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Asian Spread Option Pricing Models and Computation

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ABSTRACT

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In the commodity and energy markets, there are two kinds of risk that traders and analysts are concerned a lot about: multiple underlying risk and average price risk. Spread options, swaps and swaptions are widely used to hedge multiple underlying risks and Asian (average price) options can deal with average price risk. But when those two risks are combined together, then we need to consider Asian spread options and Asian-European spread options for hedging purposes.

For an Asian or Asian-European spread call option, its payoff depends on the difference of two underlyings' average price or of one average price and one final (at expiration) price. Asian and Asian-European spread option pricing is challenging work. Even under the basic assumption that each underlying price follows a log-normal distribution, the average price does not have a distribution with a simple form. In this dissertation, for the first time, a systematic analysis of Asian spread option and Asian-European spread option pricing is proposed, several original approaches for the Black-Scholes-Merton model and a special stochastic volatility model are developed and some numerical computation tests are conducted as well.

Keywords: Asian spread option, Asian-European spread option, option pricing, stochastic volatility model, affine structure

Contents

Chapter 1. Introduction

The ongoing subprime mortgage crisis which started in 2007 makes "risk" a very popular word. There are all kinds of risk in the market, such as credit risk, business risk, operational risk, basis risk and so on. Different markets may have different risks related to the market structure. In the commodity and energy markets, there are two kinds of risk that traders and analysts are concerned a lot about: multiple underlying risk and average price risk.

Multiple underlying risk arises from uncertainty of multiple underlyings in the market. This is quite common in the energy markets. For example, for an energy company owning natural gas fired power plants, they have the risk exposure not for the natural gas or power price only but the spread of these two underlyings' price. In most circumstances, natural gas and power prices are highly correlated, so we need combine these two underlyings together to quantify the risk.

Average price risk is related to the average underlying price. For example, many commodities and types of energy are delivered over a period of time, so the buyer and seller are both exposed to the underlying price risk for this whole period. To continue with the energy market example, if an energy company wants to sell its power in the day-ahead or day-of market for one month, then it has risk exposure to the average daily power price for that

month. Also, for an energy company, average risk plays a very important role in long term planning and analysis.

Commodity and energy markets are huge: according to BIS (-Bank of International Settlements) statistics, only the notional value of OTC (-over-the-counter) commodity derivatives contracts is $6.4 trillion in mid 2006; in the energy market, crude oil approaches $2 trillion in annual trade [2]. Because of the need to hedge risk, similarly to the way we buy car insurance to protect from possible future accidents, traders in the market can buy options to hedge the risk. An option, as a derivative, gives the holder the right but not the obligation to buy or sell some particular asset at a previously agreed price in the future. Such a buy right, is a call option; a sell right is a put option. The previously agreed price is the strike price. Traders need to pay some amount of money to have such a right; this is called the premium of the option and corresponds to the value of the option. For example, a utility company can buy a call option on natural gas if it needs to buy a certain amount of gas next summer but it has a concern that the price will be too high. Some people may think trading derivatives equals speculation. One counter example will be for utility companies, which are not allowed to speculate but, because of the huge exposure on market risk, still need to trade a huge amount of different kinds of options to hedge the risk. People use spread options, swaps and swaptions to hedge multiple underlying risk and Asian options to deal with average price risk. But what if there is the combination of these two kinds of risk? In this case, Asian spread options and Asian-European spread options will be ideal ways to hedge it. The pricing problem is, given the nature of an option, which is a right in the future, to determine the current "true" value when this option is transacted in the market.

Before we go through the technique of pricing Asian or Asian-European spread options, let's take a look at European spread options and Asian options.

A spread option is an option where the payoff is dependent on the difference between two market variables. As Eydeland and Wolyniec point out in [10]: "It is impossible to underestimate the significance of spread options in the energy markets. Practically every energy asset and every structured deal has a spread option embedded in it." For example, there are crack spread options which consider the difference price of crude oil and the refineries; there are spark spread options which are about the difference price of electricity and the power-generating fuel. In fact, for an energy company owning natural gas fired power plants, the value for the power generation, from the financial point of view, is holding a spread call option. When this energy company is making the decision about whether to use this plant to generate power, it would like to see this spread option's payoff at expiration whether it ends up with in the money, which means it is economical to generate, or out of the money, which means it is not economic to generate.

The payoff of an European spread option is $max\{S_2(T) - S_1(T) - K, 0\}$, where T is the expiration time, K is the strike price and S_1 and S_2 are two underlyings in this option. The first result about pricing European spread options came from Margrabe in [19] in 1978; he gave the closed-form formula for underlyings that are forward contracts with $K = 0$ under the classic Black-Scholes-Merton model. For nonzero strike price, at present there is no simple closed-form formula but there are many ways to approximate the price of European

spread options under Black-Scholes-Merton models. For example Kirk in [18] absorbed strike price into the second underlying's price to use Margrabe's formula and Pearson in [20] used semi-analytical techniques to change the two-dimensional problem (double integrals) into a one-dimensional problem (single integral). For other examples see [5], [15] and [21].

An Asian option is an option whose payoff depends on the average price, so it is also called an average price option. For a fixed-strike arithmetic Asian call it has payoff $max\{\frac{1}{T}\int_0^T S(t)dt - K, 0\}$; for a floating-strike Asian call it has payoff $max\{S(T) - \frac{1}{T}\int_0^T S(t)dt, 0\}$. As Wengler mentions in [25], "Many energy contracts are European-type options but with strike prices that are averages for a period. (The averaging effect is so common that most energy options can be classified as Asian...)". Here we only discuss the fixed-strike Asian call since it naturally leads to the Asian spread option. To my best knowledge, there is no analytic solution about arithmetic average Asian options yet. The closest one is from Geman and Yor in [13] where they obtained the semi-analytic solution under the Black-Scholes-Merton model. Other valuable approaches include Monte Carlo simulation by Kemna and Vorst in [17]; upper and lower bounds by Curran in [6] and Rogers and Shi in [22]; partial differential equation by Vecer in [24] and Fouque and Han in [11]; Laplace transform by Fu and Madan and Wang in [12]; and moment-matching method by Zhou and Wang in [27].

One may have the concern that right now in the market there are not many real Asian spread options or Asian-European spread options being traded everyday, so what's the point for industry to care about them? Well, let's use the European style power and natural gas spread options as an example. Even for this usual spread option in the energy market, they

are OTC derivatives. They are not as liquid as stock options, so you may not be able to sell deep "out of the money" calls, and actually no one in the market will be interested in buying your cheap but almost useless options. But for an energy company owning natural gas fired power plants, if the plant is old and not as efficient as the average plants in the market, as we just discussed, this plant is like an "out of the money" spread call option. It still has its value even if you can not "sell" it directly in the market. What you can do is build a replicate portfolio using future and forward contracts of natural gas and power for this "out of the money" spread option. Delta hedging will play an important role in this process, by dynamically changing this portfolio based on natural gas and power market price changes; this portfolio will help you to get the true value (premium) of the "out of the money" spread option without trading it directly (you can't do that). Understanding pricing spread options is the basis of delta hedging it. This example can easily go to Asian spread options and Asian-European spread options.

Another area closely related to option pricing is so called real option analysis (ROA) which applies call and put option valuation techniques to valuate different business strategies. It is an increasingly active topic extending to "real life" decision making under uncertainty, especially for physical asset pricing. Right now in industry, more and more companies use option pricing methods to help them make the optimal decision about some business which may have nothing to do with derivative trading. From valuing a generation unit for a utility company to deciding the investment amount of a certain project for updating a factory's facility, this kind of analysis actually need techniques for pricing Asian or Asian-European spread options.

Asian and Asian-European spread option pricing is a challenging work. For an Asian or Asian-European spread call with strike price K it has payoff $\max\{\frac{1}{T}\int_0^T [S_1(t)-S_2(t)]dt-K, 0\}$ or $\max\{\frac{1}{T}\int_0^T S_1(t)dt - S_2(T) - K, 0\}$ respectively. Even under the basic assumption that for each underlying price it follows a log-normal distribution, the average price does not have a distribution of a simple form. And how about for some complicated stochastic volatility model, what is the distribution for the average price? In this dissertation, for the first time, a systematic analysis of Asian spread option and Asian-European spread option pricing is proposed. Several original approaches for the Black-Scholes-Merton model and a special stochastic volatility model are developed and some numerical computation tests are conducted as well. In Chapter 2, we review methodologies for pricing European spread options and Asian options which will lead our approach later. Then the Asian spread option is discussed in Chapter 3, and the Asian European spread option is discussed in Chapter 4. Numerical computation tests for the proposed approaches are in Chapter 5. We conclude with directions for future research in Chapter 6.

Chapter 2. Background Information

2.1 European Spread Option

Here we consider a spread European call option with the payoff related to two underlying assets' price, S_1, S_2. The payoff at maturity T of this option with strike value K is the amount

$$max[(S_2(T) - S_1(T) - K), 0].$$

So the pricing problem is to compute the expectation

$$V(t) = E_{\mathbb{Q}}[e^{-r(T-t)}max[(S_2(T) - S_1(T) - K), 0]|\mathcal{F}(t)], 0 \leq t \leq T, \quad (2.1.1)$$

where $E_{\mathbb{Q}}$ is the expectation under the risk neutral measure $\mathbb{Q}$, and $\mathcal{F}(t)$ is the $\sigma-$algebra generated by the stochastic process $S_1(t), S_2(t)$.

Consider the Black-Scholes-Merton model:

$$
\begin{aligned}
dS_1 &= S_1[(r - \delta_1)dt + \sigma_1 dW_1], \\
dS_2 &= S_2[(r - \delta_2)dt + \sigma_2 dW_2],
\end{aligned}
\quad (2.1.2)
$$

where W_1, W_2 are standard Brownian motions with correlation coefficient ρ, r is the risk free rate and δ_1, δ_2 are the instantaneous dividend yields. Then the solution of stochastic

differential equation (2.1.2) is log-normal. Letting $s_i := log(S_i)$, by Ito's formula, we have

$$ds_1 = (r - \delta_1 - \frac{1}{2}\sigma_1^2)dt + \sigma_1 dW_1,$$
$$ds_2 = (r - \delta_2 - \frac{1}{2}\sigma_2^2)dt + \sigma_2 dW_2.$$

By the independence of the Brownian motion increments $W(T) - W(t)$ and $\mathcal{F}(t)$, we have

$$s_1(T) = s_1(t) + (r - \delta_1 - \frac{1}{2}\sigma_1^2)(T - t) + \sigma_1\sqrt{(T - t)} \cdot w_1$$
$$s_2(T) = s_2(t) + (r - \delta_2 - \frac{1}{2}\sigma_2^2)(T - t) + \rho\sigma_2\sqrt{(T - t)} \cdot w_1 + \sqrt{1 - \rho^2}\sigma_2\sqrt{(T - t)} \cdot w_2$$

where w_1, w_2 are independent standard normal random variables. So conditional on time t value, $(s_1(T), s_2(T))^T$ is a bivariate normally distributed random variable with mean μ and covariance matrix Σ where

$$\mu = \begin{pmatrix} s_1(t) + (r - \delta_1 - \frac{1}{2}\sigma_1^2)\tau \\ s_2(t) + (r - \delta_2 - \frac{1}{2}\sigma_2^2)\tau \end{pmatrix}, \Sigma = \begin{pmatrix} \sigma_1^2\tau & \rho\sigma_2\sigma_2\tau \\ \rho\sigma_2\sigma_2\tau & \sigma_2^2\tau \end{pmatrix}.$$

Let $\phi(s_1, s_2)$ be the density function of a bivariate normal random variable with mean μ and covariance matrix Σ; we compute the spread option price (2.1.1) as

$$V(t) = e^{-r(T-t)}E_{\mathbb{Q}}[[(S_2(T) - S_1(T) - K)]^+|\mathcal{F}(t)]$$
$$= e^{-r(T-t)}E_{\mathbb{Q}}[[(e^{s_2(T)} - e^{s_1(T)} - K)]^+|\mathcal{F}(t)]$$
$$= e^{-r(T-t)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[e^{s_2} - e^{s_1} - K]^+\phi(s_1, s_2; t, T)ds_2 ds_1.$$

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Here we used Lemma 3.3.1, which is given in section 3.3. Since ϕ is known under this model, $V(t)$ can be computed numerically.

Because of the existence of the volatility smile [16], a better model was used to describe the assets' price S_1, S_2. In [15], Hong proposed a method based on [7] and [14] to compute European spread options under the so called stochastic volatility model:

$$
\begin{aligned}
ds_1 &= (r - \delta_1 - \frac{1}{2}\sigma_1^2 \nu)dt + \sigma_1 \nu^{\frac{1}{2}}dW_1, \\
ds_2 &= (r - \delta_2 - \frac{1}{2}\sigma_2^2 \nu)dt + \sigma_2 \nu^{\frac{1}{2}}dW_2, \\
d\nu &= \kappa(\mu - \nu)dt + \sigma_\nu \nu^{\frac{1}{2}}dW_\nu,
\end{aligned}
\qquad (2.1.3)
$$

where

$$
\begin{aligned}
E_{\mathbb{Q}}[dW_1 dW_2] &= \rho dt, \\
E_{\mathbb{Q}}[dW_1 dW_\nu] &= \rho_1 dt, \\
E_{\mathbb{Q}}[dW_2 dW_\nu] &= \rho_2 dt.
\end{aligned}
$$

For simplicity, we consider the time 0 spread call option price. The key idea of Hong's method is the following.

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For any $(k_1, k_2) \in \mathbb{R}^2$, we define

$$\Pi_1(k_1, k_2) := \int_{k_1}^{\infty} \int_{k_2}^{\infty} (e^{s_2} - e^{s_1}) q_T(s_1, s_2) ds_2 ds_1,$$

where $q_T(\cdot, \cdot)$ is the joint risk neutral density of $s_1(T)$ and $s_2(T)$ conditional on $s_1(0), s_2(0)$ and $\nu(0)$. Thus we can use $\Pi_1(k_1, k_2)$ for different values of k_1, k_2 to give an approximation of $V(0)$. Applying the two dimensional Fourier transform to the following modified integral

$$\pi_1(k_1, k_2) := e^{\alpha_1 k_1 + \alpha_2 k_2} \Pi_1(k_1, k_2), \quad \alpha_1, \alpha_2 > 0,$$

we obtain

$$
\begin{aligned}
\chi_1(v_1, v_2) &:= \hat{\pi}_1(k_1, k_2) \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(v_1 k_1 + v_2 k_2)} \pi_1(k_1, k_2) dk_2 dk_1 \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{(\alpha_1 + iv_1)k_1 + (\alpha_2 + iv_2)k_2} \int_{k_2}^{\infty} \int_{k_1}^{\infty} (e^{s_2} - e^{s_1}) q_T(s_1, s_2) ds_2 ds_1 dk_2 dk_1 \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (e^{s_2} - e^{s_1}) q_T(s_1, s_2) \int_{-\infty}^{s_2} \int_{-\infty}^{s_1} e^{(\alpha_1 + iv_1)k_1 + (\alpha_2 + iv_2)k_2} dk_2 dk_1 ds_2 ds_1 \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (e^{s_2} - e^{s_1}) q_T(s_1, s_2) \frac{e^{(\alpha_1 + iv_1)s_1 + (\alpha_2 + iv_2)s_2}}{(\alpha_1 + iv_1)(\alpha_2 + iv_2)} ds_2 ds_1 \\
&= \frac{\phi_T(v_1 - \alpha_1 i, v_2 - (\alpha_2 + 1)i) - \phi_T(v_1 - (\alpha_1 + 1)i, v_2 - \alpha_2 i)}{(\alpha_1 + iv_1)(\alpha_2 + iv_2)}, \quad\quad (2.1.4)
\end{aligned}
$$

where

$$\phi_T(u_1, u_2) \quad := \quad E_{\mathbb{Q}}[\exp(iu_1 s_1(T) + iu_2 s_2(T))|s_1(0), s_2(0), \nu(0)]$$
$$= \quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i(u_1 s_1 + u_2 s_2)} q_T(s_1, s_2) ds_2 ds_1,$$

is the characteristic function of the joint risk neutral density of $s_1(T), s_2(T)$ conditional on $s_1(0), s_2(0)$ and $\nu(0)$.

So, as long as we know the characteristic function $\phi_T(u_1, u_2)$, we can compute $\chi_1(v_1, v_2)$. Then by using the two-dimensional inverse fast Fourier transform, we get the value for $\Pi_1(k_1, k_2)$, hence $V(0)$.

Now the question is how to find the characteristic function $\phi_T(u_1, u_2)$ for the stochastic volatility model (2.1.6).

Since model (2.1.6) is an affine structure model, by using the affine property (see section 3.4 later), the characteristic function has an exponential affine form. By solving a Riccati ordinary equation, Hong obtained the closed-form expression for the characteristic function $\phi_T(u_1, u_2)$:

$$\phi_T(u_1, u_2) \quad = \quad E_{\mathbb{Q}}[\exp(iu_1 s_1(T) + iu_2 s_2(T))|s_1(0), s_2(0), \nu(0)]$$
$$= \quad \exp[\sum_{j=1,2} i[s_j(0) + (r - \delta_j)T] \cdot u_j + (\frac{2\zeta(1 - e^{-\theta T})}{2\theta - (\theta - \gamma)(1 - e^{-\theta T})}) \cdot \nu(0)$$
$$- \frac{\kappa\mu}{\sigma_\nu^2}[2 \cdot \log(\frac{2\theta - (\theta - \gamma)(1 - e^{-\theta T})}{2\theta}) + (\theta - \gamma)T]], \qquad (2.1.5)$$

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where

$$\zeta := -\frac{1}{2}[(\sigma_1^2 u_1^2 + \sigma_2^2 u_2^2 + 2\rho\sigma_1\sigma_2 u_1 u_2) + i(\sigma_1^2 u_1 + \sigma_2^2 u_2)],$$

$$\gamma := \kappa - i(\rho_1\sigma_1 u_1 + \rho_2\sigma_2 u_2)\sigma_\nu,$$

$$\theta := \sqrt{\gamma^2 - 2\sigma_\nu^2\zeta}.$$

2.2 ASIAN OPTION

Here we will introduce the Monte Carlo simulation method by Kemna and Vorst[17], then the partial differential equation method from Vecer[24].

Under the Black-Scholes-Merton model, the underlying asset price satisfies the stochastic differential equation

$$dS(t) = rS(t)dt + \sigma S(t)dW(t)$$

where r is the interest rate, σ is the volatility of the asset and $W(t)$ is the standard Brownian motion. The payoff at time T for this Asian call option is

$$V(T) = (\frac{1}{T}\int_0^T S(t)dt - K)^+.$$

Here

$$\frac{1}{T}\int_0^T S(t)dt$$

is the arithmetic average of the asset price.

Since we can solve this stochastic differential equation, we can use standard Monte Carlo simulation

$$S(t+h) = S(t)\exp\{rh + \sigma\sqrt{h}w - \frac{\sigma^2 h}{2}\}$$

to update the asset price, where h is the step size and w is a standard normally distributed random variable generated at each time step. As long as we get a whole sample path about the asset price from time 0 to time T, we can compute the arithmetic average $\frac{1}{T}\int_0^T S(t)dt$. Hence we get the option payoff $V(T) = (\frac{1}{T}\int_0^T S(t)dt - K)^+$ for this sample path. We need to repeat this process to get enough sample paths, then the present value of the average payoff of these sample paths will be a good approximation of the value of this Asian call option.

Because of the low accuracy and high computing price of the standard Monte Carlo simulation, Kemna and Vorst used the geometric average of $S(t)$ as a control variable. At first they approximated the geometric average by $G(T) = (\prod_{i=0}^n S(t_i))^{1/(n+1)}$, then they substituted $G(T)$ for the arithmetic average $\frac{1}{T}\int_0^T S(t)dt$ in the payoff. Since $S(t)$ is log-normally distributed, so is $G(T)$. They showed the mean and variance of $\log(G(T))$ is $\frac{1}{2}(r - \frac{1}{2}\sigma^2)T + \log(S(0))$ and $\frac{1}{3}\sigma^2 T$. So this kind of "geometric" payoff has a closed-form

expression. Hence the price of the Asian call option at time 0 is

$$
\begin{aligned}
V(0) &= E_{\mathbb{Q}}[e^{-rT}(\frac{1}{T}\int_0^T S(t)dt - K)^+|\mathcal{F}(0)] \\
&= e^{-rT}E_{\mathbb{Q}}[(\frac{1}{T}\int_0^T S(t)dt - K)^+ - (G(T) - K)^+|\mathcal{F}(0)] \\
&\qquad\qquad + e^{-rT}E_{\mathbb{Q}}[(G(T) - K)^+|\mathcal{F}(0)] \\
&= e^{-rT}E_{\mathbb{Q}}[(\frac{1}{T}\int_0^T S(t)dt - K)^+ - (G(T) - K)^+|\mathcal{F}(0)] \\
&\qquad\qquad + e^{-rT}[e^{d*}S(0)N(d) - KN(d - \sigma\sqrt{\frac{1}{3}T})],
\end{aligned}
$$

where N is the cumulative standard normal distribution function, and d and d^* are defined

as

$$
\begin{aligned}
d^* &= \frac{1}{2}(r - \frac{1}{6}\sigma^2)T \\
d &= \frac{\log(S(0)/K) + \frac{1}{2}(r + \frac{1}{6}\sigma^2)T}{\sigma\sqrt{\frac{1}{3}T}}.
\end{aligned}
$$

Then the only part unknown is the difference of "arithmetic" and "geometric" payoff

$$
e^{-rT}E_{\mathbb{Q}}[(\frac{1}{T}\int_0^T S(t)dt - K)^+ - (G(T) - K)^+|\mathcal{F}(0)]
$$

which is computed by the standard Monte Carlo simulation.

The other classic method to deal with the Asian option price problem is the partial differential method. The basic method is using the Feynman-Kac theorem, see [23] for a good reference.

Let $Y(t) = \int_0^t S(u)du$ be the running average of the asset price; then the payoff at time T is

$$V(T) = \left(\frac{1}{T}\int_0^T S(t)dt - K\right)^+ = \left(\frac{1}{T}Y(T) - K\right)^+.$$

There exists a function $v(t,x,y)$ such that $v(t, S(t), Y(t)) = E_{\mathbb{Q}}[e^{-r(T-t)}(\frac{1}{T}\int_0^T S(u)du - K)^+|\mathcal{F}(t)]$ and that satisfies a partial differential equation

$$v_t(t,x,y) + rxv_x(t,x,y) + xv_y(t,x,y) + \frac{1}{2}\sigma^2 x^2 v_{xx}(t,x,y) = rv(t,x,y)$$

and boundary conditions

$$v(t,0,y) = e^{-r(T-t)}\left(\frac{y}{T} - K\right)^+, \quad 0 \le t < T, y \in \mathbb{R},$$

$$\lim_{y \to -\infty} v(t,x,y) = 0, \quad 0 \le t < T, x \ge 0,$$

$$v(T,x,y) = \left(\frac{y}{T} - K\right)^+, \quad x \ge 0, y \in \mathbb{R}.$$

Since this equation is not easy to solve, Vecer in [24] used a change of numeraire method to simplify the equation.

Let

$$X(t) = \int_0^t S(u)du - K,$$

and introduce the new process

$$Y(t) = \frac{X(t)}{S(t)};$$

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then we have

$$
\begin{aligned}
dY(t) &= d[(e^{-rt}X(t))(e^{-rt}S(t))^{-1}] \\
&= -\sigma Y(t)dW(t) + \sigma\gamma(t)dW(t) + \sigma^2 Y(t)dt - \sigma^2\gamma(t)dt \\
&= \sigma(\gamma(t) - Y(t))(dW(t) - \sigma dt),
\end{aligned}
$$

where

$$
\gamma(t) = \frac{1}{rT}(1 - e^{-r(T-t)}).
$$

Here we use the changing measure method.

Letting

$$
\begin{aligned}
Z(t) &= \exp\{\sigma W(t) - \frac{1}{2}\sigma^2 t\} \\
\hat{W}(t) &= W(t) - \sigma t,
\end{aligned}
$$

by Girsanov's Theorem, we can define a new probability measure $\hat{P}$ such that

$$
\hat{P}(A) = \int_A Z(T)d\mathbb{Q}
$$

for all measurable set $A \in \mathcal{F}$, where $\mathbb{Q}$ is the original risk neutral measure and $\hat{W}(t)$ is a Brownian motion under the new measure $\hat{P}$. So we obtain

$$
dY(t) = \sigma(\gamma(t) - Y(t))d\hat{W}(t),
$$

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which shows $Y(t)$ is a martingale under this new measure. Then for $V(t)$,

$$
\begin{aligned}
V(t) &= E_{\mathbb{Q}}[e^{-r(T-t)}(\frac{1}{T}\int_0^T S(u)du - K)^+|\mathcal{F}(t)] \\
&= \frac{S(t)}{Z(t)}E_{\mathbb{Q}}[Z(T)Y^+(T)|\mathcal{F}(t)] \\
&= S(t)E_{\hat{P}}[Y^+(T)|\mathcal{F}(t)].
\end{aligned}
$$

By using the Feynman-Kac Theorem there exists a function $g(t, y)$ such that

$$
g(t, Y(t)) = E_{\hat{P}}[Y^+(T)|\mathcal{F}(t)].
$$

So the Asian call option price is

$$
V(t) = S(t)g(t, Y(t))
$$

and this function $g(t, y)$ satisfies the partial differential equation

$$
g_t(t, y) + \frac{1}{2}\sigma^2(\gamma(t) - y)^2 g_{yy}(t, y) = 0, \quad 0 \le t < T, y \in \mathbb{R},
$$

with boundary conditions

$$
\lim_{y \to -\infty} g(t, y) = 0, \quad 0 \le t \le T,
$$

$$
\lim_{y \to \infty} [g(t, y) - y] = 0, \quad 0 \le t \le T,
$$

$$
g(T, y) = y^+, \quad y \in \mathbb{R}.
$$

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After this transformation with the new process $Y(t)$, the number of the variables of the partial differential equation is decreased by 1 which is much easier to compute numerically.

Chapter 3. Asian Spread Option

In this chapter, we will study pricing Asian spread option. In section 1, we introduce the Asian spread option. In section 2 to 4, we present three different methods under two different stochastic models for pricing the Asian spread option.

3.1 Introduction of Asian Spread Option

The Asian spread option is an option about two assets' price spread with an Asian style payoff. So far there is no theoretical result on the Asian spread option. Part of the reason is that we still have no very efficient way to deal with the Asian option. However, it is an important problem in energy markets. Here we only consider the arithmetic average Asian style and time 0 call option since geometric average Asian style is quite simple to deal with. Suppose S_1, S_2 are the price of two assets; then the payoff for this Asian spread call option with strike price K is

$$\max\{\frac{1}{T}\int_0^T S_1(t) - S_2(t)dt - K, 0\}.$$

Compared to the European spread option, the payoff of the Asian spread option contains

$$\frac{1}{T}\int_0^T S_1(t) - S_2(t)dt$$

which is the average of the assets' price difference. This average of the assets' price difference is the key source of difficulty for pricing.

For example, if S_1, S_2 are under the Black-Scholes-Merton model, we know $S_1(T)$ and $S_2(T)$ are log-normal given the value of $S_1(0), S_2(0)$, but $\int_0^T S_1(t)dt$ and $\int_0^T S_2(t)dt$ are not. Actually, in section 3.3, we will show how to price the Asian spread option with the density function method.

As with the relationship between the European option and the European spread option, the key difference between the Asian option and the Asian spread option is that there are two assets involved in the pricing problem. Since under the same model, the Asian option pricing problem already has one more dimension compared to European option, i.e., the average price from payoff, all together, there are two more dimensions for the Asian spread option compare to the European option. That means, instead of solving a 2 dimensional stochastic differential equation of the Black-Sholes-Merton model on the European spread option, we have 3 dimensions for the Asian spread option.

The following three sections discuss three different methods, under two different stochastic models, to treat the pricing problem of the Asian spread option.

3.2 MARTINGALE APPROACH

In this section, we consider the Black-Scholes-Merton model and will use the martingale approach to price the Asian spread options.

Consider a time 0 Asian spread call option of two assets whose price processes are the

solutions of the stochastic differential equations

$$dS_1(t) = rS_1(t)dt + \sigma_1 S_1(t)dW_1(t),$$

$$dS_2(t) = rS_2(t)dt + \sigma_2 S_2(t)dW_2(t),$$

where r is the interest rate, σ_1, σ_2 are volatilities of the assets, and W_1 and W_2 are the standard Brownian motions with correlation coefficient ρ, i.e., $E_\mathbb{Q}[dW_1 dW_2] = \rho dt$. The payoff at time T for this spread call option is

$$V(T) = (\frac{1}{T} \int_0^T [S_1(t) - S_2(t)]dt - K)^+.$$

Here

$$\frac{1}{T} \int_0^T [S_1(t) - S_2(t)]dt$$

is the arithmetic average of the difference of two assets' price.

Our approach is based on an idea from [23], where the Asian option was considered.

We start with the option price

$$V(t) = E_\mathbb{Q}[e^{-r(T-t)}V(T)|\mathcal{F}(t)].$$

By multiplying by e^{-rt} to $V(t)$

$$e^{-rt}V(t) = E_\mathbb{Q}[e^{-rT}V(T)|\mathcal{F}(t)],$$

we have $e^{-rt}V(t)$ is a martingale under the risk neutral measure $\mathbb{Q}$. Define $Y(t)$ be the average difference of the two assets' price

$$Y(t) = \int_0^t S_1(u) - S_2(u)du$$

i.e.,

$$dY(t) = S_1(t)dt - S_2(t)dt.$$

Here is the very unique factor $Y(t)$ of Asian style option pricing. The reason we add this additional stochastic process to the model is that the special payoff

$$V(T) = (\frac{1}{T}\int_0^T [S_1(t) - S_2(t)]dt - K)^+ = (\frac{1}{T}Y(T) - K)^+$$

of the Asian spread option depends on $Y(T)$ instead of $S_1(T), S_2(T)$. Since here $Y(t)$ itself is not a Markov process, we use $(S_1(t), S_2(t), Y(t))$ together to constitute a 3-dimensional Markov process.

Since $V(T) = (\frac{1}{T}Y(T) - K)^+$, by the Feynman-Kac Theorem, there exists a function v, such that

$$
\begin{aligned}
v(t, S_1(t), S_2(t), Y(t)) &= E_{\mathbb{Q}}[e^{-r(T-t)}(\frac{1}{T}Y(T) - K)^+|\mathcal{F}(t)] \\
&= E_{\mathbb{Q}}[e^{-r(T-t)}V(T)|\mathcal{F}(t)] \\
&= V(t).
\end{aligned}
$$

By the Ito formula, we have

$$d(e^{-rt}v(t, S_1(t), S_2(t), Y(t)))$$

$$= e^{-rt}[-rvdt + v_t dt + v_{x_1} dS_1 + v_{x_2} dS_2 + v_y dY$$

$$+ \frac{1}{2} v_{x_1 x_1} dS_1 dS_1 + \frac{1}{2} v_{x_2 x_2} dS_2 dS_2 + \frac{1}{2} v_{x_1 x_2} dS_1 dS_2]$$

$$= e^{-rt}[-rv + v_t + rS_1(t)v_{x_1} + rS_2(t)v_{x_2} + (S_1(t) - S_2(t))v_y$$

$$+ \frac{1}{2}\sigma_1^2 S_1(t)^2 v_{x_1 x_1} + \frac{1}{2}\sigma_2^2 S_2(t)^2 v_{x_2 x_2} + \sigma_1\sigma_2 S_1(t)S_2(t)v_{x_1 x_2}\rho]dt$$

$$+ e^{-rt}(\sigma_1 S_1(t)v_{x_1} dW_1(t) + \sigma_2 S_2(t)v_{x_2} dW_2(t)).$$

Since $e^{-rt}V(t)$ is a martingale under $\mathbb{Q}$, letting dt term equal 0, we get the partial differential equation for the function v:

$$v_t + rx_1 v_{x_1} + rx_2 v_{x_2} + (x_1 - x_2)v_y + \frac{1}{2}\sigma_1^2 x_1^2 v_{x_1 x_1} + \frac{1}{2}\sigma_2^2 x_2^2 v_{x_2 x_2} + \sigma_1\sigma_2\rho x_1 x_2 v_{x_1 x_2} = rv, \quad (3.2.1)$$

under the boundary conditions

$$v(T, x_1, x_2, y) = (\frac{y}{T} - K)^+,$$

for $x_1, x_2 \geq 0, y \in \mathbb{R}$;

$$\lim_{y \to -\infty} v(t, x_1, x_2, y) = 0,$$

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for $0 \le t < T, x_1, x_2 \ge 0$; and

$$v(t, 0, 0, y) = e^{-r(T-t)}(\frac{y}{T} - K)^+,$$

for $0 \le t < T, y \in \mathbb{R}$.

3.3 DENSITY FUNCTION METHOD

In the last section we changed the pricing problem to solving partial differential equation (3.2.1). Because of the complicated form and the fact that there is no efficient way to simplify it, it's not easy to solve even numerically. In this section we'll use the density function method.

The density function method is quite natural: the price of option, $V(t)$, is the conditional expectation of payoff on $\mathcal{F}(t)$ which is a random variable that is $\mathcal{F}(t)$ measurable. If we have the density function of this random variable, the pricing problem will be some integral problem involving this density function as we reviewed about the European spread option in section 2.1.

Consider the Black-Scholes-Merton model:

$$dS_1(t) = rS_1(t)dt + \sigma_1 S_1(t)dW_1(t),$$
$$dS_2(t) = rS_2(t)dt + \sigma_2 S_2(t)dW_2(t),$$

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where r is the interest rate, σ_1, σ_2 are volatilities of the assets, and W_1 and W_2 are standard Brownian motions with correlation coefficient ρ, i.e., $E_{\mathbb{Q}}[dW_1 dW_2] = \rho dt$.

For the European spread option, since the solutions of these stochastic differential equations follow a two dimensional log-normal distribution, so we can use the known density function to compute European spread option value $V(t)$. Similarly, in this section we try to find out the density function of $Y(t)$ to compute $V(t)$.

We know the solution of this stochastic differential equation is a log bivariate normal vector conditional on its value at time t; moreover, $S_1(T)$ is also a log-normal random variable conditional on $S_2(T)$, $S_1(t)$ and $S_2(t)$:

$$\log(S_1(T)|_{S_1(t), S_2(t), S_2(T)}) \sim N(\mu_{1|2}, \sigma_{1|2}^2),$$

where

$$
\begin{aligned}
\mu_{1|2} &= \log(S_1(t)) + (r - \frac{1}{2}\sigma_1^2)(T - t) \\
&\quad + \frac{\rho\sigma_1}{\sigma_2}(\log(S_2(T)) - (\log(S_2(t)) - (r - \frac{1}{2}\sigma_2^2)(T - t))), \\
\sigma_{1|2} &= \sqrt{1 - \rho^2}\sigma_1\sqrt{T - t}.
\end{aligned}
$$

$V(T)$ can be computed as follows:

$$
\begin{aligned}
V(T) &= (\frac{1}{T} \int_0^T [S_1(u) - S_2(u)]du - K)^+ \\
&= (\frac{1}{T} \int_0^t [S_1(u) - S_2(u)]du + \frac{1}{T} \int_t^T [S_1(u) - S_2(u)]du - K)^+ \\
&= (\frac{1}{T} \int_t^T [S_1(u) - S_2(u)]du - (K - \frac{1}{T} \int_0^t [S_1(u) - S_2(u)]du))^+ \\
&=: (\frac{1}{T} \int_t^T [S_1(u) - S_2(u)]du - K')^+,
\end{aligned}
$$

where

$$
K' = K - \frac{1}{T} \int_0^t [S_1(u) - S_2(u)]du.
$$

Here K' is determined by the price of S_1, S_2 from time 0 to t, so it is known at time t. So from the solution of the stochastic differential equation, we have

$$
\begin{aligned}
V(T) &= (\frac{1}{T} \int_0^T [S_1(u) - S_2(u)]du - K)^+ \\
&= [\frac{1}{T} S_1(t) \int_0^{T-t} \exp(\sigma_1 \hat{W}_1(s) + (r - \sigma_1^2/2)s)ds \\
&\quad - \frac{1}{T} S_2(t) \int_0^{T-t} \exp(\sigma_2 \hat{W}_2(s) + (r - \sigma_2^2/2)s)ds - K']^+,
\end{aligned}
$$

where

$$
\hat{W}_i(s) = W_i(t + s) - W_i(t),
$$

for $i = 1, 2$, $0 \leq s \leq T - t$ are both new Brownian motions independent of $\mathcal{F}(t)$. Here we need the following result.

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Lemma 3.3.1

Let $(\Omega, \mathcal{F}_0, P)$ be a probability space , $\mathcal{F}$ be a sub σ-algebra of $\mathcal{F}_0$, and X, Y be two random variables with $E|X| < \infty, E|Y| < \infty$. Suppose X is measurable in $\mathcal{F}$, Y is independent of $\mathcal{F}$, $f(x, y) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is measurable and $E|f(X, Y)| < \infty$. Then $E(f(X, Y)|\mathcal{F}) = E(f(x, Y))|_{x=X}$.

Proof:

Here we use five steps to prove the Lemma. we start with the case where the function f is an indicator function, then the case for a product indicator function, the case for a simple function and the case for a nonnegative measurable function and at last for the general measurable function.

Step 1: Indicator function

Suppose A and B are measurable in $\mathbb{R}$, i.e., $A \in \mathcal{B}(\mathbb{R})$ and $B \in \mathcal{B}(\mathbb{R})$, $\{X \in A\} \in \mathcal{F}$, $\{Y \in B\}$ is independent of $\mathcal{F}$, and $f = 1_{A \times B}$; then

$$
\begin{aligned}
E(f(X, Y)|\mathcal{F}) &= E(1_{A \times B}(X, Y)|\mathcal{F}) \\
&= E(1_A(X)1_B(Y)|\mathcal{F}) \\
&= 1_A(X)E(1_B(Y)) = E(f(x, Y))|_{x=X}.
\end{aligned}
$$

Step 2: Product indicator function

Let $f = \sum_{m=1}^{n} c_m 1_{A_m \times B_m}$, where $c_m \in \mathbb{R}$, A_m and B_m are measurable in $\mathbb{R}$, $\{X \in A_m\} \in \mathcal{F}$, and $\{Y \in B_m\}$ is independent of $\mathcal{F}$; then the linearity of the conditional expectation and the result for indicator functions gives

$$
\begin{aligned}
E(f(X,Y)|\mathcal{F}) &= \sum_{m=1}^{n} E(c_m 1_{A_m \times B_m}|\mathcal{F}) \\
&= \sum_{m=1}^{n} c_m 1_{A_m}(X) E(1_{B_m}(Y)) = E(f(x,Y))|_{x=X}.
\end{aligned}
$$

Step 3: Simple function

Let $f = \sum_{m=1}^{n} c_m 1_J$ where J is a product measurable set in $\mathbb{R} \times \mathbb{R}$.

Claim: $E(1_J(X,Y)|\mathcal{F}) = E(1_J(x,Y))|_{x=X}$.

If the claim is true, then by linearity of the conditional expectation, for $f = \sum_{m=1}^{n} c_m 1_J$, we have $E(f(X,Y)|\mathcal{F}) = E(f(x,Y))|_{x=X}$.

To prove the claim, notice that if

$\mathcal{J} := \{J \in \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}) | E(1_J(X,Y)|\mathcal{F}) = E(1_J(x,Y))|_{x=X}\}$,

then $\mathcal{J}$ is a λ-system. Since by the result of indicator functions, all the sets of the form $A \times B$ where $A \in \mathcal{B}(\mathbb{R})$, $B \in \mathcal{B}(\mathbb{R})$, i.e., $\mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R})$, are in $\mathcal{J}$, and $\mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R})$ is a π-system, then by Dynkin's $\pi - \lambda$ theorem, $\sigma(\mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R})) \in \mathcal{J}$.

Hence for $f = \sum_{m=1}^{n} c_m 1_J$ where J is a product measurable set in $\mathbb{R} \times \mathbb{R}$, $E(f(X,Y)|\mathcal{F}) = E(f(x,Y))|_{x=X}$.

Step 4: Nonnegative measurable function

Suppose $f \geq 0$, and let $f_n(x, y) = ([2^n f(x, y)]/2^n) \wedge n$; then each $f_n(x, y)$ is a simple function, such that $f_n(x, y) \nearrow f(x, y)$ as $n \to \infty$. Then by the Monotone Convergence theorem of the conditional expectation and the result for simple functions we have

$$
\begin{aligned}
E(f(X, Y)|\mathcal{F}) &= \lim_{n \to \infty} E(f_n(X, Y)|\mathcal{F}) \\
&= \lim_{n \to \infty} E(f_n(x, Y))|_{x=X} \\
&= E(f(x, Y))|_{x=X}.
\end{aligned}
$$

Step 5: General measurable function

Write $f = f^+ - f^-$; the conclusion is proved by the linearity of the conditional expectation and the result for nonnegative functions. $\blacksquare$

By Lemma 3.3.1, we can change the conditional expectation to the regular expectation

$$
\begin{aligned}
V(t) &= E_{\mathbb{Q}}[e^{-r(T-t)}V(T)|\mathcal{F}(t)] \\
&= E_{\mathbb{Q}}[e^{-r(T-t)}\{\frac{1}{T}S_1(t)\int_0^{T-t}exp(\sigma_1\hat{W}_1(s)+(r-\sigma_1^2/2)s)ds \\
&\quad -\frac{1}{T}S_2(t)\int_0^{T-t}exp(\sigma_2\hat{W}_2(s)+(r-\sigma_2^2/2)s)ds - K'\}^+|\mathcal{F}(t)] \\
&= E_{\mathbb{Q}}[e^{-r(T-t)}\{\frac{1}{T}S_1(t)\int_0^{T-t}exp(\sigma_1\hat{W}_1(s)+(r-\sigma_1^2/2)s)ds \\
&\quad -\frac{1}{T}S_2(t)\int_0^{T-t}exp(\sigma_2\hat{W}_2(s)+(r-\sigma_2^2/2)s)ds - K'\}^+] \\
&= e^{-r(T-t)}\int_0^{\infty}\int_0^{\infty}[a_1-a_2-K]_+f_{1|2}(a_1|a_2)f_2(a_2)da_1da_2 \\
&= e^{-r(T-t)}\int_0^{\infty}\int_{a_2+K}^{\infty}[a_1-(a_2+K)]f_{1|2}(a_1|a_2)da_1f_2(a_2)da_2 \\
&=: \int_0^{\infty}F(a_2)f_2(a_2)da_2,
\end{aligned}
$$

where

$$
\begin{aligned}
F(a_2) : &= e^{-r(T-t)}\int_{a_2+K}^{\infty}[a_1-(a_2+K)]f_{1|2}(a_1|a_2)da_1, \\
A_1 : &= \frac{1}{T}\int_0^T S_1(u)du, \\
A_2 : &= \frac{1}{T}\int_0^T S_2(u)du.
\end{aligned}
$$

$f_2(\cdot)$, $f_{1|2}(\cdot|a_2)$ represent the density function of A_2 and the conditional density function of A_1 given value of A_2 respectively.

Note that $F(a_2)$ is the price of the Asian call option on S_1 with strike price $a_2 + K$. There is no closed form formula to compute Asian call option price. The best result we can use is given by Geman and Yor in [13]:

$$F(a_2) := \frac{e^{-r(T-t)}}{T}(\frac{4S_1(t)}{\sigma^2_{1|2}})C^{(\nu)}(h,q),$$

where

$$\nu = \frac{2r}{\sigma^2_{1|2}} - 1; \quad h = \frac{\sigma^2_{1|2}}{4}(T-t); \quad q = \frac{\sigma^2_{1|2}}{4S_1(t)}[(a_2+K)T - \int_0^t S_1(u)du];$$

$$C^{(\nu)}(h,q) := E_{\mathbb{Q}}[(\int_0^h exp[2(W_s+\nu s)]ds - q)^+].$$

By using that result, the Laplace transform of $C^{(\nu)}(h,q)$ with respect to the variable h is

$$\int_0^\infty e^{-\lambda h}C^{(\nu)}(h,q)dh = \frac{\int_0^{1/2q} e^{-x}x^{(\mu-\nu)/2-2}(1-2qx)^{(\mu+\nu)/2+1}dx}{\lambda(\lambda-2-2\nu)\Gamma((\mu-\nu)/2-1)},$$

where $\mu = \sqrt{2\lambda + \nu^2}$, Γ is the gamma function, and we can get the value of $C^{(\nu)}(h,q)$ via the inverse Laplace transform.

Now we turn to the question of finding $V(t)$.

Now

$$
\begin{aligned}
A_2 &= \frac{1}{T}\int_0^T S_2(u)du \\
&= \frac{1}{T}\Big(\int_0^t S_2(u)du + S_2(t)\int_0^{T-t}\exp(\sigma_2\hat{W}(s) + (r - \sigma_2^2/2)s)ds\Big) \\
&= \frac{1}{T}\Big(\int_0^t S_2(u)du + S_2(t)\frac{4}{\sigma_2^2}\int_0^{\frac{(T-t)\sigma_2^2}{4}}\exp(2\hat{W}(s) + \frac{4(r - \sigma_2^2/2)s}{\sigma_2^2})ds\Big) \\
&=: \frac{1}{T}\Big(\int_0^t S_2(u)du + S_2(t)\frac{4}{\sigma_2^2}A_{(T-t)\sigma_2^2/4}\Big) =: A_2(A),
\end{aligned}
$$

let the density function of $A_{(T-t)\sigma_2^2/4}$ be $f(\cdot)$; then we have

$$
V(t) = \int_0^\infty F(A_2(u))f(u)du. \tag{3.3.1}
$$

Letting $P(\int_0^\tau \exp(2B_s)ds \in dz | B_\tau = x) = a_\tau(x,z)dz$, Yor showed in [26] that

$$
P\Big(\int_0^\tau \exp(2(B_s + \nu s))ds \in dz | B_\tau + \nu\tau = x\Big) = a_\tau(x,z)dz,
$$

and

$$
\frac{1}{\sqrt{2\pi\tau}}\exp(-\frac{x^2}{2t})a_\tau(x,z) = \frac{1}{z}\exp(-\frac{1}{2z}(1 + \exp(2x)))\theta_{e^x/z}(\tau).
$$

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so we can plug into (3.3.2) and obtain the option price $V(t)$

$$
\begin{aligned}
V(t) &= \int_{-\infty}^{\infty} dx \int_{0}^{\infty} dz F(A_2(z)) a_{\tau}^{\nu}(x, z) \\
&= \int_{-\infty}^{\infty} dx \int_{0}^{\infty} dz \\
&\quad F(\frac{1}{T}(\int_{0}^{t} S_2(u) du + S_2(t) \frac{4}{\sigma_2^2} z)) \frac{1}{z} \exp(-\frac{1}{2z}(1 + \exp(2x))) \theta_{e^x/z}(\tau),
\end{aligned}
$$

$$
(3.3.2)
$$

where

$$
\begin{aligned}
\tau &= \frac{\sigma_2^2(T-t)}{4}, \quad \nu = \frac{2(r - \sigma_2^2/2)}{\sigma_2^2}, \\
\theta_r(u) &= \frac{r}{(2\pi^3 u)^{\frac{1}{2}}} \exp(\frac{\pi^2}{2u}) \int_{0}^{\infty} \exp(-y^2/2u) \exp(-r(\cosh y))(\sinh y) \sin(\frac{\pi y}{u}) dy.
\end{aligned}
$$

This is so far the best semi-analytic solution for pricing Asian spread options under Black-Scholes-Merton model.

From (3.3.2), it's quite clear that this semi-analytic solution is really difficult to compute numerically. This is the basic Black-Scholes-Merton model, which shows the difficulty of the Asian spread option pricing problem.

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3.4 CHARACTERISTIC FUNCTION METHOD

In this section, based on Duffie, Pan and Singleton's powerful result of Affine structure in [8], and more general result from Duffie, Filipovic and Schachermayer in [9], we will propose an analytic and computable result for an affine structure stochastic volatility model.

The stochastic model is

$$
\begin{aligned}
dS_1(t) &= rS_1(t)dt + \sigma_1\sqrt{\gamma(t)}dW_1(t), \\
dS_2(t) &= rS_2(t)dt + \sigma_2\sqrt{\gamma(t)}dW_2(t), \\
d\gamma(t) &= \kappa(\mu - \gamma(t))dt + \sigma_\gamma\sqrt{\gamma(t)}dW_\gamma(t), \\
dY(t) &= (S_1(t) - S_2(t))dt, \qquad\qquad (3.4.1)
\end{aligned}
$$

where

$$
E[dW_1(t)dW_2(t)] = \rho dt,
$$
$$
E[dW_1(t)dW_\gamma(t)] = \rho_1 dt,
$$
$$
E[dW_2(t)dW_\gamma(t)] = \rho_2 dt.
$$

Taking the state vector $X_t = (S_1(t), S_2(t), \gamma(t), Y(t))^T$, this is an affine diffusion model,

$$
dX_t = \Theta(X_t)dt + \Sigma(X_t)dw(t),
$$

where

$$\Theta(X_t) = \begin{pmatrix} r & 0 & 0 & 0 \\ 0 & r & 0 & 0 \\ 0 & 0 & -\kappa & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix} X_t + \begin{pmatrix} 0 \\ 0 \\ \kappa\mu \\ 0 \end{pmatrix} =: K_1 X_t + K_0,$$

$$\Sigma(X_t) = \begin{pmatrix} \sqrt{1-\rho_1^2}\sigma_1\sqrt{\gamma(t)} & 0 & \rho_1\sigma_1\sqrt{\gamma(t)} & 0 \\ \frac{\rho-\rho_1\rho_2}{\sqrt{1-\rho_1^2}}\sigma_2\sqrt{\gamma(t)} & \frac{\sqrt{1-\rho_1^2-\rho_2^2-\rho^2+2\rho_1\rho_2\rho}}{\sqrt{1-\rho_1^2}}\sigma_2\sqrt{\gamma(t)} & \rho_2\sigma_2\sqrt{\gamma(t)} & 0 \\ 0 & 0 & \sigma_\gamma\sqrt{\gamma(t)} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and $w(t) = (w_1(t), w_2(t), w_3(t), w_4(t))^T$ is a standard 4-dimensional Brownian motion. In this model, we call $\sigma_1\sqrt{\gamma(t)}$ and $\sigma_2\sqrt{\gamma(t)}$ the volatility of S_1 and S_2 respectively. The existence and uniqueness of the general regular affine process is proved by Duffie, Filipovic and Schachermayer in [9].

Note that

$$\Sigma(X_t)\Sigma(X_t)^T = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 & \rho_1\sigma_1\sigma_\gamma & 0 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 & \rho_2\sigma_2\sigma_\gamma & 0 \\ \rho_1\sigma_1\sigma_\gamma & \rho_2\sigma_2\sigma_\gamma & \sigma_\gamma^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \gamma(t) =: H_1\gamma(t).$$

The stochastic model (3.4.1) is similar to the stochastic volatility model (2.1.3). But here S_1, S_2's volatility is determined only by the process $\gamma(t)$ instead of $\gamma(t)$ and S_1, S_2. That

means under this model, the volatility of S_1 and S_2 have a constant ratio. This assumption is quite reasonable since the spread option always involves two highly related assets. Also, the same as model (2.1.3), model (3.4.1) can deal with the volatility smile problem.

The payoff of the Asian spread option with strike price c is

$$(\frac{1}{T}Y_T - c)^+,$$

if we let $b = (0,0,0,\frac{1}{T})^T$, then the price of this Asian spread option at time 0 is

$$
\begin{aligned}
C(X_0, c, 0, T) &= E(\exp(-rT)(b \cdot X_T - c)^+ | \mathcal{F}(0)) \\
&= E(\exp(-rT)(b \cdot X_T - c)\mathbf{1}_{b \cdot X_T \geq c} | \mathcal{F}(0)).
\end{aligned}
$$

$$(3.4.2)$$

If we follow the similar approach in [8], define the "generalized expected present value" functions by

$$
\begin{aligned}
G^{(1)}_{a,b}(y; X_t, t, T) &= E(\exp(-r(T-t)) \exp(a \cdot X(T))\mathbf{1}_{b \cdot X(T) \leq y} | \mathcal{F}(t)), \\
G^{(2)}_{a,b,d}(y; X_t, t, T) &= E(\exp(-r(T-t))(a \cdot X(T)) \exp(d \cdot X_T)\mathbf{1}_{b \cdot X(T) \leq y} | \mathcal{F}(t)),
\end{aligned}
$$

where $a \in \mathbb{R}^4$, $b \in \mathbb{R}^4$ and $d \in \mathbb{R}^4$.

Then Asian spread option at time 0 is

$$C(X_0, c, 0, T) = G^{(2)}_{b,-b,\vec{0}}(-c; X_0, 0, T) - cG^{(1)}_{\vec{0},-b}(-c; X_0, 0, T). \qquad (3.4.3)$$

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Also we define

$$\psi_1(u, X_t, t, T) = E(\exp(-r(T-t))\exp(u \cdot X_T)|\mathcal{F}(t)),$$

where $u = (u_1, u_2, u_3, u_4)^T \in \mathbb{C}^4$; and

$$\psi_2(v, u, X_t, t, T) = E(\exp(-r(T-t))(v \cdot X_T)\exp(u \cdot X_T)|\mathcal{F}(t)),$$

where $v = (v_1, v_2, v_3, v_4)^T \in \mathbb{R}^4$. Here ψ_1 and ψ_2 are called the "characteristic" function and the "extended characteristic" function respectively. Then from [8], by using the inverse Fourier transform, we have

$$
\begin{aligned}
C(X_0, c, 0, T) &= G^{(2)}_{b,-b,\vec{0}}(-c; X_0, 0, T) - cG^{(1)}_{\vec{0},-b}(-c; X_0, 0, T) \\
&= [\frac{\psi_2(b, \vec{0}, X_0, 0, T)}{2} - \frac{1}{\pi}\int_0^\infty \frac{Im[\psi_2(b, -ivb, X_0, 0, T)\exp(ivc)]}{v}dv] \\
&\quad - c[\frac{\psi_1(\vec{0}, X_0, 0, T)}{2} - \frac{1}{\pi}\int_0^\infty \frac{Im[\psi_1(-ivb, X_0, 0, T)\exp(ivc)]}{v}dv]. \quad (3.4.4)
\end{aligned}
$$

So the Asian spread option pricing problem under model (3.4.1) becomes finding the "characteristic" function and the "extended characteristic" function ψ_1 and ψ_2. Since [8] and [9] proved the general affine process characteristic functions property, for the special case model (3.4.1), we have the corresponding results.

"Characteristic" function is obtained by

$$\psi_1(u, X_t, t, T) = \exp(\alpha(t, T, u) + \beta(t, T, u) \cdot X_t),$$

where $\beta(t, T, u) =: \beta_t = (\beta_t^{(1)}, \beta_t^{(2)}, \beta_t^{(3)}, \beta_t^{(4)})^T, \alpha(t, T, u) := \alpha_t$ satisfy the following complex ordinary differential equations

$$
\dot{\beta}_t = \begin{pmatrix} \dot{\beta}_t^{(1)} \\ \dot{\beta}_t^{(2)} \\ \dot{\beta}_t^{(3)} \\ \dot{\beta}_t^{(4)} \end{pmatrix}
$$

$$
= -\begin{pmatrix} r & 0 & 0 & 1 \\ 0 & r & 0 & -1 \\ 0 & 0 & -\kappa & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \beta_t - \frac{1}{2} \begin{pmatrix} 0 \\ 0 \\ \sum_{i,j} \beta_t^{(i)} H_1(i,j) \beta_t^{(j)} \\ 0 \end{pmatrix} \tag{3.4.5}
$$

$$
\dot{\alpha}_t = r - \kappa\mu\beta_t^{(3)}, \tag{3.4.6}
$$

under the boundary conditions

$$
\beta_T = u
$$

$$
\alpha_T = 0.
$$

"Extended characteristic" function is obtained by

$$
\psi_2(v, u, X_t, t, T) = \psi_1(u, X_t, t, T)(A(t, T, v, u) + B(t, T, v, u) \cdot X_t),
$$

where $B(t, T, v, u) =: B_t = (B_t^{(1)}, B_t^{(2)}, B_t^{(3)}, B_t^{(4)})^T$, $A(t, T, v, u) =: A_t$ satisfy the following complex ordinary differential equations

$$
\dot{B_t} = \begin{pmatrix} \dot{B_t^{(1)}} \\ \dot{B_t^{(2)}} \\ \dot{B_t^{(3)}} \\ \dot{B_t^{(4)}} \end{pmatrix}
$$

$$
= -\begin{pmatrix} r & 0 & 0 & 1 \\ 0 & r & 0 & -1 \\ 0 & 0 & -\kappa & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} B_t - \begin{pmatrix} 0 \\ 0 \\ \sum_{i,j} \beta_t^{(i)} H_1(i,j) B_t^{(j)} \\ 0 \end{pmatrix} \tag{3.4.7}
$$

$$
\dot{A_t} = -\kappa \mu B_t^{(3)}, \tag{3.4.8}
$$

under the boundary condition

$$
B_T = v
$$

$$
A_T = 0.
$$

To find the value of $\alpha_t, \beta_t, A_t, B_t$, we need solve boundary value ordinary differential equations (3.4.5), (3.4.6), (3.4.7) and (3.4.8). We can only solve part of these differential equations analytically and some of them need to be solved numerically.

For $u = (u_1, u_2, u_3, u_4)$,

$$
\begin{aligned}
\beta_t^{(1)} &= u_1 \exp(r(T-t)) + \frac{u_4}{r}[\exp(r(T-t)) - 1] \\
\beta_t^{(2)} &= u_2 \exp(r(T-t)) - \frac{u_4}{r}[\exp(r(T-t)) - 1] \\
\beta_t^{(4)} &= u_4.
\end{aligned}
$$

For $\beta_t^{(3)}$,

$$
\dot{\beta}_t^{(3)} = s_t \beta_t^{(3)2} + q_t \beta_t^{(3)} + p_t, \tag{3.4.9}
$$

where

$$
\begin{aligned}
s_t &= -\frac{1}{2}\sigma_\gamma^2 \\
q_t &= \kappa - \rho_1 \sigma_1 \sigma_\gamma \beta_t^{(1)} - \rho_2 \sigma_2 \sigma_\gamma \beta_t^{(2)} \\
p_t &= -\frac{1}{2}[\sigma_1^2 \beta_t^{(1)2} + 2\rho \sigma_1 \sigma_2 \beta_t^{(1)} \beta_t^{(2)} + \sigma_2^2 \beta_t^{(2)2}].
\end{aligned}
$$

Equation (3.4.9) is a Riccati ordinary equation, and we can solve it numerically and then solve (3.4.6) for α_t numerically. Then we use the result of α_t, β_t to solve for A_t, B_t. In section 5.3, we present some numerical results for this method.

Chapter 4. Asian-European Spread Option

In this chapter, we will focus on pricing the Asian-European spread options. In section 1, we introduce the Asian-European spread option. In section 2, 3, 4, we use three different methods under two different stochastic models to price Asian-European spread options.

4.1 Introduction of Asian-European Spread Option

The Asian-European spread option is an option about two assets' price spread with one side Asian and the other side European style payoff.

Similar as Asian spread option we introduced in Chapter 3, here we only consider the arithmetic average Asian style and time 0 call option. Suppose S_1, S_2 are the price of two assets, and we take Asian style payoff for S_1; then the payoff for this Asian-European spread call option with strike price K is

$$\max\{\frac{1}{T} \int_0^T S_1(t)dt - S_2(T) - K, 0\}.$$

Although it seems this kind of spread option is less complicated than the Asian spread option, it still contains the average of asset price $\frac{1}{T} \int_0^T S_1(t)dt$ which will be the key difficulty of pricing.

4.2 MARTINGALE APPROACH

In this section, we start with the classic Black-Scholes-Merton model and use a martingale approach to price Asian-European spread options.

Consider a time 0 Asian-European spread call option of two assets whose price processes are the solutions of stochastic differential equations

$$dS_1(t) = rS_1(t)dt + \sigma_1 S_1(t)dW_1(t),$$

$$dS_2(t) = rS_2(t)dt + \sigma_2 S_2(t)dW_2(t),$$

where r is the interest rate, σ_1, σ_2 are volatilities of the assets, and W_1 and W_2 are the standard Brownian motions with correlation coefficient ρ, i.e. $E_{\mathbb{Q}}[dW_1 dW_2] = \rho dt$. Suppose the Asian style is for S_1. Two kinds of payoff at time T for this spread call option are

$$V(T) = (\frac{1}{T}\int_0^T S_1(t)dt - S_2(T) - K)^+$$

or

$$V(T) = (S_2(T) - \frac{1}{T}\int_0^T S_1(t)dt - K)^+.$$

In the payoff, $\frac{1}{T}\int_0^T S_1(t)dt$ is the Asian style, the arithmetic average of the asset price S_1; $S_2(T)$ is the European style payoff. So we call this spread option an Asian-European spread option.

Consider the first of the two kinds of payoff at time T:

$$V(T) = (\frac{1}{T} \int_0^T S_1(t)dt - S_2(T) - K)^+;$$

actually, the other payoff is very similar to this one. Let option price be

$$V(t) = E_{\mathbb{Q}}[e^{-r(T-t)}V(T)|\mathcal{F}(t)],$$

Similar as the Asian spread option case, $e^{-rt}V(t)$ is a martingale under the risk neutral measure $\mathbb{Q}$. Let $Y_1(t)$ be the average price of asset 1,

$$Y_1(t) = Y_1(0) + \int_0^t S_1(u)du$$

i.e.,

$$dY_1(t) = S_1(t)dt.$$

The average price is the "trade mark" of Asian style options. As before, we add this additional stochastic process to the model because the payoff

$$V(T) = (\frac{1}{T} \int_0^T S_1(t)dt - S_2(T) - K)^+ = (\frac{1}{T}Y_1(T) - S_2(T) - K)^+$$

contains $Y_1(T)$. So we use $(S_1(t), S_2(t), Y_1(t))$ together to constitute a 3-dimensional Markov process.

Since $V(T) = (\frac{1}{T}Y_1(T) - S_2(T) - K)^+$, by the Feynman-Kac Theorem, there exists a function v, such that

$$
\begin{aligned}
v(t, S_1(t), S_2(t), Y_1(t)) &= E_{\mathbb{Q}}[e^{-r(T-t)}(\frac{1}{T}Y_1(T) - S_2(T) - K)^+ | \mathcal{F}(t)] \\
&= E_{\mathbb{Q}}[e^{-r(T-t)}V(T) | \mathcal{F}(t)] \\
&= V(t).
\end{aligned}
$$

By the Ito formula, we have

$$
\begin{aligned}
&d(e^{-rt}v(t, S_1(t), S_2(t), Y_1(t))) \\
&= e^{-rt}[-rvdt + v_t dt + v_{x_1}dS_1 + v_{x_2}dS_2 + v_{y_1}dY_1 \\
&\quad + \frac{1}{2}v_{x_1 x_1}dS_1 dS_1 + \frac{1}{2}v_{x_2 x_2}dS_2 dS_2 + \frac{1}{2}v_{x_1 x_2}dS_1 dS_2] \\
&= e^{-rt}[-rv + v_t + rS_1(t)v_{x_1} + rS_2(t)v_{x_2} + S_1(t)v_{y_1} \\
&\quad + \frac{1}{2}\sigma_1^2 S_1(t)^2 v_{x_1 x_1} + \frac{1}{2}\sigma_2^2 S_2(t)^2 v_{x_2 x_2} + \sigma_1\sigma_2 S_1(t)S_2(t)v_{x_1 x_2}\rho]dt \\
&\quad + e^{-rt}(\sigma_1 S_1(t)v_{x_1}dW_1(t) + \sigma_2 S_2(t)v_{x_2}dW_2(t)).
\end{aligned}
$$

Since $e^{-rt}V(t)$ is a martingale under $\mathbb{Q}$, letting the dt term equal 0 we get a partial differential equation for the function v:

$$
v_t + rx_1 v_{x_1} + rx_2 v_{x_2} + x_1 v_{y_1} + \frac{1}{2}\sigma_1^2 x_1^2 v_{x_1 x_1} + \frac{1}{2}\sigma_2^2 x_2^2 v_{x_2 x_2} + \sigma_1\sigma_2\rho x_1 x_2 v_{x_1 x_2} = rv, \quad (4.2.1)
$$

for $0 \leq t < T, x_1, X_2 \geq 0, y_1 \in \mathbb{R}$,

with boundary conditions

$$v(T, x_1, x_2, y_1) = (\frac{y_1}{T} - x_2 - K)^+,$$

for $x_1, x_2 \geq 0, y_1 \in \mathbb{R}$;

$$\lim_{y_1 \to -\infty} v(t, x_1, x_2, y_1) = 0,$$

for $0 \leq t < T, x_1, x_2 \geq 0$; and

$$v(t, 0, 0, y_1) = e^{-r(T-t)}(\frac{y_1}{T} - K)^+,$$

for $0 \leq t < T$. If the payoff is of the form

$$V(T) = (S_2(T) - \frac{1}{T} \int_0^T S_1(t)dt - K)^+,$$

then we have the same differential equation (4.2.1), under different boundary conditions though:

$$v(T, x_1, x_2, y_1) = (x_2 - \frac{y_1}{T} - K)^+,$$

for $x_1, x_2 \geq 0, y_1 \in \mathbb{R}$;

$$\lim_{y_1 \to \infty} v(t, x_1, x_2, y_1) = 0,$$

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for $0 \leq t < T, x_1, x_2 \geq 0$; and

$$v(t, x_1, 0, y_1) = 0,$$

for $0 \leq t < T, x_1 \geq 0, y_1 \in \mathbb{R}$.

4.3 DENSITY FUNCTION METHOD

In this section we'll use a density function method to price the Asian-European spread options.

Consider the Black-Scholes-Merton model:

$$dS_1(t) = rS_1(t)dt + \sigma_1 S_1(t)dW_1(t),$$

$$dS_2(t) = rS_2(t)dt + \sigma_2 S_2(t)dW_2(t), \qquad (4.3.1)$$

where r is the interest rate, σ_1, σ_2 are the volatilities of the assets, and W_1 and W_2 are standard Brownian motions with correlation coefficient ρ, i.e. $E_{\mathbb{Q}}[dW_1 dW_2] = \rho dt$. Suppose the Asian style is for S_1; at first consider the payoff at time T for this spread call option is

$$V(T) = (\frac{1}{T} \int_0^T S_1(t)dt - S_2(T) - K)^+.$$

The solution of the stochastic differential equation (4.3.1) is a log bivariate normal vec-

tor conditional on its value at time t; moreover, $S_1(T)$ is also a log-normal random variable conditional on $S_2(T)$, $S_1(t)$ and $S_2(t)$:

$$\log(S_1(T)|_{S_1(t),S_2(t),S_2(T)}) \sim N(\mu_{1|2}, \sigma_{1|2}^2),$$

where

$$
\begin{aligned}
\mu_{1|2} &= \log(S_1(t)) + (r - \frac{1}{2}\sigma_1^2)(T-t) \\
&\quad + \frac{\rho\sigma_1}{\sigma_2}(\log(S_2(T)) - (\log(S_2(t)) - (r - \frac{1}{2}\sigma_2^2)(T-t))), \\
\sigma_{1|2} &= \sqrt{1-\rho^2}\,\sigma_1\sqrt{T-t}.
\end{aligned}
$$

$V(T)$ can be computed as follows:

$$
\begin{aligned}
V(T) &= (\frac{1}{T}\int_0^T S_1(u)du - S_2(T) - K)^+ \\
&= (\frac{1}{T}\int_0^t S_1(u)du + \frac{1}{T}\int_t^T S_1(u)du - S_2(T) - K)^+ \\
&= (\frac{1}{T}\int_t^T S_1(u)du - S_2(T) - (K - \frac{1}{T}\int_0^t S_1(u)du))^+ \\
&=: (\frac{1}{T}\int_t^T S_1(u)du - S_2(T) - K')^+,
\end{aligned}
$$

where

$$K' = K - \frac{1}{T}\int_0^t S_1(u)du.$$

Here K' is determined by the price of S_1 from time 0 to t, so it is known at time t. So from the solution of the stochastic differential equation, we have

$$
\begin{aligned}
V(T) &= (\frac{1}{T}\int_0^T S_1(u)du - S_2(T) - K)^+ \\
&= [\frac{1}{T}S_1(t)\int_0^{T-t}\exp(\sigma_1\hat{W}_1(s) + (r - \sigma_1^2/2)s)ds \\
&\quad - S_2(t)\exp(\sigma_2\hat{W}_2(s) + (r - \sigma_2^2/2)s) - K']^+,
\end{aligned}
$$

where

$$
\hat{W}_i(s) = W_i(t + s) - W_i(t),
$$

for $i = 1, 2$, $0 \le s \le T - t$ are both new Brownian motions independent of $\mathcal{F}(t)$. Then by the Lemma 3.3.1 we can change the conditional expectation to the regular expectation

$$
\begin{aligned}
V(t) &= E_{\mathbb{Q}}[e^{-r(T-t)}V(T)|\mathcal{F}(t)] \\
&= E_{\mathbb{Q}}[e^{-r(T-t)}\{\frac{1}{T}S_1(t)\int_0^{T-t}\exp(\sigma_1\hat{W}_1(s) + (r - \sigma_1^2/2)s)ds \\
&\quad - S_2(t)\exp(\sigma_2\hat{W}_2(s) + (r - \sigma_2^2/2)s) - K'\}^+|\mathcal{F}(t)] \\
&= E_{\mathbb{Q}}[e^{-r(T-t)}\{\frac{1}{T}S_1(t)\int_0^{T-t}\exp(\sigma_1\hat{W}_1(s) + (r - \sigma_1^2/2)s)ds \\
&\quad - S_2(t)\exp(\sigma_2\hat{W}_2(s) + (r - \sigma_2^2/2)s) - K'\}^+] \\
&= e^{-r(T-t)}\int_0^\infty\int_0^\infty [a_1 - s_2 - K]_+ f_{1|2}(a_1|s_2)f_2(s_2)da_1ds_2 \\
&= e^{-r(T-t)}\int_0^\infty\int_{s_2+K}^\infty [a_1 - (s_2 + K)]f_{1|2}(a_1|s_2)da_1 f_2(s_2)ds_2 \\
&=: \int_0^\infty F(s_2)f_2(s_2)ds_2,
\end{aligned}
$$

where

$$F(s_2) = e^{-r(T-t)} \int_{s_2+K}^{\infty} [a_1 - (s_2 + K)] f_{1|2}(a_1|s_2) da_1,$$

$$A_1 = \frac{1}{T} \int_0^T S_1(u) du.$$

Here $f_2(\cdot)$ represents the density function of $S_2(T)$ given the value of $S_2(t)$ which is a log-normal density function, and $f_{1|2}(\cdot|s_2)$ is the conditional density function of A_1 given the value of $S_2(T)$.

Note that $F(s_2)$ is the price of the Asian call option on S_1 with strike price $s_2 + K$. We use the result from Geman and Yor in [13] again to get

$$F(s_2) =: \frac{e^{-r(T-t)}}{T} \left(\frac{4S_1(t)}{\sigma_{1|2}^2} \right) C^{(\nu)}(h, q),$$

where

$$\nu = \frac{2r}{\sigma_{1|2}^2} - 1; \quad h = \frac{\sigma_{1|2}^2}{4}(T - t); \quad q = \frac{\sigma_{1|2}^2}{4S_1(t)}[(s_2 + K)T - \int_0^t S_1(u) du];$$

$$C^{(\nu)}(h, q) := E_{\mathbb{Q}}[(\int_0^h exp[2(W_s + \nu s)] ds - q)^+].$$

By using that result the Laplace transform of $C^{(\nu)}(h, q)$ with respect to the variable h is

$$\int_0^{\infty} e^{-\lambda h} C^{(\nu)}(h, q) dh = \frac{\int_0^{1/2q} e^{-x} x^{(\mu-\nu)/2-2} (1 - 2qx)^{(\mu+\nu)/2+1} dx}{\lambda(\lambda - 2 - 2\nu)\Gamma((\mu - \nu)/2 - 1)},$$

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where $\mu = \sqrt{2\lambda + \nu^2}$, and Γ is the gamma function, and we can get the value of $C^{(\nu)}(h, q)$ via inverse Laplace transform.

If the payoff of this spread option is of the form

$$V(T) = (S_2(T) - \frac{1}{T}\int_0^T S_1(t)dt - K)^+,$$

we consider that $S_2(T)$ is also a log-normal random variable conditional on $S_1(T)$ and $S_1(t), S_2(t)$:

$$\log(S_2(T)|_{S_1(t),S_2(t),S_1(T)}) \sim N(\mu_{2|1}, \sigma_{2|1}^2),$$

where

$$
\begin{aligned}
\mu_{2|1} &= \log(S_2(t)) + (r - \frac{1}{2}\sigma_2^2)(T - t) \\
&\quad + \frac{\rho\sigma_2}{\sigma_1}(\log(S_1(T)) - (\log(S_1(t)) - (r - \frac{1}{2}\sigma_1^2)(T - t))), \\
\sigma_{2|1} &= \sqrt{1 - \rho^2}\sigma_2\sqrt{T - t}.
\end{aligned}
$$

In this case, we have the option price

$$
\begin{aligned}
V(t) &= E_{\mathbb{Q}}[e^{-r(T-t)}V(T)|\mathcal{F}(t)] \\
&= E_{\mathbb{Q}}[e^{-r(T-t)}\{S_2(t)\exp(\sigma_2\hat{W}_2(s) + (r - \sigma_2^2/2)s) \\
&\quad - \frac{1}{T}S_1(t)\int_0^{T-t}\exp(\sigma_1\hat{W}_1(s) + (r - \sigma_1^2/2)s)ds - K'\}^+|\mathcal{F}(t)] \\
&= E_{\mathbb{Q}}[e^{-r(T-t)}\{S_2(t)\exp(\sigma_2\hat{W}_2(s) + (r - \sigma_2^2/2)s) \\
&\quad - \frac{1}{T}S_1(t)\int_0^{T-t}\exp(\sigma_1\hat{W}_1(s) + (r - \sigma_1^2/2)s)ds - K'\}^+] \\
&= e^{-r(T-t)}\int_0^\infty\int_0^\infty [s_2 - a_1 - K]_+ f_{2|1}(s_2|a_1)f_1(a_1)ds_2 da_1 \\
&= e^{-r(T-t)}\int_0^\infty\int_{a_1+K}^\infty [s_2 - (a_1 + K)]f_{2|1}(s_2|a_1)ds_2 f_1(a_1)da_1 \\
&=: \int_0^\infty F(a_1)f_1(a_1)da_1,
\end{aligned}
$$

where

$$
\begin{aligned}
F(a_1) &= e^{-r(T-t)}\int_{a_1+K}^\infty [s_2 - (a_1 + K)]f_{2|1}(s_2|a_1)ds_2, \\
A_1 &= \frac{1}{T}\int_0^T S_1(u)du.
\end{aligned}
$$

Here $f_1(\cdot)$ represents the density function of A_1, and $f_{2|1}(\cdot|a_1)$ is the conditional density function of $S_2(T)$ given the value of $S_1(T)$ which is a log-normal density function. Hence $F(a_1)$ is the Black-Scholes-Merton price of a call option on S_2 with strike price $a_1 + K$ and we have the closed form for it

$$
F(a_1) = e^{-r(T-t)}[\exp(\mu_{2|1} + \frac{1}{2}\sigma_{2|1})N(d_1(a_1)) + (a_1 + K)N(d_2(a_1))],
$$

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where

$$d_1(a_1) = \frac{\mu_{2|1} - \log(a_1 + K) + \sigma_{2|1}^2}{\sigma_{2|1}}$$

$$d_2(a_1) = d_1(a_1) - \sigma_{2|1}.$$

Then we can use the density function of A_1 to find out $V(t)$.

$$
\begin{aligned}
A_1 &= \frac{1}{T}\int_0^T S_1(u)du \\
&= \frac{1}{T}(\int_0^t S_1(u)du + S_1(t)\int_0^{T-t} \exp(\sigma_1\hat{W}(s) + (r - \sigma_1^2/2)s)ds) \\
&= \frac{1}{T}(\int_0^t S_1(u)du + S_1(t)\frac{4}{\sigma_1^2}\int_0^{\frac{(T-t)\sigma_1^2}{4}} \exp(2\hat{W}(s) + \frac{4(r - \sigma_1^2/2)s}{\sigma_1^2})ds) \\
&= \frac{1}{T}(\int_0^t S_1(u)du + S_1(t)\frac{4}{\sigma_1^2}A_{(T-t)\sigma_1^2/4}) := A_1(A).
\end{aligned}
$$

If we let the density function of $A_{(T-t)\sigma_1^2/4}$ be $f(\cdot)$, then we obtain

$$
\begin{aligned}
V(t) &= \int_0^\infty F(A_1(u))f(u)du \\
&= \int_{-\infty}^\infty dx \int_0^\infty dz F(A_1(z))a_\tau^\nu(x,z) \\
&= \int_{-\infty}^\infty dx \int_0^\infty dz F(A_1(z))\frac{1}{z}\exp(-\frac{1}{2z}(1 + \exp(2(x + \nu\tau))))\theta_{e^{x+\nu\tau}/z}(\tau) \\
&= \int_{-\infty}^\infty dx \int_0^\infty dz \\
&\quad F(\frac{1}{T}(\int_0^t S_1(u)du + S_1(t)\frac{4}{\sigma_1^2}z))\frac{1}{z}\exp(-\frac{1}{2z}(1 + \exp(2x)))\theta_{e^x/z}(\tau)
\end{aligned}
$$

where

$$\tau = \frac{\sigma_1^2(T-t)}{4}, \quad \nu = \frac{2(r - \sigma_1^2/2)}{\sigma_1^2},$$

$$\theta_r(u) = \frac{r}{(2\pi^3 u)^{\frac{1}{2}}} \exp(\frac{\pi^2}{2u}) \int_0^\infty \exp(-y^2/2u) \exp(-r(\cosh y))(\sinh y) \sin(\frac{\pi y}{u}) dy.$$

4.4 CHARACTERISTIC FUNCTION METHOD

In this section, we consider a similar affine structure stochastic volatility model as in section 3.4 to propose an analytic and computable result. Let's assume the Asian style payoff is for S_1.

The stochastic model is

$$dS_1(t) = rS_1(t)dt + \sigma_1\sqrt{\gamma(t)}dW_1(t),$$

$$dS_2(t) = rS_2(t)dt + \sigma_2\sqrt{\gamma(t)}dW_2(t),$$

$$d\gamma(t) = \kappa(\mu - \gamma(t))dt + \sigma_\gamma\sqrt{\gamma(t)}dW_\gamma(t),$$

$$dY(t) = S_1(t)dt,$$

where

$$E[dW_1(t)dW_2(t)] = \rho dt,$$

$$E[dW_1(t)dW_\gamma(t)] = \rho_1 dt,$$

$$E[dW_2(t)dW_\gamma(t)] = \rho_2 dt.$$

Taking the state vector $X_t = (S_1(t), S_2(t), \gamma(t), Y(t))^T$, this is an affine diffusion model,

$$dX_t = \Theta(X_t)dt + \Sigma(X_t)dw(t),$$

where

$$\Theta(X_t) = \begin{pmatrix} r & 0 & 0 & 0 \\ 0 & r & 0 & 0 \\ 0 & 0 & -\kappa & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} X_t + \begin{pmatrix} 0 \\ 0 \\ \kappa\mu \\ 0 \end{pmatrix} =: K_1 X_t + K_0,$$

$$\Sigma(X_t) = \begin{pmatrix} \sqrt{1-\rho_1^2}\sigma_1\sqrt{\gamma(t)} & 0 & \rho_1\sigma_1\sqrt{\gamma(t)} & 0 \\ \frac{\rho-\rho_1\rho_2}{\sqrt{1-\rho_1^2}}\sigma_2\sqrt{\gamma(t)} & \frac{\sqrt{1-\rho_1^2-\rho_2^2-\rho^2+2\rho_1\rho_2\rho}}{\sqrt{1-\rho_1^2}}\sigma_2\sqrt{\gamma(t)} & \rho_2\sigma_2\sqrt{\gamma(t)} & 0 \\ 0 & 0 & \sigma_\gamma\sqrt{\gamma(t)} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and $w(t) = (w_1(t), w_2(t), w_3(t), w_4(t))^T$ is a standard 4-dimensional Brownian motion. In this section, we call $\sigma_1\sqrt{\gamma(t)}$ and $\sigma_2\sqrt{\gamma(t)}$ the volatility of S_1 and S_2 respectively.

Note that

$$
\Sigma(X_t)\Sigma(X_t)^T = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 & \rho_1\sigma_1\sigma_\gamma & 0 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 & \rho_2\sigma_2\sigma_\gamma & 0 \\ \rho_1\sigma_1\sigma_\gamma & \rho_2\sigma_2\sigma_\gamma & \sigma_\gamma^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \gamma(t) =: H_1\gamma(t).
$$

Here we need the same assumption that the volatilities of S_1, S_2 are determined only by the process $\gamma(t)$ as we discussed in section 3.4.

Then under this model, fixing $T \in [0, \infty)$, Duffie's "characteristic" function is

$$
\begin{aligned}
\psi_1(u, X_t, t, T) &= E(\exp(-r(T-t))\exp(u \cdot X_T)|\mathcal{F}(t)) \\
&= \exp(\alpha(t, T, u) + \beta(t, T, u) \cdot X_t),
\end{aligned}
$$

where $u = (u_1, u_2, u_3, u_4)^T \in \mathbb{C}^4$, and $\beta(t, T, u) =: \beta_t = (\beta_t^{(1)}, \beta_t^{(2)}, \beta_t^{(3)}, \beta_t^{(4)})^T, \alpha(t, T, u) =: \alpha_t$ satisfy the following complex ordinary differential equations

$$
\dot{\beta}_t = \begin{pmatrix} \dot{\beta}_t^{(1)} \\ \dot{\beta}_t^{(2)} \\ \dot{\beta}_t^{(3)} \\ \dot{\beta}_t^{(4)} \end{pmatrix}
$$

$$
= -\begin{pmatrix} r & 0 & 0 & 1 \\ 0 & r & 0 & 0 \\ 0 & 0 & -\kappa & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \beta_t - \frac{1}{2} \begin{pmatrix} 0 \\ 0 \\ \sum_{i,j} \beta_t^{(i)} H_1(i,j) \beta_t^{(j)} \\ 0 \end{pmatrix}
$$

$$
\dot{\alpha}_t = r - \kappa \mu \beta_t^{(3)},
$$

under the boundary condition

$$
\beta_T = u
$$

$$
\alpha_T = 0.
$$

For the "extended characteristic" function, we have

$$
\begin{aligned}
\psi_2(v, u, X_t, t, T) &= E(\exp(-r(T-t))(v \cdot X_T) \exp(u \cdot X_T)|\mathcal{F}(t)) \\
&= \psi_1(u, X_t, t, T)(A(t, T, v, u) + B(t, T, v, u) \cdot X_t),
\end{aligned}
$$

where $v = (v_1, v_2, v_3, v_4)^T \in \mathbb{R}^4$, and $B(t, T, v, u) =: B_t = (B_t^{(1)}, B_t^{(2)}, B_t^{(3)}, B_t^{(4)})^T$, $A(t, T, v, u) =:$ A_t satisfy the following complex ordinary differential equations

$$
\dot{B}_t = \begin{pmatrix} \dot{B}_t^{(1)} \\ \dot{B}_t^{(2)} \\ \dot{B}_t^{(3)} \\ \dot{B}_t^{(4)} \end{pmatrix}
$$

$$
= - \begin{pmatrix} r & 0 & 0 & 1 \\ 0 & r & 0 & 0 \\ 0 & 0 & -\kappa & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} B_t - \begin{pmatrix} 0 \\ 0 \\ \sum_{i,j} \beta_t^{(i)} H_1(i,j) B_t^{(j)} \\ 0 \end{pmatrix}
$$

$$
\dot{A}_t = -\kappa \mu B_t^{(3)},
$$

under the boundary condition

$$
B_T = v
$$

$$
A_T = 0.
$$

These differential equations are a little bit different from the ones in section 3.4; also we can solve part of them numerically.

As long as we have these "characteristic functions", we can get the "generalized expected

present value" functions by

$$
\begin{aligned}
G^{(1)}_{a,b}(y; X_t, t, T) &= E(\exp(-r(T-t))\exp(a \cdot X(T))\mathbf{1}_{b \cdot X(T) \leq y}|\mathcal{F}(t)), \\
G^{(2)}_{a,b,d}(y; X_t, t, T) &= E(\exp(-r(T-t))(a \cdot X(T))\exp(d \cdot X_T)\mathbf{1}_{b \cdot X(T) \leq y}|\mathcal{F}(t)),
\end{aligned}
$$

where $a \in \mathbb{R}^4, b \in \mathbb{R}^4$ and $d \in \mathbb{R}^4$.

Note that the payoff of the Asian-European spread option with strike price c is of the form

$$
(b \cdot X_T - c)^+,
$$

where $b = (0, -1, 0, \frac{1}{T})^T$ for the payoff

$$
V(T) = (\frac{1}{T}\int_0^T S_1(t)dt - S_2(T) - c)^+,
$$

and $b = (0, 1, 0, -\frac{1}{T})^T$ for the payoff

$$
V(T) = (S_2(T) - \frac{1}{T}\int_0^T S_1(t)dt - c)^+.
$$

So we can compute the value of the function

$$
\begin{aligned}
C(X_0, b, c, 0, T) &= E(\exp(-rT)(b \cdot X_T - c)^+|\mathcal{F}(0)) \\
&= E(\exp(-rT)(b \cdot X_T - c)\mathbf{1}_{b \cdot X_T \geq c}|\mathcal{F}(0)) \\
&= G^{(2)}_{b,-b,\vec{0}}(-c; X_0, 0, T) - cG^{(1)}_{\vec{0},-b}(-c; X_0, 0, T).
\end{aligned}
$$

Recall section 3.4 for the way of computing $G^{(1)}, G^{(2)}$ when the "characteristic" functions ψ_1, ψ_2 are given:

$$
\begin{aligned}
G_{a,b}^{(1)}(y; X_t, t, T) &= \frac{\psi_1(a, X_t, t, T)}{2} - \frac{1}{\pi} \int_0^\infty \frac{Im[\psi_1(a + ivb, X_t, t, T)\exp(-ivy)]}{v} dv \\
G_{a,b,d}^{(2)}(y; X_t, t, T) &= \frac{\psi_2(a, d, X_t, t, T)}{2} - \frac{1}{\pi} \int_0^\infty \frac{Im[\psi_2(a, d + ivb, X_t, t, T)\exp(-ivy)]}{v} dv.
\end{aligned}
$$

Chapter 5. Numerical Computations

In this chapter, we show the numerical results of pricing based on the methods introduced in previous chapters. In section 1, we discuss the Monte Carlo simulation method result used in pricing Asian-European spread options. In section 2, we implement the density function method to compute the Asian-European spread option based on section 2.3. In section 3, we implement the characteristic function method to price Asian spread options under a stochastic volatility model based on section 3.4. All the code is written in Visual C++ and the computations are conducted on an Intel Pentium 4 2.40 GHz CPU with 768 MB RAM.

5.1 Monte Carlo Simulation Method

In this section, general Monte Carlo simulation is used to price Asian-European spread options under the Black-Scholes-Merton model:

$$dS_1(t) = rS_1(t)dt + \sigma_1 S_1(t)dW_1(t),$$

$$dS_2(t) = rS_2(t)dt + \sigma_2 S_2(t)dW_2(t), \tag{5.1.1}$$

where r is the interest rate, σ_1, σ_2 are volatilities of the assets, and W_1 and W_2 are standard Brownian motions with correlation coefficient ρ, i.e. $E_{\mathbb{Q}}[dW_1 dW_2] = \rho dt$. Suppose the Asian style is for S_1; at first consider the payoff at time T for this spread call option is

$$V(T) = (\frac{1}{T} \int_0^T S_1(t)dt - S_2(T) - K)^+.$$

This option price at time 0 is therefore

$$V(0) = E_{\mathbb{Q}}[e^{-rT}V(T)|\mathcal{F}(0)].$$

Researchers have employed Monte Carlo simulation methods in single asset Asian option pricing, see [3],[4],[12] and [17].

The solution of equation (5.1.1) is log-normal, that is, letting $s_i(t) := \log(S_i(t))$ for $i = 1, 2$, we have

$$ds_1(t) = (r - \frac{1}{2}\sigma_1^2)dt + \sigma_1 dW_1(t),$$
$$ds_2(t) = (r - \frac{1}{2}\sigma_2^2)dt + \sigma_2 dW_2(t),$$

and hence

$$s_i(T) = s_i(t) + (r - \delta_i - \frac{1}{2}\sigma_i^2)(T - t) + \sigma_i[W_i(T) - W_i(t)]$$

for $i = 1, 2$. By the independence of the Brownian motion increments $W(T) - W(t)$ and $\mathcal{F}(t)$, given the value $s_1(t), s_2(t)$, we have

$$s(T) = \mu(t) + \Sigma(t) \cdot w \qquad (5.1.2)$$

where

$$\mu(t) \;=\; \begin{pmatrix} s_1(t) + (r - \frac{1}{2}\sigma_1^2)(T-t) \\[2mm] s_2(t) + (r - \frac{1}{2}\sigma_2^2)(T-t) \end{pmatrix},$$

$$\Sigma(t) \;=\; \begin{pmatrix} \sigma_1\sqrt{T-t} & 0 \\[2mm] \rho\sigma_2\sqrt{T-t} & \sqrt{1-\rho^2}\sigma_2\sqrt{T-t} \end{pmatrix},$$

$$w \;=\; \begin{pmatrix} w_1 \\[2mm] w_2 \end{pmatrix},$$

and $s(T) = (s_1(T), s_2(T))^T$, w_1, w_2 are independent standard normal random variables. So based on (5.1.2), we use

$$S_1(t+h) \;=\; S_1(t)\exp\{rh + \sigma_1\sqrt{h}w_1 - \frac{\sigma_1^2 h}{2}\}$$

$$S_2(t+h) \;=\; S_2(t)\exp\{rh + \sigma_2\sqrt{h}\rho w_1 + \sigma_2\sqrt{h}\sqrt{1-\rho^2}w_2 - \frac{\sigma_2^2 h}{2}\}$$

to update the assets' price.

Table 5.1 shows the numerical result of Monte Carlo simulation. M is the total number of replications of sample path, N is the number of price reading per day. "Time" is the computation time with unit second.

Full results are shown in figure 5.1. Since when $M = 10000$, the result value is too far away from the "converged value", figure 5.2, which has the results for $M \geq 50000$, gives a clearer picture.

N=1			N=10			N=100		
M	Value	Time	M	Value	Time	M	Value	Time
10000	13.2842	1	10000	13.2260	10	10000	13.0188	94
50000	13.1763	5	50000	13.1594	47	50000	13.2095	468
100000	13.1866	10	100000	13.1428	93	100000	13.2045	937
200000	13.1862	18	200000	13.1918	188	200000	13.1895	1864
500000	13.1773	47	500000	13.1892	468	500000	13.1932	4678
1000000	13.1810	95	1000000	13.1932	1048	1000000	13.1896	9707

Table 5.1 Monte Carlo simulation results.

$$S_1(0) = 100, S_2(0) = 80, K = 10, T = 0.4\text{year}$$

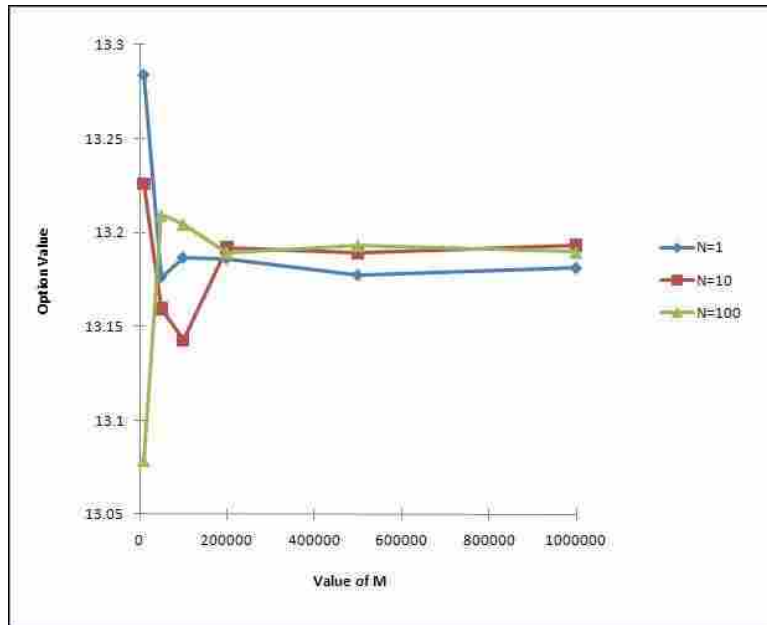$$r = 0.09, \sigma_1 = 0.2, \sigma_2 = 0.4, \rho = 0.3$$
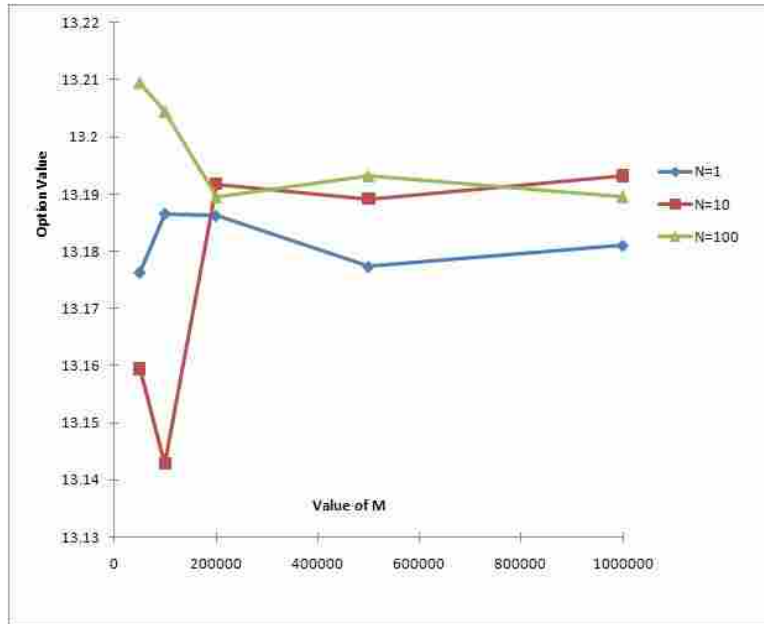

Figure 5.1 Full result

Figure 5.2 M≥50000 result

From the table and the graph, it is easy to see the result has not converged very well. For $N = 1$, i.e. one reading data daily case, even one million sample paths gives the result 13.1810 which is 1 cent away from the result 13.1932 when daily data reading frequency is 10 and the number of sample paths is one million and the result 13.1896 when daily data reading frequency is 100 and the number of sample paths is one million. In other words, the variation of the Monte Carlo simulation is too big to converge quickly. Besides, the computation time cost is expensive. From table 5.1, to get within 1 cent, the simulation result at least needs $N = 10$ and $M = 200000$, i.e. 10 daily data reading frequency and 200000 sample paths which need computation time about 200 seconds.

Actually, this is not a strange result. In [17], Kemna and Vorst got a similar result when using Monte Carlo simulation for pricing an Asian option of a single asset. Now that we are

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dealing with Asian-European spread option about two assets, it's natural to find the volatile property of the Monte Carlo simulation result. Also in [17], Kemna and Vorst applied a variance reduction method to solve this volatile problem. They used the geometric average of the stock price as a control variable to reduce the variance of the Monte Carlo simulation. Here, for the first time, we use the geometric average of S_1 as a control variable to get a much better Monte Carlo simulation method for pricing the Asian-European spread option.

Under the Black-Scholes-Merton model (5.1.1), Asian-European spread option pricing is to find the value of conditional expectation

$$V(0) = E_{\mathbb{Q}}[e^{-rT}(\frac{1}{T}\int_0^T S_1(t)dt - S_2(T) - K)^+|\mathcal{F}(0)].$$

Then we have

$$
\begin{aligned}
V(0) &= E_{\mathbb{Q}}[e^{-rT}(\frac{1}{T}\int_0^T S_1(t)dt - S_2(T) - K)^+|\mathcal{F}(0)] \\
&= e^{-rT}E_{\mathbb{Q}}[(\frac{1}{T}\int_0^T S_1(t)dt - S_2(T) - K)^+ - (G(T) - S_2(T) - K)^+|\mathcal{F}(0)] \\
&\quad + e^{-rT}E_{\mathbb{Q}}[(G(T) - S_2(T) - K)^+|\mathcal{F}(0)]
\end{aligned}
$$

where $G(T)$ is the geometric average of S_1 and is approximated by $(\prod_{i=0}^n S_1(t_i))^{1/(n+1)}$. Since $G(T)$ is lower than $\frac{1}{T}\int_0^T S_1(t)dt$, $e^{-rT}E_{\mathbb{Q}}[(G(T) - S_2(T) - K)^+|\mathcal{F}(0)]$ is therefore a lower bound of the Asian-European spread option price and we name it $V_G(0)$. Furthermore, this lower bound is actually a price of a European spread option which can be computed efficiently. So we just need to use Monte Carlo simulation to find the value of

$e^{-rT}E_{\mathbb{Q}}[(\frac{1}{T}\int_0^T S_1(t)dt - S_2(T) - K)^+ - (G(T) - S_2(T) - K)^+|\mathcal{F}(0)]$, which is the difference of the Asian-European spread option price and its lower bound $V_G(0)$, then add it to the the lower bound value to get the Asian-European spread option value.

For the value of lower bound $V_G(0)$, we obtain

$$
\begin{aligned}
V_G(0) &= e^{-rT}E_{\mathbb{Q}}[(G(T) - S_2(T) - K)^+|\mathcal{F}(0)] \\
&= e^{-rT}\int_0^\infty \int_0^\infty [g - s_2 - K]^+ f_{1,2}(g, s_2)dgds_2 \\
&= e^{-rT}\int_0^\infty \int_0^\infty [g - s_2 - K]^+ f_{1|2}(g|s_2)f_2(s_2)dgds_2 \\
&= e^{-rT}\int_0^\infty \int_{s_2+K}^\infty [g - (s_2 + K)]f_{1|2}(g|s_2)dgf_2(s_2)ds_2 \\
&=: \int_0^\infty F(s_2)f_2(s_2)ds_2, \quad\quad\quad\quad (5.1.3)
\end{aligned}
$$

where

$$
F(s_2) := e^{-rT}\int_{s_2+K}^\infty [g - (s_2 + K)]f_{1|2}(g|s_2)dg,
$$

$f_{1,2}(\cdot, \cdot)$ is the joint density function of $G(T)$ and $S_2(T)$ conditional on $S_1(0)$ and $S_2(0)$; $f_2(\cdot)$ represents the density function of $S_2(T)$ given the value of $S_2(t)$ which is a known log-normal density function. $f_{1|2}(\cdot|s_2)$ is the conditional density function of G given value of S_2 which is also a known log-normal density function. Note that $F(s_2)$ is a European option value with strike price $s_2 + K$ for the geometric average of S_1 as the underlying. So we can easily use the Black-Scholes formula to find the value of $F(s_2)$ (this value is a function of s_2).

First, conditional on $S_2(T)$, $\ln S_1(T)$ is normally distributed with mean and standard devi-

ation

$$\mu_{1|2} = \ln S_1(0) + (r - \frac{1}{2}\sigma_1^2)T + \frac{\rho\sigma_1}{\sigma_2}(\ln \frac{S_2(T)}{S_2(0)} - (r - \frac{\sigma_2^2}{2})T)$$

$$\sigma_{1|2} = \sqrt{1 - \rho^2}\sigma_1\sqrt{T}.$$

Then we have

$$F(s_2) = e^{-rT}[\exp(\mu_G + \frac{1}{2}\sigma_G^2)N(d_1) - K_G N(d_1 - \sigma_G)]$$

where

$$
\begin{aligned}
\mu_G &= \ln S_1(0) + (r + \frac{\rho\sigma_1(\ln \frac{S_2(T)}{S_2(0)} - (r - \frac{\sigma_2^2}{2})T)}{\sigma_2 T} - \frac{\sigma_1^2}{2})\frac{T}{2} \\
\sigma_G &= \sqrt{\frac{1 - \rho^2}{3}}\sigma_1 T \\
K_G &= K + s_2 \\
d_1 &= \frac{\mu_G - \ln K_G + \frac{1}{2}\sigma_G^2}{\sigma_G}.
\end{aligned}
$$

$N(\cdot)$ is the cumulative distribution function of the standard normal. See [10] and [17] for the detail of geometric average Asian options. At last we use Gauss-Legendre quadrature method to approximate the value of the righthand side of (5.1.3).

N=1			N=10			N=100		
M	Value	Time	M	Value	Time	M	Value	Time
10000	13.1813	1	10000	13.1801	11	10000	13.1783	113
50000	13.1801	6	50000	13.1799	57	50000	13.1795	554
100000	13.1799	11	100000	13.1799	115	100000	13.1796	965
200000	13.1803	23	200000	13.1797	225	200000	13.1795	1953
500000	13.1803	57	500000	13.1796	486	500000	13.1796	4822
1000000	13.1806	113	1000000	13.1798	975	1000000	13.1797	9320

Table 5.2 Improved Monte Carlo simulation results.

$$S_1(0) = 100, S_2(0) = 80, K = 10, T = 0.4\text{year}$$

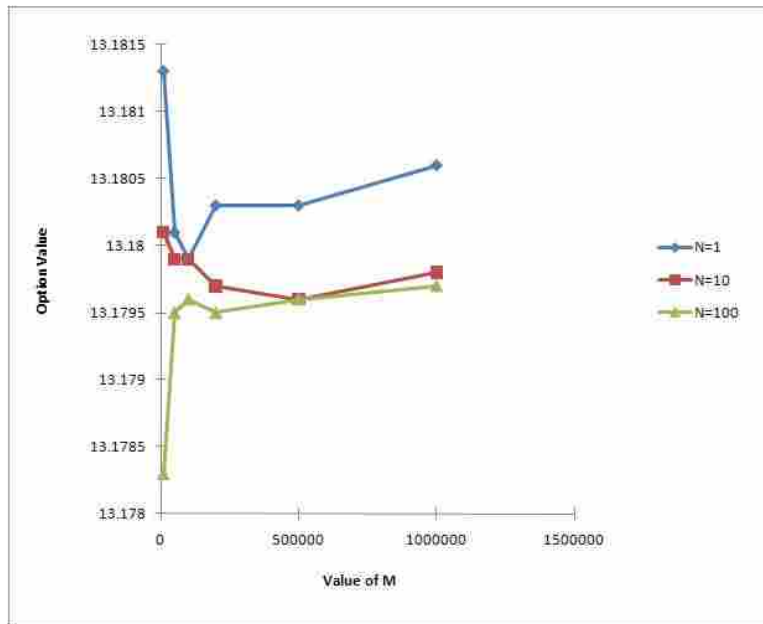$$r = 0.09, \sigma_1 = 0.2, \sigma_2 = 0.4, \rho = 0.3$$

Figure 5.3 Improved Monte Carlo Result

Full numerical results of this improved Monte Carlo simulation is in Table 5.2.

The graph of table 5.2 is figure 5.3. This improved Monte Carlo method converges so well that all the results' difference is less than 1 cent. Notice that this value, 13.18, is 1 cent away from the original Monte Carlo simulation result, 13.19. Later you will see 13.18 will match the result computed by the density function method in section 5.2.

The comparison of the original Monte Carlo simulation and the improved Reduced Variance Monte Carlo simulation is in figure 5.4, figure 5.5 and figure 5.6. It's clear that the improved Monte Carlo method is much more efficient and accurate than the original one. Figure 5.7 and 5.8 are the surface graph of option price for different values of T when $N = 1, M = 10000$. You can compare them with figure 5.9 which is just the payoff value since $T = 0$.
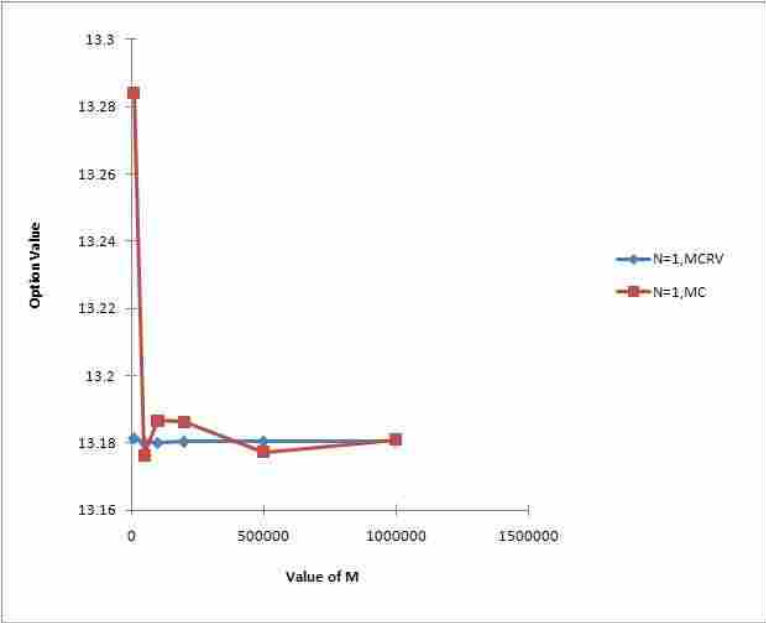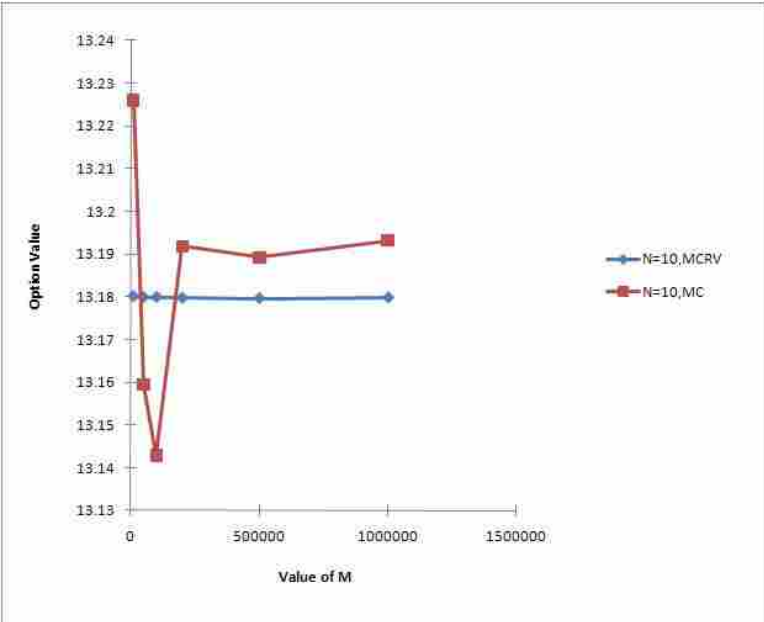
Figure 5.4 MC and Improved MC N=1
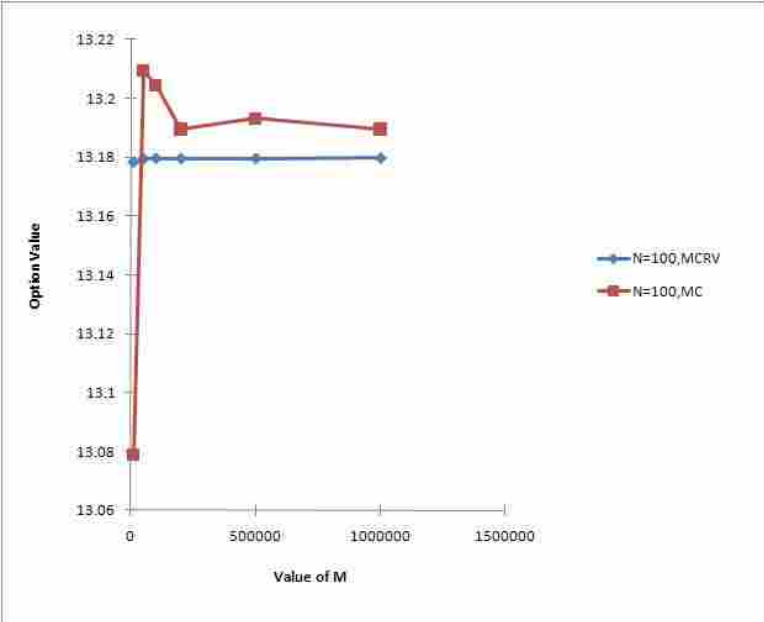

Figure 5.5 MC and Improved MC N=10
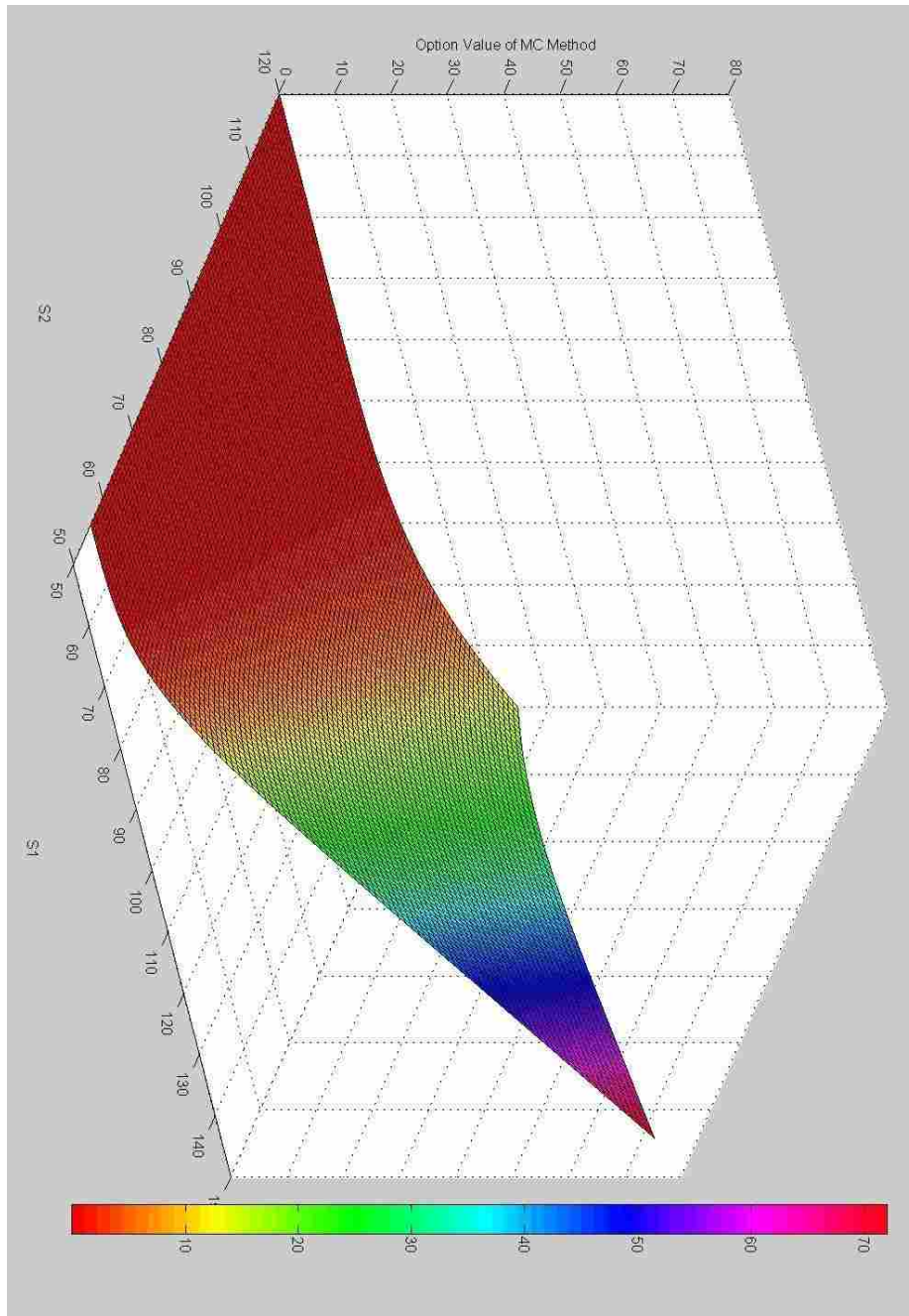

Figure 5.6 MC and Improved MC N=100

Figure 5.7 Improved Monte Carlo Result, $T = 0.4$ year.

$r = 0.09; \sigma_1 = 0.3; \sigma_2 = 0.3; \rho = 0.3; K = 20.$
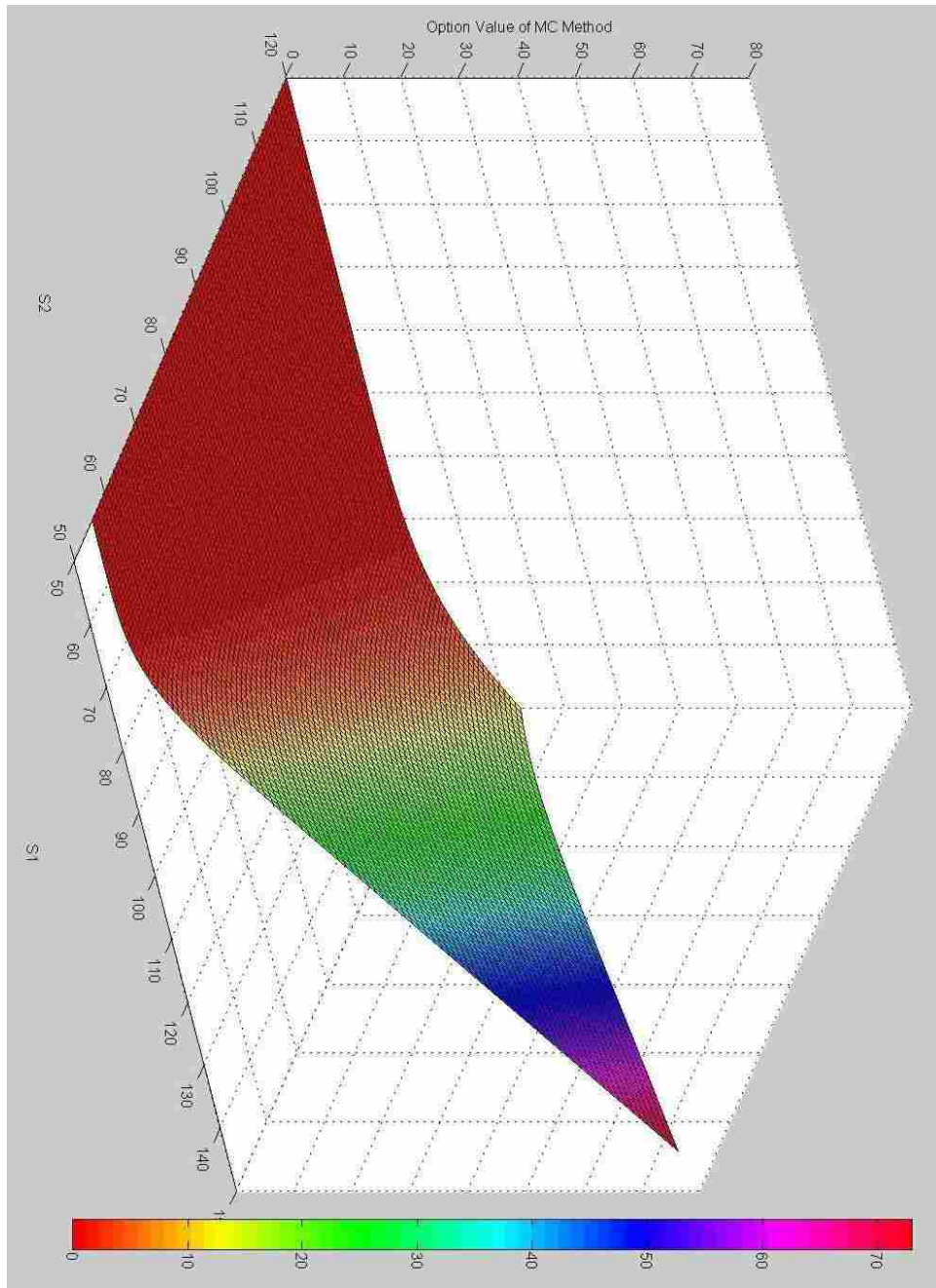
Figure 5.8 Improved Monte Carlo Result, $T = 0.2$ year.

$r = 0.09; \sigma_1 = 0.3; \sigma_2 = 0.3; \rho = 0.3; K = 20.$
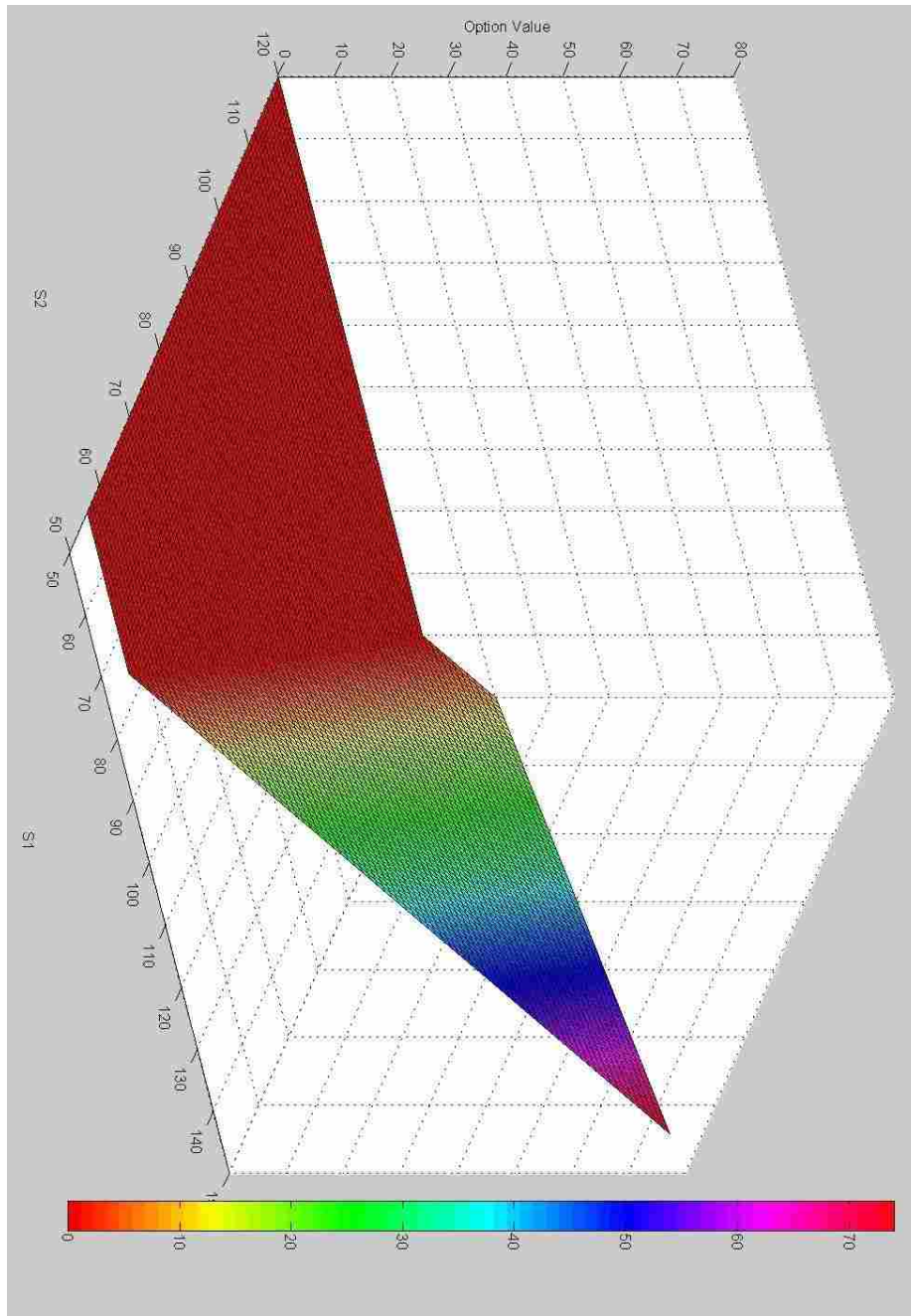
Figure 5.9 Option Value Result, $T = 0$ year.

$r = 0.09; \sigma_1 = 0.3; \sigma_2 = 0.3; \rho = 0.3; K = 20.$

Figure 5.10 Improved Monte Carlo Option Value Result (in the money), $K = 15$.

$r = 0.03; T = 0.2; \rho = 0.5; S_1(0) = 100; S_2(0) = 80.$

Figure 5.11 Improved Monte Carlo Option Value Result (at the money), $K = 20$.

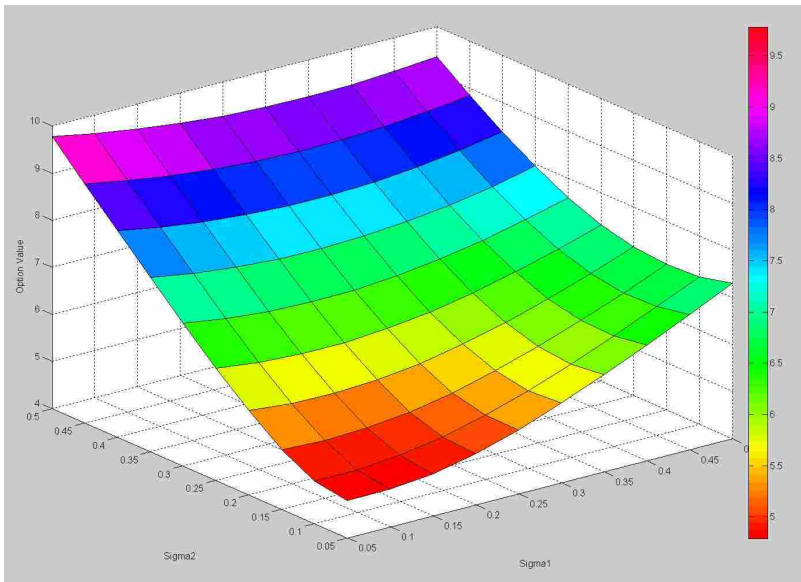$r = 0.03; T = 0.2; \rho = 0.5; S_1(0) = 100; S_2(0) = 80.$

Figure 5.12 Improved Monte Carlo Option Value Result (out of the money), $K = 25$.

$r = 0.03; T = 0.2; \rho = 0.5; S_1(0) = 100; S_2(0) = 80.$

Figure 5.10 is the option value for different value of volatilities σ_1 and σ_2 when the option is in the money. Figure 5.11 is the option value when the option is at the money. Figure 5.12 is the option value when the option is out of the money.

Figure 5.13 is the option value for different values of the interest rate r and correlation coefficient ρ when the option is in the money. Figure 5.14 is the option value when the option is at the money. Figure 5.15 is the option value when the option is out of the money. Figure 5.16 is the option value for different values of the strike price K and expire time T.

Figure 5.13 Improved Monte Carlo Option Value Result (in the money), $K = 15$.

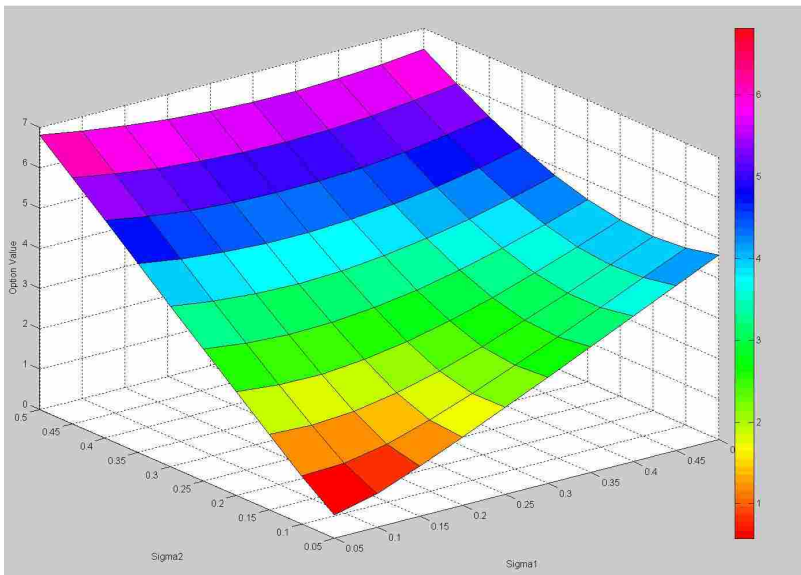$\sigma_1 = 0.3; \sigma_2 = 0.3; T = 0.2; S_1(0) = 100; S_2(0) = 80.$
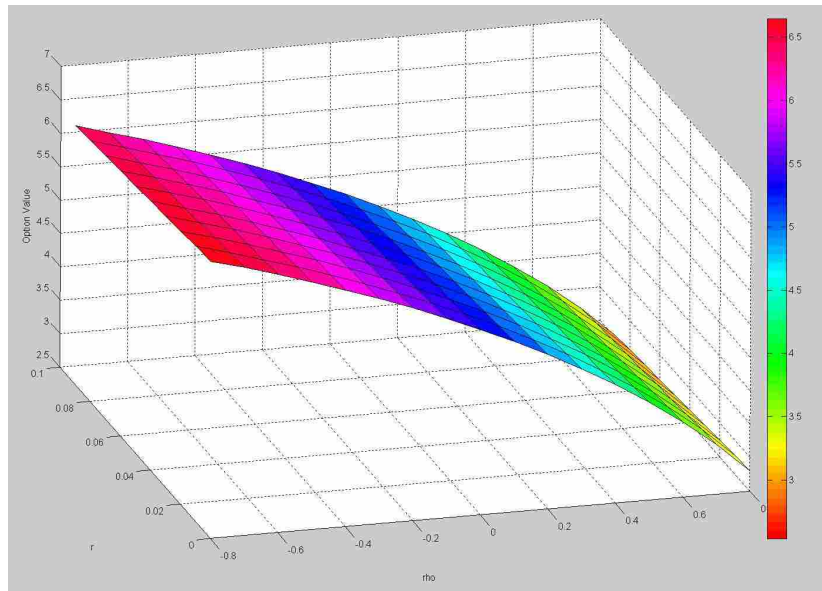
Figure 5.14 Improved Monte Carlo Option Value Result (at the money), $K = 20$.

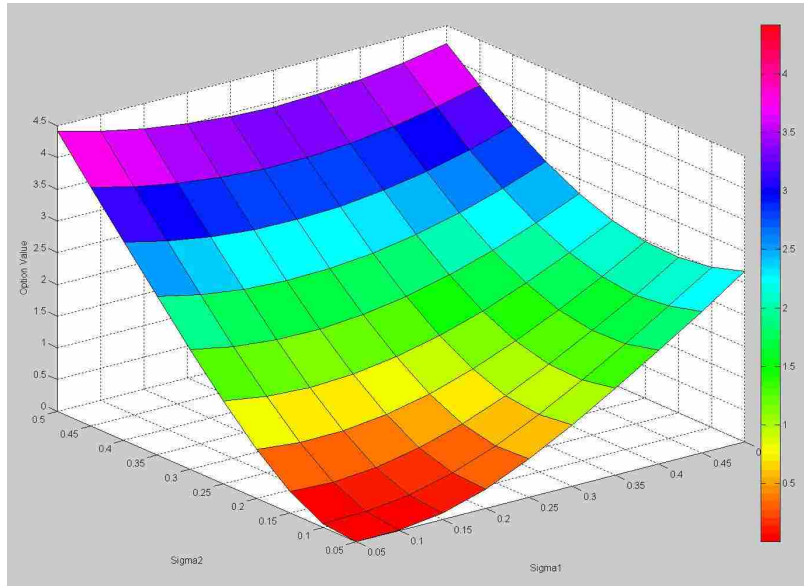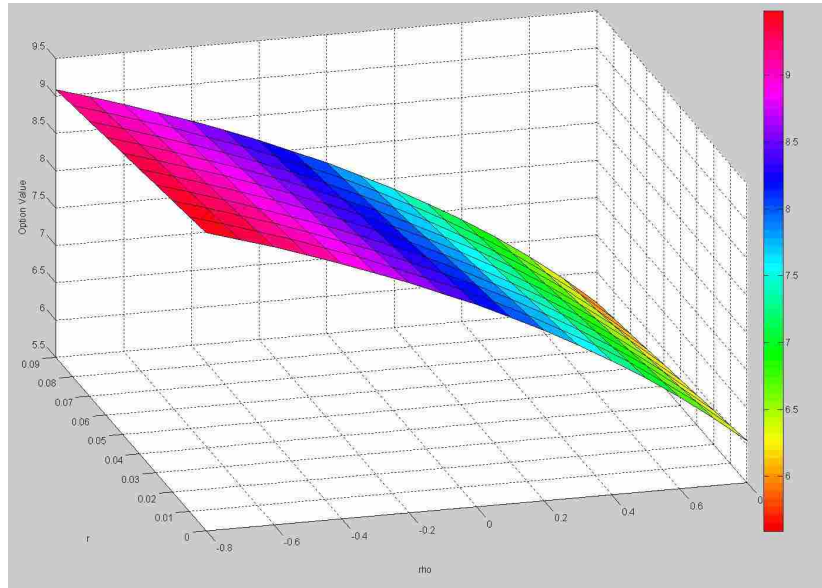$\sigma_1 = 0.3; \sigma_2 = 0.3; T = 0.2; S_1(0) = 100; S_2(0) = 80.$
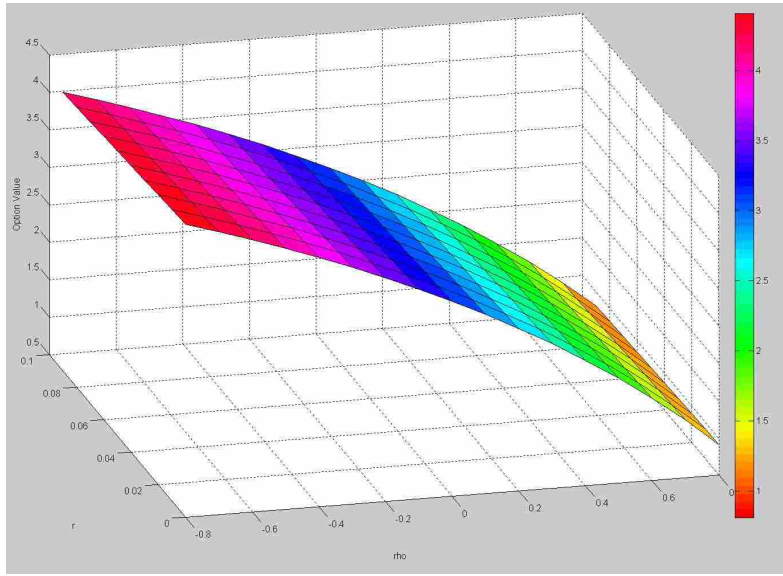
Figure 5.15 Improved Monte Carlo Option Value Result (out of the money), $K = 25$.

$\sigma_1 = 0.3; \sigma_2 = 0.3; T = 0.2; S_1(0) = 100; S_2(0) = 80.$
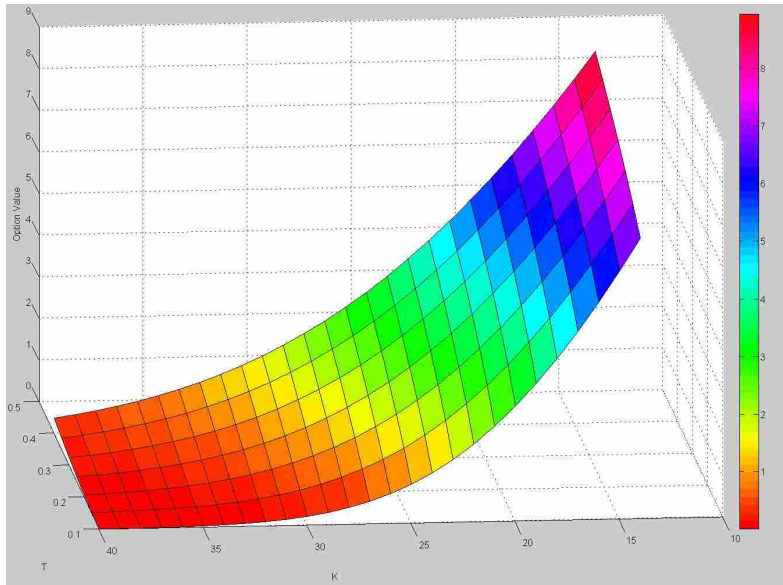
Figure 5.16 Improved Monte Carlo Option Value Result.

$r = 0.03; \sigma_1 = 0.3; \sigma_2 = 0.3; \rho = 0.5; S_1(0) = 100; S_2(0) = 80.$

5.2 Density Function Method Numerical Computation

Recall from section 4.3, the Black-Scholes-Merton model:

$$dS_1(t) = rS_1(t)dt + \sigma_1 S_1(t)dW_1(t),$$

$$dS_2(t) = rS_2(t)dt + \sigma_2 S_2(t)dW_2(t),$$

where r is the interest rate, σ_1, σ_2 are volatilities of the assets, and W_1 and W_2 are standard Brownian motions with correlation coefficient ρ, i.e. $E_{\mathbb{Q}}[dW_1 dW_2] = \rho dt$. Suppose the Asian style is for S_1, the European style is for S_2; then the payoff at time T for this Asian-European spread option is

$$V(T) = (\frac{1}{T}\int_0^T S_1(t)dt - S_2(T) - K)^+. \tag{5.2.1}$$

The price of this Asian-European spread option at time 0 is therefore

$$
\begin{aligned}
V(0) &= E_{\mathbb{Q}}[e^{-rT}V(T)|\mathcal{F}(0)] \\
&= e^{-rT}\int_0^\infty \int_{s_2+K}^\infty [a_1 - (s_2+K)]f_{1|2}(a_1|s_2)da_1 f_2(s_2)ds_2 \\
&=: \int_0^\infty F(s_2)f_2(s_2)ds_2, \tag{5.2.2}
\end{aligned}
$$

where

$$
\begin{aligned}
F(s_2) &= e^{-r(T-t)}\int_{s_2+K}^\infty [a_1 - (s_2+K)]f_{1|2}(a_1|s_2)da_1, \\
A_1 &= \frac{1}{T}\int_0^T S_1(u)du.
\end{aligned}
$$

$f_2(\cdot)$ represents the density function of $S_2(T)$ given the value of $S_2(0)$, which is a log-normal density function, and $f_{1|2}(\cdot|s_2)$ is the conditional density function of A_1 given the value of $S_2(T)$.

Then we use the fact that $F(s_2)$ is the price of the Asian call option on S_1 with strike price $s_2 + K$. From Geman and Yor in [13]:

$$F(s_2) =: \frac{e^{-r(T-t)}}{T}(\frac{4S_1(t)}{\sigma_{1|2}^2})C^{(\nu)}(h,q), \tag{5.2.3}$$

where

$$\nu = \frac{2r}{\sigma_{1|2}^2} - 1; \quad h \;=\; \frac{\sigma_{1|2}^2}{4}(T-t); \quad q = \frac{\sigma_{1|2}^2}{4S_1(t)}[(s_2+K)T - \int_0^t S_1(u)du];$$

$$C^{(\nu)}(h,q) \;:=\; E_{\mathbb{Q}}[(\int_0^h \exp[2(W_s + \nu s)]ds - q)^+].$$

By using that result the Laplace transform of $C^{(\nu)}(h,q)$ with respect to the variable h is

$$\int_0^\infty e^{-\lambda h}C^{(\nu)}(h,q)dh = \frac{\int_0^{1/2q} e^{-x}x^{(\mu-\nu)/2-2}(1-2qx)^{(\mu+\nu)/2+1}dx}{\lambda(\lambda-2-2\nu)\Gamma((\mu-\nu)/2-1)},$$

where $\mu = \sqrt{2\lambda + \nu^2}$, Γ is the gamma function, and we can get the value of $C^{(\nu)}(h,q)$ via inverse Laplace transform.

The key difficulty here is this inverse Laplace transform. So here we need an efficient way to numerically compute it with high accuracy. In [12], Fu, Madan and Wang compared several methods to numerically compute this kind of Asian option price. Based on their comparison

results and recommendation, we use the method of Euler and Post-Widder from [1] by Abate and Whitt:

If $\hat{f}(\lambda)$ is the Laplace transform of $f(y)$, then we can approximate $f(y)$ by

$$f(y) \cong \sum_{k=0}^{m} C(m, k) 2^{-m} s_{n+k}(y),$$

where

$$
\begin{aligned}
s_n(y) &= \frac{e^{A/2}}{2y} Re\{\hat{f}(\frac{A}{2y})\} + \frac{e^{A/2}}{y} \sum_{k=1}^{n}(-1)^k a_k(y) \\
a_k(y) &= Re\{\hat{f}(\frac{A + 2k\pi i}{2y})\},
\end{aligned}
$$

and $C(m, k) = \frac{m!}{k!(m-k)!}$ is the combination number; the choices of the constant m, n and A are $m = 11, n = 15$ and $A = 18.4$. After we have the value of $C^{(\nu)}(h, q)$, we plug it into (5.2.3) to get $F(s_2)$, then use a Gauss-Legendre quadrature method in (5.2.2) to get the price of this Asian-European spread option at time 0.

Tables 5.3 through 5.5 are some numerical results. In the tables, V_d is the Density Function method result, V_{MC1} is the improved Monte Carlo method result with $N = 1, M = 10000$, V_{MC2} is the improved Monte Carlo method result with $N = 1, M = 50000$, and V_{MC3} is the improved Monte Carlo method result with $N = 10, M = 10000$. The time spent for V_d is less than 2 seconds (which is similar to V_{MC1}), for V_{MC2} is 6 seconds, and for V_{MC3} is 11 seconds. From the tables, the results from the Density Function method and the improved Monte Carlo methods match very well. In most cases, the difference is less than 1 cent.

ρ	K	V_d	V_{MC1}	V_{MC2}	V_{MC3}
$r = 1.03$					
	10	14.0766	14.0772	14.0787	14.0794
0.2	20	7.86881	7.87023	7.87020	7.87035
	30	3.56675	3.56783	3.56774	3.56641
	10	13.8506	13.8548	13.8548	13.8544
0.3	20	7.60510	7.60850	7.60788	7.60702
	30	3.32650	3.33006	3.32891	3.32841
	10	13.6139	13.6227	13.6208	13.6225
0.4	20	7.32801	7.33222	7.33387	7.33242
	30	3.07605	3.07987	3.07866	3.07841
$r = 1.06$					
	10	13.7383	13.7387	13.7402	13.7411
0.2	20	7.67582	7.67721	7.67727	7.67748
	30	3.48465	3.48580	3.48554	3.48438
	10	13.5105	13.5146	13.5146	13.5143
0.3	20	7.41278	7.41607	7.41541	7.41464
	30	3.24679	3.25053	3.24918	3.24853
	10	13.2718	13.2804	13.2786	13.2802
0.4	20	7.13644	7.14038	7.14224	7.14043
	30	2.99900	3.00278	3.00150	3.00132
$r = 1.09$					
	10	13.4061	13.4059	13.4075	13.4086
0.2	20	7.48637	7.48781	7.48788	7.48813
	30	3.40391	3.40499	3.40480	3.40383
	10	13.1762	13.1802	13.1802	13.1798
0.3	20	7.22407	7.22712	7.22668	7.22589
	30	3.16846	3.17218	3.17081	3.17003
	10	12.9356	12.9437	12.9422	12.9440
0.4	20	6.94857	6.95242	6.95436	6.95231
	30	2.92336	2.92703	2.92583	2.92566

Table 5.3 Comparison of different methods(1).
$S_1(0) = 100, S_2(0) = 80, T = 0.4$year
$\sigma_1 = 0.2, \sigma_2 = 0.4.$

σ_2	K	V_d	V_{MC1}	V_{MC2}	V_{MC3}
$r = 1.03$					
	10	11.1607	11.1663	11.1693	11.1727
0.2	20	4.87133	4.87479	4.87595	4.87592
	30	1.53131	1.53089	1.53418	1.53098
	10	12.4184	12.4290	12.4271	12.4262
0.3	20	6.23150	6.23482	6.23496	6.23381
	30	2.42793	2.43177	2.43036	2.43057
	10	14.0061	14.0168	14.0123	14.0154
0.4	20	7.86221	7.86605	7.86887	7.86740
	30	3.64298	3.64442	3.64497	3.64344
$r = 1.06$					
	10	10.7871	10.7927	10.7955	10.7996
0.2	20	4.68736	4.69062	4.69184	4.69188
	30	1.47254	1.47222	1.47532	1.47226
	10	12.0657	12.0763	12.0744	12.0732
0.3	20	6.04313	6.04654	6.04635	6.04500
	30	2.35638	2.36040	2.35875	2.35889
	10	13.6676	13.6782	13.6735	13.6766
0.4	20	7.66916	7.67259	7.67565	7.67411
	30	3.55928	3.56072	3.56107	3.55970
$r = 1.09$					
	10	10.4211	10.4265	10.4291	10.4339
0.2	20	4.50840	4.51129	4.51268	4.51295
	30	1.41549	1.41545	1.41823	1.41558
	10	11.7197	11.7305	11.7283	11.7268
0.3	20	5.85892	5.86205	5.86204	5.86001
	30	2.28639	2.29082	2.28873	2.28870
	10	13.3350	13.3448	13.3407	13.3435
0.4	20	7.47964	7.48278	7.48600	7.48415
	30	3.47696	3.47847	3.47866	3.47708

Table 5.4 Comparison of different methods(2).
$S_1(0) = 100, S_2(0) = 80, T = 0.4\text{year}$
$\sigma_1 = 0.3, \rho = 0.3.$

σ_1	K	V_d	V_{MC1}	V_{MC2}	V_{MC3}
$r = 1.03$					
	10	12.1298	12.1326	12.1341	12.1357
0.2	20	5.77679	5.78026	5.77965	5.77991
	30	1.96243	1.96439	1.96405	1.96351
	10	12.4184	12.4290	12.4271	12.4262
0.3	20	6.23150	6.23482	6.23496	6.23381
	30	2.42793	2.43177	2.43036	2.43057
	10	12.9141	12.9264	12.9245	12.9250
0.4	20	6.94438	6.94508	6.94976	6.95039
	30	3.14225	3.14081	3.14253	3.14051
$r = 1.06$					
	10	11.7723	11.7751	11.7765	11.7783
0.2	20	5.58982	5.59310	5.59266	5.59295
	30	1.89850	1.90041	1.90006	1.89951
	10	12.0657	12.0763	12.0744	12.0732
0.3	20	6.04313	6.04654	6.04635	6.04500
	30	2.35638	2.36040	2.35875	2.35889
	10	12.5685	12.5808	12.5787	12.5789
0.4	20	6.75394	6.75437	6.75912	6.75978
	30	3.06105	3.05972	3.06142	3.05921
$r = 1.09$					
	10	11.4215	11.4240	11.4256	11.4273
0.2	20	5.40723	5.41057	5.41018	5.41019
	30	1.83614	1.83805	1.83766	1.83733
	10	11.7197	11.7305	11.7283	11.7268
0.3	20	5.85892	5.86205	5.86204	5.86001
	30	2.28639	2.29082	2.28873	2.28870
	10	12.2292	12.2411	12.2393	12.2394
0.4	20	6.56737	6.56782	6.57248	6.57328
	30	2.98138	2.98013	2.98173	2.97947

Table 5.5 Comparison of different methods(3).
$S_1(0) = 100, S_2(0) = 80, T = 0.4\text{year}$
$\sigma_2 = 0.3, \rho = 0.3.$

Figure 5.17 Density Method Result, $T = 0.4$ year.

$r = 0.09; \sigma_1 = 0.3; \sigma_2 = 0.3; \rho = 0.3; K = 20.$
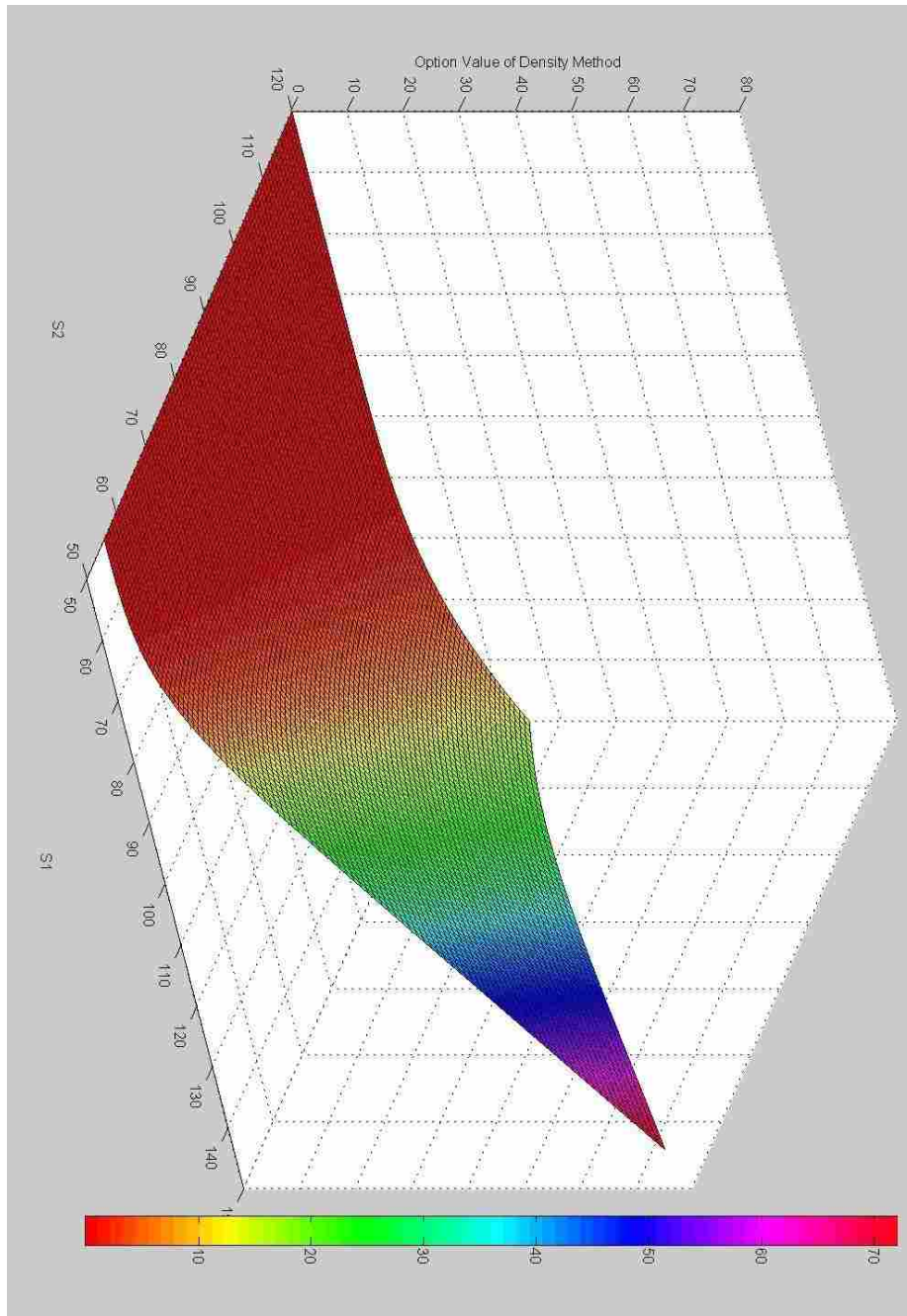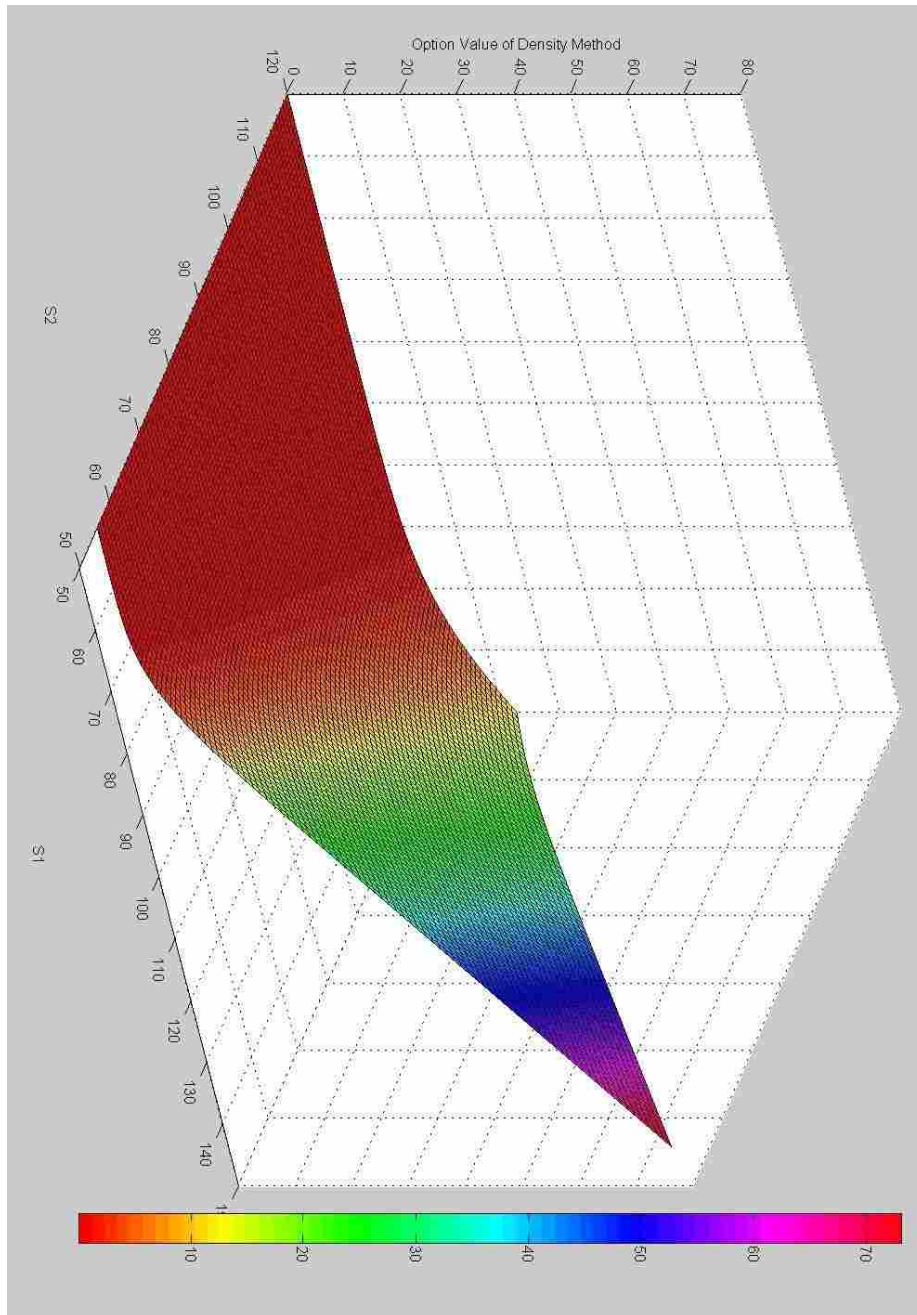
Figure 5.18 Density Method Result, $T = 0.2$ year.

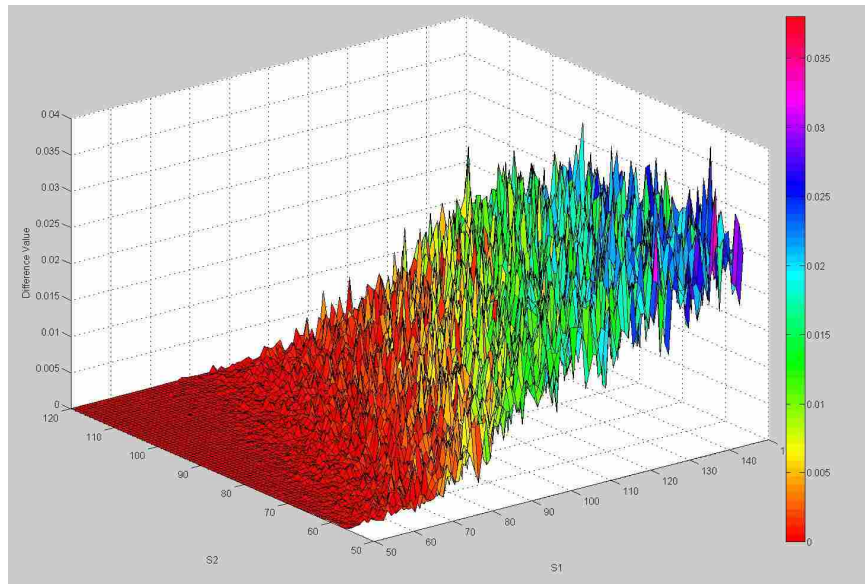$r = 0.09; \sigma_1 = 0.3; \sigma_2 = 0.3; \rho = 0.3; K = 20.$

Figure 5.19 Option Value Difference, $T = 0.4$ year.

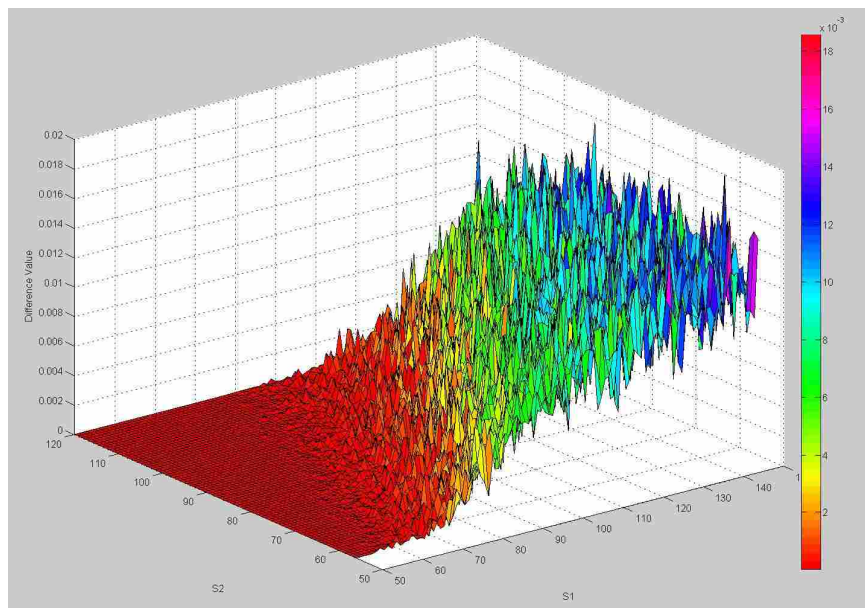$r = 0.09; \sigma_1 = 0.3; \sigma_2 = 0.3; \rho = 0.3; K = 20.$

Figure 5.20 Option Value Difference, $T = 0.2$ year.

$r = 0.09; \sigma_1 = 0.3; \sigma_2 = 0.3; \rho = 0.3; K = 20.$

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Figure 5.17, 5.18 are surface graphs of the Density Function method option price for different values of T. Figure 5.19, 5.20 are the difference graphs of the Density Function method and improved Monte Carlo method. From Figure 5.20, all the differences are less than 2 cents. From Fig 5.19, most points' difference is less than 2 cents, but there are some few positions where the difference is between 2 to 4 cents. Notice that those are the values where the option is deep in the money, i.e., in this case, $S_1(0)$ is really big and $S_2(0)$ is really small. Actually, those in the money option are so "deep" that the moneyness index $\frac{K}{S_1 - S_2}$ from [15] is already less than 0.3 which is considered to be out of the plausible range commonly encountered in practice.

If the Asian style is still for S_1, the European style is still for S_2, but the order of the spread changes to $S_2 - S_1$, then the payoff at time T for this Asian-European spread option is

$$V(T) = (S_2(T) - \frac{1}{T}\int_0^T S_1(t)dt - K)^+.$$

The price of this Asian-European spread option at time 0 is therefore

$$V(0) = E_{\mathbb{Q}}[e^{-rT}V(T)|\mathcal{F}(0)].$$

From section 4.4, we can numerically compute this value by

$$
\begin{aligned}
V(0) &= \int_{-\infty}^{\infty} dx \int_0^{\infty} dz \\
&\quad F(\frac{1}{T}(S_1(0)\frac{4}{\sigma_1^2}z))\frac{1}{z}\exp(-\frac{1}{2z}(1+\exp(2x)))\theta_{e^x/z}(\tau)
\end{aligned}
$$

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where

$$\tau = \frac{\sigma_1^2 T}{4}, \quad \nu = \frac{2(r - \sigma_1^2/2)}{\sigma_1^2},$$

$$\theta_r(u) = \frac{r}{(2\pi^3 u)^{\frac{1}{2}}} \exp(\frac{\pi^2}{2u}) \int_0^\infty \exp(-y^2/2u) \exp(-r(\cosh y))(\sinh y) \sin(\frac{\pi y}{u}) dy,$$

$$F(a_1) = e^{-r(T-t)} [\exp(\mu_{2|1} + \frac{1}{2}\sigma_{2|1}) N(d_1(a_1)) + (a_1 + K) N(d_2(a_1))],$$

$$d_1(a_1) = \frac{\mu_{2|1} - \log(a_1 + K) + \sigma_{2|1}^2}{\sigma_{2|1}},$$

$$d_2(a_1) = d_1(a_1) - \sigma_{2|1},$$

$$\mu_{2|1} = \log(S_2(0)) + (r - \frac{1}{2}\sigma_2^2)T$$
$$+ \frac{\rho\sigma_2}{\sigma_1}(\log(S_1(T)) - (\log(S_1(0)) - (r - \frac{1}{2}\sigma_1^2)T)),$$

$$\sigma_{2|1} = \sqrt{1 - \rho^2}\sigma_2\sqrt{T}.$$

Clearly this requires a painful computation. Now we propose another method to solve this problem much easier.

Rename the payoff

$$V_2(T) := (S_2(T) - \frac{1}{T} \int_0^T S_1(t)dt - K)^+,$$

and define

$$W(T) := S_2(T) - \frac{1}{T} \int_0^T S_1(t)dt - K;$$

then we have

$$
\begin{aligned}
W(T) - V_2(T) &= \min\{S_2(T) - \frac{1}{T}\int_0^T S_1(t)dt - K, 0\} \\
&= -(\frac{1}{T}\int_0^T S_1(t)dt - S_2(T) + K)^+ \\
&=: -V_1(T).
\end{aligned}
$$

So we obtain

$$
V_2(T) = W(T) + V_1(T).
$$

From the linearity of conditional expectation, we have

$$
\begin{aligned}
V_2(0) &= E_{\mathbb{Q}}[e^{-rT}V_2(T)|\mathcal{F}(0)] \\
&= E_{\mathbb{Q}}[e^{-rT}(W(T) + V_1(T))|\mathcal{F}(0)] \\
&= E_{\mathbb{Q}}[e^{-rT}W(T)|\mathcal{F}(0)] + E_{\mathbb{Q}}[e^{-rT}V_1(T)|\mathcal{F}(0)].
\end{aligned}
$$

Notice that $V_1(T)$ is just the payoff in (5.2.1) if we replace the strike price K by $-K$, so $E_{\mathbb{Q}}[e^{-rT}V_1(T)|\mathcal{F}(0)]$ is the price of an Asian-European spread option of the previous form, $\frac{1}{T}\int_0^T S_1(t)dt - S_2(T)$, with strike price $-K$ which we can use the previous Density Function method or the improved Monte Carlo method to numerically compute easily.

As for $E_{\mathbb{Q}}[e^{-rT}W(T)|\mathcal{F}(0)]$, we have

$$E_{\mathbb{Q}}[e^{-rT}W(T)|\mathcal{F}(0)] = E_{\mathbb{Q}}[e^{-rT}(S_2(T) - \frac{1}{T}\int_0^T S_1(t)dt - K)|\mathcal{F}(0)]$$

$$= E_{\mathbb{Q}}[e^{-rT}S_2(T)|\mathcal{F}(0)] - E_{\mathbb{Q}}[e^{-rT}\frac{1}{T}\int_0^T S_1(t)dt|\mathcal{F}(0)] - e^{-rT}K$$

$$= e^{-rT}\{S_2(0)e^{rT} - \frac{1}{T}S_1(0)E_{\mathbb{Q}}[\int_0^T \exp(\sigma_1 W_s + (r - \frac{\sigma_1^2}{2})s)ds] - K\}$$

$$= e^{-rT}\{S_2(0)e^{rT} - \frac{1}{T}S_1(0)\frac{4}{\sigma_1^2}E_{\mathbb{Q}}[\int_0^{\frac{\sigma_1^2}{4}T} \exp 2(W_s + \frac{2}{\sigma_1^2}(r - \frac{\sigma_1^2}{2})s)ds] - K\}$$

$$= e^{-rT}[S_2(0)e^{rT} - \frac{S_1(0)}{rT}(e^{rT} - 1) - K].$$

Here we use Lemma 3.3.1 and the fact in [13]:

If we let

$$A_t^{(\gamma)} = \int_0^t \exp[2(W_s + \gamma s)]ds,$$

then its expectation is

$$E(A_t^{(\gamma)}) = \frac{1}{2(\gamma + 1)}[\exp(2(\gamma + 1)t) - 1].$$

So the price for the Asian-European spread option with payoff

$$V_2(T) = (S_2(T) - \frac{1}{T}\int_0^T S_1(t)dt - K)^+$$

at time 0 is

$$V_2(0) = e^{-rT}[S_2(0)e^{rT} - \frac{S_1(0)}{rT}(e^{rT} - 1) - K] + V_1(0)$$

where $V_1(0)$ is the price of an Asian-European spread option of the previous form, $\frac{1}{T}\int_0^T S_1(t)dt - S_2(T)$, with strike price $-K$ at time 0.

5.3 ASIAN SPREAD OPTION NUMERICAL COMPUTATION

In section 3.4, we considered the following stochastic volatility model

$$
\begin{aligned}
dS_1(t) &= rS_1(t)dt + \sigma_1\sqrt{\gamma(t)}dW_1(t), \\
dS_2(t) &= rS_2(t)dt + \sigma_2\sqrt{\gamma(t)}dW_2(t), \\
d\gamma(t) &= \kappa(\mu - \gamma(t))dt + \sigma_\gamma\sqrt{\gamma(t)}dW_\gamma(t), \\
dY(t) &= (S_1(t) - S_2(t))dt, \quad\quad\quad\quad\quad (5.3.1)
\end{aligned}
$$

where

$$
E[dW_1(t)dW_2(t)] = \rho dt,
$$

$$
E[dW_1(t)dW_\gamma(t)] = \rho_1 dt,
$$

$$
E[dW_2(t)dW_\gamma(t)] = \rho_2 dt.
$$

And this is an affine diffusion model,

$$
dX_t = \Theta(X_t)dt + \Sigma(X_t)dw(t),
$$

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where

$$X_t = (S_1(t), S_2(t), \gamma(t), Y(t))^T,$$

$$\Theta(X_t) = \begin{pmatrix} r & 0 & 0 & 0 \\ 0 & r & 0 & 0 \\ 0 & 0 & -\kappa & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix} X_t + \begin{pmatrix} 0 \\ 0 \\ \kappa\mu \\ 0 \end{pmatrix} =: K_1 X_t + K_0,$$

and

$$\Sigma(X_t) = \begin{pmatrix} \sqrt{1-\rho_1^2}\sigma_1\sqrt{\gamma(t)} & 0 & \rho_1\sigma_1\sqrt{\gamma(t)} & 0 \\ \frac{\rho-\rho_1\rho_2}{\sqrt{1-\rho_1^2}}\sigma_2\sqrt{\gamma(t)} & \frac{\sqrt{1-\rho_1^2-\rho_2^2-\rho^2+2\rho_1\rho_2\rho}}{\sqrt{1-\rho_1^2}}\sigma_2\sqrt{\gamma(t)} & \rho_2\sigma_2\sqrt{\gamma(t)} & 0 \\ 0 & 0 & \sigma_\gamma\sqrt{\gamma(t)} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and $w(t) = (w_1(t), w_2(t), w_3(t), w_4(t))^T$ is a standard 4-dimensional Brownian motion.

For the Asian spread price with strike price K under model (5.3.1), we have the following result:

$$
\begin{aligned}
C(X_t, b, K, t, T) &= E(\exp(-r(T-t))(b \cdot X_T - K)^+ | \mathcal{F}(t)) \\
&= E(\exp(-r(T-t))(b \cdot X_T - K)\mathbf{1}_{b \cdot X_T \geq K} | \mathcal{F}(t)) \\
&= G^{(2)}_{b,-b,\vec{0}}(-K; X_t, t, T) - KG^{(1)}_{0,-b}(-K; X_t, t, T) \\
&= \frac{\psi_2(b, \vec{0}, X_t, t, T)}{2} - \frac{1}{\pi}\int_0^\infty \frac{Im[\psi_2(b, -ivb, X_t, t, T)\exp(ivK)]}{v}dv \\
&\quad - K\frac{\psi_1(\vec{0}, X_t, t, T)}{2} + K\frac{1}{\pi}\int_0^\infty \frac{Im[\psi_1(-ivb, X_t, t, T)\exp(ivK)]}{v}dv
\end{aligned}
$$

$$(5.3.2)$$

where $b = (0, 0, 0, \frac{1}{T})^T$. So we just need to numerically solve differential equations (3.4.5), (3.4.6), (3.4.7) and (3.4.8) under corresponding boundary conditions to get characteristic functions ψ_1 and ψ_2, then plug into (5.3.2) to get the Asian spread price $C(X_t, b, K, t, T)$. We compare the numerical result with Monte Carlo simulation and find that the characteristic function method result is satisfactory. Tables 5.6 and 5.7 show the numerical result of Monte Carlo simulation. M is the total number of replications of sample path, and N is the number of price reading per day. "Time" is the computation time with unit second. For the case in table 5.6, the characteristic function method has value 5.34029 with computing time 3 seconds. For the case in table 3.7, the characteristic function method has value 5.84394 with computing time 3 seconds.

N=1			N=10			N=100		
M	Value	Time	M	Value	Time	M	Value	Time
10000	5.29894	1	10000	5.28762	7	10000	5.24064	60
50000	5.32906	3	50000	5.35315	30	50000	5.36688	299
100000	5.33322	6	100000	5.34027	59	100000	5.35201	602
200000	5.33033	12	200000	5.34466	120	200000	5.33864	1197
500000	5.33538	31	500000	5.34445	300	500000	5.34428	2992
1000000	5.34076	61	1000000	5.34168	599	1000000	5.34171	5988

Table 5.6 Monte Carlo simulation results.

$S_1(0) = 100, S_2(0) = 80, \gamma(0) = 400, K = 15, r = 0.05, \kappa = 1, \mu = 400, T = 0.2$year

$\sigma_1 = 0.5, \sigma_2 = 1.0, \sigma_\gamma = 4, \rho = 0.5, \rho_1 = -0.5, \rho_2 = 0.25$

N=1			N=10			N=100		
M	Value	Time	M	Value	Time	M	Value	Time
10000	5.70156	1	10000	5.76834	13	10000	5.79358	132
50000	5.82701	7	50000	5.80755	66	50000	5.83950	657
100000	5.83101	13	100000	5.82654	131	100000	5.85096	1314
200000	5.83442	26	200000	5.85004	263	200000	5.83733	2653
500000	5.84696	66	500000	5.83943	657	500000	5.84240	6558
1000000	5.84562	132	1000000	5.84914	1315	1000000	5.84095	13130

Table 5.7 Monte Carlo simulation results.

$S_1(0) = 100, S_2(0) = 80, \gamma(0) = 400, K = 15, r = 0.05, \kappa = 1, \mu = 400, T = 0.4\text{year}$

$\sigma_1 = 0.5, \sigma_2 = 1.0, \sigma_\gamma = 4, \rho = 0.5, \rho_1 = -0.5, \rho_2 = 0.25$

CHAPTER 6. CONCLUSION

Now we have a better understanding about Asian spread option and Asian-European spread option pricing. For the Asian spread option pricing, under the Black-Scholes-Merton model, we have the semi-analytic solution which contains triple integrals; or we can try to numerically solve partial differential equation under boundary conditions; under a special stochastic volatility model, we have the analytic and numerically computable solution. For Asian-European spread option pricing, under the Black-Scholes-Merton model, not only do we have the similar semi-analytic solution and partial differential equation under the boundary conditions as in the Asian spread option case, but also an improved Monte Carlo simulation method and the numerical computation method for the semi-analytic solution. Both numerical methods are efficient and accurate. Under the special stochastic volatility model, we have the similar analytic and computable solution as well. We also established an easy way to combine Asian-European spread option and European-Asian spread option pricing problems together.

There are still many interesting questions for the future research about Asian spread option though. Here all the Asian spread options are actually European style, that is, you need wait until the expire date to exercise. How about American style Asian spread option or other strange style payoff options pricing?

We considered the Black-Scholes-Merton model and the special affine structure stochas-

tic volatility model, but how about other stochastic models? In real markets, jump is a very common behavior for underlying price, how about stochastic models containing jumps?

For the special affine structure stochastic volatility model, how can we efficiently calibrate and estimate the ten parameters? Furthermore, in real energy and commodity markets, many exotic options are traded over-the-counter which usually lack liquidity [2] [10]; this will make usual statistical methods difficult to conduct. How to reflect this property in the model setting?

Correlation structure is also an important issue. We made the assumption that the two underlyings are constantly correlated. But here the underlying price is followed over a period of time; it is quite natural to doubt this assumption. How can we describe the possible changing correlation during this period of time and how will this affect the pricing problem?

Bibliography

[1] Abate, J., W. Whitt(1995). Numerical Inversion of Laplace Transforms of Probability Distributions. *ORSA Journal of Computing*, **7**, 36-43.

[2] Alexander, C. and E, Sheedy(2008). *The Professional Risk Managers' Guide to Financial Markets*. McGraw-Hill.

[3] Boyle, P. (1977). Options: A Monte Carlo Approach. *Journal of Financial Economics*, **4**, 323-338.

[4] Broadie, M. and P. Glasserman (1996), Estimating Security Price Derivatives Using Simulation. *Management Science*, **42**, 269-285.

[5] Carmona, R. and V. Durrleman (2003). Pricing and hedging spread options, *SIAM REVIEW* **45**(4), 627-685.

[6] Curran, M. (1992). Beyond Average Intelligence. *Risk*, **5**, no.10, 60-63.

[7] Deng, S. (2000) Stochastic models of energy commodity prices and their applications: mean-reversion with jumps and spikes. Working paper of POWER, Georgia Institute of Technology.

[8] Duffie, D., J. Pan and K. Singleton (2000). Transform analysis and asset pricing for affine jump-diffusions. *Econometrica* **68** 1343-1376.

[9] Duffie, D., D. Filipovic and W. Schachermayer (2003). Affine Processes and Applications in Finance *The Annals of Applied Probability* **13**, No. 3, 984-1053.

[10] Eydeland, A. and K, Wolyniec (2002). *Energy and Power Risk Management: New Developments in Modeling, Pricing, and Hedging.* John Wiley & Sons, Hoboken, New Jersey.

[11] Fouque, J. and C., Han (2003). Pricing Asian options with stochastic volatility. *Quantitative Finance*, **3**, 353-362.

[12] Fu M.C. and D. B. Madan and T. Wang (1998). Pricing continuous Asian options: a comparison of Monte Carlo and Laplace transform inversion methods. *Journal of Computational Finance*, **2**, 49-74.

[13] Geman, H. and M. Yor (1993). Bessel processes, Asian options, and perpetuities. *Mathematical Finance* **3**(4), 349-375.

[14] Heston, S. (1993). A closed-form solution for options with stochastic volatility, with applications to bond and currency options. *Review of Financial Studies* **6** 327-343.

[15] Hong, S. G. (2001). *Pricing and hedging of spread options with stochastic component correlation.* Ph.D. Thesis.

[16] Hull, J. (2002) *Options, futures, and other Derivative securities*, 5th ed., Prentice-Hall, Englewood Cliffs, NJ.

[17] Kemna, A.G.Z. and A.C.F. Vorst (1990). A Pricing Method for Options Based on Average Asset Values. *Journal of Finance*, **14**, 113-129.

[18] Kirk, E. (1995). *Correlation in the energy markets, in managing energy price risk.* Risk Publications and Enron.

[19] Margrabe, W. (1978) The value of an option to exchange one asset for another. *Journal of Finance* **33**(no.1), 125-144.

[20] Pearson, N. D. (1995). An efficient approach for pricing spread options. *Journal of Derivatives.* Fall. 76-91.

[21] Ravindran, K. (1993). Low-fat spreads. *RISK* **6**(10) 56-57.

[22] Rogers, L. and Z., Shi (1995). The value of an Asian option. *Journal of Applied Probability*, 32, 1995, 1077-1088.

[23] Shreve, S. (2004). *Stochastic calculus for finance II Continuous-Time Models.* Springer, New York.

[24] Vecer, J. (2001) A new PDE approach for pricing arithmetic average Asian options, *J. Comput. Fin.* **4**(4), 105-113.

[25] Wengler, J. (2001). *Mannaging Energy Risk: a nontechnical guide to markets and trading.* Penn Well Publishing Company, Oklahoma.

[26] Yor, M. (1992). On some exponential functionals of Brownian motion. *Adv. Appl. Prob.* **24**, 509-531.

[27] Zhou, J. and X., Wang (2008). Accurate closed-form approximation for pricing Asian and basket options, *Appl. Stochastic Models Bus. Ind.* **24**, 343-358.