2019-07-01

Developing Understanding of the Chain Rule, Implicit Differentiation, and Related Rates: Towards a Hypothetical Learning Trajectory Rooted in Nested Multivariation

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A thesis submitted to the faculty of
Brigham Young University
in partial fulfillment of the requirements for the degree of

Master of Arts

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ABSTRACT

Developing Understanding of the Chain Rule, Implicit Differentiation, and Related Rates: Towards a Hypothetical Learning Trajectory Rooted in Nested Multivariation

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There is an overemphasis on procedures and manipulation of symbols in calculus and not enough emphasis on conceptual understanding of the subject. Specifically, students struggle to understand and correctly apply concepts in calculus such as the chain rule, implicit differentiation, and related rates. Students can learn mathematics more deeply when they make connections between different mathematical ideas. I have hypothesized that students can make powerful connections between the chain rule, implicit differentiation, and related rates through the mathematical concept of nested multivariation. Based on this hypothesis, I created a hypothetical learning trajectory (HLT) rooted in nested multivariation for students to develop an understanding of these three concepts. In this study, I explore my HLT through a small-scale teaching experiment with individual first-semester calculus students using tasks based on the HLT.

Based on the teaching experiment, nested multivariational reasoning proved to be critical in understanding how the variables within a function composition change together and in developing intuition and understanding for the multiplicative nature of the chain rule. Later, nested multivariational reasoning was mostly important in recognizing the existence of a nested relationship and the need to use the chain rule in differentiation. Overall, through the HLT, students gained a connected and conceptual understanding for the chain rule, implicit differentiation, and related rates. I also discuss how the HLT might be adjusted and improved for future use.

Keywords: calculus, chain rule, implicit differentiation, related rates, multivariation, covariation, hypothetical learning trajectory

ACKNOWLEDGEMENTS

I want to thank everyone who has supported me through this journey. I thank my advisor, Dr. Steven R. Jones, who was the most influential person to me in the creation of my thesis and in my pursuit of a master's degree. He encouraged me, showed patience towards me, and treated me like a peer. I thank my committee for their time and feedback to improve the quality of this thesis. I thank my friends and family who always encouraged me. Last, and most important of all, I thank my husband, Brookston: He loved me every day and helped me to see my potential, even when I doubted myself and wanted to give up the most. I could not have finished my thesis without these people and their support and love.

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CHAPTER ONE: INTRODUCTION

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Emphasis on Procedures in Calculus

Conceptual understanding of concepts in calculus like function, covariation, infinity, limit, derivative, and integral is often lacking in calculus students (Byerley, Hatfield, & Thompson, 2012; Davis & Vinner, 1986; Jones, 2015; Kolar & Cadez, 2012; Oehrtman, 2009). Additionally, it has been reported that students struggle to solve, interpret, and apply calculus problems (Martin, 2000; McDermott, Rosenquist, & van Zee, 1987). Selden, Selden, and Mason (1994) found that calculus students struggle to apply calculus creatively in nonroutine problems. The authors found this was true even for calculus students who received an A or B grade in the class and performed well on tests of the knowledge base of relevant calculus skills. This is problematic as calculus concepts are relevant, not only in various sectors of mathematics, but also in a variety of other disciplines including engineering, business, economics, psychology, computer science, biology, chemistry, and other natural sciences. About 71% of students enrolled in calculus are studying engineering, biological sciences, and physical and computer sciences (Bressoud, Carlson, Mesa, & Rasmussen, 2013). Yet, students may not have a strong enough understanding of calculus to know how to powerfully apply calculus concepts in their respective fields.

Students struggle to understand and appreciate calculus concepts could, in part, be related to the fact that calculus students, whether secondary or in college, are placing too much emphasis on procedures and manipulation of symbols without conceptual understanding (Tall, 1992; Ferrini-Mundy & Gaudard, 1992; Rasmussen & Marrongelle, 2014; White & Mesa, 2014). For example, White and Mitchelmore (1996) conducted a study exploring first year college students' conceptual knowledge in introductory calculus. The authors provided 24 hours of concept-based calculus instruction to students who had already learned calculus in high school. The authors collected post-instruction student responses and found that students' performance with rates of change, an important concept for derivatives, in complex situations was weak. The authors found that the students treated "variables… as symbols to be manipulated rather than as quantities to be related" (p. 79). They found that many of the students' difficulties generally came from their "manipulation focus" (p. 88), where the students focused on which procedure to apply and didn't attend to the meaning of the symbols with which they were working.

Coinciding with students' focus on procedures, textbooks also place a lot of emphasis on procedures when introducing different concepts in calculus. For example, Stewart (2016), which is by far the most commonly used textbook in the United States, introduces the chain rule as a procedure and does not provide opportunities for the student to explore why the chain rule works or why it makes sense, thereby gaining a deeper, conceptual understanding. The author begins the section by simply stating the rule: if you have a composition of functions $f \circ g$, "the derivative of the composite function $f \circ g$ is the product of the derivatives of f and g" (p. 198). Although the textbook does provide a short explanation of some intuition behind the chain rule, there are no problems or examples to really help students discover or understand this for themselves. Instead, the book proceeds to provide a multitude of examples and homework problems that require students to practice using the rule rather than explore why it works or makes sense. Stewart (2016) gives a formal proof at the end of the chapter for why the rule works, but it is a purely symbolic explanation; there is no meaningful context to help the students develop intuition for the rule before it is abstracted.

An overemphasis on procedures in calculus can be problematic. Overemphasizing procedures and calculations can make it hard for students to make sense of mathematics, remember what they learn, and apply it appropriately in the future (NRC, 2001; Pesek & Kirshner, 2000; Skemp, 2006; Thompson, Philipp, Thompson, & Boyd, 1994). It is possible that calculus students, who are focusing on procedures (Tall, 1992; Ferrini-Mundy & Gaudard, 1992; Rasmussen & Marrongelle, 2014; White & Mesa, 2014) and meaningless manipulation of symbols (White and Mitchelmore, 1996) may never gain the conceptual understanding they could have potentially gained if they focused on gaining a conceptual understanding *from the beginning*. Pesek and Kirshner (2000) studied two groups of students: one group received instruction focused only on conceptual understanding and the other group first received instruction focused mathematical procedures and then received instruction focused on conceptual understanding. They found that the first group of students, who received only instruction focused on conceptual understanding, outperformed the other group.

I am not arguing that procedures are bad. A complete understanding of a mathematical concept includes an interplay of both procedural and conceptual understanding (NRC, 2001; Pettersson and Scheja, 2008). Pettersson and Scheja explain that focusing solely on the procedural aspects of a concept might delay a more conceptual understanding, yet they point out that a complete understanding of a mathematical concept includes both a conceptual and procedural component.

Connections Between Mathematical Concepts

Another part of gaining a solid conceptual understanding of mathematics is to make connections among and between various mathematical concepts. Schoenfeld (1988) says, "thinking mathematically consists not only of mastering various facts and procedures, but also in understanding connections among them" (p. 164). Additionally, some researchers claim that the degree of our understanding of mathematics can be determined by the number and strength of connections we make to other mathematical knowledge (Brownell, 1935; Hiebert & Carpenter, 1992; Hiebert et al., 1997). In looking at the typical calculus curriculum for concepts that could be taught more conceptually and connectedly, I became more interested in the chain rule, implicit differentiation and related rates. As a calculus student, tutor, and teaching assistant I have seen how students often struggle to understand these three concepts and fail to see how they are related. Research has also shown that students struggle to understand these concepts (Cottrill, 1999; Infante, 2007).

Although researchers have begun to explore way in which the chain rule, related rates, and implicit differentiation are related to one another (Clark et al., 1997; Cottrill, 1999; Infante, 2007; Martin, 2000), they tend to examine these concepts in isolation and they have not conducted a serious investigation into how these concepts could be taught in a way that connects them together. Cottrill (1999) says that the chain rule is the underlying concept of implicit differentiation and related rates. To me, the chain rule, implicit differentiation, and related rates are different applications of the *same* underlying concept: nested multivariation. Later, in Chapter 3, I more deeply describe my conceptual framework for nested multivariation. In my study, I explore the question of whether or not nested multivariation can be used as a construct to help students understand the connectedness of these three concepts.

CHAPTER TWO: LITERATURE REVIEW

My study deals with the chain rule, implicit differentiation, and related rates. In this chapter, I present the limited research that has been done in these three areas. The use of meaningful contexts is central to the tasks I have created based on the HLT, so I also discuss what research has said about the importance of using meaningful contexts to explore mathematics. Lastly, the concept of infinitesimals is also central to the HLT and so I end by discussing what research says about infinitesimals and how they can be a useful conceptual tool for students in understanding calculus.

Research on Related Rates

Austin, Barry, and Berman (2000) describe the history behind related rates problems in first-year calculus. The authors explain that even back in the early 19th century, Rev. William Ritchie (1836) noticed that students struggled to understand and recognize the power of calculus. He attempted to reform the way calculus was taught by being one of the first to include related rates problems in his text. He did this in order to help calculus become more accessible and powerful to the "ordinary, non-university student" (p. 3). Related rates problems can potentially be a great way for students to experience mathematical reasoning. For example, they create a way for students to model meaningful contexts, use calculus to better understand that model, and evaluate and interpret their results in a meaningful way.

Although related rates problems can potentially be a great way for students to experience the power of calculus, students struggle to understand or solve related rates problems (Infante, 2007; Martin, 2000).

In his textbook, when Stewart (2016) introduces related rates, he says "the *procedure* is to find an equation that relates the two quantities and then use the Chain Rule to differentiate both sides with respect to time" (p. 245, emphasis added). He then provides step-by-step instructions for the students to follow as they attempt to solve related rates type problems. He states it as if it is a simple "procedure", yet students continue to struggle to conceptualize and solve these problems (Infante, 2007; Martin 2000).

My study builds on Infante's (2007) dissertation exploring student understanding of related rates problems. In her study, she interviewed mathematicians and conducted a teaching experiment with first-semester calculus students in order to better understand the processes and cognitive constructions necessary to solve related rates problems. Infante (2007) and White & Mitchelmore (1996) have identified students' inadequate (and generally procedural) understanding of the chain rule as one of the major obstacles in a student's ability to solve related rates problems.

Infante (2007) found that it is especially difficult for students to recognize the implicit variable of time within related rates problems. She found that using a dynamic computer program, which illustrated the related rates problems the students were working through, helped students to recognize the implicit variable of time and apply the chain rule. She also found that if students used the chain rule in a related rates problem, it became a way for them to create a delta equation (see Chapter 4 for more details on a "delta equation"), which helped them to better see the relationship between the given and unknown rates in the problem. Visualizing this relationship in the equation ultimately helped them to successfully solve the related rates problem. In other words, she found that the chain rule was a main factor for solving a related rates problem.

In Chapter 4, I more thoroughly explain exactly how my study builds on and extends Infante's dissertational work.

Research on the Chain Rule and Implicit Differentiation

In general, students struggle to understand and appropriately use the chain rule (Clark et al., 1997; Cottrill, 1999; Infante, 2007). Cottrill (1999) explains that in current pedagogical practices, there is not a visual representation of the chain rule, which may contribute to students' inability to conceptualize and successfully use the chain rule.

Speer and Kung (2016) explain that research on implicit differentiation is basically missing from mathematics education research. In response to this observation, Mirin and Zazkis (2019) presented a conceptual basis for differentiating an equation, in an effort to make implicit differentiation more explicitly defined. They explain that when performing implicit differentiation, it is important to clearly define any implicit functions and then recognize that the two functions set equal to one another in an equation may only be equal to one another on a restricted domain. They explain that only once it is clear that the two functions are equal to each other can one understand why their derivatives, with respect to the chosen independent variable, are also equal to each other. They explain that the legitimacy of taking the derivative of both sides of the equation is nontrivial for and remains unclear to some calculus students. Jones (2017) found that students in his study often thought that implicit differentiation must be required for all applied derivatives.

Stewart's (2016) textbook primarily presents both the chain rule and implicit differentiation as procedures and steps rather than providing students with opportunities to understand these ideas. For example, he does not help students to conceptually understand why implicit differentiation works. Instead, he only provides a few examples that help students to understand when and why they might use implicit differentiation. For the most part, he explains that when faced with an implicit equation, one simply chooses which variable will be independent and then applies the chain rule to find the derivative.

Using Meaningful Contexts

My study makes use of meaningful contexts through which students can build understandings for the chain rule and implicit differentiation. My work builds off Infante's (2007) findings, but my study is different than hers in that I observe whether and how students might *develop* meanings for the chain rule and implicit differentiation through meaningful contexts. In her study, Infante revisited the chain rule in the second teaching episode, after her students had already learned the rule in their individual classes. She hoped that this would help them to ultimately conceptualize and solve related rates problems. In my study, instead of conducting a teaching experiment where students relearn the chain rule, I explore how students might develop understandings of the chain rule *through* meaningful contexts. When I use the term meaningful contexts, I mean contexts that allow the students to visualize or use intuition to make sense of the mathematics.

Dienes (2006) explains that students must first *experience* mathematics in order to understand or make mental constructions of abstract mathematical concepts. That is, a student must form mental images of mathematical ideas so that when she/he abstracts that idea, the student understands what the abstraction means and why it works the way it does. Exploring mathematics through meaningful contexts can help students form mental images for abstract mathematical concepts like those found in calculus.

Schwalbach and Dosemagen (2000) looked at the practice of one high school teacher who provided students with the opportunity to explore calculus and experience mathematics through meaningful contexts. They found that these students explained the processes they were using to solve calculus problems. They also found that these students were able to explain their reasoning in multiple ways (e.g. numerically, algebraically, graphically, and verbally) and they made significant connections between concepts in physics and calculus.

Other researchers make a call for educators to give more importance to context and visual images in helping students to understand challenging mathematical concepts, like those in calculus (Dray & Manogue, 2005; Redish, 2005; Weber, Tallman, Byerly, & Thompson, 2012). Dray and Manogue (2005), a mathematician and physicist respectively, emphasize the importance of context. They explain that "the mathematics we teach tends to be about formal manipulation of symbols according to well-defined rules, whereas the mathematics we use always has a context" (p. 3). They have seen many students struggle to apply mathematics and suggest that we use more contextually rich problems in our teaching.

Using Infinitesimals

The HLT that I have constructed relies on a conception of infinitesimals by interpreting a derivative dy/dx as representing how many times as large the change in y is than an infinitesimal change in x. Although there is a historical debate about whether or not the idea of infinitesimals is mathematically rigorous, researchers have argued that infinitesimals are robust and viable and that using infinitesimals is conceptually beneficial for students (Dray & Manogue, 2010; Ely, 2010; Ely, 2017; Jones, 2015). Ely (2010) demonstrates that although infinitesimal quantities are nonstandard, they could be used to "build a cognitive structure as powerful and consistent as the standard conceptual structure of the real number line" (p. 120). He explains that for more than a century, Leibniz' conceptions of infinitesimals were used in coherent, powerful systems of mathematical thought.

The concept of infinitesimals is complex, and students may have varying and complex ways of thinking about them. For this study, I encourage students to think of an infinitesimal as a very small, or infinitely small, quantity. I want to emphasize that when taking the limit of the difference quotient in finding a derivative, the change in the independent never collapses to 0 (see Oehrtman, 2009 for more on “collapsing”). Instead, there always remains a very small, infinitesimal amount of the quantity within the tiny change.

CHAPTER THREE: CONCEPTUAL AND THEORETICAL FRAMEWORK

My study makes use of two frameworks: a conceptual framework for the concept under investigation and a theoretical framework for how I think students might come to understand that concept and its applications. In this section, I describe both my conceptual framework for nested multivariation and my theoretical framework for hypothetical learning trajectories. I end the section by describing the hypothetical learning trajectory I developed for this study that is rooted in nested multivariation.

Nested Multivariation

As I thought more about the way that covariation plays into the chain rule, implicit differentiation, and related rates, and discussed these ideas with my advisor, I began to be more convinced that these concepts are connected. My advisor and I began to hypothesize that the concept that underlies these three concepts is Jones' (2018) "nested multivariation."

In this section, I describe the results of the conceptual analysis I conducted for nested multivariation and how I see it being the concept that underlies these three concepts. I first describe and define nested multivariation. Then, I describe how I see the chain rule, implicit differentiation, and related rates as being different applications of the nested multivariation concept. I end by describing the framework for the mental actions students might go through as they employ nested multivariational reasoning. For brevity, I refer to nested multivariation, the conceptual structure, as NM and nested multivariational reasoning, the reasoning about that structure, as NMR.

Infante (2007) explains that covariational reasoning is important in helping students to understand and solve related rates. Specifically, she explains that what appears to be the most important aspect of the chain rule that helps students to solve related rates problems is the ability to coordinate relationships in the amount of change. She explains that when students coordinate relationships in amounts of change, they understand the multiplicative nature of the chain rule (i.e. for $(g(x))$, $\frac{df}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}$).

Nested multivariation (NM) is a specific extension of covariation. There has been a lot of focus in mathematics education research on covariation (e.g. Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Moore, Paoletti, & Musgrave, 2013; Moore Stevens Paoletti, & Hobson, 2016; Oehrtman, Carlson, & Thompson, 2008; Thompson, 1994). Covariational reasoning deals with students' understanding of and ability to coordinate the change between two related quantities. Not that the term "quantity" refers to a numeric value, but it usually implies a measurable quality of an object (Thompson, 1994), as compared to a decontextualized numeric value. Because it is useful within mathematics to think of both quantities and decontextualized numbers, I use "variable" as a generic umbrella term for both quantity and decontextualized number. Jones' (2018) framework extends the idea of covariation to the idea of "multivariation," which is the coordination of *multiple* variables that are related to one another. In his paper, he identifies four distinct types of multivariation which he refers to as independent multivariation, dependent multivariation, nested multivariation, and vector multivariation.

I see nested multivariation (NM) as the concept that underlies the chain rule, implicit differentiation, and related rates. As Jones (2018) explains, NM refers to the way in which one might conceptualize the change in relationships between variables in a function composition structure. The term "nested" refers to the function composition and the term "multivariation" refers to the coordination of the changes of more than two variables. To better understand nested multivariation, consider the function composition $f(g(x))$. One can use covariational reasoning to think about the way in which changes in *x* affect changes in *g*. Similarly, one can use covariational reasoning to conceptualize the way that changes in *g* affect changes in *f.* However, nested multivariational reasoning occurs as soon as one considers all three of these variables (x, f, and g) at once or conceptualizes the way that all three of these variables change together.

In a function composition such as $f\big(g(x)\big)$, it is possible to conceptualize direct, two-variable covariation, between *x* and *f,* or it is possible to conceptualize three-variable multivariation, between *x, f,* and *g*. Thus, Jones (2018) points out that the difference between covariation and multivariation is not inherently dependent on the structure of a function or a given context. Instead, the difference between covariation and multivariation is in how one conceptualizes the changes taking place. For example, consider the function $f(g(x)) = \mathrm{e}^{\sin(x)}$. Covariational reasoning is imagining how x and f change directly with each other. NMR, on the other hand, is imagining how *x, g,* and *f* change directly with each other. In my example, it would be imagining that as x increases, from say 0 to $\frac{\pi}{6}$, $g = \sin(x)$ increases from 0 to $\frac{1}{2}$, and then $f = \mathrm{e}^{\sin(x)}$ simultaneously increases from 1 to $\sqrt{e}$. Generally, NMR is imagining how the independent variable causes changes in a second variable which in turn causes changes in a third variable and continues to affect as many variables as exist in the function composition structure.

One can also use NMR to conceptualize the way in which three quantities in a nested relationship in real-world context vary simultaneously together. Consider, for example, a car driving from one place to another. NMR might involve imagining the way in which time increases and causes a simultaneous increase in the total distance traveled which also causes a simultaneous increase in the total gallons of gas burned.

The Chain Rule as an Application of NM

The chain rule says that given a function $f(g(x))$, $\frac{df}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}$. This multiplicative nature of the chain rule (Infante, 2007) is the product of NM by conceptualizing the way that *x* affects *g,* and how *g* affects *f* simultaneously. For example, suppose *g* changes 2 times as fast as *x* ($\frac{dg}{dx} = 2$), meaning that for every small (or infinitesimal) change of *x,* the corresponding change in *g* is two times as large. Suppose also that *f* changes 3 times as fast as *g* ($\frac{df}{dg} = 3$), meaning that for every small (or infinitesimal) change of *g,* the corresponding change in *f* is three times as large. Thus, this means that for every small (or infinitesimal) change of *x,* the corresponding change in *f* is $3 \cdot 2 = 6$ times as large ($\frac{df}{dx} = 6$).

I see the resultant product (i.e. $\frac{df}{dg} \cdot \frac{dg}{dx} = 3 \cdot 2 = 6$), or the multiplicative nature of the chain rule, as encompassing NM because it represents how the changes of the three variables within the function composition are affecting each other *simultaneously* within the product. That is, the multiplicative nature of the chain rule is the result of employing NMR to understand the meaning of the derivative of a composition of functions. Note that throughout the paper, I also use the term "multiplicative nature of the chain rule" to describe the *result* of NM in taking the derivative of a composition of functions.

Thompson (1994) explains the concept of rate and the difference between an additive and a multiplicative comparison. He says that when comparing two quantities additively, the result is a difference and that when comparing two quantities multiplicatively, the result is a ratio. He explains that when students are comparing multiplicatively, they could be comparing one of two ways: the first is to compare the two quantities (e.g. 3:2) and the second is to compare one quantity measure in units of the other (e.g. 1.5:1), or to create a unit rate. In this study, I primarily make sense of the chain rule and derivatives in general using the second, unit rate conception. For example, if $\frac{dy}{dx} = 1.5$ miles/hr., this means that for a small (infinitesimal) change in *x,* the change in *y* is 1.5 times as large. Although the derivative occurs at one moment, for an infinitesimal change in x, and this derivative may not be constant for larger and different changes in x, the derivative could be conceptualized as being equal to a rate of 1.5 miles for every 1-hour unit.

Implicit Differentiation as an Application of NM

Before proceeding, let me define what I mean by an explicit and an implicit function. In my definitions, I am only concerned with functions of one variable. An explicit function is one in which the dependent variable is explicitly defined in terms of its independent variable. For example, the dependent quantity is isolated and on one side of an equation and is set equal to some rule composed of the independent quantity (e.g. $f(x) = x^2 + 1$). To have an implicit function, it is necessary to first have an equation that defines a relationship between two variables, but where one of the variables is not explicitly written as a function of the other. Also, one of the variables needs to be identified as an "input" variable and the other as the "output" variable, such as thinking of y as a function of x, $y(x)$, or x as a function of y, $x(y)$. In order to conceptualize one variable as a function of another, it may be necessary to restrict the domain or range so that every input value maps to exactly one output value (e.g. $x^2 + [y(x)]^2 = 9,\ 0 \leq y \leq 3$). I define implicit differentiation to be the act of differentiating an implicit function.

I see NM as being the underlying concept for implicit differentiation. For example, consider the equation for a circle: $x^2 + y^2 = r^2$. Suppose we want to know how y changes with respect to x, or we want to find dy/dx. In differentiating both sides of the equation with respect to x, one must first recognize that y can be conceptualized as an implicit function of x, perhaps on a restricted domain of x $(-3 \leq x \leq 3)$ and range for y $(0 \leq y \leq 3)$. Once y is conceptualized as an implicit function of x, then one can conceptualize $x^2 + y^2$ and r^2 as being two different but equal functions which implies that their derivatives must also be equal.

I recognize that when conceptualizing the derivative of an implicit function, one could employ NMR at one time in the derivative process and employ covariational reasoning at another time in the derivative process. For example, in reasoning through the differentiation of the left side of the equation ($x^2 + y^2$), and recognizing that the derivative of a sum is equal to the sum of derivatives, one can consider the derivative of the x^2 part, the y^2 part separately. As for the right side of the equation (r^2), it can be conceptualized as a simple constant function of *x.* To make sense of the derivative of the x^2 part with respect to *x*, one may employ covariational reasoning to think about the way x^2 covaries with *x.* Similarly, to make sense of the derivative of the r^2 part of the implicit equation, one may employ covariational reasoning to think about the way r^2 does not change as *x* changes. NMR occurs when making sense of the y^2 part of the implicit equation with respect to *x.* That is, one must recognize that y^2 is an implicit composition of functions (i.e. $[y(x)]^2$) and conceptualize the way that y^2 varies with *x* which entails conceptualizing the way *x* affects *y,* and consequently the way that *y* affects y^2. Through NM, it becomes apparent that in order to take the derivative of y^2 (i.e. $[y(x)]^2$), the multiplicative nature of the chain rule is required. That is, $\frac{d}{dx}(y^2) = 2y \cdot \frac{dy}{dx}$ and the derivative of the entire implicit function with respect to *x* is $2x + 2y \cdot y' = 0$.

I see implicit differentiation as a more generalized application of NMR because it is possible to conceptualize any one of the variables in an implicit equation as being the independent variable and thus take the derivative with respect to that chosen independent variable. For example, if we consider again the equation for a circle, $x^2 + y^2 = r^2$, one can use NMR to understand what it means to take the derivative of the implicit function with respect to *x,* as explained in the previous paragraph, or with respect to *y.* In taking the derivative with respect to *y,* NMR is used to recognize *x* as an implicit function of *y* and conceptualizing the way that the changes in *y* affect the changes in *x* and consequently how the changes in *x* affect x^2. Thus, NMR is recognizing the x^2 part of the function as an implicit composition of functions (i.e. $[x(y)]^2$).

Related Rates as an Application of NM

Before proceeding, I define a "function of an implicit variable" as a variable that can be inferred to be a function of some other variable that is not explicitly present in the equation. For example, in the equation $x^2 + y^2 = r^2$, if *x* and *y* are both functions of time, *t,* then *x(t)* and *y(t)* can be thought of as functions of an implicit variable and the original equation can be conceptualized as $[x(t)]^2 + [y(t)]^2 = r^2$. Note that an "implicit function" is not the same thing as a "function of an implicit variable." Implicit functions are those in which one variable present in the equation is conceptualized as a function of the other variable present in the equation. Functions of an implicit variable are those in which a present variable is conceptualized as a function of a variable that is not present in the equation. Related rates problems involve finding a rate when another, related, rate is known. Related rates require differentiation of an equation with function(s) of an implicit variable, which is often in typical calculus problems. Notice that the word "implicit" appears in both of the terms "implicit function" (as defined in the previous section) and "function of an implicit variable." This is important because although both terms and situations are slightly different, they both involve implicit, or hidden functions that are inferred based on the context. This is another way in which I see implicit differentiation and related rates as being connected to one another.

Related rates are also connected to both the chain rule and implicit differentiation through NM. I see related rates problems as simply applying NM in meaningful contexts. Infante (2007) says that "related rates problems require the student to investigate the relationship(s) between two or more changing quantities, one of which is unknown and needs to be found" (p. 23). I agree with Infante, except that related rates problems require students to investigate relationships with *more* than two changing quantities. Investigating the relationships between just two quantities in a meaningful context would be classified as an applied derivative, according to my interpretation. For example, investigating the relationship between time elapsed in seconds and calories burned for a person's workout one day. Related rates problems are different in that they typically involve some sort of composition of functions, often with an implicitly-defined variable, meaning there are more than two changing variables. It is true that solving related rates problems draws on a variety of mathematical knowledge, like geometry and variable (Infante, 2007). However, in order to ultimately solve a related rates problem, the multiplicative nature of the chain rule must be applied. NMR is interpreting and understanding the ways in which the changes in different variables in a related rates problem affect the changes in the other variables and leads to understanding how or why to apply the chain rule in order to solve the problem.

To understand how NM can help make sense of how or why to apply the chain rule or implicit differentiation in order to ultimately solve a related rates problem, consider the following related rates problem in Figure 1: y represents the distance between the school and the base of the ladder, x represents the height of the top of the ladder from the ground, and the ladder itself forms the hypotenuse of the right triangle and is 25 feet long. There are two ways in which to make sense of and solve this problem.

The first way to use NM to make sense of this problem is to consider the way in which the different variables affect one another. In this context, as time changes, the length of y increases. Subsequently, as the length of y increases, the length of x decreases (the student in the problem is falling to the ground). That is, NMR is recognizing that the following function composition of functions exists: $x(y(t))$. In order to find the rate that x is changing with respect to time (i.e. $\frac{dx}{dt}$), we can take the derivative of this composition of functions with respect to time: $\frac{dx}{dt} = \frac{dx}{dy} \cdot \frac{dy}{dt}$ (i.e. $x' = x'(y) \cdot y'(t)$). We know that $\frac{dy}{dt} = y'(t) = 2$. In order to find $\frac{dx}{dy}$, we need to find an equation for x in terms of y. Through calculation, we can also determine how fast x changes with t, dx/dt, leading ultimately to the equation $\frac{dx}{dt} = \frac{dx}{dy} \cdot \frac{dy}{dt}$. In sum, NMR is directly involved in this context because of how t affects y, which in turn affects x.

A student is painting the high school and standing at the top of a 25-ft. ladder that sits perfectly against the wall. He is horrified to discover that the ladder begins to slide away from the base of the school at a constant rate of 2 ft./second. At what rate is the top of the ladder carrying him toward the ground when the base of the ladder is 16 feet away from the school?

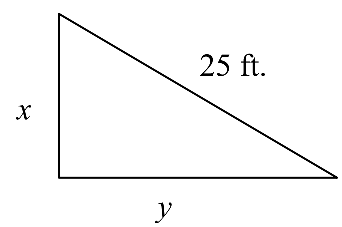


Figure 1. Typical ladder-sliding related rates problem.

The previous explanation shows how NM is connected to related rates through a function composition perspective, essentially involving the same concept behind the chain rule. Related rates problems may also involve implicit functions, meaning it would involve NM through the implicit differentiation perspective. For example, through the Pythagorean Theorem, x and y can be related to one another with $x^2 + y^2 = 25^2$. However, unlike implicit differentiation involving *dy/dx* or *dx/dy,* this time we have a “hidden” implicit variable time and we want to know $\frac{dx}{dt}$, or the rate at which the top of the ladder is approaching the ground with respect to time. I believe NM is foundation for conceptualizing both *x* and *y* changing with time and that with the x^2 and with y^2 parts of the equation are compositions of functions with time as the independent variable: $x(t)^2 + y(t)^2 = 25^2$. By differentiating both sides of the equation with respect to time, we get $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$. In sum, NMR leads to recognizing that *x* and *y* are functions of the implicit variable of time and is used to make sense of the derived equation ($2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$).

Mental Actions of NMR

In this section, I describe what the mental actions might look like for NM. I created a conceptual framework (see Table 1) for possible mental actions of NMR using a combination of Jones’ (2018) framework for NMR mental actions, that he extended from Carlson et al.’s (2002) framework for covariation, and an extension of Thompson and Carlson’s (2017) framework for mental actions of covariational reasoning.

Jones (2018) explains that in extending covariational reasoning to NM, each corresponding NM mental action involves “chained reasoning” (p. 6). Chained reasoning refers to the conceptualization of how, within a composition of functions structure, changes in the independent variable affect changes in the next outer variable within the composition. Changes in one variable in the composition of functions in turn affect changes in the next outer variable in the composition and so on and so forth in a sequence of variables, from the innermost variable to the outermost variable.

I initially tried creating my conceptual framework by only extending ideas from Thompson and Carlson’s (2017) (newer) covariational reasoning framework to NMR, but there were aspects of the (older) framework (Carlson et al., 2002) that I still felt were relevant and I anticipated seeing in my interviews in the form of NMR. Although I describe this framework in hierarchal levels, it is possible that some mental actions, which in the framework hypothetically come “before” others in students’ reasoning, might actually exhibit themselves in a different order in students’ reasoning.

Table 1

NMR Mental Action Framework

NMR Mental Action Level	**Description**
NM Relationship (RE)	Recognize the existence of a composition of functions. Recognize independent and dependent relationships within the composition of functions.
NM Pre-Coordination (PC)	Conceptualize that given a composition of functions, changes in the independent variable will cause changes in the second variable which will cause changes in the third variable and so on.
NM Increase/Decrease (ID)	Given a composition of functions, coordinate the change in the independent variable with whether the second variable increases or decreases and coordinating whether increases or decreases in the second correspond with increases or decreases in the third, and so on.
NM Coordination of Values (CV)	Given a composition of functions, coordinate values of the independent variable with corresponding values in the second and third variables and so on, creating sets of values.
NM Amount (AMT)	Given a composition of functions, coordinate how much each variable increases or decreases.
NM Chunky Continuous (CC)	Given a composition of functions, coordinate how much each variable increases or decreases for successive intervals in the independent variable.
NM Smooth Continuous (SC)	Given a composition of functions, envision changes in all of the variables smoothly, continuously, and simultaneously.

Consider the composition of functions, $f(g(x)) = \mathrm{e}^{\sin(x)}$, to illustrate this framework.

The first mental action for NMR is “NM Relationship” (RE). Although Jones (2018) uses the term “relationship” in his hypothetical NMR mental action framework, my use of the term “relationship” is slightly different. Here, someone sees the function $f(g(x)) = \mathrm{e}^{\sin(x)}$, and recognizes the existence of a function composition. She/he understands that given such a function, *f* is dependent on *g* and *g* is dependent on *x*. Here, a person recognizes the dependent relationship, but she/he does not explicitly think about, or talk about, the *changes* in the value of *x* that will cause *changes* in the value of *g* which will, in turn, cause *changes* in the value of *f*. Someone employing this mental action may say, "*f* depends on *g* which depends on *x*" or "given an *x,* you will have a *sin(x)* value and a corresponding $e^{\sin(x)}$ value."

The second mental action for NMR is "NM Pre-Coordination" (PC), which is an extension of Thompson and Carlson's (2017) framework and is similar to Jones' (2017) description of "relationship." Here, a person envisions the three variables in a function composition changing, but asynchronously. She/he recognizes that, given a composition of functions *f(g(x)), x* can change which will cause changes in *g* which will causes changes in *f.* For example, someone employing this mental action might say, "as *x* changes, there will be changes in the value of *sin(x)* and corresponding changes in the value of $e^{\sin(x)}$."

The third mental action is analogous to Thompson and Carlson's (2017) "Gross Coordination of Values" and Jones' (2018) "Increase/Decrease." Similarly, I refer to this mental action as "NM Increase/Decrease" (ID). Jones (2018) explains the second mental action level of NM, *increase/decrease*, "may consist of coordinating the change in the first variable with whether the second variable increases or decreases and coordinating whether increases or decreases in the second correspond with increases or decreases in the third, and so on" (p. 7) In our example, the increase/decrease mental action level of NM might be imagining that as *x* increases, $\sin(x)$ increases and then decreases and increases again, which would then make it so that $e^{\sin(x)}$ increases and decreases, but between different values. Here, the person is still conceptualizing these increases or decreases asynchronously.

The fourth mental action extends from Thompson and Carlson's framework and is "NM Coordination of Values" (CV). Here, there is coordination of values in one variable (*x*) with values of the other variables (*g,* or *sin(x)* and *f,* or $e^{\sin(x)}$), resulting in sets of values. For example, someone employing this mental action is tracking values of *x* as it changes ($x = 0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$) and then tracking corresponding values of *sin(x)* ($\sin(x) = 0, \frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, 1$) and then tracking corresponding values of $e^{\sin(x)}$ $\left(1, e^{\frac{1}{2}}, e^{\frac{\sqrt{2}}{2}}, e^{\frac{\sqrt{3}}{2}}, e\right)$. Here, there is no conceptualizing of the sizes of the changes between the values but simply mentally keeping track of the corresponding values.

The fifth mental action is an extension of Carlson et al.'s (2002) original framework and is termed "amount" in Jones' (2017) framework: "NM Amount" (AMT). Here, there is coordination of *how much* each variable in the composition of functions increases or decreases. In our example, this mental action level might consist of imagining that as *x* increases, from say 0 to $\frac{\pi}{6}$, imagining $\sin(x)$ increasing from 0 to $\frac{1}{2}$, and then $e^{\sin(x)}$ consequently also simultaneously increases from 1 to $\sqrt{e}$.

The sixth mental action is extended from Thompson and Carlson's (2017) framework and is called "NM Chunky Continuous" (CC). This mental action is similar to the previous mental action (AMT) but extends that reasoning to successive intervals. For example, someone employing this mental action might think about *x* increasing from 0 to $\frac{\pi}{6}$, then from $\frac{\pi}{6}$ to $\frac{\pi}{4}$, and then from $\frac{\pi}{4}$ to $\frac{\pi}{3}$, all the while coordinating the corresponding changes in *sin(x)* and $e^{\sin(x)}$. As stated earlier, nested multivariational reasoning occurs as soon as one considers all three of these variables (x, f, and g) at once or conceptualizes the way that all three of these variables change together.

The seventh, and last, hypothetical NMR mental action is also extended from Thompson and Carlson's (2017) framework and is called "NM Smooth Continuous" (SC). Here, someone envisions increases or decreases in the independent variable's value as happening simultaneously with changes in the other variable's values in the composition of functions as happening simultaneously. These changes are happening smoothly, and simultaneously. In our example, someone may think about x increasing and $sin(x)$ simultaneously oscillating between -1 and 1 and $e^{\sin(x)}$ oscillating between $\frac{1}{e}$ and e.

Carlson et al.'s (2002) framework progresses to conceptualizing changing rates of change; how the rate of change itself varies. In my thesis, I only focus on the first three mental action levels as applied to NM. Future research might explore how NM extends to conceptualizing changing rates of change in the contexts of chain rule, implicit differentiation, and related rates.

Hypothetical Learning Trajectory

Because of my approach to the concept of chain rule, implicit differentiation, and related rates, this study is focused on how students may come to develop understanding for these concepts through the common underlying concept of NM. For my study, I have created a hypothetical learning trajectory (Simon, 1995) rooted in NM for how students might develop an understanding of these concepts. Simon (1995) explains that a hypothetical learning trajectory (HLT) is made up of three components: "the learning goal that defines the direction, the learning activities, and the hypothetical learning process – a prediction of how the students' thinking and understanding will evolve in the context of the learning activities" (p. 136).

Infante (2007) created an HLT that is focused on helping students solve related rates problems by developing a deeper understanding of average rate of change and instantaneous rate of change and revisiting and rebuilding a deeper understanding of the chain rule. The HLT I have constructed builds on and extends Infante's HLT but is different in that it hopefully leads to the students' own *development* of the multiplicative nature of the chain rule through NMR. The HLT that Infante constructs assumes that students have already learned the chain rule, while the HLT I have constructed assumes that students have never been exposed to the chain rule, implicit differentiation, or related rates. I conducted a small-scale teaching experiment based on tasks intended to support the development of the schemes in the HLT.

Steffe and D'Ambrosio (1995) point out that Simon's (1995) descriptions of HLTs and mathematical learning is primarily in terms of the concepts and operations of the teacher. They wonder if "Simon intends to go further and explain the mathematics of his students and mathematical learning in terms of the students' schemes of action and operations" (p. 153). My study is focused on exploring how students might develop an understanding of the chain rule, implicit differentiation, and related rates. I am not focused on the knowledge and processes of a teacher in reaction to these student understandings. Thus, I use an HLT as a tool for better understanding and exploring how students develop schemes for these three calculus concepts.

I explore the possibility of an HLT rooted in NM for developing understandings for the chain rule, implicit differentiation, and related rates. I am not claiming that this proposed HLT is the only way, or necessarily the correct way, for a student to develop understandings for these three concepts. However, I hypothesize that this HLT is a *possible* way for a student to develop powerful understandings of these three concepts were she/he to have them.

The way in which I constructed and explored this HLT is similar to how Weber and Thompson (2014) did in their study. Weber and Thompson first conducted a conceptual analysis for students' images of graphs and used this to inform their construction of an HLT for how students might extend their understanding of graphs of one-variable functions to graphs of two-variable functions. In describing their HLT, they listed the sequence of mental actions through which students might go to visualize the graphs for two-variable functions. I too performed a conceptual analysis for NM (see Nested Multivariation in Chapter 3) and used findings from pilot studies to inform the construction of an HLT for developing understanding of the chain rule, implicit differentiation, and related rates. My HLT consists of learning goals, learning activities and key questions, both of which were informed by a conceptual analysis for NM. The learning goals also describe a hypothetical learning process through which the students might go to develop conceptual and connected understandings for the chain rule, implicit differentiation, and related rates.

The HLT Rooted in NM

The HLT I created for the teaching experiment consists of learning goals, learning activities, and key questions. Table 2 describes the learning goals I hypothesized might help the student develop a deep and connected understanding of these three concepts. It is categorized into five main stages, each having a major learning goal. Each stage is subdivided into smaller sub-goals intended to reach the overall goal for that stage. The HLT is based on all I had learned in thought experiments, conversations with my advisor, extensions from Infante's (2007) dissertation, and pilot interviews. Later, in my methods section, I explain in detail how Infante's (2007) dissertation and the pilot interviews helped inform the creation of the HLT.

In this section, I present the learning activities and key questions of the HLT and explain how they are intended to help the student reach the learning goals of the HLT. I do not discuss all of the questions and discussion points in the full interview protocol because not all of them are fundamental to the HLT. That is, some questions and discussion points were intended to help students re-think about the meaning of derivatives, functions, and compositions of functions so that I could assess whether or not students had some fundamental understandings for developing understanding for the chain rule, implicit differentiation, and related rates. Additionally, some questions were intended to help me better understand how students were using NMR as they progressed through the HLT. In order to see the entire interview protocol, including those additional questions and discussion points, see the Appendix. In this section, I also explain how the learning activities and key questions came from the conceptual analysis I conducted for NM (see Nested Multivariation in Chapter 3).

Table 2

The Hypothetical Learning Trajectory Learning Goals

Stage	Description of Goal
Stage 1	**Develop the multiplicative nature of the chain rule.**
1a	Given a function composition *f(g(x))* that models a meaningful context, interpret *df/dx* as how many times as large the change in *f* is than an infinitesimal change in *x*.
1b	Given a function composition *f(g(x))* that models a meaningful context, interpret *df/dg* as how many times as large the change in *f* is than an infinitesimal change in *g*.
1c	Given a function composition *f(g(x))* that models a meaningful context, interpret *dg/dx* as how many times as large the change in *g* is than an infinitesimal change in *x*.
1d	Given a function composition *f(g(x))* that models a meaningful context, conceptualize how changes in *x* affect changes in the other two variables simultaneously.
1e	Given a function composition *f(g(x))* that models a meaningful context, and after finding specific values of *dg/dx* and *df/dg,* construct that $\frac{df}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}$ at a specific point.
Stage 2	**Generalize the chain rule and gain procedural fluency.**
2a	Continue to construct the multiplicative nature of the chain rule with specific examples.
2b	Formalize the chain rule for any function composition *f(g(x)).*
2c	Practice the chain rule with different compositions of functions to gain procedural fluency with its application.
Stage 3	**Develop the idea of variables being functions of the implicit variable of time and recognize subsequent existence of compositions of functions.**
3a	When variables change with time, conceptualize them as functions of time and represent them accordingly (e.g. if *r* changes with time, it can be conceptualized and written as *r(t)*).
3b	NMR is used in recognizing the need to use the chain rule in related rates problems where quantities can be conceptualized as functions of the implicit variable of time, creating subsequent compositions of functions.
Stage 4	**Develop the idea of implicit functions in an equation and recognize subsequent existence of compositions of functions.**
4a	Given an equation with variables *x* and *y,* one can conceptualize *y* as an implicit function of *x* or *x* as an implicit function of *y.* These implicit functions can be represented accordingly (e.g. *y(x)* or *x(y)*).
4b	Given equations with implicit functions, and subsequent compositions of functions, recognize the need for the chain rule in taking the derivative with respect to either implicit independent variable.
Stage 5	**Extend all of these ideas to more complicated implicit differentiation and related rates contexts.**
5a	Gain procedural fluency with more complicated implicit differentiation problems.
5b	Within equations that model more complicated related rates contexts, recognize functions of the implicit variable of time and subsequent compositions of functions and the need for the chain rule in taking the derivative of the equation with respect to time.

Stage 1: Develop the Multiplicative Nature of the Chain Rule

After discussing with the student, the meaning of the derivative in general and how to interpret it multiplicatively (see The Chain Rule as an Application of NM in Chapter 3), I introduce the "Chocolate Context" (see Figure 2).

> Let's say you make $9/hr. at your job and that you're OBSESSED with chocolate. You want to spend every penny that you make on chocolate. You can buy .15 lbs. of chocolate per dollar. Let us, for now, ignore tithing, taxes, etc.
>
> $D(h) = 9h$ and
> $c(D) = .15D$
>
> Where *D* is dollars, *h* is hours you have worked, and *c* is amount of chocolate (lbs.)

Figure 2. The chocolate context.

I then ask the students the following key questions for this context:

- What is the value of *dD/dh*?
- What is the meaning of *dD/dh* in our context? What are the units of *dD/dh?*
- What is the value of *dc/dD*?
- What is the meaning of *dc/dD* in our context? What are the units of *dc/dD?*
- What would be the meaning of *c(D(h))*?
- How does hours worked affect the amount of chocolate you can buy?
- If we could find *dc/dh,* what would that mean in our context?
- What is the value of *dc/dh*? How do you know?

This Chocolate Context consists of simple and continuous linear functions so that the students can focus on the meaning of the derivatives and not get confused by messy calculations of differentiation. The first four questions are intended to help the student reach the sub-goals of

1a and 1b in Stage 1. That is, the questions encourage the student to interpret the "outside" and "inside" derivatives of *c(D(h))* multiplicatively. If the student does not initially interpret it multiplicatively, I ask her/him what the derivative tells us if there is a small change in *h* or a small change in *D,* thus encouraging her/him to think about how many times as large the change in the output is than the input within the derivative, thus covering sub-goals 1a and 1b in this stage of the HLT. I ask the student what the units of the derivatives are so they can remember the meaning of the derivatives they are interpreting within the context.

Asking the student what the meaning of the composition of functions is and how the hours worked affects the amount of chocolate one can buy is intended to help her/him employ NMR, thus beginning to develop the multiplicative nature of the chain rule. The fifth and sixth key question are meant to elicit RE or PC NMR mental actions, which involve recognizing the independent and dependent relationships within the composition of functions and/or conceptualizing how a change in the independent variable will cause changes in the second and third variables. In so doing, these questions are meant to help the student reach sub-goal 1d.

The second to last question is meant to help the student to reach the sub-goal of 1c and interpret the overall derivative of the composition of functions multiplicatively. Again, if the student does not initially interpret the derivative multiplicatively, I ask her/him what the derivative tells us if there is a small change in *h*. The last question is intended to help the student reach sub-goal of 1e in Stage 1 of the HLT by encouraging her/him to think about how they might find *dc/dh,* potentially considering using the "inside" and "outside" derivatives to find the derivative of the entire composition of functions at a specific point. This simple Chocolate Context, which involves only linear functions, is intended to create a non-complex place where a student can begin to coordinate how much each variable increases or decreases within the derivative (AMT in the NMR mental action framework), thus beginning to develop the multiplicative nature of the chain rule and reach the sub-goal 1e during this stage.

Next, I introduce the "Carnival Context" (see Figure 3).

The following graphs show two functions, *f* and *g*. The input of function *g* is temperature in degrees Fahrenheit and the output is the expected attendance at a neighborhood carnival. The input of function *f* is number of people attending the carnival and the output is the expected revenue earned by the carnival.

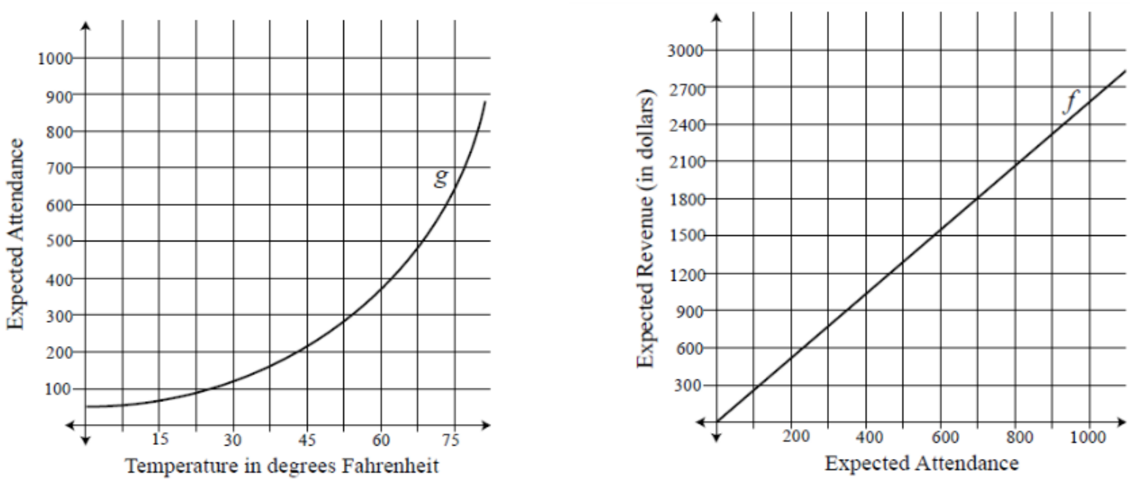


Figure 3. Carnival context (Carlson, 2016, p. 71).

I then ask the student the following key questions for this context:

- If we calculated *dg/dx*, what would that mean in our context? What would be the units of *dg/dx*?
- If we calculated *df/dg,* what would that mean in our context? What would be the units of *dg/dx?*
- What does the function *f(g(x))* mean to you in this context?
- How would you describe the way in which temperature (*x*) affects revenue (*f*)?

- If we calculated *df/dx,* what would that mean in our context? What would be the units of *df/dx*?
- How does *df/dx* relate to *f(g(x))*? How does this relate to your description of how temperature (*x*) affects revenue (*f*)?

This Carnival Context and the accompanying questions are meant to help students reach the sub-goals of this stage of the HLT in a similar way to the Chocolate context but is different in that it is only a graphical representation with no equations or quantities for the derivatives. Thus, this context elicits a more theoretical interpretation and provides an opportunity for the student to connect their graphical understanding of derivatives to NMR and to their development of the multiplicative nature of the chain rule.

The first two key questions are meant to help the student reach sub-goals 1a and 1b, but more theoretically and within a graphical context. Again, if the student does not interpret the derivative multiplicatively, I ask her/him what the derivative tells us about small changes in the independent variable. The next two questions are intended to help the student employ NMR and consider the way in which the variables within the composition of functions are related to and change with one another, thus intending to help her/him reach sub-goal 1d. The last two questions are intended to help the student interpret the overall derivative or reach sub-goal 1c. The last question, specifically, is intended to help the student connect that *df/dx* is the derivative of the composition of functions and the only way to find that is to simultaneously consider *df/dg* and *dg/dx*, thus employing NMR.

Next, I introduce the "Running Context" (see Figure 4), where the graphs are supposed to model the student's hypothetical run.

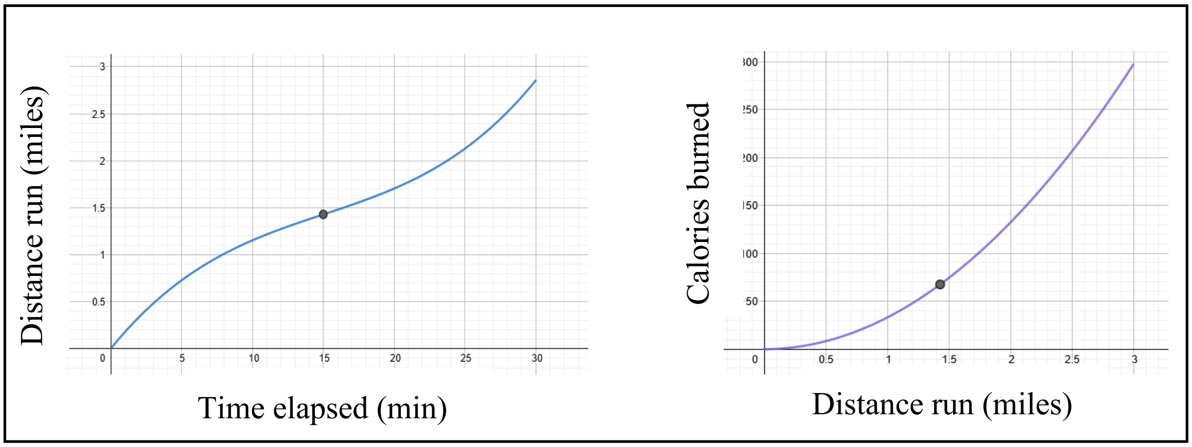


Figure 4. Running context.

I then ask the students the following key questions:

- Say right at 15 minutes, *dD/dt* is .1 (or 1/10). What is the meaning of *dD/dt*=1/10 in our context?
- Say at that moment, you have gone 1.5 miles, and so *dc/dD* at that moment (or at 1.5 miles) is 100. What is the meaning of *dc/dD*=100 in our context?
- What would *c(D(t))* mean in this context? How does times elapsed affect calories burned?
- What would the derivative of that function, or *dc/dt* mean in our context?
- How can you use what you know to find *dc/dt?* Explain your thinking.
- Say right at 25 minutes, you are running .12 miles/minute, and you are burning 110 calorie/mile. How could you use that information to find *dc/dt* at that moment? How do you know?
- In general, if you wanted to find *dc/dt*, how could you use *dD/dt* (or miles per minute) and *dc/dD* (calories per mile) at any given moment/location in order to find *dc/dt* (calories per minute) at that moment? How do you know?

This Running Context is meant to build on the previous two contexts by utilizing graphs as well as given quantities for the derivatives. The student is given the derivatives at two different points on the graph so that she/he can begin to recognize patterns and generalize her/his thinking. Also, it is simple to calculate *dc/dt* when *dD/dt*=.1 (or 1/10) and *dc/dD*=100: A student may find *dc/dt* by coordinating 1/10 (1 mile per 10 minutes) and 100 (100 calories per 1 mile) and see that because the 1's in the two fractions represent the same quantity, and 100 calories must correlate with 10 minutes. Thus, by giving the student .12 and 110 for *dD/dt* and *dc/dD* respectively, it forces her/him to stretch their thinking a little bit further and encourages them to develop the multiplicative nature of the chain rule by coordinating amounts of change in the variables, or by using AMT in the NMR mental action framework.

In a similar way as the Chocolate and Carnival contexts, the first two key questions are intended to help the student reach sub-goals 1a and 1b. Also, the third question is intended to elicit NMR and help the student reach sub-goal 1d. The fourth question is meant to help the student reach sub-goal 1c. The last three questions are meant to encourage the student to reach sub-goal 1e by developing the multiplicative nature of the chain rule at two specific points and then generalizing that to *any* specific point.

Stage 2: Generalize the Chain Rule and Gain Procedural Fluency

Next, I introduce the "Running Context with Equations" (see Figure 5).

Let's say in a perfect world, you run at a constant rate of .1 miles/minute. That is, let

$D(t) = .1t$ and
$c(D) = 20D^2 + 40D$

Where *t* is time in minutes, *D* is distance traveled in miles, and *c* is calories burned.

Figure 5. Running context with equations.

I explain that we're going to calculate some of her/his running rates right at 20 minutes, but first, we're going to calculate how far she/he has gone at 20 minutes. I ask the student the following key questions. The answers to questions marked with an asterisk are the ones that I write on a separate sheet of paper so that the student can recognize patterns in their thinking and work and ultimately generalize the chain rule. I explain why I re-write the students responses on a separate sheet of paper in the Pilot Studies section in Chapter 4 (see Figure 11):

- How far have you traveled at 20 minutes?
- *What is an equation for *dD/dt?*
- What is *dD/dt* at 20 minutes? How do you know?
- *What is an equation for *dc/dD*?
- What is *dc/dD* at that same moment? How do you know?

I remind the student that our ultimate goal is to be able to find *dc/dt* which is the derivative of *c(D(t)).*

- *What is an equation for *c(D(t))*?
- What is *dc/dt* at 20 minutes?

These first seven questions are intended to help the student reach the first sub-goals in Stage 2 of the HLT by continuing to construct the multiplicative nature of the chain rule within specific examples. Here, the student begins to understand that she/he can find the derivatives at a specific point using the given equations. I then ask the next two questions:

- *How can you use what you have found so far to write a general equation that will give you *dc/dt* at any time *t?* How do you know?
- *What is an equation for *dc/dt* in terms of only *t* (instead of D and t)*?*

After the student answers these questions, I point out to her/him what she/he has found is the derivative of the composition of functions they created. These last two questions are meant to help the student construct the multiplicative nature of the chain rule at a point and at any point, with a general equation, thus reaching sub-goal 2a in this stage of the HLT. The purpose of the last question is to help the student generalize the chain rule by being able to see the derivative of the "outside" and "inside" functions from the original composition of functions directly. Otherwise, there would be two variables in the derivative when the original function had only one independent variable.

Next, I introduce the "Dash from the Incredibles Context" (see Figure 6).

Say that now, you are "Dash" from The Incredibles, and you are running at an incredible pace.
$D(t) = 2^t$ and
$c(D) = \mathrm{D}^3$

Figure 6. Dash from the Incredibles context.

I then ask the student the following key questions:

- How far have you traveled at 5 minutes?
- *What is an equation for *dD/dt?*
- What is *dD/dt* at 5 minutes?
- *What is an equation for *dc/dD*?
- What is *dc/dD at that same moment?*
- *What is an equation for *c(D(t))*?
- What is *dc/dt* at 5 minutes?

- *How can you use what you have found so far to write a general equation for *dc/dt?*
- *What is an equation for *dc/dt* in terms of only *t* (instead of D and t)?

After the student has answered all of the questions, I point out to her/him that they have found the derivative of the composition of functions they created. These questions are structured the same as the Running Context with Equations and have the same purpose.

Once I re-write all of their answers to the questions with an asterisk, I box the two equation for the compositions of function and their corresponding derivatives. I then ask the students the following questions:

- What patterns do you notice? Why does it all make sense?
- Given any function for *c(D(t)),* what is *dc/dt?*

As I explain in the Pilot Studies section in Chapter 4, in order to help the student think about the chain rule in a slightly different way and generalize it to any composition of functions, I found that by my re-writing down their conclusions about the multiplicative nature of the chain rule from the Running Context, or the answers to the questions marked with an asterisk, that they had a more organized and consolidated picture of their thinking from which they could more easily generalize the chain rule. That is, the student can shift her/his thinking from $\frac{dD}{dt} \cdot \frac{dc}{dD} = \frac{dc}{dt}$ to $\frac{dc}{dt} = \frac{dD}{dt} \cdot \frac{dc}{dD}$. If the student can begin to shift her/his thinking in this way, she/he begins to reach the sub-goal 2b for Stage 2 of the HLT by generalizing the chain rule for any function composition.

Next, I have the student try to use the chain rule to solve the following problems:

- Let $g(x) = sin(f(x))$ Just so you know, the derivative of *sin(x)* is *cos(x)*. Do you have a hypothesis of what the derivative of this function might be?

- Let $h(x) = sin(x^2)$. What would be the derivative of this composition of functions? In other words, what is $\frac{dh}{dx}$?
- Let $j(x) = [\sin(x)]^2$. What would be the derivative of this composition of functions? In other words, what is $\frac{dj}{dx}$?
- Say you have a general composition of functions *f(g(x))*. Can you write a rule for how you could find the derivative of a composition of functions?

The first three questions are meant to help the student begin to practice the chain rule with different compositions of functions to gain procedural fluency with its application, thus reaching sub-goal 2c of this Stage in the HLT. The last question is meant to more completely generalize the chain rule for any function composition and help the student solidify everything she/he learned in the first two stages of the HLT.

Stage 3: Develop the Idea of Variables Being Functions of the Implicit Variable of Time and Recognize Subsequent Existence of Compositions of Functions

Next, I introduce the "Snowman Context" (see Figure 7).

The body of a snowman is in the shape of a sphere whose radius is melting at a rate of .25 ft./hr. Assuming the body stays spherical, how fast is the volume changing when the radius is equal to 2 ft.? Remember that for a sphere, $V = \frac{4}{3}\pi r^3$.

Figure 7. The snowman context.

After having the student draw a picture of the situation and make note of both what we know and what we are looking for, I ask the student what things are changing with time. I then explain that we can write those as functions of time. I write *V* as *V(t)* and *r* as *r(t)* and I explain that we can write the variables in this way to help us remember that they are changing with time. By talking about this with the student, I help her/him to understand that when variables change with time, they can be conceptualized as functions, thus reaching the sub-goal 3a in Stage 2. I then ask the student if it is fair to write $V(t) = \frac{4}{3}\pi[r(t)]^3$. By discussing with them this new equation, the student becomes one step closer to sub-goal 3b in Stage 3 of the HLT by conceptualizing *V* and *r* as functions of the implicit variable of time, creating a subsequent composition of functions. I then ask the student the following key questions:

- What are we trying to find? How can you represent it as a derivative?
- The radius of the snowman is melting at a rate of .25 ft./hr. How can you represent that as a derivative?

If the student writes *dr/dt*=.25 *ft./hr.,* I know that they are not thinking about how the snowman melting implies a negative change and so the derivative should actually be negative. Thus, I ask the student, "if the snowman's body was growing at a rate of .25 ft./hr., would you represent that as a derivative?"

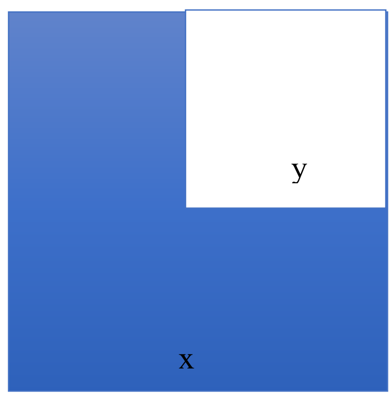
I finish by asking the student:

- How does $[r(t)]^3$ relate to what we have been doing before?
- How can we find *dV/dt*?

The first of these last two questions helps the student to officially reach sub-goal 3b in this stage of the HLT by recognizing the hidden composition of functions that exists in the equation. By so doing, the student is employing RE in the NMR mental action framework by recognizing the nested relationship and thus the need to use the chain rule when differentiating the equation. The last question is meant to guide the student to solve the problem by using all of the knowledge that she/he has built so far.

Stage 4: Develop the Idea of Implicit Functions in an Equation and Recognize Subsequent Existence of Compositions of Functions

First, I explain the difference between explicit and implicit equations with some examples (see the Appendix for details). Then, I introduce implicit differentiation to the student (see Figure 8).



The area of the square with side length *x* subtract the area of the square with side length *y* must always be 1. That is, $x^2 - y^2 = 1$ or the blue area will always be equal to 1. The question we will attempt to answer is: As *x* changes, by how much does *y* have to change?

Figure 8. Introduction of implicit differentiation.

Next, I explain that we're going to say that *y* is an implicit function of *x* and re-write the equation as $x^2 - [y(x)]^2 = 1$. By so doing, the student is beginning to see how when *y* is a function of *x* it can be represented accordingly, which is sub-goal 4a in the HLT. I then ask the student following key questions:

- Can you write what we are trying to find as a derivative?
- How does $[y(x)]^2$ relate to what we have been discussing over the past interviews?

- As *x* changes, by how much does the blue area have to change? In other words, what is $\frac{d[blue\ area]}{dx}$?

The first question is meant to help focus the student on the purpose of the problem and encourage her/him to continue to interpret the meaning of derivatives. The second question is meant to help the student begin to reach sub-goal 4b by recognizing the nested relationship (RE of the NMR mental action framework) and thus the need to use the chain rule in differentiation. Then, I explain that if $\frac{\mathrm{d[blue\ area]}}{\mathrm{dx}} = 0$ and $[\mathrm{blue\ area}] = \mathrm{x}^2 + [\mathrm{y(x)}]^2$, that implies that $\frac{\mathrm{d(x^2 - [y(x)]^2)}}{\mathrm{dx}} = 0$. After the student answers the third question and explains that as *x* changes, the blue area does not change I remind the student that we are trying to find how *y* changes with *x*, or *dy/dx*. I explain that in their calculus class, the student should have learned that the derivative of a sum or difference is equal to the sum or difference of the derivative. That is, I explain that $\frac{\mathrm{d(x^2 - [y(x)]^2)}}{\mathrm{dx}}$ is equal to $\frac{\mathrm{d(x^2)}}{\mathrm{dx}} - \frac{\mathrm{d\,[y(x)]^2}}{\mathrm{dx}}$, which means that $\frac{\mathrm{d(x^2)}}{\mathrm{dx}} - \frac{\mathrm{d\,[y(x)]^2}}{\mathrm{dx}} = 0$. I then ask the following key questions:

- What is the difference between $\frac{d(x^2)}{dx}$ and $\frac{d\,[y(x)]^2}{dx}$?
- What is the derivative of each of these?

The first question is meant to elicit NMR by encouraging the student to recognize the nested relationship and thus the need to use the chain rule in differentiation. That is, the first question is meant to further help the student to reach sub-goal 4b. The second question is meant to solidify the student's understanding of sub-goal 4b by actually using the chain rule in differentiating the equation. I end by asking, "What question are we trying to answer?" and "How can we use this information to answer it?" If the student struggles to solve for *dy/dx*, I explain that since we know that *y* is an implicit function of *x*, we can re-write the equation as

$2x - 2y \cdot \frac{dy}{dx} = 0.$

Then, in order to help student think not only about *y* as an implicit function of *x,* but also about *x* being an implicit function of *y,* I ask the student to take the same original equation and answer the question, "if *y* changes by some amount, how much will *x* change? In other words, what is the derivative of *x* with respect to *y?* They go through the entire process again, re-solidifying their understanding of sub-goals 4a and 4b.

I end this stage of the HLT with one more implicit differentiation problem to help the student further solidify their understanding of Stage 4 in the HLT. I give them the following question: If $x^2 + [f(x)]^3 = 9$ and $f(1) = 2$, find $f'(1)$.

Stage 5: Extend All of These Ideas to More Complicated Implicit Differentiation and Related Rates Contexts

Next, I introduce the "Shuttle Launch Context" (see Figure 9).

A camera is filming a shuttle launch and needs to stay focused on the shuttle by increasing the angle as the shuttle ascends. The camera is 2 miles from the launch-pad. If the shuttle is traveling vertically at 0.2 miles/second, how fast should the angle be increasing when the shuttle is 3 miles above the ground?

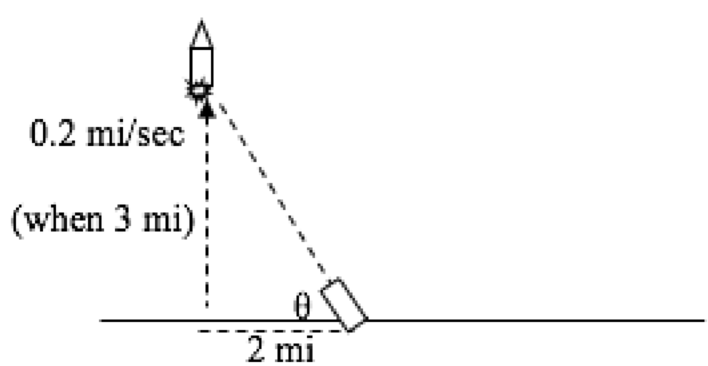


Figure 9. The shuttle launch context.

I first have the students draw a picture of the situation and make note of the things that they know as well as what they are looking for. I then ask them the following key questions:

- What are we looking for? Can you represent it as a derivative?
- What does it mean for the shuttle to be traveling vertically at .2 miles/second? Can you represent that as a derivative? (Labeling your picture may help in doing this)
- What are the quantities that change with time?
- Can you relate those quantities with an equation?
- Can you represent the quantities that are changing with time as functionss of time?

The first two questions are intended to help the student focus on the goal of the problem as well as provide her/him with an opportunity to practice interpreting derivatives. The last three questions are meant to help the student to model the problem and recognize functions of the implicit variable of time, hopefully taking one step towards recognizing subsequent compositions of functions and the need to use the chain rule and towards reaching sub-goal 5b of Stage 5 in the HLT.

If the student struggles, I remind her/him what we had done before with the Snowman Context. That is, we re-wrote *r* as *r(t)* to help us remember that it is a function of time. I then ask, "How can you find $\frac{d\theta}{dt}$? " And "What is $\frac{d\theta}{dt}$? "

I then give the student a more complex implicit differentiation problem, and I see what they can do on their own without any probing questions. However, if the students get stuck, depending on their needs, I ask them different questions to help move them forward. The problem I give them to solve is: Let $\sqrt{x+y} = x^4 + y^4$. Find *dy/dx* or *y'*. This problem provides an opportunity for the student to reach sub-goal 5a and gain procedural fluency with a more complicated implicit differentiation problem (see Figure 10).

> Lastly, if there is time, I give the students the "Airplane Context" (see Figure 10).
> A plane flying horizontally at an altitude of 1 mi and a speed of 500 mi/h passes directly over a radar station. Find the rate at which the distance from the plane to the station is increasing when it is 2 mi away from the station.

Figure 10. The airplane context.

Similarly, by this point, I do not ask the student any specific questions. I simply see what they can do with the problem and if they get stuck, I ask them questions according to their needs.

Purpose of My Study and Research Questions

In order to explore the HLT rooted in NM, I conducted a small-scale teaching experiment with four first-semester calculus students. Typically, the chain rule and implicit differentiation are first taught as procedures and then applied to related rates problems. The nature of my HLT is distinct in that it aims to help students develop conceptual understanding of the chain rule through meaningful contexts, similar to those used with related rates problems, and then continue to help them build on this understanding to develop conceptual understanding for implicit differentiation and related rates problems. Overall, my teaching experiment was guided by the following three research questions:

1. How was nested multivariation used as the first-semester calculus students progressed through the HLT?
2. What kind of understandings did the first-semester calculus students develop for these three concepts within each major stage of the HLT?
3. Where in the teaching experiment did students struggle in a way that suggested a needed revision to the HLT?

CHAPTER FOUR: METHODS

In my study, I explored students' cognitive processes in developing understandings for the chain rule, implicit differentiation, and related rates. Because I studied *processes,* qualitative methods were appropriate for this study (Maxwell, 2013). I created an HLT rooted in NM for developing an understanding for these three calculus concepts. In order to explore the viability of this HLT, I conducted small-scale teaching experiment using tasks intended to support development of the schemes in the HLT.

I conducted a series of pilot studies to help me create and refine my HLT as well as more effectively critique Infante's (2007) study to find where it needs extension. In this section, I first describe my pilot studies and how the pilot study results affected the creation of an HLT rooted in NM. Next, I describe how my pilot studies helped me to identify places in Infante's study that need extension and how this relates to the HLT. I end this section by outlining the resultant HLT and the methodology for the teaching experiment and analysis of the data for this study.

The Pilot Studies

The pilot studies that I conducted were invaluable in the development of the HLT rooted in NM. Once I had my initial ideas for an HLT sketched out, I went through an iterative cycle where I created learning activities, tested them, and edited them according to my experiences. It was only after I solidified the learning activities and processes through which students might go that I was able to identify and articulate the learning goals for the HLT. Creating and refining the learning activities helped me better understand what the goals might be and through what processes a student might go to develop connected understandings for these three concepts. In this section, I describe the important parts of this iterative process in creating my HLT.

The first draft of my pilot study only consisted of learning activities and questions intended to help a student develop the multiplicative nature of the chain rule. Developing the multiplicative nature of the chain rule was the most laborious part of my creation of the HLT. Once I better established how a student might develop the chain rule, I was able to work on the parts of the HLT more focused on implicit differentiation and related rates. The first draft was created through thought experiments and conversations with my advisor. In this first draft, I did not yet have explicit learning goals because I was simply exploring what learning activities might help a student develop the multiplicative nature of the chain rule her/his self.

The first draft consisted of the "Running Context" (see The HLT Rooted in NM in Chapter 3) and a "Square Context" where the students explored the relationship between quantities that describe a square changing with time. I incorporated the Square Context because I knew that in order for the students to eventually generalize the chain rule to any composition of functions, the students needed to explore the chain rule with equations for compositions of functions and corresponding derivative equations, as opposed to only derivatives at a point. The equations for the square context were $A = l^2$ and $l = 2t$, where A is the area of the square in square inches, l is the length of the side of the square in inches, and t is time elapsed in seconds.

To explore this first draft, I interviewed one mathematics education graduate student and one undergraduate business student. In interviewing them, I found that the composition of functions, $A(l(t))$, and the corresponding derivatives for this context were too simple for a student to clearly recognize a pattern, or the multiplicative nature of the chain rule, by looking at the original composition and its corresponding derivative. Thus, I changed the HLT to continue to explore the Running Context but add equations, resulting in the "Running Context with

Equations" (see the HLT Rooted in NM in Chapter 3). Throughout this first exploration, the key questions were very similar to those in the final HLT. I only tweaked the order of the questions.

After testing out this first draft, I felt that the HLT needed to begin with an even simpler context than the Running Context so that a student might have an additional opportunity to use intuition to develop the multiplicative nature of the chain rule which might help her/him to generalize it to equations when they got to the Running Context with Equations. Thus, I added the "Gumball Context," which was exactly the same as the "Chocolate Context" (see The HLT Rooted in NM in Chapter 3) except with different numbers and with number of gumballs instead of lbs. of chocolate per dollar. Again, the questions were very similar to the questions in the final HLT.

Testing out this second draft of the HLT on another mathematics education graduate student helped me realize that number of gumballs is a discrete quantity and although it didn't obviously affect a student's ability to develop the multiplicative nature of the chain rule, I realized that differentiating gumballs as a function of hours worked was mathematically incorrect. Thus, I changed number of gumballs, a discrete quantity, to lbs. of chocolate, a continuously changing quantity.

The second thing that testing out this second draft of the HLT helped me realize is that in order for a student to generalize the chain rule to any composition of functions, it is important to re-write the student's conclusions about the multiplicative nature of the chain rule from previous learning activities on a separate sheet of paper so that she/he can more clearly recognize patterns in their reasoning. That is, after the Chocolate Context and the Running Context, a student begins to see that for different compositions of functions $f\big(g(x)\big)$, $\frac{df}{dg} \cdot \frac{dg}{dx} = \frac{df}{dx}$. She/he begins to understand that given the "outside" and "inside" derivatives, they can find the overall derivative by multiplying them together. However, in order to generalize the chain rule, slightly different thinking is required: in order to find the overall derivative, she/he needs to *find* the “outside” and “inside” derivatives and multiply them together. That is, she/he needs to realize that given a composition of functions $f\big(g(x)\big)$, $\frac{df}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}$.

In order to help a student to think about the chain rule in this slightly different way and generalize it to any composition of functions, I found that by my re-writing her/his conclusions about the multiplicative nature of the chain rule from the Running Context, they had a more organized and consolidated picture of their thinking from which they could more easily generalize the chain rule. For the Running Context, after two times of being given the values for $\frac{dD}{dt}$ and $\frac{dc}{dD}$ and being asked to find $\frac{dc}{dt}$, I have the student calculate the equations for $\frac{dD}{dt}$ and $\frac{dc}{dD}$ by using the equations for D(t) and $c(D)$. Then, I prompt her/him to use two derivative equations to find the equation for $\frac{dc}{dt}$. I have the student do this twice, with two different sets of equations for D(t) and $c(D)$, and then I re-write their conclusions, as well as the equation they created for $c\big(D(t)\big)$, on a separate sheet of paper (see Figure 11). Then, after their conclusions are more clearly laid out for them, I ask them how they might find $\frac{dc}{dt}$ given any equation for $c\big(D(t)\big)$. This allows for the students to change their thinking from $\frac{dD}{dt} \cdot \frac{dc}{dD} = \frac{dc}{dt}$ to $\frac{dc}{dt} = \frac{dD}{dt} \cdot \frac{dc}{dD}$.

In my pilot studies, before I re-wrote the student’s conclusions about the multiplicative nature of the chain rule on a separate sheet of paper, their work and their thinking was scattered across many different sheets of paper and it was difficult for them to know which parts to focus on in order to generalize the multiplicative nature of the chain rule. By making this simple adjustment, the student had an easier time seeing patterns in their reasoning and generalizing the chain rule to any composition of functions.

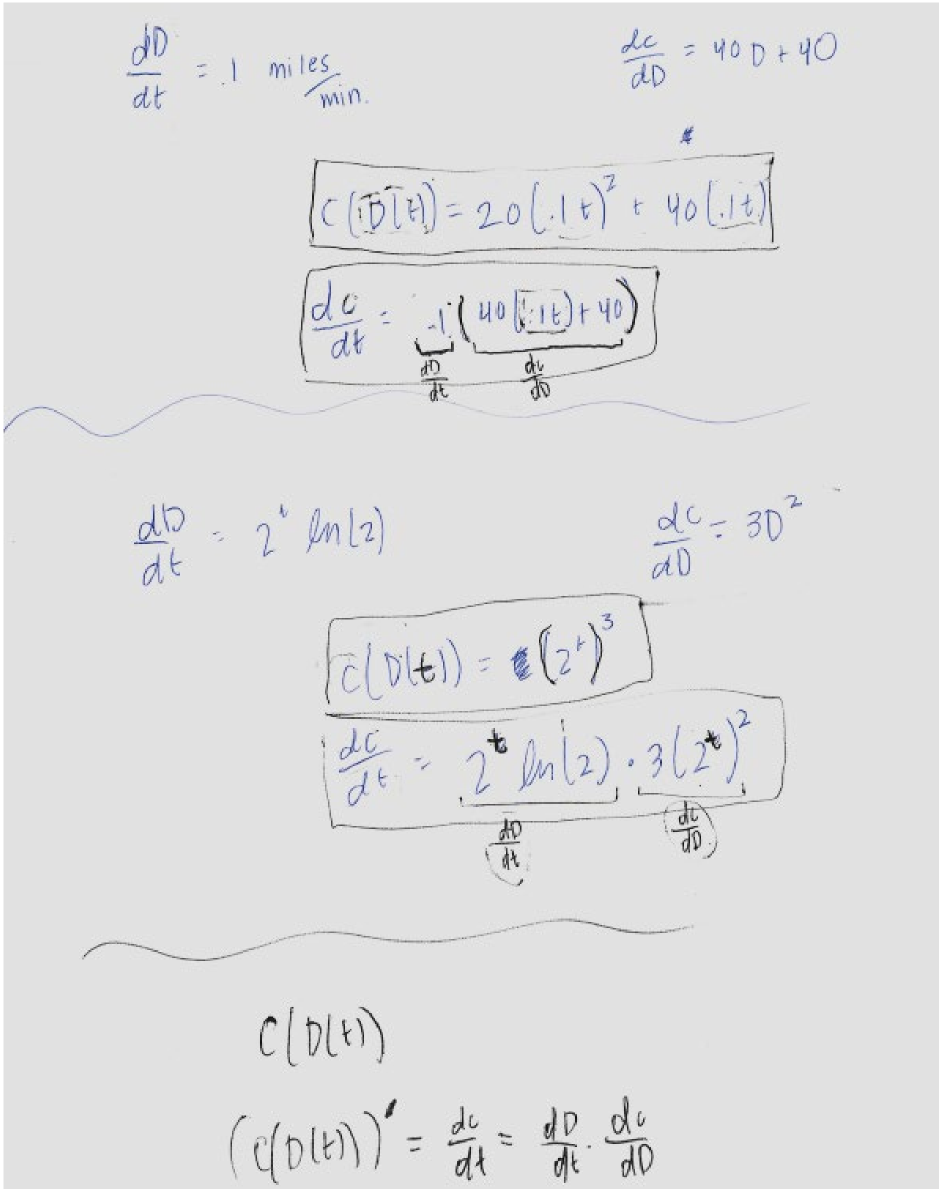


Figure 11. The re-written page of Student B's work so that he could generalize the chain rule.

After testing out this draft of the HLT, I also realized that students tend to come into the interviews with a lacking knowledge of compositions of functions. Thus, I added the "Carnival Context" (see The HLT Rooted in NM in Chapter 3) to give a student one more opportunity to employ NMR and develop the multiplicative nature of the chain rule. However, what makes the Carnival Context unique is that it is purely graphical and there are no equations or quantities given. I felt that by incorporating this graphical context, students were given an opportunity to build on their previous knowledge of derivatives and compositions of functions and how they relate to graphs.

Once I developed the learning activities for developing the multiplicative nature of the chain rule, I added some simple derivative of compositions of functions to the learning activities to give students the opportunity to gain some procedural fluency with the chain rule. My advisor and I then brainstormed together about how to introduce implicit differentiation and related rates. We knew that we wanted to incorporate a visual context in the development of implicit differentiation, because implicit differentiation tends to be taught in such an abstract way. Thus, we developed introducing implicit differentiation through the changing blue square (see The HLT Rooted in NM in Chapter 3). The questions we developed to ask with this introduction of implicit differentiation are the same as the final draft of the HLT. We also decided to pick two interesting related rates problems that my advisor had created for his own calculus class and use the next set of pilots as a way to explore what questions I might ask to help the students develop understanding for implicit differentiation and related rates.

I explored this version of the HLT with three calculus students who were in my lab at the time and volunteered to pilot with me. These pilots helped me understand how important it is to help students interpret the meaning of a derivative in meaningful contexts. Specifically, I realized that helping the students in my pilot studies to interpret the meaning of Leibniz' notation (i.e. $\frac{dy}{dx}$) in meaningful contexts helped them eventually use NMR to coordinate relationships in the amount of change (Infante, 2007) and develop the multiplicative nature of the chain rule. For example, I had my students interpret $\frac{dr}{dt} = 3\ cm/s$ as meaning that at the specific moment, the change in *r* is 3 times as large as the small, infinitesimal, change in units of time. In the pilot studies, I realized that if students were unable to interpret the meaning of a derivative by coordinating the change in two quantities, they were unable to coordinate relationships in amounts of change in more than two quantities (NMR). Conversely, in the pilot studies, I found that when students were able to interpret derivatives by coordinating the change in two quantities, they were able to employ NMR to ultimately develop the multiplicative nature of the chain rule themselves.

In these pilots with three calculus students in my lab, I saw that after they developed the multiplicative nature of the chain rule, they later recognized that for a function composition, such as $f(g(x))$, $\frac{df}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}$ that the *dg's* can "divide out" or "cancel out" (e.g. $\frac{df}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx} = \frac{df}{d\cancel{g}} \cdot \frac{\cancel{dg}}{dx} = \frac{df}{dx}$). Here, students use the idea of infinitesimals or thinking that "*dg*" represents an infinitely small amount of the variable "*g*." Recognizing that the *dg's* divide out helped students to be surer of their intuition and ideas about the multiplicative nature of the chain rule that they had developed so far. If we give *dg* the intuitive meaning of being a small change in *g,* then it is important to note that dividing out the *dg*'*s* is accurate inasmuch as *dg* does not equal 0. I realize that there is a deeper intricacy in the rigorous mathematical proof to account for what *df/dg* and *dg/dx* mean in accordance to the formal definition, and to incorporate the case $dg = 0$, but the intuition of dividing out the *dg's* does agree with the formal mathematics otherwise.

These pilots also helped me to realize the importance of focusing on NM throughout the entire interview. When I was just beginning to create my interview protocol and conduct pilot studies, I asked students questions to guide them toward the multiplicative nature of the chain rule, but they did not recognize where my questions were taking them. In the beginning, the students did not grasp the main idea that was connecting all of their work and thinking and towards the end of their interview, they did not realize that a function composition had anything to do with what we had explored so far. I realized that from the beginning, I needed to focus on

NM by asking them to create a composition of functions to model the meaningful context of the task and then ask them how changes in the independent variable affect changes in the outermost variable of the function. By doing so, the students consistently employed NMR and recognized the theme of the teaching experiment.

In the pilots with the three calculus students, I did not have enough time to get to related rates problems. So, I decided to take what I learned from them and test the HLT on one more first-semester calculus student during summer semester, to whom I will refer as the "July Student," and I made final adjustments to the HLT. Before interviewing him, I took the latest version of the learning activities for the HLT and drafted learning goals. At this point, the learning activities and questions were nearly identical to the final draft, except for I did not yet have clear questions to ask for the related rates problems.

My pilot study with the July Student helped me to realize places in the interview protocol where I needed to scaffold student thinking a little bit more. For example, after the July Student developed understanding of the multiplicative nature of the chain rule, I immediately had him try to practice the chain rule by differentiating the function $h(x) = \sin(x^2)$ with respect to *x*. He had some difficulty identifying which functions were the "inside" and "outside" functions. Part of this difficulty came from his discomfort with trigonometric functions. I simply told him that the derivative of $sin(x)$ is $cos(x)$. I knew, because of the timing of my study with students enrolled in first-semester calculus, they too would not have learned derivative rules for trigonometric functions, and they might also be uncomfortable practicing the chain rule on such a composition of functions. Thus, I decided to precede that problem with the problem of differentiating $g(x) = \sin(f(x))$, so that the students could more easily identify the "inside" and "outside" functions as well as get some practice using the new trigonometric differentiation rule I taught them. Then, I thought students could transition to differentiating $h(x) = \sin(x^2)$ with respect to *x* more easily.

I realized that the July Student also might have benefited from one more practice problem to develop procedural fluency of the chain rule, so I added the problem of differentiating $j(x) = [\sin(x)]^2$ to give the students more practice during the interviews. I also figured that changing $\sin(x)$ from the "outside" function in *g(x) and h(x)* to the "inside" function in *j(x)* would stretch the students just enough, while still being accessible to their current understanding.

My pilot study with the July Student also helped me realize that adjusting the original wording of the "Snowman Context" (see Figure 12) might help future students in the study more smoothly connect what they had learned about the multiplicative nature of the chain rule to this simple related rates context. With the July Student, the Snowman Context was originally worded as seen in Figure 12.

The body of a snowman is in the shape of a sphere and is melting at a rate of $2\,\frac{ft^3}{hr}$. How fast is the radius changing when the body is 3 ft. in radius (assuming that the body stays spherical)?

Figure 12. Original wording of snowman context.

The July Student struggled more than I anticipated with this problem. I realized that until that point, we had used NMR to think about how changes in the independent variable creates changes in the second variable which creates changes in the third variable, going from the inside out in a composition of functions. With the problem written as seen above, it was difficult for the July Student to conceptualize time causing changes in the volume, causing changes in the radius. Especially with the equation modeling the problem: $V(t) = \frac{4}{3}\pi r(t)^3$. The July Student struggled to understand how to move forward in solving the problem because he didn't know if he needed to solve for *r(t)* in terms of *V(t)* or if he could solve the problem as was, which was written as a composition of functions *V(r(t))*. Thus, to avoid this unnecessary difficulty, I adjusted the problem to provide the rate of change of the radius with respect to time and prompted the student to find the rate of change of the volume with respect to time (see "Snowman Context" in The HLT Rooted in NM in Chapter 3). That way, the equation for *V(t)* in terms of *r(t)* more easily supported the NMR that the students had employed until that point.

Additionally, the July Student helped me understand that in order to help a student solve the related rates problems, I needed to encourage her/him to draw a picture of the problem and label what she/he knows and doesn't know. That way, the student has time to explicitly digest the context, what quantities are involved, and their goal in solving the problem. I also realized that I needed to ask the student which quantities were changing with time so that they knew which quantities to relate in an equation and I could encourage them interpret those quantities as functions of time. Then, the student can recognize the hidden composition of functions in their equation that models the problem and correctly differentiate the equation with respect to time. After better understanding what questions to ask for the related rates problems, I added the last two goals to the learning goals of the HLT (see The HLT Rooted in NM in Chapter 3).

After the adjustments I made to the HLT based on all of the pilot interviews, I felt confident about conducting the study.

Building on and Extending Infante's Research

My study builds on and extends Infante's (2007) research. In this section, I explain the interplay between my pilot studies and Infante's work during the process of creating my HLT. There are four main ways in which my study extends her research: by exploring (1) how students develop and not just understand the multiplicative nature of the chain rule, as opposed to only how they understand it at a given point in time, (2) how students develop interrelated meanings for these concepts, rather than simply applying a procedure to related rates, (3) how students understand role of time, and (4) how students can think more deeply about the given context.

First, I conducted a teaching experiment to explore an HLT for students to not just understand, but also *develop* the multiplicative nature of the chain rule themselves. In order to help Infante's (2007) students understand and solve related rates problems, she first revisited the chain rule with them, attempting to help them understand the multiplicative nature of the chain rule. In my study however, the students had not yet been exposed to the chain rule, implicit differentiation, or related rates problems. I focused on having students develop the multiplicative nature of the chain rule themselves and then generalize that to other problems.

In order for Infante (2007) to help the students in her experiment develop a more conceptual understanding of the chain rule, after they had already learned it in their class, Infante (2007) introduced a related rates problem of a ball being thrown into a lake (see Figure 13).

In the very first drafts of the learning activities for the HLT, before I tested it in any pilots, I use this ball thrown into a lake related rates problem to bring out the multiplicative nature of the chain rule. However, I realized that I wanted to develop this same idea through an

A ball is thrown into a lake, creating a circular ripple that travels outward at a speed of 3 cm per second. How fast is the area of the circular ripple growing with respect to time?

Figure 13. A ball thrown into a lake related rates context (Infante, 2007, p. 180).

even simpler related rates problem, where students could really develop the multiplicative nature of the chain rule themselves. I realized that I needed to scaffold the students much more than Infante had because the students I interviewed had not yet learned the chain rule. Additionally, I realized that this problem required the symbol π. Dorko and Speer (2015) showed that sometimes students subconsciously perceive π as a unit instead of as a constant or an irrational number. I did not want the symbol π to be a distraction from NM and the big ideas that I wanted to develop with the students, thus I decided not to use this problem altogether.

Second, I hypothesized that my proposed HLT would allow students to develop deeper connections between related rates problems, the chain rule, and implicit differentiation. All of the students in Infante's study learned to create a delta equation in order to relate the changes in the quantities in the related rates context in order to eventually take the derivative to solve the problem. To better understand delta equation, consider again the ladder problem from Chapter 3.

The delta equation that could model this particular problem is $\frac{\Delta x}{\Delta t} = \frac{\Delta x}{\Delta y} \cdot \frac{\Delta y}{\Delta t}$. Infante (2007) explains that "the 'delta equation' [is] a statement of the chain rule as it [applies] to the problem at hand" (p. 255). That is, this delta equation represents the way in which the variables of the function composition *x(y(t))* change together. However, the delta equation shows how finite changes in each of the quantities affect one another, whereas instantaneous changes would be represented by the equation $\frac{dx}{dt} = \frac{dx}{dy} \cdot \frac{dy}{dt}$, which comes from taking the derivative of the composition of functions with respect to *t*.

The delta equation that could model this particular problem is related to the first of two ways to make sense of this problem (as discussed in Chapter 3). That is, the first way to make sense of this problem is to recognize that the composition of functions $x(y(t))$ exists, and recognize that in order to find $\frac{dx}{dt}$ one can take the derivative of the composition of functions with respect to time. This first way of making sense of the problem is employing RE from the NMR mental action framework (Chapter 3). Later in Infante's teaching experiment, when students were asked to describe how they might teach a friend how to solve related rates problems, they responded with the phrase “draw a delta equation” (p. 255). She explains that drawing a delta equation was perhaps a strictly procedural part of the process as a result of the chain rule discussions. She said that she believes this understanding is related to their understanding of function composition, yet she said they never used the term composition and so there was no evidence to justify this belief. If, in creating a delta equation, a student recognizes the existence of a composition of functions and the way in which the variables within multivary, then she/he is also employing RE from the NMR mental action framework (Chapter 3). However, if a student has not employed RE, it is possible that she or he is just reproducing a procedure.

As a teaching assistant for a first-semester calculus class, I saw one of my students create a delta equation, even though this method was never formally taught. Intrigued, I asked her why the delta equation made sense to her or from where the delta equation came. She could not answer my question. She simply said it was a method she had learned and memorized when she took math in a different country. So, although creating a delta equation might be a good way for students to be able to solve the related rates problem, it is possible that students do not fully understand why the delta equation works or that it comes from the derivative of a composition of functions.

I hypothesized that the HLT rooted in NM leaves the related rates problem solving process open for a student to apply the multiplicative nature of the chain rule themselves in whatever way makes most sense to her/him. I am not claiming that any one method is superior or that learning to create a delta equation is disadvantageous. However, I do think that a student’s understanding of the multiplicative nature of the chain rule should support whichever method is most comfortable and intuitive to her/him.

The third way my proposed HLT builds on Infante's (2007) study is by encouraging students to explicitly consider all of the changing quantities within a given problem and thus recognize and understand the role of time. Infante explains that her students struggled to develop an understanding of the role of time in related rates problems. She says it was especially difficult for her students to recognize that a variable was a function of time when it was not explicitly written as a function of time (e.g. $x(t)$). However, through my thought experiments and pilot studies, I came to believe that a focus on NM naturally invites students to consider time because they must cognize all of the changing quantities within a given context and think about how they change together when employing NMR.

The fourth way that my study builds on and extends Infante's (2007) study is by exploring how students can more deeply think about the given context. Infante (2007) explains that when a mathematician solves a related rates problem, she/he consistently refers back to the diagram that models the problem. In her study, she found that although referring back to a diagram was very important to the mathematicians' problem-solving process, students rarely referred back to the diagram they had created when trying to understand and solve a related rates problem. She explains that students were not adept in recognizing the way in which their diagram could help them to conceptualize and solve related rates problems. I think this is part of a larger issue of simply thinking deeply about the context. I hypothesized that an HLT rooted in NM would help students to actively consider the way in which the different variables affect each other in a related rates problem.

The Teaching Experiment

The final interview protocol (see the Appendix) consists of a set of scaffolded learning activities meant to accomplish the goals in each stage of the HLT rooted in NM.

In my study, the tasks introduce the concepts of the chain rule, implicit differentiation, and related rates in a different order than they are typically taught in first-semester calculus. Typically, in a first-semester calculus class, students are first taught the chain rule and implicit differentiation separately and as memorized procedures. Then, related rates problems are later introduced as a way to apply the chain rule in meaningful contexts (Stewart, 2018). Differently, the learning activities in the HLT of this study *begin* with meaningful context where students can develop and make sense of the multiplicative nature of the chain rule. Then, students generalize the chain rule to abstract compositions of functions. Next, students explore implicit differentiation through a meaningful context and then generalize those ideas to more abstract implicit functions. Lastly, students extend their understanding of the multiplicative nature of the chain rule to more complicated related rates and implicit differentiation problems.

The teaching experiment consisted of four, 50-minute interviews with four first-semester calculus students. The teaching experiment was done one-one-one with each student, so that I could follow that student and adapt accordingly. Doing this study with individual students creates limitations, in that I have not examined the HLT in a full classroom. However, these one-on-one interviews were critical in testing assumptions within the HLT, in observing how the students' understanding developed, and in identifying places where the students struggled more than anticipated. Based on this small-scale teaching experiment, many parts of the HLT were validated, but in a few instances, revisions to the HLT were seen as necessary. One result of this study was the creation of a finalized HLT (see Chapter 6) which is now ready for a full-scale implementation in a regular classroom setting.

Due to my pilot studies, I came to realize that in this study it would be important for students to already have certain base understandings, including an ability to interpret derivatives in meaningful contexts and a solid understanding of multiplicative comparisons in general. I also knew it would be better to have students who were willing to freely talk their thinking during the tasks, because I needed to examine their understanding carefully. Thus, in order to ensure I had students who met these characteristics, I conducted the first 50-minute interview as a screening interview with six students. These students were recruited from a group of volunteers from a first-semester calculus class. Based on these interviews, I selected four students who seemed to have the strongest base understanding of derivatives and multiplicative comparisons and were also willing to talk about their thinking. I continued with those four students for the three other 50-minute interviews.

In order to explore the viability of the HLT, these students had not yet been exposed to the chain rule, implicit differentiation, or related rates in their classes. These students had, however, seen the limit definition for the derivative, explored derivatives as slopes, and learned the power-rule for taking derivatives of polynomials. I only selected students who had never taken calculus before the semester they were enrolled, whether in high school or college. All of the students received monetary compensation for their time.

I had certain material in the interview protocol (see Appendix) I aimed to cover by the end of each interview, yet I was flexible about how much material I covered in each interview depending on the needs and understanding of each individual student. For the first interview, I aimed to help the student interpret derivatives in meaningful contexts and begin Stage 1 of the HLT by beginning to develop understandings for the multiplicative nature of the chain rule. By the end of the second interview, I aimed to finish both Stage 1 and Stage 2 of the HLT by developing understanding for the multiplicative nature of the chain rule, generalizing the chain rule for any composition of functions, and gaining procedural fluency with it. By the end of the third interview, I aimed to finish Stage 3 and Stage 4 of the HLT by developing the idea of variables being implicit functions of time and the idea implicit functions, recognizing subsequent existence of compositions of functions. By the end of the fourth and last interview, I wanted to get as far as I could with each student in Stage 5 of the HLT by helping the students to explore more complicated related rates and implicit differentiation problems. I asked the questions in the interview protocol at a pace according to the students' needs. I asked additional questions throughout in order to clarify the students' thinking and guide them towards the learning goals of the HLT. After each interview, if any of the participants had special needs, I adjusted the interview protocol accordingly.

Analysis

My analysis was founded on Braun and Clarke's (2006) framework for phases of thematic analysis. However, I modified the phases of their thematic analysis framework in order to better help me answer my research questions. My analysis consisted of five general phases: 1) familiarized myself with the data, 2) generated initial codes for the data, 3) reviewed codes and searched for themes across and differences between the students' data, 4) defined themes and outlined results, and 5) produced the report.

First, I familiarized myself with the data. During this phase, I simultaneously transcribed and broke up the transcript into "idea units" (Jacobs & Morita, 2002; Sherin & van Es, 2009). An idea unit is a segment in which one particular idea is discussed at one point in time. Because my interview protocol consists of very specific questions that scaffold the students' thinking, most responses to each question counted as one idea unit. However, there were times when a student discussed multiple ideas within a response to a single question, in which case that response was broken up into multiple idea units as appropriate. One idea unit sometimes contained responses to the original question in the interview protocol as well as any follow-up or clarifying questions that involved the same idea. If a student came back to the same idea they had discussed earlier, I counted it as a new idea unit because they were at different points in time. During this phase, I made note of any initial impressions I had about how the students were using NMR, what understandings they were developing through the HLT, and places where students struggled that suggested a needed revision to the HLT.

Second, I generated initial codes for the data. In order to answer my first research question, about how students used NMR throughout the HLT, I coded the data according to the NMR mental action framework (see Chapter 3), also remaining open for any other types of NMR mental actions that did not fit in the original framework. However, I did not find different NMR mental actions that were not previously anticipated in the framework. I was the only one to code my data and so in order to ensure internal consistency when I coded, I created a key for each code where I clearly defined what constituted receiving each code with specific indicators. As I realized certain idea units fit, or didn't fit, the pre-existing codes, I kept track of that in my key for the codes.

During this phase, in order to answer my second research question and identify understandings students developed as they progressed through the HLT, I identified all of the units of data related to the different learning goals and sub-goals in the HLT. I marked units that evidenced the student reaching the learning goal as well as units that suggested the student had an incorrect understanding within the learning goal.

During this second phase, in order to answer my third research question, I identified places where students struggled to understand the learning goals of the HLT or move forward through the associated tasks. In addition to identifying these places, I began to take notes of how I might revise the HLT in the future to address the students' struggles.

For the third phase in my analysis, I reviewed codes and searched for themes across and differences between the students' data. I went through all of the data a second time and double-checked that my codes were consistent and fit the key for the pre-existing NMR codes across all four students. I went through each student's data, one at a time, and searched for themes in the way that they used, or didn't use NMR, in their progression through the HLT.

Next, I broke each student's data into the different stages of the HLT and re-examined the data, marking any additional pieces of data that exhibited the student's understanding of the learning goals and sub-goals of the HLT. For example, I marked anytime they expressed a meaning for something or explained why something works. When they were giving an explanation, I also marked the different ideas that were in that explanation. I also looked for additional pieces of data that suggested the student either had an incorrect understanding of the learning goals and sub-goals or was inhibited in reaching the goals of the HLT. I identified and recorded specific parts of their understanding that seemed to be linked to these inhibitions.

Lastly, based on what I found for students' use, or non-use, of NMR and understandings of these concepts, I repeatedly marked specific places that needed to be addressed in the interview protocol and HLT for each individual student; places where there seemed to be something missing or something that could be adjusted to help students more easily move through the HLT and develop understandings for these concepts.

For the fourth phase in my analysis, I more clearly defined themes and differences between the students and outlined the results. After searching for themes one student at a time, I searched for and clearly defined themes across and differences between the four students' NMR throughout the HLT. For each student, I looked at the entirety of pieces of data that were linked to their understanding of the learning goals and sub-goals of the HLT. For each sub-goal, I characterized the student as either having a “complete”, “incomplete”, or “missing understanding” of that sub-goal. I characterized the student as having a complete understanding if they had multiple units of data that suggested they understood that sub-goal. I characterized the student as having an incomplete understanding if they had some evidence of reaching the learning sub-goal but they also had some evidence of misunderstanding of or continual inhibitions in reaching the sub-goal. I characterized the student as having a missing understanding if there was no evidence the student understood the learning sub-goal. Lastly, after looking at places in the HLT that needed to be addressed for each student, I more clearly defined places where all of the students struggled which suggested a needed revision to the HLT.

In the fifth phase in my analysis, I used the outline I had created in the previous phase to produce a report of the results. Here, I selected idea unit extracts that demonstrated and helped clarify the results. I tried to include extracts that helped tell the students’ stories more vividly. In producing the report, I tried to clearly answer all three of my research questions. During this phase, I also wrote the discussion where I related the results back to existing research and discussed ideas for future research.

Limitations

There are three limitations I address in this section. The first is my bias that nested multivariation is indeed the mathematical concept underlying the chain rule, implicit differentiation, and related rates. The second is my solo role in coding the data and my effort to maintain internal consistency. The third is the generalizability of this small-scale teaching experiment to full-sized classrooms. The fourth is that I selected students who seemed to have a strong base understanding of derivatives.

First, my biases about nested multivariation and the chain rule, implicit differentiation, and related rates are fundamental to my study. I have stated clearly that I see nested multivariation as being the underlying mathematical concept of the chain rule, implicit differentiation, and related rates. The HLT that I constructed for coming to understand these three concepts is necessarily influenced by this bias. I am not claiming that my belief about nested multivariation is necessarily true and I am not claiming that this HLT is the right or only way that a student might develop understandings for these concepts. Instead, I am exploring the possibilities of nested multivariation being the underlying mathematical concept of these three concepts and the viability of such an HLT. It is possible that other researchers might believe or find that these three concepts are more fundamentally related in some other way or that there exists a stronger HLT for coming to understand these concepts in more than a procedural way.

Second, there are limitations in using solo-analysis. To answer my first research question, the specific NMR mental action codes simply helped me identify NMR when I may have otherwise missed it. It turned out that the *specific* NMR codes weren't as important as identifying NMR in general. Nevertheless, in order to ensure internal consistency when I coded, I created a key for each code where I clearly defined what constituted receiving each code with specific indicators. As I realized certain idea units fit, or didn't fit, the pre-existing codes, I kept track of that in my key for the codes. Additionally, in answering my second research question, I have provided many excerpts in the results from the students to help illustrate my conclusions. In this way, the reader becomes a co-validator with me as I invite her/him to challenge or critique my findings based on the data shown from the students.

Third, this study will provide insights into the cognitive processes of only a few individual students at one university. As such, my study is preliminary to a more large-scale and generalizable study in the future. With only a few students in my sample, I used purposeful selection (Maxwell, 2013) to recruit students with the characteristic of having base understandings of derivatives and multiplicative comparisons as well as being willing to talk about their thinking. Due to my sample size, I am unable to sufficiently vary other characteristics such as gender, major, and general interests, which might affect the viability of the proposed HLT. A larger-scale teaching experiment is needed to gather results that are more generalizable to calculus students at large. Although my study is not generalizable, it has done much in terms of providing deeper insights into the cognitive processes of a few students, which appeared to validate several aspects of the HLT and identify remaining issues to address.

Fourth, as I explained in my methods section, I conducted a screening interview with six students. I selected four students who seemed to have the strongest base understanding of derivatives and multiplicative comparisons. This potentially could have been a limitation to my study because I only explored the HLT with stronger students and my results could have been limited to students with the same base understandings of derivatives and multiplicative comparisons. However, as I discuss in my results, one of the students (Student D) ended up having a weak understanding of derivatives. That is, he often got them confused with average rates of change. This did cause him to struggle to move forward in the HLT and caused me to believe that it is important that students have a strong understanding of the meaning of the derivative before attempting to develop understandings for the chain rule, implicit differentiation, and related rates through the proposed HLT.

CHAPTER FIVE: RESULTS

In this section, I use the analysis of my data to answer my three research questions. Throughout this section, I reference the different learning activities and contexts that I used in the HLT. To see these learning activities and contexts in their entirety, please see either the HLT Rooted in NM in Chapter 3 or the Appendix.

Use of Nested Multivariational Reasoning

My first research question is: How was nested multivariation used as the first-semester calculus students progressed through the HLT? In this section, I explain how students employed NMR in developing understanding of the chain rule, in developing understanding of implicit differentiation, and in developing understanding of related rates problems. Overall, I saw much more explicit NMR in developing understanding for the chain rule and then the students tended to use that foundational understanding as they applied the chain rule to developing understanding of implicit differentiation and related rates problems.

Nested Multivariational Reasoning in Developing the Chain Rule

I first describe some general trends in the use of NMR in the data for the four students in developing understanding of the chain rule, and then I provide some specific examples to illustrate what this NMR looked like in the interviews. Throughout this section, I refer to the NMR mental action framework which can be found in the Nested Multivariation section in Chapter 3.

Overall, the four students tended to focus on how all three variables were related to one another in a function composition before they focused on how all three variables changed together. Towards the beginning, students often employed RE, where they thought about the ways in which the variables in the function composition are related to and depend on one another. Some students employed PC, where they wouldn't just think about how the quantities were related to and depend on one another, but they would consider the fact that as one variable changes, it causes changes in the other two variables in a chain reaction.

The most explicit and crucial use of NMR was in developing understanding of the multiplicative nature of the chain rule. All of the students employed AMT, where students coordinated amounts of change with the three variables in the function composition, in order to make sense of the multiplicative nature of the chain rule. They explicitly used AMT in making sense of the multiplicative nature of the chain rule when looking at the rates of change of the variables at one specific moment within different meaningful context. Once they made sense of the multiplicative nature of the chain rule within different moments in multiple examples, the students extended that idea to the derivatives as a function; they used this understanding to generalize chain rule to any composition of functions. Following this initial development of the chain rule, once the students built procedural fluency with it, the students tended to only use RE to recognize the existence of a composition of functions, or nested relationship, thus recognizing that they needed to use the chain rule in order to take the derivative of the composition of functions.

I now discuss specific examples to illustrate these more general themes I identified in the data.

First, all students exhibited RE at some point in their development of the chain rule. In the beginning, most students tended to rely on RE to recognize the way in which the variables in the composition of functions are related and depend on each other. For example, when asked how the variables in the composition of functions in the Chocolate Context (see Appendix) were related to one another, Student A replied, "The chocolate is dependent on the amount of dollars which is dependent on the hours." Later on, I asked this same student what the units of the output of the composition of functions would be. Because she had thought about the way in which the variables were related to one another, she replied, "It wouldn't be very pretty, but it would be like lbs. per dollar per hour."

Here, she recognized the nested relationship but incorrectly understood how the variables in the composition of functions actually relate to one another. To deal with this, she calculated the value of $c\big(D(3)\big)$. She eventually realized that the unit of the output is only lbs. of chocolate. This realization helped prepare her to more correctly understand the derivative of the composition of functions and the meaning of said derivative.

Student C also struggled to understand the units of the output of the function of a composition of functions. In the Carnival Context, where the composition of functions was created with the temperature, attendance, and revenue of a carnival, when asked what the units of the output of the composition of functions would be, Student C said, "So, this would be the amount of money that we would make depending on what temperature it is. So, it would be dollars per Fahrenheit." Afterwards, when this same student was asked how the variables were related to one another in the composition of he replied, "The amount of money we would make for the amount of people that came for [pause] that depends on I guess the temperature at that point."

Here, it is interesting that the student did not first think about the independent variable and then how the other two variables relate in a chain reaction. Similar to Student A, it wasn't until we did a specific example where he calculated the value of *f(g(45))* that he finally understood the units of the output of the composition of functions and was thus prepared to make sense of the meaning of the derivative of the composition of functions.

As stated earlier, all students used AMT in order to understand the multiplicative nature of the chain rule. They coordinated the amount of change between the variables within a specific moment in multiple contexts in order to reason about the multiplicative nature of the chain rule.

For example, some students thought about how to coordinate the amounts in the different rates by thinking about how they could get a common amount to compare. In the Chocolate Context, Student D immediately realized that in order to get the derivative of the composition of functions, or the lbs. of chocolate per hour, he could multiply the two other derivatives. I asked him why it made sense to multiply the other two derivatives:

> Student D: You would just multiply this one [pointing to dD/dt, or 9 dollars per hour] and that one [pointing to dc/dD, 15 lbs. per dollar] together. Um, I see it as a ratio. Um, so I have $9 per 1 hour and I also have 15 lbs. or .15 lbs. sorry, [laughs] for every one dollar. And so, I see this relationship right here where for every hour I work, I have $9 and for every $9 that I have, it would be as simple as just multiplying this here [referencing dc/dD, or .15 lbs. per dollar] by 9 to figure out how that is. And in order to keep the ratio the same you would have to multiply both sides by 9 [pointing to both the numerator and denominator of dc/dD] and so you would end up with 9 times .15 lbs.

In this specific problem it is true that both of these rates are constant over time in the given context; this is a simple situation with simple derivatives. Because both of the rates are constant, he can correctly say that for every given hour he works, he would make $9 more and could buy $9 \times .15$ lbs. more of chocolate. This cannot always be said for more complicated situations where the rate is not constant. In fact, some students would even use this same type of wording (e.g. “for every” or “for any”) even though the amount, or rate, to which they were referring only applied for that specific moment. However, it still helped the students to reason through and understand the meaning of the rate at the moment of interest. For example, as recorded above, Student D realized that .15 lbs. per $1 is equivalent to 1.35 lbs. per $9. Thus, like Student D, the other students coordinated the amounts of change with the different variables in order to make sense of the multiplicative nature of the chain rule.

A couple of students made sense of the multiplicative nature of the chain rule through seeing how the units "canceled out" in order to get the desired units for the derivative they were trying to find. For example, in the Running Context when the students were told to pretend that at one moment, they are running .1 miles per minute and burning 100 calories per mile. They are then asked to find the derivative of $c(D(t))$, or the calories per minute. At this point in the interview, Student B reasoned through how to find this derivative:

> Student B: …because each mile we are going to burn 100 calories, but if we are .1 per minute, then I just, I need to multiply them in order to get calories per minute. So, I am running .1 miles per minute and every mile I am burning 100, but if I would like to know what is actually my calories per minute, I need to multiply again [pause] to get rid of the miles and then get the calories per minute.

This may not be a deep understanding of the multiplicative nature of the chain rule, yet it still helped the students to feel that their intuition was correct because they got an answer that made sense within the context. Similarly, Dorko & Speer (2015) explain that understanding units can be very important for deeper understanding of mathematical concepts.

Later on, students used this understanding of the multiplicative nature of the chain rule to generalize it to any composition of functions. Eventually, the students only used NMR to recognize a nested relationship and thus recognize the need to use the chain rule in differentiating of a composition of functions. In this way, students were employing reasoning based on RE, even if it had become more automatic to them by this point.

In trying to gain procedural fluency of the chain rule with different functional examples, a couple of the students struggled to identify the “inner-most” function and the “outer-most” function. In doing so, they incorrectly applied the chain rule. For example, when invited to find the derivative of the function $h(x) = [\sin(x)]^2$, Student B said that $h'(x) = 1 \cdot 2\cos(x)$. Below are his words as he was finding the derivative:

> Student B: Ok so *j(x)* equals [pause] the inside is going to be 1 times [pause] this is going to be 2 times 1 times *cos(x),* so then if I multiply that it is going to be 2 times 2 *cos(x).* That is what I am guessing.
>
> PI: So, at first you wrote 1, where did that come from?
>
> Student B: From this *x* [pointing to *x,* the independent variable].
>
> PI: And then why did you put 2 here and then *cos(x)* here?
>
> Student B: Because I brought [pause] So I, first I analyzed the inside, so the derivative of the *x* is 1 and then the derivative of the entire thing is *cos(x)*, and then this is affecting both of them, so then I decided to take the derivative of that one, of the entire function [using the power rule to bring down the 2, but incorrectly applying the chain rule], so it is going to be two times the entire thing. So that’s how [pause] I treated it as a power function at the end like the whole thing.

It seems that his struggles in correctly taking the derivative of *h(x)* come more from his weak understanding of functions and function compositions in general than they do from a lack of NMR. Eventually, I re-wrote the problem as $h(x) = [\mathrm{j(x)}]^2$. He correctly found that $h'(x) = 2j(x) \cdot j'(x)$. After writing *h’(x)* in this general form, he extended his thinking to the original problem and correctly find $h'(x)$. Although he struggled to correctly apply his NMR when the specific function example was originally presented, it seems like he used RE to correctly recognize the need to take the derivative of the inside function times the derivative of the outside function when given a more general representation of that function.

In summary, most students initially used RE to initially understand how the variables were related to one another in the function composition. Then, all four students used AMT to make sense of the multiplicative nature of the chain rule when finding the derivative of a composition of functions at a specific time with given rates at that time. Once they built understanding of the chain rule and generalized it to derivatives as functions and to general compositions of functions, they relied on RE to simply recognize the existence of a nested relationship and the subsequent need to employ the chain rule in finding the derivative.

Nested Multivariational Reasoning in Developing Implicit Differentiation

I first describe some general trends in the use of NMR in the data for the four students in developing understanding of implicit differentiation, and then I provide some specific examples to illustrate what this NMR looked like in the interviews.

In developing understanding of implicit differentiation, there were overall fewer instances of NMR than there were in developing understanding of the chain rule. Once students used NMR to build intuition for the multiplicative nature of the chain rule and developed procedural fluency with it, they relied on their existing understanding of the chain rule, and concomitant NMR, in making sense of implicit differentiation. That is, they weren't explicitly exhibiting NMR because they weren't unpacking it anymore, but they were using knowledge based on previous NMR. There were a few instances of NMR during the students' development of their understanding of implicit differentiation, and they all consisted of RE. RE was the mental action that came up because once students recognized the nested relationship, they jumped straight to the other NM knowledge they had built without unpacking it and reasoning about it every time.

When recognizing an implicit function of one variable in terms of another, all four students used RE in recognizing subsequent compositions of functions, or nested relationships. Once they recognized an implicit nested relationship within an equation, they correctly applied the chain rule to that nested relationship when employing implicit differentiation.

For example, when first introduced to implicit differentiation through the equation $x^2 - y^2 = 1$, I talked to each of the students about how fast *y* changes as x changes, or find $\frac{dy}{dx}$, we can think about the fact that *y* is dependent on *x,* or *y* is an implicit function of *x.* Then, we re-wrote the equation as $x^2 - [y(x)]^2 = 1$ to help the students remember the implicit function *y(x).* When I asked the students how this new equation relates to what they had previously learned about the chain rule, they all mentioned that a composition of functions existed or there was a need to take the derivative of the inside times the derivative of the outside.

For example, when I asked Student A how this new equation relates to what we had learned previously, she said, "… you could take the derivative of this [referencing $y(x)$] and multiply it to the derivative of this [referencing $x^2 - [y(x)]^2$] to get the derivative of the whole thing." Although she was incorrect in thinking that the "outside" function was the entire left side of the equation, instead of just the $[y(x)]^2$ part, she recognized a nested relationship and the need for the chain rule. It seems as though her not understanding what specifically was the "outside" function comes from her lack of exposure to non-explicit equations. After talking more together about taking the derivative of the equation with respect to *x,* she came to understand that we can take the derivative of x^2 with respect to *x,* and then subtract the derivative of $[y(x)]^2$ with respect to *x*.

In conclusion, all four students exhibited RE in the development of their understanding of implicit differentiation by recognizing the existence of an implicit nested relationship and the subsequent need to use the chain rule in taking the derivative of the equation.

Nested Multivariational Reasoning in Developing Related Rates

I first describe some general trends in the use of NMR in the data for the four students in developing understanding of related rates, and then I provide some specific examples to illustrate what their NMR looked like for related rates.

Compared to their exploration of implicit differentiation, there were both more instances of NMR and greater diversity in those instances in students' development of their understanding of related rates. The students used four NMR mental actions in making sense of related rates. First, similar to implicit differentiation, the most common NMR was RE, which all four students in the study employed in making sense of related rates problems. Just as with implicit differentiation, the students used RE to recognize a nested relationship and the subsequent need to use the chain rule in taking a derivative. Second, two students employed PC three times and a third student employed PC six times in their exploration of related rates. They used PC by recognizing the fact that the three variables in the composition of functions change together in a chained relationship which helped them eventually realize the need to use the chain rule when taking the derivative of the equation that modeled the problem. Third, two students employed ID one time in making sense of how the quantities in the situation were increasing and/or decreasing together in a chained relationship which helped move them closer to creating an equation to model the problem. Fourth, two students employed AMT one to two times in coordinating changes of the quantities in the nested relationship while actually taking the derivative at the moment of interest.

I now provide specific examples to illustrate how students used the first three NMR mental actions. I do not provide an example of how the two students used AMT because they used it in the same way they did to originally makes sense of the multiplicative nature of the chain rule.

First, just as with the chain rule and implicit differentiation, the most common NMR mental action students used was RE in recognizing the existence of a nested relationship and the subsequent need to use the chain rule in differentiating a composition of functions. For example, when asked to summarize her thinking after completing the Snowman Context, Student A said, "What we're looking for is as time changes how does the volume change. Because the volume is a function of time. And so, the volume is actually a function of the radius which is a function of time."

Sometimes, in solving the related rates problems, students would forget to use the chain rule because they failed to use RE. For example, Student A had made this mistake. Later on, when I asked her to review the mistake and explain why it was incorrect, she said, "Oh, I thought this was the derivative, but this is like theta [pause] theta prime. So, you can take the derivative of what's inside the parentheses and then the derivative of what's outside the parentheses with the tangent." Here, she corrected her mistake by employing RE and recognizing the function of the implicit variable time and subsequent nested relationship.

Second, the three students, who used PC, used the mental action similar to how they used RE; by recognizing an existing nested relationship and thus the need to use the chain rule. However, PC was slightly more advanced because students were not just recognizing a nested relationship, but they were also considering the fact that the variables within the nested relationship change together in a chained relationship. For example, while working on the Snowman Context, I asked Student C which quantities in the situation were changing with time and he replied, "So, the volume is changing with time which makes [um] the radius also change with time." This helped the student recognize the function of the implicit variable time within the equation $V = \frac{4}{3}\pi r^3$, that modeled the situation. That is, it helped the student understand the equation as $V(t) = \frac{4}{3}\pi r^3(t)$, making the function of the implicit variable of time explicit, which ultimately helped him to correctly take the derivative and solve the problem. When I asked this same student to reflect on his solving and understanding the problem, he exhibited PC again by saying, "…I knew that the volume changes with *r* as the radius changes and the radius changes with time. So, we made this chain rule thing." It should be known that afterwards, we talked about how the equation as written did not technically constitute a "chain rule thing," but instead a composition of functions, and we used the chain rule to take the derivative of that composition of functions.

Third, unlike the students' use of RE and PC, the two students who used ID did not use it to directly help them apply the chain rule to the related rates problems. Instead, these two students seemed to rely on ID to simply think about how the quantities were increasing and/or decreasing together in order to better understand the problem in general. ID seemed to help these two students to visualize the problem and better grasp how the different quantities in the problem were changing with time. Although this didn't seem to directly help them correctly take the derivative of the equation that modeled the situation, it did help them to understand the problem in a way that they could start to understand how they might use the given derivative to find the derivative of interest. That is, it helped the students to better understand which quantities they might need to use in creating an equation to model the problem in the first place, thus getting one step closer to actually taking the derivative and solving the problem.

For example, Student D was struggling to create an equation to model the Shuttle Launch Context. At one point, he knew that he was supposed to relate θ, the angle measure between the ground and the camera's view to the shuttle, to *d,* the vertical distance from the ground to the shuttle in miles. When asked how he might relate these two variables he wrote (see Figure 5):

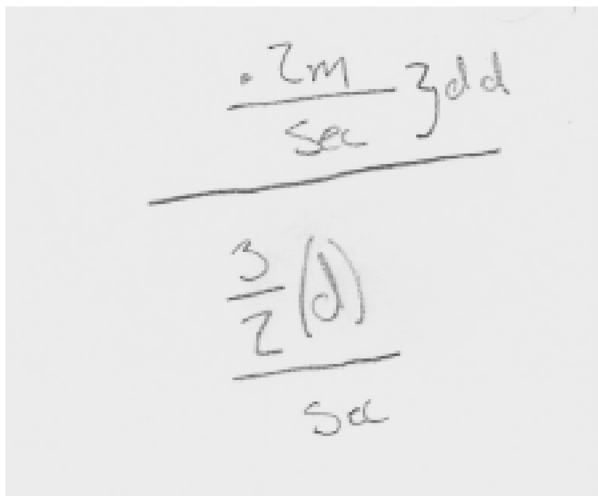


Figure 5. Student D creating a proportion in an effort to model the Shuttle Launch Context.

Then, he and I talked about this ratio that he formed in an effort to relate θ and *d:*

Student D: I am not comfortable with that answer.

PI: What does the $\frac{3}{2}d$ represent?

Student D: It's a poor guess [um] this is a poor guess because I just took a ratio that I knew would be happening at one point in time and just said, "alright well I will make it so that it will be 3/2 of the distance for any given distance per second." Which is, like I said, a bad guess, because it is more of like a generalization of how things should be changing. When in reality, I can see how at the very beginning, the camera angle has to change super-fast, and then as it gets higher and higher [using his hands to model the rocket's distance from the ground increasing], you can just see that it is slowing down.

Here, we can see that although he first provided a "poor guess," it was through his ID mental action that he made sense of why his guess was incorrect for relating θ and *d.* That is, by thinking about how the angle increased with time which was dependent on how the vertical distance changed with time, he realized that the speed of the angle changing would not be a constant ratio. He realized that it would not be constant because he visualized how θ and *d* increased together and he realized that over time, the speed of the angle of the camera from the ground decelerates as the shuttle's distance from the ground increases. So, by employing ID he understood why his thinking was incorrect and move forward towards correctly modeling the situation and solving the problem.

This NMR seemed to be an effective way to address the common mistake that students make in plugging in values for variables too early in equations modeling related rates contexts. Doing so encourages students to treat variables as constants and incorrectly conclude that their derivative is "0" when trying to solve related rates problems. Thus, asking questions to encourage NMR might help students to address this common mistake and understand that when variables are changing in the context, they will affect the derivative and so values should not be plugged in for those variables until after the derivative was calculated.

Because I am trying to answer how NMR was, or was not, used in developing understanding of related rates, it should be noted that the last student I interviewed, Student D, struggled the very most with related rates problems. By the end of the last interview, we were unable to finish the second related rates problem, the Shuttle Launch Context, and he was not making much progress. It seemed he struggled so much because of his discomfort with functions in general and with coming up with his own equation to model a related rates context. However, it should be noted that, of all the students, he is also the student that exhibited the very least amount of NMR in exploring related rates. During their exploration of related rates, Student A and student C had 9 idea units with NMR and Student B had 19. Differently, during his exploration of related rates, Student D only had 3 instances of NMR. Although I cannot be sure, it does seem like he was not reasoning more about the quantities in the nested relationships in the Shuttle Launch Context, or employing more NMR, he was unable to move forward towards correctly modeling the context and solving the problem. Yet, his struggles could also be stemmed from his lack of understanding of function in general because that seemed to be a theme throughout his interviews. Nevertheless, the other three students finished the Shuttle Launch Context and they all employed more NMR than him.

In conclusion, the most common NMR mental action that was used in the students' exploration of related rates was RE in recognizing the nested relationship and thus recognizing the need to use the chain rule in taking the derivative by recognizing the composition of function structure. PC was used a few times to recognize the need to use the chain rule in taking the derivative. If ID was used, it was used in making sense of how the quantities in the context increased and/or decreased together in a chained relationship to better understand how to correctly model the context with an equation. If AMT was used, it was to coordinate changes of the quantities in the nested relationship while actually taking the derivative at the moment of interest; similar to how all the students used AMT in first developing the multiplicative nature of the chain rule.

Understandings for the Chain Rule, Implicit Differentiation, and Related Rates

In the previous section, I explained how students used NMR in developing understanding for the chain rule, implicit differentiation, and related rates. In this section, I shift to focus on my second research question: What kind of understandings did the first-semester calculus students develop for these three concepts within each major stage of the HLT? I describe the evidence of "complete," "incomplete," or "missing" understanding for sub-goal in each stage of the HLT (see the Analysis section in Chapter 4 for more details).

First Stage of HLT

The first stage of the HLT was focused on developing intuition for the multiplicative nature of the chain rule. During this stage, all four students first employed NMR to conceptualize how changes in the independent variable cause changes in the sequence of variables in the composition of functions, from the independent to the outermost dependent variable. The students also exhibited understanding of how to interpret the different rates that exist within a composition of functions. Given values for different rates in each context, they correctly interpreted the rates as comparisons between pieces of the independent and dependent variable.

For example, for the Chocolate Context, I asked Student D to calculate the derivative of *D* (dollars) with respect to *h* (hours worked at a job). He found that $dD/dh=9$. We discussed the meaning of his answer:

PI: What is the meaning of 9 in our context?

Student D: Because the derivative is basically the slope of that point, it means that at that certain point, of which we took the derivative of, so 9, the slope is 9.

PI: What does the slope mean in our context?

Student D: It means the change of h and x…if you go one more to the right or to the left you are increasing by 9 or decreasing by 9.

PI: What would be the units?

Student D: \$9 per hour.

Later on, I asked him to calculate the derivative of *c* (lbs. of chocolate one can buy) with respect to D (dollars). He found that *dc/dD*=.15 lbs. per dollar. I asked him to explain the meaning of *dc/dD* = .15, and he said, "…for every dollar that I have… I would be able to get .15."

These two examples illustrate typical thinking for all of the students during the beginning stage of the HLT. The students correctly identified the units of the derivative and described the meaning of the derivative by comparing the dependent and independent quantities. However, Student D was unique because from the beginning, he struggled to understand the difference between the derivative and an average rate of change for a function between two values. A few times, when I asked him to calculate the derivative at a point, he found two points nearby and calculated the slope, finding the average rate of change. At these points, we would pause, and I would explain to him the difference between average rate of change and derivative; that the average rate of change is a way to estimate the slope at one point, but the derivative is the exact slope or rate of change at one point. He began to understand the difference, but his lack of understanding of the difference between derivatives and average rate of change, along with his lack of NMR towards the end of the interviews, proved to make his progress through the HLT more difficult.

All four students gained at least two different understandings for the multiplicative nature of the chain rule. The first understanding that all students exhibited during at least one of the tasks during this first stage in the HLT was reasoning about how the quantities of the given unit rates change together. For example, Student A developed intuition behind the multiplicative nature of the chain rule with the Chocolate Context. I introduced the problem to her, where she was hypothetically making $9/hr. at her job and wanted to spend every penny on chocolate costing .15 lbs./dollar. I asked her how she could use these two rates to find how much chocolate she could buy per hour of work. She explained:

> Student A: "…Well you would multiply the $9 by the .15, and that would give you over one hour how much chocolate per hour.
>
> PI: Why did you say multiply?
>
> Student A: …so you know that you get .15 lbs. per every [pause] per every dollar. But you can't go from pounds to hours because there is a medium ground you have to hit first before you can get to that… this would just symbolize 1/9 of an hour [pointing to the .15 lbs./dollar], and then you would multiply it by 9 so it would give you, for the full hour, how much.

Here, she realized that in this chained relationship, dollars were the "medium ground" to get from lbs. of chocolate to hrs. of work. She reasoned that in order to get the rate of lbs. of chocolate per hrs. of work, she could multiply the rates, using the medium ground of dollars to relate the two rates to each other. Similar to Student A, all of the students exhibited similar reasoning in making sense of the multiplicative nature of the chain rule.

After exploring different examples, a second way in which all of the students strengthened their understanding of the multiplicative nature of the chain rule was through the units in the multiplication. Student A related this canceling of units to "unit analysis" that she had learned in her chemistry experience and Student D called it "dimensional analysis." For example, when Student A was working on the Running Context, where the students were hypothetically running .1 miles/minute and burning 100 calories/mile, she explained that, “We know that .1 mile is equal to one minute [writes down the rates she knows with units]. So that would like cancel out the units and give you calories over minutes, so you multiply 100 calories by .1 mile.” Based on both Student A and Student D’s other comments and coordination of the amounts of change within the composition of functions, I did not see this cancellation of units as a procedure of “drawing lines through things that are the same on the top and bottom of the fraction.” Instead, both of the students talked about the quantities that they were coordinating and this connection to their previous experience with unit and dimensional analysis helped strengthen their understanding of the multiplicative nature of the chain rule.

A third way in which two of the four students developed understanding of the multiplicative nature of the chain rule was by recognizing that the infinitesimal values in the two derivatives would cancel out in order to get the desired derivative. For example, when talking about the Running Context, Student D explained “I know that the derivative of *c* is equal to *dc* over *dD,* and I know that the derivative of *D* is equal to *dD* over *dt.* And so right off the bat, I can see how a certain aspect will cancel, leaving me with the desired terms.” Here, the student is coordinating the amounts of change within the composition of functions, but the amounts are infinitesimal.

In summary, all of the students exhibited a lot of evidence for achieving the learning sub-goals for the first stage of the HLT by correctly interpreting the different rates within a composition of functions, using NMR within compositions of functions, and constructing the multiplicative nature of the chain rule (see Table 3). Although Student D was successful in reaching these learning sub-goals, it was not without struggle because of his lack of understanding of the derivative and how it is different from average rate of change.

Table 3

Summary of Evidence Students Achieved Learning Sub-goals of First Stage of HLT

Learning sub-goals	Complete	Incomplete	Missing
1a) Given a function composition *f(g(x)),* interpret *df/dx*	A, B, C, D		
1b) Given a function composition *f(g(x)),* interpret *df/dg*	A, B, C, D		
1c) Given a function composition *f(g(x)),* interpret *dg/dx*	A, B, C, D		
1d) Given a function composition *f(g(x)),* use NMR	A, B, C, D		
1e) Construct that for *f(g(x)),* $\frac{df}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}$ at a point	A, B, C, D		

Second Stage of HLT

The second stage of the HLT was focused on generalizing the chain rule and gaining procedural fluency with it. After strengthening their understanding of the chain rule with derivatives at specific points, most of the four students smoothly extended that understanding to taking the derivative of general compositions of functions. For example, in the Running Context, when the students were no longer given the two rates at a specific point, but were instead asked to calculate *dD/dt* and *dc/dD* as functions, they all eventually realized that they would multiply the two functions for *dD/dt* and *dc/dD* to find *dc/dt*. During this part of the interview, I re-recorded their own conclusions from the Running Context and finding the derivative of the composition of functions using general derivative functions for *dD/dt* and *dc/dD.* I re-recorded their own conclusions on a separate piece of paper (see Figure 14 for an example of how I did this with Student A) so that the students could more easily see patterns in what they had explored with general derivative functions and generalize that to any composition of functions.

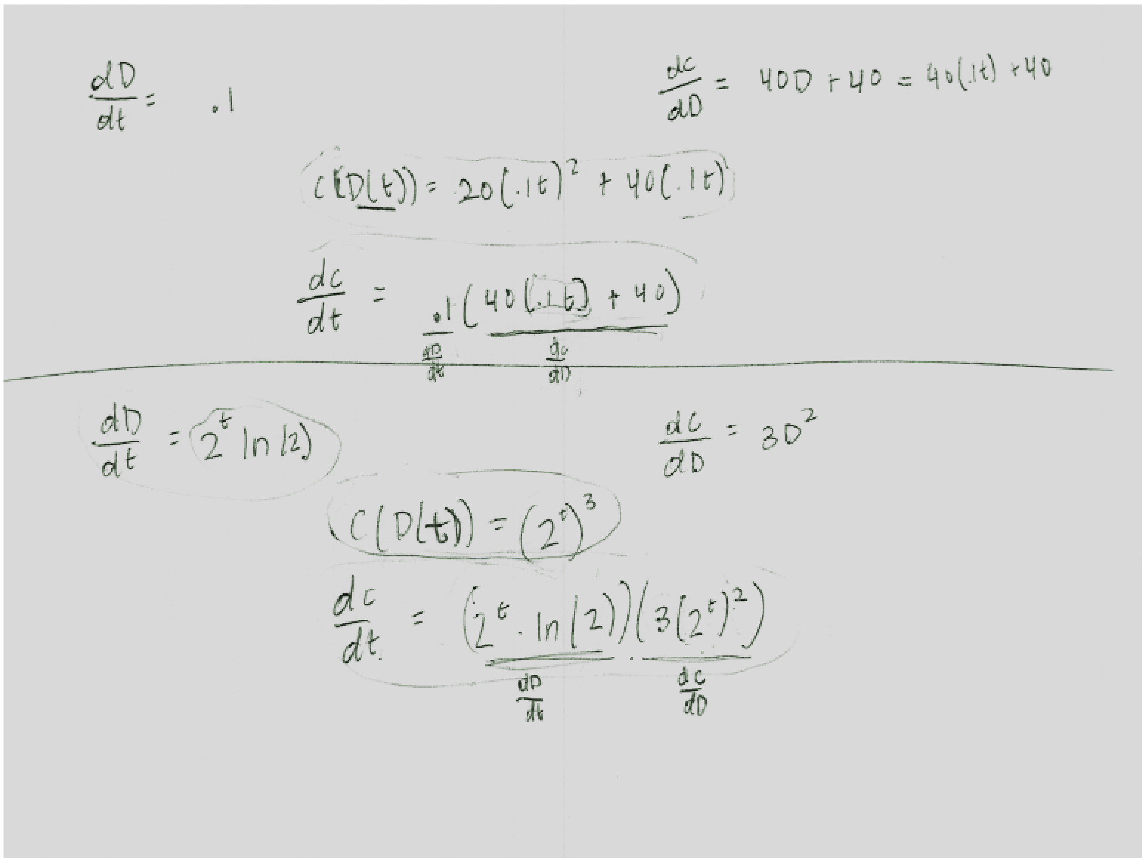
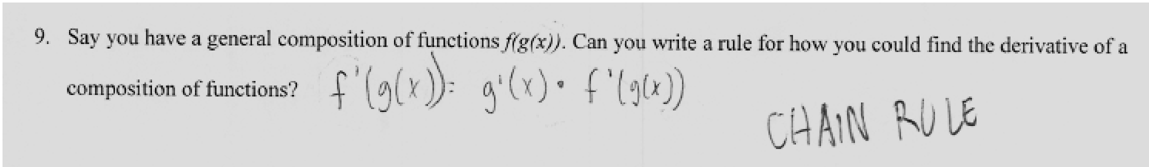


Figure 14. Student A's conclusions within the running context.

In the bottom half of Figure 14, Student A pointed out that $2^t \cdot \ln(2)$ came from *dD/dt,* and $3(2^t)^2$ came from *dc/dD.* Thus, I wrote down *dD/dt* and *dc/dD* below each part she referenced. She then said, "if you took the derivative of c(D(t)), you would get this number [pointing to the resulting product]." Through her work and this comment, we see that she found the derivative of *c(D(t))* and understood why the derivative was what it was. This helped her to understand the patterns in her work and allowed her to ultimately generalize the chain rule to any composition of functions (see Figure 15).


9. Say you have a general composition of functions *f(g(x))*. Can you write a rule for how you could find the derivative of a composition of functions? $f'(g(x)) = g'(x) \cdot f'(g(x))$

CHAIN RULE

Figure 15. Student A's generalization of the chain rule.

Like Student A, all of the students eventually recognized the patterns in what they had done in the previous examples and generalized the chain rule. However, Student B demonstrated some incorrect conclusions about the chain rule; conclusions which proved to make his understanding of related rates more difficult later on. After generalizing the chain rule, he reflected on his experience:

> Student B: If they [teacher or textbook presumably] ask me ok, so we know that we want to know the acceleration and the position, but they don't give me the velocity and then I just get lost because I don't make that connection. But now, plugging these a function inside a function I can make the relation is like ok, so position has velocity as a derivative, and the derivative of velocity is going to be acceleration, so in between that, there is I can say that the entire thing, the function itself can be combined, position and acceleration and I can find derivatives.

Although Student B demonstrated understanding of the chain rule before this comment, his words signaled that he had made some incorrect generalizations about the chain rule. It seems as though he had concluded that anytime there were three variables that were related to one another, they could automatically be placed into a composition of functions and the chain rule could be used to find a missing rate. This demonstrated his lack of understanding of compositions of functions in general. Later, in discussing student understandings in Stage 5 of the HLT, I explain the repercussions of this incorrect conclusion in his solving of related rates problems, as well as what I did to help him address this misunderstanding and move forward in the HLT.

Additionally, although Student C eventually generalized the chain rule, he struggled the most to transition from the simpler chain rule examples, where the values of the derivatives were given at specific points, to general derivative functions. The following is his reflection on the process:

> Student C: "This took me a little bit. I think this took me way long… it does make sense… that the derivative of the inside times the derivative of the outside gives me the derivative of the entire thing, because that's, I feel like, that's basically what I was doing earlier with I guess less complex [pause] It made more sense with just numbers because I had like units to cancel out…but like when it just became variables I wasn't sure if I understood that you know? Well enough. But now, I understand more.

Not only did Student C feel uncomfortable with "variables" and general function representations of derivatives, but he also felt uncomfortable with trigonometric functions which made it difficult for him to differentiate between the "inside" and "outside" functions within a composition. In fact, given $h(x) = \sin(x^2)$, he said that he saw *h(x)* as "one thing" and he explained that he didn't recognize *h(x)* as a composition of functions at all. In order to help him recognize it as a composition of functions, I compared it to the previous problem, where he found the derivative of $g(x) = \sin(f(x))$ and I explained that x^2 in *h(x)* could be thought of as *f(x)* in *g(x)*. Additionally, we talked about how, given an *x* value, in order to calculate an output for *h(x)*, he would first need to square that *x* value and then plug it into the *sine* function. I explained how this two-step process implied the existence of a composition of functions. This, along with other conversations, helped him recognize compositions of functions and ultimately gain more procedural fluency in performing the chain rule.

In the tasks aimed at helping the students gain more procedural fluency with the chain rule, the other three students were challenged, but with some work, became more comfortable with it. Most of them made a few mistakes along the way, but after we compared their efforts to their previous developments of the chain rule, the students quickly identified their mistakes and became more fluent in applying the chain rule to various, abstract functions.

In summary, Student A, C, and D demonstrated multiple pieces of evidence in achieving the first two sub-goals of the second stage of the HLT. Student B correctly generalized the chain rule but made some incorrect conclusions about when one can use the chain rule and about compositions of functions in general. All but Student C showed multiple pieces of evidence in achieving the third sub-goal of the second stage of the HLT in developing procedural fluency of the chain rule. Student C exhibited some procedural fluency, but his discomfort with general function representations of derivatives and trigonometric functions in general slightly inhibited him. See Table 4 for a summary of these results.

Table 4

Summary of Evidence Students Achieved Learning Sub-goals of Second Stage of HLT

Learning sub-goals	Complete	Incomplete	Missing
2a) Continue constructing multiplicative nature of chain rule	A, B, C, D		
2b) Generalize the chain rule to any composition of functions	A, C, D	B	
2c) Procedural fluency of the chain rule	A, B, D	C	

Third Stage of HLT

The third stage of the HLT was aimed at developing the idea of variables being functions of the implicit variable of time and recognizing subsequent existence of compositions of functions. During this stage, and given related rates problems in general, it proved very important for students to identify, in the problem, what derivative was given and what derivative they were trying to find. This helped the students to remember their goal and ultimately solve the problem. After doing so, the students identified which quantities in the given situation changed with time and could therefore be conceptualized as functions of time. For example, in the Snowman Context, Student D explained, "the two things that are changing with time is the radius and the volume. Because in order to find out volume you have to know radius." We talked about how because these quantities are changing with time, they can be conceptualized as functions of time. I then prompted the students to relate these quantities that are changing with time.

At this point in the HLT, Student C understood that V could related to r through $V = \frac{4}{3}\pi r^3$, and he understood that this same equation could be conceptualized as $V(t) = \frac{4}{3}\pi[r(t)]^3$, but he did not understand how he could get from the equation that gives the value of volume for any value of radius to the actual rate of change of the volume. He identified the given rate, $\frac{dr}{dt}$, and the rate that he was trying to find, $\frac{dv}{dt}$. In order to find $\frac{dv}{dt}$, he recognized that he could multiply $\frac{dr}{dt}$ by $\frac{dV}{dr}$, creating a sort of "delta equations" (Infante, 2007), $\frac{dV}{dt} = \frac{dr}{dt} \cdot \frac{DV}{dr}$. In this way, he reached the second sub-goal in the third stage of the HLT, recognizing the need for the chain rule. He used this delta equation to solve the problem by finding $\frac{dV}{dr}$ and multiplying it to the given value of $\frac{dr}{dt}$. Afterwards, I explained to him that we could have also solved the problem by taking the derivative of *V* with respect to *t,* and he seemed to make the connection between the equation relating *V* and *r* to the equation relating $\frac{dV}{dr}$ and $\frac{dr}{dt}$.

Similarly, Student D was struggling to understand how he could use the equation for volume in terms of radius or in terms of time to find $\frac{dv}{dt}$. Having recognized the given rate, $\frac{dr}{dt}$, he also figured that $\frac{dv}{dt} = \frac{dr}{dt} \cdot \frac{dV}{dr}$, creating a delta equation like Student C. However, unlike Student C, Student D's lack of understanding of the derivative made it difficult for him to use the delta equation to solve the problem. When I originally asked him how he might find $\frac{dv}{dt}$, he responded by saying, "I definitely would like to plot a whole bunch of points in order to do that." This comment suggests that he leans towards graphically calculating a derivative, which in this case would be impossible to calculate exactly. The comment also suggests he does not yet fully understand the tools he has learned in class for finding derivatives in general: the limit definition or derivative rules. As stated earlier in the first stage of the HLT, Student D often demonstrated that he did not know the difference between a derivative and an average rate of change. In an effort to solve the problem by using the delta equation, he tried to find $\frac{dV}{dr}$ by simply plugging in 2 for *r* into the equation $V = \frac{4}{3}\pi r^3$. We paused and talked again about both the meaning of the derivative and how we can find it; either through the limit definition or the derivative rules he had learned in class.

Instead of going back to the delta equation and correctly calculating $\frac{dV}{dr}$, we talked about the original equation that modeled the context: $V(t) = \frac{4}{3}\pi[r(t)]^3$. I asked him how he might take the derivative of this equation and he responded with the power rule, getting $V(t) = \frac{4}{3}3\pi[r(t)]^2$. This demonstrated that he did not recognize the nested relationship and the subsequent need to use the chain rule to solve the problem. I then pointed out to him the existence of a composition of functions and he continued to struggle to differentiate the equation because of the constants multiplied to the front. I simplified the problem for him, removing $\frac{4}{3}\pi$ momentarily, leaving $V(t) = [r(t)]^3$. Eventually, with some help, he correctly took the derivative and solved the problem.

Student D recognized the way in which the chain rule could help solve the problem by creating a delta equation. However, partly because I drew his focus back to the original equation, he did not end up using the delta equation to solve the problem. He also did not use NMR to recognize the existence of a composition of functions and the subsequent need to use the chain rule in the original equation that modeled the context. I was the one to point out the existence of a composition of functions in that equation. It might have been more effective and helpful had I prompted him with questions encouraging him to use NMR to recognize the need for the chain rule. However, I simply pointed it out to him, most likely out of impatience more than anything. With help he eventually solved the problem.

After creating an equation that related volume, radius, and time, both Student A and Student B recognized the existence of an implicit composition of functions and thus, the need to use the chain rule in taking the derivative. For example, at that point in the interview, Student A pointed to the $[r(t)]$ part of the equation and explained that we needed to take the "derivative of what is inside and then what is outside." Student B did likewise. This recognition is necessarily what helped them solve the problem and begin to develop understanding for related rates.

Overall, the students began to develop understandings for simple related rates problems by recognizing that when quantities change with time, they can be conceptualized as functions of time. Both Student C and Student D created a "delta equation," recognizing the need for the chain rule. However, Student D could not solve the problem himself because of his lack of understanding of derivatives in general. He also did not show evidence of recognizing the need for the chain rule in the equation for volume in terms of time that modeled the context. On the other hand, Student A and Student B did recognize the existence of a function of the implicit variable time in that equation and they used what they previously learned about the chain rule to solve the problem. In this way, all of the Students exhibited multiple pieces of evidence for reaching the first learning sub-goal in the third stage of the HLT, but only Student A, Student B, and Student C exhibited evidence for the second learning sub-goal. Student D only demonstrated incomplete evidence through his creation of the delta equation. A summary of these findings is in Table 5.

Table 5

Summary of Evidence Students Achieved Learning Sub-goals of Third Stage of HLT

Learning sub-goals	Complete	Incomplete	Missing
3a) Conceptualize variables that change with time as functions of time	A, B, C, D		
3b) Use NMR to recognize the need for the chain rule	A, B, C	D	

Fourth Stage of HLT

The fourth stage of the HLT was focused on developing the idea of implicit functions of other existing variables in an equation and recognizing subsequent existence of compositions of functions. This stage generally went well for all of the students as they understood that one can conceptualize any of the variables in an equation as being an implicit function of another one of the variables and they recognized the subsequent existence of a composition of functions. However, for Student A, at the beginning of this stage in the HLT, even before exploring the meaning and implication of an implicit function of *y* in terms of *x* in the given equation, inconsistencies and holes in her understanding of the derivative revealed themselves.

For example, despite Student A's ability to interpret the meaning of the derivative when given the corresponding symbolic representation earlier on in the HLT, during this stage, she was unable to provide the symbolic representation when given a conceptual definition of a derivative. For example, when first introduced to implicit differentiation, she was given $x^2 - y^2 = 1$ and the following prompt: "The question we will attempt to answer is: as *x* changes, by how much does *y* have to change?" I asked her how we might write what we are trying to find as a derivative. She responded, "the derivative of that would be [pause] *dx/dy* maybe?" Thus, it appeared she did not understand that *dy/dx* would actually represent how fast *y* changes with changes in *x.* We reviewed that with infinitesimal changes in the independent variable, the derivative tells us how the dependent variable changes in response. When we did the next problem, which is the same context but now trying to find the derivative of *x* with respect to *y,* she demonstrated that she better understood the symbolic representations of derivatives by saying, "…we are taking dx over dy instead of dy over dx. So, we are looking for as y changes how much does x change."

After addressing Student A's understanding of the derivative, she built her understanding of implicit differentiation by recognizing the existence of implicit, or hidden compositions of functions, within the equations, thus making connections to her existing knowledge about the chain rule. For example, in finding *dy/dx* for the equation $x^2 - y^2 = 1$, one can think of *y* as being a function of *x* and can conceptualize the equation as being $x^2 - [y(x)]^2 = 1$. When Student A visualized the equation in this way, she commented, pointing to the $[y(x)]^2$ part of the equation, that "you can take the derivative of [the inside function] and multiply it by the derivative of [the outside function] in order to find the derivative of the whole thing." All of the students made similar comments, recognizing the existence of hidden compositions of functions and building connections between their existing knowledge of the chain rule and their experience with implicit differentiation. All of the students did same thing, differentiating the *x* with respect to *y* instead.

Along with the other students, Student B conceptualized *y* as a function of x. When I pointed to the $[y(x)]^2$ part of the equation, he mentioned "that is a function within a function." However, he did not initially apply the chain rule as he should have when he took the derivative of the equation. He said that "he guessed" the derivative of $x^2 - [y(x)]^2 = 1$, with respect to *x,* is $2x - 2[y(x)] = 0$. I do not think that this mistake came from a lack of understanding of the chain rule. Instead, I think that he had simply forgotten how to take the derivative of a composition of functions because it had been a couple of days since the last interview and, after prompting him, he looked back at his previous work and eventually fixed his mistake.

During this part of the interview, Student C expressed discomfort with "*y(x)*" representing *y* as a function of *x.* It is not conventional in mathematics to use *"y"* as the name of a function. Traditionally, "*y"* is used to represent the output variable of a function and "*f"* is used as the function name for function notation. I wanted to use "*y"* as the name of the function so that the students could remember that *y* is equivalent to *y(x).* For the purposes of the interviews, the only difference between *y* and *y(x)* is that *y(x)* is emphasizing the implicit function relationship to *x,* and *y* is not. This did not apparently bother the other students, and in the end, it didn't seem to inhibit Student C's understanding of implicit differentiation. When I had originally suggested that we write *y* as *y(x)* to remind us of the implicit functional relationship, Student C said, "It is kind of weird for me to see this [pointing to y(x)]. But [pause] I have never seen *y* as a function of *x*. Usually, it is like this version [pointing to the original problem], like *y*=*x* or *f(x)*… I haven't seen this, but it does make sense." Oppositely, Student D *preferred* the *y(x)* notation to *f(x)*. In the next problem, (If $x^2 + [f(x)]^3 = 9$, and *f(1) = 2,* find *f'(1)),* he eventually changed [*f(x)*] to be written as *y(x).* He said that with the *f(x)* there, "it is confusing" and he "likes things in *x's, y's,* and *z's."* Despite Student C's discomfort, he and all of the other students successfully conceptualized implicit functions and represented them accordingly.

In summary, all of the students showed ample evidence of conceptualizing implicit functions and representing them accordingly. All of the students also recognized hidden composition of functions and the subsequent need to use the chain rule in differentiating the equation. A summary of the results is found in Table 6.

Table 6

Summary of Evidence Students Achieved Learning Sub-goals of Fourth Stage of HLT

Learning sub-goals	Complete	Incomplete	Missing
4a) Conceptualize implicit functions	A, B, C, D		
4b) Given equations with implicit functions, and subsequent compositions, recognize need for chain rule	A, B, C, D		

Fifth Stage of HLT

The fifth and final stage of the HLT was focused on extending all of the students' previous ideas to more complicated implicit differentiation and related rates contexts. In gaining procedural fluency of implicit differentiation, all of the students solved a more abstract implicit differentiation problem (#13 in the Appendix) with little help from me. All of the students solved the problem smoothly without any inhibitions. Student A was the only one who had time to solve one more implicit differentiation problem (#15 in the Appendix) in the final interview. One interesting part of Student A's understanding of implicit differentiation revealed itself when she solved this problem (See Figure 16).

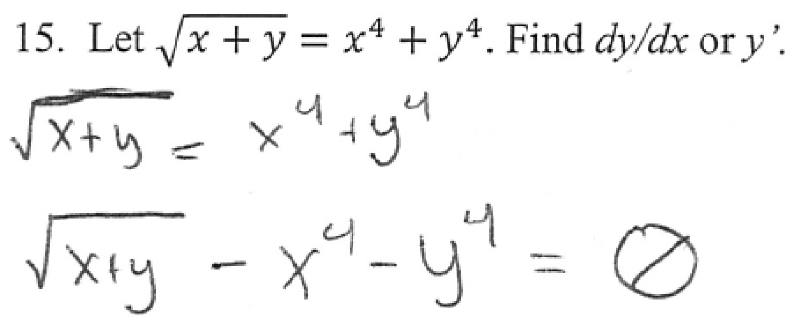


Figure 16. Student A's initial steps in solving #15 in the tasks based on the HLT.

The first thing she did in order to find *dy/dx* was subtract $x^4 + y^4$ from both sides of the equation so that she had a constant 0 on one side of the equation. This isn't necessarily wrong, and she still correctly solved the problem, yet it seemed as though she was limited in her understanding of implicit differentiation. That is, because all of the implicit differentiation examples in the tasks were set equal to a constant, she may have felt that the only way to solve an implicit differentiation problem was by setting the equation equal to a constant. The other students did not have enough time in the interviews to attempt this particular implicit differentiation problem, so I am not sure they made the same generalization. However, because the students were only exposed to these types of equations, becoming the prototype for them in their exploration of implicit differentiation, it is likely they did make this same generalization. Because Student A appeared limited in her understanding of implicit differentiation, thinking that it can only work with equations set equal to a constant, and I do not have any evidence that the rest of the students did not also have this limited understanding, I concluded that there is only incomplete evidence that the students achieved the first learning sub-goal of the fifth stage of the HLT in gaining procedural fluency with implicit differentiation.

Similarly, in the students' exploration of related rates problems later on, all of them did not seem to realize that they could take the derivative of the entire equation, that modeled the context, with respect to time. Many of them tried solving for one variable or tried to create explicit equations with one variable in terms of the other. This is not necessarily incorrect, but it did make solving the related rates problem *much* more difficult and exhibited their limited perception of the possibilities of derivative-taking and the idea that one can take the derivative of an entire equation, no matter how it is arranged or re-arranged, with respect to some variable.

For example, all of the students struggled to realize that they did not need to create an explicit equation in the Shuttle Launch Context. In the Shuttle Launch Context (see #14 in Appendix), a camera is filming a shuttle launch and needs to stay focused on the shuttle. The camera is 2 miles from the launch-pad and the shuttle is traveling vertically at .2 miles/second. The students are to find how fast the angle from the ground and the camera's line of sight should be increasing when the shuttle is 3 miles above the ground. Like the rest of the students, Student C created the equation $\sin(\theta) = \frac{D}{\sqrt{13}}$, where θ represents the angle of the camera between the ground and its line of sight to the shuttle, and *D* represents the shuttle's vertical distance from the ground in miles. He, along with all of the other students, then tried to solve for θ, so that it was by itself and in terms of *D*. Although this procedure is not incorrect, it does make taking the derivative much more complicated.

I wanted the students to understand that they could take the derivative of the equation, as was, with respect to time because it makes taking the derivative much less tedious. When the students tried to solve for θ, I simply explained to them that their thinking wasn't incorrect but that it would make solving the problem much more difficult.

Reflecting on his experience, Student C explained that he felt like his biggest problem was that he wasn't focusing on relationships. He said, "it's a lot easier… for me it's like just this thing alone with the *t*'s in it [referencing an explicit equation]. I'm just like I don't know how if I can take it [the derivative] in relation to time, but that's exactly what we just did I guess." His comments, along with the other students' discomfort with non-explicit equations, makes me wonder if we focus too much on explicit functions and equations in students' pre-calculus experience.

After understanding she did not need to solve explicitly for θ, Student A not only realized there was a hidden composition of functions, within the equation, but she also created the explicit composition of functions. Because the shuttle was moving at a constant rate of .2 miles/second, she realized that *b,* the distance of the shuttle from the ground, could be written as the function of time *t:* $b(t) = .2t$. This equation helped her remember *b* was a function of time. Then, in the equation, $\tan(\theta) = (\frac{b}{2})$, where θ is the angle between the ground and the line of sight from the camera to the shuttle, she replaced *b* with the function of time, writing $\tan(\theta(t)) = (\frac{.2t}{2})$. From here, she more directly found $d\theta/dt$.

Student B did not have as much success initially with the Shuttle Launch Context. His lack of understanding of compositions of functions and his subsequent incorrect generalization of the chain rule made his being able to solve this related rates problem more difficult. Based on what we had done in previous interviews, he knew he wanted to create a composition of functions, but he wasn't sure how. In trying to create an equation to model the situation, he explained that the velocity of the shuttle will affect the shuttle's distance above the ground which will affect the angle of the camera between the ground and its line of sight to the shuttle. Thus, he hypothesized that $c(s(v))$, where c=angle of the camera between the ground and the line of sight to the shuttle in degrees, s=vertical distance of shuttle from the ground in miles, and v=velocity of shuttle in mi/sec, could be a composition of functions to model the situation. Although the velocity of the shuttle will affect the vertical distance of the shuttle above the ground which will affect the angle of the camera, these variables cannot correctly be composed in this way. First, in this situation, the velocity is constant, and so given the velocity is .2 mi/sec, one cannot know how far the shuttle has traveled without considering time. Additionally, even if the velocity was not constant in this situation, it is unlikely that each velocity quantity would map to a unique shuttle distance output to create a function.

In order to help him reevaluate his understanding of compositions of functions, and ultimately help him move forward in the HLT, we used NMR. We talked about how, if we could compose those variables in the way he did, changes in the velocity need to determine changes in the shuttle's distances which, in turn, need to determine changes in the angle of the camera. We stepped away from the shuttle example for a moment, and we had the following conversation:

> PI: If you are driving in your car and all you know is that your velocity has changed from 50 mph to 60 mph, do you know how much your distance has changed?
>
> Student B: No, because I don't know the acceleration. [pause] So, I think I know where you are going with the distance with this kind of function [pointing to $c(s(v))$]. Because yeah, in this case, the velocity is the same, it is not going to affect anything. It is going to remain the same. That's why we should talk about the time. If the velocity is constant, then the time is going to determine the distance, because the velocity's not changing.

Here, Student B finally began to understand the constraints for a composition of functions and why we need to think about time. Later on, we talked more about which quantities were changing with time in this situation and we created an equation to relate them. With some more guidance, he eventually took the derivative of the equation with respect to time and solved the problem. At the end of interviews, he reflected on his experience:

> Student B: I feel like the most difficult part for me is to understand a composition of functions, like a function inside another one and inside another and inside another one. Like sometimes I see these [pointing at a composition of functions], "oh yeah [mumbling] I can take the derivative of that one," but then there is another one inside that and then another one inside that. And that is, I think, a concept that is hard for me to understand or to recognize right at the beginning.

Prior to this comment, although he had before recognized that one of the variables in the problem at hand was changing with time, and we had represented that variable as a function of time, he still did not apply the chain rule as he had. Earlier, he did gain a good intuition for the multiplicative nature of the chain rule, but it seems that in his application of it, he was hindered by his lack of understanding of compositions of functions: how they work and how to recognize them. Thus, understanding of function and function composition proves to be a necessary precursor for developing understanding for these concepts.

During this last phase in the HLT, although the students realized which two variables in the context were functions of the implicit variable of time, they struggled to create a useful equation to relate the two variables and model the context. For example, in the Airplane Context (#16 in Appendix), Student A knew that she needed to find a way to relate *x,* the horizontal distance of the plane from the radar station, and *d.* When asked why she thought to relate *d* and *x,* she said, "because if you can find the relationship between x and d, then you can use them to find what *dd/dt* is." She proceeded to move in the right direction for finding this relationship when she said, "So, x relates to d as [pause] kind of like the Pythagorean theorem." However, she laughed at her own suggestion in seeming insecurity and moved away from the idea. When I asked her again how she might relate x and d, she wrote down the equation in Figure 8:

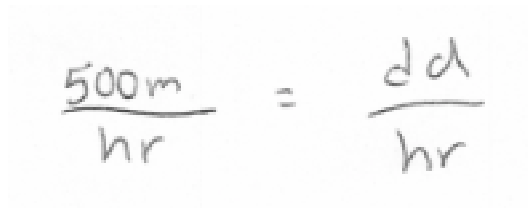


Figure 8: Student A's original equation to model the radar and plane problem.

When I asked Student A why she set these equal to one another, setting equal the rate of change of the airplane's horizontal distance and then the rate of change of the direct distance from the airplane to the radar station, she laughed and said, "it sounded good" and was unable to give a coherent justification; it seemed as if she was just guessing. In an effort to connect what she was doing to what she had previously learned with the chain rule, she then created the equation $2 = d(x(t))$, because she knew that she was interested in the time when the direct distance, d, was equal to 2 miles and that d would change as x changed and that x was also a function of t. When she tried to take the derivative, she realized that she could not solve for *dd/dt* and ultimately solve the problem.

Finally, I reviewed with her what she did in the Shuttle Launch Context, in relating θ and d through the tangent function and suggested to her that she go back to her original idea of the Pythagorean theorem. She did, and with some more discussion and prompting, successfully solved the problem, recognizing that even within the equation $d^2 - x^2 = 1$, d and x were both functions of t and she could take the derivative of the function as was with respect to time. Afterwards, she remarked that the problem "was not as hard as it seemed."

Overall, during this stage, it became clear that students may have had too narrow experience with implicit differentiation, where the only implicit differentiation problems they solved involved equations that were set equal to a constant. Because of this, I concluded there was incomplete evidence that all of the students achieved the first learning sub-goal in the fifth stage of the HLT. As far as the second learning sub-goal, all of the students exhibited evidence for recognizing functions of the implicit variable of time and subsequent existence of compositions of functions. However, this does not imply that students did not struggle with modeling and ultimately solving the related rates problems. All of the students struggled to create an equation to model the different, more difficult, related rates contexts. In the Shuttle Launch Context, they seemed to think that they needed to solve for θ explicitly, which was not wrong but would have made solving the problem much more difficult. Student B's lack of understanding of compositions of functions greatly inhibited him from correctly modeling the Shuttle Launch Context. Despite the students' difficulties in solving the related rates problems, they still exhibited that they understood how the variables changed together and solved the problems. For a summary of these results, see Table 7.

Table 7

Summary of Evidence Students Achieved Learning Sub-goals of Fifth Stage of HLT

Learning sub-goals	Complete	Incomplete	Missing
5a) Gain procedural fluency with more complicated implicit differentiation problems		A, B, C, D	
5b) Within equations that model more complicated related rates contexts, recognize functions of the implicit variable of time and subsequent composition of functions	A, B, C, D		

Areas of Student Difficulty Suggesting a Need to Revise the HLT

In the previous two sections, I presented my results for how I saw NMR being used and what types of understandings were developed as students progressed through the HLT. I now build on these results by identifying specific areas of student difficulty that would suggest a needed revision to the HLT. As I do, I insert a brief discussion of how the HLT might be changed in response to that specific difficulty. These changes are then put together into a revised HLT in the next chapter.

The first stage of the HLT was focused on developing intuition for the multiplicative nature of the chain rule. This stage went well as all of the students exhibited multiple pieces of evidence for achieving all of the learning sub-goals. There were not any areas of student difficulty suggesting a need to revise the HLT in this stage.

The second stage of the HLT was focused on generalizing the chain rule and gaining procedural fluency with it. Students also did fairly well in developing deeper understanding of the chain rule during this stage of the HLT. However, it became clear that students' struggles with generalizing the chain rule and gaining procedural fluency with it came from their weak understanding of functions and function composition. A couple of them struggled to recognize the "inside" and "outside" functions within a composition of functions. It should be noted that in order to successfully develop understanding and procedural fluency of the chain rule, students should have prerequisite understanding of functions and function compositions. That is, students should have conceptual understanding and procedural fluency with all different types of functions, including trigonometric functions, with which students in the study seemed to be particularly uncomfortable. They should also understand what it means to compose two functions, that the range of the "inside" function is equal to the domain of the "outside" function.

I hypothesize that by adding a sub-goal to the second stage of the HLT for students to practice simply identifying "inside" and "outside" functions within more abstract examples as well as in a few related rates contexts might help the students to become more comfortable with compositions of functions, gain procedural fluency of the chain rule, and solve related rates problems later in the HLT. I am not suggesting that students start to actually solve the related rates problems yet, but simply identify the "inside" and "outside" functions within equations that model such contexts. I also note that based on the data, not having a conceptual understanding of compositions of functions *before* beginning the HLT makes building understanding for these concepts very difficult. Thus, helping students build a conceptual understanding of compositions of functions in their secondary mathematics and algebra experience is important if students are to develop conceptual understanding for the concepts explored in this HLT. Additionally, although students should have a conceptual understanding of compositions of functions before the HLT, I did not want to add this sub-goal to the beginning of Stage 1 because I did not want students to get into a procedural mindset from the beginning of identifying "inside" and "outside" functions within compositions of functions. Instead, by having the additional sub-goal here, they hopefully build a conceptual understanding for chain rule and practice this identification of "inside" and "outside" functions in order to gain more procedural fluency with the chain rule and efficiency in solving implicit differentiation and related rates problems.

The third stage of the HLT was aimed at developing the idea of variables being functions of the implicit variable time and recognizing subsequent existence of compositions of functions. This stage in the HLT went well as students began to apply their understanding of the chain rule to simple related rates contexts. Students were comfortable with conceptualizing variables, that change with time, as functions of time and recognizing hidden compositions of functions within the equation that modeled the context. This allowed them to recognize the need to use the chain rule in differentiating the equation with respect to time and ultimately solve the problem. There is not anything that needs to be addressed in this stage of the HLT.

The fourth stage of the HLT was focused on developing the idea of implicit functions and subsequent existence of hidden compositions of functions. Not in this stage, but in the fifth stage of the HLT, for implicit differentiation, there was no evidence that students realized they could differentiate both sides of an equation unless it was set equal to a constant. Similarly, in the more complicated related rates problems, students continually tried to solve for one variable explicitly in terms of the other variable. Overall, it seems like students did not understand how or why they could differentiate both sides of an equation with respect to a chosen variable or with respect to time. This lack of understanding did not become apparent until the fifth stage of the HLT, but I concluded that the fourth stage of the HLT needs to be addressed in order to better prepare students to successfully move forward in the fifth stage of the HLT. I propose three ways to address this issue in the fourth stage of the HLT.

First, according to Mirin and Zazkis (2019), students struggle to understand exactly why one can take the derivative of both sides of an equation with respect to some chosen variable. Mirin and Zazkis suggest that it may be helpful to explicitly help students understand that one can conceptualize either side of the equation as a function and consider where those functions are equal. Where those functions are equal, there too will their derivatives be equal. This is not something that I made explicit with my students in the interviews and this was not a part of my original HLT. It may be important to edit the HLT to include the learning sub-goals of 1) conceptualizing both sides of an equation as different functions, but equal within a certain domain and range and 2) recognizing that where two functions are equal, their derivatives must also be equal.

Second, it became clear from Student A's experience that students may have incorrectly assumed that implicit differentiation only works with equations that have the two variables on one side of the equation and a constant on the other side of the equation. Thus, it is important to incorporate learning activities during this stage in the HLT that involve different, non-prototypical equations, so that students gain a better and more diverse understanding of implicit differentiation. This incorrect generalization may be addressed through different, non-prototypical examples during the suggested additional learning sub-goals described in the previous paragraph.

Third, by the end of the interviews, it became clear that students had experience conceptualizing functions of the implicit variable of time given a simple related rates context with a simple composition of functions (e.g. $V(t) = \frac{4}{3}\pi[r(t)]^3$ and they had experience conceptualizing simple implicit functions (e.g. $x^2 + [y(x)]^2 = 1$). However, they did not experience conceptualizing functions of the implicit variable of time with more complicated functions (e.g. $\cos\big(x(t)\big) + [x(t)]^3 = [y(t)]^2$). This may also be part of the reason that students struggled to recognize their ability to differentiate both sides of an equation in more complicated related rates contexts. Like in the Shuttle Launch Context where all of the students tried to solve the equation explicitly for θ, students often felt they needed to solve the equation so that one variable was written as an explicit function of the other variable. This reasoning is not incorrect, but it inhibited their ability to solve the problem or, at the very least, made it much more difficult than it needed to be. Thus, in order to help students better prepare for the fifth stage of the HLT, there should be an additional sub-goal in the fourth stage of the HLT where students conceptualize that given any equation with functions of an implicit variable, usually time, the derivative of both sides of that equation, with respect to the implicit variable, is also equal. By including this mental action, students are more prepared to solve more complicated related rates problems in the future as they will be more comfortable with differentiating the equation that models the context as is. They will not feel the need to solve for one variable in terms of another which can make the related rates problem more difficult than needs be for the student.

The fifth and final stage of the HLT was focused on extending all of the students' previous ideas to more complicated implicit differentiation and related rates contexts. During this stage, Student A demonstrated that she only felt comfortable solving implicit differentiation problems when the equation had a constant on one side. Additionally, during the more complicated related rates problems, all the students struggled to 1) create an equation to model the context, and 2) use that equation to find the derivative of interest. Infante (2007) also found that students in her study struggled to create an equation to model related rates contexts.

In order to address Student A's, and most likely all of the student's, narrow understanding of implicit differentiation, different and non-prototypical equations should be used in the exploration of implicit differentiation during the fourth stage of the HLT. Additionally, in the fourth stage of the HLT, the additional sub-goal of explicitly helping students understand the validity of differentiating both sides of an equation with implicit functions may also help deepen students understanding of implicit differentiation. As for the students' struggles with related rates, the additional sub-goals in the fourth stage may help students better understand how they can use the equation that models the context to find the derivative of interest and solve the related rates problem. However, it seems to be a more fundamental understanding to help students create that equation in the first place. The ability to mathematical model should be developed well before the ideas of this HLT are explored and developed.

CHAPTER SIX: DISCUSSION

In this discussion, I summarize the answers to each of my research questions, provide a revised HLT, connect my study to existing research, and discuss questions for future research.

Summarized Answers to Research Questions

My first research question is: How was nested multivariation used as the first-semester calculus students progressed through the HLT? In the beginning, students used NMR to understand the ways in which the variables in the function composition were related to and depended on one another. In order to develop intuition for the multiplicative nature of the chain rule, they first thought about the different rates within the function composition at a point and reasoned about how the quantities changed together, coordinating amounts of change. Coordinating amounts of change naturally led them to develop the multiplicative nature of the chain rule. They used this base understanding and observed patterns in the multiple examples they explored in order to generalize the chain rule. From that point on, NMR was mostly important in recognizing the existence of a nested relationship and the subsequent need of the chain rule in differentiation. It also proved important for students to use NMR to reason about how the quantities changed together within related rates contexts so they could move towards creating an equation to model the context.

My second research question is: What kind of understandings did the first-semester calculus students develop for these three concepts within each major stage of the HLT? During the first stage of the HLT, the students interpreted the meaning of the different rates within a composition of functions and developed intuition for the multiplicative nature of the chain rule. They developed intuition for the multiplicative nature of the chain rule by reasoning about how the quantities of the given unit rates change together and by using dimensional analysis. Some of the students also recognized that the infinitesimal values in the two derivatives would cancel out in order to get the desired derivative. During the second stage of the HLT, the students generalized the chain rule and gained procedural fluency with it. Most of the students smoothly extended that understanding to taking the derivative of general compositions of functions. Here, Student B made incorrect conclusions about the chain rule, generalizing it to situations beyond proper compositions of functions; I addressed this with him during the fifth stage of the HLT. Student C struggled to transition from derivatives at specific points to general derivative functions, mostly because of his discomfort with variables and general representations of derivatives. By the end, Student C felt more comfortable working with these more abstract representations.

The third stage of the HLT was aimed at developing the idea of variables being functions of the implicit variable of time and recognizing subsequent existence of compositions of functions. Here, it proved important for students to recognize what derivative was given in the problem and what derivative they were trying to find. Doing so helped them to stay focused on the goal of the problem and ultimately solve it. Students conceptualized which quantities were changing with time and then conceptualized them as functions of time. This helped them to recognize an implicit, or hidden, composition of functions and eventually differentiate the equation that modeled the problem and find the rate of interest.

The fourth stage was focused on developing the idea of implicit functions of other existing variables in an equation and recognizing subsequent existence of compositions of functions. Here, Student C expressed discomfort with the non-traditional $y(x)$ (representing y as a function of x) and $x(y)$ (representing x as a function of y) function notation. The other students didn't express discomfort with that notation and Student D even preferred it. However, as this is different from mathematical convention, it might be worth changing the interview protocol so that these functions are represented with traditional function notation (*f(x)* or *g(y)*). This is something that may need to be explored more in future research. Nevertheless, all of the students eventually conceptualized implicit functions and corresponding compositions of functions that required the chain rule when differentiating them.

The fifth and final stage of the HLT was focused on extending all of the students' previous ideas to more complicated implicit differentiation and related rates contexts. Here, Student A seemed to think that implicit differentiation could only be performed with equations that had constants on one side, because those were the only examples to which she was exposed in the interview protocol. It is possible that the other students made the same assumption. However, all of the students exhibited some evidence for gaining procedural fluency of implicit differentiation. Student B struggled to move forward in this stage because of his incorrect conclusion about generalizing the chain rule to situations beyond proper compositions of functions. During the last interview, I helped him better understand necessary stipulations for compositions of functions and how the variables within are related to one another which helped him move forward. During this stage, many of the students struggled to create equations to model the context within the more complicated related rates problem. They were also uncomfortable with differentiating the equation as was and often wanted to solve for one variable explicitly in terms of the others. Nevertheless, all of the students exhibited evidence for achieving the second learning sub-goal of this stage and recognized functions of the implicit variable of time and subsequent compositions of functions.

My third research question is: Where in the teaching experiment did students struggle in a way that suggested a needed revision to the HLT? In the first stage of the HLT, all of the students gained intuition for the multiplicative nature of the chain rule. There were not any areas of student difficult, in this stage that suggested a need to revise the HLT.

In the second stage of the HLT, students' struggle to generalize the chain rule and gain procedural fluency with it came from both their lack of understanding of functions and function composition. Understanding of functions and function compositions proved to be a necessary prerequisite knowledge for the HLT. Nevertheless, students specifically struggled to recognize the "inside" and "outside" functions within a composition of functions, during this stage and later in the HLT in their exploration of related rates. Thus, I hypothesize that it might be useful to add a sub-goal to this stage for students to gain fluency with identifying "inside" and "outside" functions within both abstract examples as well as related rates contexts.

During the third stage of the HLT, students were comfortable with conceptualizing variables, that change with time, as functions of time and recognizing hidden compositions of functions within the equation that modeled the context. There were not any student struggles that suggested a need to revise the HLT during this stage.

During the fourth stage of the HLT, it became clear that students need to understand why they can differentiate both sides of an equation with respect to some chosen variable. Thus, it may be important to include the learning sub-goals of 1) conceptualizing both sides of an equation as different functions, but equal within a certain domain and range and 2) recognizing that where two functions are equal, their derivatives must also be equal. To help students during the fifth and last stage of the HLT, it may also be important to add a learning sub-goal to the HLT that helps students understand why differentiating both sides of an equation is valid when there are functions of an implicit variable involved. Additionally, something to be adjusted in the learning activities is to make sure than non-prototypical equations are used in exploration of implicit differentiation. That is, students should not only be exposed to equations that have only a constant on one side of the equal sign.

The students' struggles during Stage 5 of the HLT may be addressed with the suggested changes to the HLT in Stage 4. However, the students' struggles to create an equation to model the related rates context suggests that students need more experience with mathematical modeling well before they enter calculus. Redish (2005) says that in our traditional approach to mathematics, we often provide students with ready-made models of the real world and we may be "exasperated – or even irritated – if they focus on details that we know to be irrelevant" (p. 7). He explains that we rarely ask students to interpret their results or evaluate whether or not their initial model is adequate. He also explains that at introductory levels, we test students on "one-step recognition, giving 'cues' so we don't require our students to recognize deep structures" (p. 7). His comments highlight the fact that we too often give our students ready-made models of contexts and we don't allow them enough opportunities to struggle and create a mathematical model themselves. This may be something that needs to change in our pre-calculus mathematics teaching if we expect students to really understand related rates problems and apply these same ideas in other meaningful contexts.

Revised HLT

The revised HLT based on the answer to my third research question is found in Table 8. The changes made to the original HLT are marked with an asterisk. Note that although the additional sub-goals 4b and 4d appear similar, they are slightly different in that the first deals with implicit functions and the second deals with functions of an implicit variable (see the Nested Multivariation section in Chapter 3 for the difference between these terms).

Table 8

Revised Hypothetical Learning Trajectory

Stage	Description of Goal
Stage 1	**Develop the multiplicative nature of the chain rule.**
1a	Given a function composition *f(g(x))* that models a meaningful context, interpret *df/dx* as how many times as large the change in *f* is than an infinitesimal change in *x*.
1b	Given a function composition *f(g(x))* that models a meaningful context, interpret *df/dg* as how many times as large the change in *f* is than an infinitesimal change in *g*.
1c	Given a function composition *f(g(x))* that models a meaningful context, interpret *dg/dx* as how many times as large the change in *g* is than an infinitesimal change in *x*.
1d	Given a function composition *f(g(x))* that models a meaningful context, conceptualize how changes in *x* affect changes in the other two variables simultaneously.
1e	Given a function composition *f(g(x))* that models a meaningful context, and after finding specific values of *dg/dx* and *df/dg,* construct that $\frac{df}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}$ at a specific point.
Stage 2	**Generalize the chain rule and gain procedural fluency.**
2a	Continue to construct the multiplicative nature of the chain rule with specific examples.
2b	Generalize the chain rule for any function composition *f(g(x)*.
2c	Practice the chain rule with different compositions of functions to gain procedural fluency with its application.
***2d**	*Gain procedural fluency in identifying the "inside" and "outside" functions within compositions of functions; both abstract and within related rates contexts
Stage 3	**Develop the idea of variables being functions of the implicit variable of time and recognize subsequent existence of compositions of functions.**
3a	When variables change with time, conceptualize them as functions of time and represent them accordingly (e.g. if *r* changes with time, it can be conceptualized and written as *r(t)*).
3b	NMR is used in recognizing the need to use the chain rule in related rates problems where quantities can be conceptualized as functions of the implicit variable of time, with subsequent compositions of functions.
Stage 4	**Develop the idea of implicit functions in an equation and recognize subsequent existence of compositions of functions.**
4a	Given an equation with variables *x* and *y,* one can conceptualize *y* as an implicit function of *x* or *x* as an implicit function of *y*. These implicit functions can be represented accordingly (e.g. *y(x)* or *x(y)*).
***4b**	* Conceptualize both sides of an equation as different functions, but equal within a certain domain and range and recognize that where two functions are equal, their derivatives must also be equal.
4c	Given equations with implicit functions, and subsequent compositions of functions, recognize the need for the chain rule in taking the derivative with respect to either implicit independent variable.
***4d**	*Given an equation with functions of an implicit variable, conceptualize both sides of an equation as different functions, but equal within a certain domain and range and recognize that where the two functions are equal, their derivatives must also be equal.
Stage 5	**Extend all of these ideas to more complicated implicit differentiation and related rates contexts.**
5a	Gain procedural fluency with more complicated implicit differentiation problems.
5b	Within equations that model more complicated related rates contexts, recognize functions of the implicit variable of time and subsequent compositions of functions and the need for the chain rule in taking the derivative of the equation with respect to time.

Connections to Existing Research

Typically, in calculus there is too much emphasis placed on procedures and manipulation of symbols without conceptual understanding (Tall, 1992; Ferrini-Mundy & Gaudard, 1992; Rasmussen & Marrongelle, 2014; White & Mesa, 2014; White & Mitchelmore, 1996). Through the tasks based on the HLT, the students gained a deeper understanding of these ideas than they might have in a typical calculus class. The tasks and the HLT focused on meaning; meaning of derivatives, the multiplicative nature of the chain rule, implicit functions, and functions of implicit variables. At the end of the fourth interview, I asked Student A if there was anything she wanted to share about her experience. She said, "It was fun. It was more interesting…it's like made me think about it a lot more in depth than I normally would have. And it was good because it made me try and figure things out instead of just saying, 'Google, what's the answer?'" In my experience as a calculus T.A., I have noticed that many students immediately turn to websites like Google or Slader to complete their homework because they do not understand the meaning of the material in their calculus class; they are focused on finding the right procedure to solve the problem and complete their homework. My hope is that my proposed HLT and associated tasks can provide information and guidance for researchers and teachers hoping to help calculus students gain a more conceptual understanding of the chain rule, implicit differentiation, and related rates.

Although researchers have hinted to the way in which the chain rule, implicit differentiation, and related rates are related to one another (Clark et al., 1997; Cottrill, 1999; Infante, 2007; Martin, 2000), the research tends to examine these concepts in isolation and has not conducted a serious investigation into how these concepts could be taught in a way that connects them together. My research is unique in asserting that not only are these concepts related, but they share the same underlying mathematical concept: nested multivariation. From the data, and from the students' use of NMR in their exploration of all three concepts, it seems that these concepts can be founded on NM. However, there are some aspects of implicit differentiation and related rates that require other foundational mathematical ideas, such as functions and mathematical modeling.

In general, students struggle to understand and appropriately use the chain rule (Clark et al., 1997; Cottrill, 1999; Infante, 2007). From the data, students used NMR to develop powerful understanding of the multiplicative nature of the chain rule. They were only inhibited in their generalization and procedural fluency of it because they lacked in understanding functions and function compositions.

Oehrtman, Carlson, and Thompson (2008) explain that it is extremely important for students to understand functions in order to understand calculus, and there have been repeated calls for school curricula to place greater emphasis on functions (NCTM, 1934, 1989, 2000). My research adds to this body of research that demonstrates the importance for understanding of functions. The students in my study struggled to generalize and gain procedural fluency with the chain rule because of their lack of understanding of functions and function composition. Additionally, Student B struggled with related rates problems because he did not understand the stipulations for composing functions and the fact that the range of the "inner" function needs to be equal to the domain of the "outer" function. Students need opportunities to gain a better understanding of functions and function compositions in order to understand the chain rule, implicit differentiation, and related rates.

Speer and Kung (2016) explain that research on implicit differentiation is largely "missing" from mathematics education research. This study adds to research important information about how students can better understand implicit differentiation. Mirin and Zazkis (2019) argue that understanding why one can differentiate both sides of an equation is nontrivial for students. In the data, I saw that students indeed struggled to understand the viability of differentiating both sides of an equation. Thus, it was important to adjust the HLT accordingly to help students understand why differentiating both sides of an equation with respect to a chosen variable is valid mathematically.

According to Austin, Barry, and Berman (2000), Ritchie (1836) introduced related rates problems into his text in an effort to help students to understand and recognize the power of calculus. In his textbook, Stewart (2016) introduces related rates with a step-by-step procedure for students to follow in order to be able to solve the problems. Nevertheless, students have struggled to understand or solve related rates problems (Infante, 2007; Martin, 2000). I believe that the HLT I have created provides a way for students to more fully understand related rates and experience the power of calculus. At the end of the last interview, Student C said, "in my experience [pause] like traditional math… I have number and I try to plug it in, and I try to plug in as much as I can until I can't plug it in anymore. But this one [referencing the related rates problem] … we are given things, but less plugging and more like there's relationships that exist and try to figure out like an equation that makes the relationship…we try to establish a relationship." Here, Student C explains that in his experience with related rates problems in the interviews, he did not see the problem as a set of steps to follow, or things to "plug in," but instead as a way to think about and model the relationships that exist within the context.

Although there is a historical debate about whether or not the idea of infinitesimals is mathematically rigorous, researchers have argued that infinitesimals are robust and viable and that using infinitesimals is conceptually beneficial for students (Dray & Manogue, 2010; Ely, 2010; Ely, 2017; Jones, 2015). My research adds to these researchers by showing how a focus on infinitesimals can help students to develop understanding of the multiplicative nature of the chain rule and extend that idea to implicit differentiation and related rates.

Lastly, as stated earlier, my study builds on Infante's (2007) dissertation. First, In Infante's study, she revisited the chain rule with her students, hoping that by helping them to better understand the chain rule that they had already learned, they would be more prepared to solve related rates problems. Instead of revisiting the already learned chain rule, my study explores what it might look like for students to develop a good understanding of the chain rule from the beginning. Second, in her study, all of the students created a delta equation to solve the related rates problem. She admits that this was perhaps a strictly procedural part of the process as a result of the chain rule discussions. In my study, students developed intuition for the multiplicative nature of the chain rule, and they were not limited to creating a delta equation to solve related rates problems. They approached the related rates problems with whatever method was most comfortable to her/him. Third, Infante (2007) found that students struggled to recognize functions of the implicit variable of time. This was an extremely important aspect of my HLT: by helping students think about which variables were changing with time and conceptualize those as functions of time, they rarely forgot about the variable of time. Fourth, Infante (2007) explains that unlike mathematicians, the students in her study did not think about the variables in the equations that modeled the related rates contexts as representations of varying quantities. She said that the students also did not "spontaneously reference their diagram once it was drawn and labeled" (p. 248). In my study, I found that for every related rates problem, each student consistently referred back to the diagram in an effort to make better sense of the problem. Students A, B, and C continued to use NMR in their exploration of related rates which allowed them to view the variables in both their diagram and equations as fundamentally representing varying quantities.

Questions for Future Research

My study opens up questions for future research. First, I have explored the HLT with only four first-semester calculus students. Although this provides deeper insights into the cognitive processes of a few students, my study is not generalizable. It would be beneficial to take the revised HLT from my study, edit the tasks accordingly, and explore it again with a few first-semester calculus students in order to gain more deep understandings of the processes of a few students. It would also be beneficial to take the revised HLT and edit the tasks to practically fit into a calculus curriculum. Then, one could explore the usefulness of the HLT with more students in an entire-class setting and gain more generalizable results. Questions that could be explored include 1) “What might be a practical lesson plan based on the HLT for a first-semester calculus classroom?”, 2) “What understandings do students in a class using this lesson develop for these three concepts?”, and 3) “How do students use NMR in the lesson and throughout the rest of their experience in the course?”

I recognize that my proposed HLT may not be the only or the best way to develop conceptual understanding of the chain rule, implicit differentiation, and related rates. Other research could explore what other HLT’s may exist for these three concepts. Additionally, in an effort to help students make deeper connections between the concepts, thus gaining a better understanding (Brownell, 1935; Hiebert & Carpenter, 1992; Hiebert et al., 1997), researchers could explore other ways in which these three concepts may be related. One could explore the question, “What other concepts are related to the chain rule, implicit differentiation, and related rates through NMR?”

My study proposes a conceptual framework for NM. One could take that framework and explore the different proposed mental action levels in one-on-one interviews with calculus students. Through questions about the way in which quantities change together in function compositions, researchers could explore the questions “How do calculus students exhibit NMR?” and “What mental action levels are present?” Additionally, my conceptual framework does not progress to mental actions students might employ in conceptualizing changing rates of change, or how the rate of change itself varies. Future research might explore how NM extends to conceptualizing changing rates of change. Future research could also explore how students might conceptualize changing rates of change within the contexts of the chain rule, implicit differentiation, and related rates.

I mentioned in my results that some students found the notation of $y(x)$, as opposed to $f(x)$, for y as a function of x as non-traditional but useful. However, Student C was uncomfortable with using $y(x)$ because it is non-traditional, and he was not used to seeing it in his mathematics classes. Other research could explore the usefulness of using $y(x)$ as opposed to $f(x)$ in helping students to remember with which variables they are working.

I invite both researchers and teachers to use this HLT in their research or classrooms in order to learn more about how we can help calculus students to develop deep and conceptual understanding of the chain rule, implicit differentiation, and related rates. Much more research needs to be done to explore how to help students make better connections; not just between the concepts explored in this study but also between other calculus concepts.

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APPENDIX: FINAL INTERVIEW PROTOCOL AND LEARNING ACTIVITIES

First Interview

Leibniz' notation

How do you interpret the symbol *dy/dx?*

Make sure they understand the following as it is crucial for the rest of the interview: In calculus, we often use the notation dy/dx to mean the derivative (or the slope) of y with respect to x. The little "d" comes from Δ*. For example, you may remember that* $\frac{\Delta y}{\Delta x}$ *is the slope of a line, where* Δy *and* Δx *just mean a piece of y and a piece of x respectively. dy and dx are both also pieces of y and x respectively but they are very, very small pieces; infinitely small pieces or "infinitesimal pieces". When we take the limit of the slope* $\frac{\Delta y}{\Delta x}$ *as* Δx *(which is originally finite) goes to 0, we are left with infinitesimal pieces of x and y, dx and dy. Say that at some point on a function, we have dy/dx = 3/1. This means that, right near that point, as x increases, y increases by 3 times as much. Or for example, if dy/dx=60 mph, that means that at that moment, the derivative, or the ratio of the tiny bit of y and tiny bit of x is equivalent to 60 miles per hour.*

Chocolate Context

Given to student:

1. Let's say you make $9/hr. at your job and that you're OBSESSED with chocolate. You want to spend every penny that you make on chocolate. You can buy .15 lbs. of chocolate per dollar. Let us, for now, ignore tithing, taxes, etc.

   $D(h) = 9h$ and

   $c(D) = .15D$

   Where *D* is dollars, *h* is hours you have worked, and *c* is amount of chocolate (lbs.).

Interview Questions:

b. How do you interpret what each of these equations in our context?
c. What is the value of dD/dh?
d. What is the meaning of *dD/dh* in our context? What are the units of *dD/dh*?
e. How is this different than D(h)?
f. What is the value *dc/dD*?
g. What is the meaning of *dc/dD* in our context*?* What are the units of *dg/dD*?
h. How is this different than c(D)?

i. What would be the meaning of *c(D(h))*?

j. How does hours worked affect the amount of chocolate you can buy?

k. If we could find dc/dh what would that mean in our context? What would be the units of dc/dh?

l. How does this relate to the original function composition c(D(h)) and how h affects c?

m. What is the value of *dc/dh*? How do you know?

<u>Carnival Context</u>

Have the students read and interpret the prompt.

Given to the student:

2. The following graphs show two functions, *f* and *g*. The input of function *g* is temperature in degrees Fahrenheit and the output is the expected attendance at a neighborhood carnival. The input of function *f* is number of people attending the carnival and the output is the expected revenue earned by the carnival.

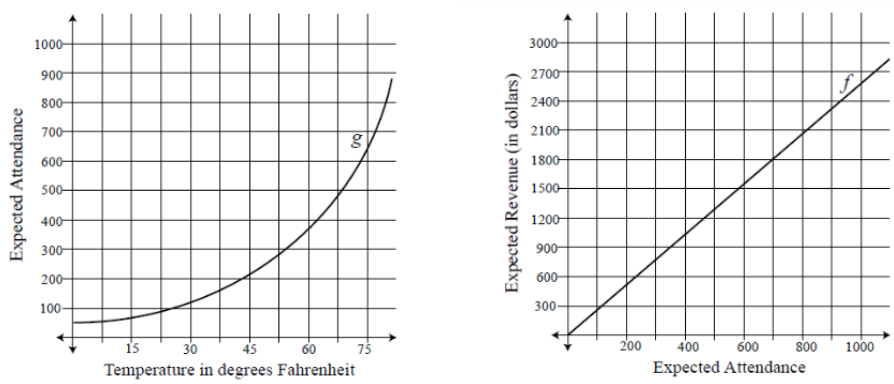


Figure 17. Carnival context (Carlson, 2016, p. 71).

Interview Questions:

a. If we calculated *dg/dx*, what would that mean in our context? What would be the units of *dg/dx?*

b. If we calculated *df/dg,* what would that mean in our context? What would be the units of *dg/dx?*

c. What does the function *f(g(x))* mean to you in this context?

d. How would you describe the way in which temperature (x) affects revenue (f)?

e. If we calculated *df/dx*, what would that mean in our context? What would be the units of *df/dx?*

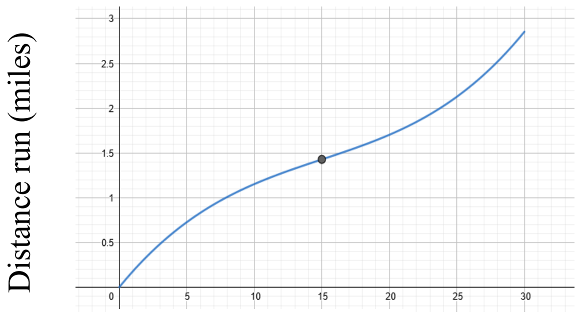
f. How does *df/dx* relate to f(g(x))?

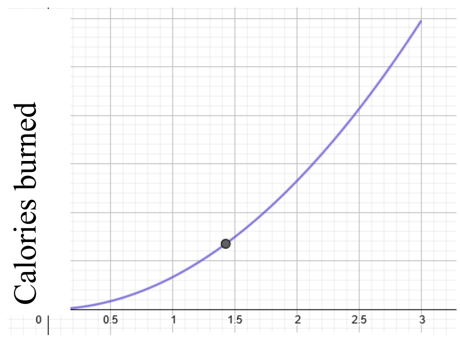
g. How does this relate to your description of how temperature (x) affects revenue (f)?

Running Context

Given to the student:

3.





Time elapsed (min) Distance run (miles)

Interview Questions:

a. Say right at 15 minutes, *dD/dt* is .1 (or 1/10). What is the meaning of *dD/dt*=1/10 in our context?

b. What are the units of *dD/dt?*

c. Say at that moment, you have traveled 1.5 miles, and so *dc/dD* at that moment (or at 1.5 miles) is 100. What is the meaning of *dc/dD=100* in our context?

d. What are the units of *dc/dD?*

e. What would c(d(t)) mean in this context? How does time elapsed affect calories burned?

f. What would the derivative of that composition of function, or *dc/dt* mean in our context? *If the student is struggling, rephrase the question as: What would be the meaning of the derivative of calories with respect to time?*

g. Say right at 15 minutes, you are running .1 miles/minute, and at that same moment, you are burning 100 calories/mile. How could you use this information to find *dc/dt*? *If the student is struggling, rephrase the question as: "How could you use this information to find the derivative of calories with respect to time?"*

h. Explain your thinking.

i. Will you burn 10 Cal/min the entire time? Why or why not?

j. Say right at 25 minutes, you are running .12 miles/minute, and you are burning 110 Cal/mile. How would you use that information to find *dc/*dt at that moment? How do you know?

k. In general, if you wanted to find *dc/dt*, how could you use *dD/dt* (or miles per minute) and d*c/dD* (calories per mile) at any given moment/location in order to find *dc/dt* (calories per minute) at that moment? How do you know?

l. Is there anything else you would like to add before ending the interview?

Second Interview

Running Context with Equations

An asterix (*) marks ideas that we will re-record on a separate sheet of paper for her/his reference.

Given to the student:

4. Let's say in a perfect world, you run at a constant rate of .1 miles/minute. That is, let

$D(t) = .1t$ and

$c(D) = 20D^2 + 40D$

Where *t* is time in minutes, *D* is distance traveled in miles, and *c* is calories burned.

Interview Questions:

We're going to calculate some of your rates right at 20 minutes. But first, let's figure out exactly how far you have gone at 20 minutes.

a. How far have you traveled at 20 minutes? (2 miles)

b. *What is an equation for *dD/dt?*

c. What is *dD/dt* at 20 minutes? How do you know?

d. *What is an equation for *dc/dD?*

e. What is *dc/dD* at that same moment? How do you know?

Our ultimate goal is to be able to find dc/dt which is the derivative of *c(D(t))*. So first,

f. *What is an equation for *c(D(t))?*

g. How does time affect calories burned?

h. So, if you know that at 20 minutes, you are traveling .1 miles/minute and you are burning 120 Cal/mile, what is *dc/dt* at 20 minutes?

i. *How can you use what you have found so far to write a general equation that will give you *dc/dt* at any time *t?* How do you know?

j. *What is an equation for *dc/dt* in terms of only *t* (instead of *D* and *t*)?

Point out to the student that what they have found is the derivative of the composition of functions they created in 4f.

<u>Dash from the Incredibles Context</u>

Given to student:

5. Say that now, you are "Dash" from The Incredibles, and you are running at an incredible pace.

   $D(t) = 2^t$ and

   $c(D) = \mathrm{D}^3$

Interview Questions:

Now we are interested in your rates at 5 minutes.

a. But first, how far have you traveled at 5 minutes? (5, 32)
b. *What is an equation for *dD/dt*?
c. What is *dD/dt* at 5 minutes?
d. *What is an equation for *dc/dD*?
e. What is *dc/dD* at that same moment?
f. *What is an equation for *c(D(t))*?
g. Say we want to find the derivative of this, or the dc/dt at 5 minutes. You are traveling 22.18 miles/minute and you are burning 3072 Cal/mile, what is *dc/dt* at 5 minutes?
h. *How can you use what you have found so far to write a general equation for dc/dt?
i. *What is an equation for dc/dt in terms of only t? (instead of D and t?)

Point out to the student that what they have found is the derivative of the composition of functions they created in 5f.

<u>Generalize the Chain Rule</u>

Look at the separate piece of paper and box the two equations for the composition of functions and their corresponding derivatives.

- What patterns do you notice? Why does that make sense?
- How does the pattern you found relate to the way we talked about how the independent variable affects the outermost function?
- Given any function for *c(D(t)),* what is *dc/dt?*

Given to student:

6. Let $g(x) = sin(f(x))$ Just so you know, the derivative of *sin(x)* is *cos(x)*. Do you have a hypothesis of what the derivative of this function might be?
7. Let $h(x) = sin(x^2)$. What would be the derivative of this composition of functions? In other words, what is $\frac{dh}{dx}$?
8. Let $j(x) = [\sin(x)]^2$. What would be the derivative of this composition of functions? In other words, what is $\frac{dj}{dx}$?
9. Say you have a general composition of functions *f(g(x))*. Can you write a rule for how you could find the derivative of a composition of functions?

Interview Questions:

a. Why would this make sense?

b. Is there anything else you would like to add before ending the interview?

Third Interview

Snowman Context

Given to student:

10. The body of a snowman is in the shape of a sphere whose radius is melting at a rate of .25 ft./hr. Assuming the body stays spherical, how fast is the volume changing when the radius is equal to 2 ft.? Remember that for a sphere, $V = \frac{4}{3}\pi r^3$.

Interview Questions:

a. Draw a picture of the situation, make note of the things we know as well as what we are looking for.

b. What are the things that are changing with time? *Let's go ahead and write V(t) and r(t) to help us remember. So, we can say* $V(t) = \frac{4}{3}\pi[r(t)]^3$. *Is that fair?*

c. What is that we're looking for?

d. How would you represent that as a derivative?

e. What does it mean for the radius of the snowman to be melting at a rate of .25 ft./hr.? Can you represent that as a derivative?

If they write dr/dt=.25 ft./hr. ask: Say the snowman's body was growing at a rate of .25ft/hr. How would you represent that as a derivative?

f. How does $[r(t)]^3$relate to what we have been doing before?

g. How can we find *dV/dt?*

h. Interpret your answer. What are the units for *dV/dt*?

i. Is there anything else you would like to add before ending the interview?

<u>Implicit Differentiation Exploration</u>

Introduce the difference between explicit and implicit equations with examples:

<u>Explicit function of y in terms of x</u>: $y = x^2 + sin(x)$

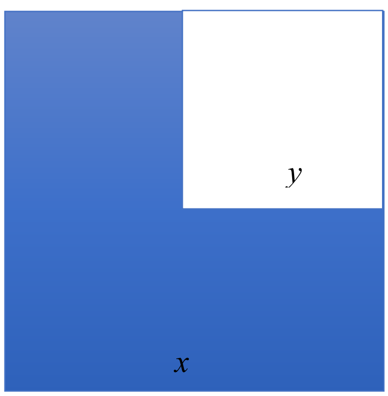
<u>Equations where y can implicitly be a function of x or x can implicitly be a function of y:</u>

$x^2 + y^2 = 9$, $5 = xy + y^2 + sin(y)$

Explain that with implicit functions we can see every point on a graph*. Pull up these equations on desmos, and explain that these two equations, when graphed, wouldn't normally pass the "vertical line test" but they can be useful in displaying all the different ways that x and y relate to one another in the equation. Explain also that* ***sometimes it is impossible to solve for y explicitly in terms of x****, like with the second equation.*

Given to student:

11.



The area of the square with side length *x* subtract the area of the square with side length y must always be 1. That is, $x^2 - y^2 = 1$ or the blue area will always be equal to 1. The question we will attempt to answer is: As *x* changes, by how much does *y* have to change?

Interview Questions:

We're going to say that *y* is an implicit function of *x* and re-write the equation as

$x^2 - [y(x)]^2 = 1$.

a. Can you write what we are trying to find as a derivative?

b. How does $[y(x)]^2$ relate to what we have been discussing over the past interviews?

c. As x changes, by how much does the blue area have to change? In other words, what is $\frac{d[blue\ area]}{dx}$?

If $\frac{d[blue\ area]}{dx}$ *= 0 and* $[blue\ area] = x^2 + [y(x)]^2$, *that implies that* $\frac{d(x^2-[y(x)]^2)}{dx}$ *= 0. We now know that as x changes, the blue area does not change. But remember, we are trying to find how y changes with x, or dy/dx. We have learned in calculus that the derivative of a sum or difference is equal to the sum or difference of the derivative. So, for example, we know that* $\frac{d(x^2-[y(x)]^2)}{dx}$ *is equal to* $\frac{d(x^2)}{dx} - \frac{d\,[y(x)]^2}{dx}$. *This means that* $\frac{d(x^2)}{dx} - \frac{d\,[y(x)]^2}{dx} = 0$.

d. What is the difference between $\frac{d(x^2)}{dx}$ and $\frac{d\,[y(x)]^2}{dx}$?

e. How does *x affection the* y^2 *term? Or the* $y(x)^2$ *term?*

f. Earlier, when we did the "Dash" running situation, we said that $C(D(t)) = (2^t)^3$, and we found that $dc/dt = 3(2^t)^2 \cdot 2^t \ln(2)$. How does that connect to finding $\frac{d\,[y(x)]^2}{dx}$?

g. What is the derivative of each of these?

We see that $2x - 2y(x) \cdot \frac{dy}{dx} = 0$. *Since we know that y is an implicit function of x I am going to re-write this as* $2x - 2y \cdot \frac{dy}{dx} = 0$.

h. What question are we trying to answer? How can we use this information to answer it?

Fourth Interview

Given to student:

12. Taking the original equation $x^2 - y^2 = 1$, if *y* changes by some amount, how much will *x* change? In other words, what is the derivative of *x* with respect to *y?*

<u>Different Implicit Differentiation Problem</u>

Given to student:

13. If $x^2 + [f(x)]^3 = 9$ and $f(1) = 2$, find $f'(1)$

Interview Questions:

a. How did you think about this problem?

b. How does this relate to what we have been doing in previous lessons?

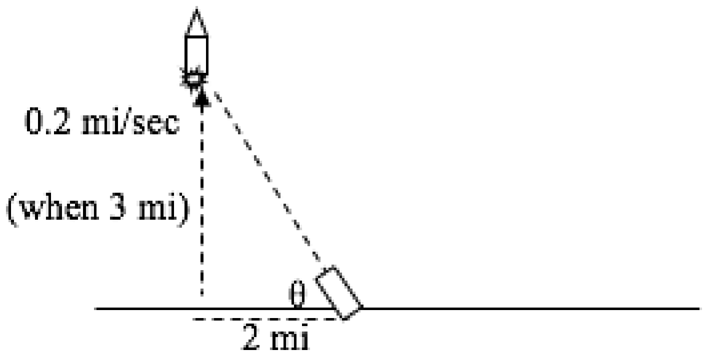
c. Is there anything else you would like to add before ending the interview?

<u>Shuttle Launch Context</u>

Given to student:

14. A camera is filming a shuttle launch and needs to stay focused on the shuttle by increasing the angle as the shuttle ascends. The camera is 2 miles from the launch-pad.

If the shuttle is traveling vertically at 0.2 miles/second, how fast should the angle be increasing when the shuttle is 3 miles above the ground?



a. Draw a picture of the situation, make note of the things we know as well as what we are looking for.
b. What are we looking for? Can you represent it as a derivative?
c. What does it mean for the shuttle to be traveling vertically at .2 miles/second? Can you write that as a derivative? (Labeling your picture may help in doing this)
d. What are the quantities that change with time?
e. Can you relate those quantities with an equation?
f. Can you represent the quantities that are changing with time as functions of time? *Before, in the snowman context, we knew that radius was a function of time and so we re-wrote the equation with [r(t)] to help us remember that. Do that same thing with the equation you have.*
g. How can you find $\frac{d\theta}{dt}$? What is $\frac{d\theta}{dt}$?

<u>More Complex Implicit Differentiation</u>

Given to student:

15. Let $\sqrt{x+y} = x^4 + y^4$. Find *dy/dx* or *y'*.

<u>Airplane Context</u>

Given to student:

16. A plane flying horizontally at an altitude of 1 mi and a speed of 500 mi/h passes directly over a radar station. Find the rate at which the distance from the plane to the station is increasing when it is 2 mi away from the station.