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The Constrained Isoperimetric Problem

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Abstract

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Let X be a space and let $S \subset X$ with a measure of set size $|S|$ and boundary size $|\partial S|$. Fix a set $C \subset X$ called the constraining set. The constrained isoperimetric problem asks when we can find a subset S of C that maximizes the Følner ratio $FR(S) = |S|/|\partial S|$. We consider different measures for subsets of $\mathbb{R}^2, \mathbb{R}^3, \mathbb{Z}^2, \mathbb{Z}^3$ and describe the properties that must be satisfied for sets S that maximize the Følner ratio. We give explicit examples.

Keywords: amenability, isoperimetric, Følner ratio, cooling function, cooling field.

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Contents

Chapter 1. Introduction

The Banach-Tarski paradox [7] is a consequence of nonamenability in group theory and motivates the study of amenability. The constrained isoperimetric problem [1] arose from the study of amenability [4].

A group G with finite generating set $C = C^{-1}$ has a geometric realization called its Cayley graph $X = X(G, C)$. The graph X has G as its vertex set. Two vertices a, b are connected by an edge e from a to b if $b = ac$ for some $c \in C$. Given a finite subset S of X, we take $|S|$ to be the number of vertices in X of S and $|\partial S|$ to be the number of edges of X with exactly one vertex in S. The *Følner ratio* of S is $FR(S) = |S|/|\partial S|$ [3]. The group G is *amenable* if there are subsets $S_1 \subset S_2 \subset \cdots \subset X$ exhausting X, with $FR(S_i) \to \infty$ as $i \to \infty$. In general there are no finite subsets $S \subset X$ with maximum Følner ratio, however, given a fixed finite set $C \subset X$, a *constraining set*, we can find a subset $S \subset C$ with maximum Følner ratio in C. This is the constrained isoperimetric problem [1].

Wherever there is a space X, a constraining subset $C \subset X$, and a way to appropriately measure the size and boundary, there exist analogous problems. We will characterize sets with maximum Følner ratios where $X = \mathbb{R}^2, \mathbb{Z}^2, \mathbb{R}^3, \mathbb{Z}^3$ and give examples of each.

Chapter 2. Technical Settings

The constrained isoperimetric problem involves a space X, a constraining set $C \subset X$, and a way to measure a set and its boundary. We will focus on the following settings:

2.1 Graphs

As a space X, we consider the case where X is the Cayley graph of the free Abelian group $\mathbb{Z}^2$ or $\mathbb{Z}^3$. We take $|S|$ to be the number of vertices in S and $|\partial S|$ to be the number of edges of X each having exactly one vertex in S.

2.2 Euclidean Space

We also consider the case when X is either $\mathbb{R}^2$ or $\mathbb{R}^3$ with $|S|$ equal to the classical area, or volume measure, respectively. For $|\partial S|$, when $X = \mathbb{R}^2$ we consider two different measures $\mathcal{L}_1$ and $\mathcal{L}_2$ of lengths; when $X = \mathbb{R}^3$ we use $\mathcal{L}_2$ measure for volume and $\mathcal{L}_1$ measure for area. These are defined in Chapter 3.

The reason for employing the $\mathcal{L}_1$ measure for the boundary length is that the problem becomes the continuous limit analogue of the group theoretic one for $\mathbb{Z}^2$.

Chapter 3. Measure

For measuring we use the following modified definitions of Hausdorff measure. For the standard definition we cite [2].

3.1 Diameter

Let $A \subset \mathbb{R}^n$ and $\pi_i : \mathbb{R}^n \to \mathbb{R}$, with $i = 1, \ldots, n$, be projection maps. Let $A_{x_i} = \pi_i(A)$. We define:

$$x_i\text{-}diam(A) = \sup\{|x - y| : x, y \in A_{x_i}\},$$

and

$$l_1\text{-}diam(A) = \sum_{i=1}^{n} x_i\text{-}diam(A),$$

while

$$l_2\text{-}diam(A) = \sup\{|x - y| : x, y \in A\}.$$

In the case $n = 2$, we have $\pi_x : \mathbb{R}^2 \to \mathbb{R}$ and $\pi_y : \mathbb{R}^2 \to \mathbb{R}$, the projections onto the x-axis and y-axis, respectively. In this case we have

$$x\text{-}diam(A) = \sup\{|x_1 - x_2| : x_1, x_2 \in A_x\},$$

$$y\text{-}diam(A) = \sup\{|y_1 - y_2| : y_1, y_2 \in A_y\},$$

and

$$l_1\text{-}diam(A) = x\text{-}diam(A) + y\text{-}diam(A).$$

Depending on the context we will use the symbol $\mathrm{diam}(A)$ to denote either the l_1 or l_2 diameter of A.

3.2 Hausdorff Measure

Definition 3.1. A *special rectangle* in $\mathbb{R}^n$ is a subset of the form $[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$ for $a_i < b_i$.

Let $S \subset \mathbb{R}^n$ be a compact set. Let $\delta > 0$. We cover S with a δ cover $\mathcal{U} = \bigcup U_i$ such that the U_i are special rectangles with diameter of $U_i \leq \delta$. Diameter here can be taken to be either l_1 or l_2. We define the *m-dimensional Hausdorff measure of* S *with respect to* δ to be

$$H^m(S, \delta) = \inf \left\{ \sum_i \mathrm{diam}(U_i)^m : \bigcup_i U_i \supset S, \mathrm{diam}(U_i) < \delta \right\}.$$

Here the infimum is taken over all covers $\mathcal{U}$. We define the *n-dimensional Hausdorff measure of* S to be

$$H^m(S) = \lim_{\delta \to 0} H^m(S, \delta).$$

We have the following theorems:

Theorem 3.2. *Let* $S \subset \mathbb{R}^n$. *Let* $0 < \alpha < \beta$. *If* $H^\alpha(S) < \infty$, *then* $H^\beta(S) = 0$.

Proof. Given a covering $\mathcal{U} = \bigcup U_i$ of S with $\operatorname{diam}(U_i) < \delta$, we have

$$\begin{aligned}
\sum_i \operatorname{diam}(U_i)^\beta &= \sum_i (\operatorname{diam}(U_i)^{\beta-\alpha})(\operatorname{diam}(U_i)^\alpha) \\
&\leq \sum_i \delta^{\beta-\alpha}(\operatorname{diam}(U_i)^\alpha) \\
&= \delta^{\beta-\alpha} \sum_i \operatorname{diam}(U_i)^\alpha.
\end{aligned}$$

So

$$\inf \left\{ \sum_i \operatorname{diam}(U_i)^\beta \right\} \leq \inf \left\{ \delta^{\beta-\alpha} \sum_i \operatorname{diam}(U_i)^\alpha \right\} \leq \delta^{\beta-\alpha} \inf \left\{ \sum_i \operatorname{diam}(U_i)^\alpha \right\}.$$

Hence

$$\begin{aligned}
0 \leq H^\beta(S) &= \lim_{\delta \to 0} \inf \left\{ \sum_i \operatorname{diam}(U_i)^\beta \right\} \\
&\leq \lim_{\delta \to 0} \delta^{\beta-\alpha} \inf \left\{ \sum_i \operatorname{diam}(U_i)^\alpha \right\} \\
&\leq \lim_{\delta \to 0} \delta^{\beta-\alpha} \lim_{\delta \to \infty} \inf \left\{ \sum_i \operatorname{diam}(U_i)^\alpha \right\} \\
&= \lim_{\delta \to 0} \delta^{\beta-\alpha} H^\alpha(S) \\
&= 0 \cdot H^\alpha(S) \\
&= 0.
\end{aligned}$$

□

Corollary 3.3. *Let $S \subset \mathbb{R}^n$. Let $0 < \alpha < \beta$. If $0 < H^\beta(S) < \infty$, then $H^\alpha(S) = \infty$.* □

In the case where the dimension $n = 1$ we define $\mathcal{L}_1(S) = H^1(S)$ and $\mathcal{L}_2(S) = H^1(S)$ to be the $\mathcal{L}_1$ and $\mathcal{L}_2$ lengths of S using the l_1 and l_2 definitions of diameter, respectively. When $n = 2$ we let *area*$(S) = H^2(S)$ using the l_2 diameter.

Note that if U_i is a special rectangle in $\mathbb{R}^2$ then

$$l_2\text{-diam}(U_i) \leq l_1\text{-diam}(U_i) \leq 2 \cdot l_2\text{-diam}(U_i).$$

We deduce the following corollary:

Corollary 3.4. *In the case where $n = 2$, the $\mathcal{L}_1$ and $\mathcal{L}_2$ lengths of S satisfy the inequality $\mathcal{L}_2(S) \leq \mathcal{L}_1(S) \leq 2 \cdot \mathcal{L}_2(S)$. If either is finite then area$(S) = 0$.* □

Chapter 4. Peano Continua

The sets that come under consideration in solving the constrained isoperimetric inequality can be very general. In the plane $\mathbb{R}^2$ this fact leads us to interesting problems in plane topology. We collect some basic results from [8].

We begin with a classical convergence theorem. Let X be a metric space with countable basis $\mathcal{U} = \{u_1, u_2, \ldots\}$. Let $N(x, \epsilon)$ denote the open ϵ-neighborhood of $x \in X$. Let $\mathcal{X}_0 = \{X_{01}, X_{02}, \ldots\}$ denote a sequence of subsets of X. We define

$$\limsup \mathcal{X}_0 = \{x \in X \mid \forall\, \epsilon > 0,\ N(x,\epsilon) \text{ intersects infinitely many } X_{0i}\text{'s}\}, \text{ and}$$

$$\liminf \mathcal{X}_0 = \{x \in X \mid \forall\, \epsilon > 0,\ N(x,\epsilon) \text{ intersects all but finitely many } X_{0i}\text{'s}\}.$$

We say that the sequence $\mathcal{X}_0$ *converges* if $\liminf \mathcal{X}_0 = \limsup \mathcal{X}_0$. This common set is called the *limit* of the sequence $\mathcal{X}_0$. This notion is defined in [5], page 5.

Lemma 4.1. *There is a subsequence of $\mathcal{X}_0$ that converges. If X is compact and if each X_{0i} is nonempty, compact, and connected, then the limit is also nonempty, compact, and connected.*

Proof. Suppose a subsequence $\mathcal{X}_j = \{X_{j1}, X_{j2}, \ldots\}$ has been chosen. If there is an infinite subsequence of $\mathcal{X}_j$ that misses the basis element u_{j+1}, then let $\mathcal{X}_{j+1}$ be such a subsequence. Otherwise, let $\mathcal{X}_{j+1} = \mathcal{X}_j$.

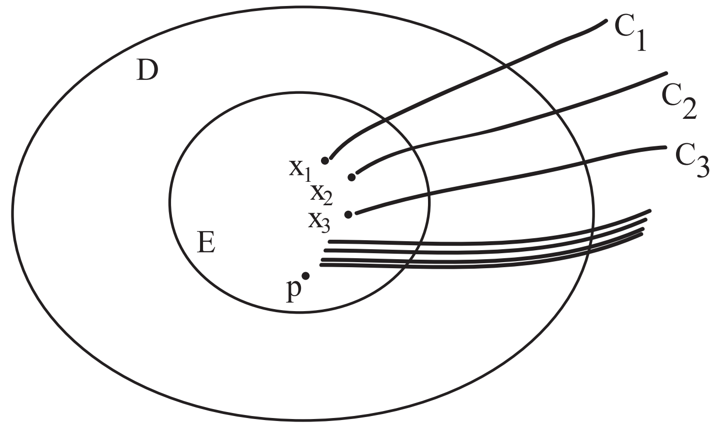
We claim that the diagonal subsequence $\mathcal{D} = \{X_{11}, X_{22}, \ldots\}$ converges. Indeed, if $x \in \limsup \mathcal{D}$, we must show that $x \in \liminf \mathcal{D}$. If not, then there is a subsequence of $\mathcal{D}$ that misses some neighborhood $N(x, \epsilon)$ of x, and thus misses some basic open set u_j about x. This means that the subsequence $\mathcal{X}_j$ also misses u_j. Hence u_j misses all X_{kk} for $k > j$, a contradiction to the inclusion $x \in \limsup \mathcal{D}$.

We next assume that each X_{ii} is nonempty, compact, and connected. Taking $x_i \in X_{ii}$, we obtain a sequence which must have a convergent subsequence, since X is compact. Thus the limit point is in $\limsup \mathcal{D}$, and hence, $\limsup \mathcal{D}$ is not empty. Since $\limsup \mathcal{D}$ is obviously closed, $\limsup \mathcal{D}$ is both nonempty and compact. It remains to show that $\limsup \mathcal{D}$ is connected. Suppose to the contrary that it is the disjoint union of nonempty compact sets A and B. Let U and V be disjoint open neighborhoods of A and B, respectively. Since each X_{ii} is connected and intersects both U and V for large i, then for such large i, X_{ii} will contain a point $x_i \in X \setminus (U \cup V)$. A limit point of the x_i's must be a point of $\limsup \mathcal{D}$ that is in neither A nor B, a contradiction. We conclude that $\limsup \mathcal{D}$ is connected. $\square$

This limit theorem allows us to characterize compact, connected, and locally connected subsets of the plane. A compact, connected metric space is called a *continuum*. If a continuum is also locally connected then it is called a *Peano continuum*.

Lemma 4.2. *Suppose that M is a continuum in the plane $\mathbb{R}^2$ that is not locally connected. Then there is an annulus A in the plane such that $M \cap A$ has infinitely many components that intersect both boundary components of A.*

Proof. Since M is not locally connected, there is a point $p \in M$ and a closed disc neighborhood D of p such that the component C of $D \cap M$ that contains p is not a neighborhood of p in M. Thus, there is a sequence $C_1, C_2, \ldots$ of components of $D \cap M$ distinct from C and points $x_i \in C_i$ that converge to p. We lose no generality in assuming that each intersects a smaller disc neighborhood E of p. Each C_i then contains a component of $M \cap (D \setminus \text{int}E)$ that intersects both boundary components of the annulus $A = D \setminus \text{int}E$.

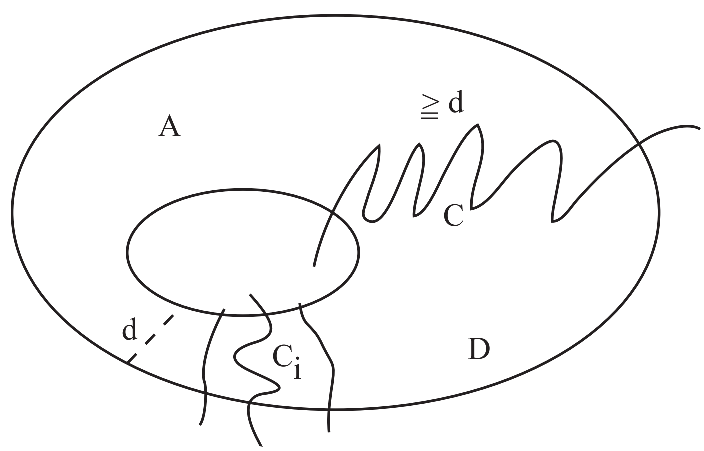


Converging continua

□

Lemma 4.3. *If M is a continuum in the plane that is not locally connected, then $\mathcal{L}_2(\partial M) = \infty$. Consequently, $\mathcal{L}_1(\partial M) = \infty$.*

Proof. Let A be an annulus in the plane such that $M \cap A$ has infinitely many components C_1, C_2, ... that intersect both boundary components of A. Let d denote the $\mathcal{L}_2$ distance between the two boundary components of A. It suffices to show that each C_i contains a portion of the boundary of M of $\mathcal{L}_1$ length $\geq d$.



Let C be one of them and use two arcs near another C_i to cut A into a disc D crossed from side to side by C. Then C separates the top and the bottom of the disc from one another in D. Hence $D \cap \partial C$ separates the top and the bottom of the disc from one another in D. By the unicoherenceof D (see [8], chapter 2, section 4 and 5), some component of $D \cap \partial C$

separates the top and the bottom of the disc from one another in D. This component must have $\mathcal{L}_1$ length $\geq \mathcal{L}_2$ length $\geq d$ since it must intersect both sides of D (recall Corollary 3.4). □

There are a number of slight modifications to the previous result. Here are two of them.

Lemma 4.4. *If M is a compact subset in the plane having infinitely many components C_1, C_2, ... of diameter $\geq \epsilon > 0$, then $\mathcal{L}_2(\partial M) = \mathcal{L}_1(\partial M) = \infty$.*

Proof. By the convergence theorem (lemma 4.1), we may assume that the C_i converge to a continuum C of diameter $\geq \epsilon$. Let p denote a point of $\limsup C_i$, and let A denote a round annulus centered at p in the $\epsilon/4$ neighborhood of p. Then, for all large i, C_i intersects both the inner and outer boundary components of A. Hence $A \cap \partial C_i$ contains a component that crosses A and therefore, has length at least as large as the distance from one component of ∂A to the other. Since there are infinitely many C's, the total length is infinite whether measured using $\mathcal{L}_1$ or $\mathcal{L}_2$. □

Lemma 4.5. *Suppose that M is a continuum in the plane $\mathbb{R}^2$ that does not separate $\mathbb{R}^2$ and that has finite boundary length. Then*

(0) The continuum M is a Peano continuum.

(1) The set ∂M is connected,

(2) The components u of $M \setminus \partial M$ form a null sequence u_1, u_2, ...

(3) The closure of each u_i is a topological disc d_i.

(4) If d_i and d_j intersect then they intersect in a single point.

(5) The area of M is the sum of the areas of the open sets u_i.

(6) The length of ∂M (using either of the $\mathcal{L}_1$ and $\mathcal{L}_2$ lengths) is greater than or equal to the sum of the (corresponding) boundary lengths of the discs d_i.

Proof. (0): Otherwise, ∂M has infinite length.

(1): If ∂M were not connected then there would be a disc D in $\mathbb{R}^2$ whose boundary misses ∂M such that ∂M intersects both the interior and exterior of D. Since M is connected, it must intersect ∂D. Since ∂D misses ∂M, we must have $\partial D \subset M$. Since M does not separate $\mathbb{R}^2$, the interior of D must also lie in M. But the interior of D intersects ∂M, a contradiction.

(2): If the components of $M \setminus \partial M$ do not form a null sequence, then we may pick arcs $A_1, A_2, \ldots$ in distinct components of $M \setminus \partial M$ with all of the A_i of diameter $\geq \epsilon$, for some fixed $\epsilon > 0$. Using the convergence theorem, we find the existence of an annulus A such that each A_i joins the two boundary components of A. These A_i are separated in A from one another by ∂M. This separation requires infinitely many distinct long components of $\partial M \cap A$, so that the length of ∂M is infinite, a contradiction.

(3): Since ∂M is a continuum of finite length, it must be locally connected. A standard result from plane topology ([8], Chapter 4, Theorem 6.7.) states that, if u is a bounded complementary domain of a locally connected continuum, then ∂u contains a simple closed curve $J(u)$ that separates u from infinity in $\mathbb{R}^2$. In the case of u_i, the simple closed curve $J(u_i)$ must contain u_i in its interior, and since M does not separate $\mathbb{R}^2$, that interior must coincide with u_i. That is, the union of u_i and $J(u_i)$ is a disc d_i that is precisely the closure of u_i.

(4): If $d_i \cap d_j$ were to contain more than one point, then the union $d_i \cup d_j$ would separate $\mathbb{R}^2$, and the bounded complementary components of the union would have to lie in M. But this would contradict the assumption that u_i and u_j are maximal components of $M \setminus \partial M$.

(5): Since the boundary of M has finite length, it also has 0 area (by Corollary 3.4). Thus the area of M is entirely carried by the open sets u_i. Thus the area of M is the sum of the areas of the u_i's.

(6): It suffices to show that, for each n, the sum of the boundary lengths of $d_1, d_2, \ldots, d_n$ is less than or equal to the boundary length of ∂M. But that is obvious since these d_i's share only finitely many points. $\square$

Chapter 5. The Unconstrained Isoperimetric problem

The classical unconstrained isoperimetric problem in the plane $\mathbb{R}^2$ has a well known solution [6]. Our problem is the $\mathcal{L}_1$ analogue of this result. We state this classical result and prove the $\mathcal{L}_1$ case.

Theorem 5.1 (Classical unconstrained isoperimetric problem)**.** *A set of $\mathcal{L}_2$ boundary length L in the plane $\mathbb{R}^2$ cannot enclose an area greater than $L^2/4\pi$. This inequality is sharp, realized by a circle of circumference L and radius $L/2\pi$.*

Theorem 5.2 (The $\mathcal{L}_1$ unconstrained isoperimetric problem)**.** *A set of $\mathcal{L}_1$ boundary length L in the plane $\mathbb{R}^2$ cannot enclose an area greater than $L^2/16$. This inequality is sharp, realized by a special square of perimeter L.*

Proof. We assume that we are given a compact set M in $\mathbb{R}^2$ with $\mathcal{L}_1$ boundary length $L < \infty$. We are to show that area$(M) \leq L^2/16$. We use the results from the previous chapter.

We may add to M any of the bounded complementary domains of M without increasing the boundary length and possibly increasing the area. We may, therefore, assume that M does not separate $\mathbb{R}^2$.

Since the boundary length is finite, the nondegenerate components M_1, M_2, … of M form a null sequence, and each M_i is a locally connected continuum that does not separate $\mathbb{R}^2$.

The bounded components of $M \setminus \partial M$ form a null sequence u_1, u_2, …, each u_i having closure d_i that is a disc. The area of M is the sum of the areas of the d_i's and the boundary length of M is greater than or equal to the sum of the boundary lengths of the d_i's.

We translate the d_i's into the plane so that they are contained in disjoint special squares Q_i in $\mathbb{R}^2$. We treat each d_i separately.

Let R_i denote the minimal special rectangle in Q_i that contains the translated d_i. Then, d_i intersects each of the four boundary edges of R_i. It is an easy matter to show that the

$\mathcal{L}_1$ boundary length of R_i is less than or equal to the boundary length of d_i. Hence, we may replace d_i by R_i without increasing the boundary length and without decreasing the area. Now among special rectangles with a given boundary length, area is maximized by the square of the same boundary length. Thus, we may replace R_i by a square $S_i \subset Q_i$ without increasing boundary length and possibly increasing area.

If we have at least two Q_i's we may place them side by side into one disc while decreasing boundary length and maintaining total area. The result may then be replaced by a single square of larger area and the same (decreased) boundary length.

By induction, we find that we may, without increasing boundary length, enclose almost as much area as the original by a single square. That is, the optimum is realized by a single square. Since the area of a square with perimeter L is $L^2/16$, our proof is complete. □

We deduce the following corollaries regarding the bound on Følner ratio of a set.

Corollary 5.3. *Let S be compact, with $\mathcal{L}_2$ boundary length L. Then*

$$FR(S) \leq \frac{1}{2\sqrt{\pi}}|S|^{1/2}.$$

Proof. By Theorem 5.1, $|S| \leq \dfrac{L^2}{4\pi}$, so $L \geq 2\sqrt{\pi}|S|^{1/2}$, and hence

$$FR(S) = \frac{|S|}{L} \leq \frac{|S|}{2\sqrt{\pi}|S|^{1/2}} = \frac{1}{2\sqrt{\pi}}|S|^{1/2}.$$

□

Corollary 5.4. *Let S be compact, with $\mathcal{L}_1$ boundary length L. Then*

$$FR(S) \leq \frac{1}{4}|S|^{1/2}.$$

Proof. By Theorem 5.2, $|S| \le \dfrac{L^2}{16}$, so $L \ge 4|S|^{1/2}$, and hence

$$FR(S) = \frac{|S|}{L} \le \frac{|S|}{4|S|^{1/2}} = \frac{1}{4}|S|^{1/2}.$$

□

Chapter 6. Characterizing Optimal Følner Sets

We have not as yet managed to show that for every compact set C, there actually exists a compact subset $S_0 \subset C$ whose Følner ratio is maximum. In the previous chapter we showed that the Følner ratios are bounded above so that there is a sequence of compact subsets with Følner ratios approaching a finite supremum. We now show that, in special cases, a subset with maximum possible Følner ratio, if it exists, may be taken to have a particular form.

Theorem 6.1. *If C is a closed disc in $\mathbb{R}^2$ and if there is a subset $S_0 \subset C$ that has maximum possible Følner ratio, then we may take S_0 to be a closed disc.*

This theorem is a corollary to the following lemma and theorem.

Lemma 6.2. *Suppose $\sum a_i$ and $\sum b_i$ are convergent series of positive numbers and that $a_i/b_i \to 0$. Then $\max(a_i/b_i) \ge (x = \sum a_i / \sum b_i)$.*

Proof. Otherwise, $\max(a_i/b_i) = \lambda x$, with $\lambda < 1$. Hence, for each i, $a_i \le \lambda x b_i$ so

$$x = \frac{\sum a_i}{\sum b_i} \le \frac{\sum \lambda x b_i}{\sum b_i} = \lambda x < x,$$

a contradiction. □

Theorem 6.3. *Suppose that S is a compact subset of $\mathbb{R}^2$ that does not separate $\mathbb{R}^2$ and that $FR(S) > 0$. Then there is a disc D in S such that $FR(D) \ge FR(S)$.*

Proof. We have seen that each component of S is locally connected, that the nondegenerate components of S form a null sequence, and that the area of S is carried by a null sequence of discs $D_1, D_2, \ldots$ in S, each pair intersecting in at most one point. Let the area of D_i be a_i, and length of ∂D_i be b_i. Then $FR(S) \leq \sum a_i / \sum b_i = x$. By lemma 6.3 there exists i, such that $FR(D_i) = a_i/b_i \geq x \geq FR(S)$. □

Theorem 6.4. *If C is a convex disc in $\mathbb{R}^2$, and if there is a subset $S_0 \subset C$ that has maximum possible Følner ratio, then we may take S_0 to be the intersection of C with special rectangle.*

Proof. We may assume S_0 is a disc by Theorem 6.1. Let t, b, l, r be top, bottom, left, and right most points of S_0, respectively. They define a special rectangle R_0 containing the points in the top, bottom, left, and right edges. Let $S_1 = R_0 \cap C$. We claim $FR(S_1) \geq FR(S_0)$. Certainly area$(S_1) \geq$ area(S_0). It suffices to show that the boundary length of S_1 is no greater than the boundary length of S_0. The path from r to b in S_1 is a geodesic by convexity of C and has length as short as the corresponding path in the boundary of S_0 and similarly for the paths from t to r, from b to l and from l to t. Thus, the boundary length of S_1 is less than or equal to the boundary of S_0. □

Theorem 6.5. *If C admits an isometry $T : C \to C$ ($\mathcal{L}_1$ or $\mathcal{L}_2$ as appropriate), and if S_0 has maximum Følner ratio in C, then $S_0 \cup T(S_0)$ also has maximum Følner ratio in C.*

Proof. Let $S_0 \subset C$ have maximum Følner ratio $\dfrac{|S_0|}{|\partial S_0|} = r$. Then $\dfrac{|T(S_0)|}{|\partial T(S_0)|} = r$. Now $\dfrac{|S_0 \cap T(S_0)|}{|\partial(S_0 \cap T(S_0))|} \leq r$, so $|S_0 \cap T(S_0)| \leq r|\partial(S_0 \cap T(S_0))|$. Hence

$$\begin{aligned}
\frac{|S_0 \cup T(S_0)|}{|\partial(S_0 \cup T(S_0))|} &= \frac{|S_0| + |T(S_0) - |S_0 \cap T(S_0)|}{|\partial S_0| + |\partial T(S_0)| - |\partial(S_0 \cap T(S_0))|} \\
&\geq \frac{r|\partial S_0| + r|\partial T(S_0)| - r|\partial(S_0 \cap T(S_0))|}{|\partial S_0| + |\partial T(S_0)| - |\partial(S_0 \cap T(S_0))|} \\
&= r.
\end{aligned}$$

□

Remark. We did not prove the existence of optimal Følner sets but rather if they existed, they would have the above descriptions. Steiner [6] also did not give a proof of existence of a solution to the classical isoperimetric inequality.

Chapter 7. Applications

We have not shown the existence of optimal Følner sets; however, if there exists an optimal Følner set, we show what form it must take in the following settings.

Theorem 7.1. *Let $|S|$ denote the area of S and $|\partial S|$ the Euclidean length of the boundary of S where $S \subset \mathbb{R}^2$. Let C be the unit square. If there is an optimal set $S_0 \subset C$, then $S_0 = \bigcup_{D_i \subset C} D_i$, where D_i are discs with radius $\frac{1}{2+\sqrt{\pi}}$. Furthermore, $FR(S_0) = \frac{1}{2+\sqrt{\pi}}$.*

Proof. We use the following results from the calculus of variations:

(1) S must be locally convex in C, otherwise we can increase the area and decrease the boundary length.

(2) Boundary of S must intersect the boundary of C tangentially, otherwise rounding the corners of S increases the Følner ratio.

(3) S may be taken to be a disc if C does not separate $\mathbb{R}^2$ by Theorem 6.1.

(4) Boundary arcs of S that miss the boundary of C must have constant curvature (classical result).

(5) S may be taken to realize the symmetries of C, because an optimal set union its image under a symmetry is also optimal by Theorem 6.5.

The only sets in the setting of Theorem 7.1 that satisfy these conditions are circular discs or a square minus the fragments cut off by four quarter circles at the four corners of C. (See the figure below.) Hence, if there is an optimal S_0, then $S_0 = \bigcup_{D_i \subset C} D_i$, where D_i are discs of uniform radius in C. Let ϵ be the radius of D_i. Then

$$|S_0| = 1 - 4(\epsilon^2 - \frac{1}{4}\pi\epsilon^2) = 1 - \epsilon^2(4 - \pi),$$

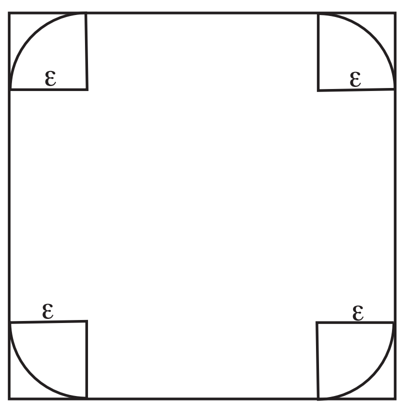
and

$$|\partial S| = 4(1 - 2\epsilon + \frac{1}{4} \cdot 2\pi\epsilon) = 4 - 2\epsilon(4 - \pi),$$

hence

$$FR(S_0) = \frac{1 - \epsilon^2(4 - \pi)}{4 - 2\epsilon(4 - \pi)},$$

which maximizes when $\epsilon = \dfrac{1}{2 + \sqrt{\pi}}$ with $FR(S_0) = \dfrac{1}{2 + \sqrt{\pi}}$.



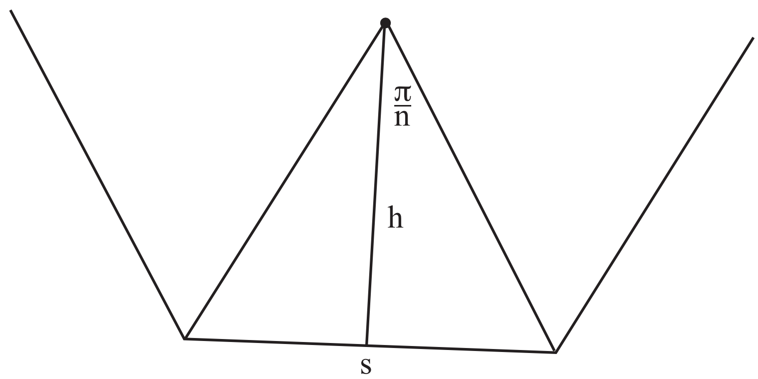
Classical Følner set.

□

Theorem 7.2. *Let the measures of sets and boundaries be the same as in Theorem 7.1, and let C be a regular n-polygon in $\mathbb{R}^2$ with side lengths 1. If there is an optimal set $S_0 \subset C$, then $S_0 = \bigcup_{D_i \subset C} D_i$, where D_i are discs with radius $\epsilon_n = \dfrac{n\tan\left(\frac{\pi}{n}\right) - \sqrt{n\tan\left(\frac{\pi}{n}\right)\pi}}{2(\tan\left(\frac{\pi}{n}\right)\left(n\tan\left(\frac{\pi}{n}\right) - \pi\right))}$. Furthermore, $FR(S_0) = \epsilon_n$.*

Proof. By the same argument as above we have $S_0 = \bigcup_{D_i \subset C} D_i$, where D_i are discs of uniform radius in C. Let ϵ be the radius of D_i. We first divide the polygon P_n with side length s

into n congruent triangles with central angel $2\pi/n$ as described below.



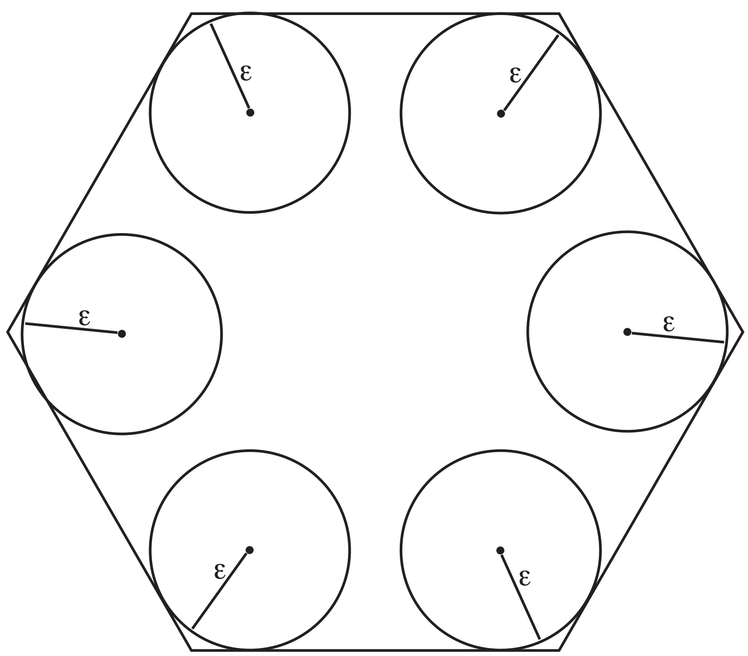
Then

$$\operatorname{area}(P_n) = n \cdot \frac{hs}{2} = n \cdot \frac{\left(\frac{s/2}{\tan(\pi/n)}\right)(s)}{2} = \frac{n}{4}\frac{s^2}{\tan(\pi/n)}.$$

In the case when $s = 1$ we have

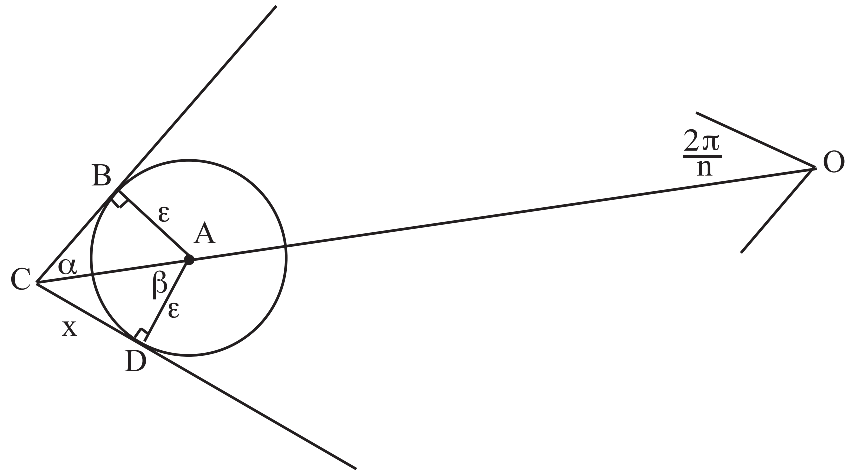
$$\operatorname{area}(P_n) = \frac{n}{4\tan(\pi/n)}.$$

We think of S_0 as P_n removing n corners as described in the figure below.



Classical Følner set in
a regular hexagon.

Now we look at one corner of the polygon P_n with center O.



Now $\alpha = (\pi - 2\pi/n)/2 = \dfrac{\pi}{2} - \dfrac{\pi}{n}$, and so

$$x = \epsilon/\tan\alpha = \frac{\epsilon}{\tan(\frac{\pi}{2} - \frac{\pi}{n})} = \frac{\epsilon}{\cot(\pi/n)} = \epsilon \cdot \tan(\pi/n).$$

Also, $\tan\beta = x/\epsilon = \dfrac{\epsilon\tan(\pi/n)}{\epsilon} = \tan(\pi/n)$. So $\beta = \pi/n$. So area of sector ABD is

$$\frac{1}{2}\epsilon^2 \cdot 2\beta = \frac{1}{2}\epsilon^2 \cdot 2\pi/n = \frac{\epsilon^2\pi}{n}.$$

Then the area of corner BCD is

$$2 \cdot \frac{1}{2}x\epsilon - \text{area}(ABD) = \epsilon \cdot \epsilon\tan(\pi/n) - \frac{\epsilon^2\pi}{n} = \epsilon^2\left(\tan\left(\frac{\pi}{n}\right) - \frac{\pi}{n}\right).$$

Then

$$\begin{aligned} |S_0| &= \text{area}(P_n) - n \cdot \text{area}(BCD) \\ &= \frac{n}{4\tan(\pi/n)} - n\epsilon^2\left(\tan\left(\frac{\pi}{n}\right) - \frac{\pi}{n}\right). \end{aligned}$$

Now

$$\begin{aligned}|\partial S_0| &= n(1-2x) + n \cdot \text{length}(BD) \\ &= n(1 - 2\epsilon \cdot \tan(\pi/n)) + 2\pi\epsilon \\ &= n - 2\epsilon\left(n \tan\left(\frac{\pi}{n}\right) - \pi\right).\end{aligned}$$

Hence, the Følner ratio of S_0 is

$$FR(S_0) = \frac{\dfrac{n}{4\tan(\pi/n)} - n\epsilon^2\left(\tan\left(\dfrac{\pi}{n}\right) - \dfrac{\pi}{n}\right)}{n - 2\epsilon\left(n\tan\left(\frac{\pi}{n}\right) - \pi\right)}.$$

The Følner ratio of S_0 is then maximized when $\epsilon = \dfrac{n\tan\left(\frac{\pi}{n}\right) - \sqrt{n\tan\left(\frac{\pi}{n}\right)\pi}}{2(\tan\left(\frac{\pi}{n}\right)\left(n\tan\left(\frac{\pi}{n}\right) - \pi\right))}$ and $FR(S_0) = \epsilon$. □

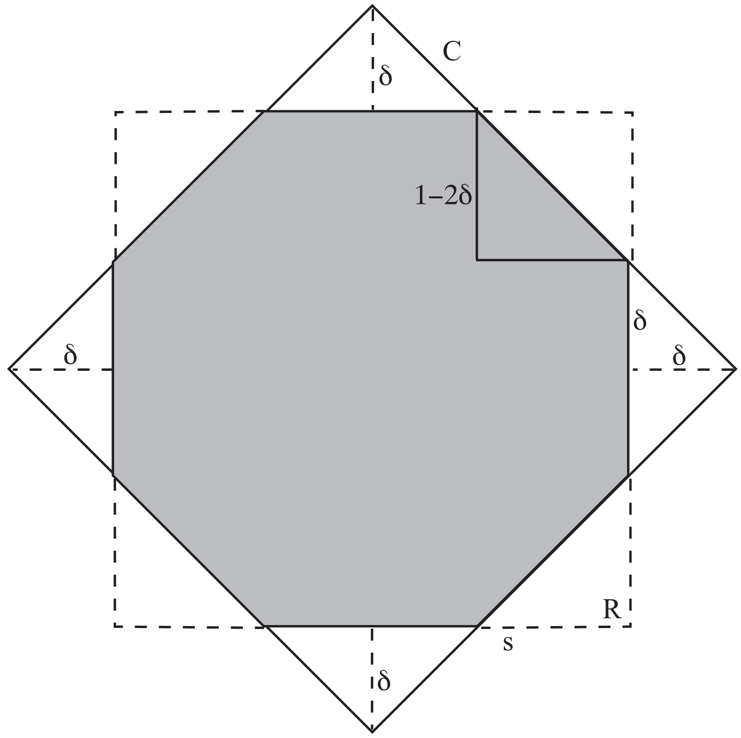
We now present the $\mathcal{L}_1$ examples.

Theorem 7.3. *Let $FR(S) = |S|/|\partial S|$ where $|S|$ denotes area of S and $|\partial S|$ denotes the $\mathcal{L}_1$ length of the boundary of S for $S \subset \mathbb{R}^2$. Let $C \subset \mathbb{R}^2$ be a special rectangle. Then $FR(C) \geq FR(S)$ for all $S \subset C$.*

Proof. Without loss of generality, assume that S is centered at the origin. From Theorem 6.4, if there is an optimal $S_0 \subset C$, then S_0 can be taken as the intersection of C and a special rectangle R centered at the origin. Any special rectangle $R \subsetneq C$ will have $FR(R) < FR(C)$, so for S_0 to be optimal, $S_0 = C \cap C = C$, and hence, C is itself optimal. □

Theorem 7.4. *Let $C \subset \mathbb{R}^2$ be a unit disc under the $\mathcal{L}_1$ norm, centered at the origin. Let $S_0 = C \cap R$, where R is a special square centered at the origin with side $\sqrt{2}$. Then $FR(S_0) \geq FR(S)$ for all $S \subset C$. Also, $FR(S_0) = \dfrac{2-\sqrt{2}}{2}$.*

Proof. By Theorem 6.4 and Theorem 6.5, we have $S_0 = C \cap R$ where R is a special square centered at the origin with side s.



The L_1 constrained isoperimetric problem

We think of S_0 as the diamond C with four cut off corners of depth δ. Then

$$|S_0| = (\sqrt{2})^2 - 4 \cdot (2\delta^2/2) = 2 - 4\delta^2,$$

and

$$|\partial S_0| = 8\delta + 4 \cdot (2 - 4\delta) = 8 - 8\delta.$$

So

$$FR(S_0) = \frac{2 - 4\delta^2}{8 - 8\delta} = \frac{1 - 2\delta^2}{4(1 - \delta)},$$

which maximizes when $\delta = \dfrac{2 - \sqrt{2}}{2}$ or $s = \sqrt{2}$ since $s = 1 - \delta$. The maximum Følner ratio is

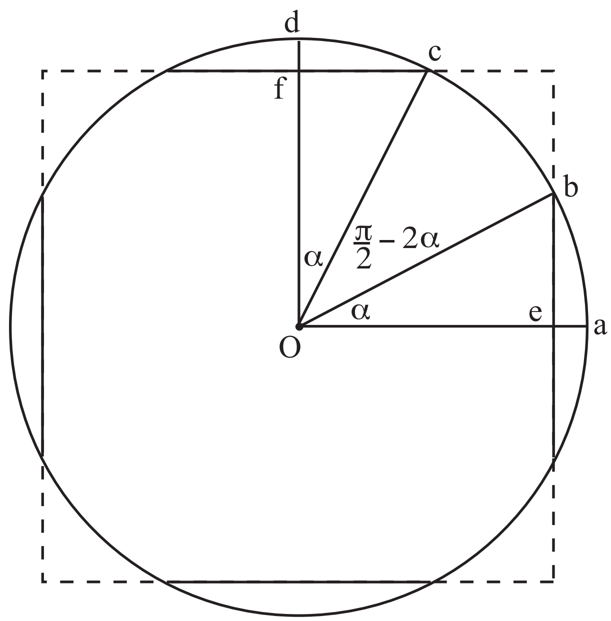
$$FR(S_0) = \frac{2 - \sqrt{2}}{2}.$$

□

Theorem 7.5. *Let $C \subset \mathbb{R}^2$ be the standard Euclidean unit disc. Equip $\mathbb{R}^2$ with the $\mathcal{L}_1$*

measure of distance. If there is an optimal $S_0 \subset C$, then $S_0 = R \cap C$, where R is a square of side length $s = 2\cos(\pi/4 - d/2)$, where d is the solution to the equation $x - \cos x = 0$.

Proof. By Theorem 6.4 and and Theorem 6.5, the optimal Følner set is the intersection with C by a special square centered at the center of the disc C.



L_1 Følner set in a circle.

Let α be the angle between Oa and Ob as in the picture, $0 < \alpha < \pi/2$. Then the area of triangles Obe and Ocf is $\dfrac{\sin\alpha \cdot \cos\alpha}{2} = \dfrac{\sin 2\alpha}{4}$. The area of the sector Obc is $\dfrac{\pi/2 - 2\alpha}{2}$. Hence, the area of the intersection S_0 is

$$|S_0| = 4 \cdot \left(2 \cdot \frac{\sin 2\alpha}{4} + \frac{1}{2}(\pi/2 - 2\alpha)\right).$$

Now the lengths of ab and cd are $\sin\alpha$. Notice here we use the $\mathcal{L}_1$ length. The $\mathcal{L}_1$ length of arc bc is

$$\mathcal{L}_1(bc) = 2(\cos\alpha - \sin\alpha).$$

Hence, the boundary length of S_0 is

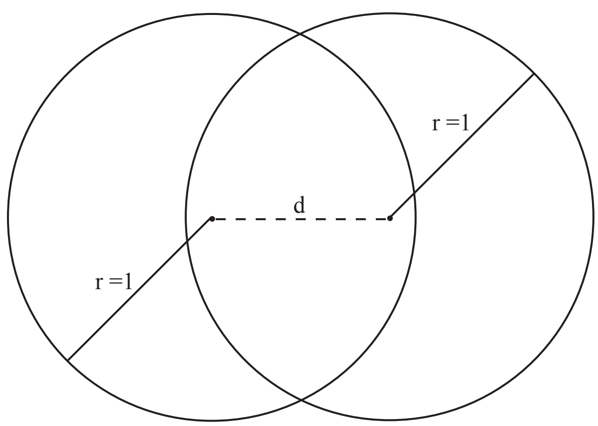
$$|\partial S_0| = 4 \cdot (2\sin\alpha + 2(\cos\alpha - \sin\alpha)) = 4 \cdot 2\cos\alpha.$$

The Følner ratio of S_0 is

$$FR(S_0) = \frac{(\sin(2\alpha) + \pi/2 - 2\alpha)}{4\cos\alpha},$$

which has derivative $FR'(S_0) = -\frac{1}{8}(4\alpha + 2\sin(2\alpha) - \pi)\tan\alpha\sec\alpha$. Thus, $FR(S_0)$ is maximized when $\alpha = \pi/4 - d/2$ where d is the solution to the equation $\cos x = x$. Insertion of α in the derivative function shows that this result is correct. The side s of the square R where $S_0 = R \cap C$ is simply $s = 2\cos(\alpha) = 2\cos(\pi/4 - d/2)$. □

Remark. The distance d is called the *Dottie number*, which is the distance between the centers of two unit circles each of which divides the area of the other in two.



We have a result for the analogous cases in $\mathbb{R}^3$ of Theorem 7.4, however it is only a heuristic result rather than a rigorous proof.

Conjecture 7.6. For $S \subset \mathbb{R}^3$, denote $|S|$ to be the volume of S and $|\partial S|$ to be the surface area of S. Let C be the unit cube. If there is an optimal Følner set $S_0 \subset C$, then $S_0 = \bigcup_{B_i \subset C} B_i$ where B_i are balls of radius ϵ. We approximate the Følner ratio of S_0 to be maximized at $FR(S_0) \approx 0.185296$ with $\epsilon \approx 0.25848315$.

Proof. By the same argument in Theorem 7.1, S_0 has the desired form. We think of S_0 as the cube removing the fragments cut off by an eighth of a sphere at eight corners of the cube as well as removing the eight cylindrical fragments along the twelve edges of the cube. Let ϵ be the radius of the balls B_i. Each corner fragment then has volume

$$\epsilon^3 \left(1 - \frac{\pi}{6}\right).$$

Each cylindrical fragment along the edges has volume

$$\epsilon^2(1 - 2\epsilon)(1 - \pi/4).$$

The volume of S_0 is then

$$\text{volume}(S_0) = 1 - 8\epsilon^3(1 - \pi/6) - 12\epsilon^2(1 - 2\epsilon)(1 - \pi/4).$$

We now compute the surface area of S_0. Once we remove the corner fragments there are eight rounded corners; each is an eighth of a sphere of radius ϵ, hence the corner area of S_0 is

$$4\pi\epsilon^2.$$

We removed the twelve fragments along the edged of the cube leaving twelve rounded edges. The area of these are

$$12 \cdot \left(\frac{1}{4} \cdot 2\pi\epsilon\right)(1 - 2\epsilon).$$

Finally, the area of the six remaining faces are

$$6 \cdot (1 - 2\epsilon)^2.$$

Thus the surface area of S_0 is

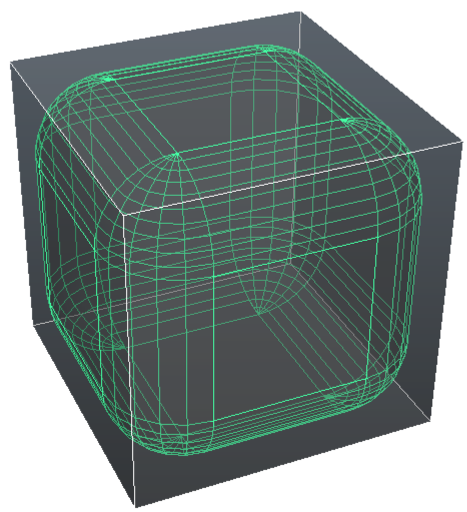
$$\text{area}(S_0) = 4\pi\epsilon^2 + 6\pi\epsilon(1-2\epsilon) + 6(1-2\epsilon)^2.$$

Hence the Følner ratio of S_0 is

$$FR(S_0) = \frac{1 - 8\epsilon^3(1-\frac{1}{6}\pi) - 12\epsilon^2(1-2\epsilon)(1-\frac{1}{4}\pi)}{4\pi\epsilon^2 + 6\pi\epsilon(1-2\epsilon) + 6(1-2\epsilon)^2}.$$

Setting the derivative of $FR(S_0)$ to zero and solving for ϵ, we obtain the maximum of $FR(S_0) \approx 0.185296$ with $\epsilon \approx 0.25848315$. □

Remark. The resulting S_0 has the following depiction.



Conjecture 7.7. Let $C \subset \mathbb{Z}^3$ be a ball of radius r under the $\mathcal{L}_1$ norm centered at the origin. For $S \subset \mathbb{Z}^3$, let $|S|$ denote the number of points in S and $|\partial S|$ denote the number of edges having exactly one vertex in S. We conjecture that if there is an optimal S_0, the it is obtained by cutting off corners with depth c and cutting along the twelve edges of the resulting octahedron with depth d, where $0 \leq c < r/2$ and $0 \leq d < c/2$. The Følner ratio of the resulting shape is

$$FR(S_0) = \frac{|S_0|}{|\partial S_0|},$$

where

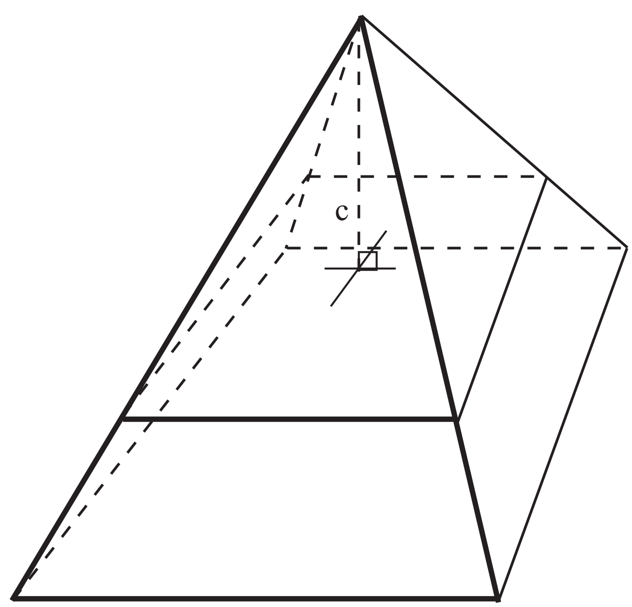
$$|S_0| = \left(\frac{4}{3}r^3 + 2r^2 + \frac{8}{3}r + 1\right) - 6\left(\frac{2}{3}c^3 + \frac{1}{3}c\right) - 12\left(\frac{2}{3}d^3 + (r - 2c + \frac{1}{2})d^2 - \frac{1}{6}d\right),$$

and

$$|\partial S_0| = 24\left(\frac{(r+1)(r+2)}{2} - 3\frac{c(c+1)}{2} - 3d\left(r - 2c + \frac{d+1}{2}\right)\right)$$
$$+24(r - 2c + d + 1)(2d - 1) + 6\left(2c^2 + 2c + 1 - 4(d^2 + 2d + 1)\right).$$

The data provided by Maple suggests that $FR(S_0)$ maximizes when $c \approx 0.4r$ and $d \approx c/3$.

Proof. Under the $\mathcal{L}_1$ norm, the ball of radius r has a shape of two pyramids with square base. We first focus on the top half of the ball.



Cut at level c.

First at each z-level set k of the ball C counting level zero at the top, there are

$$2\left(\sum_{j=0}^{k}(2j+1)\right) - (2k+1) = 2k^2 + 2k + 1 \text{ points.}$$

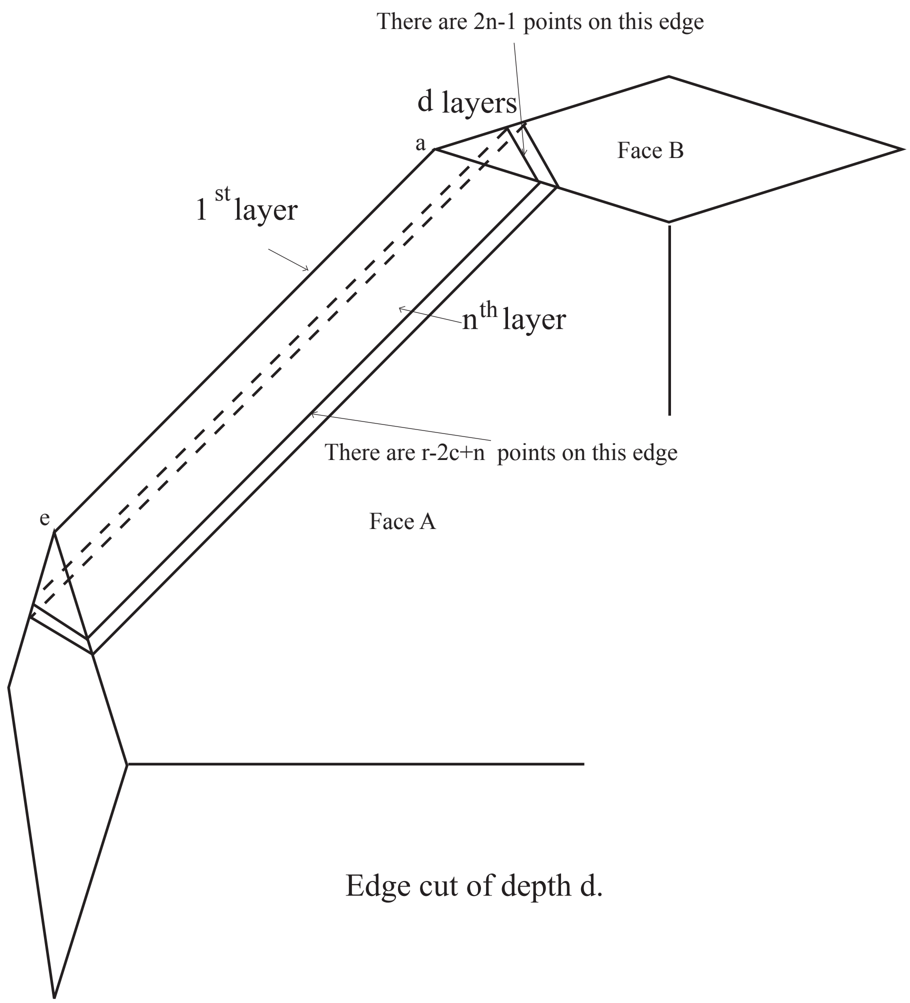
Since the ball C comprises of two pyramids each of height r we have the following formula for the number of points in C:

$$\text{volume}(C) = 2\left(\sum_{k=0}^{r}(2k^2+2k+1)\right) - (2r^2+2r+1) = \frac{4}{3}r^3 + 2r^2 + \frac{8}{3}r + 1.$$

Similarly, the volume of each corner cut CC of depth c is

$$\text{volume}(CC) = \sum_{k=0}^{c-1}(2k^2+2k+1) = \frac{2}{3}c^3 + \frac{1}{3}c.$$

We now compute the volume of each edge cut EC of depth d along each edge of the resulting octahedron. We depict the local part of the resulting shape after cutting off the corners of the pyramid as below.



Edge cut of depth d.

On face A the edge ae has $r+1-2c$ points. We think of each layer of the edge cut EC as a plane slicing parallel to the edge ae at each level $l = 1, 2, \ldots, n$. At $l = 1$ only the edge ae is sliced off. At layer $l = n$, there are $r + 1 - 2c - (1 - n) = r - 2c + n$ points on face A and there are $2n - 1$ points on face B. So the number of points being cut off at layer $l = n$ is $(r - 2c + n)(2n - 1)$ points. An edge cut EC of depth d cuts off all layers from 1 to d, so the volume of the edge cut is

$$\text{volume}(EC) = \sum_{l=1}^{d} ((r - 2c + l)(2l - 1)) = \frac{2}{3}d^3 + (r - 2c + \frac{1}{2})d^2 - \frac{1}{6}d.$$

There are 6 corner cuts and 12 edge cuts in total, so the final remaining volume is

$$\left(\frac{4}{3}r^3 + 2r^2 + \frac{8}{3}r + 1\right) - 6\left(\frac{2}{3}c^3 + \frac{1}{3}c\right) - 12\left(\frac{2}{3}d^3 + (r - 2c + \frac{1}{2})d^2 - \frac{1}{6}d\right).$$

We now proceed to compute the surface area. First, we compute the number of points on each face. Before any cut, each triangular face has

$$\sum_{i=1}^{r+1} i = \frac{(r+1)(r+2)}{2} \text{ points.}$$

The number of points lost due to corner cut is

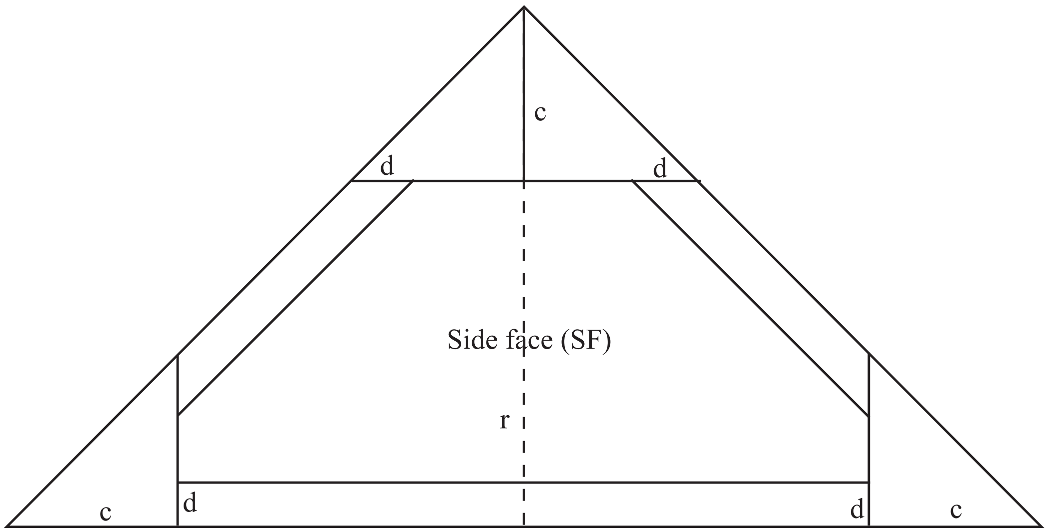
$$\sum_{i=1}^{c} i = \frac{c(c+1)}{2} \text{ points.}$$

Now we compute the number of points lost due to the edge cut. Recall the picture of edge cut of depth d. At each layer l, there are $r - 2c + l$ points on face A, so the number of points lost due to an edge cut of d layers is

$$\sum_{l=1}^{d} r - 2c + l = (r - 2c)d + \frac{d(d+1)}{2} \text{ points.}$$

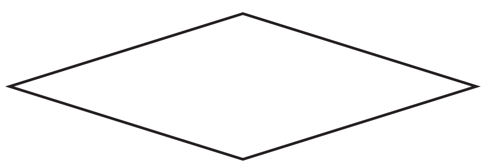
So the number of points remaining on each side face SF is

$$\frac{(r+1)(r+2)}{2} - 3\frac{c(c+1)}{2} - 3d\left(r - 2c + \frac{d+1}{2}\right).$$



Now cutting off corners results in six new corner faces CF, each of which has

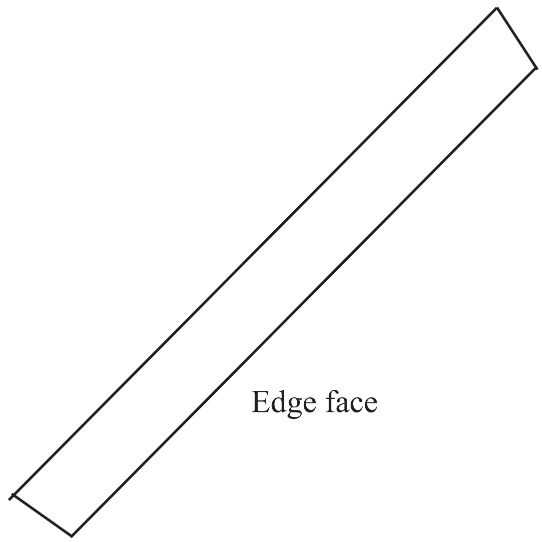
$$2\sum_{j=0}^{c}(2j+1) = 2c^2 + 2c + 1 \text{ points.}$$



Corner face (CF)

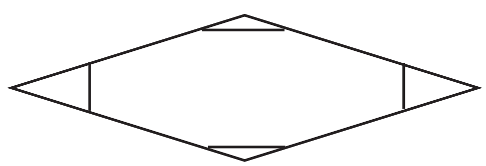
Cutting off the edges results in twelve new edge faces EF. Each has

$$(r - 2c + d + 1)(2d + 1) \text{ points.}$$



After cutting off the edges, we must recount the number of points on each corner face. Indeed, for each edge cut, $\sum_{i=1}^{d} 2l - 1 = d^2$ points have been removed from a corner face, so the number of remaining points on each corner face CF is

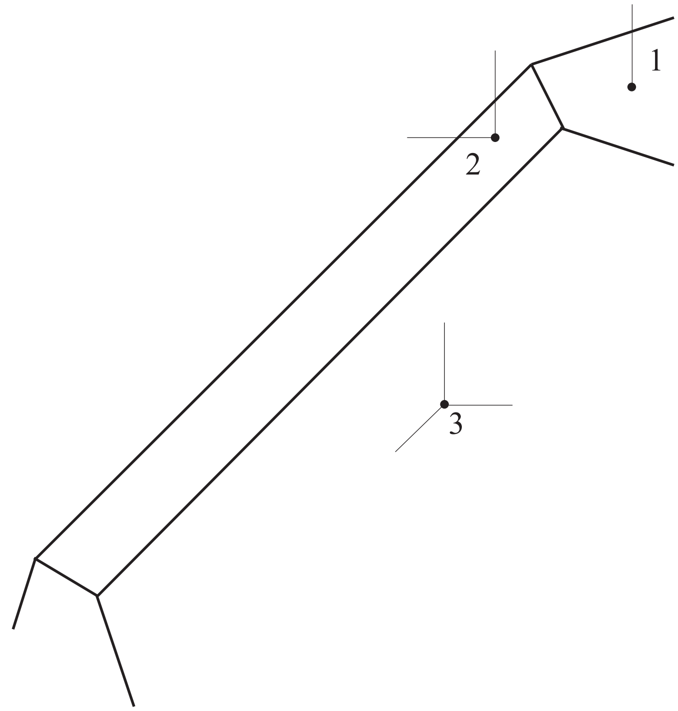
$$2c^2 + 2c + 1 - 4d^2.$$



New corner face (CF)

Remember that the surface area is not simply the number of points on the boundary faces. Rather, it is the number of edges connected to the outside of S_0 from these points. On each side face SF there are $\frac{(r+1)(r+2)}{2} - 3\frac{c(c+1)}{2} - 3d\left(r - 2c + \frac{d+1}{2}\right)$ points, each of which has three exterior edges. Now on each edge face EF there are $(r-2c+d+1)(2d+1)$ points. However, $2(r-2c+d+1)$ points have already been counted toward the side face. The remaining $(r-2c+d+1)(2d-1)$ points have two exterior edges each.

On each corner face there are $2c^2 + 2c + 1 - 4d^2$ points. $4 \cdot (2d-1)$ points have been counted toward the edge faces, and eight have been counted toward the side faces. The remaining $2c^2 + 2c + 1 - 4d^2 - 4(2d-1) - 8 = 2c^2 + 2c + 1 - 4(d^2 + 2d + 1)$ interior points has one exterior edge each.



Number of exterior edges
at points on each face

Since there are eight side faces, the surface area due to these faces is

$$8 \cdot 3 \left(\frac{(r+1)(r+2)}{2} - 3\frac{c(c+1)}{2} - 3d\left(r - 2c + \frac{d+1}{2}\right)\right).$$

There are twelve edge faces, the surface area due to these faces is

$$12 \cdot 2(r - 2c + d + 1)(2d - 1).$$

There are six corner faces, the surface area due to these faces is

$$6 \cdot \left(2c^2 + 2c + 1 - 4(d^2 + 2d + 1)\right).$$

Hence, the surface area of S_0 is

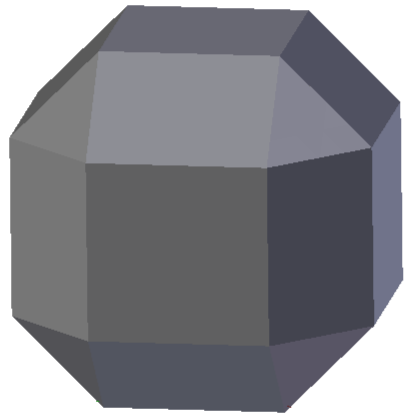
$$\text{area}(S_0) = 24\left(\frac{(r+1)(r+2)}{2} - 3\frac{c(c+1)}{2} - 3d\left(r - 2c + \frac{d+1}{2}\right)\right)$$
$$+24(r-2c+d+1)(2d-1) + 6\left(2c^2 + 2c + 1 - 4(d^2+2d+1)\right).$$

□

Maple yields the following maximum Følner ratios for the following values of r.

r	c	d	$FR(S_0)$
10	4	2	1117/750
20	9	4	7603/2670
30	14	5	8379/1978
40	19	7	≈ 5.635
50	23	8	≈ 7.036
60	28	10	≈ 8.441
70	33	11	≈ 9.847
80	38	13	≈ 11.252
90	43	14	≈ 12.66
100	47	15	≈ 14.066
200	95	30	≈ 28.148
300	144	44	≈ 42.236
400	192	58	≈ 56.325
500	240	72	≈ 70.414
600	288	86	≈ 84.503
700	336	100	≈ 98.593
800	384	114	≈ 112.683
900	432	128	≈ 126.773
1000	480	142	≈ 140.863
2000	961	283	≈ 281.763

Remark. The resulting shape has the following depiction.



Chapter 8. Cooling Functions and Cooling Fields

Let G be an infinite group with finite generating set C, and let Γ be the Cayley graph of G. Given a finite set $S \subset G$, let $E(S)$ denote the edges of Γ with at least one vertex in S; and let $\partial E(S)$ denote the edges of Γ with exactly one vertex in S. We orient each edge $e \in E(S)$ and let $i(e)$ be the initial vertex of e and $t(e)$ be the terminal vertex of e. For each edge e, we choose the orientation so that $i(e) \in S$.

We give the definition of cooling function from [1].

Definition 8.1 (Cooling Function)**.** A *cooling function* for S is a function $c : E(S) \to \mathbb{R}$ such that for all vertices $v \in S$,

$$h(v) := \sum_{i(e)=v} c(e) - \sum_{t(e)=v} c(e) \geq 1,$$

where $h(v)$ can be interpreted as the net loss of heat at each point $v \in S$. The cooling norm of c is $\|c\| = \max_{e \in E(S)} |c(e)|$.

We have the following result from [1].

Theorem 8.2 (Absolute Cooling). *Let Γ be an infinite, locally finite, connected graph with vertex set G and let $S \subset G$ be a finite set such that S together with $E(S)$ form a connected graph. Then S admits a cooling function c of minimum possible cooling norm $N = \|c\|$, and*

$$N = \|c\| = \max_{S_0 \subset S} FR(S_0) = \max_{S_0 \subset S} |S_0|/|\partial E(S_0)|.$$

In the case S is optimal i.e., $FR(S) \geq FR(S_0)$ for all $S_0 \subset S$, then we say c is an *optimal cooling function*.

The Absolute Cooling Theorem guarantees an optimal cooling function on an optimal Følner set $S \subset G$. We investigate to see whether it is possible to construct an analogue of a cooling function in the continuous case, where the space X is $\mathbb{R}^2$ with the $\mathcal{L}_1$ boundary length. We define the notion of a cooling field.

Definition 8.3 (Cooling Field). A *cooling field* for $S \subset \mathbb{R}^2$ is a differentiable function $C : S \to \mathbb{R}^2$ satisfying

$$\text{div}C((x, y)) \geq 1 \text{ for all } (x, y) \in S.$$

The *cooling norm* of C is $\|C\| = \sup_{(x,y)\in S} \{\sup_{\vec{v}_i} |\vec{v}_i \cdot C((x, y))|\}$ where $\{\vec{v}_i\}$ are standard unit vectors of $\mathbb{R}^2$.

Since the existence of an optimal cooling function is guaranteed, our goal is to construct an approximation to a cooling field from a cooling function on the integer lattice.

Let $S \subset \mathbb{R}^2$ be compact. Let $k \in \mathbb{N}$ and let the transformation map $T_k : S \to \mathbb{R}^2$ be defined as $T_k(x, y) = k(x, y) = (kx, ky)$. Let $\mathcal{Q}_k$ be a collection of squares $Q_{i,j} = [i, i + 1] \times [j, j + 1]$ with $i, j \in \mathbb{Z}$ such that $Q_{i,j} \cap T_k(S) \neq \emptyset$ and let $S_k = \mathcal{Q}_k \cap \mathbb{Z}^2$.

Let c_k be a cooling function on S_k as guaranteed by Theorem 8.2. Let $v \in S_k$ and denote the left, right, above and below edges at v to be e_l, e_r, e_a, e_b, respectively. If $v = i(e_j)$ then we define $sg(e_j) = 1$ otherwise, $sg(e_j) = -1$, where $j = \{l, r, a, b\}$. Then let $\gamma_{v_j} = sg(e_j) \cdot c_k(e_j)$ for $j = \{l, r, a, b\}$.

For each point $v \in S_k$, let $s_v \subset \mathbb{R}^2$ be the square centered at v with sides 1. We define

$f_v : s_v \to \mathbb{R}^2$ by

$$f_v(x, y) = ((1 - (x - \lfloor x \rfloor))\gamma_{v_l} + (x - \lfloor x \rfloor)\gamma_{v_r}, (1 - (y - \lfloor y \rfloor))\gamma_{v_b} + (y - \lfloor y \rfloor)\gamma_{v_a}),$$

where $\lfloor \cdot \rfloor$ denotes the floor function.

Let $f_k : T_k(S) \to \mathbb{R}^2$ be defined as follows: given $(x, y) \in T_k(S)$, there exists a point $v \in \mathbb{Z}^2$ such that (x, y) lies inside the square of side 1 centered at v. We let $f_k(x, y) = f_v(x, y)$ and let $f : S \to \mathbb{R}^2$ be defined as $f(x, y) = f_k(T_k(x, y))$.

We have the following theorem:

Theorem 8.4. *The function f is a cooling field almost everywhere on S, i.e., f exhibits the following properties:*

1. $\text{div} f = \dfrac{\partial f_1}{\partial x} + \dfrac{\partial f_2}{\partial y} \geq 1$ *almost everywhere.*
2. $\|f\| \leq FR(S) = \dfrac{H^2(S)}{H^1(\partial S)}$ *at all points where f is defined, where H^i are Euclidean i^{th} dimensional Hausdorff measures on S and ∂S.*

If S is an optimal Følner set then we obtain the following two conditions:

3. $\text{div} f = \dfrac{\partial f_1}{\partial x} + \dfrac{\partial f_2}{\partial y} = 1$ *almost everywhere.*
4. $\|f|_{\partial S}\| = FR(S)$ *at all points on the boundary of S where f is defined.*

Proof.

1,3 Let c_k be a cooling function defined on S_k and f_k and consequently f be defined as above. Let $(x, y) \in S$. Note that $T_k(x, y) = k(x, y) \in S_k$. We assume $k(x, y)$ lies inside the interior of a square of sides 1 centered at some $v \in S_k$. We show that $\text{div} f(k(x, y)) \geq 1$. Since we are concentrating on one square centered at v, without loss of generality let's assume that the lower left corner of the this square lies at the origin. Furthermore, for simplicity, let's denote $k(x,y) = (x_0, y_0)$. Then the function $f_v(x,y)$ simplifies to $f_v(x,y) = ((1-x)\gamma_{v_l} + x\gamma_{v_r}, (1-y)\gamma_{v_b} + y\gamma_{v_a})$. So

$$\begin{aligned} \mathrm{div} f(x_0,y_0) &= \mathrm{div} f_v(x_0,y_0) \\ &= \gamma_{v_r} - \gamma_{v_l} + \gamma_{v_a} - \gamma_{v_b} \\ &= h(v) = \sum_{i(e)=v} c_k(e) - \sum_{t(e)=v} c_k(e) \geq 1, \end{aligned}$$

Equality is obtained the same way for an optimal Følner set. Hence, except on the edges and corners of all squares v, conditions 1 and 3 hold.

2,4 This is a direct consequence of the properties of the cooling function c and the definition of the norm.

□

Chapter 9. Example of Cooling Functions and Cooling Fields

We would like to explicitly construct an example of a cooling field given a cooling function.

On an arbitrary $M \times N$ grid centered at the origin of 1×1 squares in $\mathbb{Z} \times \mathbb{Z}$, with M, N odd, let

$$f(x,y) = \left(\frac{N(2x+1)}{2(M+N)}, \frac{M(2y+1)}{2(M+N)} \right),$$

for (x,y) in the first quadrant. For each point (x,y) assign f_1 to the right edge and f_2 to the top edge, then with appropriate sign changes in other quadrants, f defines an optimal cooling function on the grid. From here we may think of a cooling function as being defined on the points and assign the values of f_1 to the right edge and f_2 to the top edge.

Let r be a $m \times n$ rectangle in $\mathbb{R}^2$. Let R_k be a grid of size $km \times kn$ centered at the origin of 1×1 squares in $\mathbb{Z} \times \mathbb{Z}$, with km and kn odd. Then

$$c_k(kx, ky) = \left(\frac{kn(2kx+1)}{2k(m+n)}, \frac{km(2ky+1)}{2k(m+n)} \right),$$

for (x, y) in the first quadrant and with appropriate sign changes in other quadrants defines an optimal cooling function on R_k.

We defined the scaled cooling function by dividing all lengths by the k factor

$$\begin{aligned} c_{k \text{ scaled}}(x, y) &= \left(\frac{\frac{1}{k}kn(\frac{1}{k}(2kx+1))}{\frac{1}{k}2k(m+n)}, \frac{\frac{1}{k}km(\frac{1}{k}(2ky+1))}{\frac{1}{k}2k(m+n)} \right) \\ &= \left(\frac{n(\frac{1}{k}(2kx+1))}{2(m+n)}, \frac{m(\frac{1}{k}(2ky+1))}{2(m+n)} \right) \\ &= \left(\frac{n((2x+\frac{1}{k}))}{2(m+n)}, \frac{m((2y+\frac{1}{k}))}{2(m+n)} \right). \end{aligned}$$

Then $C(x, y) = \lim_{k \to \infty} c_{k \text{ scaled}}(x, y) = \left(\frac{nx}{(m+n)}, \frac{my}{(m+n)} \right)$ is a cooling field on r.

Remark. Note that since we a have a nicely constructed cooling function on R_k, we can simply use a limiting method here rather than an interpolation method as described in the previous chapter.

On the other hand given a cooling field $C(x, y)$, we would like to create a cooling function on subsets of $\mathbb{Z} \times \mathbb{Z}$. By this we mean there exists $k > 0$ such that when we define $C_k(kx, ky) = kC\left(\frac{kx}{k}, \frac{ky}{k}\right)$ on the R_k grid, we have that $C_k(kx, ky)$ is "approximately" a cooling function. For example: Let $C(x, y) = \left(\frac{nx}{m+n}, \frac{my}{m+n} \right)$ be a cooling field on an m by n rectangle r in $\mathbb{R}^2$. On the grid R_k we define $C_k(kx, ky) = kC\left(\frac{kx}{k}, \frac{ky}{k}\right) = k\left(\frac{nx}{m+n}, \frac{my}{m+n} \right) = \left(\frac{knx}{m+n}, \frac{kmy}{m+n} \right)$. We check to see whether C_k "approximates" a cooling function.

1. Net input and output of heat at each point in R_k should be 1:

   Let $(x, y) \in r$, then $k(x, y) = (kx, ky) \in R_k$. Then the outflow of heat at the point (kx, ky) is $C_k(kx, ky) = \left(\frac{knx}{m+n}, \frac{kmy}{m+n} \right)$. The input of heat at the point (kx, ky) is composed of the x component of the outflow of heat from the point $(kx - 1, ky)$ and the y component of the outflow of heat from the point $(kx, ky-1)$. We compute those:

$$C_k(kx-1, ky) = kC\left(\frac{kx-1}{k}, \frac{ky}{k}\right) = k\left(\frac{n\frac{kx-1}{k}}{m+n}, \frac{m\frac{ky}{k}}{m+n}\right) = \left(\frac{n(kx-1)}{m+n}, \frac{mky}{m+n}\right);$$

$$C_k(kx, ky-1) = kC\left(\frac{kx}{k}, \frac{ky-1}{k}\right) = k\left(\frac{n\frac{kx}{k}}{m+n}, \frac{m\frac{ky-1}{k}}{m+n}\right) = \left(\frac{nkx}{m+n}, \frac{m(ky-1)}{m+n}\right);$$

The total net loss of heat at the point (kx, ky) is then

$$\frac{nkx}{m+n} - \frac{n(kx-1)}{m+n} + \frac{mky}{m+n} - \frac{m(ky-1)}{m+n} = \frac{n}{m+n} + \frac{m}{m+n} = 1.$$

2. We require $\|C_k(kx, ky)\|$ to be the Følner ratio of R_k for $(kx, ky) \in \partial R_k$. Since R_k is a rectangle of size $km \times kn$, the Følner ratio of R_k is $\frac{k^2mn}{2k(m+n)} = \frac{kmn}{2(m+n)}$. Points on the boundary of R_k have the form $\left(\pm\frac{km}{2}, y_1\right)$ or $\left(x_1, \pm\frac{kn}{2}\right)$, where $x_1 \leq \left|\frac{km}{2}\right|$ and $y_1 \leq \left|\frac{kn}{2}\right|$. Since we are using the sup norm it follows that $\left\|C_k\left(\pm\frac{km}{2}, y_1\right)\right\| = \left\|C_k\left(x_1, \pm\frac{kn}{2}\right)\right\| = \frac{kmn}{2(m+n)}$.

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