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Teacher Graphing Practices for Linear Functions in a Covariation-Based
College Algebra Classroom

Konda Jo Luckau

A thesis submitted to the faculty of
Brigham Young University
in partial fulfillment of the requirements for the degree of

Master of Arts

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ABSTRACT

Teacher Graphing Practices for Linear Functions in a Covariation-Based College Algebra Classroom

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Graphing is a fundamental topic in algebra that is notoriously difficult for students. Much of the past research has focused on conceptions and misconceptions. This study extends past research by looking at the mathematical practices of a practitioner, specifically one instructor of a function-based covariation-focused algebra class in the linear functions unit. Considering practices in addition to conception adds dramatically to our understanding of mathematical activity because it leads to explicit descriptions of normative purposes that are connected to particular situations or problems and also specifies how tools and symbols are coordinated to achieve these purposes. The results of this study are three levels of empirically proven practices associated with the conception of one advanced level of covariational reasoning, chunky continuous covariation. This study not only describes how practices may be described at different levels of complexity, but also demonstrates how smaller practices may be combined to form larger, more complex practices. These practices can be used to guide instruction of those who want to participate in and become practitioners in the community of teachers of function-based covariation-focused algebra curricula.

Keywords: mathematics education, mathematical practices, graphing, algebra, linear functions, covariation

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CHAPTER 1: RATIONALE

Graphing on the Cartesian coordinate plane is a fundamental component of high school mathematics and specifically a central theme in algebra. In the 2010 Common Core State Standards for Mathematics (CCSSM) graphs and graphing are explicitly mentioned in 4 of the 8 standards for mathematical practice (NGA Center and CCSSO, 2010). For example, the first standard for mathematical practice is, “Make sense of problems and persevere in solving them” (NGA Center and CCSSO, 2010). In the explanation of this mathematical practice, authors of the CCSSM describe a mathematically proficient student as one who “can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends” (NGA Center and CCSSO, 2010). Explanations of other mathematical practices mention the ability to model with mathematics by mapping a relationship using graphs, use appropriate tools strategically by analyzing graphs of functions created with a graphing calculator, attend to precision while labeling axes, and look for and express regularity in repeated reasoning while repeating slope calculations. Graphing is so fundamental that components of graphing are woven throughout the standards that govern current high school mathematics curriculum.

While much research has been done on graphing, this research is likely insufficient. Past research on graphing has either focused on identifying the knowledge, skills, and understandings students should acquire in mathematics classes (e.g., Friel, Curcio, & Bright, 2001; Leinhardt, Zaslavsky, & Stein, 1990; Moschkovich, Schoenfeld, & Arcavi, 1993), or the common misconceptions students exhibit during the process of acquiring the requisite knowledge, skills, and understandings (e.g. Leinhardt et al., 1990; Moschkovich et al., 1993; Knuth, 2000). This work, while useful for understanding cognitive processes related to graphing, does not take into

account the social nature of learning and the situated nature of knowledge, and thus is likely insufficient for informing the teaching and learning of graph creation and use, which I will explain further in the following paragraphs.

Knowledge, skills, and understandings are embedded in the practices in which they are learned (Brown, Collins, & Duguid, 1989). Practices (i.e., normative ways of acting within a particular context) in turn are tied to the particular contexts and mathematical communities in which learning takes place (Lave, 1988). The knowledge, skills, and understandings needed to engage in a practice can vary depending on the type of graph, what information is presented on a graph, and what information needs to be obtained from the graph. For example, the understanding required to find the slope of a line in an algebra class is very different from the understanding required to examine a line in a calculus class and conclude that the derivative is a constant function. As a second example, students may be able to find the slope of a line in a distance/time graph of a person walking at a constant rate by identifying two points on a line and using the slope formula. Despite this knowledge, they may be unable to answer how fast the person was walking. On the surface, these two tasks—finding the slope of a line and finding the constant speed of a person walking—are identical in that they can be solved using the exact same procedure. However, students' knowledge of slope may be situated within the practice of finding the slope of a line on a graph rather than the practice of finding the rate of change between two quantities. The two contexts are different and require the student to think about slope in very different ways. Past research on graphing does not take into account that knowledge, as described in the previous example, is situated in a specific context (Brown et al., 1989).

Because knowledge is tied to the practices and context in which learning took place, lists of knowledge, skills, and understandings about graphs are insufficient for identifying what

students must learn so that they can create and use graphs in meaningful and appropriate ways. Rather, research is needed to study the practices used by proficient members of a mathematics community in the particular context we want to be more informed about. The CCSSM focuses on graphing in the context of functions on a Cartesian coordinate plane (NGA Center and CCSSO, 2010). Kaput (1999) identified the study of functions, relations, and joint variation as one of the main strands of algebra and suggested that the concept of functions as the correspondence and variation of quantities should be taught in our schools. Other researchers (e.g. Smith & Thompson, 2008; Thompson & Carlson, 2017) support Kaput in suggesting that school mathematics should focus on fostering students' understanding of quantities and how quantities vary together, which is commonly referred to as covariation. Smith and Thompson (2008) agreed, asserting that we conceptualize representations and manipulations in algebra by reasoning about quantities. Thompson and Carlson (2017) continued by arguing that, "emerging conceptions of continuous covariation were central to the development of the mathematical idea of a function" (p. 422) and encouraged the reform of mathematics classrooms to make the study of functions from the perspective of the covariation of quantities a core component of the curricula. Understanding functions is a fundamental part of an algebra curriculum, and algebra as a course and as a mathematical subject is foundational to higher mathematics. Because understanding functions in algebra is so foundational, the context of a functions-based covariation-focused algebra class is a good context in which to begin a study of graphing practices. Instead of studying graphing practices by personal reflection or the examination of misconceptions, like previous researchers, I identified and analyzed the practices of someone who is proficient with the practices we want our students to become fluent in. To know more about graphing practices in which students should be proficient in the context of functions-based,

covariation-focused algebra class, I examined what specific graphing practices a proficient member of a function-based, covariation-focused algebra class engages in.

CHAPTER 2: BACKGROUND

Literature Review

In this section, I discuss past research that has been done in graphing. The largest body of research on graphing has been done from a cognitive perspective (e.g., Leinhardt et al., 1990; Moschkovich et al., 1993; Friel et al., 2001). I argue that focusing on knowledge and skills has both methodological and theoretical limitations. In contrast to a cognitive perspective, other researchers have researched graphing using a sociocultural perspective (e.g., Roth & Bowen, 2001). These researchers viewed graphing as a practice that teachers apprentice their students into. This allows us to think about knowledge situated in a particular context instead of generally.

Cognitive Perspectives

Early research on graphing was conducted using a cognitive focus, meaning that researchers attended to the knowledge, skills, and understandings students needed in order to create and use graphs. Much of the early work on graphing is summarized by Leinhardt et al. (1990). One contribution of this review consisted of an analysis of tasks involving functions, graphs, and graphing. In their analysis, Leinhardt et al. (1990) focused on four aspects of tasks: action, situation, variable, and focus. The first aspect of tasks consists of the actions a student should be able to perform when interpreting or constructing graphs: prediction, classification, translation, and scaling. The second aspect of tasks is situation, which refers to the mathematics class in which students encounter the task, or to whether the graph is abstract or depicts a real world situation. The third category of tasks is variable. When interpreting and constructing graphs, Leinhardt et al. (1990) noted three distinctions students should make regarding the way variables are conceptualized: static or dynamic, discrete or continuous, and abstract or

contextualized. The final category of task is focus. Focus refers to what aspect of a graph students should pay attention to when performing each task. Students are said to be proficient at graphing when they are able to perform the four actions of prediction, classification, translation, and scaling in tasks involving different situations and different uses of variables while being able to focus on different aspects of a graph.

A second major contribution of Leinhardt et al.'s (1990) literature review involved an analysis of research concerning student misconceptions about graphing. Leinhardt et al. (1990) defined misconceptions as “incorrect features of student knowledge that are repeatable and explicit” (p. 30). They noticed that misconceptions held by students regarding functions and graphs spanned many topics: what is and is not a function, correspondence, linearity, continuous versus discrete graphs, representations of functions, notation, scaling, and relative reading and interpretation. Leinhardt et al. (1990) noticed that misconceptions seemed to occur with students for three general reasons: a desire for regularity, a pointwise focus, and difficulty with abstractions. The misconceptions are catalogued as areas where students commonly lack the knowledge required to proficiently create and use functions and graphs. Identifying these misconceptions have helped teachers hypothesize about what knowledge students might be missing in regards to a particular aspect of functions and/or graphs.

A second trend in the early cognitive research on graphing focused on translating between different types of representations of functions (e.g. Janvier, 1987; Moschkovich et al., 1993; Schoenfeld, 1988). Moschkovich et al. (1993) made two contributions to the way the field thinks about functions. Their first contribution was to notice that functions, including graphs of functions, could be viewed from two different perspectives: a process view or an object view. According to Moschkovich et al. (1993), a process view involves thinking about a function as a

rule, which in terms of graphs, means to see the graph as being made up of distinct ordered pairs that are generated by applying the rule over and over again to different input values. Conversely, an object view involves thinking about a function in its entirety as an object that can be acted upon; for example, to think of a second degree polynomial function as an object would be to see it as a parabola that can be rotated and translated instead of as a set of ordered pairs. (Note: These definitions for process and object are different than the definitions for process and object used in a covariation perspective, which I describe later.) Both a process view and an object view are differentially useful, and one perspective does not necessarily precede the other (Moschkovich et al., 1993).

A second contribution of Moschkovich et al. (1993) was to consider functions in three, not necessarily comprehensive, forms: algebraic, graphical, and tabular. They noticed that teachers expected that if students could translate a function from an algebraic equation to a graph, then they could also translate a function from a graph to an algebraic equation. Instead, students saw each of these types of translation as separate rather than coinciding processes. One important finding to come out of this study was the recognition that students think about functions differently than teachers. While teachers are able to see equations, graphs, and tables as different representations of the same function, students do not naturally make this connection. Students must build an understanding of the relationship between each of these representations one connection at a time. Creating these associations between different representations of functions takes much longer than previously considered. Moschkovich et al. (1993) concluded by defining competence as being able to reason about functions as both processes and objects and to translate between and coordinate the three different representations of functions.

A third trend in early cognitive research on graphing was thinking about graphing in terms of literacy, which these researchers called graph comprehension (e.g., Curcio, 1987; Bertin, 1983; Wainer, 1992; Friel et al., 2001). They identified three levels of graph comprehension: basic, intermediate, and advanced (Olande, 2013), shown in Figure 1. A student who has acquired basic skills is able to read data (Curcio, 1987) or, as expressed by Wainer (1992), extract data from a graph. A student who has acquired skills at an intermediate level is

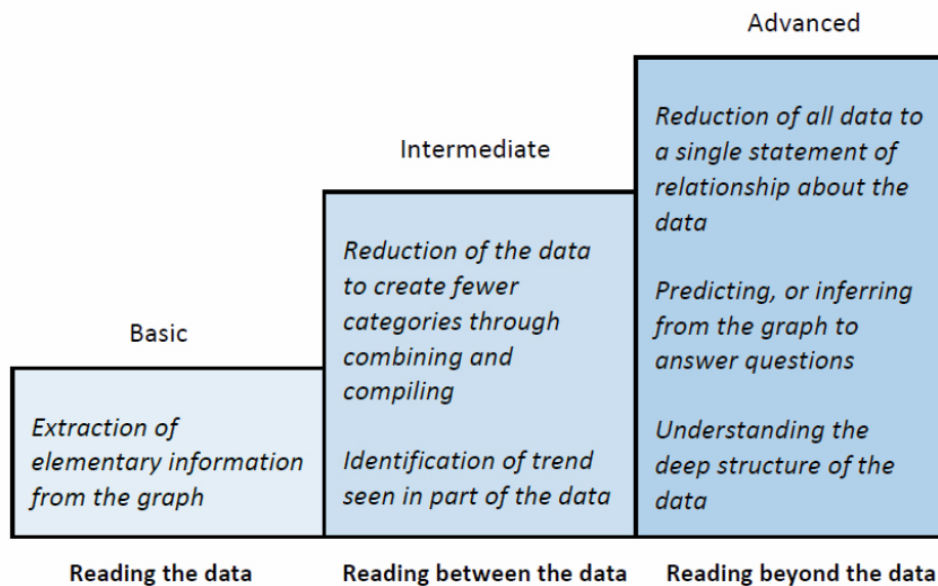


Figure 1. Three levels of graph comprehension.

able to read between the data of a graph (Curcio, 1987) or, as expressed by Wainer (1992), identify trends in parts of the data. A student reading a graph at an advanced level can read beyond the data (Curcio, 1987), or, as described by Friel et al. (2001), extrapolate from the data and analyze relationships explicit in a graph like generating and predicting. These three levels of graph comprehension categorize actions students should be able to perform using graphs in order of increasing difficulty from basic to advanced. From this research, Friel et al. (2001) recommends a progression of types of graphs for students through grade school to optimize their development of graph comprehension, or graph sense.

These three different areas of research in the cognitive research on graphing have each identified a set of knowledge, skills, and understandings students should acquire to be considered competent at graphing. Leinhardt et al. (1990) defined competency as proficiency in tasks that are part of interpreting and constructing graphs. Moschkovich et al. (1993) defined competency as being able to translate functions from one perspective (process or object) and one form (algebraic, graphical, or tabular) to another. Last, Friel et al. (2001) defined competency in terms of levels of graph comprehension. Each of these sets of competencies provides additional insight into the complexity of graphing, which validates students' difficulties with graphing. However, a finite set, or list, of knowledge, skills, and understandings suggests that a general way of creating and reading graphs exists when in reality, it is impossible to make a list of general knowledge that will satisfy every context. This point is illustrated by Roth in the next section.

Graphing from a Sociocultural Perspective

A sociocultural perspective challenges the perception that expertise or competence can be achieved by learning a body of knowledge, particularly if that knowledge is learned separately from the context in which it is to be used. An "expert" from a cognitive perspective is a person who has acquired a predetermined body of knowledge. On the other hand, an expert within a sociocultural model is often referred to as a practitioner. A practitioner is different from an expert because practitioner refers to a person who is able to engage proficiently in a particular activity in a particular setting (Brown et al., 1989), rather than someone who has acquired a body of knowledge. From a sociocultural perspective, acquiring a predetermined body of knowledge does not sufficiently account for how someone is able to engage in an activity because knowledge is always tied to an activity and the context in which it was learned (Brown et al., 1989).

One researcher who has looked at graphing from a sociocultural perspective with his colleagues is Roth. From his experiments, Roth found that even proficient graph users have trouble interpreting graphs that are unrelated to what they do every day. In one experiment, Roth and Bowen (2001) interviewed 16 practicing scientists. These scientists all had either a master's degree or a PhD and at least five years of experience in their fields. Many of them also had professional awards. These scientists can be considered experts in their fields from a cognitive perspective and a practitioner from a sociocultural perspective. Roth and Bowen (2001) showed each scientist three basic ecology graphs with which they were unfamiliar and two graphs taken from the scientist's own publications. When reading the unfamiliar graphs, all but one of the scientists struggled and made errors similar to the common misconceptions identified by Leinhardt et al. (1993). However, when reading the graphs from their own publications, the scientists effortlessly interpreted their graphs in a manner that reflected advanced graph comprehension (Friel et al., 2001). From a cognitive perspective, these scientists can be called experts. They have acquired the skills that we expect our students to have acquired. Yet, these experts struggled to interpret basic ecology graphs (Roth & Bowen, 2001), which from a cognitive perspective suggests they must be missing crucial knowledge, and thus are not experts and must now be labeled as incompetent. However, "given the training and experience of our scientist, cognitive deficit models are implausible" (Roth & McGinn, 1998). Roth and McGinn (1998) use the label cognitive deficit model to refer to a perspective that explains the errors of the scientists by suggesting that they do not have the requisite graphing knowledge to interpret the graph. As Roth and McGinn (1998) observe, it is unthinkable to use a cognitive deficit model to explain the errors of the scientists because this is equivalent to calling these scientist incompetent at reading graphs.

Having a finite list of general skills for students to attain is similarly problematic. This idea presumes that students can become proficient at graphing by acquiring a set of general skills pre-defined by a teacher. If this is the case, then how could someone who has been identified as an expert in a field that requires them to be skilled at reading graphs, struggle to read a graph (Roth & Bowen, 2001)? If a person has been identified as someone possessing a certain body of knowledge, they cannot simultaneously be identified as a person who is lacking that body of knowledge. Using a cognitive model creates this paradox. What is missing from cognitive models is a recognition that knowledge is learned in a particular context that cannot be separated from that knowledge, or as Brown et al. (1989) stated, “the activity in which knowledge is developed and deployed ... is an integral part of what is learned” (p. 32). The paradox of being simultaneously expert and incompetent is avoided because, from a sociocultural perspective, competence is associated with being able to function proficiently in a particular situation; it is possible to be proficient with graphs in one situation and struggle with graphs in a different situation.

Some might argue that a cognitive model does not inherently lead to a paradox, since a person could have general graphing knowledge that they failed to transfer (Lave, 1988) to a particular type of graph. Assuming that knowledge can be transferred from the context in which it was learned to a different context, the idea that an expert scientist could transfer their general graphing knowledge to make sense of the sophisticated graphs they use in their own research and yet be unable to transfer that same knowledge to a basic ecology graph is just as preposterous as suggesting that their errors in reading the ecology graph were caused by not having the requisite general graphing knowledge to interpret the graph in the first place.

Roth and Bowen (2001) concluded that students and practitioners can both expect to have trouble interpreting graphs when they encounter unfamiliar sociocultural conventions or when they are unfamiliar with the real world situation the graph is portraying. These two fundamental difficulties suggest that when someone struggles to interpret a graph, the struggle may come from a lack of enculturation rather than a lack of general knowledge about graphs. Using a sociocultural perspective (Brown et al., 1989), we can identify what types of social practices practitioners use in order to become better aware of what it takes to be a participant in a specific community (e.g., Lave, 1988; Hutchins, 1995).

Theoretical Framework

Rather than identifying a list of general knowledge I want my students to acquire to be successful, using a sociocultural perspective I can identify specific practices engaged in by a practitioner to understand what it looks like to be successful in a specific context. I first describe my theoretical framework by outlining the definition of practice that I will use to research graphing specifically in the context of a college algebra class with a covariation curriculum. I then explain how I use the covariation curriculum to help me understand and identify practices in this classroom.

Practice

The word practice is often used broadly and often with different meanings. Practice is used in the most general sense to refer to an observable activity (Roth & McGinn, 1995). I will be using a more detailed definition of mathematical practice proposed by Cobb, Stephan, and Bowers (2011). The detail of this definition makes the process of identifying practices easier. Cobb et al. defined a *mathematical practice* to consist of a normative act that is constrained by three related norms:

Normative purpose for engaging in mathematical activity;

Normative standards of mathematical argumentation;

Normative ways of reasoning with tools and symbols. (p. 110)

Using this definition, a mathematical practice must include a purpose, a method of argumentation, and a way using tools and symbols. This definition is grounded in social constructivism. From a social constructivist perspective, “a practice is seen to be an emergent phenomenon rather than an already-established way of reasoning and communicating into which students are to be inducted” (Cobb, Stephan, McClain, & Gravemeijer, 2001/2011). Social constructivist and sociocultural perspectives differ in their position regarding where practices originate. Social constructivists view classroom practices as something constructed by students and teachers together. Conversely, those who adopt a sociocultural perspective consider the need for students to be enculturated into a previously existing social practice. While a sociocultural perspective and social constructivism differ in how they portray students’ contribution to the construction of practices, the two perspectives are not necessarily contradictory. Cobb and Bowers (1999) described two different studies, one analyzed from a sociocultural perspective and the other analyzed from a social constructivist perspective. They said that these two analyses “complement each other in that they together portray individual students’ reasoning as acts of participating in the practices of the local classroom community and in the practices of broader communities” (Cobb & Bowers, 1999, p. 9). While these two theoretical perspectives come to different conclusions regarding the origination of practices, both perspectives are similar in their understanding of how mathematical practices are defined (Cobb & Bowers, 1999). Because of this similarity, it is not contradictory to use a definition of mathematical practice grounded in social constructivism from a sociocultural perspective.

From a sociocultural perspective, there are mathematical practices that have already been and continue to be negotiated and created in the mathematical community at large. A mathematics teacher is a member of the community of teachers of mathematics. An experienced teacher would then be considered a practitioner in the community of teachers of mathematics and would be skilled at participating in the mathematical practices that occur within a mathematics classroom. I identified and examine what specific mathematical graphing practices a teacher, as a practitioner of doing mathematics, used in the classroom. Of course, I will need to consider that the teacher's mathematical activity in the classroom may not consist merely of mathematical practices, but may contain a mixture of mathematical and pedagogical practices to allow students access to the mathematical practices. Mathematical and pedagogical practices differ in their purpose. A mathematical practice is aimed at solving a mathematical question while the purpose of a pedagogical practice is helping students develop understanding. Because the mathematical practices of the practitioner are situated in a school context, I can anticipate that her purpose would be to help her students develop authentic practices. Her pedagogical practices will then provide insight into her authentic mathematical practices. Thus, I can use both mathematical practices and pedagogical practices to consider the authentic mathematical practices of a practitioner.

Studies of proficient graphing of necessity bring into question the context in which proficiency should be studied. Because knowledge is situated in a context (Brown et al., 1989), it needs to be studied in the particular context we want to be more informed about. As previously argued, algebra is fundamental to the mathematics taught at the secondary school level and at the university level. One strand of algebra, according to Kaput (1999), is the study of functions, relations, and joint variation. He noted, "The idea of function has perhaps its deepest conceptual

roots in our sense of causality, growth, and continuous joint variation—where one quantity changes in conjunction with change in another” (Kaput, 1999, p. 146). One curriculum that exemplifies this strand is Pathways College Algebra curriculum (Carlson, 2016). This curriculum was selected because it emphasizes functions and graphing throughout the course. To know more about graphing practices in which students should be proficient in a function-based algebra class, we need to study what specific graphing practices proficient members of that particular mathematics community, or the community of a function-based algebra class, are engaged in.

Covariation

One prominent function-based algebra curriculum is Pathways College Algebra (Carlson, 2016). This curriculum was designed from research on functions from a dynamic perspective called covariation (Carlson, Jacobs, Coe, Larsen, & Hue, 2002; Thompson & Carlson, 2017) as opposed to a static object view. A dynamic process view of functions is preferable because it requires someone to envision “quantities in their conceptualized situation as having values that varied” (Thompson & Carlson, 2017, p. 425). Conversely, having a static object view of functions is undesirable because it contributes to “[students’] inability to construct meaningful formulas to represent one quantity as a function of another” (Thompson & Carlson, 2017, p. 426). Carlson et al. (2002) defined covariational reasoning as “cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other” (p. 354). In their research, Carlson et al. categorized students’ ways of coordinating the change between two varying quantities into five distinct levels that emerge developmentally that they call the five mental actions of covariation, shown in Figure 2 (Carlson et al., 2002, p. 357). For example, Mental Action 2 (MA2) requires coordinating the direction of change of one variable with changes in the other variable. Along with the mental actions,

Carlson et al. (2002) described observable behaviors, also shown in Figure 2, that, when exhibited by students, suggest that the students have the ability to reason about two varying quantities in a certain way. The observable behaviors corresponding to MA2, for example, are that students can construct an increasing straight line and verbalize an awareness of the direction of change of the output while considering changes in the input. These observable behaviors are suggestive of some of the graphing practices participants might use as they engage in covariational reasoning.

Mental action	Description of mental action	Behaviors
Mental Action 1 (MA1)	Coordinating the value of one variable with changes in the other	<ul style="list-style-type: none"> Labeling the axes with verbal indications of coordinating the two variables (e.g., y changes with changes in x)
Mental Action 2 (MA2)	Coordinating the direction of change of one variable with changes in the other variable	<ul style="list-style-type: none"> Constructing an increasing straight line Verbalizing an awareness of the direction of change of the output while considering changes in the input
Mental Action 3 (MA3)	Coordinating the amount of change of one variable with changes in the other variable	<ul style="list-style-type: none"> Plotting points/constructing secant lines Verbalizing an awareness of the amount of change of the output while considering changes in the input
Mental Action 4 (MA4)	Coordinating the average rate-of-change of the function with uniform increments of change in the input variable.	<ul style="list-style-type: none"> Constructing contiguous secant lines for the domain Verbalizing an awareness of the rate of change of the output (with respect to the input) while considering uniform increments of the input
Mental Action 5 (MA5)	Coordinating the instantaneous rate of change of the function with continuous changes in the independent variable for the entire domain of the function	<ul style="list-style-type: none"> Constructing a smooth curve with clear indications of concavity changes Verbalizing an awareness of the instantaneous changes in the rate of change for the entire domain of the function (direction of concavities and inflection points are correct)

Figure 2. Mental Actions of the Covariation Framework (Carlson et al., 2002, p. 357).

Because the behaviors that reflect particular mental actions occur on the Cartesian coordinate plane, graphing is an integral part of the curriculum developed from this framework. While these behaviors suggest graphing practices play an important role in covariational reasoning, Carlson et al. (2002) have not systematically researched what behaviors are exhibited at each level. In addition, we do not know if these behaviors could be considered normative because they do not include a description of purposes or ways of reasoning with tools and symbols. Thompson and Carlson (2017) recently updated Carlson's framework. In the updated framework, they focus on covariational reasoning independent of variational reasoning and conceptions of rates of change. In this updated framework, Thompson and Carlson continue to acknowledge that sets of behaviors accompany the different levels of their framework for covariational reasoning, but they no longer list these behaviors, leaving in question which behaviors accompany which levels of covariational reasoning. Thus, there is reason to conduct research on a teacher's graphing practices in the particular setting of a college algebra class using the function-based, covariation-focused curriculum Pathways College Algebra to study what practices are associated with advanced levels of covariational reasoning.

Research Question

In summary, much of the previous research in graphing focused on identifying the knowledge, skills, and understandings students should acquire to be considered competent at graphing, which presumes that a general way of creating and reading graphs exists. In reality, it is impossible to make a list of general knowledge that will satisfy every context. It is possible, however, to identify specific mathematical practices engaged in by a practitioner to become more aware of what it takes to be a participant in a specific community. Because function-based algebra is fundamental to the mathematics taught at the secondary school level and at the

university level, a practitioner using the function-based, covariation-focused curriculum would be the perfect subject to look for what it takes to be a participant in a function-based, covariation-focused mathematics classroom.

Thus, my research question is: What graphing practices are being used by a teacher experienced in a function-based, covariation-focused college algebra classroom?

CHAPTER 3: METHODS

In this chapter I present the research methods for my study. In the following sections I describe first the setting and participant of the study. Next, I describe how I collected, organized, and analyzed the data. Last, I describe my subjectivity and possible validity threats.

Setting and Participant

This study focused on one female university faculty member who I will refer to as Dr. Brooke Kemp. Dr. Kemp was selected for her experience in teaching algebra. She received a doctorate in curriculum and instruction with an emphasis in mathematics education and has been teaching at the high school and college levels for 20 years. In particular, Dr. Kemp has taught college algebra using the Pathways College Algebra curriculum (Carlson, 2016) at the university level for four years and is currently the course coordinator for this course. Based on her experience and training, she qualifies as a practitioner of graphing in the setting of an algebra class taught with a covariation framework. The focus of the study was specifically to identify and describe the graphing practices used by Dr. Kemp during class.

The college algebra class in which Dr. Kemp was observed met for 50-minute sessions three times a week and consisted of about 40 undergraduate university students of varied ages, genders, and mathematical backgrounds. Students took this class for a variety of reasons: as a prerequisite course for calculus, as a terminal course to fulfill a general education requirement, or as an elective course to prepare them for mathematics they might encounter in their major courses. The students sat at seven hexagonal tables and were often expected to work together in groups during class time. This arrangement matched Dr. Kemp's pedagogy and the covariation algebra curriculum of inquiry-based learning. This setting was an ideal situation for collecting data to determine Dr. Kemp's graphing practices, because not only did she engage in these

practices as she solved problems during class, but she also employed pedagogical practices aimed at making these graphing practices more visible for her students.

Data Collection

I collected two different kinds of data: video recordings and field notes. My primary source for data was videotapes of Dr. Kemp's instruction during the linear functions unit in her Pathways College Algebra class. The linear functions unit was covered in five lessons which occurred in five 50-minute class periods. I recorded the last three lessons. I did not record the first two days of the linear functions unit for two reasons. First, these lessons took place the first two days of the semester, so I was unable to get signed consent forms from each of the students for those days. Second, Dr. Kemp did not anticipate using graphs in her instruction on those days. My second source of data were field notes I created while observing each of the three class periods. I recorded these field notes using a classroom observation protocol developed during a pilot study (see the appendix for a copy of the classroom observation protocol). These field notes served as a tool to index the content of the video recordings.

During the three class periods, I observed five unique graphing instances. I defined a graphing instance to be the entirety of one task or problem assigned by Dr. Kemp that included a graph in the problem, in the process of finding a solution, or in the solution itself. Each graphing instance was recorded on a separate classroom observation protocol that initially included a recording of the problem given, a sketch of the graph drawn on the board, and time codes to mark the beginning and the end of each graphing instance. I also identified what aspect(s) of the graph were focused on during the graphing instance. Following each class period, I finished each classroom observation protocol by making sure the date, lesson, and type of graph were recorded completely. I also identified the aspect(s) of the graph with codes which were used as a reference

tool. For example, CROC was a code used to represent that the constant rate of change was part of the discussion during that particular graphing instance. Lastly, I attached a photocopy of the prompt used to begin the discussion surrounding the graphing instance to the corresponding classroom observation protocol.

While I took field notes, a research assistant recorded the class using a video camera. The video camera was set up in a fixed location in the back of the classroom. During these class periods, Dr. Kemp used only the whiteboard in the front of the classroom and the computer display which was projected onto a screen that lowered in the front of the classroom, so the video camera was mainly focused on the front whiteboard. However, Dr. Kemp walked around the room during instruction. Because her motions and gestures communicated important information regarding her graphing practices, when Dr. Kemp was moving around the classroom and speaking to the class as a whole, the video camera zoomed out to film Dr. Kemp. The location of the camera was fixed in the back of the room, but it turned and zoomed in and out as needed to capture the inscriptions of graphs and associated gestures by Dr. Kemp. The microphone on the video camera was sufficient to capture Dr. Kemp's speech, so a separate microphone was not necessary. From the video recordings, I decided to examine every observed graphing instance in detail in order to identify graphing practices that were used by Dr. Kemp in her class.

Data Analysis

I used Cobb et al.'s (2011) definition of practice to analyze my data. To use this definition, I considered two of the three aspects of normative acts that are used to recognize mathematical practices: the purpose and ways of reasoning about tools and symbols. I did not attend to the second aspect, standards of argumentation. Because I only looked at one particular practitioner, in one course, with one group of students, I assumed that I would find little variation

in the normative standards of argumentation. I used the first aspect of purpose to chunk my data into my unit of analysis—a practice—by segmenting Dr. Kemp’s mathematical activity into segments that shared the same mathematical purpose. I determined purposes by examining what mathematical information Dr. Kemp was attempting to find, construct, or determine. Examples of mathematical purposes that I observed included finding the horizontal change, constructing the vertical change, and determining if a relationship between two quantities could be characterized by a constant rate of change. I then used the aspect of normative ways of reasoning about tools and symbols to characterize and describe the practices that I observed. The specific tools and symbols I considered were graphs and their subparts on the Cartesian coordinate plane.

As I examined my data, I noticed that some of Dr. Kemp’s actions in the classroom were driven more by pedagogical purposes than mathematical purposes. These actions were usually easy to separate from Dr. Kemp’s mathematical activity because they did not seem necessary for Dr. Kemp to solve the problem herself; rather, they seemed to support student’s mathematical reasoning and understanding. These pedagogical actions, although not part of Dr. Kemp’s mathematical practice, were nonetheless useful in understanding her mathematical thinking and behavior. To illustrate, consider the following segment of instruction. Dr. Kemp introduced a task on diagramming the changes in two linearly-related variables by reading the following prompt from the student workbook: “Illustrate on the graph Δy equals negative two Δx , and it tells you that it passes through the point one, seven.” She then wrote on the board, “ $\Delta y = -2\Delta x$ ” and “(1, 7)”. After giving her students time to follow the prompt and observing the graphs that her students created, Dr. Kemp presumably felt the need to clarify the prompt. She read the equation she had written on the board as, “Change in y is ...” When she hesitated, her students responded by saying, “negative two,” to which Dr. Kemp nodded and finished, “as large as the

change in x .” While she was reading the equation, she was writing her clarification below the equation: “ Δy is -2 times as large as the Δx .” Dr. Kemp’s writing of this sentence on the board was likely more motivated by pedagogical concerns than by Dr. Kemp’s own mathematical activity. In other words, it is unlikely that Dr. Kemp needed to write this sentence in order to solve the problem herself. Instead, this action likely fulfilled the pedagogical purpose of helping her students understand how to think about the equation. This pedagogical action, in turn, suggests how Dr. Kemp was conceptualizing the equation $\Delta y = -2\Delta x$, namely as a multiplicative relationship between the changes in the two variables. Thus, Dr. Kemp’s writing on the board was not considered a part of her mathematical practice, but was used as evidence of how she was thinking mathematically while engaged in her mathematical practice.

The analysis of the data occurred in several phases. The first phase began while I was collecting data, and consisted of transcribing and segmenting data. As described previously, while taking field notes using my classroom observation protocols, I identified instances of graphing that occurred in Dr. Kemp’s instruction, which I immediately began transcribing. I eventually transcribed each of the five graphing instances using VLC Media Player. My transcriptions initially included the audible audio of each speaker (the audio of students was generally inaudible), and were segmented by speaker. These initial segments were made with the understanding that they would change upon further analysis. I next added to the transcription relevant gestures and motions that I observed, and inscriptions associated with the graph. As I added these gestures, motions, and inscriptions, my transcription became segmented further to mark a change in any of the aforementioned categories. Eventually, each of these segments were numbered sequentially and identified with a time code that marked the beginning of that segment on the video recording.

The second phase of data analysis consisted of categorizing and coding my transcripts according to the first part of my definition of practice, mathematical purpose. Beginning with the first transcript that had been segmented, I linked sequential segments that shared a common normative purpose. For each collection of segments, I inferred the mathematical purpose and used that purpose to identify and label the collection of linked segments with a code. As a result, each of the five transcripts were chunked and coded according to the mathematical purpose of observed behaviors. These codes were the beginning of my list of mathematical practices.

The third phase of data analysis began after transcribing and coding the second graphing instance as I reviewed the codes from the two graphing instance to see if I had consistently used the same code for the same purpose. To aid in this process, I created a *code book* where I compiled the codes that I had created and I recorded my reflections in a journal. In the code book, I identified the purpose of the practice associated with each code, the objects and elements that were part of the practice, the gestures associated with that code, and details that made this code distinct from other codes that might be similar. This process helped clarify my codes because I continued to add to my descriptions until each code represented a distinct practice, and I could apply the codes consistently to the rest of my data. After using the code book to clarify my codes with the first two graphing instances, I proceeded to transcribe and code the next three graphing instances, refining the codes in my code book along the way. The result of this phase was the clarification of my codes. These codes became my list of practices.

The fourth phase of data analysis began as I looked across the mathematical practices compiled in my code book to see if they could be categorized in some way. The process of analyzing the code book helped me identify relationships between codes. For example, one pattern I recognized occurred with my codes HC which represented mentioning or discussing

horizontal change and VC which represented mentioning or discussing vertical change. The code book helped me realize that Dr. Kemp always mentioned or discussed horizontal change prior to mentioning or discussing vertical change. Noticing that the codes representing horizontal change and vertical change occurred in a specific order, I looked for an order among the other codes. In this phase, instead of focusing on the accuracy of the descriptions of individual practices, I began to explore the relationships between practices to understand how Dr. Kemp coordinated her practices.

Throughout the transcription and coding process, I recorded my reflections in a journal. One purpose of journaling was to help clarify and improve the descriptions of the practices I had recorded in my code book during the third phase of analysis. For example, journaling helped me determine what level of detail was necessary to describe the crucial components of each practice. An instance of this process of clarification occurred when, in my original transcription, I transcribed Dr. Kemp's gestures in a very specific way. On one occasion, I described Dr. Kemp using the index finger and the pinky finger of her right hand to mark the ends of a horizontal vector. On another occasion, I described Dr. Kemp using two index fingers to mark the ends of a horizontal vector. After reflecting, through the journaling process, on the codes that were being developed, I realized that the gesture that was most relevant to Dr. Kemp's practice surrounding the horizontal vector was that she marked each end of the horizontal vector, and not which fingers she used to mark the ends of the vector. This enabled me to see more commonality among the different instances when she found the horizontal vector, and led to a more inclusive description of the code. It also enabled me to change my transcription process to eliminate the unimportant details of which fingers she used and record gestures using a more useful, less detailed level of description.

Journaling also played an important role in understanding the relationship between my codes. One instance when journaling helped me arose after I recognized the change diagram and struggled to understand its significance. Shortly after naming the change diagram, I wrote, “Wondering about how I want to talk about practices. I have identified the change [diagram] as a practice with multiple uses. This feels problematic because my definition says a practice has a normative purpose.” I struggled with the relationship of the change diagram to Dr. Kemp’s practices because I initially called the change diagram a practice when it was not, in fact, a practice. Through journaling, I reflected on the accuracy of my understanding of the change diagram, and after much reflection and consideration, I came to understand of the significance of the change diagram. I recorded this breakthrough in my journal:

“...the change [diagram] is NOT a practice. There are different practices associated with the change [diagram]. These practices are distinguished by purpose and ways of reasoning. I can have different practices with the same purpose that are differentiated by ways of reasoning.”

Understanding that the change diagram was not a practice led to the realization that there was a hierarchy of practices because some of Dr. Kemp’s practices focused on making sense of or constructing components of the change diagram where, in other instances, an entire change diagram was constructed as part of the practice. After realizing that I had different sizes of practices, I separated my codes first into two levels and eventually into three levels according to which practices seemed to be smaller sub-practices of other larger practices. I refer to these three general levels of practices as micro practices, intermediate practices, and advanced practices, and describe these levels, as well as the individual practices that fall into these levels, in the next

chapter. As a result of this phase of analysis, I was able to create a coherent account of how Dr. Kemp coordinated practices as she engaged in creating and making sense of graphs.

Subjectivity and Validity

Whenever conducting research, it is apropos to assess the researchers' subjectivity and possible threats to validity. I have taught mathematics, mostly in a part-time capacity, for over fifteen years in middle school and high school as well as at the university level. During that time, I have almost exclusively taught algebra courses, either pre-algebra, beginning algebra, intermediate algebra, or college algebra using many curricula. Most recently, I have taught the Pathways College Algebra curriculum for four semesters. My background brought familiarity to the course that I observed. This familiarity was helpful and also challenging. Familiarity with this course allowed me to anticipate the purposes behind many of the graphing activities I observed. The same familiarity may have caused me to assign mathematical purposes to Dr. Kemp's behavior that reflected my own purposes when engaged in similar mathematical behavior rather than Dr. Kemp's purposes. Recognizing this weakness, I used my transcripts to identify the purpose that Dr. Kemp verbally communicated to her students as well as the mathematical result of each of her practices. I deliberately considered these verbal communications and results when I assigned mathematical purposes to Dr. Kemp's behavior. In addition, having so much experience teaching algebra generally, I was concerned that my familiarity with mathematical activity while engaging with graphs in algebra might cause me to overlook crucial gestures and motions similar to ones I perform subconsciously, as well as the significance of these movements. Recognizing this possibility, I initially went out of my way to meticulously identify and record all of Dr. Kemp's movements as she moved from place to place in the classroom, as well as all of her gestures. I then carefully considered whether these

movements and gestures conveyed mathematical information in terms of emphasizing particular content or activity. As discussed above, I reflected on this in my journaling to differentiate which type of activity seemed to be mathematically important, and thus a crucial part of a mathematical practice.

In addition to my subjectivity, other threats to validity came from my intrusion into this classroom as a researcher. This intrusion had the potential of changing the classroom dynamics by affecting the teacher and/or the students. However, Dr. Kemp has had considerable experience being observed as a teacher, particularly while teaching the Pathways College Algebra curriculum. Because of this experience, I felt that my presence and the presence of a video camera had a negligible influence on her teaching. Dr. Kemp is also experienced in creating a classroom environment conducive to active student participation. Because the students in this particular class were adults and because my research assistant and I did not interact with the students, students seemed to quickly ignore our presence. Furthermore, I have had many opportunities to observe Dr. Kemp's class and teaching when there were no cameras in the room. The participatory classroom environment during video recording in this study seemed typical of Dr. Kemp's classrooms.

CHAPTER 4: RESULTS

Findings

When analyzing the codes from the transcripts, I noticed three distinct levels of practices: micro practices, intermediate practices, and advanced practices. All three levels of practice involved an image drawn on a graph that I call the *change diagram*, shown in Figure 3. Micro practices are the smallest practices and are used to create the elements of the change diagram. Intermediate practices consist of practices that involve using one instantiation of the change diagram. Advanced practices are complex practices that are composed of micro practices and intermediate practices and involve using multiple instantiations of the change diagram. I first describe the change diagram by explaining the five micro practices in which Dr. Kemp engaged as she created one change diagram. Next, I describe the three types of intermediate practices in which a single change diagram was used as part of the practice. Third, I describe an advanced practice in which multiple change diagrams were created as part of the practice. Finally, I describe practices used by students that were discredited by Dr. Kemp. These discredited practices further clarify Dr. Kemp's proficient practices.

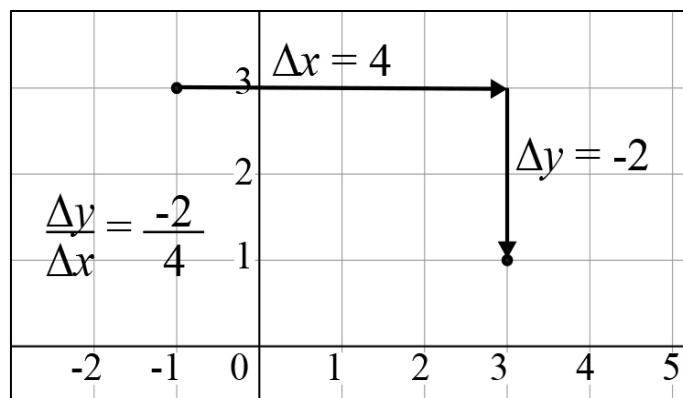


Figure 3. Change Diagram. Elements include an initial point, horizontal and vertical vectors, a final point, and a constant or average rate of change of y with respect to x .

Micro Practices Associated with the Construction of the Change Diagram

The change diagram is a diagram that is inscribed on a graph to invoke and support covariational reasoning (Thompson & Carlson, 2017). Elements of the change diagram include an initial point, horizontal and vertical vectors, final point, and the constant or average rate of change of y with respect to x written as the ratio of the vertical change to horizontal change. Each of these elements are constructed in a specific order. The construction of each of these elements on or near the graph is a distinct micro practice. Together the micro practices create one instantiation of the change diagram and its accompanying inscriptions. I will now illustrate these micro practices by describing instances where Dr. Kemp engaged in each of these micro practices, and I will also describe variations of the micro practices that I observed.

To illustrate the five micro practices, I refer to an episode from class in which Dr. Kemp used a prompt from the Pathways College Algebra Student Workbook (Carlson, 2016) to introduce the change diagram to her students. The prompt stated, “Illustrate the meaning of $\Delta y = -2 \cdot \Delta x$ on a graph that passes through the point $(1, 7)$ ” (Carlson, 2016, p. 18). Dr. Kemp read the prompt aloud to her students saying, “It says illustrate on the graph delta y equals negative two delta x , and it passes through the point one seven,” as she wrote “ $\Delta y = -2 \cdot \Delta x$ ” and “ $(1, 7)$ ” on the board. Initially, Dr. Kemp read and wrote the prompt as it appeared in the text, but after a brief moment, she rephrased the prompt in a slow, reflective manner asking, “How are you going to show that change in x and change in y ?” When she carefully rephrased the prompt using the expressions “change in x ” and “change in y ” instead of “delta x ” and “delta y ,” Dr. Kemp seemed to show that she was thinking about illustrating a change in the two quantities.

After allowing students time to work, Dr. Kemp noticed that a lot of students had interpreted the prompt incorrectly by drawing lines on their graphs, so she asked the class if there

was a difference between drawing a line and drawing changes. Dr. Kemp directed the students' attention to the word change by asking, "... remember this is talking about the *change* in y is always what?" She emphasized the word change by first pointing at Δy on the board, and then verbally prolonging the word while simultaneously circling Δy with her finger. Seemingly, to guide students' thinking about the changes in quantities, Dr. Kemp next wrote, " Δy is -2 times as large as the Δx ," while saying, "change in y is ...," nodding as students added, "negative two times," and finishing, "as large as the change in x ." With this sentence, Dr. Kemp showed that she was thinking about the original equation as a specific comparison between two changes in quantity.

Micro Practice 1: Construct an initial point. The change diagram begins from an ordered pair that is considered the initial point. This ordered pair is generally given, but it can be found if enough other information is given. An example of how an initial point can be found is described in the intermediate practice of finding a point. In the illustration of this micro practice, the initial point was given. Dr. Kemp began construction of a change diagram by drawing a point on her graph at the ordered pair $(1, 7)$, shown in Figure 4a, as specified by the prompt. This point marks the beginning of the change diagram.

Micro Practice 2: Construct a horizontal vector. From the initial point, a horizontal vector is constructed to show the distance and direction of the horizontal change. The purpose of the horizontal vector is to make visible on a graph the change that is occurring with the independent variable. This purpose can be made explicit by an inscription in the form of labeling the horizontal vector using the Greek letter delta, Δ , which means "change" in mathematics, accompanied by the variable representing the horizontal quantity and a numerical evaluation of the horizontal change. The meaning of the horizontal vector can be made more explicit through

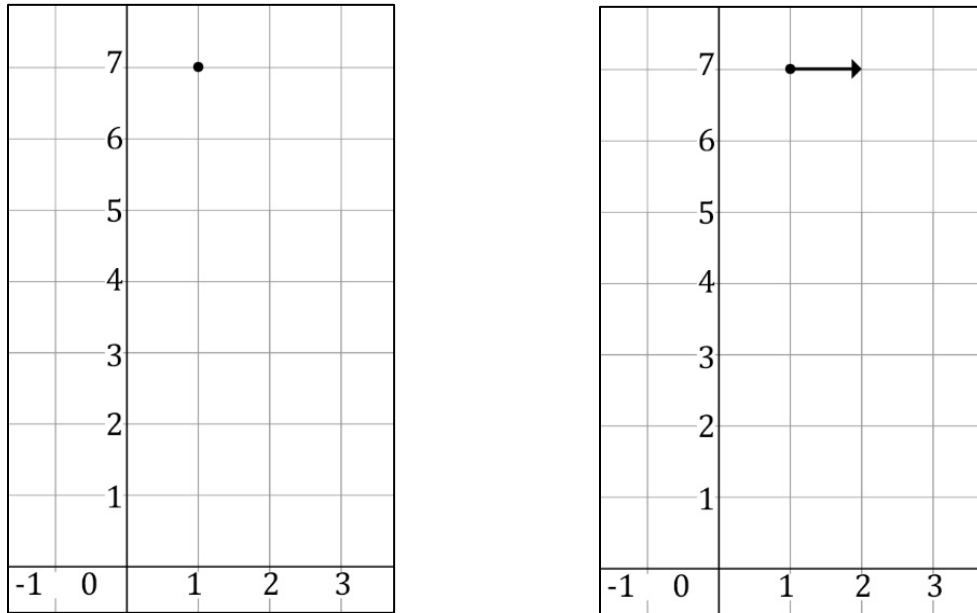


Figure 4. Micro Practices 1 and 2: a) Construct an initial point.
b) Construct a horizontal vector.

gestures pointing at the vector or identifying the ends of the vector simultaneously. The end of the horizontal change is only indicated by the head of the vector. A point is not constructed to mark the end of the horizontal change because a horizontal change is generally accompanied by a vertical change. Dr. Kemp appeared to avoid graphing a point at the end of the horizontal vector to emphasize the idea of covariational reasoning that the horizontal quantity and the vertical quantity are always changing together.

To illustrate this micro practice, we return to Dr. Kemp’s construction of a change diagram with initial point at (1, 7). After the initial point was constructed, Dr. Kemp asked her students, “How do I represent this change in y is negative two times as large as my change in x ?” One of her students suggested they “drop two,” which seemed to imply that they should draw a vertical vector down from the initial point. Dr. Kemp implied that the construction of a vertical change should come after the construction of a horizontal change when, instead of drawing a vertical vector, she asked, “From where?” and revoiced the student’s response by saying, “... so

he moved it over to the x one, right?” as she drew a horizontal vector starting at the initial point and ending one unit to the right, shown in Figure 4b. Dr. Kemp emphasized the construction of the horizontal change by asking the question, “That’s representing what?” while pointing at and tapping the horizontal vector, shown in Figure 5a, and saying, “My change in x , right?” Dr. Kemp continued to make this change explicit by adding “... ‘cause I’ve changed x one unit” while marking the magnitude of the change with her fingers along the x -axis, shown in Figure 5b, and then labeling the vector “ Δx .” Later Dr. Kemp returned to this micro practice and added the numerical evaluation “= 1” to the original label making the inscription “ $\Delta x = 1$ ” above the horizontal vector that had been constructed.

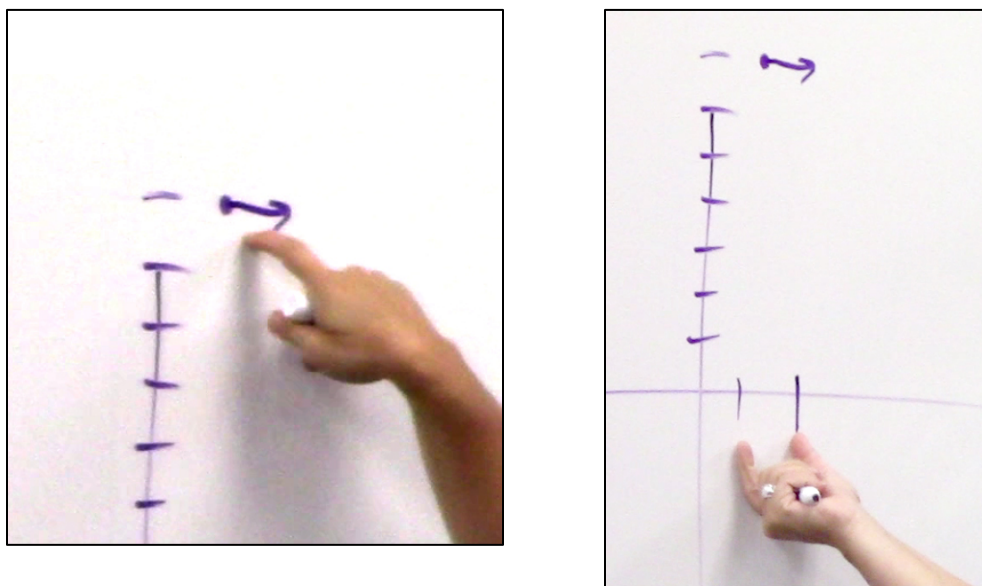


Figure 5. Gestures related to the horizontal vector: a) Pointing at the horizontal vector.
b) Marking the magnitude of the horizontal vector.

There was one important variation on the construction of the horizontal vector that I observed. This variation was engaged in when the value of the horizontal change is not known. In this case, both an initial point and a final point must be given. The horizontal change can then be constructed by calculating the difference between the two given x -coordinates. For instance, using information from the same example, had the initial point of $(1, 7)$ and the final point of

(2, 5) been given initially, we would find the horizontal change by subtracting the x -coordinate of the initial point from the x -coordinate of the final point. An accompanying inscription might be, “ $\Delta x = 2 - 1$,” possibly followed by “ $\Delta x = 1$.” Subtracting the x -coordinates constructs the quantity that is the amount that the independent variable has changed from beginning to end. The order in which the x -coordinates are subtracted is critical because when subtracted as the final point minus the initial point, the calculation will provide the direction of the change along with the distance. Both distance and direction are needed to construct the horizontal vector. Once the horizontal change has been determined, it can be represented on the graph as a horizontal vector from the initial point using the determined magnitude and direction.

Along with the variations in construction, I also observed variations in the inscriptions that accompanied the horizontal vector. In addition to the inscriptions previously described, I also observed horizontal vectors simply labeled as “ Δx ” or not labeled at all in other segments of Dr. Kemp’s instruction.

Micro Practice 3: Construct a vertical vector. Following the construction of the horizontal vector, a vertical vector is constructed. The vertical vector begins where the horizontal vector ends. The purpose of the vertical vector is to make visible on a graph the change that is occurring in the dependent variable. This construction can be made explicit by a possible inscription in the form of a label near the vertical vector using the Greek letter delta, Δ , accompanied by the variable representing the vertical quantity and a numerical evaluation of the vertical change. The meaning of the vertical vector can be made more explicit through gestures pointing at the vector or identifying the ends of the vector simultaneously. The beginning of the vertical change is indicated by the tail of the vertical vector. The end of the vertical change is indicated by the head

of the vector and is additionally designated by a point, placed at the head of the vector, which will be discussed in the next micro practice.

To illustrate this practice, we return to the ongoing instructional episode. Once Dr. Kemp had clarified that a horizontal change must occur before a vertical change, she turned her attention to the vertical change by saying, “I’m going to move twice as many in the negative direction, so that would put me right there.” while concurrently constructing a vertical vector from the end of the horizontal vector to a position two units down from the beginning, shown in Figure 6. After constructing the vertical vector, Dr. Kemp immediately drew a point at the end of

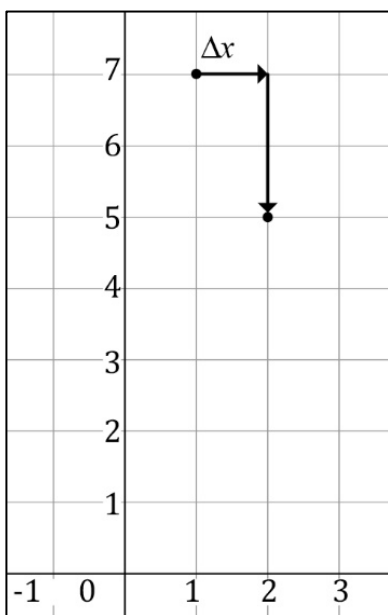


Figure 6. Micro Practice 3:
Construct a vertical vector.

the vertical vector, which I will discuss in the next section. Dr. Kemp continued to make the meaning of the vertical vector explicit by articulating, “This could be your change in y ” while writing the inscription “ Δy ” to label the vertical vector. Dr. Kemp showed that she was thinking about the vertical change as a multiple of the horizontal change by first marking the ends of the

vertical vector, shown in Figure 7a, and then carefully asking her students the following question:

Is my change in y [said while tapping her fingers marking the ends of the vertical vector on the board] negative two times [said with one more distinct tap] as large as my change in x [said while marking the ends of the horizontal vector, shown in Figure 7b]?

At this point, Dr. Kemp proceeded to make the comparison even more explicit by adding the numerical evaluation to the vector labels using the inscriptions “ $\Delta x = 1$ ” and “ $\Delta y = -2$,” shown in Figure 8. She continued to accentuate this comparison by making a downward motion, shown in Figure 9, mimicking the vertical vector and stating that because the change is negative, the vector should be going down.

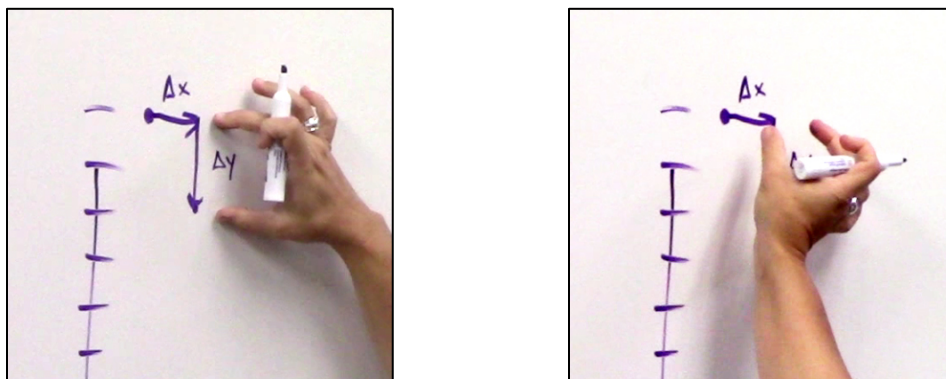


Figure 7. Gestures marking vectors: a) Marking the ends of the labeled vertical vector.
b) Marking the ends of the labeled horizontal vector.

There were two important variations on the construction of the vertical vector that I observed. One variation occurred when the value of the vertical change was constructed using a non-unit horizontal change. The vertical change is constructed by understanding that the constant rate of change means that the vertical change is always a certain multiple of the horizontal change. In this example, the constant rate of change is negative two. This means that the vertical change is always negative two times as large as the horizontal change. This relationship is

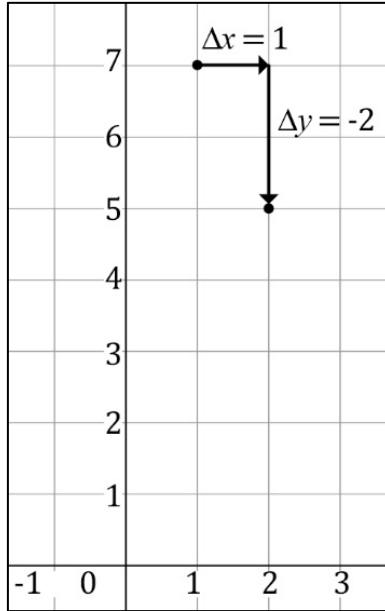


Figure 8. Micro Practice 4:
Construct a final point.

expressed using an inscription of the *change formula*, which, in this case, was given in the prompt as $\Delta y = -2\Delta x$. Then, the selected or calculated value for the horizontal change would be substituted into the change formula. If we selected a horizontal change of three, our inscription might be $\Delta y = -2(3)$. Mathematically simplifying the change formula results in $\Delta y = -6$ which is the value of the vertical change. This new quantity could then be used to construct the

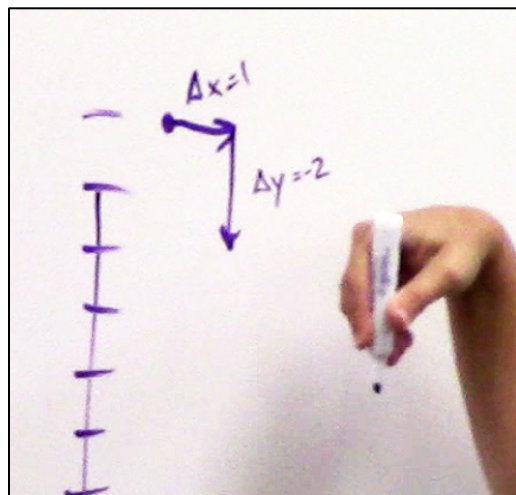


Figure 9. Making a downward motion to indicate direction.

vertical vector on the graph beginning at the end of the horizontal vector using the magnitude and direction constructed.

The second variation on the construction of the vertical vector was engaged in when the value of the vertical change was constructed using the initial and final points. In this case, both an initial point and a final point must be given so the vertical change can be constructed by calculating the difference between the two given y -coordinates. For instance, using information from the same example, had the initial point of $(1, 7)$ and the final point of $(2, 5)$ been given initially, we would find the vertical change by subtracting the y -coordinate of the initial point from the y -coordinate of the final point. An accompanying inscription might be, " $\Delta y = 5 - 7$ " possibly followed by " $\Delta y = -2$." Subtracting the y -coordinates constructs the quantity that is the distance that the dependent variable has changed from beginning to end. The order in which the y -coordinates are subtracted is critical because when subtracted as the final y -coordinate minus the initial y -coordinate, the calculation will provide the direction of the change along with the distance. Both distance and direction are needed to construct the vertical vector. Once the vertical change has been determined, it can be represented on the graph as a vertical vector from the initial point using the determined magnitude and direction.

Similar to the horizontal vector, along with the variations in construction, I also observed variations in the inscriptions that accompanied the vertical vector. I also observed vertical vectors simply labeled as " Δy " or not labeled at all.

Micro Practice 4: Construct a final point. After constructing the vertical vector, a point is constructed at the end of vertical vector. I have identified the construction of the final point as a separate micro practice because there are times when it occurs before the creation of the vertical vector, such as when two points are given instead of an initial point and the constant rate of

change. In the ongoing episode, Dr. Kemp drew this point at the end of the vertical vector immediately after constructing the vertical vector and before discussing the meaning of the vertical vector. This point is constructed at the ordered pair that marks the result of the changes that occurred with the horizontal and vertical quantities. Presumably because the prompt Dr. Kemp used to facilitate this discussion focused on illustrating horizontal and vertical change on the graph, and it did not ask about a final point, Dr. Kemp did not find the ordered pair in this example. The ordered pair representing the final point could have been found by adding the horizontal and vertical changes to the initial point of (1, 7). This would have resulted in a final x -coordinate of $1 + 1$ or 2, a final y -coordinate of $7 + (-2)$ or 5, and a resulting ordered pair of (2, 5). This ordered pair is the final result of the combined horizontal and vertical changes that occurred.

Micro Practice 5: Construct the constant rate of change. The creation of a change diagram always involves the micro practice of either constructing a constant rate of change or interpreting the meaning of a given constant rate of change. An inscription that is a written record of the calculation used to construct the constant rate of change may be found near the graph. In order to engage in this micro practice, horizontal change and vertical change must both be known values. Dr. Kemp often engaged in this practice using an inscription that showed the construction of a ratio of vertical change to horizontal change or the division of the vertical change by the horizontal change.

In the ongoing episode, the constant rate of change was given in the original prompt. Had the constant rate of change not been given, it could have been constructed using a horizontal and vertical change that were either given or constructed. Once the horizontal change and the vertical change are constructed, a ratio is created to make a comparison of how much the quantity

represented by the vertical axis changes compared to how much the quantity represented by the horizontal axis changes. A possible accompanying inscription could first be the ratio of the vertical to horizontal change, $\frac{\Delta y}{\Delta x} = \frac{-2}{1}$, followed by performing the division specified by the ratio which, in this example, would result in $\frac{\Delta y}{\Delta x} = -2$. Dr. Kemp consistently provided evidence through her comments that she interpreted the result of this process as indicating how many times the change in the dependent variable was as large as the change in the independent variable. For Dr. Kemp, finding the constant rate of change was never simply a calculation. On the contrary, finding the constant rate of change involved constructing quantities and determining the multiplicative relationship between those quantities.

Variations in the order of micro practices. Theoretically, the micro practices (MP) associated with the construction of the change diagram could be performed in any order. However, Dr. Kemp consistently engaged in these practices in a specific order. The micro practices are numbered because Dr. Kemp generally engaged in the micro practices in the order described in this section. Variations to this order occur depended on the information given in a problem. Dr. Kemp first performed the micro practices corresponding to the given information. She then worked through the remaining practices in the order described. For example, if she were given two points in the problem statement, she would graph them first before constructing the horizontal vector, then the vertical vector, and finally the constant rate of change. More detailed examples of these variations will be given in the descriptions of intermediate practices below.

Intermediate Practices Involving the Change Diagram

A practice is determined to be an intermediate practice if it meets two criteria: first, it involves finding the value of one element of the change diagram; and second, it involves constructing only one instantiation of the change diagram. I observed three intermediate

practices corresponding to the following three elements of the change diagram for which students were asked to find the value: the constant rate of change, the vertical vector, or one of the points. While Dr. Kemp could have also found the remaining element of the change diagram, the horizontal vector, I did not see such examples in her instruction.

Intermediate Practice: Finding the vertical change. One way Dr. Kemp used the change diagram was to find the vertical change. To engage in this practice, an initial point and the constant rate of change must be specified. Dr. Kemp typically did the following:

1. An initial point is constructed on a graph at the specified location (MP1).
2. A horizontal vector is constructed from the initial point on the graph (MP2).
3. The vertical change is constructed using the change formula and drawing a vertical vector that begins at the end of the horizontal vector on the graph (MP3).

One instance when Dr. Kemp used this intermediate practice occurred in response to a prompt that was a follow-up question to the prompt described in the micro practices. The original prompt read, “Illustrate the meaning of $\Delta y = -2 \cdot \Delta x$ on a graph that passes through the point (1, 7)” (Carlson, 2016, p. 18). The next part of this prompt read, “As x varies from 1 to 5.5, how does y vary? Illustrate these variations on your graph” (Carlson, 2016, p. 18). After reading the prompt aloud, Dr. Kemp inquired, “What’s it going to look like if you’re doing x varying from one to five point five? And then how do you find the y change?” These questions suggest that Dr. Kemp was thinking about what it would look like to illustrate a varying x -value on a graph and how the change in x could be used that to find a vertical change.

Because this prompt was a follow-up question, the initial point had previously been constructed (MP1). Dr. Kemp began by asking how much the x -value changed. The students quickly responded that the change was 4.5 units, and Dr. Kemp placed a mark on the graph 4.5

units to the right of the ordered pair (1, 7), drew a horizontal vector (MP2), shown in Figure 10a, from the initial point to the new mark on the graph, and confirmed, “So I’m changing all the way out to here.” In this case, she did not label the horizontal vector nor did she use gestures to explicitly refer to the horizontal vector.

Next, Dr. Kemp sketched a vertical vector down from the end of the horizontal vector past the x -axis (MP3), shown in Figure 10b, and asked her students, “I know my change is going down here, but how far does it go down?” When her students responded that the vertical change was negative nine, she labeled the vertical vector “ $\Delta y = -9?$ ” and queried, “How do I figure out that it’s negative nine? Where’d that come from?” She then recorded their response by writing “ $-2(4.5)$ ” on the board and clarified this response by pointing at the equation $\Delta y = -2 \cdot \Delta x$ on the board while reminding students,

So it goes back to this equation that we had. We are taking our constant rate of change, which was how much? Negative two [points at -2 in the equation]. We’re multiplying it by our change in x [motions in a circle in the air around Δx in the equation] which was

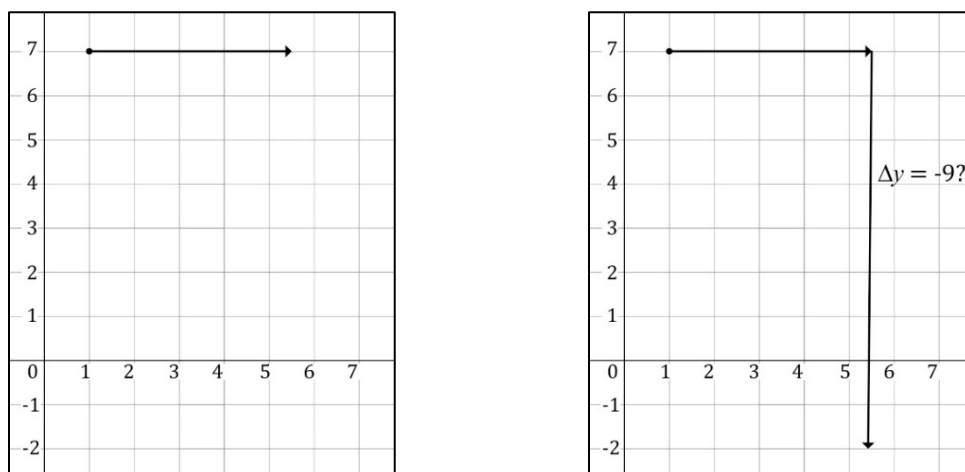


Figure 10. Construct horizontal and vertical vectors: a) Construct horizontal vector (MP2). b) Construct vertical vector (MP3).

four point five. Four point five times negative two [points at -2 in the equation] gives you negative nine [taps the -9 included in the label of the vertical vector].

Dr. Kemp's comment suggests that she was finding the value of the vertical change by multiplying the value of the horizontal change by the given value of the constant rate of change, likely because she was still thinking of the vertical change as being -2 times as large as the horizontal change.

Intermediate Practice: Finding the constant rate of change. A second way Dr. Kemp used the change diagram was to find the constant rate of change. To engage in this practice, two ordered pairs must be given. Dr. Kemp typically did the following:

1. One ordered pair is constructed as the initial point on a graph (MP1).
2. The next ordered pair is constructed as the final point on a graph (MP4).
3. A horizontal vector is constructed (MP2) from the initial point to the place on the graph directly above or below the final point.
4. A vertical vector is constructed on the graph (MP3) from the end of the horizontal vector to the final point.
5. The constant rate of change is constructed (MP5).

One example of this intermediate practice occurred when Dr. Kemp was finding the answer to a question where $(-5.1, 7.1)$ and $(0.8, 4.5)$ were given on a graph along with a prompt that read, "Suppose that the quantities whose values are represented by x and y are related by a constant rate of change of y with respect to x . What is the constant rate of change of y with respect to x ?" (Carlson, n.d.). Dr. Kemp began by drawing a graph on the board and constructing the initial point at $(-5.1, 7.1)$ (MP1) and the final point at $(0.8, 4.5)$ (MP4), as shown in Figure 11.

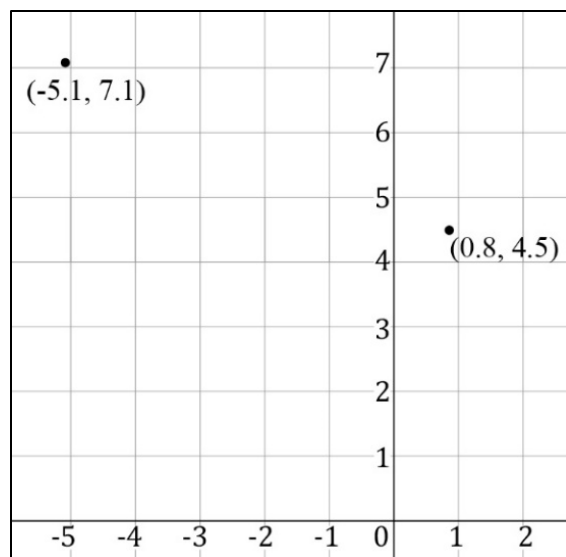


Figure 11. Start with two points.

As Dr. Kemp modeled the solution, she began: “I’m going to look at it and say I’m gonna go from this point [putting the marker on the initial point] ... that direction,” and she constructed a horizontal vector (MP2), shown in Figure 12a, from the initial point to a space on the graph directly above the final point. By constructing the horizontal vector, Dr. Kemp made the horizontal change visible. She then proceeded to make this change explicit by saying, “This is my change in x ,” and she labeled the horizontal vector “ Δx .” Only once the horizontal change was made visible and explicit did Dr. Kemp draw a vertical vector (MP3), as shown in Figure 12b, saying, “That’s my change in y ,” while labeling the vertical vector, “ Δy .”

Once the horizontal and vertical vectors were constructed, Dr. Kemp went back to determine the values for Δx and Δy . While tracing the horizontal vector from left to right and then motioning back and forth along the vector, she asked, “How do I figure out what this change in x is?” While a student explained that this value could be found by subtracting, Dr. Kemp recorded on the board what the student was saying, shown in Figure 13a, “ $\Delta x = 0.8 - -5.1$ ” and “ $\Delta x = 5.9$.” These inscriptions show the calculations for finding the value of the

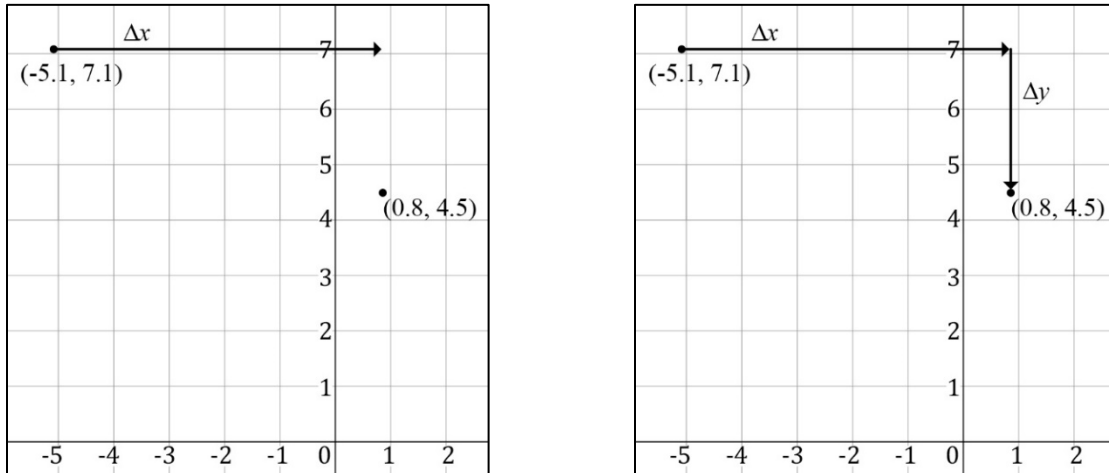


Figure 12. Construct and label horizontal and vertical vectors: a) Construct horizontal vector (MP2). b) Construct vertical vector (MP3).

horizontal vector. Dr. Kemp clarified the meaning of this value by drawing tick marks at the beginning and end of the horizontal vector and explaining, “So that means this distance right here is five point nine. That’s my change in x .” In her action of drawing tick marks and specific use of the words distance and change, Dr. Kemp showed that she was thinking of the horizontal vector as a representation of the change that had occurred in the independent variable.

Once the horizontal change was constructed, Dr. Kemp immediately turned her attention to the vertical change. She drew tick marks at the beginning and end of the vertical vector and asked the class, “How do I find my change in y ?” When a student responded that it was, “ y two minus y one,” Dr. Kemp seemed to purposefully reject this phrasing and instead wrote “ $\Delta y =$ ” on the board and asked, “What is that? Where do I get my y ’s from?” Dr. Kemp revoiced the response by saying, “So this is my y -value right here,” while underlining the y -value in each ordered pair. She then asked, “Which one do I subtract from which?” and wrote the response, “ $4.5 - 7.1$ ” next to the “ $\Delta y =$ ” already on the board, shown in Figure 13b. Dr. Kemp then explained:

We're taking our y -values [points at the y -value of the end point then the y -value of the initial point] ... this is our final value, [quickly taps four times on the y -value of the end point] ... so we're doing final [points at the y -value of the end point] minus initial [points at the y -value of the initial point].

After making it explicit which values were being used to find the value of the vertical change, Dr. Kemp continued, "So what do we get when we subtract these two? We get a change in y of what?" while pointing at the inscription " $\Delta y = 4.5 - 7.1$ " and then writing, " $\Delta y = -2.6$," shown in Figure 13b. After arriving at an answer, Dr. Kemp marked the ends of the vertical vector with her fingers and said, "This distance right here is negative two point six." She then reiterated this meaning by making a downward motion with her finger parallel to the vertical vector and repeating, "going down two point six units." Dr. Kemp's words and gestures suggest that she was thinking about the process of subtraction as a way to quantify a change that occurred in the dependent variable.

$$\Delta x = 0.8 - ^{-}5.1$$

$$\Delta x = 5.9$$

$$\Delta y = 4.5 - 7.1$$

$$\Delta y = -2.6$$

Figure 13. Inscriptions of Micro Practices 2 and 3: a) Construct horizontal change. b) Construct vertical change.

Once the values of the horizontal change and the vertical change were constructed, Dr. Kemp turned her attention to the constant rate of change. She asked her students rhetorically, "What is our constant rate of change?" and answered the question by saying, "It's our change in y divided by our change in x ," while writing, " $\frac{\Delta y}{\Delta x} =$." She finished the inscription, shown in Figure 14, as students answered with " $\frac{-2.6}{5.9}$ " (MP5). Dividing the fraction, they arrived at a final calculation that the constant rate of change of y with respect to x equal to -0.44 . While this value

$$\frac{\Delta y}{\Delta x} = \frac{-2.6}{5.9}$$
$$\frac{\Delta y}{\Delta x} = -.44$$

Figure 14. Micro Practice 5: Construct constant rate of change.

is the answer to the question originally asked in the prompt, Dr. Kemp spent the majority of time on each part of the process making the meaning of each micro practice explicit for students.

While Dr. Kemp did not make explicit to the students that -0.44 meant that the vertical change would always be -0.44 times as large as the horizontal change, she frequently interpreted the meaning of the constant rate of change as a multiplicative comparison while solving problems in the classroom, suggesting that she perhaps interpreted the -0.44 this way even though she did not explicitly describe it as such.

Intermediate Practice: Finding a point. A third way Dr. Kemp used the change diagram was to find the ordered pair that identifies either the initial point or the final point of the change diagram. To engage in this practice, the constant rate of change must be a known quantity, one point must be specified, and an x -coordinate of the other point must be a known quantity. Dr. Kemp typically did the following:

1. The known point is constructed on a graph and designated as either the initial point or the final point (MP1 or MP4).
2. The horizontal change is constructed by subtracting the x -coordinate of the initial point from the x -coordinate of the final point and drawing a horizontal vector from the initial point on the graph (MP2).
3. The vertical change is constructed using the change formula and drawing a vertical vector on the graph (MP3).

4. The y -coordinate of the unknown point is found by adding or subtracting the quantity of the vertical change from the y -coordinate of the known point and this new point is constructed on a graph (MP1 or MP4).

An example of this intermediate practice where the initial point was found occurred with an extension of the previous problem where Dr. Kemp used an intermediate practice to find the constant rate of change. She was given the points $(-5.1, 7.1)$ and $(0.8, 4.5)$ on a graph with an additional prompt that read, “Determine the missing coordinate of the point below. $(-3.1, \underline{\quad})$ ” (Carlson, n.d.). This particular problem originally appeared in the students’ online homework. A computer generated graph including the three ordered pairs was given following the prompt with the ordered pair $(-3.1, \underline{\quad})$ drawn on the graph at its correct location. However, because of the use of decimals, the exact value of the y -coordinate could not be determined by observation. Despite the three points already being drawn on a graph in the problem, Dr. Kemp redrew the graph on the board so she could physically construct each of the quantities necessary to draw the change diagram.

Dr. Kemp began this discussion by asking her students how they thought they could find the missing y -coordinate. One student suggested using the change formula to which Dr. Kemp responded, anticipating the construction of the vertical change, “Perfect! Change in y equals ... m change in x ,” while writing, “ $\Delta y = m\Delta x$ ” on the board. Dr. Kemp then mentioned that they could use their previously constructed constant rate of change (MP5) and incorrectly wrote below the general change formula, “ $\Delta y = .44(\quad)$.” (A student noticed this mistake later in the discussion, and Dr. Kemp immediately corrected the equation by adding a negative sign to the right of the equals sign.) Dr. Kemp next asked, “How do I find [the change in x]? Where am I going to go from?” When the student replied, “I used the negative three point nine,” Dr. Kemp

designated the previously constructed $(-3.9, \underline{\quad})$ to be their initial point (MP1) and subsequently designated the ordered pair $(-5.1, 7.1)$ to be their final point (MP4).

Dr. Kemp next drew a dotted horizontal vector (MP2), shown in Figure 15a, moving left from the ordered pair that had been selected as the initial point to a place on the graph directly below the final point while asking, “So if I go this direction, that’s representing what?” As she repeated the student response of, “The change in x ,” she repeatedly traced the horizontal vector with back and forth motions. Dr. Kemp added, “and it’s going that direction,” while motioning to the left with her finger. Leaving her finger pointing to the left parallel to the horizontal vector, she explained, “So this is my change in x , and it should be a negative value of something.” Dr. Kemp then labeled the horizontal vector, shown in Figure 15b, “ $\Delta x = -$ ” to make the horizontal change explicit and presumably to acknowledge that, while they had not yet determined the value of the change in x , they did know that the value they would find would be a negative value.

After constructing the horizontal vector on the graph (MP2), Dr. Kemp constructed the vertical vector on the graph (MP3), shown in Figure 16, labeled the vector “ Δy ,” and then

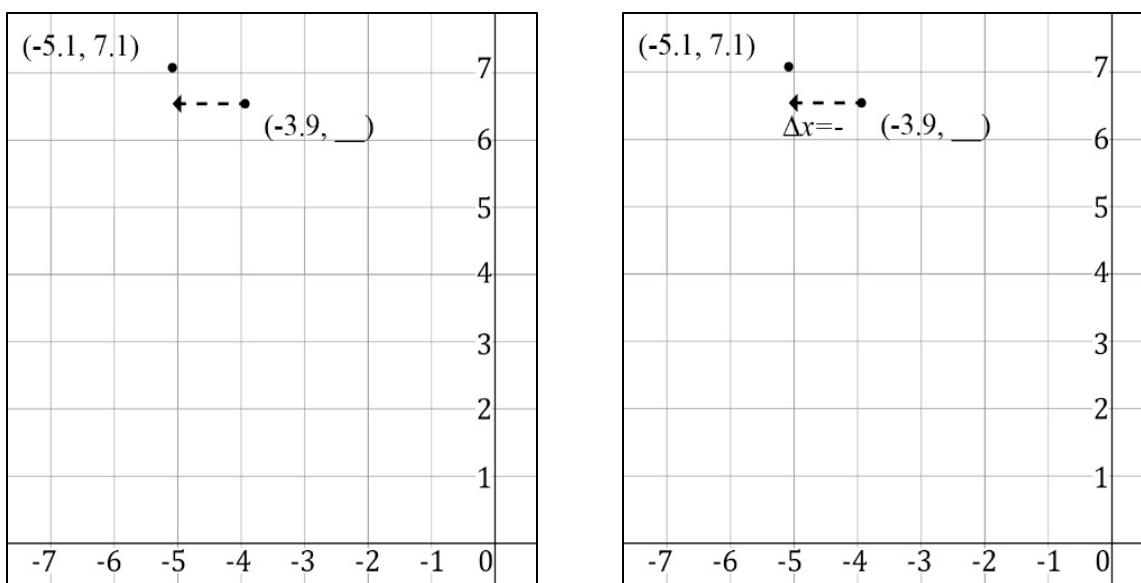


Figure 15. Construct and label horizontal vector: a) Construct horizontal vector. b) Begin to label horizontal vector.

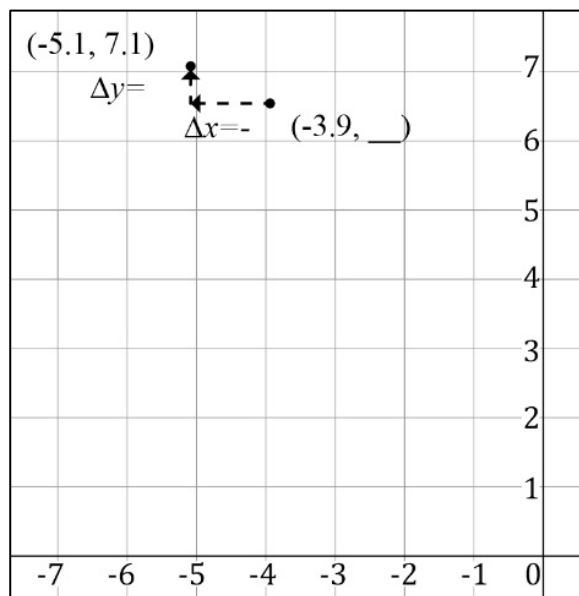


Figure 16. Construct and begin to label vertical vector (MP3).

returned to construct the horizontal change. While marking the ends of the horizontal vector and then pointing at the horizontal vector, Dr. Kemp asked how she would find the change in x . After identifying and underlining the x -coordinates in both the initial point and the final point, Dr. Kemp asked, “Now the question: Which one do I subtract from which?” Reiterating the question, she pointed at each of the x -coordinates asking, “So, which one’s my final value? Is this final, or is this one my final if I’m going that direction [pointing to the left]?” She recorded a student response on the board as the inscription, “ $\Delta x = -5.1 - -3.9$ ” and as a class they decided that this calculation would indeed yield a negative result. When the class told her the horizontal change was -1.2 , she added that into the empty set of parentheses in the change formula, so the inscription read, “ $\Delta y = -.44(-1.2)$.” They finished the calculation with a new inscription below the change formula that read, “ $\Delta y = 0.53$.”

Once the vertical change was constructed, Dr. Kemp twice underlined the value 0.53 while asking if that value was the y -coordinate. The class quickly rejected the assertion and told Dr. Kemp that it was the change in y . Making the students’ response more explicit, Dr. Kemp

rhetorically asked where the change in y was and marked the vertical vector on the graph while answering, “That’s this section right here.” Once it was clarified that 0.53 was not the y -coordinate, Dr. Kemp slowly asked, “How do I find what my actual y value is?” Dr. Kemp responded to a student’s response by saying, “Subtract this change in y ... from the final amount. Right?” She went on to explain:

So I’m going to take seven point one -- ‘cause now I’m going this direction [pointing marker downward]. I’m kind of changing my directions around [making a downward motion with the marker] -- seven point one [circles y -value in the air], and I need to go down [draws downward arrow] how far? Point five three.

Dr. Kemp spent time clarifying that the vertical change needed to be subtracted from the final y -coordinate because moving from the final point to the initial point involves a change in the opposite direction of the constructed change. She showed the final calculation with the inscription, “ $7.1 - .53 = 6.57$,” and clarified that 6.57 was the missing y -coordinate.

A second variation of this intermediate practice was used by Dr. Kemp in class to find a final point immediately following the previous example. She used this practice in response to a prompt that read:

The graph of a certain linear relationship with a constant rate of change of y with respect to x of -2.5 passes through the point $(-5, 8)$.

- a. Plot the given point on the graph, then use the constant rate of change to determine the vertical intercept (Carlson, 2016, p. 25).

Dr. Kemp began by drawing a graph and constructed the given initial point (MP1) on the graph, shown in Figure 17a. She then asked her students, “How can I use the fact that I know that I need to go this far?” and constructed the horizontal vector on the graph from the initial point to the

vertical axis (MP2), shown in Figure 17b. Dr. Kemp’s question suggests that she was thinking about using the horizontal change to find the vertical change. She then made the horizontal change explicit as she said:

That’s my delta x [labels horizontal vector ‘ $\Delta x = 5$ ’], shown in Figure 17b. That’s equal to five. And the reason I’m going to this [traces horizontal vector from left to right] is

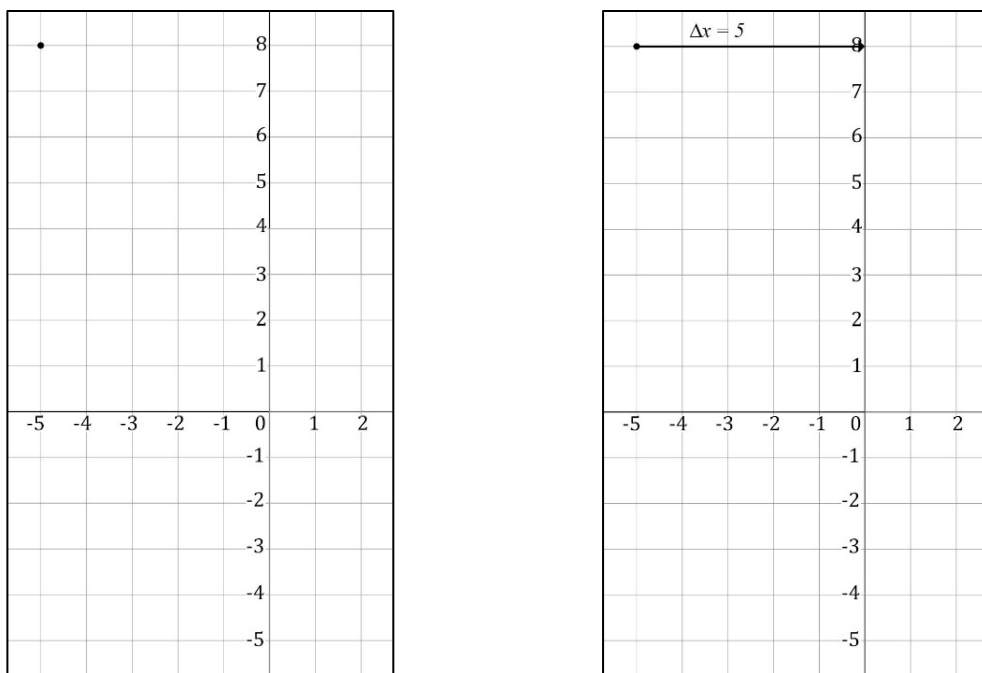


Figure 17. Construct an initial point and horizontal vector: a) Construct initial point (MP1). b) Construct and label horizontal vector (MP2).

because it’s asking me where my y -intercept is [taps on y -axis with hand].

In words and gestures, Dr. Kemp drew attention to the horizontal change as a quantity and made this change visible and explicit for her students. After constructing the horizontal vector, Dr. Kemp wrote the change formula on the board and recorded the values for m and Δx below the change formula with the inscription “ $-2.5(5)$,” and the final calculation for the vertical change in a third inscription, “ $\Delta y = -12.5$ ” (MP3), shown in Figure 18. Upon finding the value of the vertical change, Dr. Kemp asked, “What is that? Do I just go to negative twelve point five here?”

$$\Delta y = m\Delta x$$

$$-2.5(5)$$

$$\Delta y = -12.5$$

Figure 18. Inscriptions associated with constructing vertical change (MP4).

while pointing near the bottom of the y -axis on the board. Pointing back to the inscriptions, shown in Figure 18, of the constructed vertical change, Dr. Kemp answered her own question by saying, “It’s the change. So that means – what do I need to do? Go down. I’m going down because of the negative and how far am I going to go down? Twelve point five units from where? Eight.” In her explanation, Dr. Kemp made it clear that -12.5 was the value of the vertical change not the y -intercept. She continued by saying, while constructing the first part of the vertical vector, “So I’m here at eight. I’m going to go down. Where does twelve point five land me?” Dr. Kemp then drew a point at (0, -4.5) (MP4), shown in Figure 19, and asked if she had just found her y -intercept. She answered her own question in the affirmative and then labeled it on the graph “(0, -4.5),” Dr. Kemp’s final step was to finish constructing the vertical vector (MP3) on the graph, shown in Figure 19. Because she carefully distinguished between the vertical change and the y -coordinate, it is clear that to Dr. Kemp constructing a vertical change and constructing a final point are two separate practices.

Advanced Practice Involving the Change Diagram

The advanced practices are complex practices that involve micro practices and intermediate practices and multiple instantiations of the change diagram. I only observed one advanced practice: determining if a relationship has a constant rate of change.

Advanced Practice: Determining if a relationship has a constant rate of change. Dr. Kemp used multiple instantiations of the change diagram when she wanted to determine if a

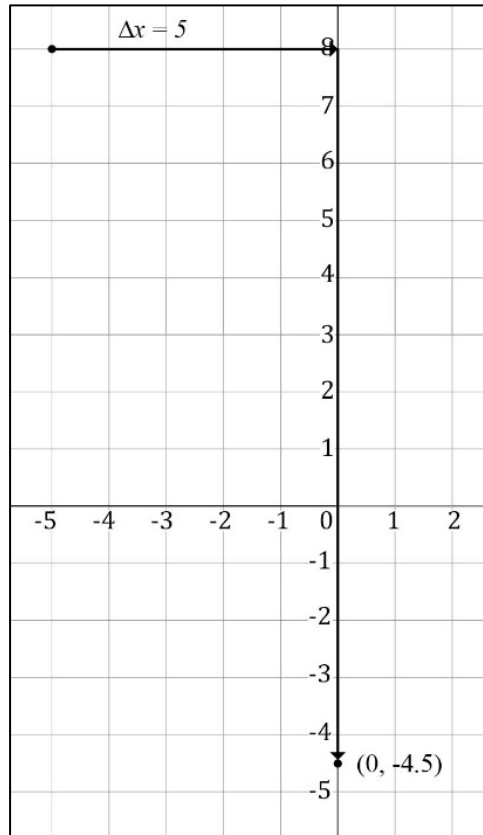


Figure 19. Construct vertical vector (MP3) and final point (MP4).

relationship has a constant rate of change. To engage in this practice, a relationship between two variables must be specified. Dr. Kemp enacted this practice in the following way:

1. Construct several ordered pairs using the given relationship.
2. Construct multiple constant rates of change using the intermediate practice of finding the constant rate of change.
3. Compare the multiple constant rates of change.

The specific instance when Dr. Kemp used this practice was in response to a multipart prompt that introduced a relationship between two variables representing distance and time. The prompt began, “A car is driving away from a crosswalk. The distance d (in feet) of the car from the crosswalk t seconds since the car started moving is given by the formula $d = t^2 + 3.5$ ”

(Carlson, 2016, p. 29). The first part required them to find the change in the car’s distance from the crosswalk as the time increased from one to three seconds and the second part required them to illustrate these changes on the graph. In response, a student constructed one change diagram on the board with (1, 4.5) designated as the initial point and with (3, 12.5) designated as the final point, shown in Figure 20, indicating with inscriptions that the time had changed by two seconds and the distance had changed by eight feet. The third question asked them to determine if the relationship, given in this example in the form of an equation, had a constant rate of change: “True or False: The car travels at a constant speed as the value of t increases from 1 to 3 seconds” (Carlson, 2016, p. 29). When students said the statement was false, Dr. Kemp asserted that it looked like a constant rate of change because, “It’s a straight line.” They proceeded to engage in this advanced practice to “prove” the students were correct. They first used the equation to find an additional ordered pair (0, 3.5), and Dr. Kemp constructed a second change

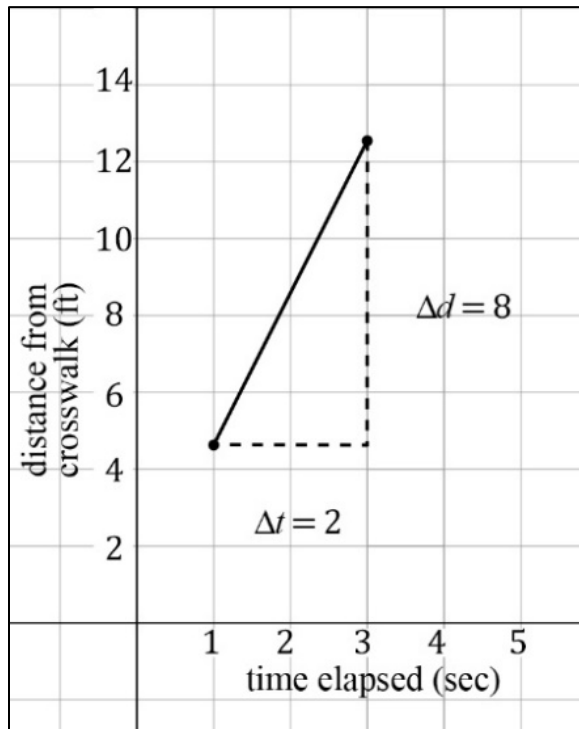


Figure 20. Change diagram drawn by a student.

diagram with (0, 3.5) designated as the initial point and (1, 4.5) designated as the final point, shown in Figure 21. A student then suggested that there was not a constant rate of change because the changes were not equal to which Dr. Kemp replied:

I would agree with you that this change in time [labels new horizontal change, “ Δt ”] and this change in distance [labels new vertical change, “ Δd ”] is not equal to this one [uses her fingers to mark the ends of the previous horizontal change and then the previous vertical change].

Dr. Kemp continued by adding values to the labels of the new horizontal and vertical changes, shown in Figure 21, and asked if that was enough to prove that the rate of change was not constant. A student suggested that the proportions had to be equal to which Dr. Kemp responded, “We talked about last time if our change in d over change in t , the ratios, are all equal, then we would have a constant rate of change” After some discussion, Dr. Kemp summarized:

Well, if I compare these ratios [quickly points at the original horizontal then vertical change] ... which would be eight over two [writes ‘ $\frac{8}{2} =$ ’], is not equal to [quickly points at the new vertical then horizontal change] one over one [adds ‘ $\frac{1}{1}$ ’ to the previous inscription].

Considering the inscription, “ $\frac{8}{2} = \frac{1}{1}$,” Dr. Kemp asked if this was a valid comparison because the intervals that created the ratios were not equal. She accepted a student’s response that the fractions represented decimals that could be equal even if the actual numbers were different, and in this case, the decimals would not be equal, regardless of the intervals they represented. When they compared the constant rates of change of each change diagram and found that each change

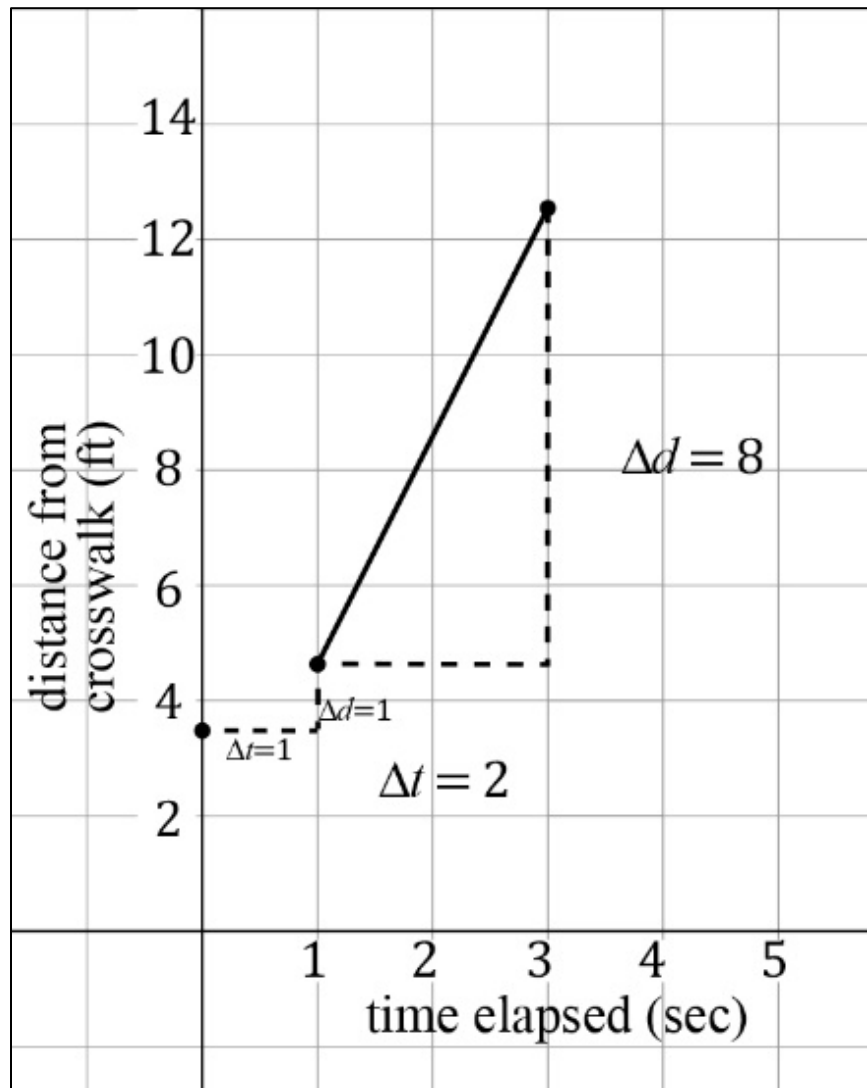


Figure 21. An additional point and change diagram are constructed.

diagram had a different constant rate of change, Dr. Kemp concluded that the relationship, or the equation, did not have a constant rate of change.

Discredited Student Practices

Throughout the course, students occasionally engaged in practices that Dr. Kemp explicitly discredited. I use the term discredit not in the sense of devaluing. Dr. Kemp did acknowledge the value and correctness of this student thinking. Rather, I use the term discredit to mean that Dr. Kemp engaged in a discussion of these practices in such a way as to lower the

status of these practices in her classroom. Her discrediting of these practices distinguishes her proficient practices from other practices. I noticed three discredited student practices: using standard formulas, drawing a line to show changes, and the stair stepper method.

Discredited Student Practice: Using standard formulas. The first discredited student practice is the use of standard linear formulas: the slope-intercept form of a line, $y = mx + b$; and the point-slope form of a line, $y - y_1 = m(x - x_1)$, which students brought up on several occasions throughout the unit on linear functions. Each time students suggested a standard formula, Dr. Kemp was quick to discredit the standard formulas in favor of constructing the change diagram and using the change formula, $\Delta y = m\Delta x$, but she was also quick to add that students were allowed to use the standard formulas should they prefer them.

One instance when Dr. Kemp discredited the standard formulas occurred immediately following the previously described intermediate practice of finding a point. Once she finished modeling her preferred practice for finding a vertical intercept, a student suggested that they could get the same answer using $y = mx + b$. Dr. Kemp responded by saying, “I personally think this [motioning toward the change diagram on the board] is easier because I don’t have to remember a lot of different formulas.” After rejecting the use of the slope-intercept formula, she worked through previous problem using the slope-intercept formula. Dr. Kemp then explained, pointing at the original inscriptions of the change diagram and the change formula, “To me, this one helps me understand what I am doing rather than just plugging in.” She discredited the slope-intercept formula by implying that it is used by “plugging in” numbers which does not promote understanding. Dr. Kemp seemed to value the meaning tied to the change diagram of constructing quantities of change and making a multiplicative comparison.

Discredited Student Practice: Drawing a line to show changes. The second discredited student practice, drawing a line to show changes, was different from the other two discredited practices. When discrediting the other two student practices, Dr. Kemp told students that those practices were not wrong, but she tried to convince her students to use her proficient practices. Conversely, she rejected this student practice because her students were drawing a line connecting two points instead of illustrating the changes in two quantities. Instead of drawing a line, students should have designated on the graph where they could see the changes in the two quantities.

An example of this student practice occurred before and after the example described in the micro practices. Recall that the original prompt stated, “Illustrate the meaning of $\Delta y = -2 \cdot \Delta x$ on a graph that passes through the point (1, 7)” (Carlson, 2016, p. 18). Many students responded to the prompt by drawing a line with a slope of negative two on a graph through the point (1, 7). Dr. Kemp suggested that this was a problem by asking, “Is there a difference between drawing a line and drawing changes?” Dr. Kemp then proceeded to model the construction of the change diagram as described in the section on micro practices. She then asked her students rhetorically if the line was the changes. Dr. Kemp answered her own question explaining, “That’s the result of using the changes, but these [pointing to the horizontal and vertical vectors on the graph] are actually the changes that we are looking for.” Dr. Kemp made the distinction between the changes in quantities which are represented by horizontal and vertical vectors and the result of the changes in quantities which are represented by the initial and final points. She continued to clarify that the line is made up of the initial point and all of the final points that are constructed as the result of the quantities varying together. Taking time to clarify

the distinction between change in quantities and the results of changes in quantities shows that Dr. Kemp values practices where changes in quantities are constructed.

Discredited Student Practice: Stair stepper method. The last discredited student practice was the practice of repeatedly counting the rise and the run of the slope of a line to find the vertical intercept. This is commonly done by first identifying the slope of a line, which in this case was negative two. Then, from an initial point, a student would count down two units and right one unit and draw a point on the graph. This process would be repeated until the point drawn lies on the y -axis, shown in Figure 22. Dr. Kemp referred to these multiple rise/run combinations as *stair steppers*.

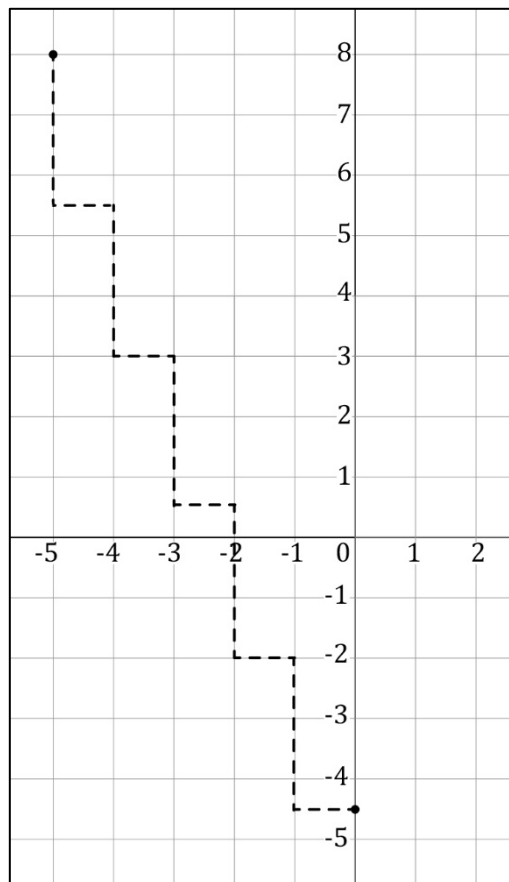


Figure 22. Discredited Student Practice: Stair stepper method.

Dr. Kemp noticed that her students were engaging in this practice in response to the prompt described in the intermediate practice of finding a point. Recall that students were asked to find the vertical intercept if the graph of a linear relationship has a constant rate of change of -2.5 and passes through the point $(-5, 8)$. After seeing students use the stair stepper method, Dr. Kemp responded with a surprised tone:

So I am seeing lots of people not use the method we just used. You're doing stair steppers. You're like, two point five, one, two point five, one, two point five, one, two point five, one [saying each 'two point five' in a high, squeaky tone and each 'one' with a normal tone]. Can you do it an easier way [throwing her hands apart]? Come on guys [in a tone of mock exasperation]!

After dramatically rejecting this student practice, Dr. Kemp asked her students how they could use their change in x and change in y and the intermediate practice of finding the initial point to help them. When students were reasoning using the stair stepper method, they were not constructing rates of change. Through a rejection of this student practice and the recommendation of the change diagram, Dr. Kemp showed that she preferred a method for finding the vertical intercept that involved the construction of horizontal change, vertical change, and a constant rate of change.

Discussion

Looking across my results, two important traits consistently emerged that characterized Dr. Kemp's practices involving the change diagram. The first trait that emerged was Dr. Kemp always constructed quantities. By constructed quantities I mean that, for Dr. Kemp, numbers always had meaning, and new quantities were created by reasoning about relationships between the existing quantities. The operations that Dr. Kemp used were never formulaic because she

selected operations that maintained the relationship between quantities and kept their meanings intact. The second trait that emerged was Dr. Kemp was always thinking about rates of change as a multiplicative comparison of the change in y to the change in x .

At every level of practice, Dr. Kemp consistently constructed quantities and never simply used a formula. Dr. Kemp always constructed physical representations of quantities of change using the change diagram. While it is likely that Dr. Kemp did not need to create physical representations in order to conceptually construct quantities, these representations are nonetheless evidence that Dr. Kemp was constructing quantities. The physical representations make explicit the meanings of each number because the quantity represented by each number is physically visible. A specific instance that showed Dr. Kemp always constructed quantities was described in the intermediate practice of finding the constant rate of change when she rejected a student suggestion of finding the vertical change by taking “ y two minus y one” and instead wrote “ Δy ” on the board. Dr. Kemp preferred a notation for the vertical change that reflected the meaning of the quantity rather than a formula that could be used devoid of meaning. Dr. Kemp consistently rejected the use of traditional formulas in favor of practices that involved constructing the quantities of the change diagram.

A second example of Dr. Kemp consistently constructing quantities occurred when she invariably constructed horizontal changes before constructing vertical changes. The order of the other micro practices of the change diagram varied depending on what information was given in the mathematical prompt, but the order of the horizontal change and the vertical change never varied. The horizontal change corresponds to the change that occurs in the independent variable, often x , while the vertical change corresponds to the change that occurs in the dependent variable, often y . Because the amount of change that occurs in the dependent variable depends on

the amount of change that occurs in the independent variable, when considering this relationship, the change of the independent variable must be considered before the change of the dependent variable. By constructing the change in the independent variable before constructing the change in the dependent variable, Dr. Kemp constructed these change quantities in an order that respected their multiplicative relationship.

The next trait that emerged as characteristic of Dr. Kemp's practices involving the change diagram was that Dr. Kemp always considered a rate of change as a relationship of how two quantities changed together. Dr. Kemp made it clear to her students that she used the value given as the constant rate of change as the multiplicative rate that the dependent variable changed in relationship to changes in the independent variable. This was significantly different than the traditional practice of talking about slope as rise over run, as in the discredited student practice Dr. Kemp referred to as stair steppers. While it is possible that students could use the stair stepper method and still view the rates of change as multiplicative comparisons, the method itself does not require that conception, and in fact discourages multiplicative reasoning about the changes in quantities. When students used the stair stepper method, they created "stairs" by iterating the composite number rise-to-run to create the total amounts of change in the dependent and independent variable. Using the composite number this way does not make explicit the multiplicative relationship between the *total* amounts of change in the two variables, namely that the *total* amount of change in the dependent variable is a certain number of times as large as the *total* amount of change in the independent variable. Furthermore, thinking about slope as rise over run requires the change of the dependent variable to be considered before the change of the independent variable, which also hinders students' thinking about the multiplicative relationship between the two change quantities.

A second limitation of the stair stepper method was that it was tied to the physical experience of traveling. Using the example that occurred in class, students traveled down two then right one and drew a point. The physical experience of traveling does not extend to the conception of two quantities changing together for two reasons. First, this travel is limited to whole number iterations of the composite number. Students were thinking of moving down two units and right one unit and were therefore thinking of the changes as the composite number two to one. While whole number iterations of the composite number do give ordered pairs which are the result of changes in two quantities, most results of changes in two quantities do not occur from whole number iterations of the created composite number. Consequently, students may struggle when they are asked to identify results of changes in two quantities that do not occur from whole number iterations of the created composite number. Second, the metaphor of travel is limited because it assumes that x and y are measured in units of distance. While many problems involving rates of change are concerned with variables that represent distance, many problems concern other units of measure. Students may struggle when the quantities associated with x and y are not distances. Thus, it is unlikely that most students used the stair stepper method with a conception of rates of change. Conversely, Dr. Kemp made her conception of changing quantities explicit in her practices using the change diagram. She avoided creating a composite number and instead consistently considered the vertical change as being a multiple of the horizontal change. For Dr. Kemp, conceptualizing rates of change as a multiplicative relationship of the changes in two variables was an intrinsic and explicit part of her practices.

Constructing quantities and thinking about a constant rate of change as a multiplicative comparison are significantly different ways of reasoning about linear functions than traditionally taught. As previously stated, traditional methods of working with linear functions using

conceptions of slope require numbers to be separated from their meanings. Conversely, by constructing quantities, numbers never need to be separated from their meanings, and new quantities are constructed using operations that preserve meaning. Because the meanings of quantities can be maintained throughout, the constant rate of change can be considered with meaning as the multiplicative comparison of the change in the dependent variable with respect to the change in the independent variable. Thus, constructing quantities and thinking about a constant rate of change as a multiplicative comparison naturally makes linear functions more accessible to students because numbers can be used with meaning.

CHAPTER 5: CONCLUSION

Graphing linear equations on the Cartesian coordinate plane is a fundamental component of junior high school, high school, and introductory college algebra courses that is well-researched from a cognitive perspective. This research, which mainly focused on identifying knowledge, skills, and understandings students should acquire, does not take into account the social nature of learning and the situated nature of knowledge. This study contributes to past research by considering the topic of graphing from a sociocultural perspective which does take into account the social and situated nature of learning and knowledge. From a sociocultural perspective, I observed one practitioner in the community of teachers of function-based covariation-focused algebra curricula as she taught three lessons on linear functions. As a result of this study, I have created a list of proficient mathematical practices regarding linear functions engaged in by this practitioner and discovered a relationship between these practices.

Contributions

This study makes a contribution to research in covariation by providing specific examples of mathematical practices that illustrate one level of Thompson and Carlson's (2017) updated framework of covariational reasoning. Thompson and Carlson (2017) indicated that their framework may be used in two ways: to describe a class of behaviors or to describe a person's capacity to reason covariationally. Dr. Kemp's practices include descriptions of behaviors and as such can be correlated to this framework. Thompson and Carlson (2017) divided conceptions of covariational reasoning into six categories. Their framework described these categories as six major levels of covariational reasoning which are intended to represent all possible levels of understanding of covariation. These levels, that Thompson and Carlson (2017) suspected are developmental, are arranged in a hierarchical order from the most naïve conception, no

coordination between the change in two variables, to the most expert conception, smooth continuous coordination of the change in two variables. In their updated framework, Thompson and Carlson (2017) provided a description of ways of thinking for each of the six major levels of covariational reasoning. While they claim they provide descriptions of the behaviors that might accompany each level of covariational reasoning, their table of descriptions does not describe observable behaviors. Instead, their updated framework describes thinking that could be inferred from observable behaviors which leaves behind an important level of detail. My research contributes to Thompson and Carlson's (2017) framework by adding not only empirically documented behaviors that correspond to one level of this framework, but also detailed descriptions regarding normative purposes for the documented behaviors and ways of reasoning with tools and symbols by someone proficient in the thinking that correspond to this level of covariational reasoning.

Dr. Kemp's mathematical practices clearly included behaviors that indicate ways of thinking described in the second highest level named chunky continuous covariation. The description of the chunky continuous covariation level is, "The person envisions changes in one variable's value as happening simultaneously with changes in another variable's value, and they envision both variables varying with chunky continuous covariation" (Thompson & Carlson, 2017, p. 441). Thompson and Carlson use the phrase chunky continuous to mean that values vary discretely, but values can be conceived of as any quantity on a continuum, First, Dr. Kemp's practices of constructing horizontal and vertical quantities together describe consistent behaviors that show changes in the values of the two variables as happening simultaneously. Second, Dr. Kemp's physical construction of horizontal and vertical vectors show discrete quantities of change, and Dr. Kemp repeatedly showed that, using the change diagram, these discrete

quantities could take on any value. Thus, Dr. Kemp's practices involving the change diagram described in this study provide specific examples of practices that can be described as chunky continuous covariational reasoning using Thompson and Carlson's framework of covariational reasoning. Because Dr. Kemp is a practitioner, I would expect that her practices would include behaviors that are described among the highest levels of Thompson's and Carlson's framework. It is not surprising, however, that her practices do not exhibit behaviors described by the smooth continuous covariational level because all of the problems Dr. Kemp considered in class examined changes in quantities and rates of change for specific values and quantities which is inherently chunky. Both smooth continuous covariation and chunky continuous covariation are levels that exhibit evidence of continuous covariational reasoning so both could be considered advanced levels of covariational reasoning. Thus, Dr. Kemp's practices described as a result of this study are empirically proven practices associated with the conception of one advanced level of covariational reasoning, chunky continuous covariation.

This study also makes a contribution concerning the usefulness of considering a practice as the unit of analysis rather than a conception. Considering mathematical practices extends Thompson and Carlson's (2017) covariational reasoning framework because these descriptions of practices include more than descriptions of behavior. By definition a mathematical practice includes a normative purpose and normative ways of reasoning with tools and symbols. Considering conceptions as the unit of analysis may involve descriptions of behavior and may include the purpose of the behavior, but considering conceptions does not involve descriptions of normative ways of reasoning with tools and symbols. Descriptions of Dr. Kemp's practices extend Thompson and Carlson's framework because they are descriptions of more than behavior. They also include descriptions of normative ways Dr. Kemp reasons with the tools and symbols

of graphs and linear functions. Thus, considering mathematical practice as the unit of analysis in this study has added to our understanding of covariation in a way that was not possible using conceptions as the unit of analysis.

Last, as a result of this study, I not only have generated a list of graphing practices used by this teacher, I also found relationships between her practices. In the lessons I observed on linear functions, all of Dr. Kemp's mathematical practices involved the change diagram and were interrelated in three levels of practices: micro practices, intermediate practices, and advanced practices. Micro practices focused on constructing components of the change diagram while intermediate practices involved constructing one instantiation of the change diagram and advanced practices involved constructing multiple instantiations of the change diagram. When Dr. Kemp engaged in either an intermediate or an advanced practice, she deliberately and consistently also engaged in each of the sub-practices. For example, when Dr. Kemp engaged in an advanced practice of determining if a relationship has a constant rate of change, she also engaged in all of the micro practices as well as the intermediate practice of finding the constant rate of change. Engaging in the micro practices and the intermediate practice as part of the process of engaging in the advanced practice are what allowed the meaning of the quantities to be retained. Thus, identifying the relationship between the three levels of practices and the care Dr. Kemp took to engage in each level of practice revealed how her mathematical practices allowed quantities to be constructed. This study not only describes how practices may be described at different levels of complexity, but also demonstrates how smaller practices may be combined to form larger, more complex practices.

Implications

This study has two types of implications: implications for research in mathematics education and implications for teaching. The first implication, for research in mathematics education, is that it is not enough to look at conceptions; we must also consider practices. Most of prior research regarding the topic of graphing looked at conceptions in the form of knowledge, skills, and understandings. Conceptions do not necessarily address normative purposes and only partially suggest normative ways of reasoning because they do not specify how tools and symbols are specifically drawn upon to support and enable particular ways of reasoning mathematically. Considering practices in addition to conception adds dramatically to our understanding of mathematical activity because it leads to explicit descriptions of normative purposes that are connected to particular situations or problems and also specifies how tools and symbols are coordinated to achieve these purposes. This added description and specification, while necessarily incomplete because there is no way to describe practices that account for all aspects of the context in which those practices are situated, nonetheless adds greatly to our understanding of what is required to engage in particular types of mathematical activity.

One implication for teaching involves the focus of instruction. Dr. Kemp is a practitioner whose mathematical graphing practices provided empirical evidence of advanced levels of covariational reasoning, which implies that Dr. Kemp's practices are ones that should be emulated and engaged in by others who want to participate in and become practitioners in the community of teachers of function-based covariation-focused algebra curricula. Dr. Kemp taught an inquiry-based curriculum and created a classroom environment conducive to active student participation. Along with student participation, Dr. Kemp modeled her practices extensively and carefully. With each problem, Dr. Kemp consistently engaged in the micro practices of the

change diagram in front of her students. A major portion of Dr. Kemp's instruction focused on micro practices rather than intermediate or advanced practices. For Dr. Kemp, micro practices were not something she engaged in to introduce a topic and then disregard. Instead, the micro practices of the change diagram were fundamental components of her intermediate and advanced practices. The care with which Dr. Kemp, a practitioner, engaged in micro practices implies that it might be useful for students if we, as teachers, are more attentive to modeling all levels of our practice.

Last, Dr. Kemp's mathematical practices have important implications for the instruction of covariation. Unlike the traditional conceptions of slope, the practices involving the change diagram allow numbers to retain their meaning as they are being used, which implies that Dr. Kemp's practices involving the change diagram potentially allow students to understand the meaning of the numbers and quantities they are working with. Understanding the meaning of quantities is crucial to reasoning about covariation in graphs. Dr. Kemp's micro practices, intermediate practices, and advanced practices of the change diagram are all explicit ways of reasoning with graphs in a way that supports advanced covariational reasoning. Because these practices support advanced covariational reasoning, it follows that teachers who would like to promote covariational reasoning in their classroom would engage in these practices as well.

Limitations and Directions

This study does have one particular limitation. I only observed one teacher in one class in one unit of instruction. Observing three days of instruction naturally brings up the question of whether the behaviors I observed could be considered normative. The consistency with which Dr. Kemp used the micro practices provides evidence to suggest that the behaviors are normative and can be considered practices. The limitation also means that my results are not comprehensive

in nature. Despite this limitation, this study extended Thompson and Carlson's (2017) framework of covariation by adding empirically documented, descriptive mathematical practices of one practitioner to one of their major levels of covariational reasoning. This research could be extended in three ways: by studying the practices of more units of instruction with the same instructor, by studying the practices of other instructors, and by studying how students engage in the practices modeled by their instructor. By studying more units of instruction with the same instructor, I would expect to find additional practices that yield insight into ways of reasoning with graphs in a manner consistent with conceptions of covariational reasoning. Studying other instructors in the same community, I would expect to see common features in their practices which would be instructive for teachers wanting to become practitioners in this community, but I may also find additional practices involving covariational reasoning about graphs that Dr. Kemp did not display. By studying the practices students engage in after seeing these practices modeled by their instructor, we could see which practices students do not adopt which could help us consider changes in instruction that might better help students adopt more of the practices.

Conclusion

Graphing is a fundamental topic in algebra and one that is notoriously difficult for students. Looking at practices may be one way to address this problem. This study looked at one teacher in one class in one unit of instruction and provided descriptions of the mathematical graphing practices of a practitioner of covariation-focused algebra curricula. These practices can be used to guide instruction to make graphing more accessible for more students. That a small study could yield such results suggests that it would be beneficial to continue studying the mathematical practices of experienced teachers.

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