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Finding Torsion-free Groups Which Do Not Have the Unique Product Property

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ABSTRACT

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This thesis discusses the Kaplansky zero divisor conjecture. The conjecture states that a group ring of a torsion-free group over a field has no nonzero zero divisors. There are situations for which this conjecture is known to hold, such as linearly orderable groups, unique product groups, solvable groups, and elementary amenable groups.

This paper considers the possibility that the conjecture is false and there is some counterexample in existence. The approach to searching for such a counterexample discussed here is to first find a torsion-free group that has subsets A and B such that AB has no unique product. We do this by exhaustively searching for the subsets A and B with fixed small sizes. When $|A| = 1$ or 2 and $|B|$ is arbitrary we know that AB contains a unique product, but when $|A|$ is larger, not much was previously known. After an example is found we then verify that the sets are contained in a torsion-free group and further investigate whether the group ring yields a nonzero zero divisor.

Together with Dr. Pace P. Nielsen, assistant math professor of Brigham Young University, we created code that was implemented in Magma, a computational algebra system, for the purpose of considering each size of A and B and running through each case. Along the way we check for the possibility of torsion elements and for other conditions that lead to contradictions, such as a decrease in the size of A or B.

Our results are the following: If A and B are sets of the sizes below contained in a torsion-free group, then they must contain a unique product.

$$|A| = 3 \text{ and } |B| \leq 16$$
$$|A| = 4 \text{ and } |B| \leq 12$$
$$|A| = 5 \text{ and } |B| \leq 9$$
$$|A| = 6 \text{ and } |B| \leq 7.$$

We have continued to run cases of larger size and hope to increase the size of B for each size of A.

Additionally, we found a torsion-free group containing sets A and B, both of size 8, where AB has no unique product. Though this group does not yield a counterexample for the Kaplansky zero divisor conjecture, it is the smallest explicit example of a non-unique product group in terms of the size of A and B.

Keywords: Group Rings, Torsion-free Groups, Zero-Divisors, Kaplansky's Zero Divisor Conjecture, Unique Product Groups

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Contents

Chapter 1. Introduction

In 1956, Irving Kaplansky presented twelve important open questions in ring theory. He explains, "I expressed the hope that these problems would help to rekindle interest in the theory of rings." [10] In this paper, we study one of these problems—the zero divisor conjecture—and search systematically for a counterexample. The conjecture states that the group ring of a torsion-free group over a field has no nonzero zero divisors.

1.1 Group Rings: Definition and Examples

To begin we must understand the basic idea of a group ring. In this section we define and explain group rings by presenting a few simple examples and noting characteristics of each.

Throughout the paper all rings are unital, but not necessarily commutative. All ring homomorphisms are unital as well. Note that identities of groups will always be denoted 1.

Definition 1.1. Let R be a ring and G be a group. The **group ring**, denoted $R[G]$, is the set of all finite formal sums of elements of G with coefficients from R.

We see that an element $\alpha \in R[G]$ is uniquely of the form $\alpha = \sum_{g\in G} \alpha_g g$ where $\alpha_g \in R$ and only finitely many α_g are nonzero. To efficiently talk about an element α of a group ring, it is useful to collect the elements of G for which α has nonzero coefficients. Therefore, we provide the following definition.

Definition 1.2. The **support** of an element $\alpha = \sum_{g\in G} \alpha_g g \in R[G]$ is the set

$$\mathrm{supp}(\alpha) = \{g \in G : \alpha_g \neq 0\}.$$

One can easily check that $R[G]$ is a ring with addition and multiplication defined in the following way. Given $\alpha = \sum_{a\in G} r_a a$ and $\beta = \sum_{b\in G} s_b b$ in $R[G]$, then

$$\alpha + \beta = \sum_{a\in G} (r_a + s_a)a$$

$$\alpha\beta = \sum_{g\in G} \left(\sum_{a,b\,:\,ab=g} (r_a s_b) \right) g.$$

We will now present a few examples of group rings. The first example demonstrates basic properties and facts of group rings.

Example 1.3. Recall that the quaternion group is

$$Q_8 = \langle x, y : x^4 = 1, x^2 = y^2, y^{-1}xy = x^{-1} \rangle.$$

Consider the group ring $\mathbb{R}[Q_8]$. An arbitrary element in this group ring is of the form

$$a_0 1 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 y + a_5 y^3 + a_6 xy + a_7 yx$$

with $a_i \in \mathbb{R}$. In multiplying two elements in this group ring we must use care in tracking the position of the group elements. For example $(3x + 7xy)(5y) = 15xy + 35xy^2 = 15xy + 35x^3$, but if we reverse the product, we get $(5y)(3x + 7xy) = 15yx + 35yxy = 15yx + 35y(yx^{-1}) = 15yx + 35x^2x^{-1} = 15yx + 35x$ which is clearly not the same.

The identity in a group ring is the identity of the ring multiplied by the identity of the group. Therefore, $1_R 1_G$ is the identity here.

If S is a subring of R, then given a group G it is easy to see that $S[G]$ is a subring of $R[G]$. In addition, if H is a subgroup of G, then $R[H]$ is a subring of $R[G]$. Therefore, $\mathbb{Z}[Q_8]$ is a subring of $\mathbb{R}[Q_8]$. Also, the center of Q_8 is $Z(Q_8) = \{1, x^2\}$. Hence, $\mathbb{R}[Z(Q_8)]$ is a subring of $\mathbb{R}[Q_8]$.

Lastly, we mention that $\mathbb{R}[Q_8]$ is an 8 dimensional real vector space.

The ring structure of the group ring often depends on some algebraic properties of the underlying group and ring. For instance, Maschke's Theorem tell us that $R[G]$ is semisimple if and only if R is the zero ring or R is a nonzero semisimple ring, G is finite, and $|G| \cdot 1$ is a unit in R, see [12, Theorem 6.1 and Proposition 6.3]. The next example shows Maschke's theorem in action.

Example 1.4. Let G be the group $\langle x : x^3 = 1 \rangle$. An arbitary element of the group ring $\mathbb{C}[G]$ is of the form $r_0 + r_1 x + r_2 x^2$ with $r_i \in \mathbb{C}$.

The complex numbers are a field; hence, $\mathbb{C}$ is semisimple. Further, $|G| = 3$ and $3 \cdot 1 = 3$ is a unit in $\mathbb{C}$. Thus, $\mathbb{C}[G]$ is a semisimple ring by Maschke's Theorem. Let ω be a cube root of 1; then one may easily verify that

$$
\begin{aligned}
e_1 &= \tfrac{1}{3}(1 + x + x^2) \\
e_2 &= \tfrac{1}{3}(1 + \omega x + \omega^2 x^2) \\
e_3 &= \tfrac{1}{3}(1 + \omega^2 x + \omega x^2)
\end{aligned}
$$

are central, orthogonal idempotents that sum to 1. Therefore, $\mathbb{C}[G] = \prod_{i=1}^{3} e_i \mathbb{C}[G] e_i \cong \mathbb{C}^3$.

Also, note that this group ring is commutative. Recall that group ring multiplication is defined so that group elements commute with coefficients. Therefore, since the group G is abelian and $\mathbb{C}$ is commutative, the group ring $\mathbb{C}[G]$ is commutative. More generally, $R[G]$ is commutative if and only if $R = 0$, or $R \neq 0$ and commutative with G abelian.

Example 1.5. Consider the group ring $\mathbb{C}[\mathbb{Z}]$. First, notice that $\mathbb{C}[\mathbb{Z}] \cong \mathbb{C}[x, x^{-1}]$. Because $\mathbb{Z}$ is an infinite cyclic group, it is torsion-free and abelian. By a similar degree argument that we do in Proposition 1.14, $\mathbb{C}[\mathbb{Z}]$ has no nontrivial idempotents. However, in general it is unknown whether a group ring over a torsion-free group yields any idempotents.

It is easy to see that this group ring is an integral domain.

Now that we have a better foundation and grasp on the concept of a group ring, we will study the Kaplansky conjecture in more depth.

1.2 The Kaplansky Conjecture

This section will be dedicated to understanding both Kaplansky's conjecture and some of the work that has been done already to prove and disprove it. The following are basic definitions needed to understand the conjecture.

Definition 1.6. Let G be a group. If for every $g \in G$, g does not have finite order ($g^n \neq 1$ for some $n \in \mathbb{N}$), then G is **torsion-free**.

Definition 1.7. Let G be a group and R be a ring. Let $\alpha \in R[G]$. If $\beta \in R[G] \setminus \{0\}$ exists such that $\alpha\beta = 0$ or $\beta\alpha = 0$, then the element α is called a **zero divisor**.

Understanding the statement of Kaplansky's conjecture is now within reach. This problem was the sixth problem that Kaplansky presented in 1956, and it is still open in spite of the work that has been done.

Conjecture 1.8. *(The Kaplansky Zero Divisor Conjecture) Let G be a torsion-free group, and let $\mathbb{F}[G]$ be the group ring over a field $\mathbb{F}$, then $\mathbb{F}[G]$ has no nonzero zero divisors.*

While initially posing this problem, Kaplansky stated that if a group can be linearly ordered, then the group ring has no nonzero zero divisors. The precise definition of these types of groups is as follows.

Definition 1.9. Let G be a group. The group G is said to be **linearly ordered** if there is a total order "$<$" such that given $a, b \in G$, $a < b$ implies that $ca < cb$ and $ac < bc$ for every $c \in G$. In other words, the order respects left and right multiplication.

The prototypical example of a linearly ordered group is the group $\mathbb{Z}$ under its canonical ordering. Thus, according to Kaplansky's observation, Conjecture 1.8 holds when $G = \mathbb{Z}$. More generally, we restate Kaplansky's observation as follows:

Proposition 1.10. *The Kaplansky conjecture holds for linearly ordered groups.*

It is known that torsion-free abelian groups can be linearly ordered, which yields the following corollary.

Corollary 1.11. *The Kaplansky Conjecture holds for abelian groups.*

Linearly ordered groups are actually part of a more general class of groups called unique product groups. These groups are an integral part of the work we have done in this paper. Therefore, we will define unique products groups below as well as prove that the Kaplansky conjecture holds in this more general setting.

Definition 1.12. Let G be a group and let $A, B \subseteq G$. An element $x \in G$ is said to be a **unique product** of the pair (A, B) if there is exactly one way to write $x = ab$ where $a \in A$ and $b \in B$.

Definition 1.13. A group G is said to be a **unique product group** if for any two nonempty finite subsets A and B of G there exists a unique product for the pair (A, B).

For convenience we sometimes write $A \cdot B$ or just AB when the two sets are clear, to mean the pair (A, B).

Now we will show that Conjecture 1.8 does hold for unique product groups and, in effect, prove Proposition 1.10 and Corollary 1.11.

Proposition 1.14. *The Kaplansky zero divisor conjecture holds for unique product groups.*

Proof. We will show, more generally, that $R[G]$ has no nonzero zero divisors if R is a domain and G is a unique product group.

Let R be a domain and G be a unique product group. Let $R[G]$ denote the group ring of G over R. Let $\alpha, \beta \in R[G] \setminus \{0\}$. Let $A = \operatorname{supp}(\alpha)$ and $B = \operatorname{supp}(\beta)$. Write $\alpha = \sum_{a \in A} r_a a$ and $\beta = \sum_{b \in B} s_b b$. Then

$$\alpha\beta = \sum_{g \in G} \left(\sum_{a \in A,\, b \in B \,:\, ab = g} (r_a s_b) \right) g.$$

Since G is a unique product group, then we know that there exists some $x \in AB$ such that $x = a_0 b_0$ where $a_0 \in A$ and $b_0 \in B$ are uniquely determined. The coefficient of x in $\alpha\beta$ is simply $r_{a_0} s_{b_0}$ and $r_{a_0} s_{b_0} \neq 0$, since $r_{a_0}, s_{b_0} \neq 0$ and R is a domain. Therefore, $\alpha\beta \neq 0$ since it has a nonzero coefficient for $x \in \operatorname{supp}(\alpha\beta)$. $\square$

Proposition 1.15. *Linearly ordered groups are unique product groups.*

Proof. Let A, B be two finite, nonempty subsets of G, a linearly ordered group. Let $|A| = n$ and $A = \{a_1, a_2 \ldots, a_n\}$, ordered so that $a_1 < a_2 < \cdots < a_n$. Let $|B| = m$ and similarly order the elements of B so that $b_1, b_2, \ldots, b_m$ are in order from least to greatest. Let $x = a_n b_m$. Since $a_i b_m < a_n b_m$ for every $i = 1, \ldots, n-1$, we then have $a_i b_j < a_i b_m < a_n b_m$ for every $j = 1, \ldots, m-1$. Thus x is a unique product of the pair (A, B). Since A and B are arbitrary, G is a unique product group. $\square$

Unique product groups are always torsion-free, but it wasn't until 1987 that the converse was proven false by Rips and Segev [16]. They produced a counterexample by showing there exist subsets A and B of a torsion-free group such that AB had no unique product with $|A| = 4$ and $|B|$ very large. This group is complicated and non-constructive. It takes the authors about eight pages to construct the group and one page to prove that it is indeed torsion-free and has no unique product for the given sets. Given the non-constructive nature of this group, it is hard to expand this work to other groups without the unique product property. However, this finding was important because a counterexample to Conjecture 1.8 would be a non-unique product group that is torsion-free. Unfortunately, it is still unknown whether the group they constructed provides a counterexample.

The following year, Promislow [14] proved that the group

$$P = \langle x, y : x^{-1} y^2 x = y^{-2}, y^{-1} x^2 y = x^{-2} \rangle$$

was a non-unique product group. This group had previously appeared in a few places in literature, including in [5] where Burns specifically inquired if this group was a unique product group. Promislow was able to produce an explicit 14 element set $S \subseteq P$ so that S^2 has no unique product. Because the set is explicit, it is easy to check that S has no unique product by calculating the 196 products in S^2. This was another step in the direction of producing a counterexample to Conjecture 1.8. However, it is noted in [6] by Carter that any group ring over Promislow's group, with coefficients from a field, is indeed a domain.

In 2013, Carter [6] was able to expand Promislow's work to create an infinite family of torsion-free, non-unique product groups namely:

$$P_k = \langle x, y : xy^{2^k}x^{-1} = y^{-2^k}, yx^2y^{-1} = x^{-2} \rangle,$$

for each integer $k \geq 1$. Taking $k = 1$ gives Promislow's group P, and for each k, these groups are distinct and do not contain P as a subgroup. In addition, each P_k contains arbitrarily large non-unique product sets. But, unlike Promislow's example, these are not explicit sets. This work shows that there are countably many finitely presented non-unique product groups which are torsion-free (up to isomorphism). These groups also fulfill Conjecture 1.8.

Most recently, in 2015, Steenbock published a paper discussing the Rips and Segev group. He defined a generalized version of the Rips-Segev presentation and constructed new torsion-free groups without the unique product property [17, Theorem 3]. Thus, we can see that these types of groups exist, but many of the examples produced so far have not been explicit. We can also see that none of the examples so far have produced a counterexample to the Kaplansky zero divisor conjecture.

1.3 Other Places the Kaplansky Conjecture Holds

In addition to unique product groups, Kaplansky's conjecture has been proven to hold for other classes of groups. Though not pertinent to our approach, we present a few additional places where the conjecture holds.

First, Lagrange and Rhemtulla [11] were able to show that having a right ordering of G

(meaning $a < b$ implies that $ac < bc$ for every $c \in G$) instead of a stronger two-sided ordering sufficed. Conrad [7] showed that there are indeed groups that can be right-ordered, but not ordered. This showed that Lagrange and Rhemtulla's argument truly was an improvement over Kaplansky's original observation.

In [9, Theorem 1] Formanek shows that when $\mathbb{F}$ is a field and G is a supersolvable group that is torsion-free, then the group ring $\mathbb{F}[G]$ has no nonzero zero divisors. Recall that a group G is supersolvable if there exists some chain of normal subgroups

$$G = G_n \trianglerighteq G_{n-1} \trianglerighteq \cdots \trianglerighteq G_1 \trianglerighteq G_0 = 1, \tag{1.1}$$

where each quotient G_{i+1}/G_i is cyclic. His theorem depends on the work of Lewin [13] "whose work was the first solution of the zero divisor question for groups which are not ordered or close relatives of ordered groups" [9].

Conjecture 1.8 also holds for group rings of polycyclic-by-finite groups over fields of characteristic zero by work of and Farkas and Snider [8]. A polycyclic group is a solvable group where every subgroup is finitely generated. A solvable group is slightly weaker than supersolvable with each G_{i+1}/G_i abelian, but not necessarily cyclic. A polycyclic-by-finite group is a group where there exists a subgroup of finite index that is polycyclic. They developed their proof by reworking some ideas from Brown, in [4], where he showed that $\mathbb{F}[G]$ is a domain when the characteristic of $\mathbb{F}$ is zero and G is an abelian-by-finite torsion-free group.

Elementary amenable groups are another class of groups for which Conjecture 1.8 holds. These groups are "constructed from abelian and finite [groups] by extending and taking homomorphic images and subgroups" [3]. Solvable groups and abelian groups (by the fundamental theorem of finitely generated abelian groups) are both examples of elementary amenable groups.

In [3] Bardakov and Petukhova present a growing list of groups for which the conjecture holds though some of the groups are a bit obscure.

1.4 Related Open Questions

We now finish this chapter by studying a few conjectures concerning group rings closely related to the Kaplansky zero divisor conjecture. These conjectures are of interest for group rings of torsion-free groups over domains.

For the first conjecture we remind the reader of the definition of a nilpotent element in a ring.

Definition 1.16. Let R be a ring. Then $r \in R$ is **nilpotent** if $r^n = 0$ for some $n \in \mathbb{N}$.

The following conjecture relates directly to the zero divisor conjecture.

Conjecture 1.17. *If G is a torsion-free group and $\mathbb{F}[G]$ is the group ring over a field $\mathbb{F}$, then $\mathbb{F}[G]$ has no nonzero nilpotents.*

It is easy to see that if Conjecture 1.8 holds, then Conjecture 1.17 is also true. For if there aren't any nonzero zero divisors in a group ring, then there certainly cannot be any nonzero nilpotents. Conversely, it is also true that if there are no nonzero nilpotents, then there are no nonzero zero divisors. Lam presents a proof in [12, Section 6], but we won't replicate it here. Thus, we see that Conjecture 1.17 is equivalent to Conjecture 1.8.

For the next conjecture we will briefly introduce the concept of Jacobson semisimplicity. We note that there is a lot of theory that can be learned about this property of rings that won't be covered here. We will, however, provide the basic definition and a few other important details. See [12, Section 4] for more details on Jacobson semisimplicity.

Definition 1.18. Given a ring R, recall that the Jacobson radical, $\operatorname{rad}(R)$, is the intersection of the maximal left ideals of R. The ring R is **Jacobson semisimple** or **J-semisimple** if $\operatorname{rad}(R) = 0$.

If $y \in \operatorname{rad}(R)$, then $1 - xy \in U(R)$ for any $x \in R$. This implies that R is J-semisimple if $y = 0$ is the only ring element that makes $1 - xy$ a unit for every x. Though it may not seem intuitive, J-semisimplicity is an important property. It is a property that generalizes semisimple rings when they are not artinian. In [12, Theorem 4.14] we see that a ring R is J-semisimple and artinian if and only if it is semisimple. Below we will give an example and a non-example of J-semisimple rings.

Example 1.19. Consider the ring $\mathbb{Z}$. First, we note that the maximal left ideals in $\mathbb{Z}$ will be maximal ideals since $\mathbb{Z}$ is commutative. Thus, $\text{rad}(\mathbb{Z})$ is the intersection of all the maximal ideals of $\mathbb{Z}$.

The prime ideals of $\mathbb{Z}$ are exactly the zero ideal and $p\mathbb{Z}$ where p is a prime. Recall that every nonzero prime ideal is maximal. Hence, we have $\text{rad}(\mathbb{Z}) = \bigcap_{p\ prime} p\mathbb{Z} = \{0\}$. Therefore, $\mathbb{Z}$ is a J-semisimple ring.

Another thing that we would like to point out is that the Jacobson radical is related to nilpotent ideals. Any nilpotent ideal is contained in the Jacobson radical. Recall that a nilpotent ideal I is any ideal such that $I^n = 0$ for some $n \in \mathbb{N}$. In fact, if R is a left artinian ring, then $\text{rad}(R)$ is the largest nilpotent left ideal and the largest nilpotent right ideal [12, Theorem 4.12].

Lam gives another nice proposition concerning the Jacobson radical in his book as follows: Let I be any ideal of R lying in $\text{rad}(R)$, then $\text{rad}(R/I) = \text{rad}(R)/I$. We will use this fact and the radical's relation to nilpotent ideals in this next example.

Example 1.20. We follow an example presented by Lam in [12, pg. 57] to give a ring that is not J-semisimple. Let $\mathbb{F}$ be a field, and let R be the ring of upper triangular $n \times n$ matrices with entries in $\mathbb{F}$, with $n \geq 2$. Let J be the subset of R consisting of matrices with zeros on the main diagonal. We will show that $J = \text{rad}(R)$.

First, we note that J is indeed an ideal of R. It is nonempty since the zero matrix is in J. It is certainly closed under addition since addition is component-wise. It is also easy to check that it is closed under multiplication from R.

Next, we note that $J^n = 0$, where n is the specific size of the matrices we are working with. This fact can be shown using an induction proof. This means that J is a nilpotent ideal of R.

Since J is a nilpotent ideal, then we know from the discussion preceding this example that $J \subseteq \mathrm{rad}(R)$. Thus, by the proposition from Lam we have that $\mathrm{rad}(R)/J = \mathrm{rad}(R/J)$. If we notice that R/J is isomorphic to the diagonal matrices, then we have that

$$R/J \cong \mathbb{F} \times \cdots \times \mathbb{F}.$$

Thus, R/J is semisimple. We know that semisimplicity implies J-semisimplicity, thus $\mathrm{rad}(R/J) = 0$. Thus, $\mathrm{rad}(R)/J = 0$ and we have that $\mathrm{rad}(R) = J$. Therefore, R is not a J-semisimple ring.

This concludes the discussion on the definition and properties of J-semisimple rings. We can now present the related conjecture.

Conjecture 1.21. *If G is a torsion-free group and $\mathbb{F}[G]$ is the group ring over a field $\mathbb{F}$, then $\mathbb{F}[G]$ is J-semisimple.*

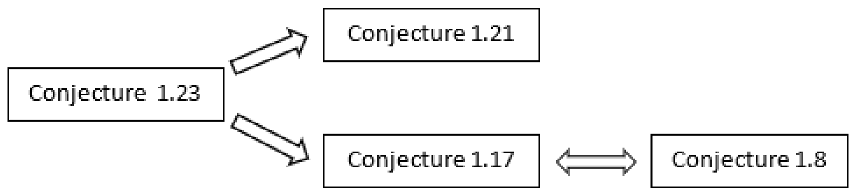
Though not obviously related to Conjecture 1.8, it is an important question in the theory of group rings. The last open question relates to all these conjectures. We will need the definition of a trivial unit in a group ring.

Definition 1.22. Let R be a domain and G be a group. If $\alpha = u \cdot g$ where u is a unit in R and $g \in G$, then x is called a **trivial unit**.

Note that the set of trivial units form a subgroup of $R[G]$. Any other unit in the group ring is called a nontrivial unit.

Conjecture 1.23. *If G is a torsion-free group and $\mathbb{F}[G]$ is the group ring over a field $\mathbb{F}$, then $\mathbb{F}[G]$ contains no nontrivial units.*

The following figure demonstrates the relationship between the three conjectures presented in this section and Conjecture 1.8.



We will prove these two implications below: Conjecture 1.23 $\Rightarrow$ Conjecture 1.17 and Conjecture 1.23 $\Rightarrow$ Conjecture 1.21.

Proposition 1.24. *An affirmative answer to Conjecture 1.23 implies an affirmative answer to Conjecture 1.17. In other words: Let G be a torsion-free group and $\mathbb{F}[G]$ be the group ring over a field $\mathbb{F}$. If $\mathbb{F}[G]$ has no nontrivial units, then $\mathbb{F}[G]$ has no nonzero nilpotents.*

Proof. We follow the proof in [12]. Let R be a domain and G be a group. Suppose contrapositively that there exists a nonzero nilpotent $r \in R[G]$. Then $r^n = 0$ for some $n \in \mathbb{N}$, $n > 1$ and minimal. Then it follows that $\alpha = r^{n-1}$ is nonzero and $\alpha^2 = 0$. Then

$$(1-\alpha)(1+\alpha) = 1-\alpha^2 = 1.$$

Thus $1-\alpha \in U(R[G])$. If this is a trivial unit then $1-\alpha = ag$ for some $a \in U(R), g \in G$. Therefore

$$0 = \alpha^2 = (1-ag)^2 = 1-2ag+a^2g^2.$$

Case (1): $g \neq 1$. Then $1, g$, and g^2 are distinct elements since G is torsion-free. Therefore $2a = 0, a^2 = 0$, and $1 = 0$. The last two inequatlities yield a contradiction because R is a domain. Therefore, case (1) cannot happen.

Case (2): $g = 1$. Since $\alpha^2 = 0$, then $(1-a\cdot 1)^2 = 0$. Hence, $(1-a)^2 = 0$ and thus $1-a = 0$ since R is a domain. In other words, $a = 1$. Then $1-\alpha = 1\cdot 1$ and so $\alpha = 0$, which is a contradiction to our assumption that α is nonzero.

Thus, we see that $1-\alpha \in U(R[G])$ and is not trivial. Therefore $R[G]$ has a nontrivial unit, as desired. $\square$

Proposition 1.25. *An affirmative answer to Conjecture 1.23 implies an affirmative answer to Conjecture 1.21. In other words: Let G be a torsion-free group and $\mathbb{F}[G]$ be the group ring over a field $\mathbb{F}$. If $\mathbb{F}[G]$ has no nontrivial units, then $\mathbb{F}[G]$ is J-semisimple.*

Proof. We again follow the proof given in [12]. Let R be a domain and G be a group. Assume that $R[G]$ has no nontrivial units. Let $\alpha \in \text{rad}(R[G])$. Since α is an element of the Jacobson radical then $1 - \alpha \cdot 1 \in U(R[G])$. This implies that $1 - \alpha = ag$ or $\alpha = 1 - ag$ for some $a \in U(R)$ and $g \in G$ by our assumption.

Case (1): $g \neq 1$. Again, since $\alpha \in \text{rad } R[G]$ we have $1 - \alpha g \in U(R[G])$. Substituting $\alpha = 1 - ag$ from above we have $1-(1-ag)g = 1-g+ag^2 \in U(R[G])$. Since G is torsion-free and $g \neq 1$ then we have that $1, g$, and g^2 are all distinct. This gives a contradiction since $1-g+ag^2$ has support greater than and thus cannot be a unit. So this case cannot happen.

Case (2): $g = 1$. Then $1 - \alpha = a \cdot 1 \Rightarrow \alpha = 1 - a$. This implies that $\alpha \in R$. Fix $h \in G - \{1\}$. Then $a - \alpha h \in U(R[G])$ since $\alpha \in \text{rad}(R[G])$. But then $\alpha = 0$ by our assumption.

Thus, $\text{rad}(R[G]) = \{0\}$ and $R[G]$ is J-semisimple. $\square$

Chapter 2. Our Approach

Sections 1.2 and 1.3 illustrate the work that has been done to attempt to solve the Kaplansky zero divisor conjecture. It has led me to believe that there is a counterexample in existence. This chapter is dedicated to describing the general strategy I have used, in joint work with Dr. Pace P. Nielsen, to construct torsion-free groups which have no unique product, with the hope that these groups might yield a counterexample.

2.1 Brute Force

One of the first questions that we asked was, "Does there exist a torsion-free group with sets A and B of some *small* fixed size where AB has no unique product?" Without loss of generality we may assume that $m = |A| \leq |B| = n$ reversing the roles of A and B as necessary. We started with A and B as small as possible. First, we deal with $|A| = 1$ and $|B|$ arbitrary.

Proposition 2.1. *Let G be a group and let $A, B \subseteq G$ such that $|A| = 1$ and $|B| = n$, where $n \geq 1$ is an integer. Then AB has a unique product.*

Proof. Let $A = \{a\}$ and $B = \{b_1, b_2, \ldots, b_n\}$ where $a, b_1, \ldots, b_n \in G$. We will show all products in AB are unique.

Any product in AB has the form ab_i for some $i \in \{1, \ldots, n\}$. Thus, given two products from AB that are equal, then $ab_i = ab_j$ where $i \neq j$. But multiplying by a^{-1} on the left gives $b_i = b_j$ which cannot be since $i \neq j$. Thus, every product in AB is unique, as desired. $\square$

Next, we will increase the size of A and deal with the case where $|A| = 2$ and $|B|$ is arbitrary using a similar argument, but first we prove a useful lemma.

Lemma 2.2. *Let $A = \{a_1, \ldots, a_m\}$ and $B = \{b_1, \ldots, b_n\}$ be finite subsets of a torsion-free group G. The product set AB has no unique product if and only if the product set $gA \cdot Bh$ also has no unique product for any $g, h \in G$.*

Proof. $(\Rightarrow)$: Let $g, h \in G$. Let A and B be finite subsets of G, a torsion-free group, with $A = \{a_1, \ldots, a_m\}$ and $B = \{b_1, \ldots, b_n\}$. Suppose contrapositively that $gA \cdot Bh$ has a unique product. Then there must exist an $x \in gA \cdot Bh$ such that $x = ga_0b_0h$ for some unique $a_0 \in A$ and $b_0 \in B$. Let $y = g^{-1}xh^{-1} = a_0b_0$. Since a_0 and b_0 are unique then y is a unique product in AB, as desired.

$(\Leftarrow)$: Let $A' = gA$ and $B' = Bh$ from above. Suppose that $A'B'$ has no unique product. Then using the argument above, we know that $g^{-1}A' \cdot B'h^{-1}$ has no unique product. But $g^{-1}A' \cdot B'h^{-1} = AB$. Thus, AB has no unique product, as desired. □

We note that this lemma is useful when choosing $g = a_1^{-1}$ and $h = b_1^{-1}$ so that we have $1 \in a^{-1}A$ and $1 \in Bb^{-1}$. Then, letting $A' = a_1^{-1}A$ and $B' = Bb_1^{-1}$, we have $A' = \{1, a_2', a_3', \ldots, a_m'\}$ and $B' = \{1, b_2', b_3', \ldots, b_n'\}$. These sets are more convenient to work with since we know the identity is one of the group elements in each set and we know by Lemma 2.2 that if $A'B'$ has no unique product, then AB has no unique product. This lemma will aid us in this next proposition as we continue to search for non-unique product groups.

Proposition 2.3. *Let G be a torsion-free group and let $A, B \subseteq G$ such that $|A| = 2$ and $|B| = n$, where $n \geq 2$ is an integer. Then AB has a unique product.*

Proof. Let G be a torsion-free group. Let $A = \{a_1, a_2\}$ and $B = \{b_1, b_2, \ldots, b_n\}$ where $a_1, a_2, b_1, \ldots, b_n \in G$. By Lemma 2.2 we can show that $a_1^{-1}A \cdot Bb_1^{-1}$ has no unique product. Thus, without loss of generality, we may assign $a_1 = 1$ and $b_1 = 1$ without affecting any existing unique products within the sets. Thus, we write $A = \{1, a_2\}$ and $B = \{1, b_2, \ldots, b_n\}$.

Suppose to the contrary that AB does not have a unique product. We will reach a contradiction by using the fact that $a^i \in B$ for all $i \in \mathbb{Z}_{\geq 0}$. We will proceed by induction to prove this fact.

For the base case it is clear that $a^0 = 1 \in B$.

For the inductive step, assume that $a^k \in B$, for some $k \in \mathbb{Z}_{\geq 0}$. Then $a \cdot a^k = a^{k+1} \in AB$. But since AB has no unique product, then there must exist another product in AB that is equal to a^{k+1}. Since the only other element from A is 1, we must have that $1 \cdot x = a^{k+1}$ for some $x \in B$; hence, $x = a^{k+1} \in B$. Thus, $B = \{1, a, a^2, \ldots\}$. As B is a finite set, this implies that a is a torsion element, contradicting the fact that G is a torsion-free group. Therefore, AB must have a unique product. $\square$

We have now proven the more trivial cases where $|A| = 1, 2$; hence, we can move on to the case when $|A| = 3$. It is still an open question if there is a torsion-free group that contains A and B with no unique product for AB when A is this size. We do know, as mentioned in chapter 1, that when $|A| = 4$ and $|B|$ is very large, that there is a non-unique product group.

Now, to motivate some of the techniques we will use in our approach to searching for non-unique product groups, we will explicitly and pedantically deal with the case $|A| = 3$ and $|B| = 3$.

Example 2.4. Let A and B be two nonempty subsets of G, a torsion-free group, with $|A| = 3$ and $|B| = 3$. Let $a_1, a_2, a_3 \in G$ be the three distinct elements of A, and let $b_1, b_2, b_3 \in G$ be the three distinct elements of B. Our goal is to ensure that AB has no unique product. By Lemma 2.2 we can say without loss of generality that $a_1 = b_1 = 1$.

First, we will deal with the product $a_1 b_1 = 1 \cdot 1$. There are eight products which we could match to $a_1 b_1$ so that it isn't unique: (1) $1 \cdot 1 = a_1 b_2$, (2) $1 \cdot 1 = a_1 b_3$, (3) $1 \cdot 1 = a_2 b_1$, (4) $1 \cdot 1 = a_2 b_2$, (5) $1 \cdot 1 = a_2 b_3$, (6) $1 \cdot 1 = a_3 b_1$, (7) $1 \cdot 1 = a_3 b_2$ and (8) $1 \cdot 1 = a_3 b_3$.

Cases (1), (2), (3), and (6) each cannot happen since we have $a_1 = b_1 = 1$. If not, then the sets A or B would not have the right number of distinct elements.

It is not hard to see that for cases (5), (7) and (8) we can simply relabel the indices to match case (4). (We will use this fact a few times to avoid doing more work than is necessary in this proof and for the code we write to model this process). Thus, the only case we will deal with is case (4).

Case (4): This is the first case that does not immediately lead to a contradiction, so suppose that $1 \cdot 1 = a_2 b_2$. Now we will continue to match the other products of AB together. Next consider the product $a_1 b_2 = 1 \cdot b_2$. There are eight products to which we could match this product: (A) $1 \cdot b_2 = a_1 b_1$, (B) $1 \cdot b_2 = a_1 b_3$, (C) $1 \cdot b_2 = a_2 b_1$, (D) $1 \cdot b_2 = a_2 b_2$, (E) $1 \cdot b_2 = a_2 b_3$, (F) $1 \cdot b_2 = a_3 b_1$, (G) $1 \cdot b_2 = a_3 b_2$ and (H) $1 \cdot b_2 = a_3 b_3$.

Again, cases (A), (B), (D), and (G) all cannot happen; otherwise, the sets A and B would not have the right amount of distinct elements. Case (C) is the earliest case for which this does not happen, so assume that $1 \cdot b_2 = a_2 b_1$, or $b_2 = a_2$. Then, substituting in our first relation, we have $1 \cdot 1 = b_2 \cdot b_2$, which means that b_2 is a torsion element, which contradicts our assumption that G is torsion-free. Thus case (C) cannot happen.

Notice that case (F) can be relabeled to be case (C); thus, it also yields a contradiction. Also, case (H) can similarly be relabeled to match case (E). Thus, we move to case (E), the earliest case that doesn't lead to a contradiction.

Case (E): Assume that $1 \cdot b_2 = a_2 b_3$. Again, we proceed by matching other products of AB together. Consider the product $a_2 b_1 = a_2 \cdot 1$. There are eight products to which we could match this product. (a) $a_2 \cdot 1 = a_1 b_1$, (b) $a_2 \cdot 1 = a_1 b_2$, (c) $a_2 \cdot 1 = a_1 b_3$, (d) $a_2 \cdot 1 = a_2 b_2$, (e) $a_2 \cdot 1 = a_2 b_3$, (f) $a_2 \cdot 1 = a_3 b_1$, (g) $a_2 \cdot 1 = a_3 b_2$ and (h) $a_2 \cdot 1 = a_3 b_3$.

Again, cases (a), (d), (e), and (f) cannot happen without reducing the number of distinct elements in A or B. For case (b) we have $a_2 = b_2$, but then a_2 is a torsion element since $a_2 b_2 = 1$. Case (c) can be relabeled to reach a similar contradiction.

Thus the earliest case that does not cause an quick contradiction is case (g). We note here that case (h) can be relabeled to match case (g). Case (g) is the only one that we have to deal with.

Thus far we have accumulated the following relations:

$$(1)\ 1 \cdot 1 = a_2 b_2$$

$$(2)\ 1 \cdot b_2 = a_2 b_3$$

$$(3)\ a_2 \cdot 1 = a_3 b_2$$

From (1) we have $b_2 = a_2^{-1}$. Substituting this into (2), we have $a_2^{-1} = a_2 b_3$; hence, $b_3 = a_2^{-2}$. Lastly, we can use (1) to simplify the third relation; hence, $a_3 = a_2^2$. Now we can write the sets A and B as follows:

$$A = \{1, a_2, a_2^2\} \text{ and } B = \{1, a_2^{-1}, a_2^{-2}\}$$

If AB has no unique product, then $a_2^2 \cdot 1$ would need to equal at least one other product. All other products are of the form a_2^i for $-2 \leq i \leq 1$. Thus, a_2 is a torsion element. But G is torsion-free, so we have a contradiction. Therefore, we can conclude that AB has a unique product.

We have now ruled out some of the smaller cardinalities for A and B. From Example 2.4 we see that it is possible to exhaustively match products until we conclude that there is no torsion-free, non-unique product group that could contain sets of small fixed sizes. In working to find a counterexample for Conjecture 1.8, this was our strategy: first, to create a non-unique product group, ensure that it is torsion-free, and finally to see if the corresponding group ring contains zero divisors or not. It might already be apparent that doing these problems by hand becomes messy and tedious as we continue to increase the size of A and B. Therefore, we turned to the help of computers to help us run through all the possibilities quickly and accurately.

2.2 Creating the Code

This section will be dedicated to describing the code that was created and then implemented in the computational algebra system, Magma V2.23-4. The approach is to exhaustively perform a search for subsets A and B of fixed small sizes that have no unique product and are contained in a torsion-free group. We will also introduce a few simplifications and strategies that we used to increase the efficiency of the program.

First, assume A and B are nonempty finite subsets of a torsion-free group G such that AB has no unique product. Let $A = \{a_1, a_2, \dots, a_m\}$ and $B = \{b_1, b_2, \dots, b_n\}$ with $|A| = m$ and $|B| = n$. Because G is a group that contains these sets, we don't necessarily care too much how the group looks, as long as it continues to be torsion-free. Thus, we may as well assume that G is generated by the elements of A and B, as this has no effect on the sizes of A and B.

In our code we use this fact and start by letting G be the free group generated the $m+n$ elements of $A \cup B$. Then, we begin setting products equal to each other to make a set of relations forcing AB to have no unique product. Next, we create the quotient group of G mod those relations. Finally, we can check the quotient group for torsion.

At the cost of (temporarily) losing the torsion-free hypothesis on G, we can also assume G is finitely presented. We can even take only the relations that force AB to have no unique product for the presentation. One of the main benefits for this is that the number of groups that are possibilities for containing A and B is now limited to a finite number. In fact, if we set $[m] := \{1, 2, \ldots, m\}$ and $[n] := \{1, 2, \ldots, n\}$ and let X be some subset, $X \subseteq ([m] \times [n])^2$, then G can be identified with

$$G_X := \langle a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n : a_i b_j = a_k b_l \text{ for all } (i, j, k, l) \in X \rangle$$

and the set $A \cup B$ is the set of generators for G_X.

Now that we have the jist of what the program does, there are some reductions we can perform on the space of possibilities to increase the efficiency of the program.

First, we can use Lemma 2.2, as seen in Proposition 2.3 and Example 2.4, and without loss of generality put $a_1 = 1$ and $b_1 = 1$. It is not hard to see that in this case we will always include the relation $1 = a_1 b_1 = a_2 b_2$ in the set of relations that allow AB to have no unique product (see Example 2.4).

Second, we can rule out quadruples from X that would result in decreasing the size of A or B. As seen in Example 2.4, if the quadruple $(2, 3, 2, 4)$ is in X, then we have the relation $a_2 b_3 = a_2 b_4$. Multiplying on the left by a_2^{-1} gives $b_3 = b_4$ which decreases the size of B. Thus, we simply discount any set X containing an analogous element as a possibility.

Third, it can happen that two different sets $X \neq Y$ give precisely the same group $G_X =$

G_Y. In such cases we treat X and Y as giving us the same structure on A and B. The process of determining if these two groups are the same is hard as the word problem in finitely presented groups is undecidable, so we don't directly work with G_X and G_Y here, but rather with large partial rewrite systems. In practice we limited the rewrite system to about 2^9 relations.

Fourth, we saw in Example 2.4 that some cases are equivalent up to relabeling the elements of A or B. Because of this we can increase the efficiency of our program by eliminating cases that are the same when we relabel elements. We can describe when two cases are the same using permutations. Each permutation $\sigma \times \tau \in S_m \times S_n$ of the set $[m]$ and $[n]$ induces an isomorphism $G_X \to G_Y$ where $a_i \mapsto a_{\sigma(i)}$ and $b_j \mapsto b_{\tau(j)}$ and

$$Y := \{(\sigma(i), \tau(j), \sigma(k), \tau(l)) : (i, j, k, l) \in X\}.$$

When G_X and G_Y are isomorphic in this way, then the case when we have Y as the subset of $([m] \times [n])^2$ is the same up to relabeling elements. Note that Lemma 2.2 is compatible when using these permutations.

Fifth, we can eliminate any set X that contain any quadruples that lead to torsion elements in the group. The Adyan-Rabin Theorem [2, 15] tells us that there is no algorithm to determine whether or not a finitely presented group is torsion-free. This adds to the difficulty of eliminating torsion inside the group. Even if we do find torsion, after factoring that torsion out, there may be new elements that become torsion as the structure of G easily changes. So instead we look for a specific type of torsion that decreases the size of A or B. In other words we look for elements of the form $a_i a_j^{-1}$ or $b_i b_j^{-1}$ and make sure they do not have small order (under a partial reduction system). If a candidate group arises which passes our initial inspection, we then use *ad hoc* methods to guarantee it is ultimately torsion-free.

In practice, we use these reductions to restrict the search space of subsets $X \subseteq ([m] \times [n])^2$. First, we run through each one element subset of $([m] \times [n])^2$ and discard those sets that fail the torsion test or are sent to a case we considered previously, etc. For each of the remaining one element sets, we consider all possible two element extensions, repeat the tests, and continue this extension process as needed.

The search was initially performed on a single personal computer with multiple cores which was sufficient resources to deal with smaller cases. Ultimately, we utilized the BYU Mary Lou supercomputer for larger cases. The code can be found in Appendix A.

The algorithm that we produced in Appendix A was used for over a year. It was successful in finding a group that contained sets with no unique product. This chapter will detail the results we have obtained and the vision we have for future work on this topic. As we continue to run the algorithm, we hope that that it will produce other examples and that we might even be able to find a relationship in these types of examples. Eventually, it may lead to a possible counterexample for Conjecture 1.8 as we had originally desired.

3.1 Main Results

We will now present one of the main results we obtained through the code we produced in Magma. This example was initially found by considering the case when the set B is restricted to be the same as the set A, though in general this is not our approach.

Theorem 3.1. *There is a torsion-free group that contains subsets A and B with $|A| = |B| = 8$, such that AB has no unique product.*

Proof. Let

$$Z := \{(1,1,2,2),(1,2,2,3),(1,3,3,2),(1,4,2,5),(1,6,2,7),(1,8,4,2),(2,1,5,5),$$
$$(2,4,5,8),(2,6,5,3),(2,8,5,7),(3,3,6,6),(4,1,7,5),(4,3,8,6),(4,1,8,2)\}.$$

and define

$$G_Z = \langle a_1,\dots,a_8,b_1,\dots,b_8 : a_1 = 1, b_1 = 1, \text{ and } a_ib_j = a_kb_l \text{ when } (i,j,k,l) \in Z\rangle.$$

Now let $A = \{a_1, a_2, \dots, a_8\}$ and $B = \{b_1, b_2, \dots, b_8\}$.

If we define $x := a_2$ and $y := b_4$, then it is not hard to write each element of A and B in terms of x and y. For instance, we can use the first quadruple in Z, $(1,1,2,2)$ to see that

$a_1b_1 = a_2b_2$. This simplifies to $1 = xb_2$, and hence $b_2 = x^{-1}$. Further, the $(1, 2, 2, 3)$ gives $a_1b_2 = a_2b_3$ or $x^{-1} = xb_3$; hence, $b_3 = x^{-2}$. Taking each quadruple in Z, we can solve for all the elements of A and B; however, the quadruples $(2, 8, 5, 7)$ and $(5, 1, 8, 2)$ are not used in that process and thus impose two further relations on our group. Therefore, the group G_Z is generated by $x := a_2$ and $y := b_4$ and is given by the presentation

$$G_Z = \langle x, y : x^{-1}y^2xy^2 = x^{-2}yx^{-2}y^3 = 1\rangle.$$

We will show that AB has no unique product, that both A and B each have eight distinct elements, and that G_Z is torsion-free.

First, we will show that there AB has no unique product. By observing the set Z of tuples, we can already see that 28 of the 64 products in AB are equal to another product in AB. For instance, $(1, 1, 2, 2)$ gives $a_1b_1 = a_2b_2$, and $(1, 2, 2, 3)$ gives $a_1b_2 = a_2b_3$. For the remaining 36 products it is slightly impractical to calculate by hand. Running the following code in Magma computes enough equal products in AB to verify that it does indeed have no unique product.

```
tuples:=[[1,1,2,2],[1,2,2,3],[1,3,3,2],[1,4,2,5],[1,6,2,7],
         [1,8,4,2],[2,1,5,5],[2,4,5,8],[2,6,5,3],[2,8,5,7],
         [3,3,6,6],[4,1,7,5],[4,3,8,6],[5,1,8,2]];
F:=FreeGroup(16);
rels:=[F.tuples[i,1]*F.(tuples[i,2]+8)=
      F.tuples[i,3]*F.(tuples[i,4]+8):i in [1..#tuples]];
G:=quo<F|rels>;
r:=RWSGroup(G:MaxRelations:=2^10);
Zmax:=[[i,j,k,l]: i in [1..8],j in [1..8],k in [1..8],
      l in [1..8]|r.i*r.(j+8) eq r.k*r.(l+8)];
Zmax;
```

We note that the first eight free group generators act as the elements of A and the second eight are the elements of B.

Next, we will use Magma as a tool to show that G_Z is torsion-free. We will follow an idea of Derek Holt, given in response to a question at MathOverflow. The group G_Z can be entered into Magma by running the following code (note that we use the second finite presentation given above, but they are exactly the same group).

```
F<a,b>:=FreeGroup(2);
G<x,y>:=quo<F|a^-1*b^2*a*b^2,a^-2*b*a^-2*b^3>;
```

Using Magma we search for a subgroup of G_Z of small index and with an especially easy rewrite system. It yielded the following subgroup which met the qualifications we required. Let H be the subgroup generated by the three elements $h_1 := yxy^{-1}x^{-1}$, $h_2 := y^2$, and $h_3 := x^4$. The group H has index 8 in G_Z, it is a normal subgroup, and it is given by the finite presentation

$$H = \langle h_1, h_2, h_3 : h_2 \text{ is central, and } h_3h_1 = h_1h_3h_2^8\rangle.$$

These facts can be verified by running the Todd-Coxeter procedure to create a coset table for G_Z/H, as can be done by executing the code

```
H:=sub<G|y^-1*x^-1*y*x,y^2,x^4>;
Index(G,H);
IsNormal(G,H);
Rewrite(G,H);
```

We will show that the subgroup H is torsion-free. First, we claim that any element of H can be written in the form $(h_1)^a(h_2)^b(h_3)^c$ for $a, b, c \in \mathbb{Z}$. We can move any h_1's past h_2 since h_2 is central, and we can move h_1 past h_3 at the cost of adding 8 extra h_2's, (see the presentation for H above). Now, when we take that general element to the power of $n \in N$, we get $((h_1)^a(h_2)^b(h_3)^c)^n$. This could be a very long word, but again all h_1's can be moved to be the first elements in the word and h_3's can be moved to the last elements in the word (adding more h_2's when necessary). Therefore, $((h_1)^a(h_2)^b(h_3)^c)^n = (h_1)^{an}(h_2)^{bn+8m}(h_3)^{cn}$ (with $m = 0$ if $a = 0$), which is not the identity unless $a = b = c = 0$ to begin with. Now, we would like to note that if G_Z were to have any torsion, it would contain an element of order 2 since the index of H in G_Z is $8 = 2^3$.

Let E be the largest elementary abelian quotient of H relative to the prime $p = 2$, and let $\varphi : H \to E$ be the natural surjective homomorphism. If we let $K := \ker(\varphi)$, then the index of K in H is 8. Let T_1 be a transversal for the cosets G_Z/H, and assume $1 \in T_1$. Similarly, let T_2 be a transversal for the cosets H/K. If G_Z were to have an element of order 2, then there would exist some $t_1 \in (T_1 - \{1\})$ and $t_2 \in T_2$ such that $(t_1t_2)^2 \in K$. We can check directly that this is not the case by running the following code:

```
E, phi:=ElementaryAbelianQuotient(H,2);
K:=Kernel(phi);
Index(H,K);
T1:=Transversal(G,H);
T2:=Transversal(H,K);
exists{t1 : t1 in T1 | t1 ne Id(G) and
   exists{t2 : t2 in T2 | (t1*t2)^2 in K}};
```

Thus, there is no $(t_1t_2)^2 \in K$ and, hence, no element with order 2. Therefore, we can conclude that G_Z is indeed torsion-free.

Lastly, we will show that A and B each have eight distinct elements. We can use the fact that $a_2 = x$ and $b_4 = y$ are generators and use the relations represented by the quadruples from Z (or even quicker by the set Z_{max}) to solve for the elements of A and B as mentioned earlier. This process gives:

$$\begin{array}{llll} a_1 = 1 & a_5 = xy^{-1}x & b_1 = 1 & b_5 = x^{-1}y \\ a_2 = x & a_6 = x^{-2}y & b_2 = x^{-1} & b_6 = y^{-1}x^{-1} \\ a_3 = x^{-1} & a_7 = x^{-2}yx^{-1} & b_3 = x^{-2} & b_7 = x^{-1}y^{-1}x^{-1} \end{array}$$

$$a_4 = x^{-1}y^2x \quad a_8 = x^{-1}y^3 \quad b_4 = y \quad b_8 = x^{-1}z^2.$$

Using the coset enumeration given previously, we can check directly that $a_ia_j^{-1} \notin K$ and that $b_ib_j^{-1} \notin K$ for $i \neq j$. Thus, $|A| = 8$ and $|B| = 8$, as desired. $\square$

We succeeded in finding a torsion-free group with subsets A and B where AB has no unique product and $|A| = |B| = 8$! The motivation for searching for such a group was to find possible counterexamples for Conjecture 1.8. Unfortunately, this group is not a counterexample for the Kaplansky zero divisor conjecture. We will prove this now.

Theorem 3.2. *If G_Z is the group from Theorem 3.1 and R is a domain, then the group ring $R[G_Z]$ has no nonzero zero divisors.*

Proof. Let G_Z be the group defined in Theorem 3.1, and let R be a domain. Assume by way of contradiction that there exists nonzero $\alpha, \beta \in R[G_Z]$ such that $\alpha\beta = 0$. Let $\alpha = \sum_{i=1}^{8} r_ia_i$, $\beta = \sum_{i=1}^{8} s_ib_i \in R[G_Z]$ with $r_i, s_i \in R - \{0\}$ for each i. Further, let A be the support of α and B be the support of β. Looking at the supports of elements in the product $\alpha\beta = 0$ and combining terms that have equal products in AB, we have the following thirty-one relations:

$$\begin{array}{llll}
r_1s_1 = -r_2s_2, & r_2s_1 = -r_5s_5, & r_3s_1 = -r_1s_2 - r_2s_3, & r_4s_1 = -r_7s_5, \\
r_5s_1 = -r_8s_2, & r_6s_1 = -r_3s_5, & r_7s_1 = -r_6s_2, & r_8s_1 = -r_4s_5, \\
r_3s_2 = -r_1s_3, & r_4s_2 = -r_1s_8, & r_5s_2 = -r_8s_3, & r_7s_2 = -r_6s_3, \\
r_3s_3 = -r_6s_6, & r_4s_3 = -r_8s_6, & r_5s_3 = -r_2s_6, & r_7s_3 = -r_4s_6, \\
r_1s_4 = -r_2s_5, & r_2s_4 = -r_5s_8, & r_3s_4 = -r_1s_5, & r_4s_4 = -r_7s_8, \\
r_5s_4 = -r_8s_5, & r_6s_4 = -r_3s_8, & r_7s_4 = -r_6s_5, & r_8s_4 = -r_4s_8 - r_7s_7, \\
r_1s_6 = -r_2s_7, & r_3s_6 = -r_1s_7, & r_5s_6 = -r_8s_7, & r_7s_6 = -r_6s_7, \\
r_3s_7 = -r_6s_8, & r_4s_7 = -r_8s_8, & r_5s_7 = -r_2s_8. &
\end{array}$$

Throughout this argument, when we write a relation in R, we treat the monomial on the left side of the equality as the term that needs to be reduced, and we replace it with the quantity on the right side of the equality. For example, any monomial containing the word r_5s_7, we replace that string with $-r_2s_8$. Also, we can apply the augmentation map ε (which sends all elements of G_Z to 1 and is the identity on R) to both sides of the equation $\alpha\beta = 0$ which forces the equality

$$\left(\sum_{i=1}^{8} r_i\right)\left(\sum_{i=1}^{8} s_i\right) = 0.$$

As R is a domain, one of these sums must be equal to zero. Because both cases are similar, it is sufficient to consider when $\sum_{i=1}^{8} r_i = 0$. Solving for r_8 gives us

$$r_8 = -\sum_{i=1}^{7} r_i.$$

After using this value of r_8 in the relation $r_8s_4 = -r_4s_8 - r_7s_7$ and we get

$$-r_1s_4 - r_2s_4 - r_3s_4 - r_4s_4 - r_5s_4 - r_6s_4 - r_7s_4 = -r_4s_8 - r_7s_7.$$

Reduce using the list of reductions and we obtain

$$r_2s_5 + r_5s_8 + r_1s_5 + r_7s_8 + r_8s_5 + r_3s_8 + r_6s_5 = -r_4s_8 - r_7s_7$$

Now use $r_8 = -\sum_{i=1}^{7} r_i$ again and simplify to obtain

$$r_3s_5 = r_3s_8 + r_4s_8 + r_5s_8 + r_7s_8 - r_4s_5 - r_5s_5 - r_7s_5 + r_7s_7.$$

A short computation using the reductions given so far leads to the zero product

$$(r_1 + r_3 + r_5 + r_7)(s_1 - s_4) = 0.$$

Because R is a domain, we have the following two cases.

Case (1): Assume $s_1 = s_4$. We can simplify the relation $r_1 s_4 = -r_2 s_5$ to be $r_1 s_1 = -r_2 s_5$. But we already have the relation $r_1 s_1 = -r_2 s_2$. Therefore we can set the two relations equal and gain that $r_2(s_2 - s_5) = 0$. As $r_2 \neq 0$, we get $s_2 = s_5$. But now the relation $r_3 s_1 = -r_1 s_2 - r_2 s_3$ gives $r_2 s_3 = -r_3 s_1 - r_1 s_2$ or $r_2 s_3 = -r_3 s_4 - r_1 s_5$. Using the relation $r_3 s_4 = -r_1 s_5$, we get that $r_2 s_3 = 0$, which is a contradiction.

Case (2): Assume $r_1 = -r_3 - r_5 - r_7$. Using our assumption in the relation $r_1 s_6 = -r_2 s_7$, we get $-r_3 s_6 - r_5 s_6 - r_7 s_6 = -r_2 s_7$. Using the list of reductions above, we can simplify this to be $r_1 s_7 + r_8 s_7 + r_6 s_7 + r_2 s_7 = 0$. Now use $r_8 = -\sum_{i=1}^{7} r_i$ and simplify to get the following equation:

$$-r_3 s_7 - r_4 s_7 - r_5 s_7 - r_7 s_7 = 0.$$

The list of reductions then gives

$$r_6 s_8 + r_8 s_8 + r_2 s_8 - r_7 s_7 = 0.$$

Again, use $r_8 = -\sum_{i=1}^{7} r_i$ and simplify to get the relation $r_7 s_7 = -r_4 s_8$. But replacing $r_7 s_7$ in the relation $r_8 s_4 = -r_4 s_8 - r_7 s_7$ gives $r_8 s_4 = 0$, which is a contradiction. This finishes the proof. $\square$

For our last theorem we summarize all the sizes for A and B that we have checked through running our code on Magma.

Theorem 3.3. *Let A and B be subsets of a torsion-free group such that $|A| \leq |B|$. Assuming AB has no unique product, the follwing hold:*

- *If $|A| = 3$, then $|B| \geq 16$.*
- *If $|A| = 4$, then $|B| \geq 12$.*
- *If $|A| = 5$, then $|B| \geq 9$.*
- *If $|A| = 6$, then $|B| \geq 7$.*

Refer to Appendix A for the code and Section 2.2 for the outline of the computations involved in proving the inequalities in Theorem 3.3. By the usual leading term argument, the numbers in Theorem 3.3 also act as lower bounds on the sizes of supports of zero divisors in a group ring $R[G]$, with G torsion-free and R a domain. In particular, if $\alpha, \beta \in R[G]$ with $\alpha\beta = 0$ and $|\text{supp}(\alpha)| = 3$, then $|\text{supp}(\beta)| \geq 16$. This not only improves the bound $|\text{supp}(\beta)| \geq 10$ given in a paper by Abdollahi and Taheri [1], but it applies to more general coefficient rings. They do have the bound $|\text{supp}(\beta)| \geq 20$, but only for the special case when $R = \mathbb{F}_2$.

This concludes the results we have obtained so far in our work to find torsion-free groups with sets having no unique product.

3.2 Final Thoughts

Now that I've shared the main results we have obtained through our efforts on this problem, I will now explain some of the hopes we have for future results, work that we plan to do, and also how our work could be used in other ways.

So far, the code in Appendix A has led to one example. We hope that the code will lead to many other examples of torsion-free groups containing sets with no unique product of *small* size. In particular, finding a such a group for $|A| = 3$ would be profitable. Similarly, finding a group for $|A| = 4$ is profitable as it would give effective bounds on the size of the example produced by Rips and Segev. We plan to continue to run the code for small sizes of A and increase the size of B as we rule out smaller cases. At the very least we hope to improve Theorem 3.3 by increasing the bounds for the set B.

Another thing that we hope to discover as we produce more of these groups is whether there are patterns or relationships between the groups or the sizes of A and B. Perhaps the sizes of A and B will always have a relation, or there could possibly be a lower bound for $|A| + |B|$.

The main thing that we hope in running this algorithm, as mentioned throughout this paper, is that we find a counterexample to Kaplansky's zero divisor conjecture. Though we have yet to succeed, I believe it is possible. This would be an amazing result as many mathematicians have worked on this problems for many years.

As the sizes of the sets A and B are increased, it only makes sense that the time spent running the code becomes longer. The time for each case has been relatively short up to this point, but some of the larger cases we predict could possibly take many months and perhaps even years. The code we produced has already seen many revisions for the purpose of increasing the speed of the algorithm. However, there may still be work for us to do so that the larger cases are plausible for us to complete.

Though we have approached finding these groups with motivation from Conjecture 1.8, there may be other reasons that our results may be beneficial. For instance, if the Kaplansky zero divisor conjecture is false, then proving Conjecture 1.23 is false may be easier. Therefore, our groups could be considered as candidates for counterexamples for the other conjectures presented in this paper.

We also hope that our work may be beneficial to any others working with non-unique product groups and other related open questions. Perhaps the fact that we have an explicit small example of a non-unique product group will aid others in their research.

In conclusion, there is still much to be done in learning about torsion-free groups that contain sets with no unique product with the Kaplansky zero divisor conjecture as the main motivation. There is also much to be gained by searching for more examples similar to the ones we were able to produce in this paper.

Appendix A. The Code

The code below uses $|A| = M = 3$ and $|B| = N = 7$ though these values can be changed as we run over every possibility for the sizes of A and B.

```
ShiftingPairs:=[[[1,1],[2,2]]];

M:=3;

N:=7;

counter:=1;

PairSeq:=[[i,j]:j in [1..N],i in [1..M]];

staging:=0;

while ShiftingPairs ne [] do

   if staging eq 0 then

      if (ShiftingPairs[#ShiftingPairs,1,1] eq ShiftingPairs[#ShiftingPairs,2,1])

          or (ShiftingPairs[#ShiftingPairs,1,2] eq

          ShiftingPairs[#ShiftingPairs,2,2]) then

         staging:=1;

      else

         Elems1:={ShiftingPairs[i,1,1]:i in [1..#ShiftingPairs]} join

             {ShiftingPairs[i,2,1]:i in [1..#ShiftingPairs]};

         mm1:=Max(Elems1);

         if mm1 gt #Elems1 then

            staging:=1;

         else

            Elems2:={ShiftingPairs[i,1,2]:i in [1..#ShiftingPairs]} join

                {ShiftingPairs[i,2,2]:i in [1..#ShiftingPairs]};

            mm2:=Max(Elems2);

            if mm2 gt #Elems2 then

               staging:=1;
```

```
        end if;
      end if;
    end if;
  end if;
  if staging eq 0 then
    f:=FreeGroup(M+N);
    rels:=Append(Append([f.ShiftingPairs[i,1,1]*f.(ShiftingPairs[i,1,2]+M)=
      f.ShiftingPairs[i,2,1]*f.(ShiftingPairs[i,2,2]+M):i in
      [1..#ShiftingPairs]],f.1=f.0),f.(M+1)=f.0);
    g:=quo<f|rels>;
    r:=RWSGroup(g:MaxRelations:=2^9,Warning:=false);
    if #{[i,j]:i in [1..M],j in [1..M]|i lt j and r.i eq r.j} ne 0 then
      staging:=1;
    else
      if #{[i,j]:i in [1..N],j in [1..N]|i lt j and
        r.(M+i) eq r.(M+j)} ne 0 then
          staging:=1;
      end if;
    end if;
  end if;
  i:=1;
  while staging eq 0 and i le 20 do
    j:=1;
    while staging eq 0 and j le M-1 do
      k:=j+1;
      while staging eq 0 and k le M do
        if (r.j*r.k^-1)^i eq r.1^0 then
          staging:=1;
        end if;
```

```
        k:=k+1;
      end while;
      j:=j+1;
   end while;
   i:=i+1;
end while;
i:=1;
while staging eq 0 and i le 20 do
   j:=M+1;
   while staging eq 0 and j le M+N-1 do
      k:=j+1;
      while staging eq 0 and k le M+N do
         if (r.j*r.k^-1)^i eq r.1^0 then
            staging:=1;
         end if;
         k:=k+1;
      end while;
      j:=j+1;
   end while;
   i:=i+1;
end while;
if staging eq 0 then
   SPC:=1;
   PartialPerms:=[[]];
   equalPairs:=[[[i,j],[k,l]]: l in [1..mm2], k in [1..mm1], j in [1..mm2], i
      in [1..mm1]|([[i,j],[k,l]] in ShiftingPairs) or ([[k,l],[i,j]] in
      ShiftingPairs) or (r.i*r.(M+j) eq r.k*r.(M+l) and i ne k)];
end if;
while staging eq 0 and SPC le #ShiftingPairs do
```

```
PPC:=1;
NewSeqOfPerms:=[];
while staging eq 0 and PPC le #PartialPerms do
   EPC:=1;
   while staging eq 0 and EPC le #equalPairs do
      ActivePerm:=PartialPerms[PPC];
      ActiveEqualPair:=equalPairs[EPC];
      canExtend:=1;
      if [ActiveEqualPair[1,1],ShiftingPairs[SPC,1,1]] notin ActivePerm then
         if ActiveEqualPair[1,1] in {ActivePerm[i,1]: i in [1..#ActivePerm]}
             or ShiftingPairs[SPC,1,1] in {ActivePerm[i,2]: i in
             [1..#ActivePerm]} then
            canExtend:=0;
         else
            ActivePerm:=Append(ActivePerm,[ActiveEqualPair[1,1],
                  ShiftingPairs[SPC,1,1]]);
         end if;
      end if;
      if canExtend eq 1 and[ActiveEqualPair[1,2]+M,ShiftingPairs[SPC,1,2]+M]
          notin ActivePerm then
         if ActiveEqualPair[1,2]+M in {ActivePerm[i,1] : i in
             [1..#ActivePerm]} or ShiftingPairs[SPC,1,2]+M in
             {ActivePerm[i,2] : i in [1..#ActivePerm]} then
            canExtend:=0;
         else
            ActivePerm:=Append(ActivePerm,[ActiveEqualPair[1,2]+M,
                  ShiftingPairs[SPC,1,2]+M]);
         end if;
      end if;
```

```
if canExtend eq 1 and [ActiveEqualPair[2,1],ShiftingPairs[SPC,2,1]]
    notin ActivePerm then
   if ActiveEqualPair[2,1] in {ActivePerm[i,1] : i in
       [1..#ActivePerm]} then
      canExtend:=0;
      if #({[ActiveEqualPair[2,1],i] : i in
          [1..ShiftingPairs[SPC,2,1]-1]} meet {ActivePerm[i] : i in
          [1..#ActivePerm]}) ge 1 then
         staging:=1;
      end if;
   else
      if #({i : i in [1..ShiftingPairs[SPC,2,1]-1]} meet
          {ActivePerm[i,2] : i in [1..#ActivePerm]}) lt
          ShiftingPairs[SPC,2,1]-1 then
         canExtend:=0;
         staging:=1;
      else
         if #({ShiftingPairs[SPC,2,1]} meet {ActivePerm[i,2] : i in
             [1..#ActivePerm]}) ge 1 then
            canExtend:=0;
         else
            ActivePerm:=Append(ActivePerm,[ActiveEqualPair[2,1],
                    ShiftingPairs[SPC,2,1]]);
         end if;
      end if;
   end if;
end if;
if canExtend eq 1 and
    [ActiveEqualPair[2,2]+M,ShiftingPairs[SPC,2,2]+M] notin ActivePerm
```

```
 then
if ActiveEqualPair[2,2]+M in {ActivePerm[i,1] : i in
    [1..#ActivePerm]} then
  canExtend:=0;
  if #({[ActiveEqualPair[2,2]+M,i] : i in
      [M+1..ShiftingPairs[SPC,2,2]+M-1]} meet {ActivePerm[i] : i in
      [1..#ActivePerm]}) ge 1 then
    staging:=1;
  end if;
else
  if #({i : i in [M+1..ShiftingPairs[SPC,2,2]+M-1]} meet
      {ActivePerm[i,2] : i in [1..#ActivePerm]}) lt
      ShiftingPairs[SPC,2,2]-1 then
    canExtend:=0;
    staging:=1;
  else
    if #({ShiftingPairs[SPC,2,2]+M} meet {ActivePerm[i,2] : i in
        [1..#ActivePerm]}) ge 1 then
      canExtend:=0;
    else
      ActivePerm:=Append(ActivePerm,[ActiveEqualPair[2,2]+M,
              ShiftingPairs[SPC,2,2]+M]);
    end if;
  end if;
end if;
end if;
if canExtend eq 1 then
  NewSeqOfPerms:=Append(NewSeqOfPerms,ActivePerm);
end if;
```

```
      EPC:=EPC+1;
    end while;
    PPC:=PPC+1;
  end while;
  PartialPerms:=NewSeqOfPerms;
  SPC:=SPC+1;
end while;
if staging eq 0 then
  staging:=2;
end if;
if staging eq 1 then
  while ShiftingPairs ne [] and ((ShiftingPairs[#ShiftingPairs,2] eq [M,N])
     or (ShiftingPairs[#ShiftingPairs,1] eq [M,N] and
     ShiftingPairs[#ShiftingPairs,2] eq [M,N-1])) do
      ShiftingPairs:=Prune(ShiftingPairs);
  end while;
  if ShiftingPairs ne [] then
    NewPos:=Position(PairSeq,ShiftingPairs[#ShiftingPairs,2])+1;
    if PairSeq[NewPos] eq ShiftingPairs[#ShiftingPairs,1] then
      NewPos:=NewPos+1;
    end if;
    ShiftingPairs:=Append(Prune(ShiftingPairs),
      [ShiftingPairs[#ShiftingPairs,1],PairSeq[NewPos]]);
  end if;
  staging:=0;
end if;
if staging eq 2 then
  Pos:=Position(PairSeq,ShiftingPairs[#ShiftingPairs,1])+1;
end if;
```

```
while staging eq 2 do
   if Pos gt M*N then
      staging:=3;
   else
      PairToCheck:=PairSeq[Pos];
      if #{i:i in [1..#ShiftingPairs]|PairToCheck eq ShiftingPairs[i,2]} ne 0
          then
         Pos:=Pos+1;
      else
         staging:=0;
         ShiftingPairs:=Append(ShiftingPairs,[PairSeq[Pos],[1,1]]);
      end if;
   end if;
end while;
i:=21;
while staging eq 3 and i le 40 do
   j:=1;
   while staging eq 3 and j le M-1 do
      k:=j+1;
      while staging eq 3 and k le M do
         if (r.j*r.k^-1)^i eq r.1^0 then
            staging:=1;
         end if;
         k:=k+1;
      end while;
      j:=j+1;
   end while;
   i:=i+1;
end while;
```

```
i:=21;
while staging eq 3 and i le 40 do
   j:=M+1;
   while staging eq 3 and j le M+N-1 do
      k:=j+1;
      while staging eq 3 and k le M+N do
         if (r.j*r.k^-1)^i eq r.1^0 then
            staging:=1;
         end if;
         k:=k+1;
      end while;
      j:=j+1;
   end while;
   i:=i+1;
end while;
if staging eq 3 then
   r:=RWSGroup(g);
   i:=1;
   while staging eq 3 and i le 100 do
      j:=1;
      while staging eq 3 and j le M-1 do
         k:=j+1;
         while staging eq 3 and k le M do
            if (r.j*r.k^-1)^i eq r.1^0 then
               staging:=1;
            end if;
            k:=k+1;
         end while;
         j:=j+1;
```

```
      end while;
      i:=i+1;
    end while;
    i:=1;
    while staging eq 3 and i le 100 do
      j:=M+1;
      while staging eq 3 and j le M+N-1 do
        k:=j+1;
        while staging eq 3 and k le M+N do
          if (r.j*r.k^-1)^i eq r.1^0 then
            staging:=1;
          end if;
          k:=k+1;
        end while;
        j:=j+1;
      end while;
      i:=i+1;
    end while;
  end if;
  if staging eq 3 then
    print "No Contradictions";
    print ShiftingPairs;
    ShiftingPairs:=[];
    staging:=0;
  end if;
  counter:=counter+1;
    ShiftingPairs;
  end if;
end while;
```

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