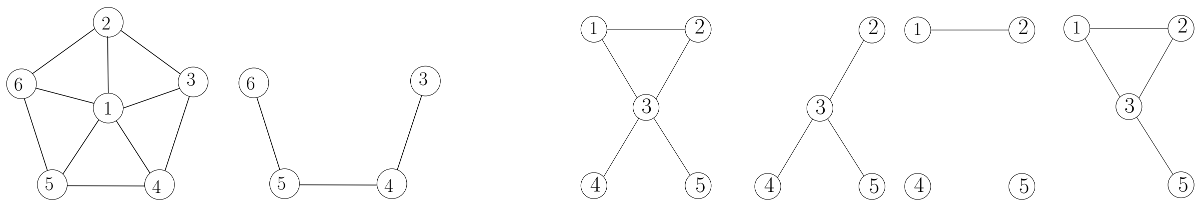
A graph H is a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

Let v be a vertex of G. Then $G - v$ is the subgraph of G with vertex set $V(G - v) = V(G) \setminus \{ v \}$ and edge set containing $\{ i, j \}$ if and only if $i, j \neq v$ and $\{ i, j \} \in E(G)$.

A subgraph H of G is said to be *induced* if it can be obtained by successively deleting vertices of G. If H is an induced subgraph of G, it is said to be induced by the set $V(H) \subseteq V(G)$.

Definition 2.2. A graph G is said to be *H-free* if H is not an induced subgraph of G.

Example 2.3. Let W_6 be the 6-wheel graph shown in Figure 2.1a. Deleting vertices 1 and 2 obtains an induced P_4. That is, $W_6 - 1 - 2 = P_4$. On the other hand, the folding stool shown in Figure 2.1b is P_4-free since deleting any single vertex results in a graph on 4 vertices that is not P_4. Deleting vertex 1 results in S_4. Deleting vertex 3 results in a disconnected induced subgraph, $K_2 \cup 2K_1$. Deleting vertex 4 results in a graph known as the paw. Vertex 2 is symmetric to vertex 1, and vertex 5 is symmetric to vertex 4.



(a) The 6-wheel and an induced P_4 (b) The folding stool and induced subgraphs

Figure 2.1: Induced Subgraphs

Definition 2.4. The *complement* of a graph G is the graph G^c with vertex set $V(G^c) = V(G)$ and edge set $E(G^c)$ given by $\{i, j\} \in E(G^c)$ if and only if $\{i, j\} \notin E(G)$ for $i, j \in V(G)$.

Definition 2.5. Given two graphs G and H with $V(G) \cap V(H) = \emptyset$, the *union* of G and H, written $G \cup H$, is the graph $(V(G) \cup V(H), E(G) \cup E(H))$.

Definition 2.6. Given two graphs G and H with $V(G) \cap V(H) = \emptyset$, the *join* of G and H, written $G \vee H$, is the graph with vertex set

$$V(G \cup H) = V(G) \cup V(H)$$

and edge set

$$E(G \cup H) = E(G) \cup E(H) \cup \{uv \mid u \in V(G) \text{ and } v \in V(H)\}.$$

Notice that the graph operation union is associative, as is the operation join. Thus it is consistent to define the union and join of multiple graphs as

$$\bigcup_{i=1}^{r} G_i = \left(\bigcup_{i=1}^{r-1} G_i\right) \cup G_r \text{ and } \bigvee_{i=1}^{r} G_i = \left(\bigvee_{i=1}^{r-1} G_i\right) \vee G_r.$$

Definition 2.7. Let G_1 and G_2 be graphs with at least two vertices, each with a vertex labeled v. The *vertex-sum at v of G_1 and G_2*, denoted $G_1 \underset{v}{\oplus} G_2$, is the graph on $|G_1|+|G_2|-1$ vertices obtained by identifying the vertex v in G_1 with the vertex v in G_2. If G is a graph that can be written as $G_1 \underset{v}{\oplus} G_2$, then v is said to be a *cut-vertex* of G.

Definition 2.8. A connected graph is k-connected if at least k vertices must be deleted before the resulting graph is disconnected or K_1.

Definition 2.9. Let G be a graph. A set of vertices of G is called *independent* (*a clique*) if the vertices are pair-wise non-adjacent (adjacent).

Definition 2.10.

- The *complete* graph on n vertices, K_n, is the graph with every pair of vertices adjacent.
- We use the shorthand notation mK_1 for $\cup_{i=1}^{m} K_1$.
- The *complete bipartite* graph with independent vertex sets of size m and size n is $K_{m,n} = mK_1 \vee nK_1$.
- The *complete tripartite* graph with independent vertex sets of size m, n, and r is $K_{m,n,r} = mK_1 \vee nK_1 \vee rK_1$.
- The *star* on n vertices, $K_{1,n-1}$, is denoted S_n.
- The *path* on n vertices, P_n, is any graph that can be labeled so that $V(P_n) = \{1, 2, \ldots, n\}$ and $E(P_n) = \{\{1,2\}, \{2,3\}, \ldots, \{n-1, n\}\}$.

Definition 2.11. A *cograph* G is defined recursively as follows:

(i) K_1 is a cograph.

(ii) If G_1 and G_2 are cographs, $G_1 \cup G_2$ is a cograph.

(iii) If G_1 and G_2 are cographs, $G_1 \vee G_2$ is a cograph.

Theorem 2.12. *[7] A graph G is a cograph if and only if G is P_4-free.*

Note that $K_n = \vee_{i=1}^{n} K_1$, $K_{m,n}$, and S_n are cographs and that P_n is a cograph if and only if $n < 4$.

Recall from Example 2.3 that the 6-wheel has an induced P_4 and that the folding stool is P_4-free. Thus by Theorem 2.12 the 6-wheel is not a cograph while the folding stool is a cograph. One can check that the 6-wheel is not the join of cographs, while the folding stool can be written as $3 \vee ((1 \vee 2) \cup 4 \cup 5)$.

Definition 2.13. A *threshold* graph is defined recursively as follows:

(i) K_1 is a threshold graph.

(ii) If G is a threshold graph, $G \cup K_1$ is a threshold graph.

(iii) If G is a threshold graph, $G \vee K_1$ is a threshold graph.

Clearly, all threshold graphs are cographs but not all cographs are threshold graphs.

Definition 2.14. Let G be a cograph and write $G = \vee_{i=1}^{r} G_i$ where the G_i cannot be further decomposed as a join of proper subgraphs. Note that for disconnected cographs, necessarily $r = 1$. Then the G_i are the *primary constituents* of G. A cograph written in this form is said to be in *standard form*, and unless otherwise noted, all cographs will be written in standard form. Occasionally I will wish to emphasize that a cograph is the join of two subgraphs, in which case I will write the cograph in the form $G \vee H$ and it should not be assumed to have only 2 primary constituents.

I use the following terminology as found in [2].

Definition 2.15. Let G be a graph. The *core* of G, denoted $\breve{G}$, is the subgraph of G induced by all non-isolated vertices. The *isolated part* of G, denoted by $\ddot{G}$, is the subgraph of G induced by all isolated vertices.

Thus $G = \breve{G} \cup \ddot{G}$.

Definition 2.16. Let G be a graph. Let H be a subgraph of G such that no subgraph of G containing H as a proper subgraph is connected. Then H is said to be a *component* of G.

Note that $\breve{G}$ and $\ddot{G}$ are not necessarily components of G. Take for example $G = K_1 \cup K_1 \cup K_2 \cup K_3$. The components of G are K_1, K_1, K_2, and K_3, while $\breve{G} = K_2 \cup K_3$ and $\ddot{G} = K_1 \cup K_1$.

2.2 Matrix Theory

Definition 2.17.

Let G be a graph on n vertices. $\mathcal{S}(G)$ is the set of all real-valued symmetric $n \times n$ matrices $A = [a_{ij}]$ such that when $i \neq j$, $a_{ij} = 0$ if and only if $\{i, j\} \notin E(G)$.

The *minimum rank* of G, $\operatorname{mr}(G)$, is the smallest rank achieved by a matrix in $\mathcal{S}(G)$. The *maximum nullity* of G, $M(G)$, is the largest nullity achieved by a matrix in $\mathcal{S}(G)$.

The *minimum rank class* of G is the set of matrices

$$\mathscr{MR}(G) = \{A \in \mathcal{S}(G) \mid \operatorname{rank} A = \operatorname{mr}(G)\}.$$

There are similar definitions for these concepts over any field. Many of the theorems I cite hold over more general fields, though I cite them only for the real field.

Observation 2.18. *If G is a graph on n vertices,* $\operatorname{mr}(G) + M(G) = n$.

Proof. For any $n \times n$ matrix A, $\operatorname{rank} A + \operatorname{nullity} A = n$. $\square$

Observation 2.19. *Let G be a disconnected graph with components $G_1, G_2, \ldots, G_k$. Then* $\operatorname{mr}(G) = \sum_{i=1}^{k} \operatorname{mr}(G_i)$ *and* $M(G) = \sum_{i=1}^{k} M(G_i)$.

Proof. Label the vertices of G so that the first $|G_1|$ are the vertices of G_1, the next $|G_2|$ are the vertices of G_2 and so on until the last $|G_k|$ are the vertices of G_k. Let $A \in \mathcal{S}(G)$.

Then A is a block diagonal matrix and we may write $A = A_1 \oplus A_2 \oplus \cdots \oplus A_k$ where each $A_i \in \mathcal{S}(G_i)$. Then $A \in \mathscr{MR}(G)$ if and only if $A_i \in \mathscr{MR}(G_i)$ for each i. Let $A \in \mathscr{MR}(G)$. Then $\operatorname{mr}(G) = \operatorname{rank} A = \sum_{i=1}^{k} \operatorname{rank} A_i = \sum_{i=1}^{k} \operatorname{mr}(G_i)$. Thus $\operatorname{mr}(G) = \sum_{i=1}^{k} \operatorname{mr}(G_i)$. To see that $M(G) = \sum_{i=1}^{k} M(G_i)$, one can use a similar argument or note that it follows from Observation 2.18. □

Definition 2.20. Let v be a vertex of a graph G. The *rank-spread* of v in G is $r_v(G) = \operatorname{mr}(G) - \operatorname{mr}(G - v)$.

Proposition 2.21. *[12] Let G be a graph and let v be a vertex of G. Then*

$$\operatorname{mr}(G - v) \leq \operatorname{mr}(G) \leq \operatorname{mr}(G - v) + 2.$$

Equivalently,

$$0 \leq r_v(G) \leq 2.$$

Observation 2.22. *If H is an induced subgraph of G, then* $\operatorname{mr}(H) \leq \operatorname{mr}(G)$.

Proof. By the proposition, $\operatorname{mr}(G - v) \leq \operatorname{mr}(G)$. Since induced subgraphs can be obtained by sucessively deleting vertices, the observation follows by induction. □

Theorem 2.23. *[11] Let G_1 and G_2 be graphs on at least two vertices each with a vertex labeled v and let $G = G_1 \underset{v}{\oplus} G_2$. Then*

$$\operatorname{mr}(G) = \min\{\operatorname{mr}(G_1) + \operatorname{mr}(G_2), \operatorname{mr}(G_1 - v) + \operatorname{mr}(G_2 - v) + 2\}.$$

Definition 2.24. The *join minimum rank* of a graph G is $\operatorname{jmr}(G) = \operatorname{mr}(G \vee K_1)$.

Definition 2.25. Let $G = \vee_{i=1}^{r} G_i$ be a cograph. Then G is said to be *anomalous* if

(i) for each i, $\operatorname{jmr}(G_i) \leq 2$; and

(ii) $K_{3,3,3}$ is an induced subgraph of G.

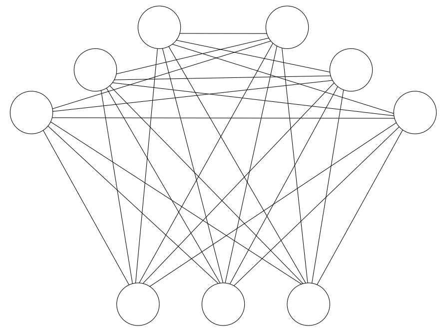


Figure 2.2: $K_{3,3,3} = 3K_1 \vee 3K_1 \vee 3K_1$

The following two results are fundamental to my arguments throughout the paper.

Proposition 2.26. *[2, Proposition 3.6] For any graph G,*

$$\mathrm{jmr}(G) = \begin{cases} \mathrm{mr}(G) & \text{if and only if } G \text{ has no isolated vertices.} \\ \mathrm{mr}(G)+1 & \text{if and only if } G \text{ has exactly one isolated vertex.} \\ \mathrm{mr}(G)+2 & \text{if and only if } G \text{ has two or more isolated vertices.} \end{cases}$$

More compactly,

$$\mathrm{jmr}(G) = \mathrm{mr}(G) + \min\{|\ddot{G}|, 2\}.$$

Theorem 2.27. *[2, Theorem 4.5] Let $G = \vee_{i=1}^{r} G_i$, $r > 1$, be a connected cograph. Then*

$$\mathrm{mr}(G) = \begin{cases} \max_i\{\mathrm{jmr}(G_i)\} & \text{if } G \text{ is not anomalous;} \\ 3 & \text{if } G \text{ is anomalous.} \end{cases}$$

Example 2.28. Throughout this paper it will be necessary to know the minimum rank of some basic graphs. To equip the reader I will establish the minimum rank of various basic graphs here, using methods varying from the definition of minimum rank to high-level methods such as Theorem 2.27.

Recall that K_n is the complete graph on n vertices, where every pair of vertices is adjacent. Thus the $n \times n$ all ones matrix is in $\mathcal{S}(K_n)$. Thus $\mathrm{mr}(K_n) \leq 1$. If $n \geq 2$, $\mathrm{mr}(K_n) = 1$ since every matrix in $\mathcal{S}(K_n)$ has an off-diagonal entry that must be nonzero. If $n = 1$, however,

$\mathrm{mr}(K_1) = 0$ since diagonal entries are unrestricted and $[0] \in \mathcal{S}(K_1)$.

Recall that the star $S_n = K_1 \vee (n-1)K_1$. We will use Proposition 2.26 to compute $\mathrm{mr}(S_n)$. First consider the case $n \geq 3$. Then

$$\mathrm{mr}(S_n) = \mathrm{jmr}((n-1)K_1) = \mathrm{mr}((n-1)K_1) + 2 = 0 + 2 = 2,$$

where Observation 2.19 shows that $\mathrm{mr}((n-1)K_1) = \mathrm{mr}(\cup_{i=1}^{n-1} K_1) = \sum_{i=1}^{n-1} \mathrm{mr}(K_1) = \sum_{i=1}^{n-1} 0 = 0$. Now consider $n = 2$. Since $S_2 = K_1 \vee K_1 = K_2$, $\mathrm{mr}(S_2) = 1$.

Any matrix $A \in \mathcal{S}(P_n)$ has a tridiagonal structure, or we can relabel so that it does. Write $A = \begin{bmatrix} d_1 & a & 0 & & \dots & 0 \\ a & d_2 & b & 0 & \dots & 0 \\ 0 & b & d_3 & c & & \vdots \\ & 0 & c & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & d_{n-1} & z \\ 0 & 0 & \dots & 0 & z & d_n \end{bmatrix}$ where $ab\cdots z \neq 0$. By deleting the first row and last column of A we obtain an upper triangular matrix with all nonzero entries on the diagonal. Thus $\operatorname{rank} A \geq n-1$. By choosing $A = \begin{bmatrix} 1 & -1 & 0 & & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ 0 & -1 & 2 & -1 & & \vdots \\ & 0 & -1 & \ddots & \ddots & 0 \\ \vdots & \vdots & & \ddots & 2 & -1 \\ 0 & 0 & \dots & 0 & -1 & 1 \end{bmatrix}$, the Laplacian matrix of P_n, we see that it is possible to choose A so that $\operatorname{rank} A < n$. Thus $\mathrm{mr}(P_n) = n-1$.

To illustrate the definition of anomalous, which is so important to Theorem 2.27, I prove some simple facts about anomalous graphs and then characterize their structure.

Observation 2.29. *$K_{3,3,3}$ is anomalous.*

Proof. $K_{3,3,3} = 3K_1 \vee 3K_1 \vee 3K_1$ and so $K_{3,3,3}$ is a cograph. The primary constituents are all $3K_1$, and $\operatorname{jmr}(3K_1) = \operatorname{mr}(S_4) = 2$. ☐

Observation 2.30. *$K_{3,3,3}$ can be an induced subgraph of a non-anomalous graph.*

Proof. Consider $G = 3K_1 \vee 3K_1 \vee 3K_1 \vee (P_3 \cup K_1)$. Certainly $K_{3,3,3}$ is an induced subgraph. Since $P_3 = 2K_1 \vee K_1$ is a cograph, G is a cograph. Since $\operatorname{jmr}(P_3 \cup K_1) = \operatorname{mr}(P_3) + 1 = 3 > 2$, G is not anomalous. ☐

Proposition 2.31. *Let $G = \vee_{i=1}^{r} G_i$ be an anomalous graph. Then for at least three distinct i, G_i is the union of three or more isolated vertices.*

Proof. Since G is anomalous, $K_{3,3,3}$ is an induced subgraph of G and $\operatorname{jmr}(G_i) \leq 2$ for each i. It is sufficient to prove the following three statements concerning the independent sets of the induced $K_{3,3,3}$.

(i) Each of the independent sets is contained in a single primary constituent.

(ii) Distinct independent sets cannot be contained in the same primary constituent.

(iii) Each primary constituent containing one of the independent sets is equal to its isolated part.

The first statement is verified by noticing that vertices from different primary constituents are adjacent in G.

The second statement follows from the first and the fact that if a constituent contained two of the independent sets it must then contain $K_{3,3}$ as an induced subgraph in order for $K_{3,3,3}$ to be an induced subgraph of G. Since primary constituents cannot be decomposed further into a join they must either be K_1 or be disconnected. Thus a primary constituent containing two independent sets would have at least two components, one of which would contain an induced $K_{3,3}$. An induced subgraph of this constituent would then be $K_1 \cup K_{3,3}$. So $K_1 \vee (K_1 \cup K_{3,3})$ would be an induced subgraph of this constituent join K_1. So the join minimum rank of this constituent would be at least $\operatorname{mr}(K_1 \vee (K_1 \cup K_{3,3})$. By Proposition 2.26, $\operatorname{mr}(K_1 \vee (K_1 \cup K_{3,3}) = \operatorname{mr}(K_{3,3}) + 1 = 3$ since $\operatorname{mr}(K_{3,3}) = \operatorname{mr}(3K_1 \vee 3K_1) = \operatorname{jmr}(3K_1) =$

$\mathrm{mr}(S_4) = 2$ by Theorem 2.27. But all the primary constituents must have join minimum rank at most 2 since G is anomalous. Thus no primary constituent contains more than one of the independent sets.

Lastly, suppose a primary constituent contained an independent set (of three or more vertices) and component(s) that were not isolated vertices. Then the minimum rank of the constituent would be at least 1, and by Proposition 2.26 the join minimum rank of the constituent would be at least 3. If so, G would not be anomalous. $\square$

Corollary 2.32. *A cograph G is anomalous if and only if $G = G_1 \vee \cdots \vee G_k \vee H_1 \vee \cdots \vee H_\ell$ where the G_i and the H_i together are the primary constituents of G, $\ddot{G}_i = G_i$ and $\ddot{H}_i \neq H_i$ for each i, $k \geq 3$, $|G_1|, |G_2|, |G_3| \geq 3$, and $\mathrm{mr}(H_i) + |\ddot{H}_i| \leq 2$ for each i.*

Proof. Assume G is anomalous. The conditions $k \geq 3$ and $|G_1|, |G_2|, |G_3| \geq 3$ are necessary by the proposition. It remains to show that any primary constituent with join minimum rank at most 2 can appear as a G_i or an H_i. If J is a graph such that $\mathrm{jmr}(J) = 1$, $J = K_m$ for some $m \geq 1$ and so J satisfies $\mathrm{mr}(J) + |\ddot{J}| \leq 2$. If $m > 1$, J can appear as an H_i. If $m = 1$, J can appear as a G_i. If $\mathrm{jmr}(J) = 2$, either $\mathrm{mr}(J) = 0$ and $|\ddot{J}| \geq 2$, or $\mathrm{mr}(J) = 1$ and $|\ddot{J}| = 1$, or $\mathrm{mr}(J) = 2$ and $|\ddot{J}| = 0$. In the case $\mathrm{mr}(J) = 0$ and $|\ddot{J}| \geq 2$, J can appear as a G_i. The last two cases satisfy $\mathrm{mr}(J) + |\ddot{J}| \leq 2$ and J can appear as an H_i.

Now assume G is of the form specified in the corollary. Then $K_{3,3,3}$ is an induced subgraph; for appropriate m_i, $\mathrm{jmr}(G_i) = \mathrm{mr}(S_{m_i}) \leq 2$; and $\mathrm{jmr}(H_i) \leq \mathrm{mr}(H_i) + |\ddot{H}_i| \leq 2$. Thus G is anomalous. $\square$

Chapter 3. Nil Vertices

I begin this chapter with a discussion of nil vertices as defined in [6] and prove some basic results, some of which are review from [6] and some of which are new to this work and relevant to cographs. I then prove that cographs with a degree 2 non-simplicial nil vertex must have two other nil vertices symmetric to the first. I discuss one of the earliest results on nil vertices, which classifies them in graphs with minimum rank two [4]. Since $\operatorname{mr}(P_4) = 3$, graphs with minimum rank two are P_4-free and thus a subset of cographs. This result and my original result lead us to some open questions about nil vertices in cographs, as well as a counterexample.

3.1 Basic Results

Definition 3.1. [6] Let v be a vertex of a graph G. If the diagonal entry corresponding to v is zero (nonzero) in every matrix in $\mathscr{MR}(G)$, then v is said to be a *nil* (*nonzero*) vertex of G, or *nil* (*nonzero*) in G. A vertex that is neither nil nor nonzero is called *neutral.*

The following illustrate the definition of nil vertices.

Observation 3.2. *Every isolated vertex of a graph is nil.*

Proof. Let v be an isolated vertex of a graph G and label v as the first vertex. Then every matrix in $\mathcal{S}(G)$ is of the form $A = \begin{bmatrix} d_1 & 0^T \\ 0 & B \end{bmatrix}$, where $B \in \mathcal{S}(G - v)$. In order for A to have the smallest possible rank, we must have $d_1 = 0$. $\square$

Proposition 3.3. *[6, Example 2.13] Let $k \geq 4$. Then the pendent vertices of S_k are nil and the central vertex is neutral.*

Proof. We first consider $k = 4$. Label the vertices of S_4 by $V(S_4) = \{1, 2, 3, 4\}$ and $E(S_4) = \{\{1, 2\}, \{1, 3\}, \{1, 4\}\}$. Every matrix in $\mathscr{MR}(S_4)$ is of the form

$$A = \begin{bmatrix} d_1 & a & b & c \\ a & d_2 & 0 & 0 \\ b & 0 & d_3 & 0 \\ c & 0 & 0 & d_4 \end{bmatrix}$$

where a, b and c are not zero and d_2, d_3, d_4 correspond to the pendent vertices of S_4. Since $\operatorname{mr}(S_4) = 2$, rank $A = 2$. If any of d_2, d_3, or d_4 is not 0, then rank A is greater than 2. To see this, consider the following cases. If all three are nonzero, A has a 3×3 principal submatrix with rank 3. If exactly two are nonzero, the 3×3 principal submatrix corresponding to row 1, the row containing the $d_i = 0$, and a row containing a nonzero d_i has rank 3. If only one is nonzero we may similarly choose a rank 3 principal submatrix. Hence, every pendent vertex of S_4 is a nil vertex. Further, both $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ are in $\mathscr{MR}(S_4)$ and thus 1 is a neutral vertex. This argument is clearly generalizable to $k > 4$. □

Proposition 3.4. *Let v be a vertex of a graph G. Let H be the component of G containing v. Then v is nil in H if and only if v is nil in G.*

Proof. This is clear if $G = H$, or in other words, if G is connected. Now suppose that G has more than one component. Then $G - V(H)$ is not the empty graph. A matrix in $\mathcal{S}(G)$ has the form $M = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$, where $A \in \mathcal{S}(G - V(G))$ and $B \in \mathcal{S}(H)$. Since this is a block diagonal matrix, M has the smallest possible rank exactly when each block has the smallest possible rank. Thus $M \in \mathscr{MR}(G)$ if and only if $A \in \mathscr{MR}(G - V(H))$ and $B \in \mathscr{MR}(H)$. Since v is a vertex of H, the diagonal entry of M corresponding to v is in B. Thus v is nil in G if and only if the diagonal entry corresponding to v is zero in every matrix in $\mathscr{MR}(G)$ if and only if the diagonal entry corresponding to v is zero in every matrix in $\mathscr{MR}(H)$ if and only if v is nil in H. □

Observation 3.5. *All vertices of P_3 are neutral.*

Proof. Consider $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, and $\begin{bmatrix} 0 & 1 & -1 \\ 1 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix} \in \mathcal{S}(P_3)$. Each matrix has rank 2, hence each is in $\mathscr{MR}(P_3)$. Thus P_3 has all neutral vertices. □

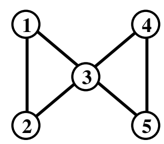
Though only the first mentions nil vertices, both of the following two theorems can be used to determine information about nil vertices of a graph when information is known about nil vertices of certain induced subgraphs.

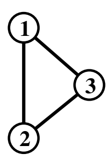
Theorem 3.6. *[4, Theorem 5.2] If H is an induced subgraph of a graph G with* $\mathrm{mr}(H) = \mathrm{mr}(G)$*, then if v is a nil (nonzero) vertex in H, v is also a nil (nonzero) vertex in G.*

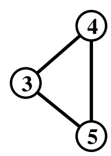
Theorem 3.6 shows that graphs retain a property from certain types of subgraphs, the reverse of the usual type of inheritance theorem. In fact, the converse of Theorem 3.6 is false. Note that $\mathrm{mr}(P_3) = 2 = \mathrm{mr}(S_4)$ and that P_3 is an induced subgraph of S_4. The pendent vertices of S_4 are nil by Proposition 3.3, but the pendent vertices of P_3 are neutral by Observation 3.5.

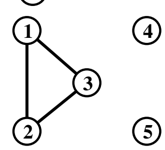
Definition 3.7. [3] Given a proper subgraph H of a graph G, let $\widetilde{H}$ be the graph with vertex set $V(G)$ and edge set $E(H)$.

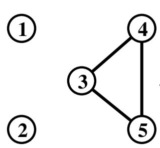
The following example illustrates and motivates the definition.

Example 3.8. [3, see Example 1.1] Let $\bowtie$ be the bowtie graph 

obtained by identifying $G_1 =$ 

and $G_2 =$ 

at vertex 3. Then $\widetilde{G_1} =$ 

and $\widetilde{G_2} =$ 

. Clearly a matrix in $\mathscr{MR}(K_3)$ is the all-ones matrix $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and it is natural to embed this matrix in two 5×5 matrices and add them to obtain a matrix in $\mathscr{MR}(\bowtie)$. Let

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

Then $\operatorname{rank}(A+B) = 2 = \operatorname{mr}(G)$ so that $A + B \in \mathscr{MR}(\bowtie)$. Note that $A \in \mathscr{MR}(\widetilde{G_1})$ and $B \in \mathscr{MR}(\widetilde{G_2})$.

While it is not always possible to so naturally construct a matrix in $\mathscr{MR}(G)$ from matrices in minimum rank classes corresponding to subgraphs of G, the following theorem tells us that that we can always do this for graphs with a cut-vertex. There is a similar theorem in [3] for graphs with a 2-separation. Since we do not need the theorem for 2-separations in our discussion I do not include it here.

Theorem 3.9. *[3, Corollary 3.3] Let G be the vertex-sum at v of G_1 and G_2, and let S_{k+1} be the star subgraph of G formed by the degree k vertex v and all of its neighbors.*

(i) If $r_v(G_1) + r_v(G_2) < 2$, then

$$\mathscr{MR}(G) = \mathscr{MR}(\widetilde{G_1}) + \mathscr{MR}(\widetilde{G_2}).$$

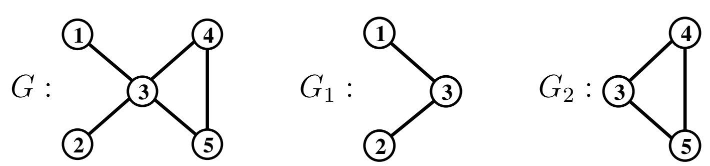
(ii) If $r_v(G_1) + r_v(G_2) > 2$, then

$$\mathscr{MR}(G) = \mathscr{MR}(\widetilde{G_1 - v}) + \mathscr{MR}(\widetilde{G_2 - v}) + \mathscr{MR}(\widetilde{S_{k+1}}).$$

(iii) If $r_v(G_1) + r_v(G_2) = 2$, then

$$\mathscr{MR}(G) = \left(\mathscr{MR}(\widetilde{G_1}) + \mathscr{MR}(\widetilde{G_2})\right) \cup \left(\mathscr{MR}(\widetilde{G_1 - v}) + \mathscr{MR}(\widetilde{G_2 - v}) + \mathscr{MR}(\widetilde{S_{k+1}})\right).$$

Example 3.10. As an illustration of the above theorem, consider the graphs



noting that G is the vertex-sum at vertex 3 of G_1 and G_2. Since $r_3(G_1) + r_3(G_2) = 2 + 0 = 2$, G illustrates statement 3. Thus any matrix in $\mathscr{MR}(G)$ is in either $\mathscr{MR}(\widetilde{G_1}) + \mathscr{MR}(\widetilde{G_2})$ or

$\mathscr{MR}(\widetilde{G_1 - 3}) + \mathscr{MR}(\widetilde{G_2 - 3}) + \mathscr{MR}(\widetilde{S_5})$, and any matrix obtained as one of these sums is in $\mathscr{MR}(G)$. We demonstrate each type of decomposition with a matrix in $\mathscr{MR}(G)$. Note that $\operatorname{mr}(G) = \operatorname{mr}(K_1 \vee (K_2 \cup K_1 \cup K_1)) = \operatorname{jmr}(K_2 \cup K_1 \cup K_1) = \operatorname{mr}(K_2) + 2 = 3$. Consider the matrices in $\mathscr{MR}(G)$

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 \\ 1 & -1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

We can decompose A as

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

where the two matrices are respectively in $\mathscr{MR}(\widetilde{G_1})$ and $\mathscr{MR}(\widetilde{G_2})$.

We can decompose B as

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

where the three matrices are respectively in $\mathscr{MR}(\widetilde{G_1 - 3})$, $\mathscr{MR}(\widetilde{G_2 - 3})$ and $\mathscr{MR}(S_5)$.

The following lemma which will be used later provides an example of how Theorem 3.9 is used to prove facts about nil vertices.

Lemma 3.11. *Let G be a graph and v be a nil vertex such that if v is isolated, $|\ddot{G}| \geq 3$. Then v is also nil in $G \vee K_1$.*

Proof. We begin by proving two trivial cases.

First, suppose that G has no isolated vertices. Then $\mathrm{mr}(G) = \mathrm{mr}(G \vee K_1)$ by Proposition 2.26. So G is an induced subgraph of $G \vee K_1$ with $\mathrm{mr}(G) = \mathrm{mr}(G \vee K_1)$. Thus any vertex that is nil in G is nil in $G \vee K_1$ by Theorem 3.6.

Second, suppose that G has no edges. In other words, G has only isolated vertices. Then $G = \ddot{G}$ and so v must be isolated. Then by assumption $|G| = |\ddot{G}| \geq 3$. So $G \vee K_1$ is a star on $|G| + 1$ vertices. Since $|G| \geq 3$, by Proposition 3.3 v is nil in $G \vee K_1$.

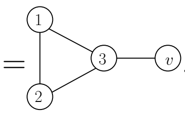
Lastly, consider the nontrivial case in which G has one or more isolated vertices and one or more edges. Then G has at least two components. Let u be the vertex in $G \vee K_1$ that comes from the K_1. Then u is a cut-vertex. Let G_1 be the subgraph of $G \vee K_1$ induced by u and $\ddot{G}$. Let G_2 be the subgraph of $G \vee K_1$ induced by u and $\breve{G}$. Then $G \vee K_1 = G_1 \underset{u}{\oplus} G_2$.

Case 1. Assume that v is not isolated in G. Hence v must be a vertex in $G_2 - u$. Note that $G_2 - u$ is the union of the components of G of size greater than one. Let H be the component of G containing v. Then H is also a component of $G_2 - u$. By Proposition 3.4, since v is nil in G, v is nil in H. Again by Proposition 3.4, since v is nil in H, v is nil in $G_2 - u$. Since $G_2 = (G_2 - u) \vee u$ and $G_2 - u$ has no isolated vertices, by the first case above, v is also nil in G_2.

Case 2. Assume that v is one of at least three isolated vertices of G. Then v is a vertex in $G_1 - u$. By Observation 3.2 v is nil in $G_1 - u$. Since $G_1 - u$ consists of all the isolated vertices of G, $G_1 = S_k$ where $k = |\ddot{G}| + 1 \geq 4$. Then v is nil in G_1 by Proposition 3.3.

Thus in the case that v is a vertex of G_i, v is nil in G_i and in $G_i - u$. By Theorem 3.9, any minimum rank matrix for G is in either in $\mathscr{MR}(\widetilde{G_1}) + \mathscr{MR}(\widetilde{G_2})$ or in $\mathscr{MR}(\widetilde{G_1 - u}) + \mathscr{MR}(\widetilde{G_2 - u}) + \mathscr{MR}(\widetilde{S_{k+1}})$ where $k = |G|$. Consider $A = [a_{i,j}] \in \mathscr{MR}(\widetilde{G_1})$ and $B = [b_{i,j}] \in \mathscr{MR}(\widetilde{G_2})$. Let $M = [m_{i,j}] = A + B$. Then $m_{v,v} = a_{v,v} + b_{v,v}$. First assume that v is a vertex of G_1. Then $a_{v,v} = 0$ since v is nil in G_1. Since the only vertex shared by G_1 and G_2 is $u \neq v$,

$b_{v,v} = 0$. Thus $m_{v,v} = 0$. The case where v is a vertex of G_2 is symmetric. Now consider $C = [c_{i,j}] \in \mathscr{MR}(\widetilde{G_1 - u})$, $D = [d_{i,j}] \in \mathscr{MR}(\widetilde{G_2 - u})$, and $S = [s_{i,j}] \in \mathscr{MR}(\widetilde{S_{k+1}})$. Let $N = [n_{i,j}] = C + D + S$. Then $n_{v,v} = c_{v,v} + d_{v,v} + s_{v,v}$. For variety here we will assume that v is a vertex of G_2. The case in which v is a vertex of G_1 is again symmetric. Since v is nil in $G_2 - u$, $d_{v,v} = 0$. Since v is not a vertex in $G_1 - u$, $c_{v,v} = 0$. Thus $n_{v,v} = s_{v,v}$. Note that v is a pendent vertex of S_{k+1}. If v is an isolated vertex of G, by assumption $|G| \geq 3$. If v is not an isolated vertex of G, then G must contain at least two non-isolated vertices. Since we are assuming that G does contains one or more isolated vertices, then $|G| \geq 3$. Thus in both cases $k + 1 \geq 4$ and so the pendent vertices of S_{k+1} are nil vertices by Proposition 3.3. Hence $n_{v,v} = s_{v,v} = 0$. Thus v is nil in $G \vee u$. $\square$

The condition that if v is an isolated vertex then $|\ddot{G}| \geq 3$ is necessary. For an example in which the lemma fails and $|\ddot{G}| = 1$, consider $G = K_2 \cup v$. While v is nil in $K_2 \cup v$ by Observation 3.2, v is not nil in $(K_2 \cup v) \vee K_1 =$ 

. In fact, v is nonzero. To see this, note that a matrix in $\mathcal{S}((K_2 \cup v) \vee K_1)$ is of the form $A = \begin{bmatrix} d_1 & a & b & 0 \\ a & d_2 & c & 0 \\ b & c & d_3 & d \\ 0 & 0 & d & d_v \end{bmatrix}$. If $d_v = 0$, rows 2,3, and 4 are linearly independent. Thus $\operatorname{rank} A \geq 3$. However, by Proposition 2.26 $\operatorname{mr}((K_2 \cup v) \vee K_1) = \operatorname{jmr}(K_2 \cup v) = \operatorname{mr}(K_2) + 1 = 2$. Thus $d_v \neq 0$ in any minimum rank matrix. Hence v is nonzero.

For an example in which $|\ddot{G}| = 2$, consider $G = K_2 \cup v \cup K_1$. Note that v is nil in $K_2 \cup v \cup K_1$. However, v is not nil in the folding stool $(K_2 \cup v \cup K_1) \vee K_1$ as will be shown in Example 3.24.

3.2 Structure of Cographs with an Edge Subdivision

In this section I will show that if a cograph G has a nil vertex produced by edge-subdivision, then in fact G has three nil vertices in an independent set.

Definition 3.12. If G is a graph and $e = \{v, w\} \in E(G)$, subdividing e is the action of creating a new graph G_e from G by adding a new vertex u, and adjusting the edge set as shown:

$$G_e = (V(G) \cup \{u\}, (E(G) \setminus \{v, w\}) \cup \{\{u, v\}, \{u, w\}).$$

Theorem 3.13. *[6, Theorem 4.1] Let G be a connected graph, and G_e be the graph obtained from G by subdividing an edge $e \in E(G)$. Let v be the vertex created by subdividing the edge e. Then $\mathrm{mr}(G_e) = \mathrm{mr}(G)$ if and only if v is a nil vertex.*

Proposition 3.14. *If G is a connected cograph that can be obtained by edge subdivision, then G is of the form $(2K_1) \vee (v \cup H)$, where H is a cograph and v is the vertex created by the edge subdivision.*

Proof. If G is a cograph that can be obtained by edge subdivision, then the vertex v created by the edge subdivision has exactly two neighbors, call them u and w. Then u and w are not adjacent.

We first consider the vertices that are adjacent to either u or v. If x is adjacent to u, then $x - u - v - w$ is a path of length 4. Since G is a cograph, this cannot be an induced subgraph by Theorem 2.12. Since x is not adjacent to v, then x must be adjacent to w. Thus every vertex in G adjacent to u is also adjacent to w. Similarly, any vertex adjacent to w is also adjacent to u.

We now show that all vertices of G are adjacent to u and w. If a vertex existed that was not adjacent to u or w, since G is connected we can choose such a vertex y adjacent to a neighbor x of u and w. Then $y - x - u - v$ is an induced path of length 4. Thus no such vertices exist. Thus G consists of u, v, w and some subgraph H composed of the remaining vertices. Every vertex of H is adjacent to both u and w, as is v. Thus G is the join of v and H to the vertices u and w. Or, $G = (2K_1) \vee (v \cup H)$. Since H is an induced subgraph of a cograph, H must be a cograph. □

Corollary 3.15. *If G is a connected cograph with a non-simplicial degree 2 vertex v, then G is of the form $(2K_1) \vee (v \cup H)$ where H is a cograph.*

Proof. If v is a non-simplicial degree 2 vertex, then v could be obtained by edge subdivision. □

Lemma 3.16. *Let H be a cograph. Then $G = (2K_1) \vee (H \cup K_1)$ is not anomalous.*

Proof. Since both $2K_1$ and $H \cup K_1$ are disconnected graphs, these are exactly the primary constituents of G. First suppose that H is anomalous. Then $K_{3,3,3}$ is an induced subgraph of H. By Observation 2.22

$$\operatorname{jmr}(H \cup K_1) = \operatorname{mr}((H \cup K_1) \vee K_1) \geq \operatorname{mr}(H \cup K_1) \geq \operatorname{mr}(H) \geq \operatorname{mr}(K_{3,3,3}) = 3.$$

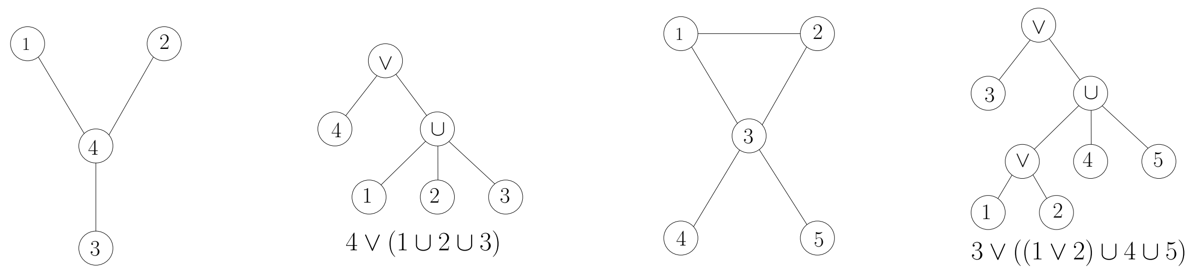
Since the join minimum rank of one of the primary constituents of G is greater than 2, G is not anomalous.

Now suppose that H is not anomalous. If $K_{3,3,3}$ is an induced subgraph of H, then by the argument above, G is not anomalous. If $K_{3,3,3}$ is not an induced subgraph of H, then $K_{3,3,3}$ is also not an induced subgraph of G. To see this consider each of the vertices of G that are not vertices of H. Recall $G = (2K_1) \vee (H \cup K_1)$. Label the vertices so that $G = (u \cup v) \vee (H \cup w)$. Then u and v each are adjacent to all the remaining vertices, so that neither vertex can be in an independent set of size three or greater. Thus neither u nor v are vertices of an induced $K_{3,3,3}$. On the other hand, w is only adjacent to u and v. Every vertex of $K_{3,3,3}$ is adjacent to 6 vertices. Thus w is not in an induced $K_{3,3,3}$. Thus any induced $K_{3,3,3}$ lies entirely inside H, which we've assumed does not contain an induced $K_{3,3,3}$. Thus G does not contain an induced $K_{3,3,3}$ and so is not anomalous. □

Before stating the next theorem we need some definitions. The motivation for this will be made clear in Section 3.3.

Definition 3.17. [10] The *composition tree* of a cograph G is a root tree whose leaves are the vertices of G. Nodes that are not leaves are labeled to indicate either a join or a union, and taking joins and unions as indicated results in the graph G.

Definition 3.18. Let G be a cograph. An independent set of vertices of G is called a *fundamental* independent set if the vertices are leaves off of the same union node in the composition tree of G.



(a) S_4 and its composition tree

(b) The folding stool and its composition tree

Figure 3.1: Cographs and Composition Trees

Example 3.19. Consider the cographs and corresponding composition trees in Figure 3.1. While vertices 1,4, and 5 form an independent set in the folding stool, they do not form a fundamental independent set. Vertices 1,2, and 3 do form a fundamental independent set in S_4.

Theorem 3.20. *If G is a cograph with a nil vertex v obtained by edge-subdivision, then G has at least 3 nil vertices. These 3 nil vertices occur as an independent set of an induced $K_{2,3}$. Furthermore, the independent set is a fundamental independent set of G.*

Proof. If the component of G containing v satisfies the statement then G satisfies the statement. Thus we may assume G to be connected. By Proposition 3.14, we may write $G = (2K_1) \vee (v \cup H)$, where H is non-empty since v is a nil vertex and $2K_1 \vee v = P_3$ has no nil vertices by Observation 3.5. The graph from which G was obtained by edge subdivision can be written as $K_2 \vee H$. Since v is nil, by Theorem 3.13, $\mathrm{mr}((2K_1) \vee (v \cup H)) = \mathrm{mr}(K_2 \vee H)$. By Lemma 3.16, G is not anomalous. So by Theorem 2.27, $\mathrm{mr}(G)$ is the maximum of the join minimum ranks of the primary constituents of G. The primary constituents are $2K_1$ and $v \cup H$. Thus $\mathrm{mr}(G) = \max\{2, \mathrm{jmr}(v \cup H)\}$. Since P_3 is an induced subgraph of $K_1 \vee (v \cup H)$, $\mathrm{jmr}(v \cup H) \geq \mathrm{mr}(P_3) = 2$ and we have simply $\mathrm{mr}(G) = \mathrm{jmr}(v \cup H)$. By definition, $\mathrm{jmr}(H) = \mathrm{mr}(K_1 \vee H)$ and $\mathrm{jmr}(K_1 \vee H) = \mathrm{mr}(K_1 \vee K_1 \vee H)$. Since $K_1 \vee H$ has no isolated vertices, $\mathrm{jmr}(K_1 \vee H) = \mathrm{mr}(K_1 \vee H)$. Thus $\mathrm{jmr}(H) = \mathrm{jmr}(K_1 \vee H) = \mathrm{mr}(K_1 \vee K_1 \vee H) = \mathrm{mr}(K_2 \vee H)$.

Thus we have that $\mathrm{jmr}(K_1 \cup H) = \mathrm{mr}(G) = \mathrm{mr}(K_2 \vee H) = \mathrm{jmr}(H)$. By Proposition 2.26 this occurs exactly when H has at least 2 isolated vertices. Since these two isolated vertices of H, considered as vertices of G, are only adjacent to the neighbors of v, by symmetry the two isolated vertices of H are also nil vertices in G. The subgraph of G induced by these two vertices, v, and the two neighbors of v is a $K_{2,3}$.

By considering the components of H, we may write $v \cup H = H_1 \cup \cdots \cup H_k$ where H_1 is the vertex v and all isolated vertices of H and each H_i for $i > 1$ is a connected component of H that has more than one vertex. Thus we have that $G = (K_1 \cup K_1) \vee (\cup_{i=1}^{k} H_i)$. Note that the vertices of H_1 form a fundamental independent set of size three or greater. $\square$

3.3 Nil Vertices: Open Questions and a Counterexample

We begin with some background. The following theorems characterize minimum rank 2 graphs and characterize the nil, nonzero, and neutral vertices of minimum rank 2 graphs. The graph called the *dart* is the graph $(P_3 \cup K_1) \vee K_1$. The graph $\ltimes$ is $K_1 \vee (K_2 \cup 2K_1)$ (see Example 3.10).

Theorem 3.21. *[5, Theorem 9] Let G be a connected graph. Then the following are equivalent:*

(i) $\mathrm{mr}(G) \leq 2$.

(ii) G^c *can be expressed as the union of at most 2 complete graphs and of complete bipartite graphs.*

(iii) G *is* $(P_4, \text{dart}, \ltimes, K_{3,3,3})$*-free.*

Theorem 3.22. *[4, Theorem 5.7] Let* G *be a connected graph with* $\mathrm{mr}(G) = 2$ *and write* $G = (K_k \cup K_\ell \cup K_{m_1,n_1} \cup \cdots \cup K_{m_r,n_r} \cup sK_2 \cup tK_1)^c$, *where* $k, \ell \in \{0, 3, 4, 5, ...\}$, $s, t \geq 0$, *and none of the* K_{m_i,n_i} *are* K_2*'s or* K_1*'s. Then the vertices of* G *corresponding to*

- K_k *and* K_ℓ *are nil vertices.*
- *the* K_{m_i,n_i}*'s are nonzero vertices.*
- *the* K_2*'s and* K_1*'s are nonzero vertices if* $k, \ell \geq 3$ *and neutral vertices otherwise.*

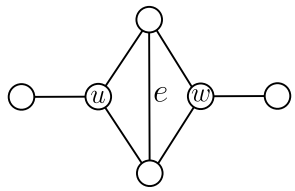
Theorem 3.22 shows that nil vertices in minimum rank two graphs come in sets of three or more. Does this result carry over to the larger class of cographs? Theorem 3.13 classifies vertices that arise from an edge-subdivision as nil exactly when the minimum rank did not change after the edge-subdivision. The following example demonstrates that we can use this to produce graphs with exactly one nil vertex. Since Theorem 3.20 says that a cograph with a nil vertex produced by edge-subdivision in fact must have three nil vertices, the example is not a cograph. While it might be possible for a cograph to have a single nil vertex not arising from edge-subdivision, no such example is known.

Question 1: Is there a cograph with exactly one nil vertex?

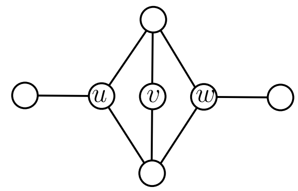
Question 1a: Is there a cograph with exactly two nil vertices?

Example 3.23. In Figure 3.2 the graph G_e is obtained from G by subdividing the edge e. The vertex obtained by edge-subdivision is v. We first show that $\mathrm{mr}(G_e) = \mathrm{mr}(G)$. To aid us we will compute the minimum ranks of some subgraphs of G. By Theorem 2.27

$$\mathrm{mr}(\mathrm{paw}) = \mathrm{mr}((K_2 \cup K_1) \vee K_1) = \mathrm{mr}(K_2 \cup K_1) + 1 = 2.$$



(a) G



(b) G_e

Figure 3.2: A Single Nil Vertex from Edge-subdivision

Applying Theorem 2.23 to w and using Theorem 2.27 we see that

$$\operatorname{mr}([(u \cup w) \vee K_2] \underset{w}{\oplus} K_2) = \min \left\{ \begin{array}{l} \operatorname{mr}((u \cup w) \vee K_2) + \operatorname{mr}(K_2) = 2 + 1 = 3 \\ \operatorname{mr}(u \vee K_2) + \operatorname{mr}(K_1) + 2 = 1 + 0 + 2 = 3 \end{array} \right\} = 3.$$

Using this, applying Theorem 2.23 to u, and using Theorem 2.27 we see that

$$\operatorname{mr}(G) = \min \left\{ \begin{array}{l} \operatorname{mr}(K_2) + \operatorname{mr}([(u \cup w) \vee K_2] \underset{w}{\oplus} K_2) = 1 + 3 = 4 \\ \operatorname{mr}(K_1) + \operatorname{mr}(\text{paw}) + 2 = 0 + 2 + 2 = 4 \end{array} \right\} = 4.$$

We now compute the minimum rank of some subgraphs of G_e. Applying Theorem 2.23 to w we find that

$$\operatorname{mr}([(v \cup w) \vee 2K_1] \underset{w}{\oplus} K_2) = \min \left\{ \begin{array}{l} \operatorname{mr}((v \cup w) \vee 2K_1) + \operatorname{mr}(K_2) = 2 + 1 = 3 \\ \operatorname{mr}(P_3) + \operatorname{mr}(K_1) + 2 = 2 + 0 + 2 = 4 \end{array} \right\} = 3.$$

and

$$\operatorname{mr}([(u \cup v \cup w) \vee 2K_1] \underset{w}{\oplus} K_2) = \min \left\{ \begin{array}{l} \operatorname{mr}((u \cup v \cup w) \vee 2K_1) + \operatorname{mr}(K_2) = 2 + 1 = 3 \\ \operatorname{mr}((u \cup v) \vee 2K_1) + \operatorname{mr}(K_1) + 2 = 2 + 0 + 2 = 4 \end{array} \right\} = 3.$$

Using this, Theorem 2.23 applied to u, and Theorem 2.27, we see that

$$\operatorname{mr}(G_e) = \min \left\{ \begin{array}{l} \operatorname{mr}(K_2) + \operatorname{mr}([(u \cup v \cup w) \vee 2K_1] \underset{w}{\oplus} K_2) = 1 + 3 = 4 \\ \operatorname{mr}(K_1) + \operatorname{mr}([(v \cup w) \vee 2K_1] \underset{w}{\oplus} K_2) + 2 = 0 + 3 + 2 = 5 \end{array} \right\} = 4.$$

So $\mathrm{mr}(G) = 4 = \mathrm{mr}(G_e)$. Thus by Theorem 3.13 v is a nil vertex. We show that v is the only nil vertex. Write $G_e = K_2 \underset{u}{\oplus} H_e \underset{w}{\oplus} K_2$. Note that $r_u(K_2) = \mathrm{mr}(K_2) - \mathrm{mr}(K_1) = 1 - 0 = 1$ and that $r_u(H_e \underset{w}{\oplus} K_2) = \mathrm{mr}([u \cup v \cup w) \vee 2K_1] \underset{w}{\oplus} K_2) - \mathrm{mr}([(v \cup w) \vee 2K_1) \underset{w}{\oplus} K_2) = 3 - 3 = 0$. So $r_u(K_2) + r_u(H_e \underset{w}{\oplus} K_2) = 1 + 0 = 1$. Thus by Theorem 3.9, every minimum rank matrix for G_e is in $\mathscr{MR}(\widetilde{K_2}) + \mathscr{MR}(\widetilde{H_e \underset{w}{\oplus} K_2})$.

We have $r_w(H_e) = \mathrm{mr}((u \cup v \cup w) \vee 2K_1) - \mathrm{mr}((u \cup v) \vee 2K_1) = 2 - 2 = 0$ and $r_w(K_2) = \mathrm{mr}(K_2) - \mathrm{mr}(K_1) = 1 - 0 = 1$. So $r_w(H_e) + r_w(K_2) = 0 + 1 = 1$. Thus by Theorem 3.9 every minimum rank matrix for $H_e \underset{w}{\oplus} K_2$ is in $\mathscr{MR}(\widetilde{H_e}) + \mathscr{MR}(\widetilde{K_2})$. It follows that every minimum rank matrix for G_e is in $\mathscr{MR}(\widetilde{K_2}) + \mathscr{MR}(\widetilde{H_e}) + \mathscr{MR}(\widetilde{K_2})$, and conversely. Let $A = [a_{i,j}] \in \mathscr{MR}(\widetilde{K_2})$, $B = [b_{i,j}] \in \mathscr{MR}(\widetilde{H_e})$, and $C = [c_{i,j}] \in \mathscr{MR}(\widetilde{K_2})$. Let $M = [m_{i,j}] = A + B + C$. Since every vertex of K_2 is nonzero, we can choose $a_{u,u}$ so that $m_{u,u} = a_{u,u} + b_{u,u} + c_{u,u} \neq 0$. Thus u is not nil in G_e. Similarly w is not nil in G_e. If x is a vertex other than u or w, then at most one of $a_{x,x}, b_{x,x}$ and $c_{x,x}$ is nonzero since x is a vertex in at most one of K_2, H_e and K_2. Thus x is nil in G_e if and only if x is nil in the appropriate subgraph. Recall that K_2 has no nil vertices. By Theorem 3.21 the only nil vertices of $H_e = K_{2,3}$ are u, v and w. Thus v is nil in G_e. We've shown that u and w are not nil in G_e. Thus G_e is a graph with exactly one nil vertex.

One way to answer Questions 1 and 1a would be to produce a cograph with exactly one or exactly two nil vertices. Of course, another way to answer the questions would be by proving that nil vertices in cographs come in groups of three or more. We would be looking to extend Theorem 3.22 in some way to cographs. For minimum rank 2 graphs, the nil vertices are exactly those vertices in an independent set of size 3 or greater. A simple minimum rank 3 cograph serves to shows that this is not true more generally.

Example 3.24. Consider the folding stool (see Figure 3.1b), $3 \vee ((1 \vee 2) \cup 4 \cup 5)$. Vertices $1, 4$, and 5 form an independent set of size three. However, they are not nil vertices. By Proposition 2.26, $\mathrm{mr}(\text{folding stool}) = \mathrm{mr}(3 \vee ((1 \vee 2) \cup 4 \cup 5)) = \mathrm{jmr}((1 \vee 2) \cup 4 \cup 5) =$

$\operatorname{mr}(1 \vee 2) + 2 = 1 + 2 = 3$. Thus $\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 3 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \in \mathscr{MR}(\text{folding stool})$. Since all the diagonal entries are nonzero, in fact none of the vertices of the folding stool are nil.

Notice that the choice of independent set was somewhat arbitrary. We could have just as easily been considering the independent set composed of vertices $2, 4$, and 5. Thus there is something less structured about the independent sets in the folding stool as compared to the independent sets in minimum rank 2 graphs. If we could select only those independent sets that were "structured enough," we might have more hope that their vertices would be nil. This motivates the definition of a fundamental independent set, which we recall here with the concept of a composition tree. See Figure 3.1 and Example 3.19.

Definition 3.25. The *composition tree* of a cograph G is a root tree whose leaves are the vertices of G. Nodes that are not leaves are labeled to indicate either a join or a union, and taking joins and unions as indicated results in the graph G.

Definition 3.26. Let G be a cograph. An independent set of vertices of G is called a *fundamental* independent set if the vertices are leaves off of the same union node in the composition tree of G.

Proposition 3.27. *Let G be a connected graph with $\operatorname{mr}(G) = 2$. Then every nil vertex in G is in a fundamental independent set of size three or greater.*

Proof. By Theorem 3.21, we may write $G = (K_k \cup K_\ell \cup K_{m_1,n_1} \cup \cdots \cup K_{m_r,n_r} \cup sK_2 \cup tK_1)^c$, where $k, \ell \in \{0, 3, 4, 5, ...\}$, $s, t \geq 0$, and none of the K_{m_i,n_i} are K_2's or K_1's. By Theorem 3.22 we have that the nil vertices of G correspond to the vertices of the K_k and K_ℓ. If G has at least one nil vertex, then k and ℓ are not both 0 and so we may assume that $k \geq 3$. Certainly the vertices corresponding to K_k form an independent set of size three or greater in G. Since for any graphs G_1 and G_2 we have that $(G_1 \cup G_2)^c = G_1^c \vee G_2^c$, $G = K_k^c \vee H^c = kK_1 \vee H^c$ where $H = K_\ell \cup \cdots \cup tK_1$. Thus the nil vertices corresponding to K_k form a fundamental independent set of size three or greater. Similarly, if there are other vertices nil in G, they will correspond to K_ℓ and will form a fundamental independent set of size three or greater. $\square$

Theorem 3.20 shows that in a cograph with a nil vertex produced by edge subdivision, that nil vertex is part of a fundamental independent set of size three or greater of nil vertices. Our most basic example of a cograph with nil vertices is S_4, and here again the nil vertices are in a fundamental independent set. If the answer to Questions 1 and 1a were that every cograph with any nil vertices at all must have at least three nil vertices, we might ask ourselves the following.

Question 2: Is every nil vertex in a cograph part of a fundamental independent set of size 3 or greater?

While no answer is known yet, we can produce a counterexample to the converse. That is, not every vertex of a fundamental independent set of size 3 or greater in a cograph is nil.

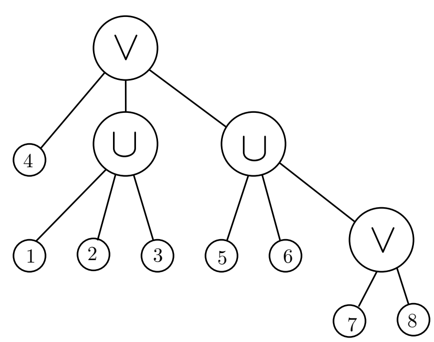


Figure 3.3: A Counterexample

Example 3.28. Let $G = (1 \cup 2 \cup 3) \vee 4 \vee (5 \cup 6 \cup (7 \vee 8))$ (see Figure 3.3). Then G is a cograph and $\{1, 2, 3\}$ is a fundamental independent set. We will show that none of $1, 2, 3$ are nil. By Theorem 2.27 and Proposition 2.26,

$$\mathrm{mr}(G) = \max\{\mathrm{jmr}(K_1), \mathrm{jmr}(3K_1), \mathrm{jmr}(2K_1 \cup K_2)\} = \mathrm{jmr}(2K_1 \cup K_2) = \mathrm{mr}(K_2) + 2 = 3.$$

Let

$$M = \left[\begin{array}{ccc|c|cccc} 1 & 0 & 0 & 2 & 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 2 & 2 & 2 & 1 & 1 \\ 0 & 0 & 1 & -2 & 3 & 1 & 1 & 1 \end{array}\right].$$

Let

$$A = M^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} M = \left[\begin{array}{ccc|c|cccc} 1 & 0 & 0 & 2 & 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 2 & 2 & 2 & 1 & 1 \\ 0 & 0 & -1 & 2 & -3 & -1 & -1 & -1 \\ \hline 2 & 2 & 2 & 4 & 12 & 4 & 6 & 6 \\ \hline 1 & 2 & -3 & 12 & -4 & 0 & 0 & 0 \\ -1 & 2 & -1 & 4 & 0 & 4 & 0 & 0 \\ 1 & 1 & -1 & 6 & 0 & 0 & 1 & 1 \\ 1 & 1 & -1 & 6 & 0 & 0 & 1 & 1 \end{array}\right].$$

Then $\operatorname{rank} A = 3$. Since $A \in \mathcal{S}(G)$ and $\operatorname{mr}(G) = 3$, $A \in \mathscr{MR}(G)$. The first, second, and third diagonal entries are nonzero, showing that vertices 1,2, and 3 are not nil.

While unfortunately we cannot identify nil vertices in cographs by identifying fundamental independent sets from the composition tree, we can say something about how nil vertices behave relative to the join and union structure of cographs. Notice in this example the fundamental independent set does not occur in a primary constituent whose join minimum rank is the minimum rank of the whole graph. When this occurs the situation is much simpler.

Proposition 3.29. *Let $G = \vee_{i=1}^{r} G_i$ be a cograph and let $\operatorname{mr}(G) = \operatorname{jmr}(G_1)$. If v is a nil vertex of G_1 such that either v is not isolated in G_1 or $|\ddot{G}_1| \geq 3$, then v is nil in G.*

Proof. If $r = 1$, $G = G_1$ and the statement is trivially true. Assume that $r > 1$. Let u be any vertex of $\vee_{i=2}^{r} G_i$. Then $\operatorname{mr}(G_1 \vee u) = \operatorname{jmr}(G_1) = \operatorname{mr}(G)$. Thus $G_1 \vee u$ is an induced subgraph of G having the same minimum rank. Since v is nil in G_1, v is nil in $G_1 \vee u$ by Lemma 3.11. Thus by Theorem 3.6 v is nil in G. □

While it is not true in general that if v is nil in a cograph G and H is another cograph that v is nil in $G \vee H$, Proposition 3.29 gives sufficient conditions under which this does occur. By Proposition 3.4 we see that if v is nil in G then v is always nil in $G \cup H$. Thus if v is nil in a cograph G such that either v is not isolated or $|\ddot{G}| \geq 3$, and G' is any cograph that is formed by taking unions with G and joining cographs with no greater join minimum rank, then by repeatedly applying Propositions 3.29 and 3.4 we see that v is nil in G'. While this does help us to identify nil vertices in an infinite though rather limited class of cographs, we would prefer to have a method that involves examining only graph theoretic properties.

The next theorem replaces the condition on the join minimum ranks of the primary constituents with a restrictions on the number of vertices. Thus if v is a nil vertex in a cograph G such that either v is not isolated in G or $|\ddot{G}| \geq 3$, then v is also nil in any cograph built up from G by taking unions with cographs or joins with small enough cographs.

Theorem 3.30. *Let $G = G_1 \vee G_2$ be a connected cograph with $|G_2| \leq 2$. If v is a nil vertex of G_1 such that either v is not isolated in G_1 or $|\ddot{G}_1| \geq 3$, then v is nil in G.*

Proof. Note that here we are not requiring G_1 and G_2 to be primary constituents of G.

If G_2 has only one vertex, then $G = G_1 \vee K_1$. Then v is nil in G by Lemma 3.11. Now suppose that G_2 has exactly two vertices. Either $G_2 = 2K_1$ or $G_2 = K_2$. Then $\text{jmr}(G_2) = 1$ or 2. Since graphs with minimum rank one do not have nil vertices, $\text{jmr}(G_1) \geq 2$. For the moment assume that G is not anomalous. Then $\text{mr}(G) = \text{jmr}(G_1)$ by Theorem 2.27. Thus if u is any vertex in G_2, $G_1 \vee u$ is an induced subgraph of G with $\text{mr}(G_1 \vee u) = \text{mr}(G)$. By Theorem 3.6, any nil vertex in $G_1 \vee u$ is also nil in G. By Lemma 3.11, v is nil in $G_1 \vee u$. Thus v is nil in G.

We now consider the case where G is anomalous. Then $\text{mr}(G) = 3$ and $K_{3,3,3}$ is an induced subgraph of G. Note that none of the vertices of G_2 can be included in the induced $K_{3,3,3}$, since they are each adjacent to every vertex in G_1. Thus $K_{3,3,3}$ is an induced subgraph of G_1. Write G_1 in the standard form $G_1 = \vee_{i=1}^{r} H_i$ where each H_i is a primary constituent of G_1 and thus also of G. Since G is anomalous, $\text{jmr}(H_i) \leq 2$ for each i. Thus G_1 is itself anomalous. So G_1 is an induced subgraph of G with $\mathrm{mr}(G_1) = 3 = \mathrm{mr}(G)$. Thus by Theorem 3.6 v is nil in G. □

Corollary 3.31. *Let $G = G_1 \vee G_2$ be a connected cograph that is not $3-$connected. Let v be a nil vertex in G_1 such that either v is not isolated in G_1 or $|\ddot{G}_1| \geq 3$. Then v is nil in G.*

Proof. Since G is not 3-connected, G_1 and G_2 cannot both have 3 or more vertices. The only nil vertices in graphs on 1 or 2 vertices are isolated vertices. Since v is in G_1, it follows that G_2 must have fewer than 3 vertices. Thus v is nil in G by Theorem 3.30. □

Chapter 4. Zero Forcing Number of Cographs

In this section we show that the zero forcing number as defined in [1] is equal to the maximum nullity for cographs without an induced $K_{3,3,3}$. We begin with some basic definitions and examples.

Definition 4.1.

- Color-change rule: If G is a graph with each vertex colored either white or black, u is a black vertex of G, and exactly one neighbor v of u is white, then change the color of v to black.

- Given a coloring of G, the derived coloring is the result of applying the color-change rule until no more changes are possible.

- A zero forcing set for a graph G is a subset of vertices Z such that if initially the vertices in Z are colored black and the remaining vertices are colored white, the derived coloring of G is all black.

- $Z(G)$ is the minimum of $|Z|$ over all zero forcing sets $Z \subset V(G)$.

Example 4.2. We show that $Z(S_n) = n - 2$ for $n \geq 3$. Let Z be the set of all the pendent vertices of S_n except one. See Figure 4.1 for the case $n = 5$. The central vertex of S_n is the only white neighbor of any of the vertices in Z. Applying the color-change rule changes the central vertex to black. Now, the remaining white pendent vertex is the only white neighbor of the central vertex. Applying the color-change rule again changes this pendent vertex black. Thus Z is a zero forcing set for S_n. So $Z(S_n) \leq n - 2$. Now suppose that $Z \subset V(G)$ and that $|Z| < n - 2$. Then at least two pendent vertices are not included in Z. The only neighbor of a pendent vertex is the central vertex. In order for the color-change rule to apply to color one of the white pendent vertices black, the white pendent vertex must be the only white neighbor of the central vertex. So Z is not a zero forcing set. Thus $Z(S_n) = n - 2$.

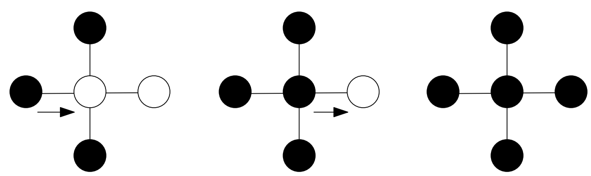
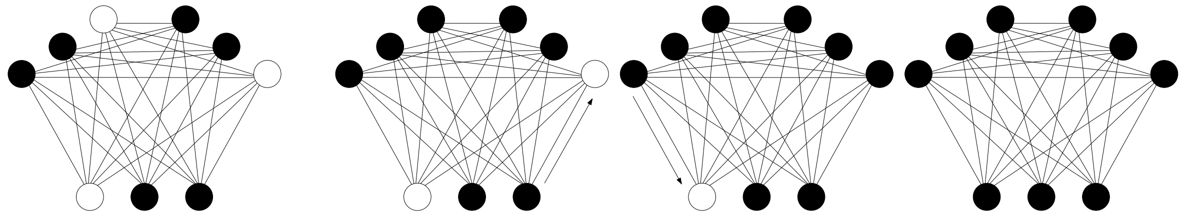


Figure 4.1: A minimal zero forcing set for S_5

Theorem 4.3. *[1, See Proposition 2.4] For any graph G, $M(G) \leq Z(G)$.*

Example 4.4. Strict inequality can occur in Theorem 4.3. By Observation 2.29 and Theorem 2.27, $\mathrm{mr}(K_{3,3,3}) = 3$. So $M(K_{3,3,3}) = 9 - 3 = 6$. However, we will show that $Z(K_{3,3,3}) = 7$. Consider one of the independent sets of $K_{3,3,3}$. Call the three vertices of this independent set u, v, and w. Each of u, v, w is a neighbor of every vertex not in the same independent set. Thus if u and v are white, no vertex not in the independent set will be able to force either u or v. Similarly for any two vertices of u, v, w. Thus if Z is a zero forcing set for $K_{3,3,3}$, Z must contain at least two vertices from each independent set, as pictured in Figure 4.2a. However, if Z only contains two vertices from each independent set, each vertex of Z will have two white neighbors. Hence any zero forcing set for $K_{3,3,3}$ must contain more than 6 vertices. Thus $Z(K_{3,3,3}) \geq 7$. In fact $Z(K_{3,3,3}) = 7$ as Figure 4.2b shows.



(a) Not a zero forcing set (b) A zero forcing set

Figure 4.2: $Z(K_{3,3,3}) = 7$

While $Z(G)$ is not always equal to $M(G)$, equality does hold for many classes of graphs, including for all trees [1, Proposition 4.2]. I show that $Z(G) = M(G)$ for another infinite class of graphs, cographs without an induced $K_{3,3,3}$.

Observation 4.5. *Let $G = \bigcup_{i=1}^{k} G_i$. Then $Z(G) = \sum_{i=1}^{k} Z(G_i)$.*

Proof. Let Z be a zero forcing set for G. If $v \in Z$ is a vertex of G_i, v cannot force any vertex of G_j for $j \neq i$. Thus Z must contain a zero forcing set for G_j for each j. So $Z(G) \geq \sum_{i=1}^{k} Z(G_i)$. However, clearly if Z contains a zero forcing set for G_j for each j then Z is a zero forcing set for G. Thus $Z(G) \leq \sum_{i=1}^{k} Z(G_i)$. Therefore $Z(G) = \sum_{i=1}^{k} Z(G_i)$. □

Lemma 4.6. *Let $G = \cup_{i=1}^{k} G_i$. If $Z(G_i) = M(G_i)$ for $i = 1, 2, \ldots, k$, then $Z(G) = M(G)$.*

Proof. By Observation 2.19 and Observation 4.5

$$M(G) = M(\cup_{i=1}^{k} G_i) = \sum_{i=1}^{k} M(G_i) = \sum_{i=1}^{k} Z(G_i) = Z(\cup_{i=1}^{k} G_i) = Z(G).$$

□

Definition 4.7. Let $G = \vee_{i=1}^{r} G_i$ be a non-anomalous cograph. If G is connected, a *maximum primary constituent of* G is a primary constituent G_i such that $\operatorname{jmr}(G_i) = \operatorname{mr}(G)$. If G is disconnected, the *maximum primary constituent of* G is G.

Lemma 4.8. *Let $G = \vee_{i=1}^{r} G_i$ be a connected non-anomalous cograph with maximum primary constituent G_1. If $Z(G_1) = M(G_1)$, then $Z(G) = M(G)$.*

Proof. Since the zero forcing number is an upper bound on maximum nullity, it suffices to prove that $Z(G) \leq M(G)$. We do this by demonstrating a forcing set of size $M(G)$. Since G is not anomalous, the minimum rank of G is the largest of the join minimum ranks of its primary constituents. Thus the condition that G_1 is a maximum primary constituent is merely a matter of choice of labeling. Let $H = \vee_{i=2}^{r} G_i$. Then $G = G_1 \vee H$.

Case 1. Assume $|\ddot{G}_1| = 0$. Then $\operatorname{jmr}(G_1) = \operatorname{mr}(G_1)$ and so $\operatorname{mr}(G) = \operatorname{mr}(G_1)$. Then

$$M(G) = |G| - \operatorname{mr}(G) = |G_1| + |H| - \operatorname{mr}(G_1) = M(G_1) + |H| = Z(G_1) + |H|.$$

Let Z be a minimal zero forcing set for G_1. Then $|Z| = Z(G_1)$. The set $Z \cup V(H)$ is a forcing set for $G \vee H$ since the only white neighbors of vertices in G_1 are in G_1. Thus $Z(G) \leq Z(G_1) + |H| = M(G)$.

Case 2. Assume $|\ddot{G}_1| = 1$. Then $\operatorname{jmr}(G_1) = \operatorname{mr}(G_1) + 1$ and so $\operatorname{mr}(G) = \operatorname{mr}(G_1) + 1$. Then

$$M(G) = |G| - \operatorname{mr}(G) = |G_1| + |H| - (\operatorname{mr}(G_1) + 1) = M(G_1) + |H| - 1 = Z(G_1) + |H| - 1.$$

Let Z be a minimal zero forcing set for G_1. Let v be the isolated vertex of G_1. Since no other vertex of G_1 is adjacent to v, $v \in Z$. Consider $(Z \setminus \{v\}) \cup V(H)$. This is a set of size $M(G)$. We show that this is a forcing set. Since all vertices of H are colored and v is not a neighbor of any vertex in $G_1 - v$, all white vertices of $G_1 - v$ can be forced. Then any vertex of H can force v. Thus $Z(G) \leq Z(G_1) - 1 + |H| = M(G)$.

Case 3. Assume $|\ddot{G}_1| \geq 2$. Then $\operatorname{jmr}(G_1) = \operatorname{mr}(G_1) + 2$ and so $\operatorname{mr}(G) = \operatorname{mr}(G_1) + 2$. Then

$$M(G) = |G| - \operatorname{mr}(G) = |G_1| + |H| - (\operatorname{mr}(G_1) + 2) = M(G_1) + |H| - 2.$$

Let Z be a minimal forcing set for G_1. Let u and v be two isolated vertices of G_1. Then $u, v \in Z$. Let w be any vertex of H. Consider $(Z \setminus \{v\}) \cup (V(H) \setminus \{w\})$. This is a set of size $M(G)$. We show that this is a forcing set. Since u is adjacent to all vertices of H and no other vertices of G_1, u can force w. Then as in the preceding case, all vertices of $G_1 - v$ can be forced. Then any vertex in H can force v. Thus $Z(G) \leq Z(G_1) - 1 + |H| - 1 = M(G)$. $\square$

Theorem 4.9. *Let G be a non-anomalous cograph with a $K_{3,3,3}$-free maximum primary constituent. Then $Z(G) = M(G)$.*

Proof. We proceed by induction on $n = |G|$. Clearly, $Z(K_1) = 1 = M(K_1)$. Thus every cograph on a single vertex satisfies $Z(G) = M(G)$. Now assume that every non-anomalous cograph G on fewer than n vertices with a $K_{3,3,3}$-free maximum primary constituent satisfies $Z(G) = M(G)$.

Let G be a non-anomalous cograph on n vertices with a $K_{3,3,3}$-free maximum primary constituent H.

First assume that G is connected. Then G has at least two primary constituents and so $|H| < n$. Since H is $K_{3,3,3}$-free, by the inductive hypothesis $Z(H) = M(H)$. By Lemma 4.8, $Z(G) = M(G)$.

Now assume that G is disconnected and write $G = \bigcup_{i=1}^{k} G_i$. Then by assumption G is $K_{3,3,3}$-free. So each component G_i of G is $K_{3,3,3}$-free. Since $|G_i| < n$ for each i, by the inductive hypothesis $Z(G_i) = M(G_i)$ for each i. By Lemma 4.6 $Z(G) = M(G)$.

This completes the induction and we see that for every non-anomalous cograph G with a $K_{3,3,3}$-free maximum primary constituent, $Z(G) = M(G)$. $\square$

Corollary 4.10. *Let G be a $K_{3,3,3}$-free cograph. Then $Z(G) = M(G)$.*

Corollary 4.11. *Let G be a threshold graph. Then $Z(G) = M(G)$.*

Proof. Recall that threshold graphs are built up from K_1 by consecutively taking the join or union with a single vertex. Thus deleting a vertex from a threshold graph is the same as skipping the corresponding join or union in the process of building the graph. Hence deleting a vertex from a threshold graph results in a threshold graph. Therefore every induced subgraph of a threshold graph is a threshold graph.

Every connected threshold graph has a vertex adjacent to every other vertex. Thus $K_{3,3,3}$ is not a threshold graph and so every threshold graph must be $K_{3,3,3}$-free. □

Chapter 5. Maximum Minimum Rank of Cographs

The following proposition and theorem are well known results about minimum rank.

Proposition 5.1. *Let G be a graph on n vertices. Then* $\mathrm{mr}(G) \leq n-1$.

Proof. The Laplacian matrix for G is in $\mathcal{S}(G)$, has -1's in every off-diagonal entry corresponding to an edge, and for every vertex v has the degree of v in the diagonal entry corresponding to v. Thus the rows sum to 0 and the Laplacian matrix is singular. □

Theorem 5.2. *[9] The path on n vertices, P_n, is the only graph on n vertices with minimum rank $n-1$.*

Proposition 5.3. *Let G be a graph on n vertices with k components. Then* $\mathrm{mr}(G) \leq n-k$.

Proof. Let $G_1, G_2, \ldots, G_k$ be the components of G and let n_i be the number of vertices of G_i for each i. By Proposition 5.1 $\mathrm{mr}(G_i) \leq n_i - 1$ for each i. Thus by Observation 2.19 $\mathrm{mr}(G) = \mathrm{mr}(\cup_{i=1}^{k} G_i) = \sum_{i=1}^{k} \mathrm{mr}(G_i) \leq \sum_{i=1}^{k}(n_i - 1) = n-k$. □

The bound on the minimum rank of disconnected graphs improves upon the bound for all graphs by only an additive constant. In this section I prove that the minimum rank of a cograph on n vertices is bounded by $\left\lfloor \frac{2n}{3} \right\rfloor$, improving on the bound for all graphs by a multiplicative constant.

Lemma 5.4. *Let G be a connected non-anomalous cograph. Then there exists a cograph on $|G|$ vertices of the form $K_1 \vee H$ such that* $\mathrm{mr}(K_1 \vee H) \geq \mathrm{mr}(G)$.

Proof. Write $G = \vee_{i=1}^{r} G_i$ with labeling so that G_1 is a maximum primary constituent of G. Thus $\mathrm{mr}(G) = \mathrm{jmr}(G_1)$. Since G is connected, $r \geq 2$. Let $m = \sum_{i=2}^{r} |G_i|$. Then $|(G_1 \cup (m-1)K_1) \vee K_1| = |G|$, and $\mathrm{mr}((G_1 \cup (m-1)K_1) \vee K_1) = \mathrm{jmr}(G_1 \cup (m-1)K_1) \geq \mathrm{jmr}(G_1) = \mathrm{mr}(G)$. □

Lemma 5.5. *Let G be a disconnected cograph on n vertices. Then there is a connected cograph on n vertices with minimum rank at least as large.*

Proof. Let $G_1, G_2, \ldots, G_k$ be the components of G and let $n_i = |G_i|$ for each i. First assume that G has an isolated vertex, say $G_1 = K_1$. Then

$$\mathrm{mr}(K_1 \vee (\cup_{i=2}^{k} G_i) \geq \mathrm{mr}(\cup_{i=2}^{k} G_i) = \mathrm{mr}(K_1 \cup (\cup_{i=2}^{k} G_i)) = \mathrm{mr}(G).$$

Now assume that G does not have any isolated vertices and that one of the components is not anomalous. Let G_1 be a non-anomalous component. Applying Lemma 5.4 we may obtain a cograph on n_1 vertices of the form $K_1 \vee H$ such that $\mathrm{mr}(K_1 \vee H) \geq \mathrm{mr}(G_1)$. Thus

$$\mathrm{mr}(K_1 \vee (H \cup (\cup_{i=2}^{k} G_i)) = \mathrm{mr}(H \cup (\cup_{i=2}^{k} G_i)) + \min\{|\ddot{H}|, 2\}$$

$$= \mathrm{mr}(H) + \sum_{i=2}^{k} \mathrm{mr}(G_i) + \min\{|\ddot{H}|, 2\} = \mathrm{mr}(K_1 \vee H) + \sum_{i=2}^{k} \mathrm{mr}(G_i)$$

$$\geq \mathrm{mr}(G_1) + \sum_{i=2}^{k} \mathrm{mr}(G_i) = \mathrm{mr}(G).$$

Assume every component of G is anomalous. Then by definition and Proposition 2.31 every primary constituent of G_1 has join minimum rank at most 2, and three of the primary constituents consist entirely of (three or more) isolated vertices. Let u and v be two isolated vertices of the same one of these primary constituents. If $G_1 - u - v$ is still anomalous, then $\mathrm{mr}(G_1 - u - v) = 3$. If $G_1 - u - v$ is not anomalous, $\mathrm{mr}(G_1 - u - v)$ can be computed by Theorem 2.27 by finding the maximum of the join minimum ranks of the primary constituents. The join minimum rank of each primary constituent is still at most 2, and there is still a primary constituent with at least 3 isolated vertices. Thus $\mathrm{mr}(G_1 - u - v) = 2$.

Since each G_i is anomalous, no G_i is an isolated vertex. Also note that $G_1 - u - v$ is connected and is not a single vertex. So by Proposition 2.26, $\mathrm{mr}(u \vee (G - u - v \cup v)) = \mathrm{mr}(G - u - v) + 1 = \mathrm{mr}(G_1 - u - v) + 1 + \sum_{i=2}^{k} \mathrm{mr}(G_i)$. Since $\mathrm{mr}(G_1 - u - v) \geq 2$ and $\mathrm{mr}(G_1) = 3$, $\mathrm{mr}(u \vee (G - u - v \cup v)) \geq 2 + 1 + \sum_{i=2}^{k} \mathrm{mr}(G_i) = \mathrm{mr}(G_1) + \sum_{i=2}^{k} \mathrm{mr}(G_i) = \mathrm{mr}(G)$.

Thus in all cases there is a connected cograph on n vertices with minimum rank at least as large as $\mathrm{mr}(G)$, where G is disconnected. $\square$

Theorem 5.6. *Let G be a cograph on n vertices. Then $\mathrm{mr}(G) \leq \left\lfloor \dfrac{2n}{3} \right\rfloor$. Furthermore, for each n there is a threshold graph for which equality is achievable.*

Proof. We first show that the proposed bound is achievable by a threshold graph. We proceed by induction on n. For $n = 1, 2, 3$, P_n is a threshold graph with $\mathrm{mr}(P_n) = n - 1 = \left\lfloor \frac{2n}{3} \right\rfloor$. For $n = 4, 5, 6$, $K_1 \vee (P_{n-3} \cup 2K_1)$ is a threshold graph with $\mathrm{mr}(K_1 \vee (P_{n-3} \cup 2K_1)) = \mathrm{jmr}(P_{n-3} \cup 2K_1) = \mathrm{mr}(P_{n-3}) + 2 = ((n-3) - 1) + 2 = n - 2 = \left\lfloor \frac{2n}{3} \right\rfloor$.

Assume that the proposed bound is achievable by a connected threshold graph for all $i < n$. Let H_i be a connected threshold graph on i vertices that achieves the proposed bound for each i. Let $H_n = K_1 \vee (H_{n-3} \cup 2K_1)$. Since union with K_1 and joining K_1 preserve threshold graphs, H_n is a threshold graph. By the inductive hypothesis $\mathrm{mr}(H_{n-3}) = \left\lfloor \frac{2(n-3)}{3} \right\rfloor$. By Proposition 2.26 and since H_{n-3} has no isolated vertices, $\mathrm{mr}(H_n) = \mathrm{jmr}(H_{n-3} \cup 2K_1) = \mathrm{mr}(H_{n-3}) + 2 = \left\lfloor \frac{2(n-3)}{3} \right\rfloor + 2 = \left\lfloor \frac{2n}{3} - 2 \right\rfloor + 2 = \left\lfloor \frac{2n}{3} \right\rfloor$.

We now show that the proposed bound is actually an upper bound. For $n = 1, 2, 3$, $\left\lfloor \frac{2n}{3} \right\rfloor = n - 1$, which is an upper bound for the minimum rank of any graph on n vertices by Proposition 5.1. For $n = 4, 5, 6$, $\left\lfloor \frac{2n}{3} \right\rfloor = n - 2$, which is an upper bound for the minimum rank of any graph on n vertices that is not the path P_n by Theorem 5.2. Let $n \geq 6$. Then $\left\lfloor \frac{2n}{3} \right\rfloor \geq 4$. Thus any cograph on n vertices that achieves or exceeds the proposed bound is not anomalous.

We now prove the following claim. The maximum minimum rank achievable by a cograph on n vertices can be achieved by a cograph of the form $K_1 \vee H$ where H has at most 2 isolated vertices. Let G be a cograph on n vertices of maximum minimum rank. Since $n \geq 6$ and since we've shown that the proposed bound is achievable, $\mathrm{mr}(G) \geq 4$. Thus G is not anomalous. By Lemma 5.5, since G has maximum minimum rank, we may choose G to be connected. By Lemma 5.4 we can obtain a cograph of the form $K_1 \vee H$ which also has maximum minimum rank. Now suppose that H has more than 2 isolated vertices. Then $\mathrm{mr}(K_1 \vee H) = \mathrm{jmr}(H) = \mathrm{mr}(\breve{H}) + 2$. Let v be one of the isolated vertices. Then $\mathrm{mr}(H) = \mathrm{mr}(H - v)$. Thus $\mathrm{mr}(K_1 \vee H - v) = \mathrm{jmr}(H - v) = \mathrm{mr}(\breve{H - v}) + 2 = \mathrm{mr}(\breve{H}) + 2$. Since $H - v$ is disconnected, by Proposition 2.31 $K_1 \vee v \vee (H - v)$ is not anomalous. Thus by Theorem 2.27, $\mathrm{mr}(K_1 \vee v \vee (H - v)) = \max\{1, 1, \mathrm{jmr}(H - v)\} = \mathrm{jmr}(H - v) = \mathrm{mr}(\breve{H}) + 2$. Let H' be $v \vee (H - v)$. Then $\mathrm{mr}(K_1 \vee H') = \mathrm{mr}(K_1 \vee H)$ and H' has no isolated vertices. Therefore, we may assume that $G = K_1 \vee H$ has maximum minimum rank and $|\ddot{H}| \leq 2$.

Note that for $i = 4, 5, 6$, H_i constructed above is a connected cograph of maximum minimum rank equal to $\lfloor \frac{2i}{3} \rfloor$. Thus we may assume that there is an $n > 6$ such that for every $i < n$ there is a connected cograph with maximum minimum rank equal to $\lfloor \frac{2i}{3} \rfloor$ on i vertices. We've shown that we can pick a cograph G on n vertices with maximum minimum rank of the form $K_1 \vee H$ where $|\ddot{H}| \leq 2$. It remains only to show that $\mathrm{mr}(G) = \lfloor \frac{2n}{3} \rfloor$. Since $G = K_1 \vee H$, $|H| = n - 1$. Since $|\ddot{H}| \leq 2$

$$\mathrm{mr}(G) = \mathrm{mr}(K_1 \vee H) = \mathrm{jmr}(H) = \mathrm{mr}(H) + |\ddot{H}| = \mathrm{mr}(\breve{H}) + |\ddot{H}|.$$

Also since $|\ddot{H}| \leq 2$, $n - 3 \leq |\breve{H}| \leq n - 1$. Since $\breve{H}$ is a cograph on fewer than n vertices, we can apply the inductive hypothesis in cases as follows.

$$\mathrm{mr}(G) = \mathrm{mr}(\breve{H}) + |\ddot{H}| \leq \begin{cases} \left\lfloor \frac{2(n-1)}{3} \right\rfloor = \left\lfloor \frac{2n}{3} - \frac{2}{3} \right\rfloor & \text{if } |\ddot{H}| = 0 \\ \left\lfloor \frac{2(n-2)}{3} \right\rfloor + 1 = \left\lfloor \frac{2n}{3} - \frac{1}{3} \right\rfloor & \text{if } |\ddot{H}| = 1 \\ \left\lfloor \frac{2(n-3)}{3} \right\rfloor + 2 = \left\lfloor \frac{2n}{3} \right\rfloor & \text{if } |\ddot{H}| = 2. \end{cases}$$

Recall that G has maximum minimum rank. Thus $\breve{H}$ must achieve maximum minimum rank since otherwise we could choose a different graph and increase the minimum rank of G.

Thus we actually have the equality $\mathrm{mr}(G) = \left\lfloor \frac{2n}{3} \right\rfloor$ and we see that regardless of the value of n, the maximum minimum rank may always be achieved when H has exactly 2 isolated vertices. Depending on the value of n, the maximum minimum rank may also be achieved when H has fewer than 2 isolated vertices. $\square$

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