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Existence and multiplicity of solutions in frictional contact mechanics. Part I: A simplified criterion

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ABSTRACT

The work presented hereafter deals with an important issue arising in frictional contact problems involving flexible bodies: the occurrence of more than one solution, with an emphasis on the quasi-static incremental problem in the presence of rectilinear obstacle and in two dimensions. The conditions for the existence of multiple solutions to the quasi-static incremental problem, with an intrinsic combinatorial character, are presented for several criteria. A simplified criterion is proposed that avoids the exponential character of the problem. An algorithm is proposed for the computation of all the solutions of the incremental problem and to verify the sharpness of the frictional coefficient estimates corresponding to the several criteria. The contributions may be summarized as follows: (*i*) new simplified (sufficient) criterion for uniqueness of solution based on the solution of an optimization problem, avoiding the exponential character of the (necessary and sufficient) complete criterion of Alart or of the (sufficient) criterion due to Andersson; the proposed criterion assumes that the onset of multiplicity is associated with a mode involving sliding in the whole contact candidate region. (*ii*) The use of the suggested algorithm to compute all the solutions of the quasi-static incremental problem (for a given loading) in a finite element version of Klarbring's two degree of freedom model. For some lumped mass examples, all the solutions were calculated and their dependencies on some parameters were discussed. The conditions under which a problem may have multiple solutions were also discussed for some lumped models.

1. Introduction

Existence and/or uniqueness of solution belong to the most basic aspects that a researcher ought to address in order to understand and to further develop his/her knowledge on any mechanical/mathematical problem posed to him/her. The mathematical notions of non-existence or non-uniqueness of solution are intimately related with the intuitive mechanical notions of bifurcation or instability.

A case of non-convergence of an algorithm may have three causes: (1) *absence of solution* (for the set of data, the mathematical model is unable to produce a response for the system), (2) *solution multiplicity* (for the set of data, multiple solutions coexist competing for an algorithm's convergence) or (3) *inability of the algorithm to find a solution* (the use of other algorithms should be considered). To be in the possession of a good estimation of the conditions (namely the value of the coefficient of friction) for which a frictional contact problem fails to exhibit a solution permits us (*a*) to discard scenario (3) when the conditions for solution existence are not met, or, (*b*) to avoid losses of time when the conditions for the existence of solution are met and the algorithm being used does not perform satisfactorily. Consequently, the development of sharp estimates for the conditions under which a problem may have no solution has a very useful practical application, as pointed out by Acary et al. (2011). Convergence difficulties (cycling) in the generalized Newton method applied to the solution of the quasi-static incremental problem were reported in (Alart and Curnier, 1987; Alart, 1997)in connection with the occurrence of multiple solutions. Agwa and Pinto da Costa (2009), Agwa (2011), Agwa et al. (2012), Andersson et al. (2016) investigated the conditions for which the quasi-static incremental problem may exhibit more than one solution.

In spite of the truth that the right formulation for the frictional law of Coulomb's should involve the change in tangential displacement with respect to time, its usual substitution using incremental ratio is beneficial because it leads to a suitable formulation for the quasi-static *incremental* problem, this formulation is appropriate for computational implementation. Despite the possibility that the solution obtained for each load step of the incremental problem does not depend on the loads evolution during this step, the progression of solutions for the quasi-static incremental problem after multiple steps leads to a good

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approximation for the correct quasi-static evolution problem which depend on the method of load variation in the course of previous steps (Klarbring, 1988, 1990a,b). Alternatively stated, the progression of solution for the incremental problem take into account the dissipative/irreversible nature of Coulomb's friction law, this implies that the configuration and the obstacle reactions at the end of load steps depend on the total history of loading and the prior frictional contact states. Furthermore, using smaller load step for the progression of the incremental problem yields better solutions closer to the quasi-static evolution problem.

Alart and Curnier (1987) and Alart (1993) studied the multiple solutions for the incremental problem, by taking into account its intrinsic non-associated (nonsymmetric) character. On the basis of conewise properties (for the case of two dimensional contact problems with friction) and raywise properties (for the case of three dimensional contact problems) nonsmooth operators, Alart (1993), Alart et al. (1995) gave the necessary and sufficient conditions for the uniqueness of solution and carried out numerical experiments with solids using finite element method. Klarbring and Pang (1998) addressed the quasi-static incremental problem for finite dimensional spatial problems with positive semidefinite (semicoerciveness) stiffness matrices; the authors considered also piecewise linearizations for the friction cone. In (Andersson, 1999; Andersson et al., 2014) a Theorem is proved guaranteeing (*i*) the existence of solution, independently of the coefficient of friction and (*ii*) the solution such that the friction coefficient remains less than or equal to a threshold calculated from the resolution of combinatorial optimization; these references present a new fundamental frictional parameter to deduce assessments of the friction coefficient for the existence and uniqueness for the quasi-static (incremental and rate) evolution problems. The computation of that frictional parameter using Andersson's procedure is analogous to the resolution of a set of exponentially growing generalized eigenvalue problems (Holmgren, 1999) Hassani et al. (2003) also studied the quasi-static incremental problem and provided examples in which branches with infinite number of solutions involving the slip of all the contact candidate nodes are calculated analytically/numerically using eigenvalue analyzes.

The aim of the current work is to study the occurrence of more than one solution for the two dimensional quasi-static incremental problem in the presence of unilateral frictional rectilinear obstacle. The conditions for the existence of multiple solutions to the plane quasi-static incremental problem are presented for several criteria. A simplified criterion is proposed that avoids the exponential character of the problem. An algorithm is proposed for the computation of all the solutions of the incremental problem and to verify the sharpness of the frictional coefficient estimates corresponding to the several criteria. For the two dimensional version of some lumped mass examples, all the solutions were calculated and their dependencies on some parameters were discussed. The conditions under which a problem may have multiple solutions were discussed for several two dimensional lumped models.

2. Contact kinematics and Coulomb's law

In this study we generally consider elastic bodies subjected to static loads, possibly establishing frictional contact with a rigid surface. The body represented in Fig. 1 occupies a domain $\Omega \in \mathbb{R}^2$ with a sufficiently regular boundary Γ. The boundary Γ is decomposed in three parts, Γ_D, Γ_F and Γ_C, where Γ_D is the part on which displacements are prescribed, Γ_F is the part on which tractions are applied and Γ_C is the part that may be in frictional contact with flat obstacle. This part of the boundary is subjected to (*i*) a unilateral condition of non-penetration and (*ii*) the local version of Coulomb's friction law (Agwa and Pinto da Costa, 2011, Pinto da Costa and Agwa, 2011, 2013; Agwa and da Costa, 2015; Agwa et al., 2016; Domenico et al., 2017).

We also consider finite element discretizations of linear elastic solids (Fig. 1) or lumped mass systems. For systems with a finite number of

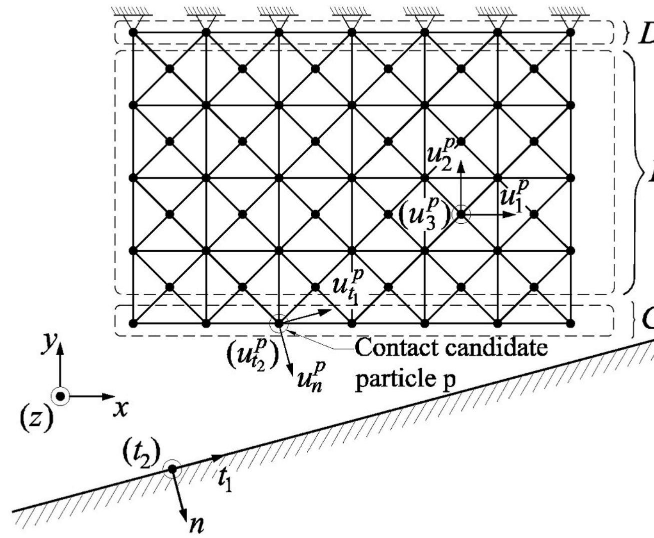


Fig. 1. A mechanical system with a finite number of degrees of freedom. Rigid body motions are prevented by a constant set of prescribed displacements. The generalized coordinates of the nodes that may establish frictional contact with flat obstacles are normal and tangent with respect to the obstacle's surface.

degrees of freedom the configuration at each time $t \geq 0$ is defined by the displacements $u_i(t)$, $1 \leq i \leq N$. These include the prescribed Displacements $\mathbf{u}_D$, the displacements $\mathbf{u}_F$ corresponding to the n_F particles that are Free from kinematic constraints (their labels form set $\mathscr{P}_F$), and the displacements $\mathbf{u}_C$ of the n_C Contact candidate nodes (their labels form set $\mathscr{P}_C$) that are *normal* ($\mathbf{u}_n$) and *tangential* ($\mathbf{u}_t$) to rigid obstacles. We assume the following decomposition of the vector of displacements at time t

$$\mathbf{u}(t) = \begin{Bmatrix} \mathbf{u}_D(t) \\ \mathbf{u}_F(t) \\ \mathbf{u}_C(t) \end{Bmatrix} = \begin{Bmatrix} \mathbf{u}_D(t) \\ \mathbf{u}_F(t) \\ \begin{Bmatrix} \mathbf{u}_n(t) \\ \mathbf{u}_t(t) \end{Bmatrix} \end{Bmatrix} \in \mathbb{R}^N. \tag{1}$$

The vector of reactions is

$$\mathbf{r}(t) = \begin{Bmatrix} \mathbf{r}_D(t) \\ \mathbf{r}_F(t) \\ \mathbf{r}_C(t) \end{Bmatrix} = \begin{Bmatrix} \mathbf{r}_D(t) \\ 0 \\ \begin{Bmatrix} \mathbf{r}_n(t) \\ \mathbf{r}_t(t) \end{Bmatrix} \end{Bmatrix}, \tag{2}$$

where $\mathbf{r}_D$ contains the external reactions acting at the kinematically constrained degrees of freedom, the null vector $\mathbf{0}$ denotes the reactions at the nodes which are kinematically unconstrained and $\mathbf{r}_C$ contains the reactions from the (frictional) obstacles. A decomposition analogous to (1) and (2) holds for the vector $\mathbf{v}$ of the system's velocities.

For all practical purposes, we consider in the sequel that the number of degrees of freedom is twice the number of nodes or lumped masses that are free from any kinematical constraint or that are candidate to contact: $N = 2(n_F + n_C)$.

For any particle $p \in \mathscr{P}_C$ of the system, the normal (n) and tangential (t) components of its displacement vector $\mathbf{u}^p(t) = \{u_n^p(t)\ u_t^p(t)\}^T$, of its velocity vector $\dot{\mathbf{u}}^p(t) = \{\dot{u}_n^p(t)\ \dot{u}_t^p(t)\}^T$ and of its reaction vector $\mathbf{r}^p(t) = \{r_n^p(t)\ r_t^p(t)\}^T$ satisfy the (Signorini) unilateral contact conditions and the friction law of Coulomb. For flat obstacles the unilateral contact law may be written in the form

$$u_n^p(t) - d_n^p \leq 0, \qquad r_n^p(t) \leq 0, \qquad (u_n^p(t) - d_n^p) r_n^p(t) = 0, \tag{3}$$

where d_n is the initial normal gap between particle p and the obstacle (Fig. 2(a)). Coulomb's friction law (Fig. 2(b)) may alternatively be written as a conjunction of the following two nonsmooth conditions

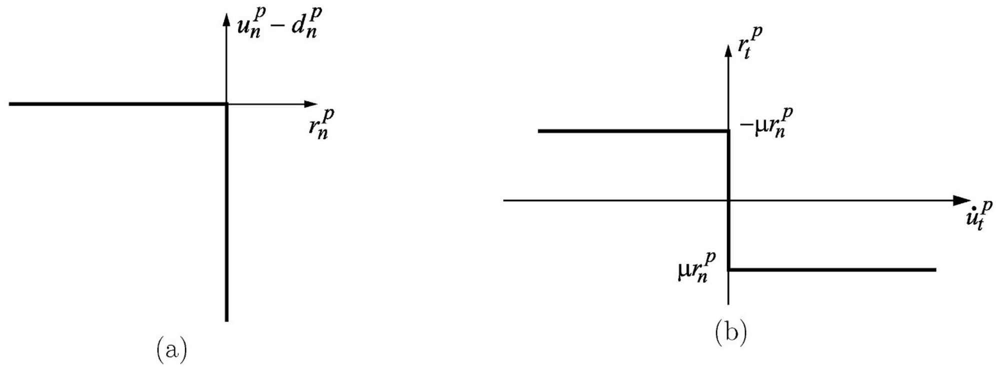


Fig. 2. Graphs of (a) the unilateral contact law and (b) the friction law of Coulomb. d_n: normal gap between the node p and the obstacle.

$$\left|r_t^p(t)\right| + \mu r_n^p(t) \le 0, \qquad \left|\dot{u}_t^p(t)\right| r_t^p(t) - \mu\left(\dot{u}_t^p(t)\right) r_n^p(t) = 0. \tag{4}$$

Factor μ denotes the (constant) coefficient of friction, $(\dot{.})$ denotes derivation with respect to time. During smooth dynamic evolutions, the balance of momentum equation is satisfied together with the laws (3) and (4) and with the appropriate initial conditions on the configuration and on the velocity.

Since the cardinal number of set $\mathscr{P}_C$ is n_C, and because each contact candidate particle has 2 qualitatively different contact statuses (in contact with non-vanishing reaction or out of contact with vanishing reaction), a discrete unilateral contact problem has 2^{n_C} potential different unilateral contact patterns; this fact puts in evidence the combinatorial character of the (discrete) problems to be dealt with in this work, a consequence of the complementarity condition $(3)_3$.

3. Admissible sets

Since the position $\mathbf{x}$ of each particle at any time t may be defined in terms of its displacement $\mathbf{u}$, an obstacle's surface may, for the sake of simplicity, be defined by $\phi(\mathbf{u}) = 0$. It becomes clear from the previous section that the *set of admissible displacements* is defined by

$$\mathscr{U} \doteq \left\{\mathbf{u} \in \mathbb{R}^N : \phi^p(\mathbf{u}) \le 0, \quad p \in \mathscr{P}_C\right\}. \tag{5}$$

Another fundamental set to be defined is the *set of admissible reactions*

$$\begin{aligned} \mathscr{R} = \Big\{ & r \in \mathbb{R}^N : r^p = 0, \qquad p \in \mathscr{P}_F; \\ & r_n^p \le 0, \quad \left|r_t^p\right| + \mu r_n^p \le 0, \quad p \in \mathscr{P}_C \Big\} \end{aligned} \tag{6}$$

For each $\mathbf{u} \in \mathbb{R}^N$, we consider the following partition of the set $\mathscr{P}_C$ of the n_C contact candidate nodes

$$\mathscr{P}_C = \mathscr{P}_f(\mathbf{u}) \cup \mathscr{P}_c(\mathbf{u}), \tag{7}$$

where

$$\mathscr{P}_f(\mathbf{u}) \doteq \{p \in \mathscr{P}_C : \phi^p(\mathbf{u}) < 0\}$$

$$\left[\text{particles currently out of contact } (\underline{\text{f}}\text{ree}), \# \mathscr{P}_f = n_f\right]$$

$$\mathscr{P}_c(\mathbf{u}) \doteq \{p \in \mathscr{P}_C : \phi^p(\mathbf{u}) \ge 0\}$$

$$[\text{particles currently in } \underline{\text{c}}\text{ontact}, \# \mathscr{P}_c = n_c].$$

A visualization of the admissible right rates of change of u_n, r_n and r_t is shown in Fig. 3. A particle in $\mathscr{P}_s$ has 2 qualitatively different possibilities of smooth evolution in the near future (see Fig. 3):

- it may become stuck (*sd*), which is represented by an arrow into the interior of the friction cone;
- it may remain in a state of impending slip (*ss*), which is represented by an arrow on the border of the friction cone.

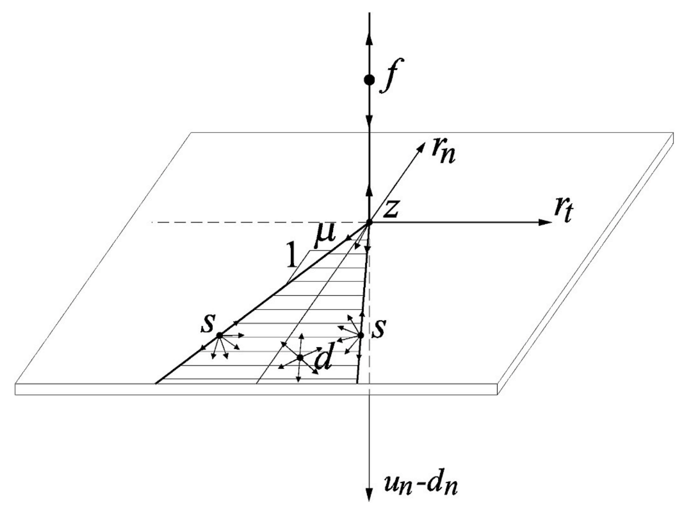


Fig. 3. Schematic representation of some admissible first order displacement rates and reaction rates.

A particle in $\mathscr{P}_z$ has 4 qualitatively different possibilities of smooth evolution in the near future (see Fig. 3):

- it may become out of contact (*zf*), represented by the arrow pointing to the negative part of the $u_n - d_n$ axis;
- it may become stuck (*zd*), represented by an arrow pointing to the interior of the friction cone;
- it may remain become in forward sliding (zs^+), represented by an arrow pointing to the border of the friction cone corresponding to negative tangential reactions ($r_t = \mu r_n$);
- it may remain become in backward sliding (zs^-), represented by an arrow pointing to the border of the friction cone corresponding to positive tangential reactions ($r_t = -\mu r_n$).

Consequently, in an equilibrium state, where n_z particles are in $\mathscr{P}_z$ and n_s particles are in $\mathscr{P}_s$, the system has $4^{n_z} \times 2^{n_s}$ potential different tangent behaviors, which is a manifestation of the combinatorial character of the frictional contact rate type problems. The enlarged versions, corresponding to an immovable obstacle, of the sets' definitions in this section may be found in (Martins et al., 1999, Martins and Pinto da Costa, 2000; Agwa and Pinto da Costa, 2008; Pinto da Costa and Agwa, 2009; Agwa, 2019).

4. Formulations of the incremental problem

The numerical computation of quasi-static evolutions is, in practice, accomplished by solving the quasi-static incremental problem, which is obtained from the quasi-static evolution problem after substitution of the tangential velocity present in the friction law by an incremental ratio (quotient between an increment of tangential displacement and a time interval corresponding to the time discretization). The choice of the quasi-static incremental problem is due to its relevance in the determination of quasi-static evolutions or equilibrium states in problems with practical relevance and to its similarity with the incremental dynamic problem (Jourdan et al., 1998; Pang and Stewart, 1999).

Given a sequence of external forces and prescribed displacements, the quasi-static incremental problem consists in the resolution of the system of algebraic equilibrium equations plus the unilateral contact conditions and a time discretized version of Coulomb's law. The two dimensional incremental problem without viscous damping may be written in terms of a mixed complementarity - algebraic inclusion formulation:

Given the force $\mathbf{f}^{k+1}$ *applied at the current increment, the initial distances* d_n^p *in the normal direction between the contact candidate particles and the obstacle, and the tangential displacements* $\mathbf{u}_t^k$ *in the end of the previous increment, compute the current displacements* $\mathbf{u}^{k+1}$ *and reactions* $\mathbf{r}^{k+1}$ *such that*

$$\mathbf{K}\mathbf{u}^{k+1} = \mathbf{f}^{k+1} + \mathbf{r}^{k+1}, \tag{8}$$

$$0 \geq u_n^{p,k+1} - d_n^p \perp r_n^{p,k+1} \leq 0, \tag{9}$$

$$r_t^{p,k+1} \in \mu \mathrm{Sign}\left(\frac{u_t^{p,k+1} - u_t^{p,k}}{\Delta t}\right) r_n^{p,k+1}, \tag{10}$$

for all $p \in \mathscr{P}_C$.

In the above formulation, $\mathbf{u}^{k+1}$ is the vector of absolute displacements, $u_n^{p,k+1} - d_n^p$ is the non-positive distance between particle p and the obstacle at the current increment, $u_t^{p,k}$ is the absolute tangential displacement of the particle *evaluated in the end of the previous load increment*, and Δt is the time increment.

An elegant way to formulate and solve numerically non-associated frictional contact problems is the one presented in (De Saxće and Feng, 1991; Feng, 1995, De Saxće and Feng, 1998; Joli and Feng, 2008); the use of the so-called bi-potential concept enables one to formulate non-associated problems in terms of a minimization principle. Their approach corresponds to deal with unilateral contact and friction as a single variational inequality, which leads to shorter computer execution times. The bi-potential method applies equally well to dynamic problems.

4.1. A mixed complementarity-inclusion formulation

Assuming vanishing viscous damping, the system of equilibrium equations after the prescription of essential boundary conditions $\mathbf{u}_D \equiv \overline{\mathbf{u}}_D$ is

$$\begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_{FF} & \mathbf{K}_{FC} \\ \mathbf{0} & \mathbf{K}_{CF} & \mathbf{K}_{CC} \end{bmatrix} \begin{Bmatrix} \mathbf{u}_D \\ \mathbf{u}_F \\ \mathbf{u}_C \end{Bmatrix} = \begin{Bmatrix} \overline{\mathbf{u}}_D \\ \mathbf{f}_F - \mathbf{K}_{FD}\overline{\mathbf{u}}_D \\ \mathbf{f}_C - \mathbf{K}_{CD}\overline{\mathbf{u}}_D \end{Bmatrix} + \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{r}_C \end{Bmatrix}, \tag{11}$$

following the partition of the degrees of freedom indicated in Section 2. The stiffness matrix in (11) is nonsingular. After solving (11) with respect to the displacement vector and using the partition of the degrees of freedom of the contact candidate nodes in normal and tangent components, the quasi-static incremental problem has the following mixed complementarity-inclusion formulation

Find $(\mathbf{u}_n, \mathbf{u}_t) \in \mathbb{R}^{2n_C}$ *and* $(\mathbf{r}_n, \mathbf{r}_t) \in \mathbb{R}^{2n_C}$ *such that*

$$\begin{Bmatrix} \mathbf{u}_n - \mathbf{d}_n \\ \mathbf{u}_t - \overline{\mathbf{u}}_t \end{Bmatrix} = \begin{Bmatrix} \mathbf{u}_n^* - \mathbf{d}_n \\ \mathbf{u}_t^* - \overline{\mathbf{u}}_t \end{Bmatrix} + \begin{bmatrix} \mathbf{F}_{nn} & \mathbf{F}_{nt} \\ \mathbf{F}_{tn} & \mathbf{F}_{tt} \end{bmatrix} \begin{Bmatrix} \mathbf{r}_n \\ \mathbf{r}_t \end{Bmatrix} \tag{12}$$

$$\mathbf{0} \geq \mathbf{u}_n - \mathbf{d}_n \perp \mathbf{r}_n \leq 0, \tag{13}$$

$$r_t^p \in \mu r_n^p \mathrm{Sign}\left(u_t^p - \overline{u}_t^p\right), \quad \forall p \in \mathscr{P}_C. \tag{14}$$

Vector $\mathbf{d}_n$ denotes initial distances to the obstacle, $\overline{\mathbf{u}}_t$ groups the displacements of the contact candidate nodes that are tangent to the obstacle, evaluated at the end of the previous load increment, $\mathbf{F}_{nn}, \mathbf{F}_{nt} = \mathbf{F}_{tn}$ and $\mathbf{F}_{tt}$ denote square blocks of the flexibility matrix and $\mathbf{u}_n^*$ and $\mathbf{u}_t^*$ are the normal and tangent components of

$$\mathbf{u}_C^* = \mathbf{F}_{CF}(\mathbf{f}_F - \mathbf{K}_{FD}\overline{\mathbf{u}}_D) + \mathbf{F}_{CC}(\mathbf{f}_C - \mathbf{K}_{CD}\overline{\mathbf{u}}_D). \tag{15}$$

In formulation (12)–(14), $(\mathbf{u}_n, \mathbf{u}_t)$ and $(\mathbf{r}_n, \mathbf{r}_t)$ denote respectively the absolute displacements and the obstacles' reactions at the current increment.

4.2. A full complementarity formulation

In a manner similar to what was done in (Pinto da Costa et al., 2004, Pinto da Costa and Agwa, 2006; Agwa and Pinto da Costa, 2006, 2007; Sitzmann et al., 2015) for directional instability problem, the quasi-static incremental problem may be written in a full complementarity way. For that purpose one uses a bijective single valued non-smooth map $g : \mathbb{R}^N \rightarrow \mathbb{R}^N$ built from the local maps $g^p : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$\begin{aligned} g^p(\mathbf{v}^p) &= \begin{Bmatrix} v_n^p \\ v_t^p \end{Bmatrix}, & p \in \mathscr{P}_F \\ g^p(\mathbf{v}^p) &= \begin{Bmatrix} v_n^p - \mu|v_t^p| \\ v_t^p \end{Bmatrix}, & p \in \mathscr{P}_C. \end{aligned} \tag{16}$$

For any vector $\mathbf{v} \in \mathbb{R}^N = \mathbb{R}^{2(n_F+n_C)}$,

$$g(\mathbf{v}) = \begin{Bmatrix} g^1(\mathbf{v}^1) \\ \cdot \\ \cdot \\ \cdot \\ g^{n_F+n_C}(\mathbf{v}^{n_F+n_C}) \end{Bmatrix}. \tag{17}$$

In the case of a flat obstacle, the set $\mathscr{U}$ of admissible displacements (5) may be viewed as the cartesian product of the sets of admissible displacements for each node

$$\mathscr{U}^p = \begin{cases} \mathbb{R}^2, & p \in \mathscr{P}_F \\ \left\{\mathbf{u}^p \in \mathbb{R}^2 : u_n^p - d_n^p \leq 0\right\} & p \in \mathscr{P}_C. \end{cases} \tag{18}$$

and the set of admissible reactions $\mathscr{R}$ (6) is equal to the cartesian product of the sets

$$\mathscr{R}^p = \begin{cases} \{(0,0)\}, & p \in \mathscr{P}_F \\ \left\{\mathbf{r}^p \in \mathbb{R}^2 : r_n^p \leq 0, \quad |r_t^p| + \mu r_n^p \leq 0\right\}, & p \in \mathscr{P}_C. \end{cases} \tag{19}$$

Sets $\mathscr{U} = \mathscr{U}^1 \times \ldots \times \mathscr{U}^{n_F+n_C}$ and $\mathscr{R} = \mathscr{R}^1 \times \ldots \times \mathscr{R}^{n_F+n_C}$ are closed convex cones since they are the cartesian product of closed convex cones (Hiriart-Urruty and Lemaréchal, 1996).

The transformation of cone $\mathscr{U}^p$ by map g^p defined above is

$$g^p(\mathscr{U}^p) = \begin{cases} \mathbb{R}^2, & p \in \mathscr{P}_F \\ \left\{g^p \in \mathbb{R}^2 : g_n^p + \mu|g_t^p| \leq 0\right\}, & p \in \mathscr{P}_C, \end{cases} \tag{20}$$

which one recognizes to be the dual cone of $\mathscr{R}^p$ (see Fig. 4) i.e., $g^p(\mathscr{U}^p) = (\mathscr{R}^p)^+$.

Since the dual of a cartesian product of closed convex cones is the cartesian product of the duals, one also has

$$g(\mathscr{U}) = \mathscr{R}^+. \tag{21}$$

It is thus possible to rewrite the quasi-static incremental problem in terms of an Explicit Complementarity Problem (*ECP*) as

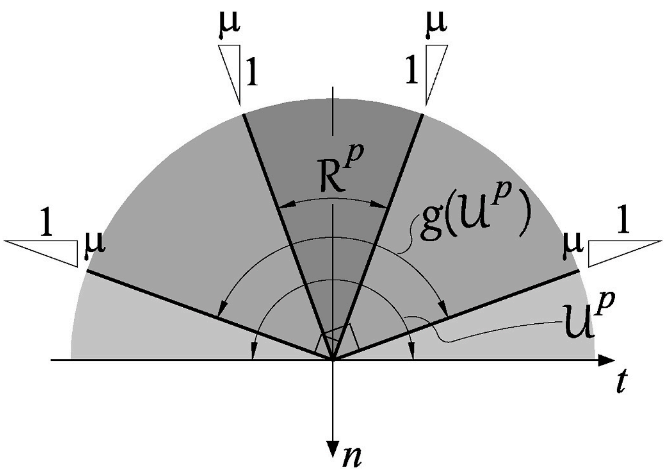


Fig. 4. Illustration of cones $\mathscr{U}^p, g(\mathscr{U}^p)$ and $\mathscr{R}^p$, for $p \in \mathscr{P}_C$.

Find $(\mathbf{u}_n, \mathbf{u}_t) \in \mathbb{R}^{2n_C}$ *and* $(\mathbf{r}_n, \mathbf{r}_t) \in \mathbb{R}^{2n_C}$ *such that*

$$\left\{ \begin{array}{c} \mathbf{u}_n - \mathbf{d}_n \\ \mathbf{u}_t - \bar{\mathbf{u}}_t \end{array} \right\} = \left\{ \begin{array}{c} \mathbf{u}_n^* - \mathbf{d}_n \\ \mathbf{u}_t^* - \bar{\mathbf{u}}_t \end{array} \right\} + \begin{bmatrix} \mathbf{F}_{nn} & \mathbf{F}_{nt} \\ \mathbf{F}_{tn} & \mathbf{F}_{tt} \end{bmatrix} \left\{ \begin{array}{c} \mathbf{r}_n \\ \mathbf{r}_t \end{array} \right\}, \tag{22}$$

$$\mathscr{R} \ni \left\{ \begin{array}{c} \mathbf{r}_n \\ \mathbf{r}_t \end{array} \right\} \perp g\left(\left\{ \begin{array}{c} \mathbf{u}_n - \mathbf{d}_n \\ \mathbf{u}_t - \bar{\mathbf{u}}_t \end{array} \right\} \right) \in g(\mathscr{U}) = \mathscr{R}^+. \tag{23}$$

The previous formulation involves a number of unknowns equal to $4n_C$ and a symmetric matrix. However it also involves admissible cones with a not so simple structure depending on the magnitude of the coefficient of friction.

4.3. A linear complementarity formulation

The introduction of the non-negative complementarity variables

$$\boldsymbol{\xi}_n = \mathbf{d}_n - \mathbf{u}_n, \tag{24}$$

$$\boldsymbol{\xi}_t^+ = (\mathbf{u}_t - \bar{\mathbf{u}}_t)_+ = \max\{\mathbf{u}_t - \bar{\mathbf{u}}_t, \mathbf{0}\}, \tag{25}$$

$$\boldsymbol{\xi}_t^- = (\mathbf{u}_t - \bar{\mathbf{u}}_t)_- = \max\{-(\mathbf{u}_t - \bar{\mathbf{u}}_t), \mathbf{0})\} \tag{26}$$

$$\boldsymbol{\psi}_n = -\mathbf{r}_n, \tag{27}$$

$$\boldsymbol{\psi}_t^+ = \mathbf{r}_t - \mu \mathbf{r}_n, \tag{28}$$

$$\boldsymbol{\psi}_t^- = -\mathbf{r}_t - \mu \mathbf{r}_n, \tag{29}$$

enables writing the quasi-static incremental problem in terms of the following Linear Complementarity Problem (*LCP*)

Find $\left(\boldsymbol{\xi}_n, \boldsymbol{\xi}_t^-, \boldsymbol{\psi}_t^+\right) \in \mathbb{R}^{3n_C}$ *and* $\left(\boldsymbol{\psi}_n, \boldsymbol{\psi}_t^-, \boldsymbol{\xi}_t^+\right) \in \mathbb{R}^{3n_C}$ *such that*

$$\left\{ \begin{array}{c} \boldsymbol{\xi}_n \\ \boldsymbol{\xi}_t^- \\ \boldsymbol{\psi}_t^+ \end{array} \right\} = \left\{ \begin{array}{c} \mathbf{d}_n - \mathbf{u}_n^* \\ \bar{\mathbf{u}}_t - \mathbf{u}_t^* \\ \mathbf{0} \end{array} \right\} + \begin{bmatrix} \mathbf{F}_{nn} - \mathbf{F}_{nt}\mathbf{U} & \mathbf{F}_{nt} & \mathbf{0} \\ \mathbf{F}_{tn} - \mathbf{F}_{tt}\mathbf{U} & \mathbf{F}_{tt} & \mathbf{I} \\ 2\mathbf{U} & -\mathbf{I} & \mathbf{0} \end{bmatrix} \left\{ \begin{array}{c} \boldsymbol{\psi}_n \\ \boldsymbol{\psi}_t^- \\ \boldsymbol{\xi}_t^+ \end{array} \right\}, \tag{30}$$

$$\left\{ \begin{array}{c} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{array} \right\} \le \left\{ \begin{array}{c} \boldsymbol{\xi}_n \\ \boldsymbol{\xi}_t^- \\ \boldsymbol{\psi}_t^+ \end{array} \right\} \perp \left\{ \begin{array}{c} \boldsymbol{\psi}_n \\ \boldsymbol{\psi}_t^- \\ \boldsymbol{\xi}_t^+ \end{array} \right\} \ge \left\{ \begin{array}{c} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{array} \right\}, \tag{31}$$

where $\mathbf{U} = \mathrm{diag}(\mu, \ldots, \mu) \in \mathbb{R}^{n_C \times n_C}$. A *linear* complementarity problem involving $6n_C$ unknowns and a nonsymmetric matrix due to the non-associated character of Coulomb's law is now obtained. The structure of the cones is very simple: the self-dual first orthants, $(\mathbb{R}^{3n_C})^+ \equiv \mathbb{R}^{3n_C}$.

For sufficiently small coefficients of friction the matrix in the previous *LCP* is copositive, a necessary condition for Lemke's method to compute a solution in a finite number of steps (Cottle et al., 1992; Pfeiffer and Bremer, 2017).

A linear complementarity formulation with a copositive coefficient matrix regardless of the magnitude of μ (but involving $8n_C$ unknowns) was considered by Klarbring (1999), Brogliato (2016), Barber (2018), inspired by the works (Stewart and Trinkle, 1996) and (Anitescu and Potra, 1997) on the dynamic incremental problem.

4.4. A generalized linear complementarity formulation

In order to find *all the solutions* of the quasi-static frictional contact problem for a given load increment one may reformulate the above *LCP* as a Generalized Linear Complementarity Problem (*GLCP*) and use the algorithm of De Moor and Vandewalle (1987), De Moor (1988), De Moor et al. (1992) to solve it. The *GLCP* formulation of the *LCP* presented in equations 30 and 31 is

Given $\mathbf{d}_n$, $\bar{\mathbf{u}}_t$, $\mathbf{u}_C^* = (\mathbf{u}_n^*, \mathbf{u}_t^*)$, *find* $\left(\boldsymbol{\xi}_n, \boldsymbol{\xi}_t^-, \boldsymbol{\psi}_t^+, \boldsymbol{\psi}_n, \boldsymbol{\psi}_t^-, \boldsymbol{\xi}_t^+\right) \in \mathbb{R}^{6n_C}$ *such that*

$$\begin{bmatrix} -\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{F}_{nn} - \mathbf{F}_{nt}\mathbf{U} & \mathbf{F}_{nt} & \mathbf{0} & \mathbf{d}_n - \mathbf{u}_n^* \\ \mathbf{0} & -\mathbf{I} & \mathbf{0} & \mathbf{F}_{tn} - \mathbf{F}_{tt}\mathbf{U} & \mathbf{F}_{tt} & \mathbf{I} & \bar{\mathbf{u}}_t - \mathbf{u}_t^* \\ \mathbf{0} & \mathbf{0} & -\mathbf{I} & 2\mathbf{U} & -\mathbf{I} & \mathbf{0} & \mathbf{0} \end{bmatrix} \left\{ \begin{array}{c} \boldsymbol{\xi}_n \\ \boldsymbol{\xi}_t^- \\ \boldsymbol{\psi}_t^+ \\ \boldsymbol{\psi}_n \\ \boldsymbol{\psi}_t^- \\ \boldsymbol{\xi}_t^+ \\ \varrho \end{array} \right\} = \left\{ \begin{array}{c} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{array} \right\}, \tag{32}$$

$$\left\{ \begin{array}{c} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{array} \right\} \le \left\{ \begin{array}{c} \boldsymbol{\xi}_n \\ \boldsymbol{\xi}_t^- \\ \boldsymbol{\psi}_t^+ \end{array} \right\} \perp \left\{ \begin{array}{c} \boldsymbol{\psi}_n \\ \boldsymbol{\psi}_t^- \\ \boldsymbol{\xi}_t^+ \end{array} \right\} \ge \left\{ \begin{array}{c} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{array} \right\}, \quad \varrho \ge 0. \tag{33}$$

The algorithm of De Moor and Vandewalle (1987), De Moor (1988), De Moor et al. (1992) is geometrically inspired method, non-iterative, the algorithm does not involve inversion of matrices. It executes geometric intersections among the objects that represent the solution sets. This algorithm is also eligible for the calculation of all solutions of a GLCP but the time of execution grows exponentially as the number of variables increase. The implementation of the De Moor's algorithm to a discretized solid in the framework of finite elements is found in reference Pinto da Costa and Martins (2004).

5. Existence of solution

One easily recognizes that problem (22)-(23) is written in the canonical form of an *ECP*:

$$\textit{Find } \mathbf{x} \textit{ such that } \mathscr{K} \ni \mathbf{x} \perp f(\mathbf{x}) \in \mathscr{K}^+, \tag{34}$$

where $\mathscr{K}$ is a closed convex cone. Formulation (22)–(23) of the quasi-static incremental problem is then appropriate to invoke corollary 4.13 in (Hyers et al., 1997) which guarantees the existence of solution provided the vectorial function $f(\mathbf{x})$ is completely continuous and satisfies

$$\lim_{\substack{\mathbf{x} \in \mathscr{K} \\ \|\mathbf{x}\| \to +\infty}} \frac{\mathbf{x}.f(\mathbf{x})}{\|\mathbf{x}\|} = +\infty. \tag{35}$$

Function g, presented in (23), is completely continuous because it is continuous and the problem is finite dimensional (Martins et al., 2002). The coercivity condition (35) leads to

$$\lim_{\substack{(\mathbf{r}_n,\mathbf{r}_t)\in\mathscr{R}\\ \|(\mathbf{r}_n,\mathbf{r}_t)\|\to+\infty}} \frac{\begin{Bmatrix}\mathbf{r}_n\\ \mathbf{r}_t\end{Bmatrix}.g\begin{pmatrix}\mathbf{u}_n-\mathbf{d}_n\\ \mathbf{u}_t-\overline{\mathbf{u}}_t\end{pmatrix}}{\|(\mathbf{r}_n,\mathbf{r}_t)\|} = \lim_{\substack{(\mathbf{r}_n,\mathbf{r}_t)\in\mathscr{R}\\ \|(\mathbf{r}_n,\mathbf{r}_t)\|\to+\infty}} \frac{1}{\|(\mathbf{r}_n,\mathbf{r}_t)\|}\left(\begin{Bmatrix}\mathbf{r}_n\\ \mathbf{r}_t\end{Bmatrix}.\begin{Bmatrix}\mathbf{u}_n^*-\mathbf{d}_n\\ \mathbf{u}_t^*-\overline{\mathbf{u}}_t\end{Bmatrix}\right.$$
$$\left.+\begin{bmatrix}\mathbf{F}_{nn} & \mathbf{F}_{nt}\\ \mathbf{F}_{tn} & \mathbf{F}_{tt}\end{bmatrix}\begin{Bmatrix}\mathbf{r}_n\\ \mathbf{r}_t\end{Bmatrix}.\begin{Bmatrix}\mathbf{r}_n\\ \mathbf{r}_t\end{Bmatrix}-\sum_{\mathscr{P}_C}\mu r_n\left|\mathbf{u}_t^*-\overline{\mathbf{u}}_t+[\mathbf{F}_{tn}\quad \mathbf{F}_{tt}]\begin{Bmatrix}\mathbf{r}_n\\ \mathbf{r}_t\end{Bmatrix}\right|\right)=$$
$$+\infty \tag{36}$$

because the flexibility matrix $\mathbf{F}_{CC}$ is strictly copositive in the closed convex cone $\mathscr{R}$. A matrix $\mathbf{M}$ is strictly copositive in a closed convex cone $\mathscr{R}$ if and only if $\mathbf{x}\cdot\mathbf{M}\mathbf{x}>0,\ \forall\mathbf{x}\in\mathscr{R},\mathbf{x}\neq 0$. Note that since $\mathbf{F}_{CC}$ is symmetric positive definite (*SPD*), it is also strictly copositive. The sum $-\sum_{\mathscr{P}_C}\mu r_n\left|\mathbf{u}_t^*-\overline{\mathbf{u}}_t+[\mathbf{F}_{tn}\quad \mathbf{F}_{tt}]\begin{Bmatrix}\mathbf{r}_n\\ \mathbf{r}_t\end{Bmatrix}\right|$ is nonnegative because $-\ r_n\geq 0$. This is a Proof of solution existence for the quasi-static incremental problem, alternative to several others published previously, for example in (Alart, 1993; Alart et al., 1995; Andersson, 1999; Klarbring, 1999).

6. Criteria to estimate the onset of multiplicity

The three methods used in this study to detect solution multiplicity of the quasi-static incremental problem and to compute approximations of the conditions at the onset of multiplicity are described in this section. The first two methods considered next involve the computation of the determinants of frictionally affected matrix families, while the third one involves the resolution of an optimization problem.

6.1. Bijectiveness of a conewise linear (CL) operator

Alart and Curnier (1987, 1991), Alart and Lebon (1995), Neto et al. (2016), Charroyer et al. (2018) describe unilateral contact and Coulomb's friction by the unconstrained nonsmooth equations

$$r_n^p=\mathrm{proj}_{\mathbb{R}_-}\left(r_n^p-\alpha\left(u_n^p-d_n^p\right)\right), \tag{37}$$

$$r_t^p=\mathrm{proj}_{\mathscr{D}^p\left(\mu\mathrm{proj}_{\mathbb{R}_-}\left(r_n^p-\alpha\left(u_n^p-d_n^p\right)\right)\right)}\left(r_t^p-\alpha\left(u_t^p-\overline{u}_t^p\right)\right), \tag{38}$$

respectively, where $p\in\mathscr{P}_C$, $\alpha>0$ and, for $y<0$, $\mathscr{D}^p(y)=[y,-y]$. By defining $\mathscr{C}$ as the cartesian product of the n_C intervals $]-\infty,0]$ and $\mathscr{D}$ as the cartesian product of the n_C intervals $\mathscr{D}^p(\mu\mathrm{proj}_{\mathbb{R}_-}(r_n^p-\alpha(u_n^p-d_n^p))$, the quasi-static incremental problem may be written as a system of unconstrained nonsmooth equations after a condensation on the degrees of freedom of the contact candidate nodes,

$$\mathbf{H}(\mathbf{u},\mathbf{r})=\left\{\begin{matrix}\begin{bmatrix}\mathbf{K}_{nn} & \mathbf{K}_{nt}\\ \mathbf{K}_{tn} & \mathbf{K}_{tt}\end{bmatrix}\begin{Bmatrix}\mathbf{u}_n\\ \mathbf{u}_t\end{Bmatrix}-\begin{Bmatrix}\mathbf{f}_n\\ \mathbf{f}_t\end{Bmatrix}-\begin{Bmatrix}\mathrm{proj}_{\mathscr{C}}(\mathbf{r}_n-\alpha(\mathbf{u}_n-\mathbf{d}_n))\\ \mathrm{proj}_{\mathscr{D}}(\mathbf{r}_t-\alpha(\mathbf{u}_t-\overline{\mathbf{u}}_t))\end{Bmatrix}\\ -\dfrac{1}{\alpha}[\mathbf{r}_n-\mathrm{proj}_{\mathscr{C}}(\mathbf{r}_n-\alpha(\mathbf{u}_n-\mathbf{d}_n))]\\ -\dfrac{1}{\alpha}[\mathbf{r}_t-\mathrm{proj}_{\mathscr{D}}(\mathbf{r}_t-\alpha(\mathbf{u}_t-\overline{\mathbf{u}}_t))]\end{matrix}\right\}=\mathbf{0}. \tag{39}$$

According to Alart and Curnier (1987), Alart (1993), Alart et al. (1995), the above system of nonsmooth equations is conewise linear (CL). Consequently, on the borders of the cones of linearity, the classical jacobian based on the Fréchet derivative is not defined. The tangent behavior of a nonsmooth vectorial function is then characterized by the *generalized jacobian* (Facchinei and Pang, 2003) which, on a border between cones of linearity is *set valued*, taking any value in the convex hull formed by the (classical) jacobians defined in the regions of linearity adjacent to that border. Moreover, the set of infinitely many jacobian matrices formed the generalized jacobian at a point of non-differentiability in the sense of Fréchet is completely defined by the finitely many jacobian matrices at the regions of linear behavior adjacent to that point. This minimum set of jacobian matrices able to characterize the generalized jacobian is called the base of the generalized jacobian. Alart et al. (1995) proved that (39) has a unique solution if and only if the left hand side is a homeomorphic mapping. A mapping is called a homeomorphism when it is bijective ("one to one" and "onto") and continuous with a continuous inverse (Seymour Lipschutz, 1965). The possible source of non-bijectivity in (39) is the projection operator necessary to characterize unilateral contact and friction. Kojima and Saigal (1979), Alart et al. (1995) showed that in 2D, the conewise linear *function* (39) *is a homeomorphism if and only if all the determinants of the matrices forming the base of its generalized jacobian have the same sign* (see Theorem 10 in (1993)). It suffices then to check the determinants of a *finite* number of jacobian matrices to verify if a conewise linear vector mapping is a homeomorphism.

To illustrate, consider the one particle system shown in Fig. 5. Four states are possible for that mechanical system: *out of contact* (*f*), *stick* (*d*), *forward slip* (s^+) and *backward slip* (s^-). Each one of those four states represents a region of linear behavior of $\mathbf{H}\in\mathbb{R}^4$, in the interior of which $\mathbf{H}$ is differentiable in the sense of Fréchet, the jacobian being a matrix in $\mathbb{R}^{4\times4}$. For the above mentioned two degree of freedom system, function (39), its four jacobians forming the base of the generalized jacobian and the corresponding determinants are listed next:

- Out of contact (*f*)

$$\mathbf{H}_f(\mathbf{u},\mathbf{r})=\left\{\begin{matrix}\begin{bmatrix}k_{nn} & k_{nt}\\ k_{tn} & k_{tt}\end{bmatrix}\begin{Bmatrix}u_n\\ u_t\end{Bmatrix}-\begin{Bmatrix}f_n\\ f_t\end{Bmatrix}\\ -\dfrac{1}{\alpha}r_n\\ -\dfrac{1}{\alpha}r_t\end{matrix}\right\}, \tag{40}$$

$$\mathbf{J}_f=\begin{bmatrix}k_{nn} & k_{nt} & 0 & 0\\ k_{tn} & k_{tt} & 0 & 0\\ 0 & 0 & -\dfrac{1}{\alpha} & 0\\ 0 & 0 & 0 & -\dfrac{1}{\alpha}\end{bmatrix},\quad \det(\mathbf{J}_f)=\frac{k_{nn}k_{tt}-k_{nt}^2}{\alpha^2}; \tag{41}$$

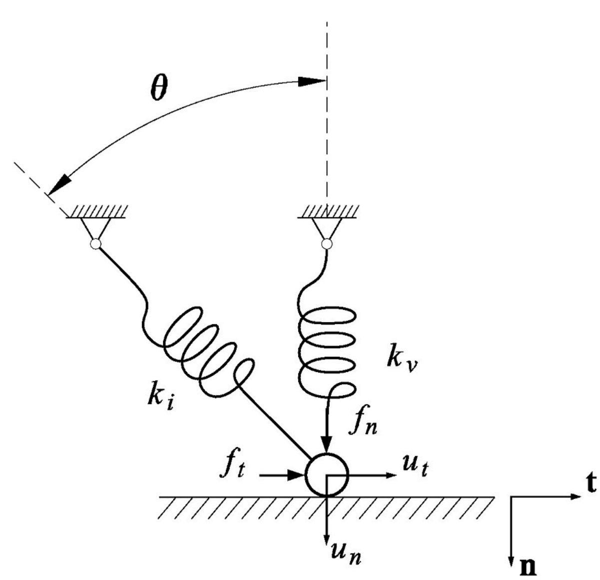


Fig. 5. An elastically restrained particle in the presence of an obstacle with friction. The angle between the inclined spring and the vertical is $\theta\in\ ]0,\frac{\pi}{2}[$.

- Sticking (d)

$$\mathbf{H}_d(\mathbf{u},\mathbf{r}) = \left\{ \begin{array}{c} \begin{bmatrix} k_{nn} & k_{nt} \\ k_{tn} & k_{tt} \end{bmatrix} \begin{Bmatrix} u_n \\ u_t \end{Bmatrix} - \begin{Bmatrix} f_n \\ f_t \end{Bmatrix} - \begin{Bmatrix} r_n - \alpha(u_n - d_n) \\ r_t - \alpha(u_t - d_t) \end{Bmatrix} \\ -(u_n - d_n) \\ -(u_t - d_t) \end{array} \right\}, \tag{42}$$

$$\mathbf{J}_d = \begin{bmatrix} k_{nn}+\alpha & k_{nt} & -1 & 0 \\ k_{tn} & k_{tt}+\alpha & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad \det(\mathbf{J}_d) = +1; \tag{43}$$

- Forward slip (s^+)

$$\mathbf{H}_{s^+}(\mathbf{u},\mathbf{r}) = \left\{ \begin{array}{c} \begin{bmatrix} k_{nn} & k_{nt} \\ k_{tn} & k_{tt} \end{bmatrix} \begin{Bmatrix} u_n \\ u_t \end{Bmatrix} - \begin{Bmatrix} f_n \\ f_t \end{Bmatrix} - \begin{Bmatrix} r_n - \alpha(u_n - d_n) \\ \mu(r_n - \alpha(u_n - d_n)) \end{Bmatrix} \\ -(u_n - d_n) \\ -\dfrac{1}{\alpha}(r_t - \mu r_n) + \mu(u_n - d_n) \end{array} \right\}, \tag{44}$$

$$\mathbf{J}_{s^+} = \begin{bmatrix} k_{nn}+\alpha & k_{nt} & -1 & 0 \\ k_{tn}+\mu\alpha & k_{tt} & -\mu & 0 \\ -1 & 0 & 0 & 0 \\ -\mu & 0 & \dfrac{\mu}{\alpha} & -\dfrac{1}{\alpha} \end{bmatrix}, \quad \det(\mathbf{J}_{s^+}) = \frac{k_{tt} - \mu k_{nt}}{\alpha}; \tag{45}$$

- Backward slip (s^-)

$$\mathbf{H}_{s^-}(\mathbf{u},\mathbf{r}) = \left\{ \begin{array}{c} \begin{bmatrix} k_{nn} & k_{nt} \\ k_{tn} & k_{tt} \end{bmatrix} \begin{Bmatrix} u_n \\ u_t \end{Bmatrix} - \begin{Bmatrix} f_n \\ f_t \end{Bmatrix} - \begin{Bmatrix} r_n - \alpha(u_n - d_n) \\ -\mu(r_n - \alpha(u_n - d_n)) \end{Bmatrix} \\ -(u_n - d_n) \\ -\dfrac{1}{\alpha}(r_t + \mu r_n) + \mu(u_n - d_n) \end{array} \right\}, \tag{46}$$

$$\mathbf{J}_{s^-} = \begin{bmatrix} k_{nn}+\alpha & k_{nt} & -1 & 0 \\ k_{tn}-\mu\alpha & k_{tt} & \mu & 0 \\ -1 & 0 & 0 & 0 \\ \mu & 0 & -\dfrac{\mu}{\alpha} & -\dfrac{1}{\alpha} \end{bmatrix}, \quad \det(\mathbf{J}_{s^-}) = \frac{k_{tt} + \mu k_{nt}}{\alpha}. \tag{47}$$

For the jacobians to have the same sign

$$\mathrm{Sign}\big(\det(\mathbf{J}_f)\big) = \mathrm{Sign}(\det(\mathbf{J}_d)) = \mathrm{Sign}(\det(\mathbf{J}_{s^+})) = \mathrm{Sign}(\det(\mathbf{J}_{s^-})) = +1, \tag{48}$$

and since $\alpha > 0$, $k_{nn}k_{tt} - k_{nt}^2 > 0$, it is sufficient that $\mu < \frac{k_{tt}}{|k_{nt}|}$, which, for the specific spring arrangement in Fig. 5, corresponds to $\mu < |\tan\theta|$. One concludes that, for Klarbring's example, the solution to the quasi-static incremental problem is unique if and only if the effective stiffnesses $k_{tt} \mp \mu k_{nt} = k_i \sin\theta \cos\theta(\tan\theta \mp \mu)$ are strictly positive. The geometrical interpretation of these conditions corresponds to slope positivity in equilibrium trajectories. Note that the only possible sources of non-uniqueness are the contact states corresponding to forward or backward slip. The out of contact and sticking states have tangent stiffness that do not depend on μ, so that these two states can not by themselves be the cause of solution multiplicity of the quasi-static incremental problem. Since n_C represents the number of candidate nodes, the verification of solution uniqueness by this method involves the evaluation of the determinants of 4^{n_C} matrices with dimensions $4n_C \times 4n_C$. As the computational effort grows exponentially with the number of the contact candidate particles n_C, the full (necessary and sufficient) condition of computing the 4^{n_C} determinants can only be implemented up to $n_C = 9$ or 10 (in this case, more than one million determinants of 40 × 40 matrices ought to be computed).

Alart et al. (1995) applied their method to finite element discretized elastic bodies, mainly squares with the top side fixed and the bottom side candidate to contact. These authors detected a constant pattern of frictional contact states at the onset of solution multiplicity of the quasi-static incremental problem for several coarse and moderately refined meshes. They also concluded that the frictional contact pattern leading to the minimum value of μ for solution multiplicity involves the impending slip of all the contact candidate nodes. Numerical experiments conducted for more refined meshes (up to $n_C = 17$) and restricting the possible states of all the contact candidate nodes to impending slip (s^+ or s^-), revealed that the critical values of μ at the onset of multiplicity converged with mesh refinement (Alart et al., 1995). This is an indication that, at least for certain geometries, the frictional contact pattern corresponding to the onset of solution multiplicity of the quasi-static incremental problem is associated to impending slip of all the contact candidate regions. In fact, this conclusion, taken for square shaped solids, is not so surprising since the jacobian matrices include more friction affected elements as the number of nodes assumed to be in impending slip increases. Consequently, by assuming just 2 possible frictional states out of 4, one has to compute a total of 2^{n_C} determinants (instead of 4^{n_C}), which enables the consideration of more refined meshes. The objective of achieving good quality estimates of μ at the onset of multiplicity with the limitation of the number of possible frictional contact states to $\{s^+, s^-\}$ is further explored in this work in the context of an optimization based method.

6.2. P property of matrices

The sufficient condition guaranteeing solution uniqueness presented in this section has its roots in (Lötstedt, 1981; Trinkle et al., 1995), where dynamic problems were addressed. The criterion for solution uniqueness derived next is based on a classical argument: two solutions are assumed to exist and the conditions for them to be equal are derived.

Suppose that the quasi-static incremental problem has two solutions $(\mathbf{u}_n^A, \mathbf{u}_t^A, \mathbf{r}_n^A, \mathbf{r}_t^A)$ and $(\mathbf{u}_n^B, \mathbf{u}_t^B, \mathbf{r}_n^B, \mathbf{r}_t^B)$ corresponding to the same data set $\mathbf{d}_n$, $\bar{\mathbf{u}}_t$, $\mathbf{u}_n^*$and $\mathbf{u}_t^*$. From the equilibrium equation (12) and after noticing that $\mathbf{r}_t^A - \mathbf{r}_t^B \in \mu\mathbb{I}(\mathbf{r}_n^A - \mathbf{r}_n^B) + \mu(\mathbf{S}^A - \mathbf{S}^B)\mathbf{r}_n^B$ where $\mathbb{I} = \mathrm{diag}([-1, 1])$ is a $n_C \times n_C$ diagonal interval matrix, $\mathbf{S}^A = \mathrm{diag}(\mathrm{Sign}(u_t^{Ap} - \bar{u}_t^{Ap}), p \in \mathscr{P}_C)$ and $\mathbf{S}^B = \mathrm{diag}(\mathrm{Sign}(u_t^{Bp} - \bar{u}_t^{Bp}), p \in \mathscr{P}_C)$ one concludes that the difference between the two solutions satisfies the algebraic inclusion

$$\begin{Bmatrix} \mathbf{u}_n^A - \mathbf{u}_n^B \\ \mathbf{u}_t^A - \mathbf{u}_t^B \end{Bmatrix} \in \begin{bmatrix} \mathbf{F}_{nn} & \mathbf{F}_{nt} \\ \mathbf{F}_{tn} & \mathbf{F}_{tt} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mu\mathbb{I} & \mathbf{I} \end{bmatrix} \begin{Bmatrix} \mathbf{r}_n^A - \mathbf{r}_n^B \\ \mu(\mathbf{S}^A - \mathbf{S}^B)\mathbf{r}_n^B \end{Bmatrix}. \tag{49}$$

The coefficient matrix in the previous inclusion is non-symmetric, again a manifestation of the dependence of the tangential reactions on the normal reactions (non-associated law).

In the previous inclusions the usual set-valued algebraic operations hold, namely $\mu[-1, +1] = [-\mu, \mu]$ for any $\mu \geq 0$ (Moore, 1979; Neumaier, 1990). The following property holds between the vectors in both sides of inclusion (49).

Property (Trinkle et al., 1997)

$$\left(\begin{Bmatrix} \mathbf{r}_n^A - \mathbf{r}_n^B \\ \mu(\mathbf{S}^A - \mathbf{S}^B)\mathbf{r}_n^B \end{Bmatrix}\right)_i \left(\begin{Bmatrix} (\mathbf{u}_n^A - \mathbf{d}_n) - (\mathbf{u}_n^B - \mathbf{d}_n) \\ (\mathbf{u}_t^A - \bar{\mathbf{u}}_t) - (\mathbf{u}_t^B - \bar{\mathbf{u}}_t) \end{Bmatrix}\right)_i \leq 0, \quad \forall i \in \{1, \dots, 2n_C\}. \tag{50}$$

Proof:

(1) For the first set of n_C pairs of homologous components

$$\left(\mathbf{r}_n^A-\mathbf{r}_n^B\right)_i\left[\left(\mathbf{u}_n^A-\mathbf{d}_n\right)-\left(\mathbf{u}_n^B-\mathbf{d}_n\right)\right]_i=\left(\mathbf{r}_n^A\right)_i\left(\mathbf{u}_n^A-\mathbf{d}_n\right)_i-\left(\mathbf{r}_n^A\right)_i\left(\mathbf{u}_n^B-\mathbf{d}_n\right)_i-\left(\mathbf{r}_n^B\right)_i\left(\mathbf{u}_n^A-\mathbf{d}_n\right)_i+\left(\mathbf{r}_n^B\right)_i\left(\mathbf{u}_n^B-\mathbf{d}_n\right)_i\leq 0 \tag{51}$$

(2) Let s_i^A and s_i^B be the i-th elements of the main diagonals of matrices $\mathbf{S}^A$ and $\mathbf{S}^B$. Then

$$\begin{aligned}&\left(\mu\left(\mathbf{S}^A-\mathbf{S}^B\right)\mathbf{r}_n^B\right)_i\left[\left(\mathbf{u}_t^A-\overline{\mathbf{u}}_t\right)-\left(\mathbf{u}_t^B-\overline{\mathbf{u}}_t\right)\right]_i=\\ &\quad\mu\left(\mathbf{r}_n^B\right)_i\left(s_i^A-s_i^B\right)\left[\left(\mathbf{u}_t^A-\overline{\mathbf{u}}_t\right)_i-\left(\mathbf{u}_t^B-\overline{\mathbf{u}}_t\right)_i\right]=\\ &\mu\left(\mathbf{r}_n^B\right)_i\left[s_i^A\left(\mathbf{u}_t^A-\overline{\mathbf{u}}_t\right)_i-s_i^A\left(\mathbf{u}_t^B-\overline{\mathbf{u}}_t\right)_i-s_i^B\left(\mathbf{u}_t^A-\overline{\mathbf{u}}_t\right)_i+s_i^B\left(\mathbf{u}_t^B-\overline{\mathbf{u}}_t\right)_i\right]=\\ &\mu\left(\mathbf{r}_n^B\right)_i\left[s_i^A\left(\mathbf{u}_t^A-\overline{\mathbf{u}}_t\right)_i+s_i^B\left(\mathbf{u}_t^B-\overline{\mathbf{u}}_t\right)_i-s_i^B\left(\mathbf{u}_t^A-\overline{\mathbf{u}}_t\right)_i-s_i^A\left(\mathbf{u}_t^B-\overline{\mathbf{u}}_t\right)_i\right]=\\ &\mu\left(\mathbf{r}_n^B\right)_i\left[\left|\left(\mathbf{u}_t^A-\overline{\mathbf{u}}_t\right)_i\right|+\left|\left(\mathbf{u}_t^B-\overline{\mathbf{u}}_t\right)_i\right|-s_i^B\left(\mathbf{u}_t^A-\overline{\mathbf{u}}_t\right)_i-s_i^A\left(\mathbf{u}_t^B-\overline{\mathbf{u}}_t\right)_i\right]\leq 0,\end{aligned} \tag{52}$$

This means that the products of homologous components of the above mentioned vectors can not be strictly positive. It is now convenient to introduce the following classical

Theorem. (Cottle et al., 1992) Let $\mathbf{M}\in\mathbb{R}^{n\times n}$. The following statements are equivalent:

(a) Matrix $\mathbf{M}$ is of class P.
(b) Matrix $\mathbf{M}$ does not change the sign of any non-vanishing vector, i.e.,

$$\left[z_i(\mathbf{M}\mathbf{z})_i\leq\mathbf{0},\ \forall i\in\{1,\dots,n\}\right]\Rightarrow[\mathbf{z}=\mathbf{0}] \tag{53}$$

(c) All the real eigenvalues of $\mathbf{M}$ and of all its principal submatrices are strictly positive.

When the interval matrix in (49) is of class P, properties (50) and (53) together enable one to state the following sufficient condition.

Proposition. The quasi-static frictional contact problem has only one solution if the interval matrix

$$\begin{bmatrix}\mathbf{F}_{nn} & \mathbf{F}_{nt}\\ \mathbf{F}_{tn} & \mathbf{F}_{tt}\end{bmatrix}\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mu\mathbb{I} & \mathbf{I}\end{bmatrix}\in P. \tag{54}$$

The condition above respects the non-symmetry, contrary to what happens with criteria based on eigenvalues which consider just the symmetric part of the matrices (Doudoumis et al., 1995). In spite of being a sufficient condition, method (54) seems then able to compute sharp estimates of μ at the onset of solution multiplicity. Although condition (54) involves an infinite number of point matrices, in its numerical implementation only a finite number of point matrices are needed to be considered due to a result by (Rohn and Rex, 1996), as explained next.

Let an interval matrix $\mathbb{A}$ be represented by a set of matrices defined in the form

$$\mathbb{A}=[\underline{\mathbf{A}},\overline{\mathbf{A}}]=\{\mathbf{A}:\ \underline{\mathbf{A}}\leq\mathbf{A}\leq\overline{\mathbf{A}}\}, \tag{55}$$

where $\underline{\mathbf{A}}$ and $\overline{\mathbf{A}}$ are $n\times n$ matrices satisfying $\underline{\mathbf{A}}\leq\overline{\mathbf{A}}$ (componentwise). Matrix $\mathbb{A}$ is said to be a P-matrix if each point matrix $\mathbf{A}\in\mathbb{A}$ is a P-matrix. In (Rohn and Rex, 1996) it was shown that the number of test matrices may be reduced to 2^{n-1}. To show this, they introduced an auxiliary set

$$Z=\left\{\mathbf{z}\in\mathbb{R}^n:\ z_j\in\{-1,1\}\text{ for } j=1,\dots,n\right\}, \tag{56}$$

i.e., the set of all ±1-vectors. The cardinality of Z is obviously 2^n. For an interval matrix, matrices $\mathbf{A_z}$, $\mathbf{z}\in Z$ are defined by

$$(\mathbf{A_z})_{ij}=\frac{1}{2}\left((\underline{\mathbf{A}})_{ij}+(\overline{\mathbf{A}})_{ij}\right)-\frac{1}{2}\left((\overline{\mathbf{A}})_{ij}-(\underline{\mathbf{A}})_{ij}\right)z_iz_j \tag{57}$$

$(i,j=1,\dots,n)$. Clearly, $(\mathbf{A_z})_{ij}=(\underline{\mathbf{A}})_{ij}$ if $z_iz_j=1$ and $(\mathbf{A_z})_{ij}=(\overline{\mathbf{A}})_{ij}$ if $z_iz_j=-1$. Hence $\mathbf{A_z}\in\mathbb{A}$ for each $\mathbf{z}\in Z$, and the number of mutually different matrices $\mathbf{A_z}$ is at most 2^{n-1} (since $\mathbf{A_{-z}}=\mathbf{A_z}$ for each $\mathbf{z}\in Z$) and equal to 2^{n-1} if $\underline{\mathbf{A}}<\overline{\mathbf{A}}$. According to Theorem 2.3 in (68) matrix $\mathbb{A}$ is a P-matrix if and only if each of the finitely many matrices $\mathbf{A_z}$, $\mathbf{z}\in Z$, is a P-matrix.

The verification of property P for the interval matrix (54) is equivalent to the verification of the property P for at most 2^{n_C} of the extreme point matrices of dimension $2n_C\times 2n_C$ obtained from the original interval matrix by substitution of each interval component by its maximum or by its minimum. In addition, the verification of property P for a point matrix of dimension $2n_C\times 2n_C$ requires the verification of the signs of $2^{2n_C}-1$ principal minors (Horn and Johnson, 1985; Lancaster and Tismenetsky, 1985). Consequently, the task of verifying the property P of an interval matrix grows exponentially with the dimension of the problem. Such task becomes impossible for an interval matrix originated from a finite element model with a number of contact nodes corresponding to a satisfactorily refined mesh.

6.3. A simplified condition based on optimization techniques

Andersson (1999) defined a new fundamental parameter to compute estimates of the coefficient of friction for solution existence and uniqueness for the quasi-static evolution, incremental and rate problems. In terms of the notation used in this work Andersson's parameter may be defined as

$$c=\inf_{\substack{\mathbf{r}\in\mathbb{R}^{2n_C}\\ \|\mathbf{r}\|=1}}\ \max_{\substack{p:\ \mathbf{r}^p\neq\mathbf{0}\\ (\mathbf{F}_{CC}\mathbf{r})_t^p\neq 0}}\ \frac{r_t^p(\mathbf{F}_{CC}\mathbf{r})_t^p}{\|\mathbf{r}^p\|\,\|(\mathbf{F}_{CC}\mathbf{r})_t^p\|}, \tag{58}$$

where $\|\mathbf{r}\|=\sqrt{\sum_{p\in\mathscr{P}_C}\|\mathbf{r}^p\|^2}=1$ and $\mathbf{F}_{CC}$ is the already mentioned restriction of the flexibility matrix to the candidate contact nodes ($\mathbf{F}_{CC}\in SPD$), so that

Theorem. (Andersson, 1999) *When $\mu<\frac{c}{\sqrt{1-c^2}}$, the solution of the quasi-static incremental problem is unique.*

The resolution of problem (58) corresponds to the resolution of a number of optimization problems that grows exponentially with n_C. It may also be shown (Andersson, 2010; Holmgren, 1999) that (58) is equivalent to a set of $\frac{4^{n_C}-2^{n_C}}{2}$ classical generalized eigenvalue problems. This exponential growing of the effort to compute sharp estimates is due to the necessity of considering all the combinations of frictional contact states among the set $\mathscr{P}_C$.

Although it is wished to program the full combinatorial optimization problem (58) in the future, the will to circumvent the exponential character intrinsic to this type of estimations in the context of complementarity problems motivated the construction of a simplified method, based on Andersson's optimization criterion. It is hoped that this simplified method will lead to acceptable estimates for the value of μ at the onset of solution multiplicity in a much more economical way. The argument already invoked in Section 6.1 associates the frictional contact patterns involving active contact with impending slip at all nodes in $\mathscr{P}_C$ with (possibly) sharp estimates of μ at the onset of solution multiplicity. Based on this argument a simplified estimation of the critical value of c consists in choosing, from the many optimization problems that are implicit in (58), the one that assumes impending slip at all the contact candidate nodes. The simplified approach corresponds then to the single minmax problem

$$c = \inf_{\substack{\mathbf{r}\in\mathbb{R}^{2n_C} \\ \|\mathbf{r}\|=1}} \quad \max_{\substack{\mathbf{r}^p \neq 0 \\ p:\ (\mathbf{F}_{CC}\mathbf{r})_n^p = 0 \\ (\mathbf{F}_{CC}\mathbf{r})_t^p \neq 0}} \frac{r_t^p(\mathbf{F}_{CC}\mathbf{r})_t^p}{\|\mathbf{r}^p\|\left|(\mathbf{F}_{CC}\mathbf{r})_t^p\right|}, \tag{59}$$

where the constraints $(\mathbf{F}_{CC}\mathbf{r})_n^p = 0$ and $(\mathbf{F}_{CC}\mathbf{r})_t^p \neq 0$ have the mechanical meaning of a vanishing normal displacement and of a non-vanishing tangential displacement, respectively.

Problem (59) will be solved numerically in the next section by three different methods. The first method (denoted by Opt-SQP in several tables) consists in using the FORTRAN program FFSQP (Zhou et al., 1997; Agwa and Megahed, 2019) based on the Sequential Quadratic Programming algorithm. FFSQP includes a set of subroutines for sake of minimization/maximization of an objective functions subject to some constraints. If the initial solution for the problem is infeasible for any of the equality/inequality constraints, FFSQP creates a feasible solution for constraints and thereafter the consecutive iterations generated by the FFSQP satisfy the equality/inequality constraints.

Another method (denoted by Opt-Math) involves the built-in function FindMinMax of software MATHEMATICA (2008). Finally, problem (59) is also solved by varying the orientation of the reaction vector at each contact candidate node by small increments (usually degree by degree) such that

$$\|\mathbf{r}\| = \sqrt{\sum_{p\in\mathscr{P}_C} \|\mathbf{r}^p\|^2} = 1 \tag{60}$$

and computing the n_C objective functions for all the combinations of the reactions' orientations; this method is denoted by Opt-Def and could only be applied to check the results of the other two methods for small size problems involving one or two contact candidate nodes.

7. Onset of multiplicity: simple examples

The three methods represented earlier in order to reveal the solution multiplicity of the plane quasi-static incremental problem in the presence of rectilinear obstacle and to compute approximations of the conditions at the onset of solution multiplicity are tested in this section. Numerical experiments for the calculation of the critical value μ_I of the coefficient of friction at the onset of multiple solutions for small sized models with up to three contact candidate particles are used for sake of comparisons among all methods and to indicate the dependence of the critical values of the coefficient of friction on the structure geometry.

Table 1
Model of Klarbring ($k_i = k_v = 1$). Critical value of the friction coefficient at the onset of solution non-uniqueness of the quasi-static incremental frictional contact problem.

θ	Bijectiveness, P property, Opt-Math and Opt-Def	Opt-SQP
10	0.17632698	0.17632719
20	0.36397023	0.36397023
30	0.57735026	0.57735026
50	1.19175359	1.19175358
80	5.67128182	5.67128182

7.1. Klarbring's model

We begin by the simplest case – Klarbring's model – already described in Fig. 5. Table 1 shows that the three methods (based on bijectiveness, P property and optimization) yield the exact critical value ($\mu_{cr} = \tan\theta$) for any orientation θ of the inclined spring. The very small, may be neglected, differences in numerical values (Table 1) may be due to various round-off errors which propagate relying on the arithmetic computations.

7.2. Alart's model

The four degree of freedom model (two contact candidate particles) represented in Fig. 6(a) was first analyzed by Alart and Curnier (1987), Alart (1993, 1997). As shown in Table 2, all methods yielded the same values of μ_{cr} regardless of the model's geometry. It is interesting to observe that the values of μ_{cr} in Klarbring's and Alart's models coincide ($\mu_{cr} = \tan\theta$). In fact, Alart's model is nothing else than two (weakly) coupled models of Klarbring. The introduction of the horizontal spring in Alart's model does not affect the stiffness coupling between the directions that are normal and tangential to the obstacles. It is this stiffness coupling, solely provided by the inclined springs, that is responsible for a finite value of μ above which multiplicity of solutions may occur. For a same angle θ in both models, the inclined springs produce exactly the same normal-tangent stiffness coupling. Fig. 6(b) illustrates the variation of the critical value of μ at the onset of solution non-uniqueness of the quasi-static incremental problem as a function of the orientation of the inclined springs.

From the graph in Fig. 6(b) we see that the value of μ_{cr} vanishes for $\theta = 0$ because, for that geometry, the model has no elastic stiffness in the direction tangent to the obstacle. Consequently, for $\theta = 0$ and arbitrary small values of μ, the effective (friction affected) tangent stiffness may be negative. Recall that also in Klarbring's model, for $\theta = 0$, $k_{tt} = 0$ leading to $k_{tt} - \mu k_{nt} = -\mu k_{nt} < 0$ for arbitrary small values of μ.

A final note is due on the structure of the interval matrix for the criterion based on the P property. If one labels the left and right particles respectively 1 and 2, and chooses the displacement vector $\mathbf{u} = (u_n^1, u_t^1, u_n^2, u_t^2)$, condition (54) is

$$\mathbf{F}_{CC}\cdot\begin{bmatrix} 1 & 0 & 0 & 0 \\ [-\mu,\mu] & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & [-\mu,\mu] & 1 \end{bmatrix} \in P, \tag{61}$$

where $\mathbf{F}_{CC} \in SPD$ is a 4 by 4 point matrix.

7.3. Extended model of Alart

We consider now the system in Fig. 7(a) with three contact candidate particles restrained by a set of linear elastic springs with the same stiffness. As shown in Table 3, the two tested criteria (P property and optimization based technique) match for a wide range of values of the geometric parameter θ. The symbolic resolution with MAPLE (52) of the sufficient condition based on P property yielded the following condition

$$\mu < \frac{2+\sin^2\theta}{1+\sin^2\theta}\tan\theta, \tag{62}$$

for the interval matrix (condition (54)) to be of class P. Fig. 7(b) illustrates the estimations of μ_{cr} given by the right hand side of (62). Similarly to what happens with the two previous lumped models, the critical value of μ vanishes for $\theta = 0$ due to the possibility of a horizontal rigid body motion; in fact, for $\mu = 0$ and $\theta = 0$ the horizontal position of the three masses is indeterminate as the system has zero stiffness in the horizontal direction.

It is also interesting to note that, if the estimations of (62) for this modified version of Alart's model correspond to the exact value of μ_{cr} ,

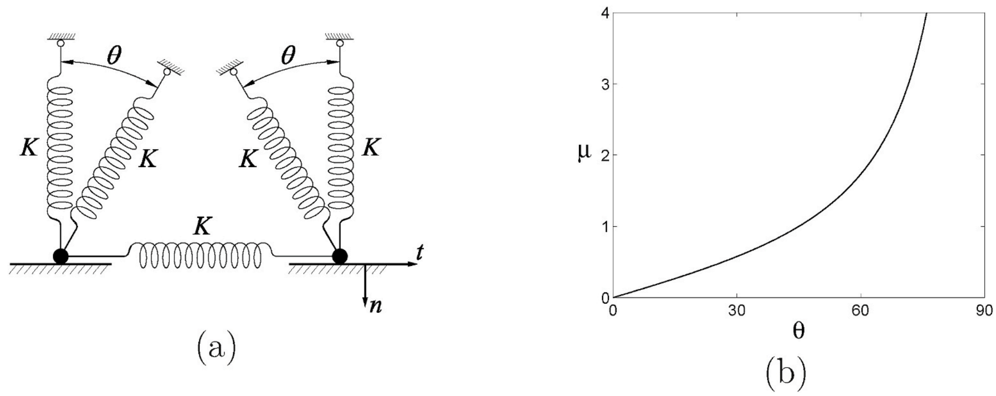


Fig. 6. Model of Alart. (a) Two particles restrained by a system of elastic springs, in the presence of unilateral frictional obstacles. (b) Coefficient of friction at the onset of solution non-uniqueness of the quasi-static incremental problem as a function of the orientation θ.

Table 2
Model of Alart. Critical value of the friction coefficient at the onset of solution non-uniqueness of the quasi-static incremental frictional contact problem using all methods (Bijectiveness, P property, Opt-SQP,Opt-Math, Opt-Def).

θ	All methods
10	0.1763269
40	0.8390996
50	1.1917535
60	1.7320508

they *do not* coincide with their homologous for the simple Alart's model and Klarbring's model. And they do not have to coincide! By inspection of the stiffness matrix of the extended Alart's model it can be observed that the normal-tangent stiffness coupling at the intermediate contact node does not exist due to the mutual canceling effects of the two inclined springs attached to that node. We have then a model where, from the three contact candidate nodes, only two have normal-tangent coupling, which explains the higher values for the estimations of μ_{cr} in the extended model of Alart with respect to the simple model of Alart and the example of Klarbring.

7.4. A lumped model with periodicity conditions

In this subsection, a model with three particles having periodicity conditions imposed on the first and third particles is studied, so that it represents an alignment of an infinite number of particles (Fig. 8(a)).

Each contact candidate particle is restrained from the exterior by mutually orthogonal springs of stiffness K_1 and K_2 with an orientation angle θ, and is connected with the two adjacent particles by springs of stiffness K_3. The symbolic treatment of the P property criterion for solution uniqueness leads to

$$\mu < \frac{\sin^2\theta + \frac{k_1}{k_2}\cos^2\theta}{\left(\frac{k_1}{k_2} - 1\right)\sin\theta\cos\theta}, \tag{63}$$

where the non-dimensional parameters $k_1 = \frac{K_1}{K_3}$ and $k_2 = \frac{K_2}{K_3}$ were used. Fig. 8(b) represents the right hand side of (63) and puts in evidence that, according to the criterion based on P property, there is no occurrence of multiple incremental solutions when the external springs are orthogonal and parallel to the obstacle or when those springs have the same stiffness, regardless of their orientation. Moreover, the minimum values of the estimations of μ_{cr} decrease with the difference between K_1 and K_2. It is also interesting to conclude that the horizontal springs of stiffness K_3 linking adjacent particles have no influence on the threshold for multiplicity. In fact, we have seen on the simple examples studied so far that the onset of solution multiplicity is due to stiffness couplings between the directions normal and tangent to the obstacle (recall the example of Klarbring in which $k_{tt} - \mu k_{nt}$ can only be negative provided k_{nt} is a non-

Table 3
Extended model of Alart. Critical value of the friction coefficient at the onset of solution non-uniqueness of the quasi-static incremental frictional contact problem.

θ	P property	Opt-SQP
10	0.34749	0.34749
40	1.43286	1.43286
50	1.9427	1.9427
60	2.7218	2.7218

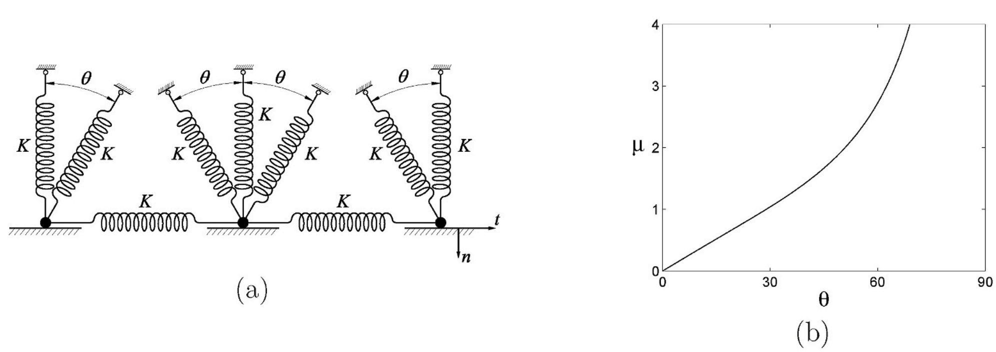


Fig. 7. Extended model of Alart. (a) Three particles of mass m each restrained by a system of elastic springs, in the presence of unilateral frictional obstacles. (b) Coefficient of friction at the onset of solution non-uniqueness of the quasi-static incremental problem as a function of the orientation of inclined springs.

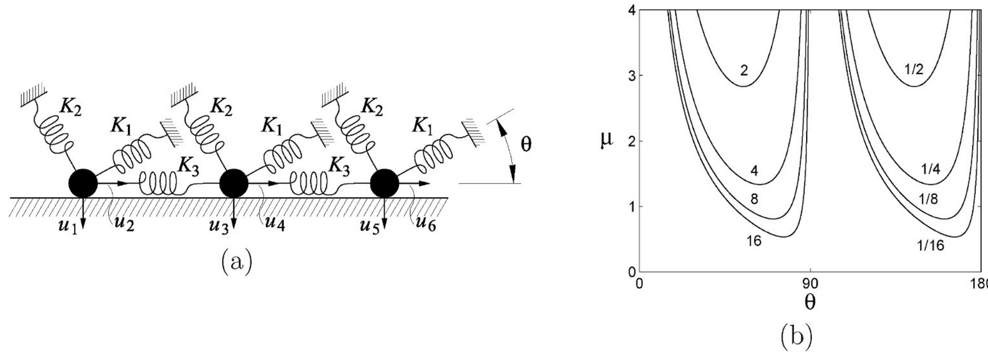


Fig. 8. Lumped model with periodicity conditions ($u_5 = u_1$, $u_6 = u_2$). (a) Geometry and the orientation of the restraining springs. (b) Coefficient of friction at the onset of solution multiplicity as a function of the inclined springs orientation for several values of $\frac{k_1}{k_2}$.

vanishing positive number). In the system represented in Fig. 8(a), the horizontal springs of stiffness K_3 have no contribution to the elements of the stiffness matrix coupling the displacements in each of the groups $\{u_1, u_2\}$, $\{u_3, u_4\}$ and $\{u_5, u_6\}$.

8. Computing all the solutions of simple examples

The following subsections are devoted to determine all the solutions of the previously described lumped examples.

8.1. The one particle model

We consider the two degrees of freedom system represented in Fig. 5. Fig. 9 shows the atlas of the solution types in the (f_n,f_t) space when the equilibrium state, at which the incremental problem is solved, is *grazing contact* (z). It is clear that, provided $\mu > \tan\theta$, the region for multiple solutions corresponds to a cone defined by $\tan\theta < \frac{f_t}{f_n} < \mu$ where three solutions coexist: stick (d), forward slip (s^+) and out of contact (f). For this one particle example, $\mu_{cr} = \tan\theta$.

Table 4 shows the solutions of the quasi-static incremental problem at a grazing contact state and the corresponding conditions on the data. In that table, $s = \sin\theta$and $c = \cos\theta$. For $\theta = 30°, 45°, 60°$, the values of the coefficient of friction at the onset of multiplicity are, respectively, $\mu_{cr} = \frac{\sqrt{3}}{3}(\simeq 0.5774), 1, \sqrt{3} \simeq 1.7321$.

We may also visualize the occurrence of solution multiplicity in the space (μ, $\tan\alpha$) where α is the angle between the applied force and the positive normal direction. Now, both springs have the same stiffness constant K. We adopt the notation $\mathbf{f} = (fc_\alpha, fs_\alpha)$with $s_\alpha = \sin\alpha$and $c_\alpha = \cos\alpha$(see Fig. 10). Table 5 shows the solution of the quasi-static incremental problem and the corresponding conditions on the data. Fig. 11 shows region by region the types of solutions of the incremental problem: in (a) when the force points downwards ($c_\alpha > 0$) and in (b) when the force points upward ($c_\alpha < 0$). Solution multiplicity (the three solutions d,f,s^+) can only occur for $c_\alpha > 0$(downward force) and (μ, $\tan\alpha$) in the shadowed cone. When $c_\alpha < 0$(force pointing upward) only two types of solutions may be observed, f or s^-, and no multiplicity occurs, as illustrated in Fig. 11(b).

By choosing conditions (μ,$\tan\alpha$) near the corners and edges in each of the regions defined by the charts in Fig. 11, the algorithm of De Moor (1988) computed correctly the set of all solutions to the quasi-static incremental problem (formulated as a *GLCP*) in each case. The algorithm of Bart de Moor computes all the solutions of a generalized linear complementarity problem, even if there are infinitely many. It requires, however, a CPU execution time that grows *exponentially* with the number of complementarity variables.

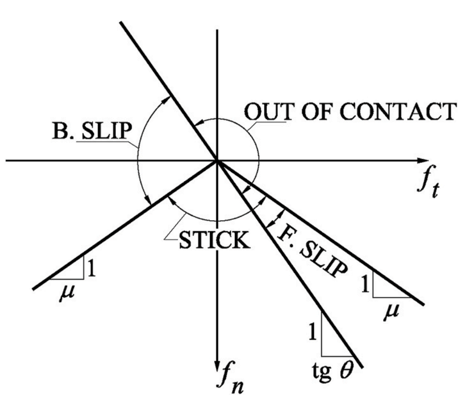


Fig. 9. Example of Klarbring. Atlas of the solution types of the quasi-static incremental problem in the (f_n, f_t) space when particle is in a state of grazing contact.

Table 4
Solutions of the quasi-static incremental problem formulated at a grazing contact state and conditions on the data for the one particle model represented in Fig. 5. Notation: $d\equiv$stick, $s^+\equiv$forward slip, $s^-\equiv$backward slip, $f\equiv$out of contact.

Contact state	Conditions on the data	Solution
d	$f_n > 0$ and $\lvert f_t \rvert < \mu f_n$	$u_n = u_t = 0$, $r_n = -f_n$ and $r_t = -f_t$
s^+	$f_t - \frac{s}{c}f_n < 0$ and $\mu < \frac{s}{c}$ $f_t - \frac{s}{c}f_n > 0$ and $\mu > \frac{s}{c}$	$u_n = 0$, $u_t = \frac{f_t - \mu f_n}{s(s-\mu c)}$, $r_n = \frac{cf_t - sf_n}{s-\mu c}$ and $r_t = \mu r_n$
s^-	$f_t + \mu f_n < 0$ and $f_t - \frac{s}{c}f_n < 0$	$u_n = 0$, $u_t = \frac{f_t + \mu f_n}{s(s+\mu c)}$, $r_n = \frac{cf_t - sf_n}{s+\mu c}$ and $r_t = -\mu r_n$
f	$f_t > \frac{s}{c}f_n$	$u_n = f_n - \frac{c}{s}f_t$, $u_t = -\frac{c}{s}f_n + \frac{1+c^2}{s^2}f_t$, $r_n = r_t = 0$

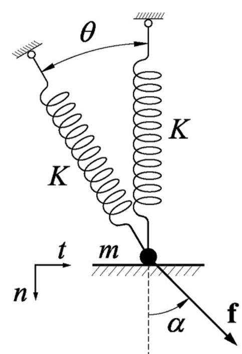


Fig. 10. The one particle model with a concentrated applied force.

Table 5
Solutions of the quasi-static incremental problem and conditions on the data for the structure with one contact candidate particle represented in Fig. 10.

Contact state	Conditions on the data	Solution
d	$\alpha \in]-\frac{\pi}{2}, \frac{\pi}{2}[$ and $\lvert\tan\alpha\rvert < \mu$	$u_n = u_t = 0$, $r_n = -c_\alpha$ and $r_t = -s_\alpha$
s^+	$\alpha \in]-\frac{\pi}{2}, \frac{\pi}{2}[$, $\mu < \tan\alpha < \frac{s}{c}$ and $\mu < \frac{s}{c}$	$u_n = 0$, $u_t = \frac{s_\alpha - \mu c_\alpha}{s(s - \mu c)}$,
	$\alpha \in]-\frac{\pi}{2}, \frac{\pi}{2}[$, $\frac{s}{c} < \tan\alpha < \mu$ and $\mu > \frac{s}{c}$	$r_n = \frac{c s_\alpha - s c_\alpha}{s - \mu c}$ and $r_t = \mu r_n$
s^-	$\alpha \in]-\frac{\pi}{2}, \frac{\pi}{2}[$ and $\tan\alpha < -\mu$	$u_n = 0$, $u_t = \frac{s_\alpha + \mu c_\alpha}{s(s + \mu c)}$,
	$\alpha \in]\frac{\pi}{2}, \frac{3\pi}{2}[$ and $\tan\alpha > \frac{s}{c}$	$r_n = \frac{c s_\alpha - s c_\alpha}{s + \mu c}$ and $r_t = -\mu r_n$
f	$\alpha \in]-\frac{\pi}{2}, \frac{\pi}{2}[$ and $\tan\alpha > \frac{s}{c}$	$u_n = \frac{s c_\alpha - c s_\alpha}{s}$, $u_t = \frac{s_\alpha(1 + c^2) - c s c_\alpha}{s^2}$,
	$\alpha \in]\frac{\pi}{2}, \frac{3\pi}{2}[$ and $\tan\alpha < \frac{s}{c}$	$r_n = 0$ and $r_t = 0$

8.2. Alart's model (with a particular loading)

The model presented in this subsection consists of two particles A and B, connected by a horizontal spring and restrained by vertical and inclined springs in a symmetric way (see Fig. 12). All the springs have the same stiffness constant K. The inclined springs make an angle θ with the vertical. We assume that both particles are in a state of *grazing contact* when the pair of forces $\mathbf{f}^{\mathrm{A}}$ and $\mathbf{f}^{\mathrm{B}}$ are applied symmetrically (both make an angle α with the vertical). The system has four degrees of freedom and the generalized displacements $\mathbf{u} = (u_n^A, u_t^A, u_n^B, u_t^B)$ are used. With respect to $\mathbf{u}$, the stiffness matrix is

$$\mathbf{K} = K \begin{bmatrix} 1+c^2 & -sc & 0 & 0 \\ -sc & 1+s^2 & 0 & -1 \\ 0 & 0 & 1+c^2 & sc \\ 0 & -1 & sc & 1+s^2 \end{bmatrix}, \tag{64}$$

where $s = \sin\theta$ and $c = \cos\theta$; the force vector is $\mathbf{f} = (f c_\alpha, f s_\alpha, f c_\alpha, -f s_\alpha)$, with $f = \left\|\mathbf{f}^{\mathrm{A}}\right\| = \left\|\mathbf{f}^{\mathrm{B}}\right\|$, $s_\alpha = \sin\alpha$ and $c_\alpha = \cos\alpha$. For this four-degree of freedom system, the quasi-static incremental problem is governed by the following system of equations

$$K \begin{bmatrix} 1+c^2 & -sc & 0 & 0 \\ -sc & 1+s^2 & 0 & -1 \\ 0 & 0 & 1+c^2 & sc \\ 0 & -1 & sc & 1+s^2 \end{bmatrix} \begin{Bmatrix} u_n^A \\ u_t^A \\ u_n^B \\ u_t^B \end{Bmatrix} = \begin{Bmatrix} f c_\alpha \\ f s_\alpha \\ f c_\alpha \\ -f s_\alpha \end{Bmatrix} + \begin{Bmatrix} r_n^A \\ r_t^A \\ r_n^B \\ r_t^B \end{Bmatrix}, \tag{65}$$

together with the unilateral contact conditions $u_n^p \le 0$, $r_n^p \le 0$, $u_n^p r_n^p = 0$, and the incremental form of Coulomb's friction law $r_t^p \in \mu r_n^p \mathrm{Sign}(u_t^p)$ ($\overline{u}_t^p$ is assumed to be zero for both particles) (Agwa and Pinto da Costa, 2011, Pinto da Costa and Agwa, 2011, 2013; Agwa and da Costa, 2015).

Table 6 shows the solutions of the quasi-static incremental problem of the above described system and the corresponding conditions on the data. All the types of solutions of this model problem were also calculated by the algorithm of De Moor and Vandewalle (1987), De Moor (1988), De Moor et al. (1992). In order to provide a graphical insight of the dependence of the solution set on the two independent parameters controlling the data, μ and α, we show in Fig. 13, region by region of the space $(\mu, \tan\alpha)$ the types of solutions of the incremental problem: in (a) when the forces point downwards ($c_\alpha > 0$) and in (b) when the forces point upwards ($c_\alpha < 0$). Fig. 13(a) is to be observed in parallel with Table 7. We observe that, for $\mu < \frac{s}{c}$ and $c_\alpha > 0$, regardless of the value of $\tan\alpha$, the solution is unique, as in Klarbring's example. Moreover, solution multiplicity can only occur for $\tan\alpha < -\frac{s}{c}$ and $c_\alpha > 0$, i.e., for

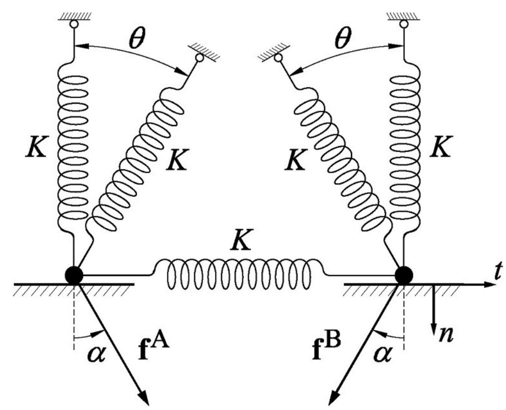


Fig. 12. Model of Alart with a symmetric force loading.

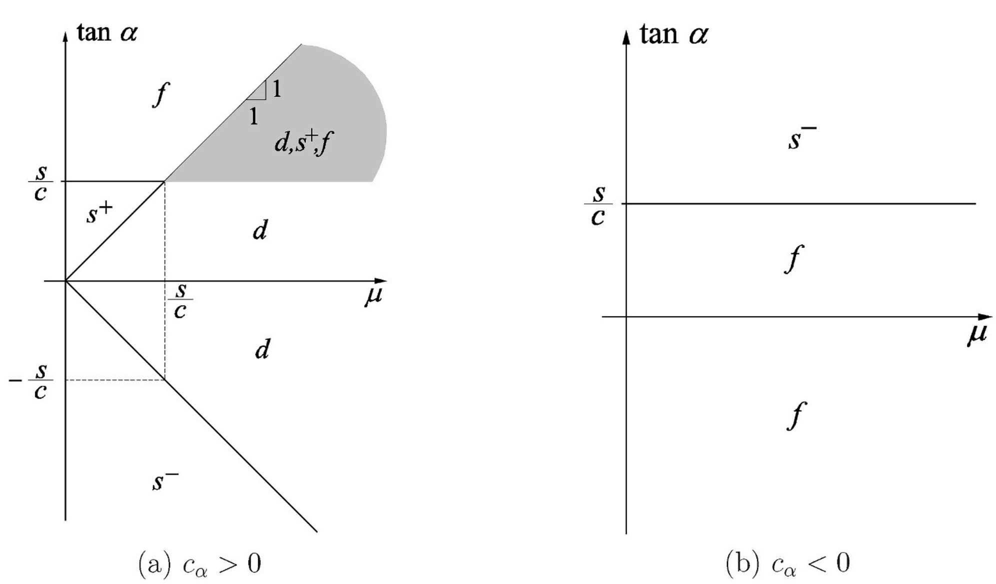


Fig. 11. The one particle model. Atlas of the solution types in the space $(\mu, \tan\alpha)$. The grey regions indicate multiplicity of solution. Notation: $d \equiv$ stick, $s^+ \equiv$ forward slip, $s^- \equiv$backward slip, $f \equiv$out of contact.

Table 6
Solutions of the quasi-static incremental problem and conditions on the data for the structure with two contact candidate particles represented in Fig. 12.

Contact state	Conditions on the data	Solution
(d,d)	$\alpha\in\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$ and $\lvert\tan\alpha\rvert<\mu$	$u_n^A=u_t^A=u_n^B=u_t^B=0$, $r_n^A=r_n^B=-c_\alpha$ and $r_t^A=-r_t^B=s_\alpha$
(s^+,s^-)	$\alpha\in\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$ and $\tan\alpha>\mu$ $\alpha\in\left]\frac{\pi}{2},\frac{3\pi}{2}\right[$, $\tan\alpha<-\frac{2+s^2}{sc}$	$u_n^A=0$, $u_t^A=\frac{-\mu c_\alpha+s_\alpha}{s(s+\mu c)+2}$, $u_n^B=0$, $u_t^B=-u_t^A$, $r_n^A=\frac{-(scs_\alpha+c_\alpha(2+s^2))}{s(s+\mu c)+2}$, $r_t^A=\mu r_n^A$, $r_n^B=r_n^A$ and $r_t^B=-\mu r_n^B$
(s^-,s^+)	$\alpha\in\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$, $-\frac{2+s^2}{sc}<\tan\alpha<-\mu$ and $\mu<\frac{2+s^2}{sc}$ $\alpha\in\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$, $-\mu<\tan\alpha<-\frac{2+s^2}{sc}$ and $\mu>\frac{2+s^2}{sc}$	$u_n^A=u_n^B=0$, $u_t^A=-u_t^B=\frac{\mu c_\alpha+s_\alpha}{s(s-\mu c)+2}$, $r_n^A=r_n^B=-\frac{scs_\alpha+c_\alpha(2+s^2)}{s(s-\mu c)+2}$, $r_t^A=-\mu r_n^A$ and $r_t^B=\mu r_n^B$
(f,f)	$\alpha\in\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$ and $\tan\alpha<-\frac{2+s^2}{sc}$ $\alpha\in\left]\frac{\pi}{2},\frac{3\pi}{2}\right[$ and $\tan\alpha>-\frac{2+s^2}{sc}$	$u_n^A=u_n^B=\frac{c_\alpha(2+s^2)+scs_\alpha}{3+c^2}$, $u_t^A=-u_t^B=\frac{s_\alpha(1+c^2)+scc_\alpha}{3+c^2}$ and $r_n^A=r_t^A=r_n^B=r_t^B=0$
(d,s^+)	$\alpha\in\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$, $\mu\in\left]\frac{s}{c},\frac{1+s^2}{sc}\right[$ and $\tan\alpha\in\left]-\frac{1+s^2}{sc},-\mu\right[$ $\alpha\in\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$, $\tan\alpha\in\left]-\mu,-\frac{1+s^2}{sc}\right[$ and $\mu>\frac{1+s^2}{sc}$	$u_n^A=u_t^A=u_n^B=0$, $u_t^B=-\frac{s_\alpha+\mu c_\alpha}{s(s-\mu c)+1}$, $r_n^A=-c_\alpha$, $r_t^A=\frac{\mu c_\alpha-s_\alpha s(s-\mu c)}{s(s-\mu c)+1}$, $r_n^B=-\frac{scs_\alpha+c_\alpha(1+s^2)}{s(s-\mu c)+1}$ and $r_t^B=\mu r_n^B$
(s^-,d)	$\alpha\in\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$, $\mu\in\left]\frac{s}{c},\frac{1+s^2}{sc}\right[$ and $\tan\alpha\in\left]-\frac{1+s^2}{sc},-\mu\right[$ $\alpha\in\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$, $\tan\alpha\in\left]-\mu,-\frac{1+s^2}{sc}\right[$ and $\mu>\frac{1+s^2}{sc}$	$u_n^A=u_n^B=u_t^B=0$, $u_t^A=\frac{s_\alpha+\mu c_\alpha}{s(s-\mu c)+1}$, $r_n^A=-\frac{scs_\alpha+c_\alpha(1+s^2)}{s(s-\mu c)+1}$, $r_t^A=-\mu r_n^A$, $r_n^B=-c_\alpha$ and $r_t^B=-\frac{\mu c_\alpha-s_\alpha s(s-\mu c)}{s(s-\mu c)+1}$
(f,d)	$\alpha\in\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$, $\frac{sc-2\mu}{s^2}<\tan\alpha<-\frac{1+s^2}{sc}$ and $\mu>\frac{s}{c}$	$u_n^A=\frac{c_\alpha(1+s^2)+scs_\alpha}{2}$, $u_t^A=\frac{s_\alpha(1+c^2)+scc_\alpha}{2}$, $u_n^B=u_t^B=0$, $r_n^A=r_t^A=0$, $r_n^B=-c_\alpha$ and $r_t^B=\frac{s(ss_\alpha-cc_\alpha)}{2}$
(d,f)	$\alpha\in\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$, $\frac{sc-2\mu}{s^2}<\tan\alpha<-\frac{1+s^2}{sc}$ and $\mu>\frac{s}{c}$	$u_n^A=u_t^A=0$, $u_n^B=\frac{c_\alpha(1+s^2)+scs_\alpha}{2}$, $u_t^B=-\frac{s_\alpha(1+c^2)+scc_\alpha}{2}$, $r_n^A=-c_\alpha$, $r_t^A=-\frac{s(ss_\alpha-cc_\alpha)}{2}$ and $r_n^B=r_t^B=0$,
(f,s^+)	$\alpha\in\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$, $\tan\alpha\in\left]-\frac{2+s^2}{sc},\frac{sc-2\mu}{s^2}\right[$ and $\mu\in\left]\frac{s}{c},\frac{3}{2}\frac{s}{c}\right[$ $\alpha\in\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$, $\tan\alpha\in\left]\frac{sc-2\mu}{s^2},-\frac{2+s^2}{sc}\right[$ and $\mu\in\left]\frac{3}{2}\frac{s}{c},+\infty\right[$	$u_n^A=\frac{(\mu c-s)(c_\alpha(2+s^2)+s_\alpha sc)}{2\mu c-3s}$, $u_t^A=\frac{(c\mu-s)(ss_\alpha(1+c^2)+c_\alpha c(1+s^2))+c_\alpha\mu}{s(2\mu c-3s)}$, $u_n^B=0$, $u_t^B=\frac{2\mu c_\alpha+s(ss_\alpha+cc_\alpha)}{s(2\mu c-3s)}$, $r_n^A=r_t^A=0$, $r_n^B=\frac{s(s_\alpha sc+c_\alpha(2+s^2))}{2\mu c-3s}$ and $r_t^B=\mu r_n^B$
(s^-,f)	$\alpha\in\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$, $\tan\alpha\in\left]-\frac{2+s^2}{sc},\frac{sc-2\mu}{s^2}\right[$ and $\mu\in\left]\frac{s}{c},\frac{3}{2}\frac{s}{c}\right[$ $\alpha\in\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$, $\tan\alpha\in\left]\frac{sc-2\mu}{s^2},-\frac{2+s^2}{sc}\right[$ and $\mu\in\left]\frac{3}{2}\frac{s}{c},+\infty\right[$	$u_n^A=0$, $u_t^A=\frac{2\mu c_\alpha+s(ss_\alpha+cc_\alpha)}{s(3s-2\mu c)}$, $u_n^B=\frac{(\mu c-s)(c_\alpha(2+s^2)+s_\alpha sc)}{2\mu c-3s}$, $u_t^B=\frac{(c\mu-s)(ss_\alpha(1+c^2)+c_\alpha c(1+s^2))+c_\alpha\mu}{s(3s-2\mu c)}$, $r_n^A=\frac{s(s_\alpha sc+c_\alpha(2+s^2))}{2\mu c-3s}$, $r_t^A=-\mu r_n^A$ and $r_n^B=r_t^B=0$
(s^+,f), (f,s^-)	–	No solution
(s^+,d), (d,s^-)	–	No solution
(s^+,s^+), (s^-,s^-)	–	No solution

applied forces $\mathbf{f}^A$ and $\mathbf{f}^B$ sufficiently inclined with respect to the vertical and pointing outwards and downwards; recall that in the example of Klarbring multiple solutions can only occur when the applied force points to the side opposite to the inclined spring and with a sufficiently large inclination. We observe from Fig. 13(a) that, like in the example of Klarbring in which the region for multiple solutions corresponds to a cone with a limited range of angles for the orientation of the applied loads, in the model of Alart (with symmetric loading) a similar situation occurs.

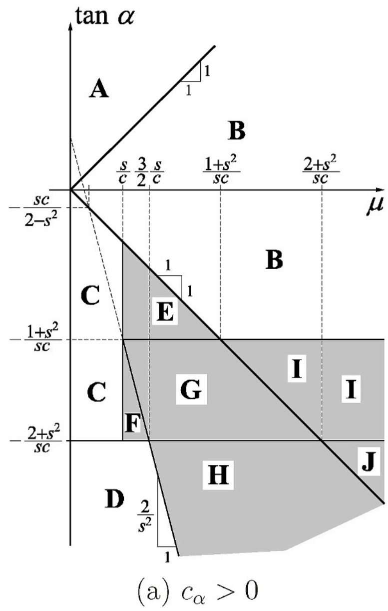
From the observation of Fig. 13(a) and Table 7 it can be seen that in regions H, I and J, corresponding to larger values of μ, the solution set has larger cardinalities. Fig. 13(a) presents a peculiarity. For any $\tan\alpha\in\left]-\frac{s}{c},-\frac{1+s^2}{sc}\right[$, as μ increases from 0, the cardinality of the solution set does not increase monotonically with μ; in fact, as regions C, E and B are crossed in the direction of growing μ, the number of solutions is, respectively, 1, 3 and 1. In this range of variation of $\tan\alpha$, the number of solutions of the incremental problem does not increase monotonically as the intuition would suggest and as it happens in other ranges of $\tan\alpha$.

When $c_\alpha<0$ (force pointing upwards) only two types of solution may be observed, (f,f) or (s^+,s^-), and no solution multiplicity occurs, as shown in Fig. 13(b). Moreover, when $\tan\alpha<-\frac{2+c^2}{sc}$ (for directions sufficiently near the horizontal) the solution is (s^+,s^-), while when $\tan\alpha>-\frac{2+c^2}{sc}$ (always with $c_\alpha<0$) the solution is (f,f).

Finally we note that, for the particular type of loading considered, the threshold for solution multiplicity occurs for $\mu_{cr}=\tan\theta$, as predicted by the three methods used in subsection 7.2 for any loading.

9. Conclusions

The present paper (Part I) is not intended to have conclusions as it



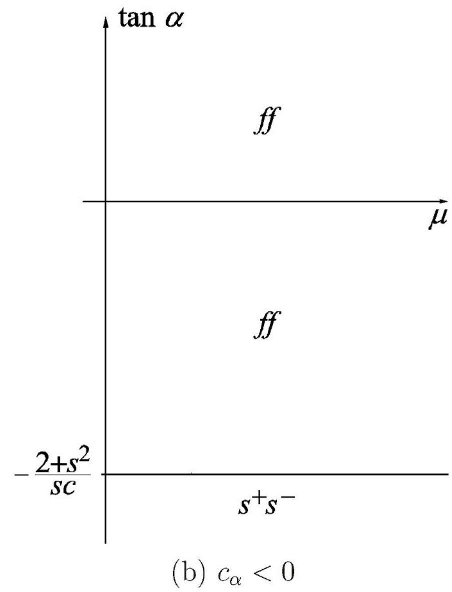


Fig. 13. Model of Alart. Atlas of the solution types in the space (μ, tanα) corresponding to the particular loading indicated in Fig. 12. The grey regions indicate solution multiplicity.

Table 7
Model of Alart with symmetric force loading. Types of solutions existing in each of the regions A to J of Fig. 13.

Region in Fig. 13	♯solutions	Types of solutions
A	1	$\{(s^+,s^-)\}$
B	1	$\{(d,d)\}$
C	1	$\{(s^-,s^+)\}$
D	1	$\{(f,f)\}$
E	3	$\{(d,s^+),(s^-,d),(s^-,s^+)\}$
F	3	$\{(f,s^+),(s^-,f),(s^-,s^+)\}$
G	3	$\{(d,f),(f,d),(s^-,s^+)\}$
H	5	$\{(f,s^+),(s^-,f),(d,f),(f,d),(f,f)\}$
I	5	$\{(d,s^+),(s^-,f),(d,f),(f,d),(d,d)\}$
J	9	$\{(f,s^+),(s^-,f),(d,s^+),(s^-,d),$ $(s^-,s^+),(d,f),(f,d),(f,f),(d,d)\}$

mainly prepares the way by laying down the notations and the definitions used in this work and by giving the formulations of the main problem used in the present study. We began with the definitions of the unilateral contact law and Coulomb friction law in 2D, and then the notation associated with the partition of the degrees of freedom was presented. The definitions of the sets of admissible displacements and admissible reactions, relevant for solving incremental problem were given. Also the definitions of the sets of admissible displacement rates and admissible reaction rates were also established.

The work presented here deals with an important issue arising in frictional contact problems involving flexible bodies: the occurrence of more than one solution, with an emphasis on the quasi-static incremental problem. The present research was exclusively dedicated to the theoretical modeling of the quasi-static problem and to the comparison between several techniques used in the determination of the conditions for which the quasi-static incremental problem may exhibit more than one solution. The two ingredients necessary for solution multiplicity are stiffness and friction; the above mentioned conditions are directly related with a sufficiently large friction coefficient and a sufficiently large stiffness coupling between the directions that are normal and tangent to the obstacle.

The conditions for existence of multiple solutions to the quasi-static incremental problem, with an intrinsic combinatorial character, are presented for several criteria. Three different criteria to assess the onset of solution multiplicity of the quasi-static incremental problem were proposed: (*i*) the necessary and sufficient condition based on bijectiveness, (*ii*) the sufficient condition based on *P* property and (*iii*) the simplified condition based on optimization. The study concerning solution existence/multiplicity of the quasi-static incremental problem led to the proposal of a simplified criterion based on an optimization problem for estimating the friction coefficient at the onset of multiplicity. This criterion avoids the combinatorial character of the necessary and sufficient condition based on the properties of a conewise linear (nonsmooth) operator. The conditions for the occurrence of multiple solutions were discussed. An algorithm was introduced for the computation of all the solutions of the incremental problem and to verify the sharpness of the frictional coefficient estimates corresponding to the several criteria.

All the solutions were calculated for some lumped mass examples and their dependencies on some parameters were also discussed. The conditions under which a problem may have multiple solutions were discussed for several lumped models. The theoretical analysis presented in this paper provides the basic formulations of the quasi-static incremental problem upon which following analytical and numerical solutions, for two dimensional finite element examples, ought to be introduced in the second part (Part II) of this series.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Supplementary data

Supplementary data to this article can be found online at https://doi.org/10.1016/j.euromechsol.2020.104062.

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