Mechanics of variable-mass systems—Part 1: Balance of mass and linear momentum

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The equations of balance of momentum and energy usually are formulated under the assumption of conservation of mass. However, mass is not conserved when sources of mass are present or when the equations of balance are applied to a non-material volume. Mass then is said to be variable for the system under consideration. It is the scope of the present contribution to review the mechanical equations of balance for variable-mass systems. Our review remains within the framework of the classical, non-relativistic continuum mechanics of solids and fluids. We present general formulations and refer to various fields of applications, such as astronomy, machine dynamics, biomechanics, rocketry, or fluid dynamics. Also discussed are the equations for a single constituent of a multiphase mixture. The present review thus might be of interest to workers in the field of heterogeneous media as well. We first summarize the general balance law and review the Reynolds transport theorem for a non-material volume. Then the latter general formulations are used to derive and to review the equations of balance of mass and linear momentum in the presence of sources of mass in the interior of a material volume. We also discuss the appropriate modeling of such sources of mass. Subsequently, we treat the equations of balance of mass and linear momentum when mass is flowing through the surface of a non-material volume in the absence of sources of mass in the interior, and we point out some analogies to the previously presented relations. A strong emphasis is given in this article to the historical evolution of the balance equations and the physical situations to which they have been applied. The equations of balance of angular momentum and energy for variable-mass systems will be treated separately, in a second part of the review, to be published later. This review article cites 96 references. [DOI: 10.1115/1.1687409]

1 INTRODUCTION

When the total of some entity, its quantity, is known to be invariant for a given volume, it is said to be conserved within the volume. When it is not conserved, then it is said to be balanced by sources of that quantity.

Only in exceptional cases, a physical meaning can be attached to the sources balancing a specific quantity. A considerable part of the history of mechanics may be understood as a gigantic struggle for the proper kinematical quantities to be balanced by physically meaningful sources of the motion of material bodies. Nowadays, the laws of balance of mass, of linear and angular momentum, and of energy are considered as the proper set of balance equations of mechanics. These balance equations are considered as fundamental, since they hold for all material bodies, be the latter modeled as a system of single mass-points, be the mass continuously distributed in solid or fluid form, or be it mixed from various constituents.

The concept of mass asserts that mass is conserved within a material volume of a body in the absence of sources of mass. A material volume possesses a closed surface that is moving with the material particles located on this surface. The fundamental equations of classical mechanics were originally formulated for the case of an invariant mass contained in a material volume. (In the Einstein special theory of relativity, which is outside the topic of the present review, a velocity-dependent relativistic mass may be introduced in formulating the equations of motion of a particle. However, the rest mass is usually assumed to be constant, see, eg, Sections 29 and 63 of Pauli [1].)

Mass is generally not conserved when sources of mass are present, or when it is not appropriate to consider a material volume at all. Mass then is said to be variable for the system under consideration. Mechanical systems with variable mass follow as the result of a problem-oriented modeling. In the mechanics of solids and mass-points, numerous applied problems can be found in which material is expelled from a reservoir, or in which material is captured and afterwards

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transported by some mechanism. Frequently, it turns out to be impossible (or it is not appropriate) to model the motion of the material which is situated inside the reservoir, or which is located within some distance of the mechanism. It is then necessary to enclose the interesting portions of the respective material body by means of a non-material volume.

A non-material volume possesses a closed surface that is moving at a velocity different from the velocity of the material particles instantaneously located on this surface. The mass, which is contained in a non-material volume, therefore needs not to be conserved. Such a non-material volume is called a control volume in the terminology of fluid mechanics. A flux of mass per unit area and time appears to take place across the control surface enclosing the control volume.

The use of the control volume concept is quite natural in fluid mechanics and has become standard in this field. In the literature on fluid mechanics, the special case of a non-material volume at rest is sometimes denoted as *control volume*, the case of a rigid control volume moving relatively to the material is referred to as *non-inertial control volume*, and the material contained in a material volume is called the *system*, see Section 4.7 of the book on the mechanics of fluids by Shames [2]. See also, eg, Section 1.2.4 of the handbook article on fluid dynamics by White [3]. In order to avoid ambiguities, we prefer to use the notion of a *non-material volume* when we discuss a volume enclosed by a control surface moving at a velocity different from the velocity of the material particles. Subsequently, we do not restrict ourselves to the case of a rigid control surface. We furthermore refer to a *spatial volume* in the special case of a volume enclosed by a rigid control surface at rest. When we write about a *system*, we mean the mechanical model consisting of material particles, control surfaces, and sources of mass.

Frequently, in solid as well as in fluid mechanics, it is appropriate to model the exchange of mass between the volume under consideration and the outside world by means of interior sources (or sinks) of mass. For instance, in the continuum theory of reactive mixtures, in the so-called theory of heterogeneous media, mass is assumed to be exchanged between the various constituents such that it appears to be not conserved for a single constituent within a volume being material for the total mixture. When writing the equations of balance for a single constituent of a reactive mixture, distributed sources of mass thus must be considered, see Section 158 of Truesdell and Toupin [4].

It is the scope of the present contribution to review some extensions of the fundamental equations of mechanics that were presented in the literature with respect to problems without conservation of mass. To a certain extent, we tried to follow the notations used in the celebrated Handbuchartikel of Truesdell and Toupin [4] on the classical field theories published in 1960, such that the reader may consult their widely distributed text if a further clarification is needed. The majority of the subsequently presented relations concerning the equations of balance of mass and of linear momentum for variable-mass systems were not worked out by Truesdell and Toupin [4]. Our review thus contains a large number of additional important contributions belonging to both the periods after and before the year 1960. The Handbuchartikel of Truesdell and Toupin [4], however, represents an important land-mark also with respect to the development of the topic under consideration.

The material of the present review is organized as follows: In Section 2, we summarize the presentation of the general balance law given in Section 157 of Truesdell and Toupin [4], and we review a general formulation of the transport theorem dating back to Reynolds [5]. In Section 3 of our review, we use the general formulations of Section 2 to derive and to review the equations of balance of mass and linear momentum when mass is supplied by sources in the interior of a material volume, and we discuss the appropriate modeling of sources of mass in some detail. In Section 4, we treat the equations of balance of mass and linear momentum when mass is flowing through the surface of a non-material volume in the absence of interior sources of mass, and we point out some analogies to the equations presented in Section 3. The equations of balance of angular momentum and energy for variable-mass systems will be treated separately, in a second part of the review.

Throughout the paper, we use the spatial description of continuum mechanics. A transition from our continuum formulation to systems of mass-points can be performed by interpreting the integrals occurring in the equations of balance in the sense of Stieltjes-integrals, see, eg, Sections 2 and 26 of Hamel [6]. For the sake of brevity, the transition to systems of mass-points will be left to the reader. We further restrict our formulations to homogeneous media, or to the single constituents of a mixture.

In writing the present review, we intended to classify results to be found here and there in the literature, not only in older expositions, but also in more recent contributions. We especially wished to bring some possibly forgotten work to the attention of the reader. However, we cannot claim that our review is complete. The literature on the topic turned out to be sparsely distributed over a wide range of different areas. Please understand that we have limited our exposition to some of the more fundamental contributions that came to our knowledge, and to some of the respective areas only.

2 GENERAL LAWS OF BALANCE AND THE GENERALIZED TRANSPORT THEOREM

We start our derivations from the generalized form of the balance law, as it was presented in Section 157 of Truesdell and Toupin [4] with reference to Prigogine [7] and Grad [8]:

$$\frac{d}{dt}\int_V \Psi \rho dv = \int_V s[\Psi]\rho dv - \int_S da \cdot i[\Psi] \tag{2.1}$$

In this relation, V represents a material volume with a closed surface S, and ρ denotes the density of the not necessarily invariant mass included within V. Time is denoted by t. The trident Ψ stands for a scalar, a vector, or a tensor of any order. The influx of Ψ through S is denoted by $i[\Psi]$, and $s[\Psi]$ is the supply of Ψ within V. The dot product operation appearing in the surface integral at the right hand side of Eq.

(2.1) is defined and explained in Section 3 of Ericksen [9]. The oriented area element da of S is a vector pointing outwards of V, see Ericksen [9], Section 25. Consequently, due to the minus sign, the last term in Eq. (2.1) models the total influx of Ψ into S. In the following, we restrict Ψ to scalars and vectors. The general balance law, Eq. (2.1), defines how the rate of change of the total of $\Psi\rho$ included in the material volume V should be balanced by an appropriate combination of influx $i[\Psi]$ and supply $s[\Psi]$.

For the subsequent applications of the general balance law, it is noted that we refer our formulations to an inertial frame throughout the paper. When we speak about the rate of change of an entity or a quantity, we thus mean the absolute entity or quantity and the absolute rate of change, measured with respect to that inertial frame. Particularly, when we talk about a relative velocity, we mean a portion of the absolute velocity, the rate of change of the relative velocity being observed in the inertial frame. For the sake of brevity, we do not explicitly utilize the representation of absolute quantities by their counterparts measured in a non-inertial frame, nor do we write down explicitly the partitioning of quantities into their local and convective portions.

The Gauss-Green-Ostrogradsky divergence theorem asserts that

$$\int_S da \cdot i[\Psi] = \int_V \mathrm{div}\, i[\Psi]\, dv \tag{2.2}$$

The divergence operator is denoted by div. For the continuity requirements necessary for Eq. (2.2) to hold, see Ericksen [9], Section 26. As was remarked in Section 157 of Truesdell and Toupin [4], the divergence theorem demonstrates the equivalence of surface and volume sources appearing in Eq. (2.1): In order to secure balance, it is sufficient to replace any continuous influx by a supply $s = -\rho^{-1}\mathrm{div}\, i$, or any supply (satisfying a Hölder condition) by an influx such that $\mathrm{div}\, i = -\rho s$. Any solenoidal (divergence-free) field may be added, such that there are infinitely many i satisfying the latter requirement. Not only is there this indeterminism in the right hand side of Eq. (2.1), it may be further said that all quantities Ψ may be balanced, by definition, since an appropriate combination of sources i and s for Ψ may always be defined. Truesdell and Toupin [4] however remarked that, ‘‘despite the tautologism of the general balance law, it is useful because in many cases we have a priori knowledge of i or s, or we can derive special forms for these quantities from an information about Ψ.’’ This is particularly the case when it is known that the right hand side of Eq. (2.1) must vanish from physical reasons. The total of $\Psi\rho$ then is said to be conserved in V.

We note that it is indeed the physical meaning of the sources balancing a specific kinematical quantity, which makes the respective equation of balance applicable in practice. For instance, Laplace in 1799 contemplated the balance of a kinematical quantity given by the product of mass and an arbitrary function of velocity, see Chapter 1 of Part IV of the book on the history of mechanics by Dugas [10]. To Laplace, the momentum of a mass-point was not necessarily proportional to its velocity, but the mass was considered as constant. It has been pointed out by Dugas [10], however, that the momentum of Laplace may be rewritten as the linear momentum of a velocity dependent mass. Notwithstanding its lack of physically meaningful sources, the generalized mechanics of Laplace thus represents an important predecessor of the mechanics of systems with a variable mass.

The general balance law, Eq. (2.1), is written for a material volume V whose boundary S is in motion at the velocity of the material particles located on this boundary. When we want to consider a non-material volume $v(t)$ bounded by a control surface $s(t)$ which is moving at a prescribed velocity u different from the particle velocity, and when $s(t)$ is instantaneously coinciding with the material surface, $s(t) = S$, we may utilize the Reynolds transport theorem in the following general form:

$$\frac{d_u}{dt}\int_{v(t)} \rho\Psi\, dv = \frac{d}{dt}\int_V \rho\Psi\, dv - \int_{s(t)} da \cdot (v-u)\rho\Psi \tag{2.3a}$$

The velocity of a material particle is $v = \dot{p}$, the position vector of the particle from a fixed origin being denoted as p. The superimposed dot denotes the rate of change. The operator d_u/dt in Eq. (2.3a) indicates that the surface $s(t)$ of the non-material volume is moving at the velocity u. The term on the left hand side of Eq. (2.3a) represents the rate of change of the total of $\Psi\rho$ contained in $v(t)$.

The general form of the transport theorem, Eq. (2.3a), rests upon an axiom given in the memoir entitled *The Sub-Mechanics of the Universe* by Reynolds [5]. In Article 9 of this memoir, Reynolds formulated: ‘‘AXIOM I: Any change whatsoever in the quantity of any entity within a closed surface can only be affected in one or other of two distinct ways: (1) it may be affected by the production or destruction of the entity within the surface, or (2) by the passage of the entity across the surface.’’ In Article 14 of the memoir, Reynolds formulated this axiom for scalar entities in three mathematical versions.

First, Reynolds [5] treated a control surface moving at a velocity different from the velocity of the particles located on the surface, and in his equation (13) he obtained a formulation which, by the divergence theorem, can be identified as being equivalent to Eq. (2.3a).

We note that the above general formulation of Reynolds for an arbitrarily moving nonmaterial volume, Eq. (2.3a), was not cited or utilized in the literature for a long time. Only recently did it appear in the literature in this latter form, see Section 1.12 of the book on fluid dynamics by Warsi [11], first published in 1993. See also equation (1.31) of the handbook article on fluid dynamics by White [3]. Some special formulations of Eq. (2.3a) for balance of mass and momentum appeared earlier and will be addressed in Section 4 of the present review. From these special formulations, the argumentation given in Chapter 7 of the book on mechanics of solids and fluids by F Ziegler [12], first published in German in 1985, follows closely the lines of Axiom I by Reynolds [5]. With respect to the balance of momentum for a control volume, Ziegler [12] wrote: ‘‘1) the mass elements within the control volume are accelerated by external forces.

2) Mass flows trough the control surface, which results in the transport of momentum.'' Ziegler then formulated the equation of balance of linear momentum for a non-material volume in his equations (7.13,7.14), see also Fig. 7.3 of Ziegler [12], showing a sketch of the control surface fixed in space or moving in a prescribed motion.

We now return to the memoir of Reynolds [5]. In equation (14) of Article 14, Reynolds applied his equation (13) to the case of a fixed surface including a spatial volume. Equivalently, putting $u=0$ and rearranging terms, it follows from Eq. (2.3a) that

$$\frac{d}{dt}\int_V \rho\Psi\, dv = \frac{d_{u=0}}{dt}\int_v \rho\Psi\, dv + \int_s da\cdot \dot{p}\rho\Psi \tag{2.3b}$$

where v denotes the spatial volume with a surface s instantaneously coinciding with the material surface S. This form of the transport theorem was attributed to Reynolds [5] in Section 81 of the Handbuchartikel by Truesdell and Toupin [4], with a further reference to Jaumann [13] and Spielrein [14]. In their presentation, Truesdell and Toupin [4] further remarked that ''to consider a volume $v(t)$ bounded by a surface $s(t)$ at a different velocity u, we need only imagine fictitious particles whose velocity is u.'' The operator on the left hand side of Eq. (2.3a) in the latter sense indicates that the time derivative of the integral is taken over the volume $v(t)$ that is assumed to be material with respect to the field u. Applying this strategy of fictitious particles to Eq. (2.3b), the following result is obtained:

$$\frac{d_u}{dt}\int_{v(t)} \rho\Psi\, dv = \frac{d_{u=0}}{dt}\int_v \rho\Psi\, dv + \int_s da\cdot u\rho\Psi \tag{2.3c}$$

where we have taken into account the instantaneous coincidence of the surfaces under consideration, see also equation (81.4) of Truesdell and Toupin [4]. Reynolds himself in [5] obtained an equivalent expression by eliminating the production terms from his equations (13,14) for the non-material and the spatial volume, respectively, and he presented the result in equations (15,16) of his memoir.

Note that the differentiation and the integration can be interchanged in the integral over the spatial volume on the right hand side of Eq. (2.3c), since the rate of change of an elementary spatial volume vanishes. Using the common notation $d_{u=0}/dt = \partial/\partial t$, we thus may write:

$$\frac{d_u}{dt}\int_{v(t)} \rho\Psi\, dv = \int_v \frac{\partial}{\partial t}(\rho\Psi)dv + \int_s da\cdot u\rho\Psi \tag{2.3d}$$

In 1972, this formulation was stated in Chapter 1.3 of the book on compressible-fluid dynamics by Thompson [15] in the form of an elegant axiom. When translated into our notation, the axiom by Thompson can be formulated as: ''Two contributions of the rate of change are recognized: (1) the value of $\Psi\rho$ may be changing with time within the volume, giving a rate of change $\partial(\rho\Psi)/\partial t$ in each volume element dv; (2) the moving surface envelops new regions of space with time, giving a rate of change $da\cdot u\rho\Psi$ at each surface element.'' Thompson [15] thus obtained an expression equivalent to Eq. (2.3d), and he attributed his axiom to Reynolds [5]. In order to return from Eq. (2.3d) back to Eq. (2.3a), we remark that the term $\partial(\rho\Psi)/\partial t$ represents the local portion of the total rate of change, $d(\rho\Psi)/dt$. Introducing the partitioning of the latter rate into its local and convective portions, using the divergence theorem of Eq. (2.2), and interchanging time differentiation and integration over the material volume instantaneously coinciding with the spatial volume, we indeed arrive at Eq. (2.3a). In order to complete this proof, a result presented subsequently, in Eq. (2.4a), is needed.

For more than half of a century, until 1960, the work of Reynolds [5] on the transport theorem remained buried in his memoir that was devoted to show ''that there is one, and only one, conceivable purely mechanical system capable of accounting for all the physical evidence, as we know it, in the Universe.'' It was the merit of Truesdell and Toupin [4] to connect the transport theorem with the name of Reynolds, and to introduce the idea of fictitious particles moving at a velocity different from the material particles.

We note that it is the idea of fictitious particles, which enables a formal mathematical proof of the various versions of the Reynolds transport theorem stated in Eqs. (2.3a–d). The reader may perform this proof by starting from Euler's expansion formula for the change of an infinitesimal volume dv carried by a material particle:

$$\frac{d}{dt}(dv) = dv\ \mathrm{div}\dot{p} \tag{2.4a}$$

We add the following equivalent expression for a later use:

$$\dot{J} = J\,\mathrm{div}\dot{p} \tag{2.4b}$$

where J denotes the Jacobian of the deformation gradient tensor. In the sense of the method of fictitious particles, the operation $d_u(dv)/dt$ in Eqs. (2.3a) and (2.3c) now can be performed by substituting the fictitious particle velocity u in Eq. (2.4a), instead of $\dot{p}$. Similarly, the rates $d_u(\rho\Psi)/dt$ and $d(\rho\Psi)/dt$ can be expressed by their local and convective portions, where both the material and the fictitious particle velocities are to be used for the convective parts. Taking into account that $d_{u=0}(\rho\Psi)/dt = \partial(\rho\Psi)/\partial t$ represents the local portion in the case of both rates, the proof can be completed by using the divergence theorem of Eq. (2.2). For the case of Eq. (2.3b), this type of proof was sketched in Section 81 of Truesdell and Toupin [4]. See also the discussion of Eq. (2.3d) given above.

Putting Eq. (2.3a) into (2.1) and rearranging terms, we arrive at the general equation of balance for a non-material volume, which we write as:

$$\frac{d_u}{dt}\int_{v(t)} \rho\Psi\, dv = \int_V \rho s[\Psi]dv - \int_S da\cdot(i[\Psi] + i_u[\Psi]) \tag{2.5a}$$

with the influx attributed to the fictitious particles

$$i_u[\Psi] = -(u-\dot{p})\rho\Psi \tag{2.5b}$$

These relations assert how the rate of change of the total of $\rho\Psi$ contained in the nonmaterial volume $v(t)$ is to be balanced by an appropriate combination of influx $i[\Psi]$ and supply $s[\Psi]$, taking into account the velocity u of the control surface $s(t)$ instantaneously coinciding with the material surface S. The motion of $s(t)$ relative to S is considered in Eqs. (2.5*a,b*) by means of the fictitious influx $i_u[\Psi]$ across S. Hence, when we add the fictitious influx $i_u[\Psi]$ to the original influx $i[\Psi]$ on the right hand side of the general balance law, Eq. (2.1), then on the left hand side we obtain the rate of change of the total of $\rho\Psi$ contained in the non-material volume $v(t)$, instead of the rate of change of this quantity contained in the material volume V. Furthermore, assuming the velocity field u to be continuous within V, the above principle of equivalence between surface and volume sources can be extended by introducing a fictitious supply $s_u = -\rho^{-1}\mathrm{div} i_u$. An interpretation complementary to the one indicated in Eq. (2.5) was presented in the book on mechanics of solids and fluids by F Ziegler [12], who did not use the idea of fictitious particles, but talked about the influx transported by the material particles through the control surface $s(t)$, see also Ziegler [16].

It is to be emphasized that the validity of the Reynolds transport theorem, Eqs. (2.3), requires that no singular surface is present in the volume under consideration. A singular surface is a surface at which the entity $\rho\Psi$ exhibits different values when approaching from the two sides of that surface. A review on the theory of singular surfaces was presented in Sections 173–176 and 180–194 of the Handbuchartikel by Truesdell and Toupin [4]. The extension of Eqs. (2.3) to volumes containing singular surfaces can be derived following the lines presented in Section 192 of Truesdell and Toupin [4] where reference was made to Thomas [17]. In short, a material volume containing a singular surface can be treated by subdividing the volume into two non-material volumes separated by the singular surface, applying Eqs. (2.3) to the latter two volumes, and adding the results. See also the comprehensive presentation given in Appendix A.II of the book on continuum mechanics of electromagnetic solids by Maugin [18]. The cited strategy for including a singular surface in the Reynolds transport theorem eventually leads to the following extension of Eq. (2.3*b*):

$$\frac{d}{dt}\int_V \rho\Psi\,dv = \frac{d_{u=0}}{dt}\int_v \rho\Psi\,dv + \int_s da\cdot\dot{p}\rho\Psi - \int_\Sigma da\cdot u[\![\rho\Psi]\!] \tag{2.6}$$

where the term $[\![\rho\Psi]\!]$ represents the jump of $\rho\Psi$ across the singular surface Σ. Usually it is supposed that the general equation of balance for a material volume, Eq. (2.1), remains valid whether or not there is a singular surface within it. Letting the volume under consideration shrink towards Σ, jump conditions are obtained, see Section 193 of Truesdell and Toupin [4]. The resulting local jump conditions across a singular surface for the special cases of balance of mass, momentum, and energy were discussed in some detail, eg, by Bednarczyk [19,20], and by Kluwick [21].

In the theory of multiphase mixtures, singular surfaces come into the play in case interfaces between the constituents have to be considered. Surface production terms reflecting the interaction of constituents at an interface were introduced by Kelly [22] and by Eringen and Ingram [23]. Momentum production at an interface in a two-phase gravity flow was considered by Hutter, Jöhnk, and Svendsen [24]. A comprehensive presentation on general interface production terms in multiphase mixtures was given by Morland and Sellers [25], who also discussed the interface cross mass flux to a constituent from the other constituents. Note that the presence of interface production terms at the singular surface leads to additional source terms not explicitly expressed in the above formulation of the general balance law, Eq. (2.1). In a forthcoming paper, Irschik [26], it is shown that surface growth terms moreover are needed in order to assure the consistency of the various forms of the equations of jump at a singular surface, eg, concerning the relation between jump conditions for balance of linear momentum and balance of kinetic energy. These surface growth terms may include surface production terms as well as the rate of change of the quantity associated with the material in the vicinity of the singular surface.

3 SOURCES OF MASS IN THE INTERIOR OF A MATERIAL VOLUME

The general equations of balance for a non-material volume, Eqs. (2.5) and (2.6), will serve as the basis of the following considerations on problems of applied mechanics without conservation of mass. Basically two physical models of sources of mass appear to be compatible with the general formulations of Section 2 above. These two models reflect the two fundamental formulations of continuum mechanics, the material and the spatial description.

The material formulation represents a continuum extension of the mechanics of masspoints, where the velocity emerges as a function of time for a given particle. It appears to be quite natural in this formulation to attach material sources (or sinks) of mass to the particles contained in a material volume, without changing the numbers of these particles (the degree-of-freedom of the system). As a practically important point from the modeling aspect, the impact velocity of continually adhering masses, or similarly the release velocity of continually separating masses, can be consistently introduced into this material model of sources of mass. In Section 197 of the book on dynamics by Tait [27], published in 1895, the idea of a material source of mass attached to a mass-point was introduced by assuming that ‘‘those particles the mass meets will adhere to it.’’ In a recent study on the mechanics of a growing tumor, Ambrosi and Mollica [28] formulated: ‘‘Here we model the growing material as a single phase continuum, in which growth is not seen as an increase in the number of particles, but as an increase of the mass of the already existing particles.’’

As an example for such a material source of mass, consider a planet, the reference mass of which is gradually changed by the steady impact of meteorites. If the planetary system is not destroyed by the impacts, it may be modeled by a fixed number of mass-points, and the change of mass of the planets may be characterized by sources of mass attached to these mass-points. Due to some concerns about the stability of the planetary system, this problem was extensively discussed at the beginning of the 20th century, see the thesis of Meshchersky defended in the year 1897, reprinted in the collection of Meshchersky's papers on the mechanics of bodies with a variable mass, Meshchersky [29]. See also Article 299 of the elementary part of the treatise on the dynamics of a system of rigid bodies by Routh [30], first published in 1905. A further reference to this problem was given in Section 27 of the Handbuchartikel on rigid body mechanics by Poeschl [31]. In celestial mechanics, the study of bodies with variable mass represents an ongoing field of research. For an extensive review, see Hadjidemetriou [32]. See also, eg, Plastino and Muzzio [33].

As a further example for the application of the material model of sources of mass, consider a solid carrier structure, which is deformed due to some loose mass that is falling down from a reservoir to the structure at a given velocity of impact, and that remains attached to the structure. The corresponding rate of change of the mass transported by the structure may be modeled by attaching some sources of mass to the particles of the carrier structure, without changing the degrees of freedom of the system. Various mechanisms that can be modeled by such sources were discussed in Chapter 1 of the book on the dynamics of machines with variable mass by Cveticanin [34]. This book contains numerous solutions of problems of this type, together with an extensive list of references.

The notion of continually impacting masses adhering to a material particle does not find a physical basis in the mechanics of fluids. That is why, in the theory of fluid mechanics, another type of source (or sink) of mass is introduced, usually in the framework of the spatial formulation. In the spatial formulation, the velocity emerges as a function of time for a given place in space. It is thus quite natural to model a single source of mass by removing an infinitesimal volume at a given place from the fluid volume under consideration, or to let the removed infinitesimal volume move along a known path at a given velocity. A flow of material particles is then assumed to take place across the surface of this infinitesimal volume without introducing a velocity of impact (or release). Hence, the notion of a source of mass in fluid mechanics is emerging from the concept of a non-material volume. In Article 56 of his treatise on hydrodynamics, first edition published in 1879, Lamb [35] defined a corresponding simple source as ''a point from which fluid is imagined to flow out uniformly in all directions.'' As was remarked in Section 49 of the Handbuchartikel on the physical foundations of fluid mechanics by Oswatitsch [36], it may be necessary to exclude some regions from the mathematical solution for simple sources in order to obtain results that are physically meaningful. The simple source model of fluid mechanics nevertheless turns out to be extremely useful. If the equations to be considered are linear, the principle of superposition allows one to construct sources that are distributed within a region of space, or that are located in line, see Section 52 of the Handbuchartikel of Oswatitsch [36].

In the following, the spatial formulation of the mechanics of continua is used. Since it is possible to rewrite any material description into a spatial formulation, at least in principle, it is usually not necessary to make a formal distinction in the notation between the above two physical models of sources of mass. It has to be noted, however, that the intensity of a source present at a given place in space at a given time can not be considered as an a priori known entity when the material model of sources attached to moving particles is utilized in the spatial formulation. The reason is that one does not follow the path of a specific particle in the spatial formulation. Moreover, as was explained above, no velocity of impacting masses is introduced in the fluid mechanics model of sources of mass.

In the remainder of the present Section 3, we discuss the case of mass supplied by sources in the interior of a material volume. The case of a non-material volume with a control surface moving at an arbitrary velocity is treated in Section 4, without considering sources of mass in the interior. It is a main goal of the present view to show that some analogous results have been obtained in the literature for the problems treated in Sections 3 and 4.

We start our considerations on a material volume with a supply of mass in the interior by studying the balance of the total mass contained in a material volume V,

$$M=\int_V \rho dv \tag{3.1}$$

When we substitute $\Psi=1$ in Eq. (2.1), and when we put the corresponding influx to zero, $i[\Psi]=0$, we arrive at the equation of balance of mass for a material volume in which mass is supplied at a rate $s[\Psi]=e$ in the interior:

$$\frac{d}{dt}\int_V \rho dv=\int_V \rho e dv=Q \tag{3.2}$$

With Eq. (3.1), this may be written as

$$\dot{M}=Q \tag{3.3}$$

where Q denotes the resultant strength of the sources of mass contained in the material volume.

In a heterogeneous medium, the mass of an individual constituent is not necessarily conserved, since mass may be exchanged among the constituents. Describing the balance of mass of a single constituent of a reacting continuum, Eq. (3.2) was stated in 1964 by Kelly [22], where ρe represents the volume supply of mass of a constituent due to chemical reactions. Note that we have suppressed the indices that usually are introduced in the literature in order to characterize specific constituents. Moreover, the rate e is to be defined with respect to the total density of the mixture, see also Section 159 of Truesdell and Toupin [4] and the fundamental contribution by Truesdell [37].

Using Eq. (2.3a), or Eqs. (2.5) with $u=0$, Eq. (3.2) may be transformed to a form valid for a spatial volume v that instantaneously coincides with the material volume V:

$$\frac{d_{u=0}}{dt}\int_v \rho dv=Q-\int_S da\cdot\dot{p}\rho \tag{3.4}$$

A discussion of this expression is given in Section 3 of the Handbuchartikel of Oswatitsch [36] on the physical foundations of fluid mechanics.

Considering Eq. (2.4a), the left hand side of Eq. (3.2) can be written as

$$\frac{d}{dt}\int_V \rho dv = \int_V (\dot{\rho} + \rho \operatorname{div} \dot{p}) dv \tag{3.5}$$

Inserting this result into Eq. (3.2), and using a standard argument, we obtain the local form of the equation of balance of mass appropriate when mass is supplied in the interior:

$$\dot{\rho} + \rho \operatorname{div} \dot{p} = \rho e \tag{3.6a}$$

Without an explicit reference to earlier contributions, Eq. (3.6a) was discussed in Section 4 of the Handbuchartikel on ideal fluids by Lagally [38] in 1927. Independently, Eq. (3.6a) was stated in a fundamental paper by Arrighi [39], published in 1933.

For the application of Eq. (3.6a) to a single constituent of a heterogeneous medium in which mass is transferred from other species by chemical reactions, see Section 159 of Truesdell and Toupin [4] and Kelly [22].

For recent discussions of Eq. (3.6a) in the framework of the growth of biomaterials, see Ambrosi and Mollica [28] and Lubarda and Hoger [40].

Arrighi [39] also expressed Eq. (3.6a) in terms of a reference state. For that sake, Eq. (3.6a) can be rewritten by means of Eq. (2.4b) in the form

$$\frac{\dot{\rho}}{\rho} + \frac{\dot{J}}{J} = e \tag{3.6b}$$

Formulated in the spatial description, Eq. (3.6b) refers to the actual position of a specific particle to who mass is supplied at a rate e at the actual time t. The rate at which mass is supplied to this same particle at time τ is denoted by E, where $t_0 \leqslant \tau \leqslant t$, and $E(t) = e$. If we use the configuration of the body at time t_0 as the reference placement, Eq. (3.6b) may be integrated along the path of a specific material particle between t_0 and the actual time t. Setting $J=1$ at the beginning of the path, the result becomes

$$\rho = \rho_0 J^{-1} \exp \int_{t_0}^{t} E d\tau \tag{3.6c}$$

where ρ_0 is the mass density of the particle in the reference configuration. The distinction between E and e has to be understood in the derivation of Arrighi [39]. For a recent discussion on the use of the material description of continuum mechanics in the present context, see the exposition on the mechanics of solids with a growing mass by Lubarda and Hoger [40]. For the case of a volume preserving deformation and growth, $J=1$ in the course of the motion, the latter authors obtained the simple result

$$\rho = \rho_0 + \int_{t_0}^{t} r_g^0 d\tau \tag{3.6d}$$

where, however, r_g^0 is defined as the product of E and the mass density at time τ of the particle under consideration. Hence, Eq. (3.6d) represents an integral equation for the mass density ρ, the solution of which is given by Eq. (3.6c) with $J=1$. On the other hand, should the time evolution of r_g^0 be given instead of ρe in Eq. (3.6a), then Eq. (3.6d) can be used directly.

We now turn to the equation of balance of linear momentum. The linear momentum contained in the material volume is

$$P = \int_V \dot{p} \rho dv \tag{3.7}$$

When we set $\Psi = \dot{p}$, $s[\Psi] = b + s + d$, $i[\Psi] = -t$ in Eq. (2.1), the equation of balance of linear momentum follows in a form appropriate when mass is supplied in the interior:

$$\frac{d}{dt}\int_V \dot{p}\rho dv = \int_V (b+s+d)\rho dv + \int_S da \cdot t \tag{3.8}$$

The Cauchy stress tensor is denoted by t, and b is the assigned body force per unit mass. The additional momentum supply per unit mass due to the supply of mass in the interior is a vector denoted by s. The vector d has been introduced in order to consider the supply of momentum due to diffusion of mass in the single constituent of a multiphase mixture.

In the material model of sources produced by impacting masses, mass is gained or lost in a particle at a velocity u that in general differs from the particle velocity $\dot{p}$. The additional momentum supply s due to the supply of mass in the interior occurring at a rate e may thus be modeled as:

$$s = eu \tag{3.9}$$

Putting Eqs. (3.7) and (3.8) into Eq. (3.9), we obtain the extension of Euler's law of balance of linear momentum appropriate when mass is supplied in the interior:

$$\dot{P} = F + \int_V \rho(eu + d) dv \tag{3.10}$$

The resultant of the surface forces and the assigned body forces is

$$F = \int_V \rho b dv + \int_S da \cdot t \tag{3.11}$$

The surface integral in Eq. (3.11) transforms to the resultant of the surface forces due to Cauchy's fundamental stress theorem.

Using the divergence theorem, Eq. (2.2), from Eqs. (3.8) and (3.9) we obtain the local form of the equation of balance of linear momentum in the presence of a supply of mass in the interior as

$$\rho \ddot{p} = \operatorname{div} t + \rho(b+d) + \rho e(u - \dot{p}) \tag{3.12}$$

where we have considered Eqs. (2.4a) and (3.6a).

We could not find the term stemming from mass supply in Eq. (3.12), $\rho e(u-\dot{p})$, to be introduced in the literature on the continuum theory of homogeneous media.

In the theory of multiphase mixtures, Eq. (3.12) is applied to a single constituent in which a supply of mass occurs in the interior. Recall that in our notation the usual indices characterizing specific constituents of heterogeneous media are suppressed. The term $\rho e(u-\dot{p})$ is absent in the expositions on multiphase mixtures given in Sections 215 and 295 of Truesdell and Toupin [4]. In this latter contribution, however, a theory of mass diffusion was developed in extension of the work of Stefan [41], published in 1871. Remarkably, in this theory of diffusion the term ρd emerges as a linear function of the difference of the velocity of the constituent under consideration and the velocities of the other species. This indicates an analogy between the supply of momentum due to diffusion, ρd, and the term $\rho e(u-\dot{p})$ in Eq. (3.12). In equation (5.4.15) of the comprehensive book on the historical development and the current state of the theory of porous media by de Boer [42], a special case of Eq. (3.12) with $u=0$ was presented, where ρd was introduced as interaction forces which belong to the volume forces, see also Sections 4.1 and 4.2 of the review article on contemporary progress in porous media theory by de Boer [43]. A formulation similar to the full version of Eq. (3.12) with $u\neq 0$ may be identified in equation (4.13) of the comprehensive study on multiphase mixtures by Morland and Sellers [25]. The term $\rho e u$ thereby has to be replaced by a weighted sum of the velocities of the other species transferring mass to the constituent under consideration. See also the exposition of Morland [44] on the flow of viscous fluid through a porous matrix.

In the continuum mechanics literature on the growth of biomaterials, where the biomaterial is understood as a single constituent of a multi-component body, the supply of momentum given in Eq. (3.9) is applied under the special assumption $u=\dot{p}$, such that the term $\rho e(u-\dot{p})$ in Eq. (3.12) is absent, see the expositions by Ambrosi and Mollica [28] and by Lubarda and Hoger [40].

In the following, we set $d=0$ and return to the extension of Euler's law of balance of linear momentum given in Eq. (3.10). In the context of this expression, we introduce the center of mass of the material volume under consideration. The instantaneous position of the center of mass is given by a vector c defined through the relation

$$cM=\int_V p\rho dv \tag{3.13}$$

Differentiating with respect to time and considering Eqs. (2.4*a*) and (3.6) gives

$$M\dot{c}+\dot{M}c=\frac{d}{dt}\int_V p\rho dv=\int_V \dot{p}\rho dv+\int_V p\rho e dv \tag{3.14}$$

This suggests introducing a second characteristic position vector $\hat{c}$ by means of

$$\hat{c}\dot{M}=\int_V p\rho e dv \tag{3.15}$$

Note that $\hat{c}=c$ when the rate e at which mass is supplied is constant throughout the body, see Eqs. (3.1), (3.2), and (3.13). The expression for the linear momentum, Eq. (3.7), appears at the right hand side of Eq. (3.14). The linear momentum therefore may be expressed as

$$P=M\dot{c}+\dot{M}(c-\hat{c}) \tag{3.16}$$

see Eq. (3.15). We furthermore introduce a mean velocity of the impacting masses w by formulating

$$w\dot{M}=\int_V u\rho e dv \tag{3.17}$$

The mean velocity w reflects the velocity u at which mass is gained or lost in a single particle of the continuum. The equation of balance of linear momentum, Eq. (3.10), then can be rewritten as

$$M\ddot{c}=F+\dot{M}\left(w-2\dot{c}+\frac{d}{dt}\hat{c}\right)-\ddot{M}(c-\hat{c}) \tag{3.18}$$

If $\hat{c}=c$, Eq. (3.18) reduces to the more simple form

$$M\ddot{c}=F+\dot{M}(w-\dot{c}) \tag{3.19}$$

In reviewing Eq. (3.18), we start with a single mass-point. For this problem, the simplified form of Eq. (3.19) does apply, where c denotes the position vector of the mass-point to which mass is gradually added with the impact velocity w. Note that $\dot{M}<0$, when mass is gradually lost.

Without considering the velocity of the impacting masses, $w=0$, Eq. (3.19) was introduced in 1890 by Painlevé in his lectures given in Lille for the case of a particle with a velocity dependent mass, see Dugas [10], Part V, Chapter 1, first published 1955. Painlevé used the projections of Eq. (3.19) on the tangent and on the principal normal of the trajectory of the particle. Following Dugas, Painlevé thus, as early as 1890, suggested a generalization of the dynamics of a particle which included the dynamics of special relativity in a given system of reference, and which suggested the notions of transverse and longitudinal mass.

In the literature, Eq. (3.19) with the special assumption of $w=0$ is often attributed to Levi-Civita [45,46], published in 1928. Levi-Civita referred to the case of a planet whose mass is gradually changed due to the fall of meteorites. Levi-Civita pointed out that in the presence of a variable mass with a vanishing impact velocity, $w=0$, one must use the formulation of Eq. (3.19), which may be written as

$$\frac{d}{dt}(M\dot{c})=M\ddot{c}+\dot{M}\dot{c}=F \tag{3.20}$$

instead of inserting the variable mass into the classical form of Newton's law,

$$M(t)\ddot{c}=F \tag{3.21}$$

Levi-Civita, however, remarked that Eq. (3.21) must be used when the velocity of the impacting mass relative to the mass of the planet vanishes, $w=\dot{c}$, see again Eq. (3.19).

In 1884, the formulation of Eq. (3.21) was considered by Gylden [47] in a study on the motion of a system of two mass-points with a variable mass subject to Newton's law of gravitation. The paper of Gylden [47] gave rise to various contributions in the literature, see Section 299 of the treatise of Routh [30]. See also the comprehensive review of Hadjidemetriou [32] on problems of celestial mechanics with a variable mass. The problem of an isotropic mass loss in celestial mechanics has been critically reviewed in a more recent exposition on the use and abuse of Newton's second law for variable mass problems by Plastino and Muzzio [33]. In the case of an isotropic mass loss, a body is assumed to loose mass with an ejection velocity of $w=\dot{c}$. The problem therefore must be described by Eq. (3.21), instead of Eq. (3.20). Plastino and Muzzio [33] pointed out that Eq. (3.20) sometimes had been used erroneously in the literature for the case of an isotropic mass loss, and that the relativity principle under Galileian transformations would be violated by the latter formulations.

The formulation of Eq. (3.19), with $w\neq 0$, was derived in 1890 by Seeliger [48], who studied steadily occurring impacts and separations of planetary masses. Seeliger did not mention any predecessors of this formulation. He considered motions of planets in 3D space where he formulated the components of vectors in an inertial Cartesian frame. He did not talk about the approximation of a mass-point, but he discussed the motion of the center of mass of the planet, which he denoted as the center of gravity. First, he discussed the motion of the center of gravity of a mass suddenly impacting with a second mass, and he formulated the equivalence of the linear momentum of the system of the two masses before and after the impact. Seeliger thereby followed the classical theory of impact as it has been established by Euler, see Kapitel V of the book on the history of the principles of mechanics by Szabo [49]. Having assumed the two bodies to be united after the impact, Seeliger [48] eventually proceeded to the new case of a stream of masses continuously impacting the planet under consideration and being united with this mass after having impacted. From a continuation argument, and tacitly assuming that $\hat{c}=c$ (or that the planet could be described as a single mass-point), Seeliger obtained Eq. (3.19). He thus extended the classical theory of dynamics with respect to the case of a body with variable mass when mass is added or separated at a mean velocity w different from the velocity of the center of mass $\dot{c}$. We therefore consider Seeliger as an important founder of the theory of systems with a variable mass.

The formulation of Seeliger [48] was reviewed, studied, and applied to various problems with a variable mass by Meshchersky in his master thesis of 1897. Meshchersky devoted a main part of his scientific work to problems with a variable mass, see the collection of Meshchersky's papers [29], published in 1949 with an introduction by Kosmodemyansky. The term $\dot{M}w$ in Eq. (3.19) is referred to as the Meshchersky reactive force in the literature, see, eg, Chapter 2 of the book of Cveticanin [34]. The case of a reactive force being colinear with the velocity of the center of mass,

$$(w-\dot{c})\dot{M}=\alpha\dot{c}\dot{M} \tag{3.22}$$

frequently occurs in practice. The solutions of various linear and nonlinear problems with α=const were reviewed in the book of Cveticanin [34]. Recently, the Duffing oscillator with an exponentially and a sinusoidal varying mass was studied by Holl, Belyaev, and Irschik [50].

Independently from the work of Seeliger and Meshchersky, some problems with a variable mass had been treated in Great Britain in the 19th century. The special problem of a mass, under no forces, which moves through a uniform cloud of little particles, which are at rest, was solved in Section 197 of the book on dynamics by Tait [27], published in 1895. In Section 198 of this treatise, the motion of a rocket fired vertically was studied, where the motive power of the rocket was attributed to the fact that a portion of mass is detached with a considerable relative velocity. In a slight linguistic ambiguity, Tait noted that ''the increase of the momentum of the rocket due to this cause is equal to the relative momentum with which the products of combustion escape.'' However, the acceleration of the rocket was derived correctly by Tait, in accordance with Eq. (3.19). Tait eventually obtained an expression for the greatest speed acquired during the flight. In the literature, the latter result is usually attributed to the Russian grammar school professor Ziolkowsky, who encountered it around the end of the 19th century, see Chapter 2 of the book on spacecraft systems by Messerschmid and Fasoulas [51]. The latter book contains an appendix with historical data about spacecraft and rockets, ranging from 3000 BC to the present time. Routh [52], in Article 149 of his treatise on the dynamics of a particle, first published in 1898, studied the equation of motion of a mass-point with a variable mass in a rectilinear motion, and he obtained the 1D version of Eq. (3.19). See also Article 300 of the treatise on the dynamics of a system of rigid bodies by Routh [30], first published in 1905.

British and French scholars of the 19th century attempted to apply the ideas expressed by Eq. (3.19) to various problems of the rectilinear motion of strings and chains which are coiled up at rest and which are continually set into motion. The study of chains set into motion started in 1857 with a fundamental study by Cayley [53], who wrote: ''There are a class of dynamical problems which, so far as I am aware, have not been considered in a general manner. The problems referred to (which might be designated as continuous-impact problems) are those in which the system is continually taking into connexion with itself particles of infinitesimal mass . . . , so as not itself to undergo any abrupt change of velocity, but to subject to abrupt changes of velocity the particles so taken into connexion. For instance, a problem of the sort arises when a portion of a heavy chain hangs over the edge of the table, the remainder of the chain being coiled or heaped up close to the edge of the table, the part hanging over constitutes the moving system, and in each element of time the system takes into connexion with itself, and sets into motion with a finite velocity an infinitesimal length of the chain.'' Without a further proof, Cayley then stated a variational formulation, which might be derived from the Seeliger formulation of Eq. (3.19) for the single mass-point with a variable mass by performing a limit to infinitesimal masses and by applying scalar multiplication with a virtual change of position. Afterwards, a summation over the particles under consideration has to be performed in order to obtain Cayley's statement. Cayley thus, as early as 1857, presented a variational formulation, which also should be valid for a fixed number of mass points with attached sources of mass. He then solved the above problem of a hanging chain by applying a single-degree-of-freedom version of his variational formulation. This procedure leads to a correct result in the case under consideration. However, the distinction between the concept of material type of sources of mass and the concept of a non-material volume comes into the play in the continuous-impact problems introduced by Cayley in [53]. Cayley noted an abrupt change of velocity of the particles set into motion. In modern terminology, a singular surface thus subdivides the portion of the chain at rest from the hanging part. Of course, the concept of a singular surface was not yet at the disposal of Cayley. A continuum mechanics based discussion of the variational statement by Cayley has been undertaken and will be reported elsewhere, Irschik [54].

In 1898, a history on solutions concerning the rectilinear motion of strings and chains was presented in Articles 148 and 150 of the treatise on the dynamics of a particle by Routh [52], see also Article 300 of the treatise on the dynamics of a system of rigid bodies by Routh [30], first published in 1905. Routh did not mention the variational formulation by Cayley [53], but the solutions presented by Routh rested directly upon the application of Eq. (3.19).

In 1905, a further review on the rectilinear motion of strings and chains was presented by Wittenbauer [55], who criticized some of the older solutions by noting that a loss of energy would occur due to impacts taking place when masses of a coiled chain are set into motion. In the second part of his paper, Wittenbauer also introduced the notion of a virtual variability of mass, an idea that turned out to be very useful in machine dynamics.

A contemporary attempt to apply Eq. (3.19) to the Cayley problem of a hanging chain was presented in Chapter 1.4 of the book on classical dynamics by Jose and Saletan [56]. We return to the problem of the rectilinear motion of chains and strings gradually set into motion in Section 4 of our review.

When written separately for each mass-point, Eq. (3.19) was frequently used in the literature as the starting point for reformulating the equations of motion of a material system consisting of a fixed number of mass-points according to D'Alembert's principle. For D'Alembert's principle applied to a system of mass-points with an invariant mass, see Section 45 of the Handbuchartikel by Synge [57]. The application of D'Alembert's principle applied to a system of mass-points with variable masses was demonstrated by Agostinelli [58], who derived the equations of Lagrange and the equation of conservation of energy according to the Levi-Civita formulation of Eq. (3.20). See also Chapter XI of the treatise on analytical dynamics by Pars [59]. Equation (3.19) with $w \neq 0$ was applied by Ge [60] to the single members of a nonlinear nonholonomic variable mass system in order to derive an extended form of the Lagrange equations. Starting from Eq. (3.19) and using D'Alembert's principle, Luo Shaokai and Mei Feng-xiang [61] discussed the principles of least action of variable mass nonholonomic nonconservative systems in a noninertial reference frame. Kane's equations for variable mass nonholonomic mechanical systems were derived by Ge and Cheng [62], and by Zhang Yueliang and Qiao Yongfeng [63]. D'Alembert's principle was applied to Eq. (3.19) in order to discuss energy change laws for systems with variable mass by Musicki [64].

For a system of mass-points with $\hat{c} \neq c$, the full version of Eq. (3.18) was derived in a fundamental contribution by Federhofer [65], published in 1922. In order to derive the continuum mechanics formulations of Eqs. (3.13–3.18), it is only necessary to repeat the proof presented by Federhofer in the context of a system of mass-points. The paper of Federhofer contained not only a derivation of Eq. (3.18), but it also presented the equations of conservation of angular momentum and of the kinetic energy for systems of mass-points and for rigid bodies with variable mass. These latter contributions will be discussed in the second part of our review. Unfortunately, Federhofer's paper [65] appeared in a journal of little international relevance, shortly after World War I, and the citation of this paper in Section 27 of the Handbuchartikel by Poeschl [31] were incomplete. Shortly after his publication on variable masses [65], Federhofer moved from Brno to Graz, where he took over the chair of his teacher, Wittenbauer. In Graz, Federhofer turned his interest to other problems of mechanics. Instead of receiving the attention it would deserve, the contribution of Federhofer [65] on the dynamics of systems with a variable mass became forgotten.

Independently, various special cases of the relations derived in the paper of Federhofer [65] were published in the literature at a later time—eg, the variation of the center of mass relative to a rigid body with variable mass was neglected in the formulations given in Chapter 2 of the book on the dynamics of machines with variable mass by Cveticanin [34]. The motion of the rigid body then is approximated by an expression equivalent to Eq. (3.19), instead of the more general expression stated in Eq. (3.18). As a practically important result, Eq. (3.19) was formulated in a non-inertial frame moving with the rigid body in Chapter 2 of the book by Cveticanin [34].

4 VARIABLE MASS CONTAINED IN A NON-MATERIAL VOLUME

In the present Section, we discuss problems in which a variable mass appears due to the use of a non-material volume $v(t)$ in the modeling of the problem at hand. The total mass contained in $v(t)$ is denoted by

$$M_u = \int_{v(t)} \rho dv \tag{4.1}$$

Recall that the control surface $s(t)$ enclosing the volume $v(t)$ moves at a velocity u, which generally differs from the velocity of the material particles instantaneously located on this surface. In the present section, we do not take into account sources of mass within $v(t)$, since such sources have been discussed in Section 3 of the present review.

When we put $\Psi = 1$ in Eq. (2.5), and when we set the corresponding influx and supply terms to zero, we arrive at the equation of balance of mass for a non-material volume,

$$\frac{d_u}{dt} \int_{v(t)} \rho dv = \int_S da \cdot (u - \dot{p}) \rho \tag{4.2}$$

where we have assumed that the surface $s(t)$ of the non-material volume instantaneously coincides with the surface S of the material volume introduced in Eq. (2.5), $S = s(t)$. With Eq. (4.1), the equation of balance of mass, Eq. (4.2), then may be written as

$$\dot{M}_u = Q_u = \int_S da \cdot (u - \dot{p}) \rho \tag{4.3}$$

Note that the rate of change Q_u of the total of the mass included in the non-material volume $v(t)$ generally differs from the rate of change of the total M of the mass included in the material volume V, even though these volumes are assumed to coincide instantaneously, $v(t)=V$. For the material volume, there is $\dot{M}=Q=0$ in the absence of sources of mass, see Eq. (3.2). Hence, the superimposed dot in Eq. (4.3) is to be understood together with the index indicating the velocity of a non-material surface. An analogous remark holds for the subsequently treated linear momentum.

For the case of a spatial volume, $u=0$, the integral form of the equation of balance of mass, Eq. (4.2), was discussed in Section 3 of the Handbuchartikel of Oswatitsch [36] on the physical foundations of fluid mechanics, and in Section 5 of the Handbuchartikel of Serrin [66] on the mathematical principles of classical fluid mechanics.

Based on the extended form of the Reynolds transport theorem, Eq. (2.3*d*), the full version of Eq. (4.2) for a non-material volume with a surface $s(t)$ moving at an arbitrary velocity u was derived in Section 1.6 of the book on compressible-fluid dynamics by Thompson [15], published in 1972. For more recent expositions, see Section 1.6 of the book of Ziegler [12] on the mechanics of solids and fluids, first published in German in 1985, and Section 2.3 of the book on fluid dynamics by Warsi [11], first published in 1993.

We now turn to the equation of balance of linear momentum for a non-material volume. The linear momentum included in the non-material volume is

$$P_u=\int_{v(t)}\rho\dot{p}\,dv \tag{4.4}$$

When we put $\Psi=\dot{p}$, $i[\Psi]=-t$, $s[\Psi]=b$ in Eq. (2.5), we obtain the equation of balance of linear momentum appropriate for a non-material volume in the absence of a supply of mass in the interior:

$$\frac{d_u}{dt}\int_{v(t)}\dot{p}\rho dv=\int_V b\rho dv+\int_S da\cdot t+\int_S da\cdot(u-\dot{p})\rho\dot{p} \tag{4.5}$$

Again, the surface integral in Eq. (4.5) transforms to the resultant of the surface forces due to Cauchy's fundamental stress theorem. Hence, see Eq. (3.11), there is

$$\frac{d_u}{dt}\int_{v(t)}\dot{p}\rho dv=F+\int_S da\cdot(u-\dot{p})\rho\dot{p} \tag{4.6}$$

where F denotes the resultant of the surface forces and the assigned body forces acting upon the material volume instantaneously coinciding with the non-material volume. Putting Eq. (4.4) into Eq. (4.6), we obtain the extension of Euler's law of balance of linear momentum appropriate for a non-material volume:

$$\dot{P}_u=F+\int_S da\cdot(u-\dot{p})\rho\dot{p} \tag{4.7}$$

Particularly, the balance of linear momentum contained in a spatial volume v reads

$$\dot{P}_{u=0}=\frac{d_{u=0}}{dt}\int_v\dot{p}\rho dv=F-\int_S da\cdot\rho\dot{p}\dot{p} \tag{4.8}$$

see Eqs. (4.6) and (4.7) with $u=0$. The tensor $\rho\dot{p}\dot{p}$ is called the momentum transfer. The rate of momentum included in the spatial volume $\dot{P}_{u=0}$ is sometimes referred to as the local or apparent rate of momentum. For a stationary motion with a steady density, the linear momentum is conserved within the spatial volume, $\dot{P}_{u=0}=0$, and the resultant force becomes

$$F=\int_S da\cdot\rho\dot{p}\dot{p} \tag{4.9}$$

When no additional material enters or leaves the spatial volume, $da\cdot\dot{p}=0$, the momentum transfer in Eq. (4.9) also vanishes, and the static relation $F=0$ is obtained. Hence, the stationary motion of a material filling up a closed spatial vessel has no reaction upon the vessel.

We now return to the full version of the equation of balance of linear momentum for a non-material volume, Eqs. (4.6) and (4.7). Let the velocity of the control surface be decomposed into the velocity of a fictitious body instantaneously coinciding with the non-material volume $v(t)$, and let $\hat{u}$ be the velocity of that fictitious body. Then

$$u=\hat{u}+\overline{d} \tag{4.10}$$

in Eq. (4.7), where $\overline{d}$ denotes the velocity of deformation of the control surface relative to the fictitious body. Analogously, we decompose the velocity of the material particles in the form

$$\dot{p}=\hat{u}+\overline{v} \tag{4.11}$$

where the velocity of the material particles relative to the fictitious body here is denoted by $\overline{v}$. Substituting this decomposition into Eq. (4.6), and putting $\Psi=\hat{u}$ in the extended form of the Reynolds transport theorem, Eq. (2.3*a*), we arrive at the relation

$$\frac{d_u}{dt}\int_{v(t)}\overline{v}\rho dv=F-\frac{d}{dt}\int_V\hat{u}\rho dv-\int_S da\cdot(\overline{v}-\overline{d})\rho\overline{v} \tag{4.12}$$

In reviewing the above formulations for the balance of linear momentum in a non-material volume, Eqs. (4.6)–(4.12), we start with the special case of a spatial volume, Eq. (4.8). For an elegant explanation of this relation, we refer to Section 14 of the Handbuchartikel on ideal fluids by Lagally [38], published in 1927. The application to the case of a stationary motion of an inviscid fluid flowing through regions with fixed rigid walls was discussed in detail in Chapter XIV of Volume 1 of the book on hydro- and aeromechanics by Prandtl and Tietjens [67], first published in 1929. The latter reference dealt with the computation of the forces exerted by the fluid upon the rigid walls, and gave an extension to motions, which are stationary in the mean. For a further discussion of Eq. (4.8), we refer to Chapter IV of the book on gas dynamics by Oswatitsch [68], see also Oswatitsch [69], and Section 4 of the Handbuchartikel of Oswatitsch [36] on the physical foundations of fluid mechanics.

Oswatitsch [36], Section 45 made reference to a formulation of the type of Eq. (4.8) when discussing the problem of the rectilinear flight of a rocket at a constant speed, without considering an internal flow. Oswatitsch assumed that fluid is detached with a constant relative velocity $\overline{v}$ from the rocket, and he replaced the momentum transfer by the relative momentum transfer through the exit plane of the rocket in order to obtain the force acting upon a non-material volume flying with the rocket. This formulation holds, since the speed of the rocket was assumed to be constant. Then a rocket-fixed inertial system may be used, with respect to which the left hand side of Eq. (4.8) disappears. We note that the cited result of Oswatitsch [36] follows also from Eq. (4.12) for a rigid control surface enclosing the rocket, $d=0$, when we assume the internal flow as well as the acceleration of the rocket to vanish, such that the contribution of the volume integrals in Eq. (4.12) disappear.

Referring to the work of Cisotti [70], some important consequences of the equation of balance of linear momentum for a spatial volume, Eq. (4.8), were presented in Sections 170 and 220 of the Handbuchartikel by Truesdell and Toupin [4]. Particularly, Truesdell and Toupin transmitted the result of Eq. (4.9) on a steady motion with a steady density, and its specialization for a closed vessel.

The developments towards the formulation of Eq. (4.12) for a non-material volume in an arbitrary motion started in 1908, with an ingenious study on the dynamics of water flowing along a moving rigid wall by von Mises, see Abschn. III, Section 9 of the treatise on the theory of water wheels by von Mises [71]. In order to obtain a relation for the force exerted by the fluid upon the moving wall, von Mises selected a closed rigid control surface, which consisted of the rigid wall and some parts with a flux of mass per unit area and time across the surface. He then applied the equation of balance of linear momentum to the material volume instantaneously coinciding with the non-material volume thus created. Having introduced the notion of a relative stream tube, von Mises succeeded to convert a part of the rate of change of the linear momentum contained in the material volume into a surface integral, ending with a practically appealing formulation. The study given in Abschn. III, Section 9 of von Mises [71], however, turned out to be far beyond the common frontiers of science at that time.

The strategy of converting parts of the volume integral appearing in the equation of balance of linear momentum for a material volume instantaneously coinciding with a non-material rigid volume was further substantiated in 1933 by Müller [72], who explicitly introduced the formulas of rigid body kinematics for the fictitious velocity $\hat{u}$, and who also presented a connection to the Kutta-Joukowski lift formula.

The strategy of a moving non-material volume however disseminated only slowly into the scientific community. Sources of further development were the studies on the rocket motion presented to the public after the end of World War II. In Section 1 of their treatise on the mathematical theory of rocket flight, published in 1947, Rosser, Newton, and Gross [73] stated as their Principle II that "the vector sum of all the exterior forces acting on (a system of particles) S is equal to the time rate of change of the total momentum of *S* plus the rate at which momentum is being transported out of *S* by the particles that are leaving *S*." The latter principle was applied by Rosser, Newton, and Gross to derive an equation for the acceleration of a rocket without internal flow, ie, for an accelerated rigid non-material volume, from which a stream of particles is ejected through the exit plane. In a shining example of scientific honesty, Rosser, Newton, and Gross [73] noted: "We have not been able to find this principle stated elsewhere, but we hesitate to call it new because no great originality was involved in its discovery. In fact it is very closely related to the momentum theorem of hydrodynamics." Rosser, Newton, and Gross then cited Chapter XIV of the book of Prandtl and Tietjens [67] on hydro- and aeromechanics. For a discussion of the accelerated flight of a rocket without internal flow, see also Section V.10 of the book on gas dynamics by Oswatitsch [68] from 1952.

As a further fundamental study on rocket motion, we cite the treatise of Rankin [74] on the mathematical theory of the motion of rotated and unrotated rockets with internal flow, published in 1948. In the case of an internal flow, the relative velocity $\overline{v}$ must not be neglected in Eq. (4.12). The comprehensive treatise of Rankin [74] contains a chapter on the equations of motion for a body of variable mass density and invariant shape, which is loosing mass from a certain plane portion of its surface. Independently, similar considerations were already presented by Gantmakher and Levin in 1947, see the book on the flight of uncontrolled rockets by Gantmakher and Levin [75], published in English in 1964.

A large amount of papers has appeared on rockets with a variable mass since the 1940s. Due to the restriction of space, we only mention the contribution of Meirowitch [76], who was the first to achieve a formulation for flexible rockets with variable mass and internal flow, see also Chapter 12 of the book on methods of analytical dynamics by Meirowitch [77]. An interesting numerical algorithm for the simulation of variable mass systems was recently presented by Djerassi [78]. This algorithm allows treating continuous-particle-ejecting systems such as rockets, as well as to study discrete-particle-ejecting systems. Djerassi showed that the results of his numerical algorithm do satisfy the equations of balance of momentum.

An influential formulation for a rigid control surface moving relatively to the motion of the material flow was presented in Chapter 5.6 of the book on the mechanics of fluids by Shames [2], first published in 1962. This latter contribution was the first textbook to include the idea of a rigid non-material volume performing a translatory and rotatory motion relative to the fluid. Shames noted that the corresponding formidable formulation might "intimidate all but the hardiest students." The equation of balance of linear momentum, Eq. (4.6), was also derived for a non-material rigid volume in the paper by Grubin [79], published in 1963, and in Chapter 4.3 of Volume 1 of the book on dynamics by Halfman [80], already published in 1962. The latter book contains various informative problems concerning bodies of variable mass.

The formulation for a rigid non-material volume by Shames [2] was further extended by Eke and Wang [81], who in 1994 presented several versions of the equations of motion of two-phase variable mass systems comprising a main solid frame and a fluid phase. The control surface including the moving two-phase system was assumed to maintain a constant shape, where mass was allowed to be continuously removed or added to the solid phase inside the rigid control surface. Thereby it was tacitly assumed that no additional contributions due to the presence of a singular surface needs to be considered within the non-material volume.

As mentioned in Section 2 of our review, the Reynolds transport theorem for a non-rigid volume with a control surface moving at an arbitrary velocity u different from the velocity of the material particles located on that surface, Eq. (2.3*c*), was brought to the attention of a wider audience in Section 81 of the Handbuchartikel by Truesdell and Toupin [4] in 1960. The implications of the latter general formulation upon the equations of balance of linear momentum for a non-material volume however were not worked out by Truesdell and Toupin [4].

Without giving any explicit reference to other literature, the formulation of Eq. (4.6) for a non-rigid control surface moving at a velocity different from the velocity of the material particles was presented by Thorpe [82] in 1962. Thorpe noted that ‘‘the derivations of the theorem of linear momentum, which are found in most textbooks, are unnecessarily restrictive in that the results are limited either to systems of constant mass or to control systems fixed in space. Furthermore, these derivations often result in a misconception of the law of momentum for a system of variable mass.’’ Certainly with legitimacy, Thorpe [82] attributed his equation (1), a relation equivalent to Eq. (2.3*d*), to the Leibniz theorem for differentiating an integral. From this theorem, he derived a relation equivalent to Eq. (4.2), his equation (8), and an equivalent of Eq. (4.6), his equation (16). Since Thorpe [82] did not introduce different notations for the various quantities and volumes to be considered, however, he was led to the conclusion that Eq. (2.3*b*) would be ‘‘valid regardless of the motion of the control volume, since the velocity of the control surface has been eliminated without making any assumptions about the motion of the control volume itself.’’ The representation of Thorpe [82], as fundamental as it appears, thus cannot be considered to be free of misconceptions itself.

With reference to Reynolds [5], the equation of balance of linear momentum for a non-rigid control surface moving at a velocity different from the velocity of the material particles, Eq. (4.6), was derived in Chapter 1.6 of the book on compressible-fluid dynamics by Thompson [15] published in 1972, see the discussion given in Section 2 above. The transition from the formulation for a spatial volume, Eq. (4.8), to the case of the extension of Euler's law of balance of linear momentum appropriate for a non-material volume, Eq. (4.7), was discussed in some detail in Chapter 7.1 of the book on mechanics of solids and fluids by Ziegler [12], first published in 1985, see also Section 2 above. Ziegler transmitted important didactic aspects of this topic in [16].

The versatile formulation of Eq. (4.12) was derived in Section 2.7 of the book on fluid dynamics by Warsi [11], first published in 1993. The latter book represents a valuable contemporary source on conservation laws and the kinetics of flow.

In an analogy to Eq. (3.13), we now introduce the center of mass of the non-material volume under consideration. The instantaneous position of the center of mass is given by a vector c_u defined through the relation

$$c_u M_u = \int_{v(t)} p \rho dv \tag{4.13}$$

Differentiating with respect to time gives

$$M_u \dot{c}_u + \dot{M}_u c_u = \frac{d_u}{dt} \int_{v(t)} p \rho dv \tag{4.14}$$

The extended transport theorem, Eq. (2.3*a*), proves that

$$\frac{d_u}{dt} \int_{v(t)} p \rho dv = \frac{d}{dt} \int_V p \rho dv + \int_S da \cdot (u - \dot{p}) \rho p \tag{4.15}$$

But, for a material volume in the absence of sources of mass in the interior, we may write

$$\frac{d}{dt} \int_V p \rho dv = \int_V \dot{p} \rho dv = \int_{v(t)} \dot{p} \rho dv = P_u \tag{4.16}$$

see Eq. (2.4*a*), and Eq. (3.6) with $e=0$. In an analogy to Eq. (3.15), we furthermore introduce the characteristic position $\hat{c}_u$ by

$$\hat{c}_u \dot{M}_u = \int_S da \cdot (u - \dot{p}) \rho p \tag{4.17}$$

Putting Eqs. (4.15)–(4.17) into Eq. (4.14), we arrive at the following relation for the linear momentum contained in the non-material volume:

$$P_u = M_u \dot{c}_u + \dot{M}_u (c_u - \hat{c}_u) \tag{4.18}$$

We furthermore introduce a mean velocity w_u of the rate of mass contained in the non-material volume by

$$w_u \dot{M}_u = \int_S da \cdot (u - \dot{p}) \rho \dot{p} \tag{4.19}$$

The equation of balance of linear momentum for the non-material volume, Eq. (4.7), then can be rewritten as

$$M_u \ddot{c}_u = F + \dot{M}_u \left(w_u - 2\dot{c}_u + \frac{d}{dt} \hat{c}_u \right) - \ddot{M}_u (c_u - \hat{c}_u) \tag{4.20}$$

In the present context of a non-material volume, we thus have arrived at a relation for the motion of the center of mass in complete analogy to the result of Federhofer [65] for a material volume with sources of mass in the interior, Eq. (3.18).

For the case of a rocket motion with internal flow, Eq. (4.20) was derived in the expositions of Rankin [74], and independently by Leitmann [83]. In 1966, it was proved by Thomson [84] that the various derivations presented by Rankin [74], Leitmann [83], Thorpe [82], Halfman [80], and others on this topic indeed are equivalent. In 1972, Belknap [85] transmitted a short note on his work on a general transport rule for variable mass dynamics. In his equation (8), Belknap [85] presented a formulation related to Eq. (4.20), where he remarked that the origin and the significance of the $\ddot{M}_u c_u$ and $2\dot{M}_u \dot{c}_u$ terms would be often questioned. Independently, Eq. (4.20) was derived by Kapoulitsas [86] for a variable system of particles, for which the mass variation is extended continuously on a part of the surface.

It has to be emphasized that the equations derived in the present section prove to be suitable not only in fluid mechanics, but also in the mechanics of solids and structures. The application of the equation of balance of linear momentum for a system with a variable mass was discussed in 1962 in the framework of the mechanics of solids in Chapter IV.7 of the 2nd edition of the book on the mechanics of solids by Parkus [87], first edition published in 1960. Parkus introduced an alternative form of Eq. (4.20). We arrive at the latter formulation by setting

$$P_u = M_u \dot{c}_s \tag{4.21}$$

in Eq. (4.7). Substituting further Eq. (4.19) in Eq. (4.7) leads to the somewhat simpler expression

$$M_u \ddot{c}_s = F + \dot{M}_u (w_u - \dot{c}_s) \tag{4.22}$$

which appears to be directly analogous to the Seeliger relation, Eq. (3.19). It must be noted, however, that the characteristic velocity $\dot{c}_s$ introduced by Parkus in general does not coincide with the velocity $\dot{c}_u$ of the center of mass, the position of which has been defined in Eq. (4.13).

The use of a non-material volume in the mechanics of solids was further substantiated in the book on technical continuum mechanics by Riemer [88]. Using the example of a deformable and rotating rod with an axially moving thin rigid disc, Riemer clearly demonstrated the possibility of the occurrence of singular surfaces due to non-material constraints in structures. As mentioned already above, a singular surface subdivides a material volume into two separated non-material volumes. When writing the equations of balance for a material volume including a singular surface, a jump term thus has to be considered, which follows from adding the equations of balance for the two non-material sub-volumes, see Eq. (2.6) and the discussion presented in Section 2 above. Accordingly, Riemer [88] developed the equations of balance of mass, linear and angular momentum, and of energy for the non-material volumes separated by a singular surface, eg, by the singular surface enforced in the rotating rod by the axial movement of the thin disc. These 3D formulations were preceded by formulations given by Wauer [89], who studied the transversal vibrations of uniaxial continua of variable length. Wauer [89] emphasized the important role of non-material boundary and transition conditions in structural problems, and he presented various valuable solutions, see also Riemer and Wauer [90].

Prior to the work of Wauer [89] and Riemer [88], the consequences that may result from the presence of singular surfaces and non-material boundary conditions were usually not taken into account in the literature on solids and structures. As an example, consider the problem of a whip formed by an inextensible string with a sharp reversed fold. The fold subdivides the string into two straight parts moving in parallel at different velocities. Each of these parts must be enclosed by a non-material volume, since the fold moves at a velocity different from the velocities of the two parts. Hence the fold represents a singular surface. This fact was not taken into account in the solution of a related problem presented in Section 123 of the book on theoretical mechanics by Hamel [91], published in 1949, nor was it considered in the solution developed in Übung 16 of Chapter IV of the introduction to mechanics by Szabo [92], first published in 1954, or in the alternative solution presented in Chapter II.D of the book on the history of the principles of mechanics by Szabo [49], first published in 1977.

Recently, the related problem of a folded falling inextensible string or chain fixed at one end was rediscovered in the literature. The equations of motion of this system were derived from balance equations by Steiner and Troger [93], where the discontinuity of velocity between the two parts of the string was taken into account by assuming Carnot loss forces to occur due to plastic impacts at the location of the fold. Recall the similar argument by Wittenbauer [55], reported in Section 3 above. The problem of a folded falling inextensible string was treated as an application of a more general formulation for bodies deploying along cables by Crellin, Janssens, Poelaert, Steiner, and Troger [94], who extended the equations of Lagrange by means of the Carnot loss forces, and obtained the result of Steiner and Troger [93]. The Lagrange equations were recently extended to account for a non-material volume by Irschik and Holl [95], where the problem of a falling folded string was reconsidered. We plan to return to the Lagrange equations in the second part of our review, when dealing with the balance of kinetic energy.

We end this part of our review by noting that the existence of singular surfaces in the form of shock fronts was recognized in fluid mechanics already in the 19th century. As was reported in an exposition on the historical developments and on recent contributions to the theory of shock waves by Kluwick [96], the possibility of a state of motion in which the fluid is divided into two parts by a surface of discontinuity, however, was sometimes doubted in the literature of the first half of the 20th century.

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