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New Computational Techniques in FJRW Theory with Applications to Landau Ginzburg Mirror Symmetry

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ABSTRACT

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Mirror symmetry is a phenomenon from physics that has inspired a lot of interesting mathematics. In the Landau-Ginzburg setting, we have two constructions, the A and B models, which are created based on a choice of an affine singularity with a group of symmetries. Both models are vector spaces equipped with multiplication and a pairing (making them Frobenius algebras), and they are also Frobenius manifolds. We give a result relating stabilization of singularities in classical singularity to its counterpart in the Landau-Ginzburg setting.

The A model comes from so-called FJRW theory and can be defined up to a full cohomological field theory. The structure of this model is determined by a generating function which requires the calculation of certain numbers, which we call correlators. In some cases the their values can be computed using known techniques. Often, there is no known method for finding their values. We give new computational methods for computing concave correlators, including a formula for concave genus-zero, four-point correlators and show how to extend these results to find other correlator values. In many cases these new methods give enough information to compute the A model structure up to the level of Frobenius manifold. We give the FJRW Frobenius manifold structure for various choices of singularities and groups.

Keywords: FJRW Theory, Moduli Space of Curves, Frobenius algebra, Frobenius manifold, cohomological field theory, genus-g, k-point correlators

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Chapter 1. Introduction

In the Landau-Ginzburg setting the structure of the A model, given by FJRW theory, is defined up to the level of cohomological field theory, but in most examples cannot be explicitly computed using known methods. We give new methods for finding the A model structure. In many cases our new methods determine the A model Frobenius manifold structure. We show the Frobenius manifold structure for most of the unknown unimodal and bimodal singularities and corresponding groups.

1.1 Background

Conjecture 1. ***The Landau Ginzburg Mirror Symmetry Conjecture.***
There should be A and B model structures, each constructed from a singularity W associated to a polynomial (see Equation 2.1) and an associated group, G (see Definition 4), such that the A model for W and G is isomorphic to the B model for W^T and G^T, where W^T and G^T are dual to the original singularity and group (see Definitions 5 and 6). Both the A and B models should be graded vector spaces, Frobenius algebras, Frobenius manifolds, and cohomological field theories. They should be isomorphic at each of these levels.

Although physics predicted its existence, a mathematical construction of the A model was unknown until 2007 when Fan, Jarvis, and Ruan, following the ideas of Witten, proposed a cohomological field theory $\mathscr{H}_{W,G}$ to satisfy Conjecture 1 [1, 2, 3]. This field theory, when restricted in various ways, produces a vector space, a Frobenius algebra, and a Frobenius manifold. A Frobenius algebra $\mathscr{A}$ is an algebra over a field (in this case $\mathbb{C}$) which also has a bilinear map $\langle \bullet, \bullet \rangle : \mathscr{A}^2 \to \mathbb{C}$ called a pairing, such that the product in the algebra respects this pairing, that is, $\langle \alpha \cdot \beta, \gamma \rangle = \langle \alpha, \beta \cdot \gamma \rangle$. A Frobenius manifold is a manifold M, such that at each point $p \in M$, the tangent space T_pM is a Frobenius algebra, and such that these Frobenius algebras vary continuously in p. A cohomological field theory is made up of a class Λ in $H^*(\overline{\mathscr{M}}_{g,k})$ for each choice of genus g, number of marked points k, and k-tuple of elements in the basis for the underlying space. These cohomology classes must also satisfy a certain composition property described in Section 2.3.6.

To construct the B model dual to $\mathscr{H}_{W,G}$, a singularity and symmetry group which are dual to W and G are needed. However, in certain cases these are still not defined. In 1993, Berglund and Hubsch first described the dual of a polynomial in [4], but it was still unclear what the right choice for a dual group would be in the construction of the B model. In 2009, Mark Krawitz [5] constructed a dual group for invertible polynomials and conjectured that it would fit in the mirror symmetry picture described in Conjecture 1. His construction involves using a unique representation of each group element given as a linear combination of the rows of the inverse of the exponent matrix (see Equation 2.2). For non-invertible polynomials, the exponent matrix is not invertible, so this construction will not work, thus a dual group construction to use in the B model is still needed.

Given an appropriate choice of singularity W^T and symmetry group G^T, the corresponding B model structure has been partially defined and computed. In 1990, Intriligator and Vafa described the B model vector space structure and claimed there should be a ring structure but did not compute it [6]. Kaufmann outlined many properties of the product, and it was described completely by Krawitz [5] following a recipe from Kaufman [7, 8, 9]. Thus the Frobenius algebra structure of the B model is now known. The Frobenius manifold and higher genus structures, however, are only known in certain cases. For example, when we choose the maximal group for the A model construction (see Definition 4), the dual group is then generated by only the identity element. In this case the B model is called *unorbifolded*, the Frobenius algebra structure is just the Milnor ring, the Frobenius manifold structure is given by the work of Saito [10], and the higher genus potential function was given by Givental [11]. In the case where we do not use the maximal group for the A model, and therefore have a nontrivial dual group, we call the B model *orbifolded*. In this case the Frobenius algebra structure is given by the orbifold Milnor ring with product structure as described by Krawitz, but the higher structure of the orbifolded B model is still unknown.

We can find a basis whose elements are consist of a pair of monomial and group element. When the group element does not act trivially on any variable of W, we call this a *narrow* element, otherwise, it is called *broad*. The structure in the A model is determined by certain structure constants called genus-g, k-point correlators, which come from the cohomology of the moduli space $\overline{\mathscr{M}}_{g,k}$ of genus-g curves with k marked points. The Frobenius algebra structure is given by the genus-zero, three-point correlators, the Frobenius manifold structure by the genus-zero, k-point correlators for $k \geq 3$, and the higher genus structure by the genus-g, k-point correlators for all nonnegative integers g and k such that $2g - 2 + k > 0$. These correlators are defined as integrals of certain cohomology classes Λ over $\overline{\mathscr{M}}_{g,k}$. They are difficult to compute, especially when they contain broad elements, so in some cases we still do not know how to compute even the A model Frobenius algebra structure. In most cases we do not know how to compute the Frobenius manifold or higher genus structures. In this dissertation we make progress along these lines.

It is generally believed that the weights of a singularity and the choice of symmetry group completely determine the A model structure. To our knowledge, no proof of this statement has been given. We give a proof for a special case.

In classical singularity theory, it is well-known that if $W \in C[x_1, \ldots, x_n]$ is a non degenerate, quasihomogeneous singularity, then W and $\tilde{W} = W + x_{n+1}^2$ are equivalent as singularities. In this paper we describe when the A and B models are equivalent for W and $\tilde{W}$ depending on the choice of symmetry group for each.

In 2010, Krawitz proved the Conjecture 1 at the level of Frobenius algebra for any invertible polynomial W and $G = G_W^{max}$, the maximal symmetry group. It is more difficult to determine the product structure in the A model when $G \neq G_W^{max}$, because of the introduction of broad elements. In 2011 Johnson, Jarvis, Francis and Suggs [12] proved the conjecture at the Frobenius algebra level for any pair (W, G) of invertible polynomial and admissible symmetry group with the following property:

Property 1. *Let W be an invertible, non degenerate, quasihomogeneous polynomial and let*

G be an admissible symmetry group for W. We say the pair (W, G) has Property 1 if

1. *W can be decomposed as $W = \sum_{i=1}^{M} W_i$ where the W_i are themselves invertible polynomials having no variables in common with any other W_j.*

2. *For any element g of G, where some monomial $\lceil m\,; g\rfloor$ is an element of $\mathscr{H}_{W,G}$, and for each $i \in \{1, \ldots, M\}$, g fixes either all of the variables in W_i or none of them.*

An important fact about invertible polynomials, found in [13], is that such polynomials can be written as a decoupled sum $W = \sum_i W_i$ where no W_i shares any common variables with any W_j for $j \neq i$ and such that each W_i is an *atomic* polynomial: a *chain*, a *loop*, or a *Fermat* (see Lemma 1).

A corollary to the theorem in [12] is that any polynomial which is a sum of loop and Fermat type polynomials or a two variable chain satisfies the Property 1, and will therefore satisfy the Landau-Ginzburg mirror symmetry conjecture. Since the Frobenius algebra structure is always computable in the B model, the above theorems allow us to determine the A model algebra structure up to isomorphism for various singularities and groups, which otherwise could not be computed.

Finding Frobenius manifold structures for each model is an important task, both as it relates to the Landau-Ginzburg mirror symmetry conjecture, and because it may lead to computing the full cohomological field theory of each model. A semi-simple Frobenius manifold is a manifold M where the Frobenius algebra T_pM at a generic point p is a semi-simple algebra. A theorem by Givental states that a semi-simple Frobenius manifold will determine the entire cohomological field theory associated to the model ([11]). Saito proved that the unorbifolded B model associated to W is semi-simple ([10]). One open topic for future research would be proving something similar for either the orbifolded B model, or for any group in the A model.

Computing the full structure of either model is difficult, and has only been done in a few cases. The easiest examples of singularities are the so-called "simple" or ADE singularities. Fan, Jarvis, and Ruan computed the full A model structure for these in [1]. The next examples come from the "Elliptic" singularities P_8, X_9, J_{10}, and their transpose singularities. Shen and Krawitz calculated the entire A model for the P_8, X_9^T and J_{10}^T, with maximal symmetry group [14].

Much of the work that has been done so far in this area has used a set of axioms which help calculate exact values of correlators in certain cases as described in Section 2.3.6. Primarily, these axioms have been used to compute genus-zero three-point correlators, but many of them can be used to find information about higher genus and higher point correlators. One of these is called the concavity axiom. When this axiom applies to a correlator, it gives a formula for the cohomology class Λ associated to a given correlator, in terms of the top Chern class of a sum of derived pushforward sheaves. In this dissertation, we use a result of Chiodo [15] to compute the Chern characters of the individual sheaves in terms of some cohomology classes in the moduli stack of W-curves $\mathscr{W}_{g,k}$ in genus zero. Then we use various properties of Chern classes to compute the top Chern class of the sum of the sheaves given the Chern character of each individual sheaf. Much is known about the cohomology $\overline{\mathscr{M}}_{g,k}$, and not a lot about $\mathscr{W}_{g,k}$, so we push down the cohomology classes in $\mathscr{W}_{g,k}$ to certain tautological classes ψ_i, κ_a and Δ_I over $\overline{\mathscr{M}}_{0,k}$. In this way, we provide a method for expressing Λ as a polynomial in the tautological classes of $\overline{\mathscr{M}}_{g,k}$. To compute our correlator values we must integrate Λ over $\overline{\mathscr{M}}_{g,k}$, which is equivalent to calculating intersection numbers. Algorithms for computing these numbers are well established, for example in [16, 17]. Code in various platforms (for example [16] in Maple and [18] in Sage) has been written which computes the intersection numbers we need. I wrote code in Sage which performs each of the steps mentioned above to find the top Chern class of the sum of the derived push forward sheaves, and then uses Johnson's intersection code [18] to find intersection numbers. This allows us to compute certain correlator values which were previously unknown. In particular, we give an explicit formula for computing any concave genus-zero four-point correlator. We follow ideas found in [1], and give a complete proof. We also describe how to compute higher point correlators, and use the reconstruction lemma to find values of non-concave correlators, with an aim to describe the full Frobenius manifold structure of many pairs (W, G) of singularities and groups. In some cases, these new methods allow us to compute Frobenius algebra structures for certain singularities and groups which were previously unknown.

There are eleven pairs (W, G) of elliptic singularity and symmetry group which have unknown Frobenius manifold structure. In this paper we use our new methods to find the full Frobenius manifold structure for seven of these eleven combinations. In [19] Arnol'd lists 33 other invertible unimodal and bimodal singularities. From these we find 36 pairs of singularities and group with unknown Frobenius manifold structure. Using our new methods, we are able to fully describe the Frobenius manifold structure for 26 of these pairs.

1.2 Overview

We begin by reviewing the construction of the A and B models in Chapter 2. After describing the vector spaces and pairings for each model, we define multiplication and other known structure information for each. This includes a brief description of the moduli space of curves and the moduli stack of W-curves. We also give a proof that the weights and group determine the FJRW structure in a special case.

In Chapter 3 we compare stabilization of singularities in classical singularity theory to what occurs in both the A and B models when we add an extra squared variable to the singularity, and make various choices for G.

Chapter 4 contains an explanation of the concavity axiom and Chiodo's formula for the Chern character of the derived pushforward sheaf. Using these, we provide a formula for the class Λ in terms of some of the tautological classes in $H^*(\overline{\mathscr{M}}_{g,k})$, a formula for computing concave genus-zero four-point correlators, a description of how to compute higher point correlators and a brief review of the reconstruction lemma.

Chapter 5 contains our results for computing the Frobenius algebra and Frobenius manifold structure for the unimodal singularities with all possible symmetry groups.

Chapter 2. Review of Quantum Ring Construction

We begin by reviewing key facts about the construction of the A and B models.

2.1 Admissible Polynomials and Symmetry Groups

The A and B models require the choice of a singularity and an associated symmetry group.

2.1.1 Singularities. A singularity is given by a polynomial $W \in \mathbb{C}[x_1, \ldots, x_n]$, where

$$W = \sum_i c_i \prod_{j=1}^{n} x_j^{a_{i,j}}. \tag{2.1}$$

Definition 1. *A polynomial W, as in Equation 2.1 is called* ***quasihomogeneous*** *if there exist positive rational numbers q_j (called weights) for each variable x_j such that each monomial of W has weighted degree one. That is, for every i where $c_i \neq 0$,*

$$\sum_j q_j a_{i,j} = 1.$$

The **central charge** of the polynomial W is

$$\hat{c} = \sum_j (1 - 2q_j).$$

Definition 2. *A polynomial $W \in \mathbb{C}[x_1, \ldots, x_n]$ is called* ***nondegenerate*** *if it has an isolated singularity at the origin, that is, if*

1. $\dfrac{\partial W}{\partial x_1} = \dfrac{\partial W}{\partial x_2} = \ldots = \dfrac{\partial W}{\partial x_n} = 0$ *has an isolated solution at the origin, and*
2. *The weights of the polynomial W are uniquely determined.*

Definition 3. *We say that a nondegenerate quasihomogeneous polynomial is* ***invertible*** *if it has the same number of variables and monomials.*

If a polynomial W is invertible, then it is possible to rescale variables to make all coefficients equal to one.

The following lemma from Kreuzer [13] is helpful in classifying the nondegenerate, quasihomogeneous, invertible singularities.

Lemma 1. *If W is a quasihomogeneous nondegenerate invertible polynomial, then it can be written as the decoupled sum $W = \sum_i W_i$, where W_i does not share any variables with any W_j for $i \neq j$, and each W_i is one of the following atomic type polynomials:*

1. ***Fermat**: x^a*
2. ***Loop**: $x_1^{a_1}x_2 + x_2^{a_2}x_3 + \ldots + x_n^{a_n}x_1$*
3. ***Chain**: $x_1^{a_1}x_2 + x_2^{a_2}x_3 + \ldots + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}$.*

To each nondegenerate quasihomogeneous polynomial, W as described in Equation 2.1, we can associate an **exponent matrix**, A_W by

$$A = [a_{i,j}]. \tag{2.2}$$

Recall the following result from [1].

Lemma 2. *If a nondegenerate quasihomogeneous polynomial W is invertible, then its exponent matrix A_W is an invertible matrix.*

Proof. The requirement that the weights of W be unique is equivalent to saying that the rank of the matrix A_W is equal to the number of variables, n. □

Example 1. *The singularity $W = J_{10}^T$ is defined by the polynomial $x^3 + y^3 + yz^2$. This is a quasihomogeneous polynomial with weights $q_x = \frac{1}{3}$, $q_y = \frac{1}{3}$, and $q_z = \frac{1}{3}$.*

$$\hat{c}_W = (1 - 2/3) + (1 - 2/3) + (1 - 2/3) = 1.$$

We can check that this is a nondegenerate singularity, since

$$\frac{\partial W}{\partial x} = 3x^2, \quad \frac{\partial W}{\partial y} = 3y^2 + z^2, \quad \frac{\partial W}{\partial z} = 2yz,$$

have a common zero only at the origin. Also, W has three variables and three monomials, so it is an invertible singularity. W is a sum of the Fermat polynomial x^3 and the chain polynomial $z^2y + y^3$.

The exponent matrix for W and its inverse are given below.

$$A_W = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 1 & 2 \end{bmatrix}; \quad A_W^{-1} = \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & -1/6 & 1/2 \end{bmatrix}.$$

2.1.2 Symmetry Groups. The construction of both the A and B models require the choice of an admissible symmetry group G for the singularity given by W.

We begin by defining the maximal symmetry group

Definition 4. *The **maximal symmetry group** G_W^{max} is the group of elements of the form $g = (g_1, \ldots, g_n) \in (\mathbb{Q}/\mathbb{Z})^n$ such that*

$$W(e^{2\pi i g_1} x_1, e^{2\pi i g_2} x_2, \ldots, e^{2\pi i g_n} x_n) = W(x_1, x_2, \ldots, x_n).$$

Lemma 3. *If W is a nondegenerate polynomial, then the maximal symmetry group G_W^{max} for W is finite.*

Proof. See [1]. □

Lemma 4. *If W is an invertible, nondegenerate, quasihomogeneous, invertible singularity, then G_W^{max} is generated by the columns ρ_i of A_W^{-1}.*

Proof. Notice that an n-tuple $g \in (\mathbb{Q}/\mathbb{Z})^n$ fixes the monomial $m_i = \prod_j x_j^{a_{i,j}}$ exactly when

$$\begin{bmatrix} a_{i,1} & a_{i,2} & \dots & a_{i,n} \end{bmatrix} \cdot g^T \in \mathbb{Z}.$$

This means an n-tuple g is in G_W^{max} if and only if

$$A \cdot g^T = \mathbf{n},$$

for some integer vector $\mathbf{n}$. For an invertible singularity, this is the same as

$$g^T = A^{-1} \cdot \mathbf{n},$$

which means that g is a linear combination of the columns ρ_i of A^{-1}.

□

Notice that if $q_1, q_2, \dots, q_n$ are the weights of W, then the element $J = (q_1, \dots, q_n)$ is an element of G_W^{max}. This element is called the **grading element**.

An FJRW ring requires the choice of a subgroup G of G_W^{max} which contains the element J. Such a group is called **admissible**. We denote $\langle J \rangle = G_W^{min}$, and any subgroup between G^{min} and G^{max} is admissible.

Example 2. *For $W = J_{10}^T$, we have*

$$A^{-1} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & -\frac{1}{6} & \frac{1}{2} \end{bmatrix}.$$

Since $\rho_3 \equiv 3\rho_2 \mod \mathbb{Z}^n$, $G_W^{max} = \langle \rho_1, \rho_2 \rangle$, so $|G_W^{max}| = 18$. On the other hand, $J = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, so $|G_W^{min}| = 3$. In this example, it turns out that there are four admissible symmetry groups:

$$\langle J \rangle, \quad \langle (1/3, 1/3, 5/6) \rangle = \langle \rho_1 + \rho_2 \rangle, \quad \langle (1/3, 0, 0), (0, 1/3, 1/3) \rangle = \langle \rho_1, 4\rho_2 \rangle, \quad \text{and } G_W^{max}.$$

We shall use

$$G = \langle (1/3, 0, 0), (0, 1/3, 1/3) \rangle, \tag{2.3}$$

for the rest of the examples in this section.

The B model, or Orbifold Milnor ring also requires a singularity and a symmetry group, and it is necessary that they be dual to the singularity and group chosen for the A model.

Definition 5 (Berglund and Hubsch)**.** *The* ***transpose singularity*** W^T *is the one determined by the transpose matrix* A_W^T*. That is, if*

$$A_W = [a_{i,j}],$$

then,

$$W^T = \sum_i c_i \prod_j x_j^{a_{j,i}}.$$

Definition 6 (Krawitz)**.** *If G is an admissible symmetry group for a polynomial W, then the* ***transpose group*** G^T *is defined by*

$$G^T = \{h = \left(A^T\right)^{-1} \mathbf{r}^T | \mathbf{r} \cdot g \in \mathbb{Z} \text{ for all } g \in G, \mathbf{r} \in (\mathbb{Z})^n\}$$

Here g is expressed as a $n \times 1$ vector, and $\mathbf{r}$ as a $1 \times n$ vector.

Example 3. *For $W = J_{10}^T$ and G as in Equation 2.3,*

$$A^T = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

So, $W^T = x^3 + y^3 z + z^2$, and G^T can be found in the following way. If

$$\gamma_1 = (1/3, 0, 0) \text{ and } \gamma_2 = (0, 1/3, 1/3), \tag{2.4}$$

then

$$G^T = \{(A^T)^{-1}\mathbf{r}^T | \mathbf{r} \in (\mathbb{Z})^3, \gamma_1\mathbf{r}^T, \gamma_2\mathbf{r}^T \in \mathbb{Z}\}.$$

So, $(A^T)^{-1}\mathbf{r}^T \in G^T$ *whenever* $\mathbf{r} = (i, j, k)$ *and* $i \equiv j + k \equiv 0 \mod 3$*. Since*

$$(A^T)^{-1} = \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & -1/6 \\ 0 & 0 & 1/2 \end{bmatrix},$$

$G^T = \langle (0, 1/2, 1/2) \rangle$.

Remark 1. *For any singularity* W*, if we choose* $G = G_W^{max}$*, then* $G^T = \langle (0, 0, \ldots, 0) \rangle$*. We can see this by observing that*

$$\mathbf{r}g^T = \mathbf{r}(A^{-1}k),$$

for some integer vector k*, and since* $\mathbf{r}g^T$ *is a scalar,* $(\mathbf{r}g^T)^T = g\mathbf{r}^T$*, so*

$$\left(\mathbf{r}(A^{-1}k)\right)^T = k^T \left(A^{-1}\right)^T \mathbf{r}^T \in \mathbb{Z}.$$

Notice that $\left(A^{-1}\right)^T \mathbf{r}^T$ *creates a general element* $h = (h_1, \ldots, h_n) \in G^T$*, and that this element must satisfy:*

$$\begin{bmatrix} k_1 & \ldots & k_n \end{bmatrix} \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix} \in \mathbb{Z},$$

for any choice of integers $k_1, \ldots k_n$*. The only element* h *that will satisfy this condition is* $\mathbf{0}$.

2.2 Vector Space Construction

Both the A and B models are graded vector spaces. The method of their construction as vector spaces is identical.

Notation 1. *We use the notation* I_g *to denote the indices of the variables which are fixed* *by an element g,*

$$Fix(g) = \{(x_1, \ldots, x_n)| \text{ such that } x_i = 0 \text{ whenever } i \notin I_g\},$$

and W_g is the polynomial obtained from W by setting the all variables whose coordinates are not in I_g equal to zero.

Lemma 5. *If W is a quasihomogeneous, non degenerate polynomial, and g is any element of an admissible symmetry group, then W_g is also a quasihomogeneous nondegenerate polynomial.*

Proof. Suppose that $I_g = (i_1, \ldots, i_r)$, and that $(a_{i_1}, \ldots, a_{i_r})$ is a nontrivial solution to

$$\frac{\partial W_g}{\partial x_{i_1}} = 0, \ldots, \frac{\partial W_g}{\partial x_{i_r}} = 0.$$

If we define $(b_1, \ldots, b_n)$ so that for each $k = i_j \in I_g$, $b_k = a_{i_j}$, and for each $k \notin I_g$, $b_k = 0$, then $(b_1, \ldots, b_n)$ is a nontrivial solution to

$$\frac{\partial W}{\partial x_1} = 0, \ldots, \frac{\partial W}{\partial x_n} = 0.$$

□

Definition 7. *The **Jacobian Ideal** of a polynomial is the ideal generated by its partial derivatives:*

$$\mathcal{J}(W) = \left(\frac{\partial W}{\partial x_1}, \frac{\partial W}{\partial x_2}, \ldots, \frac{\partial W}{\partial x_n} \right).$$

Definition 8. *The **Milnor ring** of a polynomial W is defined by*

$$\mathcal{Q}_W = \frac{\mathbb{C}[x_1, \ldots, x_n]}{\mathcal{J}(W)}.$$

The dimension of the Milnor ring is denoted by μ_W.

The dimension of the Milnor ring can be computed directly using the formula

$$\mu_W = \prod_i \left(\frac{1}{q_i} - 1 \right).$$

Definition 9. *The **Hessian** of the polynomial W is given by*

$$Hess_W = \det \left(\left[\frac{\partial^2 W}{\partial x_i \partial x_j}. \right] \right)$$

The Hessian is always a scalar multiple of the unique element of top weighted degree in the Milnor ring, and the weighted degree of this monomial is always equal to $\hat{c}$.

Notation 2. *We will use the notation $\mathscr{J}_g$, $\mathscr{Q}_g$, μ_g, and $Hess_g$ to denote the Jacobian Ideal, Milnor ring, dimension, and Hessian of the polynomial W_g, respectively.*

Example 4. *For $W = J_{10}^T$,*

$$\mathscr{J}(W) = \left(3x^2, 3y^2 + z^2, 2yz\right),$$

$$\mathscr{Q}_W = \left\langle 1, x, y, xy, y^2, xy^2, z, xz \right\rangle,$$

and $\mu = \left(\frac{3}{1} - 1\right)\left(\frac{3}{1} - 1\right)\left(\frac{3}{1} - 1\right) = 8$. The Hessian is given by

$$Hess(W) = \det \left(\begin{bmatrix} 6x & 0 & 0 \\ 0 & 6y & 2z \\ 0 & 2z & 2y \end{bmatrix} \right) = 6x(12y^2 - 4z^2).$$

Using the following Jacobian relation in the Milnor ring,

$$z^2 = -3y^2,$$

we can simplify the Hessian.

$$Hess(W) = 72xy^2 - 24xz^2 = 72xy^2 + 72xy^2 = 144xy^2,$$

which has weighted degree

$$\frac{1}{3} + 2\left(\frac{1}{3}\right) = 1 = \hat{c}.$$

If $g = \left(\frac{1}{3}, 0, 0\right)$*, then* $W_g = y^3 + yz^2$*,*

$$\mathscr{J}_g = \left(3y^2 + z^2, 2yz\right),$$

$$\mathscr{Q}_g = \left\langle 1, y, y^2, z \right\rangle, \tag{2.5}$$

and $\mu_g = \left(\frac{3}{1} - 1\right)\left(\frac{3}{1} - 1\right) = 4.$

$$Hess_g = \det\left(\begin{bmatrix} 6y & 2z \\ 2z & 2y \end{bmatrix}\right) = 12y^2 - 4z^2 = 12y^2 + 12y^2 = 24y^2. \tag{2.6}$$

The state space of the A model is defined in terms of Lefschetz thimbles, as in the following definition, but it will be of more practical use to us to use the lemma that follows it.

Definition 10. *For the singularity W with admissible group G, the A-model state space $\mathscr{H}_{W,G}$, the underlying state space of the **FJRW ring** is defined in the following way. Let $\mathscr{H}_{g,G}$ be the G-invariants of the middle-dimensional relative cohomology*

$$\mathscr{H}_{g,G} = H^{mid}(Fix(g), (W)_g^{-1}(\infty))^G,$$

where $W_g^{-1}(\infty)$ is a generic smooth fiber of the restriction of W_g, and mid is half the dimen-

sion of $Fix(g)$. The state space is given by

$$\mathscr{H}_{W,G} = \left(\bigoplus_{g \in G} \mathscr{H}_{g,G} \right).$$

The following lemma from [20, 21] will be useful in the remainder of this paper.

Lemma 6. *Let $\omega = dx_1 \wedge \ldots \wedge dx_n$, then*

$$\mathscr{H}_{\mathbf{0},G} = H^{mid}(\mathbb{C}^n, (W)^{-1}(\infty)) \cong \mathscr{Q}_W \omega,$$

as G_W-spaces, and this isomorphism respects the pairing on both. In the case of $\mathscr{Q}_W \omega$, this means that elements of G_W act on both monomials in $\mathscr{Q}_W$ as well as the volume form ω.

The isomorphism in Lemma 6 certainly will hold for the restricted polynomials W_g as well, if we say $\omega_g = dx_{i_1} \wedge \ldots \wedge dx_{i_s}$ for $i_j \in Fix(g)$. This gives us the useful fact

$$\mathscr{H}_{W,G} = \bigoplus_{g \in G} \mathscr{H}_{g,G} \cong \bigoplus_{g \in G} \left(\mathscr{Q}_g \omega_g \right)^G. \tag{2.7}$$

It is important to note that while the Milnor ring $\mathscr{Q}_W$ has a natural ring structure, $H^{mid}(\mathbb{C}^n, (W)^{-1}(\infty))$ does not. The choice of product structure for the A-model (see Section 2.3.5) will not be the same as the product in the Milnor ring.

Notation 3. *An element of $\mathscr{H}_{W,G}$ is a linear combination of basis elements, which we denote by $\lceil m\, ; g \rfloor$, where m is a monomial in $\mathscr{Q}_g$, and therefore in the subspace corresponding to the group element g. We say that $\lceil m\, ; g \rfloor$ is **narrow** if $I_g = \emptyset$.*

The Orbifold Milnor ring is defined in the same way:

Definition 11. *The B model or **Orbifold Milnor ring** state space is*

$$\mathscr{B}_{W^T, G^T} = \left(\bigoplus_{g \in G^T} \mathscr{Q}_{W_g^T} d\omega_g \right)^{G^T}.$$

Remark 2. *If $G = G_W^{max}$, the B model is just the unorbifolded Milnor ring:*

$$\mathscr{B}_{W^T,(\mathbf{0})} = (\mathscr{Q}_{W^T}\omega)^{(\mathbf{0})} \cong \mathscr{Q}_{W^T}.$$

Example 5. *For $W = J_{10}^T$, as a vector space $\mathscr{H}_{W,G}$ is generated by the basis elements*

$$\lceil 1\,; k_1\gamma_1 + k_2\gamma_2 \rfloor \text{ for } k_1 \text{ and } k_2 \text{ in } \{1,2\}, \text{ and}$$
$$\lceil y\,; k_1\gamma_1 \rfloor\,, \text{ and } \lceil z\,; k_1\gamma_1 \rfloor \text{ for } k_1 \text{ in } \{1,2\}.$$

$\mathscr{B}_{W^T,G^T}$ is generated by the basis elements

$$\lceil m\,; \mathbf{0} \rfloor \text{ for } m \in \{1, x, y^2, xy^2, yz, xyz\} \text{ and } \lceil m\,; (0, 1/2, 1/2) \rfloor \text{ for } m \in \{1, x\}.$$

2.2.1 Pairing.

Both the A and B models are equipped with a natural pairing. For each element $g \in G$ the subspace corresponding to g is equipped with a residue pairing $\langle \bullet, \bullet \rangle_g$ defined by

$$m \cdot n = \frac{\langle m, n \rangle_g}{\mu_{W_g}} Hess(W_g) + \text{lower order terms.}$$

We give each of the A and B models a pairing using inverse group elements, that is

$$\langle \lceil m\,; g \rfloor\,, \lceil h\,; n \rfloor \rangle = \begin{cases} \langle m, n \rangle_g & if \quad h = g^{-1} \\ 0 & otherwise. \end{cases}$$

Notice that this pairing is well-defined, since $W_g = W_{g^{-1}}$, and so $\mathscr{H}_g \cong \mathscr{H}_{g^{-1}}$. Thus, $\langle \bullet, \bullet \rangle_g = \langle \bullet, \bullet \rangle_{g^{-1}}$.

Example 6. *For $W = J_{10}^T$, recall that $G = \langle \gamma_1, \gamma_2 \rangle$ as in Equation 2.4. So, $(2\gamma_1 + 2\gamma_2) + (\gamma_1 + 2\gamma_2) = \mathbf{0}$, $W_{2\gamma_1+\gamma_2} = 0$, $Hess_{2\gamma_1+\gamma_2} = 1$, $\mu_{2\gamma_1+\gamma_2} = 1$, and*

$$1 \cdot 1 = \frac{\langle 1, 1 \rangle_{2\gamma_1+\gamma_2}}{1} \cdot 1 \text{ implies } \langle \lceil 1\,; 2\gamma_1 + \gamma_2 \rfloor\,, \lceil 1\,; \gamma_1 + 2\gamma_2 \rfloor \rangle = 1,$$

$(\gamma_1) + (2\gamma_1) = \mathbf{0}$, *and Equations 2.5 and 2.6 show that*

$$z^2 = \frac{\langle z, z\rangle_{\gamma_1}}{4}(24y^2) \sim \frac{\langle z, z\rangle_{\gamma_1}}{4}(-8z^2) \text{ implies } \langle \lceil z\,;\gamma_1\rfloor\,, \lceil z\,;\gamma_2\rfloor\rangle = -\frac{1}{2}.$$

But, $\langle \lceil 1\,;\gamma_1+\gamma_2\rfloor\,, \lceil y\,;\gamma_1\rfloor\rangle = 0$*, since* $\gamma_1+\gamma_2$ *and* γ_1 *aren't inverse group elements.*

By fixing an order for the basis, we can create a matrix which contains all the pairing information.

Definition 12. *The* ***pairing matrix*** η *is given by*

$$\eta = [\langle \alpha_i, \alpha_j\rangle],$$

where $\{\alpha_i\}$ *is a basis for* $\mathscr{H}_{W,G}$*.*

2.3 Frobenius Algebra Structure

In this section we define the product structure on each of the A and B models.

The product structure of the A model requires computation of certain structure constants which come from the cohomology of the moduli space of curves and the moduli space of W-orbicurves. We begin by discussing each of these spaces.

2.3.1 The Moduli Space of Curves.

Definition 13. $\overline{\mathscr{M}}_{g,k}$ *is the* ***Moduli Space of stable curves of genus*** g ***over*** $\mathbb{C}$ ***with*** k ***marked points****.*

The curves referenced above can be thought of as (possibly nodal) Riemann surfaces C with k marked points, $p_1, \ldots, p_k$, where $p_i \neq p_j$ if $i \neq j$. We require an additional stabilization condition, that the automorphism group of any such curve be finite. In fact, numerically, this means that $2g_i - 2 + k_i > 0$ for each irreducible component of C, where g_i

is the genus of the component, and k_i is the number of marked and nodal points. It turns out that $\overline{\mathscr{M}}_{g,k}$ is a smooth, compact orbifold.

The **dual graph** of a curve in $\overline{\mathscr{M}}_{g,k}$ is a graph with a node representing each irreducible component, an edge for each nodal point, and a half edge for each mark.

Example 7. *A nodal curve in* $\overline{\mathscr{M}}_{1,3}$ *and its dual graph are shown below.*

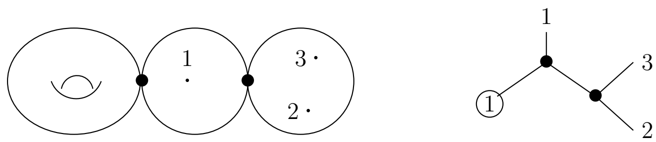


Figure 2.1: A nodal curve in $\overline{\mathscr{M}}_{1,3}$ and its dual graph.

It is worthwhile to note that $\overline{\mathscr{M}}_{0,3}$ is a single point, and $\overline{\mathscr{M}}_{0,4}$ is isomorphic to $\mathbb{P}^1$. This means that any two points on $\overline{\mathscr{M}}_{0,4}$ are cohomologous.

There is a universal curve $\mathfrak{C}_{g,k}$ over $\overline{\mathscr{M}}_{g,k}$,

$$\mathfrak{C}_{g,k} \xrightarrow{\pi} \overline{\mathscr{M}}_{g,k}.$$

2.3.2 Orbicurves. An orbicurve $\mathscr{C}$ with marked points $p_1, \ldots, p_k$ is a Riemann surface C with orbifold structure at each p_i and each node. This means that for each marked point p_i there is a local group $\mathbb{Z}/m_i$ for some positive integer m_i acting as $z \mapsto \zeta z$ for some $\zeta \in \mu_{m_i}$ (where μ_{m_i} are the m_ith roots of unity). For each node p there is again a local group $\mathbb{Z}/n_j$ whose action on one branch is inverse to the action on the other branch.

In a neighborhood of p_i $\mathscr{C}$ maps to C via the map,

$$\rho : \mathscr{C} \to C, \tag{2.8}$$

where if z is the local coordinate on $\mathscr{C}$ near p_i, and x is the local coordinate on C near p_i, then $\rho(z) = z^{m_i} = x$.

Definition 14. *Let K_C be the canonical bundle of C. The **log-canonical bundle** of C is the line bundle*

$$K_{C,log} = K \otimes \mathscr{O}(p_1) \otimes \ldots \otimes \mathscr{O}(p_k),$$

where $\mathscr{O}(p_i)$ is the holomorphic line bundle of degree one whose sections may have a simple pole at p_i.

The log-canonical bundle of $\mathscr{C}$ is defined to be the pullback to $\mathscr{C}$ of the log-canonical bundle of C:

$$K_{\mathscr{C},log} = \rho^* K_{C,log}.$$

Near a marked point p_i of C with local coordinate x, the bundle $K_{C,log}$ is locally generated by the meromorphic one-form $\frac{dx}{x}$. Then the lift $K_{\mathscr{C},log} = \rho^*(K_{C,log})$ is also locally generated by $\frac{dx}{x}$, since $x = z^{m_i}$, and

$$\frac{dx}{x} = \frac{mz^{m-1}dz}{z^m} = m\frac{dz}{z}.$$

2.3.3 W-structures on an orbicurve. A W-structure on an orbicurve $\mathscr{C}$ is basically a choice of n line bundles $\mathscr{L}_1, ..., \mathscr{L}_n$ so that for each monomial of $W = \sum_j M_j$, with $M_j = x_1^{aj,1} \ldots x_n^{a_{j,n}}$, we have an isomorphism of line bundles

$$\phi_j : \mathscr{L}_1^{\otimes a_{j,1}} \ldots \mathscr{L}_n^{\otimes a_{j,n}} \xrightarrow{\sim} K_{\mathscr{C},log}.$$

Example 8. *For $W = J_{10}^T$, this gives three isomorphisms,*

$$\mathscr{L}_x^3 \cong K_{\mathscr{C},log},$$
$$\mathscr{L}_y^3 \cong K_{\mathscr{C},log},$$
$$\mathscr{L}_y \otimes \mathscr{L}_z^2 \cong K_{\mathscr{C},log}.$$

Thus $\mathscr{L}_x$ and $\mathscr{L}_y$ are third roots of $K_{\mathscr{C},log}$, and

$$\mathscr{L}_y^3 \otimes \mathscr{L}_z^6 \cong K_{\mathscr{C},log}^3 \Rightarrow \mathscr{L}_z^6 \cong K_{\mathscr{C},log}^2.$$

So, $\mathscr{L}_z^2$ is a sixth root of $K_{\mathscr{C},log}$.

2.3.4 Moduli Spaces of W-Curves. For a nondegenerate, quasihomogeneous polynomial W, and two non-negative integers g and k satisfying $2g - 2 + k > 0$, we consider the stack of stable W-orbicurves, $\mathscr{W}_{W,g,k} = \{\mathscr{C}, p_1, \ldots, p_k, \mathscr{L}_1, \ldots, \mathscr{L}_N, \varphi_1 \ldots \varphi_s\}$, and the canonical morphism,

$$\begin{array}{c} \mathscr{W}_{g,k} \\ \big\downarrow st \\ \mathscr{M}_{g,k} \end{array}$$

from the stack of W-curves to the stack of stable curves, $\mathscr{M}_{g,k} = \{(\mathscr{C}, p_1, \ldots p_k)\}$. Fan, Jarvis and Ruan [1] showed that this morphism is flat, proper and quasi-finite (but not representable).

Recall that a genus-g, k-pointed curve is considered stable if $2g - 2 + k > 0$. If Γ is a graph, with each node n_i assigned a genus g_i, then Γ is **stable** if each node, n_i satisfies $2g_i - 2 + k_i > 0$, where k_i is the number of half edges and tails attached to the node n_i

Definition 15. *A G_W**-decorated stable graph** is a stable graph Γ with a decoration of each tail τ by a choice of $\gamma_\tau \in G_W$.*

Suppose that the local action on $\mathscr{C}$ near p_i is $z \to \zeta z$, where $\zeta \in \mu_r$ (the primitive rth roots of unity). Recall the covering map ρ in Equation 2.8.

Notice that ρ_* will take global sections to global sections, so, if $\mathscr{L}$ on $\mathscr{C}$, such that $\mathscr{L}^{\otimes r} \cong K_{\mathscr{C},\log}$, then sections of $\mathscr{L}$ locally must look like $f(z)s$ where $s^r = (\frac{dz}{z})$. Since $s^r = \frac{dz}{z}$ is locally invariant, then $\zeta(s) = \zeta^{m_i} s$ for some m_i in $\mathbb{Z}/r$.

Also, sections of $\rho_*\mathscr{L}$ are locally invariant so they must be of the form $f(z)s$, where So,

$$\zeta(f(z)s) = f(\zeta z)\zeta^{m_i} s = f(z)s.$$

This implies that $f(\zeta z) = \zeta^{r-m_i} f(z)$. Then

$$f(z) = a_0 z^{r-m_i} + a_1 z^{2r-m_i} + \ldots = z^{r-m_i}(a_0 + a_1 z^r + a_2 z^{2r} + \ldots) = z^{r-m_i} f(z^r) = z^{r-m_i} f(x).$$

Thus, sections of $\rho_* \mathscr{L}$ are of the form $f(x) z^{r-m_i} s$.

Now, $(f(x) z^{r-m_i} s)^r = f(x)^r x^{r-m_i} \frac{dz}{z} = g(x) x^{r-m_i} \frac{dx}{x}$, so

$$(\rho_* \mathscr{L})^r = K_{C,\log}(-(r-m_i)p).$$

Recall that if $K_{C,log}$ is the line bundle associated to the sheaf $\omega_{C,log}$, and if the action $\mathbb{Z}/r$ on $K_{C,log}$ is ζ^{m_i}, then the action on $\omega_{C,log}$ will be ζ^{r-m_i}.

Thus, if $\mathscr{L}^r \cong K_{\mathscr{C},log}$ on a smooth orbicurve with action of the local group on L defined by ζ^{m_i} for $m_i > 0$ at each marked point p_i, then

$$(\rho_* \mathscr{L})^r = |\mathscr{L}|^r = \omega_{C,log} \otimes \left(\bigotimes_i \mathscr{O}((-m_i)p_i) \right).$$

Example 9. *For $W = J_{10}^T$ and G as in 2.3, if, for a curve in $\overline{\mathscr{M}}_{0,3}$ the three marked points correspond to the A-model elements $\lceil 1\,; (2/3, 1/3, 1/3) \rfloor$, $\lceil y\,; (1/3, 0, 0) \rfloor$, and $\lceil y\,; (1/3, 0, 0) \rfloor$, then*

$$|\mathscr{L}_x|^3 = \omega_{C,log} \otimes \mathscr{O}((-2)p_1) \otimes \mathscr{O}((-1)p_2) \otimes \mathscr{O}((-1)p_3)$$
$$|\mathscr{L}_y|^3 = \omega_{C,log} \otimes \mathscr{O}((-1)p_1) \otimes \mathscr{O}((0)p_2) \otimes \mathscr{O}((0)p_3)$$
$$|\mathscr{L}_z|^6 = \omega_{C,log}^2 \otimes \mathscr{O}((-2)p_1) \otimes \mathscr{O}((0)p_2) \otimes \mathscr{O}((0)p_3)$$

We are now ready to define the product in the A model.

2.3.5 A Model Frobenius Algebra Structure.

The product structure of FJRW rings is given by certain so-called correlators that act like structure constants. In order to further describe the structure of the A-side, we need to define these genus-g k-point correlators.

For each pair of non-negative integers g and k, with $2g - 2 + n > 0$, the FJRW cohomological field theory produces for each k-tuple $(\alpha_1, \ldots, \alpha_k) \in \mathscr{H}_{W,G}^{\otimes k}$ a cohomology class

$\Lambda^W_{g,k}(\alpha_1, \alpha_2, ..., \alpha_n) \in H^*(\overline{\mathscr{M}}_{g,k})$. The definition of this class can be found in [1].

Definition 16. *A genus-g, k-point* ***correlator*** *with insertions* $\alpha_1, \ldots, \alpha_k \in \mathscr{H}_{W,G}$ *is defined by the integral*

$$\langle \alpha_1, \ldots, \alpha_k \rangle_{g,n} = \int_{\overline{\mathscr{M}}_{g,k}} \Lambda_{g,k}(\alpha_1, \ldots, \alpha_k).$$

Finding the values of these correlators is a difficult PDE problem, which has not been solved in general. But, we need them, in part because multiplication of basis elements requires computing certain genus-zero, three-point correlators.

If $\mathscr{A}$ is a vector space basis for $\mathscr{H}_{W,G}$, and $\alpha, \beta \in \mathscr{A}$, then define

$$\alpha \star \beta = \sum_{\sigma,\tau \in \mathscr{A}} \langle \alpha, \beta, \sigma \rangle_{0,3} \eta^{\sigma,\tau} \tau.$$

It is always possible to compute η and η^{-1}, as we have seen. But, some of the genus-zero three-point correlators necessary to determine the product structure still cannot be computed. However, there are some axioms that the correlators must satisfy which, in many cases, allow us to determine their values.

2.3.6 Axioms.

The axioms in this section come from [1].

The first axiom tells us that correlators are symmetric with respect to their insertions.

Axiom 1. ***Symmetry***

$$\langle \alpha_i, \ldots, \alpha_k \rangle_{g,k} = \left\langle \alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(k)} \right\rangle_{g,k}$$

for any $\sigma \in S_k$.

The following two axioms are sometimes called *selection rules*, because they give conditions that any nonzero correlators must satisfy.

If $\alpha = \lceil m\,; g \rfloor$ is an element in the basis of $\mathscr{H}_{W,G}$, where $g = (g_1, \ldots, g_n)$, and q_i is the weight associated to the variable x_i, then the W-degree of α is

$$\deg_W \alpha = |I_g| + 2\sum_{i=1}^{n}(g_i - q_i),$$

the $\mathbb{C}$-degree of α is one half of its W-degree:

$$deg_{\mathbb{C}}(\alpha) = \frac{1}{2}\deg_W(\alpha),$$

and the class $\Lambda_{g,k}(\alpha_1, \ldots, \alpha_k)$ has degree

$$D = \hat{c}_W(g-1) + \sum_{j=1}^{k}\deg_{\mathbb{C}}(\alpha_j).$$

Axiom 2. ***Dimension.***

$$\langle \alpha_1, \ldots, \alpha_k \rangle_{g,k} = 0$$

unless $D = 3g - 3 + k$.

Example 10. *For* $W = J_{10}^T$ *the* $\mathbb{C}$*-degrees of all basis elements are given in Table 2.1.*

Element	$\mathbb{C}$-degree
$A_1 = \lceil 1\,;\gamma_1+\gamma_2\rfloor$	0
$A_2 = \lceil 1\,;2\gamma_1+\gamma_2\rfloor$	1/3
$A_3 = \lceil y\,;\gamma_1\rfloor$	1/3
$A_4 = \lceil z\,;\gamma_1\rfloor$	1/3
$A_5 = \lceil 1\,;\gamma_1+2\gamma_2\rfloor$	2/3
$A_6 = \lceil y\,;2\gamma_1\rfloor$	2/3
$A_7 = \lceil z\,;2\gamma_1\rfloor$	2/3
$A_8 = \lceil 1\,;2\gamma_1+2\gamma_2\rfloor$	1

Table 2.1: A basis for the graded vector space of $\mathscr{H}_{J_{10}^T,G_1}$ with $\mathbb{C}$-degrees.

Since $\hat{c} = 1$*, if* $\langle \alpha_1, \alpha_2, \alpha_3\rangle_{0,3}$ *is a nonzero genus-zero three-point correlator, then*

$$1(-1) + \sum_{i=1}^{3}\deg_{\mathbb{C}}\alpha_i = 0 \Rightarrow \sum deg_{\mathbb{C}}\alpha_i = 1$$

This gives us a selection rule for nonzero correlators. The only genus-zero three-point correlators which might be nonzero are,

$$\begin{array}{cccc} \langle A_1, A_1, A_8\rangle\,, & \langle A_1, A_2, A_5\rangle\,, & \langle A_1, A_2, A_6\rangle\,, & \langle A_1, A_2, A_7\rangle\,, \\ \langle A_1, A_3, A_5\rangle\,, & \langle A_1, A_3, A_6\rangle\,, & \langle A_1, A_3, A_7\rangle\,, & \langle A_1, A_4, A_5\rangle\,, \\ \langle A_1, A_4, A_6\rangle\,, & \langle A_1, A_4, A_7\rangle\,, & \langle A_2, A_2, A_2\rangle\,, & \langle A_2, A_2, A_3\rangle\,, \\ \langle A_2, A_2, A_4\rangle\,, & \langle A_2, A_3, A_3\rangle\,, & \langle A_2, A_3, A_4\rangle\,, & \langle A_2, A_4, A_4\rangle\,, \\ \langle A_3, A_3, A_3\rangle\,, & \langle A_3, A_3, A_4\rangle\,, & \langle A_3, A_4, A_4\rangle\,, & \langle A_4, A_4, A_4\rangle \end{array}$$

For genus-zero four point correlators we have the selection rule:

$$1(-1) + \sum_{i=1}^{3} \deg_{\mathbb{C}} \alpha_i = -3 + 4 = 1 \Rightarrow \sum_{i=1}^{3} deg_{\mathbb{C}} \alpha_i = 2$$

So, the possibly nonzero correlators come in four varieties,

1. *Those with two insertions of degree 0, and 2 of degree 1. There is actually only one such correlator:* $\langle A_1, A_1, A_8, A_8\rangle$*,*

2. *Those with one insertion of degree zero, one of degree 1/3, one of degree 2/3 and one of degree 1. There are nine of these kind of correlators.*

3. *Those with one insertion of degree zero, and three of degree 2/3.*

4. *Those with three insertions of degree 1/3 and two of degree 1.*

5. *Those with two insertions of degree 1/3 and two of degree 2/3.*

The next axiom relies on the degrees of line bundles $\mathscr{L}_1, \ . \ . \ . \ , \mathscr{L}_n$ endowing an orbicurve, $\mathscr{C}$ with a W-structure, as in Section 2.3.3. Consider the class $\Lambda^W_{g,k}(\alpha_1, \alpha_2, ..., \alpha_k)$, with $\alpha_j \in (\mathscr{H}_{g^j})^G$ where $g^j = (\theta^j_1, \ldots, \theta^j_n)$, for each $j \in 1, ..., N$. Then, the degree of each associated line bundle $|\mathscr{L}_i|$, denoted l_i is given by

$$l_i = q_i(2g - 2 + k) - \sum_{j=1}^{k} \theta^j_i$$

Axiom 3. ***Line Bundle Degrees.***

$$\Lambda^W_{g,k}(\alpha_1, \alpha_2, ..., \alpha_k) = 0$$

unless $l_i \in \mathbb{Z}$ for each i.

This axiom gives us a way to determine the group element of the last insertion in any nonzero correlator.

Lemma 7. *For a genus-zero, k-point correlator with insertions $\alpha_1, \ldots, \alpha_k$, with*

$$\alpha_i = \lceil m_i\, ; g_i \rfloor\,,$$

if the correlator $\langle \alpha_1, \ldots, \alpha_k \rangle$ is nonzero,

$$g_k = (k-2) \cdot J - \sum_{i=1}^{k-1} g_i$$

Proof. If the correlator $\langle \alpha_1, \ldots, \alpha_k \rangle \neq 0$, then the class $\Lambda^W_{g,k}(\alpha_1, \alpha_2, ..., \alpha_k) \neq 0$, so $q_i(2g - 2 + k) - \sum_{j=1}^{k}(g^j)_i \in \mathbb{Z}$ for each i. This means that

$$(2g-2+k)(q_1, \ldots, q_n) - \sum_{i=1}^{k}(g^i_1, \ldots, g^i_n) \in \mathbb{Z}^n,$$

$$\Rightarrow (2g-2+k)J - \sum_{i=1}^{k} g^i \equiv \mathbf{0},$$

so,

$$g^k = (2g-2+k)J - \sum_{i=1}^{k-1} g^i.$$

□

Example 11. *Axiom 3 gives us another selection rule for possibly nonzero correlators. If we apply this to the list of correlators in the previous example, we find that only the following* *satisfy the line bundle degree axiom, and therefore may not vanish:*

$$\begin{aligned} &\langle A_1, A_1, A_8\rangle\,, \quad \langle A_1, A_2, A_5\rangle\,, \quad \langle A_1, A_3, A_6\rangle\,, \quad \langle A_1, A_3, A_7\rangle\,, \\ &\langle A_1, A_4, A_6\rangle\,, \quad \langle A_1, A_4, A_7\rangle\,, \quad \langle A_2, A_3, A_3\rangle\,, \quad \langle A_2, A_3, A_4\rangle\,, \\ &\langle A_2, A_4, A_4\rangle\,. \end{aligned} \tag{2.9}$$

Of the four point correlators discussed in Equation 10, only the following satisfy the line bundle degree axiom:

$$\begin{matrix} \langle A_1, A_3, A_5, A_8\rangle & \langle A_1, A_4, A_5, A_8\rangle & \langle A_1, A_5, A_5, A_6\rangle \\ \langle A_1, A_5, A_5, A_7\rangle & \langle A_2, A_2, A_2, A_8\rangle & \langle A_3, A_3, A_3, A_8\rangle \\ \langle A_3, A_3, A_4, A_8\rangle & \langle A_3, A_4, A_4, A_8\rangle & \langle A_4, A_4, A_4, A_8\rangle \\ \langle A_2, A_3, A_5, A_5\rangle & \langle A_3, A_3, A_5, A_6\rangle & \langle A_3, A_3, A_5, A_7\rangle \\ \langle A_3, A_4, A_5, A_6\rangle & \langle A_3, A_4, A_5, A_7\rangle & \langle A_4, A_4, A_5, A_6\rangle \\ \langle A_4, A_4, A_5, A_7\rangle & \langle A_2, A_2, A_6, A_6\rangle & \langle A_2, A_2, A_6, A_7\rangle \\ \langle A_2, A_2, A_7, A_7\rangle & & \end{matrix} \tag{2.10}$$

Axiom 4. ***Concavity*** *Suppose that all α_i are narrow insertions. If $\pi_* \left(\bigoplus_{i=1}^n \mathcal{L}_i\right) = 0$, then the cohomology class $\Lambda_{g,k}\left(\alpha_1, \ldots, \alpha_k\right)$ can be given in terms of top Chern class of the derived pushforward sheaf $R^1\pi_* \left(\bigoplus_{i=1}^t \mathcal{L}_i\right)$:*

$$\Lambda^W_{g,k}(\alpha_1, \ldots, \alpha_k) = \frac{|G|^g}{\deg(st)} PDst_* \left(PD^{-1} \left((-1)^D c_D \left(R^1\pi_* \bigoplus_{i=1}^n \mathcal{L}_i \right)\right)\right), \tag{2.11}$$

where $PD_{\mathscr{M}}$ is the Poincare dual map taking $H^*\overline{\mathscr{M}}_{g,k}$ to $H_*\overline{\mathscr{M}}_{g,k}$, and $PD_{\mathscr{W}}$ is the Poincare dual map taking $H^*\mathscr{W} g, k$ to $H_*\mathscr{W} g, k$.

We will discuss Chern classes in detail in Section 4.1.1, where we will also explore this axiom more fully.

Lemma 8. *When $g = 0$, $k = 3$, if Axioms 2 and 3 are satisfied, and if $l_j < 0$ for all j, then*

$$\langle \alpha_1, \alpha_2, \alpha_3 \rangle = 1.$$

Proof. The concavity axiom applies because each $l_j < 0$. In this case $D = 0$, and $c_0(R^1\pi_* \bigoplus_j \mathscr{L}_j) = 1 \in H^*(\mathscr{W}_{0,3})$ by a simple property of Chern classes. Since $st^*(1_{\overline{\mathscr{M}}_{0,3}}) = 1_{\mathscr{W}_{0,3}}$, we see that

$$\Lambda^W_{0,3}(\alpha_1, \alpha_2, \alpha_3) = \frac{|G|^0}{\deg(st)} PDst_* \left(PD^{-1} st^* 1 \right) = 1,$$

and since $\overline{\mathscr{M}}_{0,3}$ is a single point,

$$\langle \alpha_1, \alpha_2, \alpha_3 \rangle = \int_{\overline{\mathscr{M}}_{0,3}} 1 = 1.$$

□

If all insertions of $\langle \alpha_1, \ldots, \alpha_k \rangle$ are narrow, and if the degrees of all line bundles are negative integers, then $\pi_*(\bigoplus_j \mathscr{L}_j) = 0$, and we say that the correlator is *concave*.

Notice that this means that for each curve C in $\mathscr{W}_{g,k}(\alpha_1, \ldots, \alpha_n)$, we must check the degrees of the line bundles associated to each irreducible component of C.

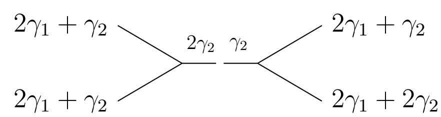
Example 12. *In the genus-zero, three-point case, there are no boundary pieces, so we need only check the line bundle degrees corresponding to the correlator itself. It turns out that the following correlators are concave:*

$$\langle A_1, A_1, A_8 \rangle \,, \quad \langle A_1, A_2, A_5 \rangle \,,$$

and hence they must be equal to 1. Of the four point correlators listed in 2.10, there is only one with all narrow insertions: $\langle A_2, A_2, A_2, A_8 \rangle$. The line bundle degrees associated to the *corresponding smooth genus-zero, four-pointed curve are*

$$l_x = 2(1/3) - 2/3 - 2/3 - 2/3 - 2/3 = -2;$$
$$l_y = 2(1/3) - 1/3 - 1/3 - 1/3 - 2/3 = -1;$$
$$l_z = 2(1/3) - 2/3 - 1/3 - 1/3 - 2/3 = -1.$$

These are all less than zero. To determine if the correlator is concave, we still need to check the line bundles associated to all nodal curves. In this case, there is only one such nodal curve (up to isomorphism). Its G-decorated dual graph is drawn below:



The line bundle degrees associated to each node of the above graph are calculated below:

Node 1	*Node 2*
$l_x = 1/3 - 2/3 - 2/3 - 0 = -1;$	$l_x = 1/3 - 2/3 - 2/3 - 0 = -1;$
$l_y = 1/3 - 1/3 - 1/3 - 2/3 = -1;$	$l_y = 1/3 - 1/3 - 2/3 - 1/3 = -1;$
$l_z = 1/3 - 1/3 - 1/3 - 2/3 = -1;$	$l_z = 1/3 - 1/3 - 2/3 - 1/3 = -1.$

So, we know that $\langle A_2, A_2, A_2, A_8 \rangle$ is a concave correlator. In Section 4.1 we will see how to compute correlators like this.

Axiom 5. ***Forgetting Tails.***

$$\Lambda_{g,k}(\alpha_1, \ldots, \alpha_{k-1}, \lceil 1\,; J \rfloor) = \Lambda_{g,k-1}(\alpha_1, \ldots, \alpha_{k-1})$$

Lemma 9. ***Pairing.***

For $\alpha_1, \alpha_2 \in \mathscr{H}_{W,G}$,

$$\langle \alpha_1, \alpha_2, \lceil 1\,; J \rfloor \rangle = \langle \alpha_1, \alpha_2 \rangle$$

Lemma 10. *If $\langle \lceil 1\,; J \rfloor, \alpha_2, \ldots, \alpha_k \rangle_{g,k}$ is correlator with $k > 3$, then $\langle \alpha_1, \ldots, \alpha_k \rangle_{g,k} = 0$*

Proof. If the degree, D of the class $\Lambda^W_{g,k}(\lceil 1\,;J\rfloor, \alpha_2, \ldots, \alpha_k)$ is equal to the degree of the class $\Lambda^W_{g,k-1}(\alpha_2, \ldots, \alpha_k)$ since the W-degree of $\lceil 1\,;J\rfloor$ is zero. So, they cannot both satisfy the dimension axiom. Since the degree of $\Lambda_{g,k-1}$ must be less than or equal to the dimension of the space $\overline{\mathscr{M}}_{g,k-1}$, which is $3g-3+(k-1)$, this means that $D \neq 3g-3+k$, and $\langle \lceil 1\,;J\rfloor, \alpha_2, \ldots, \alpha_k \rangle_{g,k} = 0$ $\square$

Example 13. *The pairing lemma shows us that*

$$\langle A_1, A_3, A_6 \rangle = \tfrac{1}{6} \quad \langle A_1, A_4, A_7 \rangle = -\tfrac{1}{2}$$
$$\langle A_1, A_1, A_8 \rangle = 1, \quad \langle A_1, A_2, A_5 \rangle = 1,$$
$$\langle A_1, A_3, A_7 \rangle = 0, \quad \langle A_1, A_4, A_6 \rangle = 0,$$

For the genus-zero four-point correlators in 2.10, Lemma 10 shows that all of the correlators with A_1 as an insertion must vanish.

Notice that for g = 0, n = 3, some correlators will satisfy both the concavity axiom. For example:

$$\langle A_1, A_2, A_5 \rangle = 1$$

Whenever this is the case, the insertions are narrow, which means that the pairing of the two elements left when $\lceil 1\,;J\rfloor$ is removed, will have trivial fixed locus, and thus will have pairing equal to 1. Thus, the two axioms will never contradict each other.

Axiom 6. ***Sums of Singularities.***
If $W_1 \in \mathbb{C}[x_1, \ldots, x_r]$ and $W_2 \in \mathbb{C}[x_{r+1}, \ldots, x_n]$ are two nondegenerate, quasihomogeneous singularities with symmetry groups G_1 and G_2, then

$$\mathscr{H}_{W_1+W_2, G_1 \oplus G_2} \cong \mathscr{H}_{W_1, G_1} \otimes \mathscr{H}_{W_2, G_2}$$

and the Λ classes are related by

$$\Lambda^{W_1+W_2}(\alpha_1 \otimes \beta_1, \ldots, \alpha_k \otimes \beta_k) = \Lambda^{W_1}(\alpha_1, \ldots, \alpha_k) \otimes \Lambda^{W_2}(\beta_1, \ldots, \beta_k),$$

where α_i and β_i are elements in $\mathscr{H}_{W_1,G_1}$ and $\mathscr{H}_{W_2,G_2}$, respectively.

Axiom 7. ***Composition.***

If $\Lambda_{g,k}(\alpha_1,\ldots,\alpha_k)$ has degree $D = 3g-4+k$, then

$$\Lambda_{g,k}(\alpha_1,\ldots,\alpha_k) = \sum_{\sigma,\tau} \Lambda_{g_1,k_1+1}(\alpha_1,\ldots,\alpha_{k_1},\sigma)\eta^{\sigma,\tau}\Lambda_{g_2,k-k_1+1}(\alpha_{k_1}+1,\ldots,\alpha_k,\sigma),$$

for any $g_1+g_2 = g$ and $k-1 \leq k$ with $2g_1 - 1 + k_1 > 0$ and $2g_2 - 1 + k - k_1 > 0$.

Lemma 11. *If $g = 0$ and $|I| = |J| = 2$, and $\Lambda_{0,4}(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$ has degree 0, all insertions are narrow, and $\pi_*(\bigoplus \mathcal{L}_i) = 0$, then*

$$1 = \sum_{\tau_1,\tau_2} \left\langle \alpha_{\sigma(1)},\alpha_{\sigma(2)},\tau_1 \right\rangle \eta^{\tau_1,\tau_2} \left\langle \tau_2,\alpha_{\sigma(3)},\alpha_{\sigma(4)} \right\rangle,$$

for any $\sigma \in S_4$.

Proof. If $\bigoplus \mathscr{L}_i$ is concave, then the concavity axiom says that $\Lambda = \frac{1}{\deg st} PDst_* PD^{-1} 1 = 1$, and since $\overline{\mathscr{M}}_{0,3}$ is a single point, $\langle \alpha_1,\alpha_2,\alpha_3\rangle = \Lambda(\alpha_1,\alpha_2,\alpha_3)$. □

Example 14. *The class $\Lambda_4(A_1,A_1,A_1,A_8)$ has degree $D = 0$, and is concave, so*

$$\Lambda_4(A_1,A_1,A_1,A_8) = 1 = \sum_{\sigma,\tau} \langle A_1,A_1,\sigma\rangle\eta^{\sigma,\tau}\langle \tau,A_1,A_8\rangle = \langle A_1,A_1,A_8\rangle(1)\langle A_1,A_1,A_8\rangle,$$

which means that $\langle A_1,A_1,A_8\rangle = \pm 1$. In fact, from the concavity axiom (or the pairing lemma) we already determine that $\langle A_1,A_1,A_8\rangle = 1$.

Axiom 8. ***Deformation Invariance*** *Let $W_t \in \mathbb{C}[x_1,\ldots,x_n]$ be a family of quasihomogeneous non degenerate polynomials depending smoothly on a parameter $t \in [a,b] \subset \mathbb{R}$. Suppose that G is the common automorphism group of W_t. The Λ class associated to (W_t, G) is independent of t.*

Correlators must also satisfy the following property.

Property 2. G^{max} ***Invariance*** *The action of any element* $g \in G_W^{max}$ *on an A model element,* α_i*, can be described by multiplication by a complex number, that is,*

$$g \cdot \lceil m\, ; h \rfloor = \frac{g(m \cdot \omega_h)}{m \cdot \omega_h} \lceil m\, ; h \rfloor = c \cdot \lceil m\, ; h \rfloor\, .$$

If the action of g *on* α_i *is determined by the complex number* c_i*, then the correlator* $\langle \alpha_1, \ldots, \alpha_k \rangle = 0$ *unless* $\prod c_i = 1$*.*

Example 15. *Certainly any correlators where all insertions are narrow will automatically satisfy this property. Let's look at the action of the generators of* G^{max} *on the remaining potentially nonzero three-point correlators, as listed in 2.9. It turns out that* $\gamma_1 = (1/3, 0, 0)$ *will have a trivial action on any basis element, and therefore any correlator. However,* $\gamma_3 = (0, 1/3, 5/6)$ *does not fix all elements. For example,*

$$\gamma_3 \lceil y\, ; \gamma_2 \rfloor = \gamma_3 \cdot \lceil y\, ; 2\gamma_2 \rfloor = e^{2\pi i(1/3(2)+5/6(1))} = e^{2\pi i(\frac{9}{6})} = e^{2\pi i(\frac{1}{2})},$$
$$\gamma_3 \lceil z\, ; \gamma_2 \rfloor = \gamma_3 \cdot \lceil z\, ; 2\gamma_2 \rfloor = e^{2\pi i(1/3(1)+5/6(2))} = e^{2\pi i(\frac{12}{6})} = 1.$$

So,

$$\gamma_3 \cdot \langle A_2, A_3, A_3 \rangle\, , = e^{2\pi i(\frac{1}{2})} \cdot e^{2\pi i(\frac{1}{2})} = 1,$$
$$\gamma_3 \cdot \langle A_2, A_3, A_4 \rangle = e^{2\pi i(\frac{1}{2})},$$
$$\gamma_3 \cdot \langle A_2, A_4, A_4 \rangle = 1,$$

which means that the correlator $\langle A_2, A_3, A_4 \rangle$ *vanishes.*

How much of the A-model Frobenius algebra structure do we know at this point? We know the values of all genus-zero three-point correlators except for $\langle A_2, A_3, A_3 \rangle$ and $\langle A_2, A_4, A_4 \rangle$.

So, we can determine some, but not all of the products:

$$
\begin{array}{lll}
A_1 \star A_1 = A_1 & A_1 \star A_2 = A_2 & A_1 \star A_3 = A_3 \\
A_1 \star A_4 = A_4 & A_1 \star A_5 = A_5 & A_1 \star A_6 = A_6 \\
A_1 \star A_7 = A_7 & A_1 \star A_8 = A_8 & A_2 \star A_3 = ??? \\
A_2 \star A_4 = ??? & A_2 \star A_5 = A_8 & A_3 \star A_3 = ??? \\
A_3 \star A_6 = \frac{1}{6}A_8 & A_4 \star A_4 = ??? & A_4 \star A_7 = -\frac{1}{2}A_8
\end{array}
$$

Next we define the product structure in the B model.

2.3.7 B Model Frobenius Algebras Structure: The Orbifold Milnor Rings.

Multiplication in the B-model is defined in the following way. If $\lceil m\,;g\rfloor \in \mathscr{B}_g$ and $\lceil n\,;h\rfloor \in \mathscr{B}_h$, then

$$\lceil m\,;g\rfloor \star_B \lceil n\,;h\rfloor = \left\lceil [\gamma_{g,h} \cdot n \cdot m]_{\mathscr{Q}_{g+h}}\,; g+h \right\rfloor,$$

where

$$\gamma_{g,h} = \begin{cases} \mu_{g\cap h} Hess(W_{g+h})/(\mu_{g+h} Hess(W_{g\cap h}) & if\, I_g \cup I_h \cup I_{g+h} = [N] \\ 0 & \text{otherwise} \end{cases} \tag{2.12}$$

Example 16. *For* $W^T = \left(J_{10}^T\right)^T = J_{10} = x^3 + y^3z + z^2$ *and* $G^T = \langle(0, 1/2, 1/2)\rangle$*. If* $\zeta = (0, 1/2, 1/2)$*, the vector space for the B-model is given by*

$B_1 = \lceil 1\,;\mathbf{0}\rfloor_B$	$B_2 = \lceil 1\,;\zeta\rfloor_B$
$B_3 = \lceil x\,;\mathbf{0}\rfloor_B$	$B_4 = \lceil y^2\,;\mathbf{0}\rfloor_B$
$B_5 = \lceil xy^2\,;\mathbf{0}\rfloor_B$	$B_6 = \lceil x\,;\zeta\rfloor_B$
$B_7 = \lceil yz\,;\mathbf{0}\rfloor_B$	$B_8 = \lceil xyz\,;\mathbf{0}\rfloor_B$

Notice first that $B_1 \star B_j = B_j$ *for any* j*. Notice also that* $\gamma_{g,h} = 1$ *unless* $g = h = \zeta$*. In this case:*

$$\gamma_{\zeta,\zeta} = \frac{2 \cdot 180xyz}{10 \cdot 6x} = 6yz.$$

We also get the following products:

$$B_2 \star B_2 = \lceil 6yz\,;\mathbf{0}\rfloor = 6B_7 \qquad B_2 \star B_3 = \lceil x\,;\zeta\rfloor = B_6 \qquad B_2 \star B_4 = \lceil y^2\,;\zeta\rfloor = 0$$
$$B_2 \star B_5 = \lceil xy^2\,;\zeta\rfloor = 0 \qquad B_2 \star B_6 = \lceil 6xyz\,;\mathbf{0}\rfloor = 6B_8 \qquad B_2 \star B_7 = \lceil yz\,;\zeta\rfloor = 0$$
$$B_3 \star B_3 = \lceil x^2\,;\mathbf{0}\rfloor = 0 \qquad B_3 \star B_4 = \lceil xy^2\,;\mathbf{0}\rfloor = B_5 \qquad B_3 \star B_5 = \lceil x^2y^2\,;\mathbf{0}\rfloor = 0$$
$$B_3 \star B_6 = \lceil x^2\,;\zeta\rfloor = 0 \qquad B_3 \star B_7 = \lceil xyz\,;\mathbf{0}\rfloor = B_8 \qquad B_3 \star B_8 = \lceil x^2yz\,;\mathbf{0}\rfloor = 0$$
$$B_4 \star B_4 = \lceil y^4\,;\mathbf{0}\rfloor = 0 \qquad B_4 \star B_5 = \lceil xy^4\,;\mathbf{0}\rfloor = -2B_8 \qquad B_4 \star B_6 = \lceil xy^2\,;\zeta\rfloor = 0$$
$$B_4 \star B_7 = \lceil y^3z\,;\mathbf{0}\rfloor = 0 \qquad B_4 \star B_8 = \lceil xy^3z\,;\mathbf{0}\rfloor = 0 \qquad B_5 \star B_5 = \lceil x^2y^4\,;\mathbf{0}\rfloor = 0$$
$$B_5 \star B_6 = \lceil x^2y^2\,;\zeta\rfloor = 0 \qquad B_5 \star B_7 = \lceil xy^3z\,;\mathbf{0}\rfloor = 0 \qquad B_5 \star B_8 = \lceil x^2y^3z\,;\mathbf{0}\rfloor = 0$$
$$B_6 \star B_6 = \lceil 6x^2yz\,;\mathbf{0}\rfloor = 0 \qquad B_6 \star B_7 = \lceil xyz\,;\zeta\rfloor = 0 \qquad B_6 \star B_8 = \lceil x^2yz\,;\zeta\rfloor = 0$$
$$B_7 \star B_7 = \lceil y^2z^2\,;\mathbf{0}\rfloor = 0 \qquad B_7 \star B_8 = \lceil xy^2z^2\,;\mathbf{0}\rfloor = 0 \qquad B_8 \star B_8 = \lceil x^2y^2z^2\,;\mathbf{0}\rfloor = 0$$

Remark 3. *Both $\star_A$ and $\star_B$ have the Frobenius property with respect to the pairing:*

$$\langle \alpha \star \beta, \gamma \rangle = \langle \alpha, \beta \star \gamma \rangle$$

*This makes both models **Frobenius algebras**.*

Each model is also equipped with higher structure.

2.3.8 Higher Structure. A **Frobenius manifold** is a family of Frobenius algebras which vary continuously. That is if M is a manifold, then TM_t will be a Frobenius algebra for each $t \in M$.

For the unorbifolded B model the Frobenius manifold is geven by the Saito Frobenius manifold for the singularity. For the unorbifolded B model the Frobenius manifold structure is still unknown.

For the A model, this structure is given by a generating function determined by the genus-zero correlators.

$$\Phi(t_1, \ldots, t_N) = \sum_k \frac{1}{k!} \sum_{i_1,\ldots,i_k} \langle t_{i_1}, \ldots, t_{i_k} \rangle t_{i_1} \ldots t_{i_k}$$

2.4 Singularities With the Same Weights

It is generally believed that for two non degenerate quasihomogeneous polynomials W_1 and W_2 with the same weight system, such that G is an admissible symmetry group for both singularities, the FJRW rings are isomorphic,

$$\mathscr{H}_{W_1,G} \cong \mathscr{H}_{W_2,G}.$$

At present, we are unaware of a proof of this statement. We have made some headway in proving this fact however, thanks to some very useful observations from Rachel Suggs.

Whenever we choose $G = \langle J \rangle$, the two FJRW vector spaces will be isomorphic. To prove this we need the following lemma from [19].

Lemma 12. *If $p_W(t)$ is the **Poincare polynomial** for a singularity W, with weights $q_1, \ldots q_n$, defined by*

$$p_W(t) = \prod_{i=1}^{n} \frac{t^{1-q_1} - 1}{t^{q_1} - 1} = \sum_w a_w t^w$$

then a_W is the number of monomials in the basis for $\mathscr{Q}_W$ of weighted degree w.

Lemma 13. *For any singularity W and symmetry group G, the number of basis elements coming from the $g = \mathbf{0}$ sector correspond to the monomials in the milnor ring with weighted degree $\equiv -\sum_i q_i \mod \mathbb{Z}$.*

Proof. $\lceil \prod x_i^{m_i}\, ; g \rfloor$ is in $\mathscr{H}_{W,G}$ if and only if $(\mathbf{m} + \mathbf{1}) \cdot (q_1, \ldots, q_n)^T$ is an integer. □

Lemma 14. *For an invertible singularity W, if $g = J^k$, then the number of basis elements coming from the g-sector is completely determined by the weights vector J.*

Proof. The case where $g = \mathbf{0}$ is taken care of in Lemma 13. On the other hand, if $Fix(g) = \emptyset$, then there is always a single basis element from this sector.

Now suppose that $Fix(g) \neq \emptyset$, then it must correspond to certain variables in chain, loop and Fermat type polynomials. Recall that if W_j is a loop or Fermat type polynomial, it has the property that for any symmetry group element g, if g fixes any of the variables of W_j, then it must fix all of them. This means that $Fix(g)$ will either capture loop and Fermat polynomials completely, or miss them completely. On the other hand, if W_j is a chain type polynomial,

$$W_j = x_{j_1}^{a_{j_1}} x_{j_2} + \ldots x_{j_{s-1}}^{a_{j_{s-1}}} x_{j_s} + x_{j_s}^{a_{j_s}}$$

Then

$$J = \left(\frac{\sum_{i=1}^{s} (-1)^{s-i} \prod_{k=i}^{s} a_{j_k}}{\prod_{i=1}^{s} a_{j_i}} \ldots, \frac{a_{j_s} a_{j_{s-1}} - a_{j_s} + 1}{a_{j_s} a_{j_{s-1}} a_{j_{s-2}}}, \frac{a_{j_s} - 1}{a_{j_s} a_{j_{s-1}}}, \frac{1}{a_{j_s}} \right)$$

And $Fix(g) \cap \{x_{j_1}, \ldots, x_{j_s}\} = \{x_{j_k}, \ldots, x_{j_s}\}$ for some $k \geq 1$. This means that

$$(W_j)_g = x_{j_k}^{a_{j_k}} x_{j_{k+1}} + \ldots x_{j_{s-1}}^{a_{j_{s-1}}} x_{j_s} + x_{j_s}^{a_{j_s}}$$

And then

$$J_{(W_j)_g} = \left(\frac{\sum_{i=j_k}^{s} (-1)^{s-i} \prod_{k=i}^{s} a_{j_k}}{\prod_{i=1}^{s} a_{j_i}} \ldots, \frac{a_{j_s} a_{j_{s-1}} - a_{j_s} + 1}{a_{j_s} a_{j_{s-1}} a_{j_{s-2}}}, \frac{a_{j_s} - 1}{a_{j_s} a_{j_{s-1}}}, \frac{1}{a_{j_s}} \right) = J_{W_j}|_{Fix(g)}.$$

So, $J_{W_g} = J_W|_{Fix(g)}$, and the number of monomials in $\mathscr{Q}_g$ with weighted degrees congruent to $-\sum_{Fix(g)} q_i \mod \mathbb{Z}$ can be determined using the poincare polynomial restricted to the weights of the variables in $Fix(g)$.

□

Corollary 1. *If W_1 and W_2 are two invertible singularities with the same weights vector, then*

$$\mathscr{H}_{W_1,\langle J \rangle} \cong \mathscr{H}_{W_2,\langle J \rangle}$$

as vector spaces.

Lemma 15. *With W_1 and W_2 as described above,*

$$\mathscr{H}_{W_1,\langle J\rangle} \cong \mathscr{H}_{W_2,\langle J\rangle}$$

as cohomological field theories.

Proof. Now, if we let M_0 be the set of monomials which W_1 and W_2 have in common, M_1 be the set of monomials unique to W_1 and M_2 the set of monomials unique to W_2, then we can apply Axiom 8 to W_t, the family of polynomials defined by

$$W_t = M_0 + tM_1 + (1-t)M_2.$$

□

Finally we extend Lemma 15 to a more general set of symmetry groups.

Lemma 16. *If $W_1 = \sum_i W_i^1$ and $W_2 = \sum_i W_i^2$ are two invertible polynomials with the same weight systems, W_i^1 and W_i^2 act on the same variables for each i, and*

$$G = \bigoplus_i \langle (0, \ldots, 0, J_i, 0, \ldots, 0) \rangle$$

Then

$$\mathscr{H}_{W_1,G} \cong \mathscr{H}_{W_2,G}.$$

Next we give new results comparing stabilization of singularities in classical singularity theory to the Landau-Ginzburg setting.

Chapter 3. Stabilization of Singularities

In classical singularity theory it is a well-known fact that if $W \in \mathbb{C}[x_1, \ldots, x_n]$ and $\tilde{W} = W + z^2$, that W and $\tilde{W}$ are analytically equivalent. It turns out that for different choices of symmetry groups, it is not always true that

$$\mathscr{H}_{W,G} \cong \mathscr{H}_{\tilde{W},G}.$$

In this sense, FJRW theory gives a finer classification of singularities.

Lemma 17. *If W is a nondegenerate quasihomogeneous polynomial, $W \in \mathbb{C}[x_1, \ldots, x_n]$, and $\tilde{W} = W + x_{n+1}^2$, G an admissible subgroup of G^{max}, and $\tilde{G} = G \times \mathbb{Z}_2$, then we have*

$$\mathscr{H}_{W,G} \cong \mathscr{H}_{\tilde{W},\tilde{G}}$$

Proof. If $V = x_{n+1}^2$ and $H = \langle \left(\frac{1}{2}\right) \rangle$, notice that $\tilde{W} = W + V$, and $\tilde{G} = G \times H$. Now, W is a Fermat type polynomial whose Milnor ring basis is just $\langle 1 \rangle$. Thus, no x_{n+1} will appear in any monomial in any sector of $\mathscr{H}_{V,H}$. Since $1 \cdot dx_{n+1}$ is not fixed under H,

$$\mathscr{H}_{V,H} = \left\langle \left\lceil 1\,;\left(\frac{1}{2}\right) \right\rfloor \right\rangle$$

Then

$$\mathscr{H}_{\tilde{W},\tilde{G}} \cong \mathscr{H}_{W,G} \times \left\langle \left\lceil 1\,;\left(\frac{1}{2}\right) \right\rfloor \right\rangle \cong \mathscr{H}_{W,G}$$

□

Example 17. *If we let $W = x^6$ and $\tilde{W} = x^6 + y^2$, we can see what happens when we choose different symmetry groups for $\tilde{W}$.*

$W = x^6$, $G = \langle \frac{1}{6} \rangle$

Element	$\mathbb{C}$-*degree*
$A_1 = \lceil 1\,;(\frac{1}{6}) \rfloor$	*0*
$A_2 = \lceil 1\,;(\frac{1}{3}) \rfloor$	*1/6*
$A_3 = \lceil 1\,;(\frac{1}{2}) \rfloor$	*1/3*
$A_4 = \lceil 1\,;(\frac{2}{3}) \rfloor$	*1/2*
$A_5 = \lceil 1\,;(\frac{5}{6}) \rfloor$	*2/3*

$\tilde{W} = x^6 + y^2$, $\tilde{G} = \langle (\frac{1}{6}, 0), (0, \frac{1}{2}) \rangle$

Element	$\mathbb{C}$-*degree*
$A_1 = \lceil 1\,;(\frac{1}{6}, \frac{1}{2}) \rfloor$	*0*
$A_2 = \lceil 1\,;(\frac{1}{3}, \frac{1}{2}) \rfloor$	*1/6*
$A_3 = \lceil 1\,;(\frac{1}{2}, \frac{1}{2}) \rfloor$	*1/3*
$A_4 = \lceil 1\,;(\frac{2}{3}, \frac{1}{2}) \rfloor$	*1/2*
$A_5 = \lceil 1\,;(\frac{5}{6}, \frac{1}{2}) \rfloor$	*2/3*

In this case we do get isomorphic FJRW Rings, because $\tilde{G} \cong G \oplus \mathbb{Z}_2$. However, if we choose $\tilde{G} = J_{\tilde{W}}$ instead, we find that

$\tilde{W} = x^6 + y^2$, $\tilde{G} = \langle (\frac{1}{6}, \frac{1}{2}) \rangle$

Element	$\mathbb{C}$-*degree*
$A_1 = \lceil 1\,;(\frac{1}{6}, \frac{1}{2}) \rfloor$	*0*
$A_2 = \lceil 1\,;(\frac{1}{2}, \frac{1}{2}) \rfloor$	*1/3*
$A_3 = \lceil x^2\,;(0, 0) \rfloor$	*1/3*
$A_4 = \lceil 1\,;(\frac{5}{6}, \frac{1}{2}) \rfloor$	*2/3*

One interesting fact is that if we choose this *wrong* kind of symmetry group for $\tilde{W}$ as in the last part of the preceding example, if $\hat{W}$ is $\tilde{W}$ plus another squared variable, and we say that $\hat{G}$ acts like $\tilde{G}$ on all variables in $\tilde{W}$, and acts identically on both of the squared variables, then $\mathscr{H}_{W,G} \cong \mathscr{H}_{\hat{W},\hat{G}}$.

Lemma 18. *If W is a non degenerate quasihomogeneous polynomial, $W \in \mathbb{C}[x_1, \ldots, x_n]$, $\tilde{W} = W + y^2 + z^2$, $\tilde{G}$ is any admissable subgroup of the maximal symmetry group for $\tilde{W}$, with the following two properties:*

1. *For all $\tilde{g} \in \tilde{G}$, $y \in Fix(\tilde{g}) \Leftrightarrow z \in Fix(\tilde{g})$*

2. *$(g_1, \ldots, g_n, 0, 0) \in \tilde{G} \Leftrightarrow (g_1, \ldots, g_n, \frac{1}{2}, \frac{1}{2}) \notin \tilde{G}$*

and G is the group generated by all elements of the form $(g_1, \ldots, g_n)$*, where* $(g_1, \ldots, g_n, g_y, g_z) \in \tilde{G}$*, then*

$$\mathscr{H}_{W,G} \cong \mathscr{H}_{\tilde{W},\tilde{G}}$$

as vector spaces.

Proof. First notice that $y^2 + z^2$ is a Fermat type polynomial whose Milnor ring basis is just $\langle 1 \rangle$. Thus, no y or z will appear in any monomial in any sector of $\mathscr{H}_{\tilde{W},\tilde{G}}$.

Now suppose that $\lceil m\,; g \rfloor \in \mathscr{H}_{W,G}$, and $g = (g_1, \ldots g_n)$. Then either $\tilde{g}_1 = (g_1, \ldots, g_n, \frac{1}{2}, \frac{1}{2})$ or $\tilde{g}_2 = (g_1, \ldots, g_n, 0, 0) \in \tilde{G}$, by our construction of G. Note that $Fix(g) = Fix(\tilde{g}_1)$, so $\lceil m\,; \tilde{g}_1 \rfloor \in \mathscr{H}_{\tilde{W},\tilde{G}^{max}}$ if $\tilde{g}_1 \in \tilde{G}$. Otherwise, the action of any element $\tilde{h}$ on the element $m \cdot d\mathbf{x}_{Fix(\tilde{g}_2)} = m \cdot d\mathbf{x}_{Fix(g)} dy dz$ will be equivalent to the action of h on $m \cdot d\mathbf{x}_{Fix(g)}$ because of the way group elements act on dy and dz simultaneously. Thus $\lceil m\,; \tilde{g}_2 \rfloor \in \mathscr{H}_{\tilde{W},\tilde{G}}$.

Let φ be the map which sends $\lceil m\,; g \rfloor$ to either$\lceil m\,; \tilde{g}_1 \rfloor$ or $\lceil m\,; \tilde{g}_2 \rfloor$ in the way we just described. It is injective by definition. Condition 2 on $\tilde{G}$ ensures that this map is well-defined. Notice that this map is surjective because of the way we constructed $\tilde{G}$, and our earlier observation that the monomials in $\mathscr{H}_{\tilde{W},\tilde{G}}$ will never contain any y or z variables.

Recall that pairings in either ring will be zero whenever the elements do not come from inverse sectors, and that $h = g^{-1} \Leftrightarrow \tilde{h} = \tilde{g}^{-1}$. If $\tilde{g}$ and $\tilde{h}$ are chosen to look like $\tilde{g}_1$ above, then the fixed locii of g and $\tilde{g}$ are the same, and do not include y or z, thus the pairing is computed in exactly the same way for each. Otherwise, if

$$\left\langle \lceil m\,; g \rfloor , \lceil n\,; g^{-1} \rfloor \right\rangle$$

is nonzero, then $\mu_g = \mu_{\tilde{g}}$ and $Hess(W_g) = \frac{1}{4} Hess(W_{\tilde{g}})$, so

$$\left\langle \lceil m\,; \tilde{g} \rfloor , \lceil n\,; \tilde{g}^{-1} \rfloor \right\rangle = \frac{mn\mu_{\tilde{g}}}{Hess(W_{\tilde{g}})} = \frac{mn\mu_g}{4Hess(W_g)} = \frac{1}{4} \left\langle \lceil m\,; g \rfloor , \lceil n\,; g^{-1} \rfloor \right\rangle .$$

Thus whenever $y, z \notin Fix(\tilde{g})$,

$$\varphi(\lceil m\,;g\rfloor) = \lceil m\,;\tilde{g}\rfloor\,,$$

otherwise, we need

$$\varphi(\lceil m\,;g\rfloor) = k\,\lceil m\,;\tilde{g}\rfloor \text{ and } \varphi(\lceil n\,;g^{-1}\rfloor) = \frac{1}{4k}\,\lceil n\,;\tilde{g}^{-1}\rfloor\,,$$

where $\lceil n\,;g^{-1}\rfloor$ is the element which pairs nontrivially with $\lceil m\,;g\rfloor$. See Example 18 and look at the pairing $\langle A_2, A_4\rangle$ in each FJRW ring. □

Example 18. *Let $W = x^6$ and $G = \langle(\frac{1}{6})\rangle$.*

$W = x^6$, $G = \langle(\frac{1}{6})\rangle$

Element	$\mathbb{C}$-*degree*
$A_1 = \lceil 1\,;\frac{1}{6}\rfloor$	*0*
$A_2 = \lceil 1\,;\frac{1}{3}\rfloor$	*1/6*
$A_3 = \lceil 1\,;\frac{1}{2}\rfloor$	*1/3*
$A_4 = \lceil 1\,;\frac{2}{3}\rfloor$	*1/2*
$A_5 = \lceil 1\,;\frac{4}{6}\rfloor$	*2/3*

Pairings
$\langle A_1, A_5\rangle = 1$
$\langle A_2, A_4\rangle = 1$
$\langle A_3, A_3\rangle = 1$

Now if we let $\tilde{W} = x^6 + y^2 + z^2$ with symmetry group $= \langle J\rangle$.

$\tilde{W} = x^6 + y^2 + z^2$, $G = \langle(\frac{1}{6}, \frac{1}{2}, \frac{1}{2})\rangle$

Element	$\mathbb{C}$-*degree*
$A_1 = \lceil 1\,;(\frac{1}{6}, \frac{1}{2}, \frac{1}{2})\rfloor$	*0*
$A_2 = \lceil 1\,;(\frac{1}{3}, 0, 0)\rfloor$	*1/6*
$A_3 = \lceil 1\,;(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})\rfloor$	*1/3*
$A_4 = \lceil 1\,;(\frac{2}{3}, 0, 0)\rfloor$	*1/2*
$A_5 = \lceil 1\,;(\frac{5}{6}, \frac{1}{2}, \frac{1}{2})\rfloor$	*2/3*

Pairings
$\langle A_1, A_5\rangle = 1$
$\langle A_2, A_4\rangle = \frac{1}{4}$
$\langle A_3, A_3\rangle = 1$

Conjecture 2. *With the same hypthoses as above,*

$$\mathscr{H}_{W,G} \cong \mathscr{H}_{\tilde{W},\tilde{G}}$$

as Frobenius Algebras.

Although we cannot currently prove this conjecture, it is interesting to look at how φ acts on correlators, since they determine product structure. Consider the correlators

$$A = \langle \lceil m_1\,; g_1 \rfloor, \lceil m_2\,; g_2 \rfloor, \lceil m_2\,; g_3 \rfloor \rangle, \text{ and } B = \langle \lceil m_1\,; \tilde{g_1} \rfloor, \lceil m_2\,; \tilde{g_2} \rfloor, \lceil m_2\,; \tilde{g_3} \rfloor \rangle$$

First, we consider the how the Dimension axiom applies to each correlator. Note that

$$\hat{c}_{\tilde{W}} = \sum_{i=1}^{n} (1-2q_i) + 1 - 2q_y + 1 - 2q_z = \sum_{i=1}^{n} (1-2q_i) + 1 - 1 + 1 - 1 = \sum_{i=1}^{n} (1-2q_i) = \hat{c}$$

Recall that for any $\tilde{g} \in \tilde{G}$ we have two cases:

1. $\tilde{g}_y = \tilde{g}_z = 0$

2. $\tilde{g}_y = \tilde{g}_z = \frac{1}{2}$.

For Case 1, $N_{\tilde{g}} = N_g + 2$, and

$$\sum_{i=1}^{n+2} \tilde{g}_i - q_i = \sum_{i=1}^{n} (g_i - q_i) + \left(0 - \frac{1}{2}\right) + \left(0 - \frac{1}{2}\right) = \sum_{i=1}^{n} g_i - q_i - 1$$

$$deg_{\tilde{W}} \lceil m\,; \tilde{g} \rfloor = N_g + 2 + 2\left(\sum_{i=1}^{n} g_i - q_i - 1\right) = deg_W \lceil m\,; g \rfloor$$

For Case 2, $N_{\tilde{g}} = N_g$ and

$$\sum_{i=1}^{n+2} \tilde{g}_i - q_i = \sum_{i=1}^{n} g_i - q_i + \left(\frac{1}{2} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{2}\right) = \sum_{i=1}^{n} g_i - q_i$$

So, $deg_{\tilde{W}} \lceil m\,; \tilde{g} \rfloor = deg_W \lceil m\,; g \rfloor$. In either case the dimension axiom is satisfied for A exactly when it is satisfied for B.

Next we consider the Integer Degree and Concavity Axioms. Note that by the definition of φ, $l_i^A = l_i^B$ for $i = 1, \dots n$, and $l_y^B = l_z^A = \frac{1}{2} - \tilde{g}_1^y - \tilde{g}_2^y - \tilde{g}_3^y$, which may be equal to

$\frac{1}{2}, 0, -\frac{1}{2}$, or 1. Recall that we can use Lemma 7 on g_3 so that the line bundle degrees are integers. So, we know the sum will be equal to 0 or -1. Whenever $l_y^B = l_z^B = -1$, the integer degree and concavity axioms will both be satisfied for A exactly when it is satisfied for B. If $l_y^B = l_z^B = 0$, then A will satisfy the integer degree axiom iff B does. However, it is possible for A to satisfy concavity, when B will not. To see this, examine Example 19 and compare the correlator $\langle A_2, A_2, A_3 \rangle$.

We can see from the discussion above that G^{max} invariance will be satisfied for A exactly when satisfied for B.

In Lemma 18, when Condition 1 fails, $\mathscr{H}_{\tilde{W},\tilde{G}}$ seems to be isomorphic to $\mathscr{H}_{W+y^2,H}$ with group $H \neq G \oplus \mathbb{Z}_2$. This may be because when Condition 1 fails we can always express $\tilde{G}$ as a product of something with $\mathbb{Z}_2$. Compare Example 20 and the third FJRW ring of Example 17.

On the other hand, when Condition 2 fails, the vector space basis for $\mathscr{H}_{\tilde{W},\tilde{G}}$ looks like a double copy of the vector space basis for $\mathscr{H}_{W,G}$. It is not clear exactly how the nonzero products and correlators of the two structures relate. Example 21 shows this case.

Example 19. *For* $W = x^6$, $\tilde{G} = \langle \left(\frac{1}{6}\right) \rangle$,

Nonzero products	
$A_1 \star A_1 = A_1$	$A_1 \star A_2 = A_2$
$A_1 \star A_3 = A_3$	$A_1 \star A_4 = A_4$
$A_1 \star A_5 = A_5$	$A_2 \star A_2 = A_3$
$A_2 \star A_3 = A_4$	$A_2 \star A_4 = A_5$
$A_3 \star A_3 = A_5$	

Correlators
$\langle A_5, A_1, A_1 \rangle = 1$
$\langle A_1, A_2, A_4 \rangle = 1$
$\langle A_1, A_3, A_3 \rangle = 1$
$\langle A_2, A_2, A_3 \rangle = 1$

For $\tilde{W} = x^6 + y^2 + z^2$, $\tilde{G} = \langle \left(\frac{1}{6}, \frac{1}{2}, \frac{1}{2}\right) \rangle$

Nonzero products	
$A_1 \star A_1 = A_1$	$A_1 \star A_2 = A_2$
$A_1 \star A_3 = A_3$	$A_1 \star A_4 = A_4$
$A_1 \star A_5 = A_5$	$A_2 \star A_2 =???$
$A_2 \star A_3 =???$	$A_2 \star A_4 = \frac{1}{4} A_5$
$A_3 \star A_3 = A_5$	

Correlators
$\langle A_0, A_1, A_1 \rangle = 1$
$\langle A_1, A_2, A_4 \rangle = \frac{1}{4}$
$\langle A_1, A_3, A_3 \rangle = 1$
$\langle A_2, A_2, A_3 \rangle =???$

Example 20. *If* $W = x^6 + y^2 + z^2$ *with symmetry group generated by* J *and* $\left(0, 0, \frac{1}{2}\right)$*, then Condition 1 of Lemma 18 fails:*

Element	$\mathbb{C}$*-degree*
$A_1 = \lceil 1\,; (\frac{1}{6}, \frac{1}{2}, \frac{1}{2}) \rfloor$	*0*
$A_2 = \lceil 1\,; (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \rfloor$	*1/3*
$A_3 = \lceil x^2\,; (0, 0, \frac{1}{2}) \rfloor$	*1/3*
$A_4 = \lceil 1\,; (\frac{5}{6}, \frac{1}{2}, \frac{1}{2}) \rfloor$	*2/3*

Pairings
$\langle A_1, A_4 \rangle = 1$
$\langle A_2, A_2 \rangle = 1$
$\langle A_3, A_3 \rangle = \frac{1}{12}$

Nonzero products
$A_1 \star A_1 = A_1$
$A_1 \star A_2 = A_2$
$A_1 \star A_3 = A_3$
$A_1 \star A_4 = A_4$
$A_2 \star A_2 = A_4$
$A_2 \star A_3 = \frac{1}{12} A_4$

Example 21. *If* $W = x^6 + y^2 + z^2$ *with symmetry group generated by* J *and* $\left(0, \frac{1}{2}, \frac{1}{2}\right)$*, then Condition 2 of Lemma 18 fails:*

Element	$\mathbb{C}$-*degree*
$A_1 = \lceil 1\,;(\frac{1}{6},0,0)\rfloor$	*0*
$A_2 = \lceil 1\,;(\frac{1}{6},\frac{1}{2},\frac{1}{2})\rfloor$	*0*
$A_3 = \lceil 1\,;(\frac{1}{3},0,0)\rfloor$	*1/6*
$A_4 = \lceil 1\,;(\frac{1}{3},\frac{1}{2},\frac{1}{2})\rfloor$	*1/6*
$A_5 = \lceil 1\,;(\frac{1}{2},0,0)\rfloor$	*1/3*
$A_6 = \lceil 1\,;(\frac{1}{2},\frac{1}{2},\frac{1}{2})\rfloor$	*1/3*
$A_7 = \lceil 1\,;(\frac{2}{3},0,0)\rfloor$	*1/2*
$A_8 = \lceil 1\,;(\frac{2}{3},\frac{1}{2},\frac{1}{2})\rfloor$	*1/2*
$A_9 = \lceil 1\,;(\frac{5}{6},0,0)\rfloor$	*2/3*
$A_{10} = \lceil 1\,;(\frac{5}{6},\frac{1}{2},\frac{1}{2})\rfloor$	*2/3*

Pairings
$\langle A_1, A_9\rangle = \frac{1}{4}$
$\langle A_2, A_{10}\rangle = 1$
$\langle A_3, A_7\rangle = \frac{1}{4}$
$\langle A_4, A_8\rangle = 1$
$\langle A_5, A_5\rangle = \frac{1}{4}$
$\langle A_6, A_6\rangle = 1$

It is interesting to observe what happens in the B-model in both of these cases.

Lemma 19. *If W, G, $\tilde{W}$, and $\tilde{G}$ are as described in Lemma 17, then*

$$\mathscr{B}_{W^T,G^T} \cong \mathscr{B}_{(\tilde{W})^T,(\tilde{G})^T}$$

as Frobenius algebras.

Proof. If $\tilde{W}$ and $\tilde{G}$ split as in Lemma 17, then it is easy to check that

$$\tilde{G}^T = \{(\theta_1,\ldots,\theta_n,0)| \quad (\theta_1,\ldots,\theta_n)\in G^T\}$$

So the map $\varphi : \mathscr{B}_{W^T,G^T} \to \mathscr{B}_{\tilde{W}^T,\tilde{G}^T}$ given by

$$\varphi\left(\lceil m\,;(\theta_1,\ldots,\theta_n)\rfloor\right) = \lceil m\,;(\theta_1,\ldots,\theta_n,0)\rfloor$$

is clearly an isomorphism of vector spaces.

Notice that if $\langle \lceil m\,;g\rfloor, \lceil n\,;h\rfloor\rangle = 0$ then $\langle\varphi(\lceil m\,;g\rfloor),\varphi(\lceil n\,;h\rfloor)\rangle$ will also vanish for the same reasons. On the other hand if $\langle \lceil m\,; g \rfloor\,, \lceil n\,; h \rfloor \rangle$ is not zero, then

$$\langle \varphi(\lceil m\,; g \rfloor), \varphi(\lceil n\,; h \rfloor) \rangle = \frac{mn\mu_{\varphi(g)}}{Hess(\tilde{W}_{\varphi(g)})} = \frac{mn\mu_g}{2Hess(W_g)} = \frac{1}{2}\langle \lceil m\,; g \rfloor\,, \lceil n\,; h \rfloor \rangle$$

Now we check that φ is a mulitplicative homomorphism. $I_g \cup I_h \cup I_{g+h} = [n]$ as in Equation 2.12, exactly whenever $I_{\varphi(g)} \cup I_{\varphi(h)} \cup I_{\varphi(g+h)} = [n+1]$. Also, $\mu_{\varphi(g)} = \mu_g$ and $2Hess(W_g) = Hess(\tilde{W}_{\varphi(g)})$ for any $g \in G^T$, so $\gamma_{g,h} = \gamma_{\varphi(g),\varphi(h)}$ for all g, h. This easily gives,

$$\varphi(\lceil m\,; g \rfloor \star_B \lceil n\,; h \rfloor) = \varphi(\lceil m\,; g \rfloor) \star_B \varphi(\lceil n\,; h \rfloor)$$

□

Lemma 20. *If W, G, $\tilde{W}$, and $\tilde{G}$ are as described in Lemma 18, then*

$$\mathscr{B}_{W^T,G^T} \cong \mathscr{B}_{(\tilde{W})^T,(\tilde{G})^T}$$

as Frobenius algebras.

Proof. Suppose G is generated by the group elements $g_1, \ldots, g_k$ for some k, then let's define the following two sets:

$$R = \{\mathbf{r} \in \mathbb{Z}^n | \mathbf{r} \cdot g^T \in \mathbb{Z} \quad \forall g \in G\}$$

$$S = \{\mathbf{s} \in \mathbb{Z}^n | \mathbf{s} \cdot g^T \in \frac{1}{2}\mathbb{Z} \quad \forall g \in G, \mathbf{s} \notin R\}$$

Then,

$$G^T = \{(A^T)^{-1}\mathbf{r} | \mathbf{r} \in R\}$$

For any $\mathbf{r} = (r_1, \ldots, r_n) \in R$, say that $\mathbf{r}_0 = (r_1, \ldots, r_n, 0, 0)$, and $\mathbf{r}_1 = (r_1, \ldots, r_n, 1, 1)$. Also, for any $\mathbf{s} \in S$, we say that $\mathbf{s}_0 = (s_1, \ldots, s_n, 0, 1)$ and $\mathbf{s}_1 = (s_1, \ldots, s_n, 1, 0)$. Then,

$$\tilde{G}^T = \{(\tilde{A}^T)\mathbf{r}_0\}_{\mathbf{r}\in R} \cup \{(\tilde{A}^T)\mathbf{r}_1\}_{\mathbf{r}\in R} \cup \{(\tilde{A}^T)\mathbf{s}_0\}_{\mathbf{s}\in S} \cup \{(\tilde{A}^T)\mathbf{s}_1\}_{\mathbf{s}\in S}$$

We define the map $\varphi : \mathscr{B}_{W^T,G^T} \to \mathscr{B}_{\tilde{W}^T,\tilde{G}^T}$ in the following way. If $\lceil \prod x_i^{m_i}\, ; g \rfloor \in \mathscr{B}_{W^T,G^T}$, then if

$$(\mathbf{m}+\mathbf{1})^g = (a_1, \dots, a_n)$$

with $a_i = m_i + 1$ whenever $i \in Fix(g)$ and $a_i = 0$ otherwise. Let $(\mathbf{m}+\mathbf{1})_0$ and $(\mathbf{m}+\mathbf{1})_1$ be defined in the same way as for $\mathbf{r}_0$ and $\mathbf{r}_1$, then for any $\mathbf{r} \in R$,

$$(A^T)^{-1}\mathbf{r} \cdot ((\mathbf{m}+\mathbf{1})^g)^T \in \mathbb{Z} \Rightarrow (\tilde{A}^T)^{-1}\mathbf{r}_i \cdot (\mathbf{m}+\mathbf{1})_j^T \in \mathbb{Z}$$

for $i = 0, 1$, and $j = 0, 1$. If, in addition $(A^T)^{-1}\mathbf{s} \cdot (\mathbf{m}+\mathbf{1}^g)^T \in \mathbb{Z}$ for all $\mathbf{s}$ in S, then we say

$$\varphi \left\lceil \prod x_i^{m_i}\, ; g \right\rfloor = \left\lceil \prod x_i^{m_i}\, ; \left(g_1, \dots, g_n, \frac{1}{2}, \frac{1}{2}\right) \right\rfloor \tag{3.1}$$

otherwise,

$$\varphi \left\lceil \prod x_i^{m_i}\, ; g \right\rfloor = \left\lceil \prod x_i^{m_i}\, ; (g_1, \dots, g_n, 0, 0) \right\rfloor. \tag{3.2}$$

This map is clearly well-defined and injective. To prove surjectivity, suppose that $\lceil \prod x_i^{m_i}\, ; \tilde{g} \rfloor \in \mathscr{B}_{\tilde{W}^T,\tilde{G}^T}$. Then $\tilde{g} = (g_1, \dots, g_n, 0, 0)$, $(g_1, \dots, g_n, \frac{1}{2}, \frac{1}{2})$, $(g_1, \dots, g_n, 0, 0)$, $(g_1, \dots, g_n, 0, \frac{1}{2})$, or $(g_1, \dots, g_n, \frac{1}{2}, 0)$. Notice that if $g = (g_1, \dots, g_n)$, then $\prod x_i^{m_i} \in \mathscr{Q}_g$. Now suppose that $\tilde{g} = (g_1, \dots, g_n, 0, \frac{1}{2})$, and $\hat{g} = (g_1, \dots, g_n, \frac{1}{2}, 0)$, then

$$\begin{aligned} \tilde{g} \cdot (\mathbf{m}+\mathbf{1})^{\tilde{g}} &= g \cdot (\mathbf{m}+\mathbf{1})^g, \text{ and} \\ \hat{g} \cdot (\mathbf{m}+\mathbf{1})^{\tilde{g}} &= g \cdot (\mathbf{m}+\mathbf{1})^g + \tfrac{1}{2} \end{aligned}$$

Which means $\prod x_i^{m_i}$ cannot be fixed by both $\tilde{g}$ and $\hat{g}$, and thus $\lceil \prod x_i^{m_i}\, ; \tilde{g} \rfloor \notin \mathscr{B}_{\tilde{W}^T,\tilde{G}^T}$. The same argument works for $\tilde{g} = (g_1, \dots, g_n, \frac{1}{2}, 0)$

Next, suppose that $\tilde{g} = (g_1, \dots, g_n, \frac{1}{2}, \frac{1}{2})$. If $\mathbf{s} \in S$, $h = (A^T)^{-1}\mathbf{s}^T$, and $\tilde{h} = (\tilde{A}^T)^{-1}\mathbf{s}_0$ then,

$$\tilde{h} \cdot (\mathbf{m}+\mathbf{1})^{\tilde{g}} = h \cdot (\mathbf{m}+\mathbf{1})^g \in \mathbb{Z} \tag{3.3}$$

and if $\hat{g} = (g_1, \ldots, g_n, 0, 0)$,

$$\tilde{h} \cdot (\mathbf{m}+\mathbf{1})^{\hat{g}} = h \cdot (\mathbf{m}+\mathbf{1})^{g} + \frac{1}{2} \notin \mathbb{Z}$$

So, $\lceil \prod x_1^{m_i} \,; \hat{g} \rfloor \notin \mathscr{B}_{\tilde{W}^T, \tilde{G}^T}$. Notice that Equation 3.3 implies that $\lceil \prod x_i^{m_i} \,; \tilde{g} \rfloor$ is the image of some element under the part of the map shown in 3.1 A similar argument shows that if $\lceil \prod x_i^{m_i} \,; \hat{g} \rfloor \in \mathscr{B}_{\tilde{W}^T, \tilde{G}^T}$, then it is the image of some element under the part of the map shown in 3.2, and that $\lceil \prod x_i^{m_i} \,; \tilde{g} \rfloor \notin \mathscr{B}_{\tilde{W}^T, \tilde{G}^T}$.

Now we need to prove that φ respects the product on both of the B-models in question. If $\lceil \prod x_i^{m_i} \,; \tilde{g} \rfloor, \left\lceil \prod x_i^{m_i} \,; \tilde{h} \right\rfloor \in \mathscr{B}_{\tilde{W}^T, \tilde{G}^T}$ correspond to $\lceil \prod x_i^{m_i} \,; g \rfloor, \lceil \prod x_i^{m_i} \,; h \rfloor \in \mathscr{B}_{W^T, G^T}$, then the condition

$$I_g \cup I_h \cup I_{g+h} = [n]$$

as in Equation 2.12 is satisfied exactly when

$$I_{\tilde{g}} \cup I_{\tilde{h}} \cup I_{\tilde{g}+\tilde{h}} = [n+2].$$

There are three interesting cases.

1. $\tilde{g} = (g_1, \ldots, g_n, 0, 0)$ and $\tilde{h} = (h_1, \ldots, h_n, 0, 0)$.

   In this case, $I_{\tilde{g} \cap \tilde{h}} = I_{g \cap h} \cup \{n+1, n+2\}$, and $I_{\tilde{g}+\tilde{h}} = I_{g+h} \cup \{n+1, n+2\}$, which means that $\gamma_{\tilde{g},\tilde{h}} = \gamma_{g,h}$, and so,

$$\varphi\left(\left\lceil \prod x_i^{m_i} \,; g \right\rfloor \star_B \left\lceil \prod x_i^{m_i} \,; h \right\rfloor\right) = \left\lceil \prod x_i^{m_i} \,; \tilde{g} \right\rfloor \star_B \left\lceil \prod x_i^{m_i} \,; \tilde{h} \right\rfloor$$

2. $\tilde{g} = (g_1, \ldots, g_n, \frac{1}{2}, \frac{1}{2})$ and $\tilde{h} = (h_1, \ldots, h_n, 0, 0)$.

   In this case, $I_{\tilde{g} \cap \tilde{h}} = I_{g \cap h}$, and $I_{\tilde{g}+\tilde{h}} = I_{g+h}$, which means that $\gamma_{\tilde{g},\tilde{h}} = \gamma_{g,h}$, and so,

$$\varphi\left(\left\lceil \prod x_i^{m_i} \,; g \right\rfloor \star_B \left\lceil \prod x_i^{m_i} \,; h \right\rfloor\right) = \left\lceil \prod x_i^{m_i} \,; \tilde{g} \right\rfloor \star_B \left\lceil \prod x_i^{m_i} \,; \tilde{h} \right\rfloor$$

3. $\tilde{g} = (g_1, \ldots, g_n, \frac{1}{2}, \frac{1}{2})$ and $\tilde{h} = (h_1, \ldots, h_n, \frac{1}{2}, \frac{1}{2})$.

In this case, $I_{\tilde{g}\cap\tilde{h}} = I_{g\cap h}$, and $I_{\tilde{g}+\tilde{h}} = I_{g+h} \cup \{n+1, n+2\}$, which means that $Hess(\tilde{W}_{\tilde{g}+\tilde{h}}) = 4Hess(W_{g+h})$, and so $\gamma_{\tilde{g},\tilde{h}} = 4\gamma_{g,h}$, and so,

$$4\varphi\left(\left\lceil \prod x_i^{m_i}; g\right\rfloor \star_B \left\lceil \prod x_i^{m_i}; h\right\rfloor\right) = \left\lceil \prod x_i^{m_i}; \tilde{g}\right\rfloor \star_B \left\lceil \prod x_i^{m_i}; \tilde{h}\right\rfloor$$

□

Chapter 4. New Computational Methods

In this chapter we give results for computing concave genus-zero correlators and then discuss how to use the reconstruction lemma to find values of other correlators.

4.1 Using the Concavity Axiom

In this section we use Chiodo's formula to give a formula for Λ as a polynomial in the tautological classes ψ_i, κ_a, and Δ_I in $H^*(\overline{\mathscr{M}}_{g,k})$. To do this we will first review some material from K-theory, discuss certain classes in the cohomology of $\overline{\mathscr{M}}_{g,k}$ and $\mathscr{W}_{g,k}$, define $Sing'$, a double cover of the the space of singular points in $\overline{\mathscr{M}}_{g,k}$. Finally, we give a formula for computing concave genus-zero four-point correlators.

Recall from Equation 2.11 that for a given genus-g, n-point correlator, when the hypotheses of the concavity axiom are satisfied,

$$\Lambda^W_{g,k}(\alpha_1, \ldots, \alpha_k) = \frac{|G|^g}{\deg(st)} PDst_*\left(PD^{-1}\left((-1)^D c_D\left(R^1\pi_*\bigoplus_{i=1}^n \mathcal{L}_i\right)\right)\right)$$

Also recall that integration of bottom dimensional cohomology classes is the same as pushing them forward to a point, so if we have a map:

$$\overline{\mathscr{M}}_{g,k} \xrightarrow{\quad p \quad} \{\bullet\}$$

Then,

$$\langle \alpha_1, \ldots, \alpha_k \rangle = p_* \Lambda(\alpha_1, \ldots, \alpha_k)$$

This pushforward map is actually the same as counting self intersection. In 1999 Carel Faber [16] wrote some code in Maple which computes intersection numbers for polynomials in the appropriate cohomology classes. In 2011 Drew Johnson [18] wrote similar code in Sage, which is faster and able to calculate more things. If we can express $(-1)^D c_D R^1 \pi_* (\mathcal{L}_1 \oplus \ldots \oplus \mathcal{L}_N)$ as a polynomial f in terms of pullbacks of ψ, κ, and Δ_I classes, then, we will have

$$\begin{aligned} \langle \alpha_1, \ldots, \alpha_k \rangle &= p_* \tfrac{|G|^g}{\deg(st)} PD st_* \left(PD^{-1} \left(st^* f(\kappa_1, \ldots, \kappa_D, \psi_1, \ldots, \psi_k, \{\Delta_I\}_{I \in \mathscr{I}}) \right) \right) \\ &= |G|^g p_* \left(f(\kappa_1, \ldots, \kappa_D, \psi_1, \ldots, \psi_k, \{\Delta_I\}_{I \in \mathscr{I}}) \right), \end{aligned}$$

and we will be able to use intersection theory to solve for these numbers.

To find the polynomial q, we begin by reviewing some material from K-theory and some properties of Chern classes.

We will use K-theory to find some important cohomology classes on the moduli space of curves. The ring $K(X)$ of a smooth variety or smooth algebraic stack is defined to be the free Abelian group generated by vector bundles on X modulo the relation $E = E' \oplus E''$ if there is an exact sequence of vector bundles,

$$0 \to E' \to E \to E'' \to 0.$$

The ring structure is given by

$$[\mathscr{E}] \cdot [\mathscr{E}'] := [\mathscr{E} \otimes \mathscr{E}'].$$

We will be working in $K \otimes_{\mathbb{Z}} \mathbb{Q}$.

4.1.1 Chern Classes.

Given a locally free sheaf $\mathscr{E}$ of rank r on a nonsingular projective variety X, then for each $i = 0, \ldots, r$, there is an i**th Chern class**, $c_i(\mathscr{E}) \in H^i(X)$. The **total Chern class** is given by

$$c(\mathscr{E}) = c_0(\mathscr{E}) + c_1(\mathscr{E}) + \ldots + c_r(\mathscr{E}).$$

The **Chern polynomial** is given by

$$c_t(\mathscr{E}) = c_0(\mathscr{E}) + c_1(\mathscr{E})t + \ldots + c_r(\mathscr{E})t^r.$$

We can think of c_i as a map from $K(X)$ to $H^i(X)$, and $c_t : K(X) \to \bigoplus_i H^i(X)t^i \subseteq H^*[[t]]$.

Chern classes satisfy the following properties:

Property 3. *1. If $f : X' \to X$ is a morphism and $\mathscr{E}$ is a locally free sheaf on X, then*

$$c_i(f^*\mathscr{E}) = f^*c_i(\mathscr{E})$$

2. $c_t(\mathscr{E}) \in \Lambda^\circ$ where

$$\Lambda^\circ = \{1 + a_1 t + a_2 t^2 + \ldots | a_i \in H^i(X)\}$$

3. $c_t\left(\bigoplus_i \mathscr{E}_i\right) = \prod_i c_t\left(\mathscr{E}_i\right)$, *so* $c_i\left(\bigoplus_j \mathscr{E}_j\right) = \sum_{\substack{\sum i_j = i \\ 0 \le i_j \le i}} \prod_j c_{i_j}(\mathscr{E}_j)$

4. $c_t(\mathscr{E}) = \dfrac{1}{c_t(-\mathscr{E})} = \dfrac{1}{1-(-c_t(-\mathscr{E}))} = \sum_{i=1}^{\infty}(-c_t(-\mathscr{E}))^i$

Also, recall that a vector bundle $\mathbb{E}$ is concave when $R^0\pi_*\mathbb{E} = 0$, so we have

$$R^\bullet\pi_*\mathbb{E} = R^0\pi_*\mathbb{E} - R^1\pi_*\mathbb{E} = -R^1\pi_*\mathbb{E}$$

Property 3 tells us that

$$c_t(R^1\pi_* \bigoplus_i \mathscr{L}_i) = \sum_{j=0}^{\infty}(-c_t(-R^1\pi_* \bigoplus_i \mathscr{L}_i))^j$$

So, if $b_k = c_k(R^\bullet\pi_* \bigoplus_i \mathscr{L}_i)$, then

$$c_D(R^1\pi_* \bigoplus_i \mathscr{L}_i) = f_1(b_1, \ldots, b_D)$$

for some polynomial f_1.

Property 3 also tells us that

$$\begin{aligned} c_k(R^\bullet\pi_* \bigoplus_i \mathscr{L}_i) &= \sum_{\sum i_j = k}\left(\prod_{j=1}^{n} c_{i_j}(R^\bullet\pi_*\mathscr{L}_j)\right) \\ &= f_2^k(d_{0,1}, \ldots, d_{D,1}, d_{0,2}, \ldots, d_{D,2}, \ldots, d_{0,n}, \ldots, d_{D,n}) \end{aligned}$$

if $d_{i,j} = c_i(R^\bullet\pi_*\mathscr{L}_j)$, for some polynomial $f_2^i \in \mathbb{C}[c_0(R^\bullet\pi_*\mathscr{L}_k), \ldots, c_D(R^\bullet\pi_*\mathscr{L}_k)]$. In other words, it is possible to find a polynomial f_2^i which will give the ith Chern class for the sum $R^\bullet\pi_* \bigoplus_i \mathscr{L}_i$ given the individual Chern classes of each $R^\bullet\pi_*\mathscr{L}_i$.

Also,

$$\begin{aligned} c_t(R^\bullet\pi_*\mathscr{L}_k) &= exp\left(\sum_{i=1}^{\infty}(i-1)!(-1)^{i-1}ch_i(R^\bullet\pi_*\mathscr{L}_k)t^i\right) \\ &\sum_{j=0}^{\infty}\frac{1}{j!}\left(\sum_{i=1}^{\infty}(i-1)!(-1)^{i-1}ch_i(R^\bullet\pi_*\mathscr{L}_k)t^i\right)^j \end{aligned}$$

So,

$$c_j(R^\bullet\pi_*\mathscr{L}_k) = f_3^{j,k}(ch_1(R^\bullet\pi_*\mathscr{L}_k), \ldots, ch_D(R^\bullet\pi_*\mathscr{L}_k))$$

for some polynomial $f_3^{j,k}$. So, $f_3^{j,k}$ is the polynomial which gives the $j$the Chern class of $R^\bullet\pi_*\mathscr{L}_k$.

If we now say that $f_2 = (f_2^1, \ldots, f_2^D)$, and

$$f_3 = (f_3^{0,1}, \ldots, f_3^{D,1}, \ldots, f_3^{0,2}, \ldots, f_3^{D,2}, f_3^{0,n}, \ldots, f_3^{D,n})$$

are tuples of polynomials, then

$$\begin{aligned} c_D(R^1\pi_* \textstyle\bigoplus_i \mathscr{L}_i) = \quad & f_1 \circ f_2 \circ f_3(ch_1(R^\bullet\pi_*\mathscr{L}_k), \ldots, ch_D(R^\bullet\pi_*\mathscr{L}_k)) \\ & f(ch_1(R^\bullet\pi_*\mathscr{L}_k), \ldots, ch_D(R^\bullet\pi_*\mathscr{L}_k)) \end{aligned} \tag{4.1}$$

In Section 4.1.3 we will see that Chiodo in [15] provides a formula for these $ch_i(R^\bullet\pi_*\mathscr{L}_k)$.

4.1.2 Some Special Cohomology Classes in $\overline{\mathscr{M}}_{g,k}$ and $\mathscr{W}_{g,k}$.

Definition 17. *For $i \in \{1, \ldots, k\}$, $\psi_i \in H^q(\overline{\mathscr{M}}_{g,k})$ is the first Chern class of the line bundle whose fiber at $(C, p_1, \ldots, p_k)$ is the cotangent space to C at p_i.*

In other words, if $\pi : \overline{\mathscr{M}}_{g,k+1} = \mathfrak{C} \to \overline{\mathscr{M}}_{g,k}$is the universal curve, and it is also the morphism obtained by forgetting the $(k+1)$-st marked point, $\omega_{\pi_{k+1}}$ is the relative dualizing sheaf, and σ_i is the section of π_{k+1} which attaches a genus-zero, three-pointed curve to C at the point p_i, and then labels the two remaining marked points on the genus-zero curve i and $k+1$,

$$\begin{array}{c} \overline{\mathscr{M}}_{g,k+1} \\ \sigma_i \uparrow \; \downarrow \pi_{k+1} \\ \overline{\mathscr{M}}_{g,k} \end{array}$$

then, $\mathbb{L} = \sigma^*(\omega_{\pi_{k+1}})$ is the *cotangent line bundle* and its first Chern class is ψ_i:

$$\psi_i = c_1(\sigma^*(\omega_{\pi_{k+1}})),$$

Definition 18. *Each partition $I \sqcup J = \{1, \ldots, k\}$ and $g_1 + g_2 = g$ of marks and genus such that $1 \in I$, $2g_1 - 2 + |I| + 1 > 0$ and $2g_2 - 2 + |J| + 1 > 0$, gives an irreducible boundary divisor, which we label $\Delta_{g_1,I}$.*

These boundary divisors are the nodal curves in $\overline{\mathscr{M}}_{g,k}$. For example, a boundary divisor in $\overline{\mathscr{M}}_{1,5}$ and its dual graph are given below.

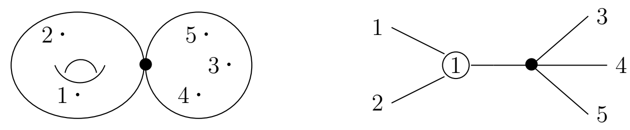


Figure 4.1: A boundary divisor in $\overline{\mathscr{M}}_{1,5}$ and its dual graph

We will use the following well-known lemma for $\overline{\mathscr{M}}_{0,k}$ for expressing ψ classes in terms of boundary divisors.

Lemma 21.

$$\psi_i = \sum_{\substack{a \in I \\ b,c \notin I}} \Delta_I$$

Definition 19. *Let $D_{i,k+1}$ be the image of σ_i in $\overline{\mathscr{M}}_{g,k+1}$, then we define*

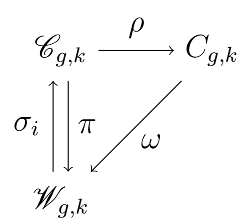
$$K = c_1 \left(\omega_{k+1} \left(\sum_{i=1}^{k} D_{i,k+1} \right) \right).$$

For $a \in \{1, \ldots, 3g-3+k\}$,

$$\kappa_a = \pi_*(K^{a+1})$$

Now we consider some cohomology classes on $\mathscr{W}_{g,k}$.

Consider the diagram:



where $\mathfrak{C}_{g,k}$ is the universal curve over $\mathscr{W}_{g,k}$ and $\bar{\rho}$ is the map which forgets the orbifold structure.

Definition 20. *Let $\tilde{\psi}_i$ be the first Chern class of the $\mathscr{C}$-cotangent line bundle on $\mathscr{W}_{g,k}$.*

$$\tilde{\psi}_i = c_1 \left(\sigma_i^*(K_{\mathscr{C}}) \right)$$

If $\bar{\psi}$ is the first Chern class of the C-cotangent line bundle on $\mathscr{W}_{g,k}$ gives the pullback of the usual ψ-class (on $\overline{\mathscr{M}}_{g,k}$).

$$\bar{\psi}_i = c_1(\sigma_i^*(K_C)) = st^*(\psi_i)$$

We define κ-classes on $\mathscr{W}_{g,k}$ in a similar way:

Definition 21.

$$\tilde{\kappa}_a = \pi_*(c_1(K_{\mathscr{C},log})^{a+1}) = \omega_*\rho_*\rho^*(c_1(K_{C,log})^{a+1}) = \kappa_a,$$

the usual κ_a on $\overline{\mathscr{M}}_{g,k}$.

Definition 22. *We define the boundary divisors $\tilde{\Delta}_I$ for $\mathscr{W}_{g,k}$ in the same way that we did for $\overline{\mathscr{M}}_{g,k}$. They correspond to points in $\mathscr{W}_{g,k}$ with singular curves over them.*

Note that there is an orbifold action attached to the node in each nodal curve, so the map st is ramified of order $\frac{1}{r}$ at the boundary points. This means that

$$st^*\Delta_I = r\tilde{\Delta}_I \tag{4.2}$$

Now, let's take a closer look at the singular curves in $\overline{\mathscr{M}}_{g,k}$. We have the universal curve over $\overline{\mathscr{M}}_{g,k}$,

$$\begin{array}{c} \mathscr{C}_{g,n} \\ \downarrow \pi \\ \overline{\mathscr{M}}_{g,n} \end{array}$$

We say that $Sing$ is the subspace of points in $\overline{\mathscr{M}}_{g,n}$ with singular curves over them. For example, for a point in $Sing$, the curve and dual graph in $\mathscr{C}$ and $\overline{\mathscr{M}}_{g,k}$ might look like:

It turns out that $Sing$ has a natural double cover obtained by choosing one side of the node to be '+' and the other to be '−' which does yield some natural line bundles. We say that $Sing' = Sing_+ \cup Sing_-$. There are natural line bundles on both $Sing_+$ and

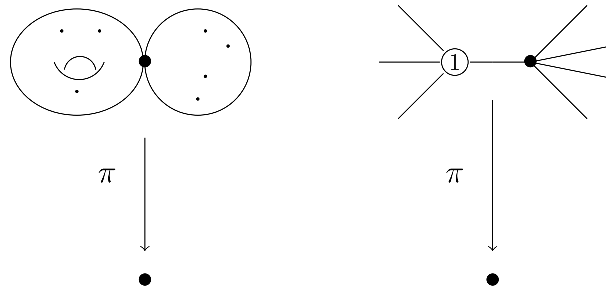


Figure 4.2: The curve and dual graph represented by a point in *Sing*

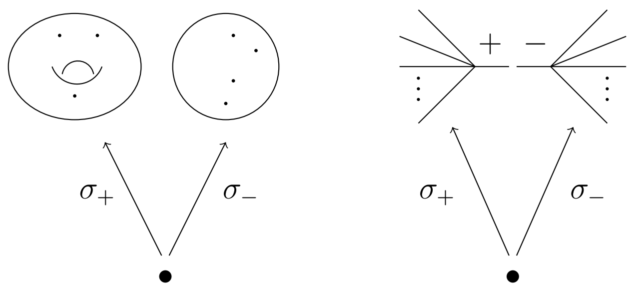


Figure 4.3: An example of *Sing*′ the double cover of *Sing*

$Sing_-$ and hence on $Sing'$. We define $\mathscr{L}_+ = (\sigma_+)^*\omega$, $\mathscr{L}_- = (\sigma_-)^*\omega$, $\psi_+ = c_1(\mathscr{L}_+)$, and $\psi_- = c_1(\mathscr{L}_-)$. The last two are just the ψ classes corresponding the marked points $+$ and $-$ on the corresponding side of the node.

Now we are ready to examine Chiodo's formula.

4.1.3 Chiodo's Formula.

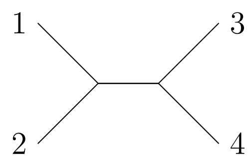
The following theorem is from Chiodo [15].

Theorem 1. *Let $s, m_1, \ldots, m_n$ be integers satisfying $(2g-2+n)s - \sum_i m_i \in r\mathbb{Z}$. Let $\mathcal{S}$ be the universal rth rooth of $(\omega^{\log})^{\otimes s}(-\sum_{i=1}^n m_i[x_i])$ on the universal r-stable curve $\mathcal{C}$ over the moduli stack $\overline{\mathscr{M}}^r_{g,n}$. The direct image $R^\bullet\pi_*\mathcal{S}$ via the universal curve $\pi : \mathcal{C} \to \bar{\mathcal{M}}^r_{g,n}$ satisfies the equation*

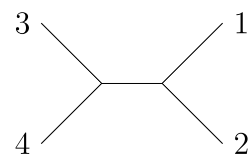
$$ch(R^\bullet\pi_*\mathcal{S}) = \sum_{d\geq 0}\left(\frac{B_{d+1}(s/r)}{(d+1)!}\kappa_d - \sum_{i=1}^n \frac{B_{d+1}(m_i/r)}{(d+1)!}\psi_i^d + \frac{1}{2}\sum_{q=0}^{r-1}\frac{rB_{d+1}(q/r)}{(d+1)!}(j_q)_*(\gamma_{d-1})\right)t^d$$

where the cycle γ_d in $A^d(Sing'_q)$ is $\sum_{i+j=d}(-\psi_+)^i(\psi_-)^j$ and we set $\gamma_d = 0$ when d is negative, *where $ch(R^1\pi_*\mathcal{S})$ is the total chern character of $R^\bullet\pi_*\mathcal{S}$, and where B_{d+1} is the $d+1$st Bernoulli polynomial.*

For the last sum in the formula, in our computations we will choose for to sum over the combinatorial ways to split a k-pointed curve into a nodal curve, instead of summing over q, the group element attached to a particular side of the node. That is, we will sum over subsets $K \subset \{1, \ldots, k\}$, where the marks in K will be the points on the '+' curve. Notice that in this way the curves with dual graphs



and



will each be counted, once for $K = \{1,2\}$ and once for $K = \{3,4\}$, even though they represent identical curves. This will always happen in genus zero. We adopt the convention of denoting each boundary graph by the set of marks on whichever side contains the mark 1. (The curve pictured above will be $\Delta_{1,2}$), Limiting our choices for K to those that contain a 1, we will count each curve only once, and will not need to multiply by $\frac{1}{2}$.

Also, recall from Equation 4.2 that because of the ramification of the map st at the boundary points, $\tilde{\Delta}_I = \frac{1}{r} st^*\Delta_I$. So,

$$ch_t(R^\bullet\pi_*\mathcal{S}) = st^*\left(\sum_{d\geq 0}\left(\frac{B_{d+1}(s/r)}{(d+1)!}\kappa_d - \sum_{i=1}^{n}\frac{B_{d+1}(m_i/r)}{(d+1)!}\psi_i^d + \sum_K \frac{B_{d+1}(\gamma_+^K)}{(d+1)!}(j_K)_*(\gamma_{d-1})\right)t^d\right)$$

Recall from Lemma 21 that in genus zero we have the relation

$$\psi_i = \sum_{\substack{i\in I,\\ a,b\notin I}} \Delta_I.$$

We can use this to rewrite ψ_+ and ψ_- in terms of boundary divisors of $\overline{\mathscr{M}}_{0,n_+}$ and $\overline{\mathscr{M}}_{0,n_-}$. This will enable us to easily push down these classes to $\overline{\mathscr{M}}_{0,n}$. This idea comes from [22], and yields the following formulas:

$$\begin{aligned}
&(j_K)_*(\psi_+) = 0 && \text{if } |K| \leq 2 \\
&(j_K)_*(\psi_+) = \textstyle\sum_{\{1,a,b\}\subseteq I\subseteq K} \Delta_K\Delta_I + \sum_{1\in I\subseteq K-\{a,b\}} \Delta_K\Delta_{I\cup K^c} && \text{if } |K| > 2 \\
&(j_K)_*(\psi_+) = 0 && \text{if } n-|K| \leq 2 \\
&(j_K)_*(\psi_+) = \textstyle\sum_{\substack{\emptyset\neq I\subset K^c \\ a,b,\notin I}} \Delta_K\Delta_{I\cup K} && \text{if } n-|K| > 2
\end{aligned} \tag{4.3}$$

Using the formulas in Equation 4.3 and the polynomial defined in Equation 4.1 we can know express Λ as a polynomial in ψ, κ and Δ classes,

$$\begin{aligned}
&\Lambda^W_{g,k}(\alpha_1,\ldots,\alpha_k) \\
&\quad = (-1)^D f\left(\left(\tfrac{B_2(s/r)}{(2)!}\kappa_1 - \textstyle\sum_{i=1}^n \tfrac{B_2(m_i/r)}{(2)!}\psi_i + \sum_K \tfrac{B_2(\gamma_+^K)}{(2)!}(j_K)_*(\psi_- - \psi_+)\right),\ldots\right. \\
&\left.\left(\tfrac{B_{D+1}(s/r)}{(D+1)!}\kappa_D - \quad \textstyle\sum_{i=1}^n \tfrac{B_{D+1}(m_i/r)}{(D+1)!}\psi_i^D + \sum_K \tfrac{B_{D+1}(\gamma_+^K)}{(D+1)!}(j_K)_*(\sum_{i+j=D-1}(-\psi_+)^i\psi_-^j)\right)\right)
\end{aligned} \tag{4.4}$$

The following lemma allows us to always choose a γ_+^K in a way that makes sense in Chiodo's formula.

Lemma 22. *If B is a degree one boundary graph with two nodes, with decoration $\boldsymbol{\gamma}_1$ for the first node and $\boldsymbol{\gamma}_2$ for the second, and genera g_1 and g_2, respectively, and if Γ is the dual graph of a smooth curve with decoration $\boldsymbol{\gamma}_1+\boldsymbol{\gamma}_2$ and genus g_1+g_2 has integer line bundle degree, then it is possible to choose a group element decoration, γ_0, for the first half edge so that the line bundle degree of the first node is integral. Moreover, the group element $-\gamma_0$, when assigned to the other half edge will force the line bundle degree of the second node to be integral.*

Proof. If the line bundle degree of the degree zero graph is integral then:

$$S_0 = J(2(g_1+g_2) - 2 + |\boldsymbol{\gamma}_1| + |\boldsymbol{\gamma}_2|) - \sum_i \gamma_1^i - \sum_j \gamma_2^j \in \mathbb{Z}^n$$

Similarly let S_1 and S_2 be the equivalent sums in $\mathbb{Q}^n$, corresponding to the nodes Γ_1 and Γ_2, respectively. To find γ_0 we take

$$\gamma_0 \equiv J(2g_1 - 2 + (|\boldsymbol{\gamma}_1| + 1)) - \sum_i \gamma_1^i$$

Then, by Lemma 7 this will force the S_1 to be an integer vector.

Also,

$$S_0 = S_1 + \gamma_0 + J(g_2 - 2 + (|\boldsymbol{\gamma}_2| + 1)) - \sum_j \gamma_2^j = S_1 + S_2$$

Which implies that $S_2 \in \mathbb{Z}$. □

We can now give a formula for concave genus-zero four-point correlators.

Lemma 23. *If $\langle \lceil 1\,; g_1 \rfloor, \lceil 1\,; g_2 \rfloor, \lceil 1\,; g_3 \rfloor, \lceil 1\,; g_4 \rfloor \rangle$ is a genus-zero, four-point correlator which satisfies Axioms 2 and 3, and if it is also satisfies the hypotheses of the concavity axiom, (all insertions are narrow, all line bundle degrees are negative, and all line bundle degrees of the nodes of the boundary graphs $\Delta_{1,2}$, $\Delta_{1,3}$, and $\Delta_{1,4}$ are all negative), then*

$$\langle \lceil 1\,; g_1 \rfloor, \lceil 1\,; g_2 \rfloor, \lceil 1\,; g_3 \rfloor, \lceil 1\,; g_4 \rfloor \rangle = \frac{1}{2} \sum_{i=1}^{N} \left(B_2(q_i) - \sum_{j=1}^{4} B_2((g_j)_i) + \sum_{k=1}^{3} B_2(\gamma_+^j) \right)$$

$$= \frac{1}{2} \sum_{i=1}^{N} \left(q_i(q_i - 1) - \sum_{j=1}^{4} \theta_j^i(\theta_j^i - 1) + \sum_{k=1}^{3} \gamma_+^j(\gamma_+^j - 1) \right)$$

where for each j $g_j = (\theta_j^1, \ldots, \theta_j^n)$, $\gamma_+^1 \equiv J - g_1 - g_2$, $\gamma_+^2 \equiv J - g_1 - g_3$, and $\gamma_+^3 \equiv J - g_1 - g_4$, and B_2 is the 2nd Bernoulli polynomial.

Proof.

$$ch_t(-R^1\pi_*\mathcal{L}_i) = \sum_{d \geq 0} \left(\frac{B_{d+1}(q_i)}{(d+1)!}\kappa_d \quad - \sum_{j=1}^{4} \frac{B_{d+1}(\theta_j^i)}{(d+1)!}\psi_j^d \right.$$
$$\left. + r \sum_K \frac{B_{d+1}((\gamma_+^K)_i)}{(d+1)!}(p_K)_* \sum_{k=0}^{d-1} (-\psi_+)^k (\psi_-)^{d-1-k} \right) t^d$$

Which means that

$$ch_1(-R^1\pi_*\mathcal{L}_i) = \frac{B_2(q_i)}{(2)!}\kappa_1 - \sum_{j=1}^{4}\frac{B_2(\theta_j^i)}{(2)!}\psi_j + r\sum_K \frac{B_2((\gamma_+)_K)}{(2)!}(p_K)_*(1)$$

Notice that $(p_K)_*(1_{\overline{\mathscr{M}}_{0,3}}) = \Delta_K$, and that for $\overline{\mathscr{M}}_{0,4}$, our choices for $K \in \Gamma_{cut}$ are just $\{1,2\}$, $\{1,3\}$, and $\{1,4\}$. Numbering these gives:

$$\begin{aligned}(\gamma_+)_1 &= J - g_1 - g_2, \quad \text{for } K = \{1,2\}\\ (\gamma_+)_2 &= J - g_1 - g_3, \quad \text{for } K = \{1,3\}\\ (\gamma_+)_3 &= J - g_1 - g_4, \quad \text{for } K = \{1,4\}\end{aligned}$$

Since $B_2(x) = x^2 - x + \frac{1}{6}$,

$$\begin{aligned}ch_1(-R^1\pi_*\mathcal{L}_i) &= \frac{1}{2}\Bigg(\left(q_i(q_i-1)+\frac{1}{6}\right)\kappa_1\\ &-\sum_{j=1}^{4}\left(\theta_j^i(\theta_j^i-1)+\frac{1}{6}\right)\psi_j\\ &+r\sum_K\left(\gamma_+^K(\gamma_+^K-1)+\frac{1}{6}\right)\tilde{\Delta_K}\Bigg)\end{aligned}$$

The psi and kappa classes in $\mathscr{W}_{0,4}$ are all pullbacks of the equivalent psi and kappa classes in $\overline{\mathscr{M}}_{,0,4}$, and the Δ_I classes are scalar multiples of the equivalent classes in $\overline{\mathscr{M}}_{,0,4}$,

$$\psi_i = st^*\psi_i, \quad \kappa_i = st^*\kappa_i, \quad \tilde{\Delta_I} = \frac{1}{r}st^*\Delta_I$$

So,

$$ch_1(-R^1\pi_*\mathcal{L}_i) = \frac{1}{2}st^*\Big(\big(q_i(q_i-1)\big)\kappa_1 - \sum_{j=1}^{4}\big(\theta_j^i(\theta_j^i-1)\big)\psi_j + \sum_K\big(\gamma_+^K(\gamma_+^K-1)\big)\Delta_K\Big)$$

Converting to chern classes, we get

$$\begin{aligned} c_t(-R^1\pi_*\mathcal{L}_i) &= exp\left(\sum_{i=1}^{\infty}(-1)^{i-1}(i-1)!ch_i(-R^1\pi_*\mathcal{L}_i)t^i\right) \\ &= \sum_{j=0}^{\infty}\frac{1}{j!}\left(\sum_{i=1}^{\infty}(-1)^{i-1}(i-1)!ch_t(-R^1\pi_*\mathcal{L}_i)t_i\right)^j \end{aligned}$$

Which means that $c_0(-R^1\pi_*\mathcal{L}_i) = 1$ and $c_1(-R^1\pi_*\mathcal{L}_i) = ch_1(-R^1\pi_*\mathcal{L}_i)$, then, by Property 3 since $D = 1$,

$$\begin{aligned} c_1(-R^1\pi_* \oplus_i \mathcal{L}_i) &= \sum_{\substack{0\leq j_1,\dots j_N \\ j_1+\dots+j_N=1}} \prod c_{j_i}(-R^1\pi_*\mathcal{L}_i) \\ &= \sum_{i=1}^{N} c_1(-R^1\pi_*\mathcal{L}_i) = \sum_{i=1}^{N} ch_1(-R^1\pi_*\mathcal{L}_i) \end{aligned}$$

Finally we recall that $c_t(R^1\pi_*\mathcal{L}_i) = -\sum_j (c_t(-R^1\pi_*\mathcal{L}_i))^j$, so,

$$c_1(R^1\pi_*\mathcal{L}_i) = -c_1(-R^1\pi_*\mathcal{L}_i) = -\sum_{i=1}^{N} ch_1(-R^1\pi_*\mathcal{L}_i)$$

And, from Equation 2.11

$$\begin{aligned} \Lambda_{0,4}(\alpha_1,\dots,\alpha_4) &= \frac{1}{deg(st)} PDst_*PD^{-1}(-1)^1(-\sum_{i=1}^{N} ch_1(-R^1\pi_*\mathcal{L}_i)) \\ &= \frac{1}{deg(st)} PDst_*PD^{-1}\sum_{i=1}^{N} ch_1(-R^1\pi_*\mathcal{L}_i)) \\ = \frac{1}{2}\Bigg(\left(q_i(q_i-1)+\frac{1}{6}\right)\kappa_1 &- \sum_{j=1}^{4}\left(\theta_j^i(\theta_j^i-1)+\frac{1}{6}\right)\psi_j + r\sum_K\left(\gamma_+^K(\gamma_+^K-1)+\frac{1}{6}\right)\tilde{\Delta_K}\Bigg) \end{aligned}$$

Next, we notice that if $p : \overline{\mathscr{M}}_{0,4} \to (\bullet)$ is the map sending all of $\overline{\mathscr{M}}_{0,4}$ to a point, then the push forward of any of the cohomology classes mentioned above is equal to 1. That is,

$$p_*\kappa_1 = p_*\psi_i = p_*\Delta_K = 1$$

So,

$$
\begin{aligned}
&\langle \lceil 1\,;g_1\rfloor , \lceil 1\,;g_2\rfloor , \lceil 1\,;g_3\rfloor , \lceil 1\,;g_4\rfloor \rangle \\
&\qquad\qquad = p_* \frac{1}{deg(st)} PDst_* PD^{-1} \sum_{i=1}^{N} ch_1(-R^1\pi_*\mathcal{L}_i)) \\
&= \frac{1}{2} p_* \sum_{i=1}^{N} \left(\left(q_i(q_i-1)\right)\kappa_1 - \sum_{j=1}^{4} \left(\theta_j^i(\theta_j^i-1)\right)\psi_j + \sum_{K} \left(\gamma_+^K(\gamma_+^K-1)\right)\Delta_K \right) \\
&= \frac{1}{2} \sum_{i=1}^{N} \left(q_i(q_i-1) + \gamma_+^1(\gamma_+^1-1) + \gamma_+^2(\gamma_+^2-1) + \gamma_+^3(\gamma_+^3-1) - \sum_{j=1}^{4} \theta_j^i(\theta_j^i-1) \right)
\end{aligned}
$$

□

4.2 Using the Reconstruction Lemma

In this section we show how to use known correlator values to find unknown correlator values. In some cases our new methods for computing concave correlators will allow us to compute all genus-zero correlators in the A model.

Recall that if $\mathscr{B}$ is a basis for a vector space V with a non degenerate pairing $\langle \bullet, \bullet \rangle$, then for each basis element $b \in \mathscr{B}$, it is possible to find a corresponding element b' in V such that

$$\langle b, b' \rangle = 1$$

We call the basis made up of these elements the **dual basis**.

The WDVV equations are a powerful tool which can be derived from the Composition axiom. Applying these equations to correlators, we get the following useful equation,

$$\langle \gamma_1, \ldots, \gamma_k \rangle \langle \alpha_1, \ldots, \alpha_l \rangle = \langle \{\gamma_i\}_{i \in I}, \{\alpha_j\}_{j \in J} \rangle \langle \{\gamma_i\}_{iinI^c}, \{\alpha_j\}_{j \in J^c} \rangle,$$

for any sets $I \subseteq [k]$, and $J \subseteq [l]$. We can know state and prove the reconstruction lemma, from [1].

Lemma 24. ***Reconstruction Lemma*** *Any genus-zero, k-point correlator of the form*

$$\langle \gamma_1, ..., \gamma_{k-3}, \alpha, \beta, \epsilon \star \phi \rangle_{0,k}$$

where $0 < \deg_W(\epsilon), \deg_W(\phi) < 2\hat{c}$*, can be rewritten as*

$$\begin{aligned}\langle \gamma_1, ..., \gamma_{k-3}, \alpha, \beta, \epsilon \cdot \phi \rangle = & \sum_{I \sqcup J = [k-3]} \sum l \left\langle \gamma_{k \in I}, \alpha, \epsilon, \delta_l \right\rangle \left\langle \delta_l', \phi, \beta, \gamma_{j \in J} \right\rangle \\ & - \sum_{\substack{I \sqcup J = [k-3] \\ J \neq \emptyset}} \sum_l \left\langle \gamma_{k \in I}, \alpha, \beta, \delta_l \right\rangle \left\langle \delta_l', \phi, \epsilon, \gamma_{j \in J} \right\rangle .\end{aligned}$$

where the δ_l *are the elements of some basis* $\mathscr{B}$ *and* δ_l' *are the corresponding elements of the dual basis* $\mathscr{B}'$*.*

Proof. First recall that the definition of the product, $\epsilon \star \phi = \sum_{\sigma,\tau} \langle \epsilon, \phi, \sigma \rangle \eta^{\sigma,\tau} \tau$, where σ and τ are elements in a basis for the vector space. If, instead, we sum over a basis which contains the element $\epsilon \star \phi$, and we let δ_0 be the corresponding element in the dual basis then we know that $\epsilon \star \phi = \langle \epsilon, \phi, \delta_0 \rangle \epsilon \star \phi$, and thus,

$$\langle \gamma_1, ..., \gamma_{k-3}, \alpha, \beta, \epsilon \star \phi \rangle_{0,k} = \langle \epsilon, \phi, \delta_0 \rangle \langle \gamma_1, ..., \gamma_{k-3}, \alpha, \beta, \epsilon \star \phi \rangle_{0,k}$$

The WDVV equations show that

$$\sum_{I \sqcup J = [k-3]} \sum_l \langle \{\gamma_i\}_{i \in I}, \alpha, \beta, \delta_l \rangle \langle \{\gamma_j\}_{j \in J}, \epsilon, \phi, \delta_l' \rangle = \sum_{I \sqcup J = [k-3]} \sum_l \langle \{\gamma_i\}_{i \in I}, \alpha, \epsilon, \delta_l \rangle \langle \{\gamma_j\}_{j \in J}, \beta, \phi, \delta_l' \rangle$$

Which means that

$$\begin{aligned}& \langle \gamma_1, ..., \gamma_{k-3}, \alpha, \beta, \epsilon \star \phi \rangle_{0,k} = \langle \epsilon, \phi, \delta_0 \rangle \langle \gamma_1, ..., \gamma_{k-3}, \alpha, \beta, \epsilon \star \phi \rangle_{0,k} \\ = & \sum_{\substack{l \\ I \sqcup J = [k-3]}} \langle \{\gamma_i\}_{i \in I}, \alpha, \epsilon, \delta_l \rangle \langle \{\gamma_j\}_{j \in J}, \beta, \phi, \delta_l' \rangle - \sum_{\substack{l \\ I \sqcup J = [k-3] \\ J \neq \emptyset}} \langle \{\gamma_i\}_{i \in I}, \alpha, \beta, \delta_l \rangle \langle \{\gamma_j\}_{j \in J}, \epsilon, \phi, \delta_l' \rangle\end{aligned}$$

□

We say that an element $\alpha \in \mathscr{H}_{W,G}$ is **non-primitive** if it can be written $\epsilon \star \phi = \alpha$ for some ϵ and ϕ in $\mathscr{H}_{W,G}$ with $0 < \deg_{\mathbb{C}} \epsilon, \deg_{\mathbb{C}} \phi < \deg_{\mathbb{C}} \alpha$. Otherwise, we say that α is **primitive**.

Corollary 2. *A genus-zero, four-point correlator containing a non-primitive insertion can be rewritten:*

$$\langle \gamma_1, \alpha, \beta, \epsilon \cdot \psi \rangle_{0.4} = \sum_l \langle \gamma_1, \alpha, \epsilon, \delta_l \rangle \langle \delta_l', \psi, \beta, \rangle + \sum_l \langle \alpha, \epsilon, \delta_l \rangle \langle \delta_l', \psi, \beta, \gamma_1 \rangle \\ - \sum_l \langle \alpha, \beta, \delta_l \rangle \langle \delta_l', \psi, \epsilon, \gamma_1 \rangle .$$

In fact, using the reconstruction lemma, it is possible to write any genus-zero k-point correlator in terms of the pairing, genus-zero three-point correlators and correlators of the form $\langle \gamma_1, \ldots, \gamma_k' \rangle$ for $k' \leq k$ where γ_i is primitive for $i \leq k' - 2$. (See [1])

We say that a correlator is **basic** if at most two of the insertions are non-primitive.

Lemma 25 ([1])**.** *If* $\deg_{\mathbb{C}}(\alpha)\hat{c}$ *for all classes* α*,* P *is the maximum complex degree of any primitive class, and* $P < 1$*, then all the genus-zero correlators are uniquely determined by the pairing and* k*-point correlators with*

$$k \leq 2 + \frac{1 + \hat{c}}{1 - P} \tag{4.5}$$

Proof. If $\langle \gamma_1, \ldots, \gamma_k \rangle$ is a nonzero basic correlator, then it must satisfy:

$$\sum_i \deg_{\mathbb{C}} \gamma_i = \hat{c} + k - 3$$

The degrees of the primitive elements are bounded by P and the degrees of the non-primitive elements are bounded by $\hat{c}$, so

$$\hat{c} + k - 3 \leq 2\hat{c} + (k-2)P$$
$$\text{so, } k - kP \leq 3 - 2P + \hat{c}$$
$$k \leq \tfrac{3-2P+\hat{c}}{1-P} = 2 + \tfrac{1+\hat{c}}{1-P}$$

□

Chapter 5. Examples

The simplest examples are the so-called "simple" singularities ADE. These were worked out in detail by Fan, Jarvis, and Ruan in [1]. The next simplest examples are the Elliptic Singularities, P_8, X_9 and its transpose, and J_{10} and its transpose. These are part of the unimodal singularities, as listed by Arnol'd [19]. Shen and Krawitz worked out all the details of the A model structure for P_8, X_9^T and J_{10}^T with maximal symmetry groups. We attempt to find the Frobenius manifold structure for each of these singularities with alternate symmetry groups, the remaining elliptic singularities and other unimodal and bimodal singularities with all choices of symmetry groups.

Much of the work shown in this section was performed in Sage using code written by this author, Tyler Jarvis, Drew Johnson, Rachel Suggs, Scott Mancuso, Julian Tay, Mark Woo, Paul Draper, and Ryan Stringham under this author's supervision. Copies of this code will soon be available upon request from the author.

5.1 Elliptic Hypersurface Singularities

5.1.1 $J_{10} = x^3 + y^3z + z^2$**.** For this singularity we find that

$$J = \left(\frac{1}{3}, \frac{1}{6}, \frac{1}{2}\right), \text{ and } G^{max} = \left\langle \left(\frac{1}{3}, 0, 0\right), \left(0, \frac{1}{6}, \frac{1}{2}\right) \right\rangle.$$

Here, J and G^{max} are the only two admissible groups.

$\mathbf{W = J_{10} = x^3 + y^3 z + z^2, G = \langle J \rangle}$:

The FJRW structure for $G = \langle J \rangle$ is a four element ring.

Element	$\mathbb{C}$-degree
$A_1 = \lceil 1\,; J \rfloor$	0
$A_2 = \lceil 1\,; \mathbf{0} \rfloor$	$\frac{1}{2}$
$A_3 = \lceil xyz\,; \mathbf{0} \rfloor$	$\frac{1}{2}$
$A_4 = \lceil 1\,; 5J \rfloor$	1

There is only one nontrivial product: $A_2 \star A_3 = \frac{1}{18} A_4$. This means that A_1, A_2, and A_3 are primitive elements. Lemma 25, tells us that the Frobenius manifold structure can be determined by k-point correlators for

$$k \leq 2 + \frac{1+1}{1-1/2} = 2 + \frac{2}{1/2} = 6. \tag{5.1}$$

The nonzero 3 pt correlators are $\langle A_1, A_1, A_4 \rangle = 1$ and $\langle A_1, A_2, A_3 \rangle = \frac{1}{18}$. The selection rules and G^{max} invariance show that there are no four or five point correlators. The only possibly nonzero six-point correlator is $\langle A_2, A_2, A_3, A_3, A_4, A_4 \rangle$. We can use the reconstruction lemma to show that this correlator vanishes. This gives the entire Frobenius manifold structure. If we let $X = A_2$, $Y = A_3$, then $XY = \frac{1}{18} A_4$, so the generating function is

$$\Phi_0(X, Y) = \frac{1}{36} t_1^2 t_{XY} + \frac{1}{18} t_1 t_X t_Y$$

$\mathbf{W = J_{10} = x^3 + y^3 z + z^2, G = G^{max}}$:

Recall that Krawitz proved that when $G = G^{max}$ the A and B model Frobenius Algebras are isomorphic. However, the specific correlator values are not necessarily known, and the Frobenius manifold structure certainly is not known, so we will try to compute all possible correlators in examples like this one. Let $\gamma_1 = \left(0, \frac{1}{6}, \frac{1}{2}\right)$ and $\gamma_2 = \left(\frac{1}{3}, 0, 0\right)$, then a vector space basis for $\mathscr{H}_{J_{10}, G^{max}}$ is given by

Element	$\mathbb{C}$-Degree
$A_1 = \lceil 1\,;\gamma_1 + \gamma_2 \rfloor$	0
$A_2 = \lceil 1\,;\gamma_1 + 3\gamma_2 \rfloor$	1/3
$A_3 = \lceil 1\,;2\gamma_1 + \gamma_2 \rfloor$	1/3
$A_4 = \lceil y^2\,;\gamma_1 \rfloor$	1/3
$A_5 = \lceil 1\,;\gamma_1 + 5\gamma_2 \rfloor$	2/3
$A_6 = \lceil 1\,;2\gamma_1 + 3\gamma_2 \rfloor$	2/3
$A_7 = \lceil y^2\,;2\gamma_1 \rfloor$	2/3
$A_8 = \lceil 1\,;2\gamma_1 + 5\gamma_2 \rfloor$	1

Nontrivial Products
$A_2 \star A_2 = A_5$
$A_2 \star A_3 = A_6$
$A_2 \star A_6 = A_8$
$A_3 \star A_4 =???$
$A_3 \star A_5 = A_8$
$A_4 \star A_4 =???$
$A_4 \star A_7 = -\frac{1}{3}A_8$

The primitive elements in this ring are A_1, A_2, A_3, A_4, and possibly A_7.

We will need to evaluate the genus-zero three-point correlators in order to determine multiplication of basis elements. It turns out that all three-point correlators can be determined by the axioms in Section 2.3.6, except $\langle A_3, A_4, A_4\rangle$.

If we use the reconstruction lemma and the formula for Λ, we find that

$$-\frac{1}{9} = -\frac{1}{3}\langle A_3, A_3, A_3, A_8\rangle = \langle A_3, A_3, A_3, A_4 \star A_7\rangle = -3\langle A_3, A_3, A_7, A_7\rangle\langle A_3, A_4, A_4\rangle$$

This shows that $\langle A_3, A_4, A_4\rangle \neq 0$, so if we say that $\langle A_3, A_4, A_4\rangle = a$, then this gives the formerly unknown products of basis elements:

$$A_3 \star A_4 = -3aA_7, \quad A_4 \star A_4 = aA_5$$

Thus, A_7 is not primitive, so Lemma 25 and Equation 4.5 tell us the the entire Frobenius manifold structure can be determined from genus-0 k-point correlators for

$$k \leq 2 + \frac{1+1}{1-1/3} = 5 \tag{5.2}$$

The reconstruction lemma and the formula for Λ give the values of all basic non-zero four-point correlators:

$$\begin{array}{ll} \langle A_2, A_2, A_2, A_8 \rangle = \frac{1}{6} & \langle A_2, A_2, A_5, A_6 \rangle = \frac{1}{6} \\ \langle A_2, A_3, A_5, A_5 \rangle = 0 & \langle A_3, A_3, A_3, A_8 \rangle = \frac{1}{3} \\ \langle A_2, A_2, A_6, A_6 \rangle = \frac{1}{3} & \langle A_2, A_4, A_4, A_8 \rangle = -\frac{a}{6}, \\ \langle A_2, A_4, A_5, A_7 \rangle = \frac{1}{18}, & \langle A_3, A_3, A_7, A_7 \rangle = \frac{1}{27a}, \\ \langle A_4, A_4, A_5, A_6 \rangle = -\frac{a}{6} & \end{array}$$

Similarly, we can find the value of the only possibly nonzero basic five-point correlator:

$$\langle A_2, A_3, A_3, A_8, A_8 \rangle = 0$$

If we say that $X = A_2$, $Y = A_3$, and $Z = A_4$, then

$$\Phi_0(X, Y, Z) = \frac{1}{2}t_X^2 t_Y + t_1 t_X t_{XY} + a t_1 t_Z t_{YZ} + \frac{a}{2} t_Y t_Z^2 + \frac{1}{2} t_Y t_{X^2}^2 + \text{higher order terms}$$

the higher order terms can be found using the pairings, three-point correlators, and the four-point correlators listed above.

5.1.2 $J_{10}^T = x^3 + y^3 + yz^2$**.** For J_{10}^T we find that

$$J = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right), \text{ and } G^{max} = \left\langle \left(\frac{1}{3}, 0, 0\right), \left(0, \frac{2}{3}, \frac{1}{6}\right) \right\rangle$$

It turns out there are four admissible symmetry groups, $\langle J \rangle$, G^{max}, $G_1 = \langle \left(\frac{1}{3}, 0, 0\right), \left(0, \frac{1}{3}, \frac{1}{3}\right) \rangle$, and $G_2 = \langle \left(\frac{1}{3}, \frac{1}{3}, \frac{5}{6}\right) \rangle$.

Shen and Krawitz in [14] determined the complete A and B model structures for for $G = G^{max}$.

$\mathbf{W = J_{10}^T = x^3 + y^3 + yz^2, G = \langle J \rangle}$

$\mathscr{H}_{W,G}$ is a four dimensional vector space.

Element	$\mathbb{C}$-degree
$A_1 = \lceil 1\,; J \rfloor$	0
$A_2 = \lceil 1\,; \mathbf{0} \rfloor$	$\frac{1}{2}$
$A_3 = \lceil xyz\,; \mathbf{0} \rfloor$	$\frac{1}{2}$
$A_4 = \lceil 1\,; J^2 \rfloor$	1

There is only one nontrivial product: $A_2 \star A_3 = \frac{1}{18}A_4$. This means that A_1, A_2, and A_3 are primitive elements. We can use Equation 5.1 and Lemma 25 to show that the Frobenius manifold structure can be determined by k-point correlators for $k \leq 6$.

The nonzero 3 pt correlators are $\langle A_1, A_1, A_4 \rangle = 1$ and $\langle A_1, A_2, A_3 \rangle = \frac{1}{18}$. The selection rules and G^{max} invariance show that there are no four or five point correlators. The only possibly nonzero six-point correlator is $\langle A_2, A_2, A_3, A_3, A_4, A_4 \rangle$. We can use the reconstruction lemma to show that this correlator vanishes. This gives the entire Frobenius manifold structure. If we let $X = A_2$, $Y = A_3$, then $XY = \frac{1}{18}A_4$, so the generating function is

$$\Phi_0(X,Y) = \frac{1}{36}t_1^2 t_{XY} + \frac{1}{18}t_1 t_X t_Y$$

$\mathbf{W = J^T_{10}, G = G_1}$:

If we let $\gamma_1 = \left(\frac{1}{3}, 0, 0\right)$ and $\gamma_2 = \left(0, \frac{1}{3}, \frac{1}{3}\right)$, then a vector space basis for $\mathscr{H}_{J^T_{10},G_1}$ is given by

Element	$\mathbb{C}$-degree
$A_1 = \lceil 1\,; \gamma_1 + \gamma_2 \rfloor$	0
$A_2 = \lceil 1\,; 2\gamma_1 + \gamma_2 \rfloor$	1/3
$A_3 = \lceil y\,; \gamma_1 \rfloor$	1/3
$A_4 = \lceil z\,; \gamma_1 \rfloor$	1/3
$A_5 = \lceil 1\,; \gamma_1 + 2\gamma_2 \rfloor$	2/3
$A_6 = \lceil y\,; 2\gamma_1 \rfloor$	2/3
$A_7 = \lceil z\,; 2\gamma_1 \rfloor$	2/3
$A_8 = \lceil 1\,; 2\gamma_1 + 2\gamma_2 \rfloor$	1

Nontrivial products
$A_2 \star A_3 =$???
$A_2 \star A_4 =$???
$A_2 \star A_5 = A_8$
$A_3 \star A_3 =$???
$A_3 \star A_6 = \frac{1}{6}A_8$
$A_4 \star A_4 =$???
$A_4 \star A_7 = -\frac{1}{2}A_8$

It turns out that using the axioms in Section 2.3.6 we can find all 3-point correlator values except for $\langle A_2, A_3, A_3\rangle$ and $\langle A_2, A_4, A_4\rangle$.

If we use the reconstruction lemma and the formula for Λ, we find that

$$\frac{1}{18} = \frac{1}{6}\langle A_2, A_2, A_2, A_8\rangle = \langle A_2, A_2, A_2, A_3 \star A_6\rangle = 6\langle A_2, A_3, A_3\rangle\langle A_2, A_2, A_6, A_6\rangle, \text{ and}$$
$$-\frac{1}{6} = -\frac{1}{2}\langle A_2, A_2, A_2, A_8\rangle = \langle A_2, A_2, A_2, A_4 \star A_7\rangle = -2\langle A_2, A_4, A_4\rangle\langle A_2, A_2, A_7, A_7\rangle$$

Which means that $\langle A_2, A_3, A_3\rangle$ and $\langle A_2, A_4, A_4\rangle$ are nonzero. If we set them equal to constants,

$$\langle A_2, A_3, A_3\rangle = a \quad \langle A_2, A_4, A_4\rangle = b$$

Then we can determine some formerly unknown products:

$$A_2 \star A_3 = 6aA_6 \quad A_2 \star A_4 = -2bA_7$$
$$A_3 \star A_3 = aA_5 \qquad A_4 \star A_4 = bA_5$$

So, we see that A_1, A_2, A_3 and A_4 are the primitive basis elements, and we can use Lemma 25 and Equation 5.2 which tell us that the Frobenius manifold structure can be determined from the genus-zero k point correlators for $k \leq 5$.

Unfortunately, determining the higher point correlators is more of a challenge. It turns out that there is only one concave four-point correlator $\langle A_2, A_2, A_2, A_8\rangle$, and there are no nonzero concave five, six, or seven-point correlators, so we are unable to find exact values for the higher point correlators. However, if we say that $\langle A_3, A_3, A_4, A_8\rangle = c$, then we can determine all the four and five point correlators:

$$\langle A_2, A_2, A_6, A_6\rangle = \frac{1}{108a} \quad \langle A_2, A_2, A_7, A_7\rangle = \frac{1}{12b} \quad \langle A_2, A_4, A_5, A_5\rangle = 0 \quad \langle A_3, A_3, A_4, A_8\rangle = c$$
$$\langle A_3, A_3, A_5, A_7\rangle = -\frac{c}{2b} \quad \langle A_3, A_4, A_5, A_6\rangle = \frac{c}{6a} \quad \langle A_4, A_4, A_4, A_8\rangle = -\frac{bc}{a} \quad \langle A_4, A_4, A_5, A_7\rangle = \frac{c}{2a}$$

and, the only nonzero basic five-point correlator:

$$\langle A_2, A_2, A_4, A_8, A_8 \rangle = 0$$

However, we cannot tell whether $c \neq 0$, so this does not fully determine the Frobenius Manifold structure.

$\mathbf{W = J_{10}^T, G = G_2}$:

If we let $\gamma = \left(\frac{1}{3}, \frac{1}{3}, \frac{5}{6}\right)$, then the vector space basis for $\mathscr{H}_{J_{10}^T, G_2}$ is given by

Element	$\mathbb{C}$-degree
$A_1 = \lceil 1\,; 4\gamma \rfloor$	0
$A_2 = \lceil 1\,; \gamma \rfloor$	1/2
$A_3 = \lceil 1\,; 5\gamma \rfloor$	1/2
$A_4 = \lceil x\,; 3\gamma \rfloor$	1/2
$A_5 = \lceil y\,; 3\gamma \rfloor$	1/2
$A_6 = \lceil 1\,; 2\gamma \rfloor$	1

Nontrivial Products
$A_2 \star A_3 = A_6$
$A_4 \star A_5 = \frac{1}{9} A_6$

In this case all genus-zero, three-point correlators can be determined from the axioms in Section 2.3.6. The primitive basis elements are A_1, A_2, A_3, A_4, and A_5, so we can use Equation 5.1 to show that the Frobenius manifold structure is determined by the genus-zero k-point correlators for $k \leq 6$.

There is only one concave four-point correlator, $\langle A_2, A_2, A_2, A_3 \rangle = -\frac{1}{6}$, which contains no non-primitive insertions. There is also one concave five-point correlator,

$$\langle A_2, A_2, A_2, A_2, A_6 \rangle = 1/9$$

which does contain a product, so we will be able to use the reconstruction lemma here. Unfortunately, it is not quite enough to allow us to solve for all the needed correlators values. In particular, if we use the reconstruction lemma on the correlator $\langle A_2, A_2, A_2, A_2, A_4 \star A_5 \rangle$, we get:

$$\frac{1}{81} = 18\langle A_2, A_2, A_4, A_5\rangle^2 + \frac{1}{3}\langle A_2, A_2, A_4, A_5\rangle$$

Thus either $\langle A_2, A_2, A_4, A_5\rangle = \frac{1}{54}$ or $-\frac{1}{27}$. If $\langle A_2, A_2, A_4, A_5\rangle = \frac{1}{54}$, we do not have enough information to determine the other basic correlator values, or even which are nonzero. If $\langle A_2, A_2, A_4, A_5\rangle = -\frac{1}{27}$, then it is the only nonzero basic four-point correlator, and there are no nonzero basic five or six-point correlators.

5.1.3 $P^8 = x^3 + y^3 + z^3$. For this singularity,

$$J = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) \text{ and } G^{max} = \left\langle \left(\frac{1}{3}, 0, 0\right), \left(0, \frac{1}{3}, 0\right), \left(0, 0, \frac{1}{3}\right) \right\rangle.$$

There are three admissible subgroups between $\langle J\rangle$ and G^{max}, and if we let $\bar{\gamma}_1 = \left(\frac{1}{3}, 0, 0\right)$, $\bar{\gamma}_2 = \left(0, \frac{1}{3}, 0\right)$, and $\bar{\gamma}_3 = \left(0, 0, \frac{1}{3}\right)$, they all look like $\tilde{G} = \langle \gamma_1, \gamma_2\rangle$, for

$$\gamma = \bar{\gamma}_{\sigma(1)}, \quad \gamma_2 = \bar{\gamma}_{\sigma(2)}\bar{\gamma}_{\sigma(3)}$$

for some permutation $\sigma \in S_3$. The full A model structure for the maximal symmetry group is known ([14]).

$\mathbf{W} = \mathbf{P_8}, \mathbf{G} = \langle \mathbf{J} \rangle$:

We again get a 4 dimensional vector space.

Element	W-degree
$A_1 = \lceil 1\,; J \rfloor$	0
$A_2 = \lceil 1\,; \mathbf{0} \rfloor$	1
$A_3 = \lceil xyz\,; \mathbf{0} \rfloor$	1
$A_4 = \lceil 1\,; J^2 \rfloor$	2

There is only one nontrivial product: $A_2 \star A_3 = \frac{1}{27}A_4$. This means that A_1, A_2, and A_3 are primitive elements. Lemma 25 and Equation 5.1 tell us that the Frobenius manifold structure can be determined by k-point correlators for $k \leq 6$.

The nonzero 3 pt correlators are $\langle A_1, A_1, A_4\rangle = 1$ and $\langle A_1, A_2, A_3\rangle = \frac{1}{27}$. The selection rules and G^{max} invariance show that there are no four or five point correlators. The only possibly nonzero six-point correlator is $\langle A_2, A_2, A_3, A_3, A_4, A_4\rangle$. We can use the reconstruction lemma to show that this correlator vanishes. This gives the entire Frobenius manifold structure. If we let $X = A_2$, and $Y = A_3$,

$$\Phi_0(1, X, Y, XY) = \frac{1}{54}t_1^2 t_{XY} + \frac{1}{27}t_1 t_X t_Y$$

$\mathbf{W = P_8, G = \tilde{G}}$:

In [12] Francis, Jarvis, Johnson, and Suggs showed that for any singularity which is a sum of loops and Fermats with any admissible subgroup, the A and B Frobenius algebras are isomorphic. This is an example of such a singularity, so the Frobenius algebra structure is already known. However, the isomorphism between A and B models will not help us find exact A model correlator values, so we will find as many of these as possible using our methods, and then use these to try to expand to higher structure constants.

The vector space basis for $\mathscr{H}_{P_8,G}$ is

Element	$\mathbb{C}$-degree
$A_1 = \lceil 1\,;\gamma_1+\gamma_2\rfloor$	0
$A_2 = \lceil 1\,;2\gamma_1+\gamma_2\rfloor$	1/3
$A_3 = \lceil y\,;\gamma_1\rfloor$	1/3
$A_4 = \lceil z\,;\gamma_1\rfloor$	1/3
$A_5 = \lceil 1\,;\gamma_1+2\gamma_2\rfloor$	2/3
$A_6 = \lceil y\,;2\gamma_1\rfloor$	2/3
$A_7 = \lceil z\,;2\gamma_1\rfloor$	2/3
$A_8 = \lceil 1\,;2\gamma_1+2\gamma_2\rfloor$	1

Nontrivial Products
$A_2 \star A_3 =???$
$A_2 \star A_4 =???$
$A_2 \star A_5 = 1A_8$
$A_3 \star A_4 =???$
$A_3 \star A_7 = 1/9 A_8$
$A_4 \star A_6 = 1/9 A_8$

All of the three-point correlators are determined from the axioms in Section 2.3.6 except $\langle A_2, A_3, A_4\rangle$. Reconstruction tells us that

$$\tfrac{1}{27} = \tfrac{1}{9}\langle A_2, A_2, A_2, A_8\rangle = \langle A_2, A_2, A_2, A_3 \star A_7\rangle = 9\langle A_2, A_3, A_4\rangle\langle A_2, A_2, A_6, A_7\rangle,$$

which means that $\langle A_2, A_3, A_4\rangle$ is nonzero. If we set it equal to a, then we find that

$$A_2 \star A_3 = 9aA_6, \quad A_2 \star A_4 = 9aA_7, \quad A_3 \star A_4 = aA_5$$

So, the primitive elements are A_1, A_2, A_3, and A_4. The Frobenius Manifold structure can be determined, using Lemma 25 and Equation 5.2, genus-zero, k-point correlators for $k \leq 5$.

Unfortunately, we cannot completely determine all basic four-point correlator values form the reconstruction lemma and the formula for Λ. Given two (possibly zero) complex numbers b and c, we can determine that

$$\begin{array}{lll} \langle A_2, A_2, A_2, A_8\rangle = \frac{1}{3} & \langle A_2, A_2, A_6, A_7\rangle = \frac{1}{243a} & \langle A_3, A_3, A_3, A_8\rangle = 9ab \\ \langle A_3, A_3, A_5, A_6\rangle = b & \langle A_4, A_4, A_4, A_8\rangle = 9ac & \langle A_4, A_4, A_5, A_7\rangle = c \end{array}$$

It turns out that there are no nonzero basic five-point correlators.

If $\sigma = id$, the G in this example is actually the same as G_1 in the J_{10}^T example, and two singularities have the same weights. Lemma 16 shows that

$$\mathscr{H}_{P_8,G} \cong \mathscr{H}_{J_{10}^T,G_1}$$

5.1.4 $X_9 = x^3y + y^2z + z^2$. For this example, $J = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{2}\right)$, and $G^{max} = \langle\left(\frac{1}{12}, \frac{3}{4}, \frac{1}{2}\right)\rangle$. So the only admissible symmetry groups are $\langle J\rangle$ and G^{max}.

$\mathbf{W = X_9, G = \langle J\rangle}$:

We again get a 4 dimensional vector space.

Element	$\mathbb{C}$-degree
$A_1 = \lceil 1\,;J \rfloor$	0
$A_2 = \lceil 1\,;\mathbf{0} \rfloor$	1/2
$A_3 = \lceil xyz\,;\mathbf{0} \rfloor$	1/2
$A_4 = \lceil 1\,;J^3 \rfloor$	1

There is only one nontrivial product: $A_2 \star A_3 = \frac{1}{12}A_4$. This means that A_1, A_2, and A_3 are primitive elements. Lemma 25 and Equation 5.1 tell us that the Frobenius manifold structure can be determined by k-point correlators for $k \leq 6$.

The nonzero 3 pt correlators are $\langle A_1, A_1, A_4 \rangle = 1$ and $\langle A_1, A_2, A_3 \rangle = \frac{1}{12}$. The selection rules and G^{max} invariance show that there are no four or six point correlators. The only possibly nonzero five-point correlator is $\langle A_2, A_2, A_3, A_3, A_4 \rangle$. We can use the reconstruction lemma to show that this correlator vanishes. This gives the entire Frobenius manifold structure. If we let $X = A_2$, and $Y = A_3$,

$$\Phi_0(1, X, Y, XY) = \frac{1}{24}t_1^2 t_{XY} + \frac{1}{12}t_1 t_X t_Y$$

$\mathbf{W = X_9, G = G^{max}}$:

Let $\gamma = (1/12, 3/4, 1/2)$, then the vector space basis for $\mathscr{H}_{X_9,G^{max}}$, is

Element	$\mathbb{C}$-degree
$A_1 = \lceil 1\,;3\gamma \rfloor$	0
$A_2 = \lceil 1\,;\gamma \rfloor$	1/3
$A_3 = \lceil 1\,;7\gamma \rfloor$	1/3
$A_4 = \lceil y\,;4\gamma \rfloor$	1/3
$A_5 = \lceil 1\,;11\gamma \rfloor$	2/3
$A_6 = \lceil 1\,;5\gamma \rfloor$	2/3
$A_7 = \lceil y\,;8\gamma \rfloor$	2/3
$A_8 = \lceil 1\,;9\gamma \rfloor$	1

Nontrivial Products
$A_2 * A_2 =???$
$A_2 * A_3 = A_6$
$A_2 * A_5 = A_8$
$A_3 * A_3 = A_5$
$A_3 * A_4 =???$
$A_3 * A_6 = A_8$
$A_4 * A_4 =???$
$A_4 * A_7 = -1/2A_8$

The only three-point correlators which are not determined by the axioms in Section 2.3.6 are

$$\langle A_2, A_2, A_2\rangle, \text{ and } \langle A_3, A_4, A_4\rangle$$

Using the reconstruction lemma we get the following equations,

$$\begin{aligned}
\langle A_2, A_6, A_6, A_6, A_2 \star A_3\rangle : &\quad \tfrac{1}{8} = \langle A_2, A_2, A_6, A_6, A_8\rangle \\
\langle A_2, A_2, A_6, A_8, A_2 \star A_3\rangle : &\quad \tfrac{1}{8} = 2\langle A_2, A_2, A_5, A_6\rangle\langle A_2, A_2, A_3, A_8\rangle \\
\langle A_2, A_6, A_3, A_2 \star A_3\rangle : &\quad 0 = \langle A_2, A_2, A_5, A_6\rangle - \langle A_2, A_2, A_3, A_8\rangle
\end{aligned}$$

So, $\langle A_2, A_2, A_5, A_6\rangle = \langle A_2, A_2, A_3, A_8\rangle = \pm\frac{1}{4}$. We also find that

$$\langle A_2, A_2, A_5, A_2 \star A_3\rangle : \quad \pm\tfrac{1}{4} = -\tfrac{1}{6}\langle A_2, A_2, A_2\rangle - \langle A_2, A_2, A_3, A_8\rangle$$

This means that , $\pm\frac{1}{2} = -\frac{1}{6}\langle A_2, A_2, A_2\rangle$, which implies that $\mp 3 = \langle A_2, A_2, A_2\rangle$. Next we find that

$$\begin{aligned}
\langle A_2, A_4, A_7, A_2 \star A_3\rangle : &\quad \langle A_2, A_4, A_6, A_7\rangle = \tfrac{1}{2}\langle A_2, A_2, A_3, A_8\rangle = \pm\tfrac{1}{8} \\
\langle A_4, A_2, A_7, A_2 \star A_3\rangle : &\quad \pm\tfrac{1}{8} = \langle A_2, A_2, A_2\rangle\langle A_3, A_4, A_5, A_7\rangle
\end{aligned}$$

So, $\langle A_3, A_4, A_5, A_7\rangle = -\frac{1}{24}$. Finally,

$$\langle A_3, A_7, A_4, A_2 \star A_3\rangle : \quad -\tfrac{1}{24} = -2\langle A_3, A_3, A_7, A_7\rangle\langle A_3, A_4, A_4\rangle + \tfrac{1}{8},$$

which means that $\langle A_3, A_4, A_4\rangle \neq 0$.

If we set $\langle A_3, A_4, A_4\rangle = a$ and $\langle A_2, A_2, A_2\rangle = b$, this gives us some new product information.

$$A_3 \star A_4 = -2aA_7, \quad A_4 \star A_4 = aA_6, \quad A_2 \star A_2 = bA_5$$

This means that A_1, A_2, A_3, and A_4 are the primitive basis elements for this example. Hence we can use Equation 5.2 to argue that the Frobenius manifold structure is determined by the genus-zero k-point correlators for $k \le 5$.

The nonzero four-point correlators are:

$$\begin{array}{lll}
\langle A_2, A_3, A_5, A_5\rangle = -\frac{1}{6} & \langle A_2, A_3, A_6, A_6\rangle = 0 & \langle A_3, A_3, A_3, A_8\rangle = \frac{1}{4} \\
\langle A_3, A_3, A_5, A_6\rangle = \frac{1}{12} & \langle A_2, A_2, A_3, A_8\rangle = -\frac{b}{12} & \langle A_2, A_2, A_5, A_6\rangle = -\frac{b}{12} \\
\langle A_2, A_2, A_7, A_7\rangle = 0 & \langle A_2, A_4, A_4, A_8\rangle = \frac{ab}{12} & \langle A_2, A_4, A_6, A_7\rangle = -\frac{b}{24} \\
\langle A_3, A_3, A_7, A_7\rangle = \frac{1}{12e} & \langle A_3, A_4, A_5, A_7\rangle = -\frac{1}{24} & \langle A_4, A_4, A_5, A_5\rangle = \frac{a}{12} \\
\langle A_4, A_4, A_6, A_6\rangle = -\frac{ab}{12} & &
\end{array}$$

The basic nonzero five-point correlators are :

$$\langle A_2, A_2, A_2, A_8, A_8\rangle = \frac{1}{8} \quad \langle A_2, A_3, A_3, A_8, A_8\rangle = \frac{b}{72} \quad \langle A_3, A_4, A_4, A_8, A_8\rangle = \frac{ab}{72}$$

If we let $X = A_2$, $Y = A_3$, and $Z = A_4$, then the degree three part of the Frobenius manifold generating function is

$$\phi_0(X, Y, Z) = \frac{1}{2}\left(at_Y t_Z^2 + bt_X^3 + t_X t_Y^2 + bt_1 t_X t_{X^2} + t_1 t_Y t_{XY} + t_1 t_Z t_{YZ}\right) + \text{higher order terms}$$

The higher order terms in the function can be found using the pairing, three-point correlators and basic four and five-point correlators listed above.

5.1.5 $X_9^T = x^3 + xy^2 + yz^2$.

$$J = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right), \quad G^{max} = \left\langle \left(\frac{1}{3}, \frac{5}{6}, \frac{1}{12}\right) \right\rangle$$

There are three admissible symmetry groups for X_9^T, $\langle J\rangle$, G^{max} and $G = \left\langle\left(\frac{2}{3}, \frac{2}{3}, \frac{1}{6}\right)\right\rangle$.

The A-model structure for $G = G^{max}$ was computed by Shen and Krawitz [14].

$\mathbf{W = X_9^T, G = \langle J\rangle}$:

We again get a 4 dimensional vector space.

Element	W-degree
$A_1 = \lceil 1\,;J \rfloor$	0
$A_2 = \lceil 1\,;\mathbf{0} \rfloor$	1
$A_3 = \lceil x^2y\,;\mathbf{0} \rfloor$	1
$A_4 = \lceil 1\,;J^2 \rfloor$	2

There is only one nontrivial product: $A_2 \star A_3 = \frac{1}{12}A_4$. This means that A_1, A_2, and A_3 are primitive elements. Lemma 25 and Equation 5.1 tell us that the Frobenius manifold structure can be determined by k-point correlators for $k \leq 6$.

The nonzero 3 pt correlators are $\langle A_1, A_1, A_4 \rangle = 1$ and $\langle A_1, A_2, A_3 \rangle = \frac{1}{12}$. The selection rules and G^{max} invariance show that there are no four or five point correlators. The only possibly nonzero six-point correlators are $\langle A_2, A_2, A_2, A_2, A_4, A_4 \rangle$, $\langle A_2, A_2, A_3, A_3, A_4, A_4 \rangle$, and $\langle A_3, A_3, A_3, A_3, A_4, A_4 \rangle$. We can use the reconstruction lemma to show that each of these correlators vanishes. This gives the entire Frobenius manifold structure. If $X = A_2$ and $Y = A_3$, then

$$\Phi_0(1, X, Y, XY) = \frac{1}{24}t_1^2 t_{XY} + \frac{1}{12}t_1 t_X t_Y$$

$\mathbf{W} = \mathbf{X_9^T}, \mathbf{G} = \left\langle \left(\tfrac{\mathbf{2}}{\mathbf{3}}, \tfrac{\mathbf{2}}{\mathbf{3}}, \tfrac{\mathbf{1}}{\mathbf{6}}\right) \right\rangle$:

Notice that the weights vector and group for this example match those for the J_{10}^T, G_2 case considered earlier, but G is not a product of $\langle J \rangle$ for any partition of the polynomial X_9^T. It is generally believed that $\mathscr{H}_{J_{10}^T, G_2} \cong \mathscr{H}_{X_9^T, G}$, but we are not aware of a proof of this statement.

If $\gamma = \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{6}\right)$, the vector space basis for $\mathscr{H}_{X_9^T, G}$, is

Element	$\mathbb{C}$-degree
$A_1 = \lceil 1\,;2\gamma \rfloor$	0
$A_2 = \lceil 1\,;5\gamma \rfloor$	1/2
$A_3 = \lceil 1\,;\gamma \rfloor$	1/2
$A_4 = \lceil x\,;3\gamma \rfloor$	1/2
$A_5 = \lceil y\,;3\gamma \rfloor$	1/2
$A_6 = \lceil 1\,;4\gamma \rfloor$	1

Nontrivial Products
$A_2 * A_3 = A_6$
$A_4 * A_4 = \frac{1}{6}A_6$
$A_5 * A_5 = -\frac{1}{2}A_6$

All three-point correlators are determine by the axioms in Seciton 2.3.6.

In this example A_1, A_2, A_3, A_4, and A_5 are all primitive elements. Lemma 25 and Equation 5.1 tell us that the Frobenius Manifold structure is determined by the genus-zero, k-point correlators for $k \leq 6$.

If we use the reconstruction lemma on the correlator $\langle A_2, A_2, A_2, A_2, A_4 \star A_4\rangle$, we find that

$$\frac{1}{54} = 12\left(\langle A_2A_2A_4A_4\rangle\right)^2 + \frac{1}{3}\langle A_2A_2A_4A_4\rangle$$

So either $\langle A_2A_2A_4A_4\rangle = \frac{1}{36}$ or $-\frac{1}{18}$. In the first case the nonzero four, five, and six point correlators are

$$\langle A_2, A_2, A_2, A_3\rangle = -\tfrac{1}{6} \qquad \langle A_2, A_2, A_4, A_4\rangle = \tfrac{1}{36} \qquad \langle A_2, A_2, A_5, A_5\rangle = -\tfrac{1}{12}$$
$$\langle A_2, A_3, A_3, A_5\rangle = 0 \qquad \langle A_3, A_3, A_3, A_3\rangle = -96c^2 \qquad \langle A_3, A_4, A_4, A_5\rangle = \tfrac{c}{3}$$
$$\langle A_3, A_5, A_5, A_5\rangle = c$$

$$\langle A_2, A_2, A_2, A_2, A_6\rangle = \tfrac{1}{9} \qquad \langle A_2, A_2, A_3, A_5, A_6\rangle = 0$$
$$\langle A_2, A_3, A_3, A_3, A_6\rangle = 16c^2 \qquad \langle A_2, A_4, A_4, A_5, A_6\rangle = \tfrac{c}{9}$$
$$\langle A_2, A_5, A_5, A_5, A_6\rangle = \tfrac{c}{3} \qquad \langle A_3, A_3, A_4, A_4, A_6\rangle = -\tfrac{8}{3}c^2$$
$$\langle A_3, A_3, A_5, A_5, A_6\rangle = 8c^2$$

$$\langle A_2, A_2, A_2, A_5, A_6, A_6\rangle = 0 \qquad \langle A_2, A_2, A_3, A_3, A_6, A_6\rangle = -\tfrac{16}{3}c^2$$

$$\langle A_2, A_3, A_4, A_4, A_6, A_6\rangle = -\tfrac{7}{18}c^2 \qquad \langle A_2, A_3, A_5, A_5, A_6, A_6\rangle = \tfrac{4}{3}c^2$$

$$\langle A_3, A_3, A_3, A_5, A_6, A_6\rangle = 0 \qquad \langle A_4, A_4, A_4, A_4, A_6, A_6\rangle = -\tfrac{2}{9}c^2$$

$$\langle A_4, A_4, A_5, A_5, A_6, A_6\rangle = \tfrac{2}{9}c^2 \qquad \langle A_5, A_5, A_5, A_5, A_6, A_6\rangle = -2c^2$$

In the second case we get the following basic four-point correlators:

$$\langle A_2, A_2, A_2, A_3\rangle = -\tfrac{1}{6} \quad \langle A_2, A_2, A_4, A_4\rangle = -\tfrac{1}{18} \quad \langle A_2, A_2, A_5, A_5\rangle = \tfrac{1}{6}, or - \tfrac{1}{12}$$

The only nonzero basic five-point correlator is $\langle A_2, A_2, A_2, A_2, A_6\rangle = \frac{1}{9}$, and there are no nonzero basic six-point correlators.

5.2 Other Unimodal Singularities

5.2.1 $E_{12} = x^3 + y^7$**.** For this singularity we find that $J = \left(\frac{1}{3}, \frac{1}{7}\right)$, and $G^{max} = \left\langle \left(\frac{1}{3}, \frac{1}{7}\right)\right\rangle$. So, $\langle J\rangle = G^{max}$ is the only admissible group. This group splits using the decomposition, $E_{12} = A_2 + A_6$. So the entire A model structure can be determined using the sums of singularities axiom, and the computations in [1].

5.2.2 $E_{13} = x^3 + xy^5$**.** For this singularity we find that

$$J = \left(\frac{1}{3}, \frac{2}{15}\right), \text{ and } G^{max} = \left\langle \left(\frac{2}{3}, \frac{1}{15}\right)\right\rangle.$$

So, $\langle J\rangle = G^{max}$ is the only admissible group.

The FJRW vector space structure for $G = \langle J\rangle = G^{max}$ is given by

Element	$\mathbb{C}$-Degree
$A_1 = \lceil 1\,;J \rfloor$	0
$A_2 = \lceil 1\,;10J \rfloor$	1/5
$A_3 = \lceil 1\,;8J \rfloor$	4/15
$A_4 = \lceil 1\,;4J \rfloor$	2/5
$A_5 = \lceil 1\,;2J \rfloor$	7/15
$A_6 = \lceil y^4\,;\mathbf{0} \rfloor$	8/15
$A_7 = \lceil 1\,;13J \rfloor$	3/5
$A_8 = \lceil 1\,;11J \rfloor$	2/3
$A_9 = \lceil 1\,;7J \rfloor$	4/5
$A_{10} = \lceil 1\,;5J \rfloor$	13/15
$A_{11} = \lceil 1\,;14J \rfloor$	16/15

Nontrivial Products	
$A_2 \star A_2 = A_4$	$A_2 \star A_3 = A_5$
$A_2 \star A_4 = A_7$	$A_2 \star A_5 = A_8$
$A_2 \star A_7 = A_9$	$A_2 \star A_8 = A_{10}$
$A_2 \star A_{10} = A_{11}$	$A_3 \star A_3 = \pm 5A_6$
$A_3 \star A_4 = A_8$	$A_3 \star A_6 = \pm A_9$
$A_3 \star A_7 = A_{10}$	$A_3 \star A_9 = A_{11}$
$A_4 \star A_4 = A_9$	$A_4 \star A_5 = A_{10}$
$A_4 \star A_8 = A_{11}$	$A_5 \star A_7 = A_{11}$
$A_6 \star A_6 = -\frac{1}{5}A_{11}$	

The primitive elements in this ring are A_1, A_2, and A_3. It turns out that all three-point correlators can be determined by the axioms in Section 2.3.6.

Lemma 25 and Equation 4.5 tell us the the entire Frobenius manifold structure can be determined from genus-0 k-point correlators for

$$k \leq 2 + \frac{1 + 16/15}{1 - 4/15} \leq 5$$

There are six basic genus-zero four-point correlators, three of which are concave. Using the formula for Λ, and the reconstruction lemma, we find that

$$\begin{array}{cc} \langle A_2, A_2, A_7, A_{11} \rangle = \frac{2}{15} & \langle A_2, A_2, A_9, A_{10} \rangle = \frac{1}{15} \\ \langle A_2, A_3, A_9, A_9 \rangle = -\frac{1}{15} & \langle A_2, A_3, A_6, A_{11} \rangle = -\frac{1}{15} \\ \langle A_3, A_3, A_5, A_{11} \rangle = \frac{1}{3} & \langle A_3, A_3, A_8, A_{10} \rangle = \frac{1}{3} \end{array}$$

There are no basic 5-point genus-zero correlators.

If we let $X = A_2$ and $Y = A_3$ then the degree three part of the Frobenius manifold generating function is

$$\begin{aligned}\phi_0(X,Y) = \;& \tfrac{1}{2}t_X^2 t_{X^2Y} + t_X t_{X^2} t_{XY} + t_X t_Y t_{X^3} + t_1 t_X t_{X^3Y} + \tfrac{1}{2}t_{X^2}^2 t_Y \\ & \pm\tfrac{5}{2}t_Y^2 t_{Y^2} + t_1 t_{X^2} t_{X^2Y} + t_1 t_{XY} t_{X^3} - \tfrac{5}{2}t_1 t_{Y^2}^2 + \text{higher order terms}\end{aligned}$$

The higher order terms in the function can be found using the pairing, three-point correlators and basic four-point correlators listed above.

5.2.3 $E_{14} = x^3 + y^8$**.** For this singularity we find that $J = \left(\frac{1}{3}, \frac{1}{8}\right)$, and $G^{max} = \left\langle\left(\frac{1}{3}, \frac{1}{8}\right)\right\rangle$. So, $\langle J\rangle = G^{max}$ is the only admissible group. This group splits using the decomposition, $E_{14} = A_2 + A_7$. So the entire A model structure can be determined using the sums of singularities axiom, and the computations in [1].

5.2.4 $Z_{11} = x^3y + y^5$**.** For this singularity we find that

$$J = \left(\frac{4}{15}, \frac{1}{5}\right), \text{ and } G^{max} = \left\langle\left(\frac{1}{15}, \frac{4}{5}\right)\right\rangle.$$

So, $\langle J\rangle = G^{max}$ is the only admissible group.

The FJRW vector space structure for $G = \langle J\rangle = G^{max}$ is given by

Element	$\mathbb{C}$-Degree
$A_1 = \lceil 1\,; J \rfloor$	0
$A_2 = \lceil 1\,; 12J \rfloor$	2/15
$A_3 = \lceil 1\,; 8J \rfloor$	4/15
$A_4 = \lceil 1\,; 6J \rfloor$	1/3
$A_5 = \lceil 1\,; 4J \rfloor$	2/5
$A_6 = \lceil 1\,; 2J \rfloor$	7/15
$A_7 = \lceil x^2\,; \mathbf{0} \rfloor$	8/15
$A_8 = \lceil 1\,; 13J \rfloor$	3/5
$A_9 = \lceil 1\,; 11J \rfloor$	2/3
$A_{10} = \lceil 1\,; 9J \rfloor$	11/15
$A_{11} = \lceil 1\,; 7J \rfloor$	4/5
$A_{12} = \lceil 1\,; 3J \rfloor$	14/15
$A_{13} = \lceil 1\,; 14J \rfloor$	16/15

Nontrivial Products	
$A_2 \star A_2 = A_3$	$A_2 \star A_3 = A_5$
$A_2 \star A_4 = A_6$	$A_2 \star A_5 = \pm 3A_7$
$A_2 \star A_6 = A_8$	$A_2 \star A_7 = \pm A_9$
$A_2 \star A_8 = A_{10}$	$A_2 \star A_9 = A_{11}$
$A_2 \star A_{11} = A_{12}$	$A_2 \star A_{12} = A_{13}$
$A_3 \star A_3 = \pm 3A_7$	$A_3 \star A_4 = A_8$
$A_3 \star A_5 =???$	$A_3 \star A_6 = A_{10}$
$A_3 \star A_9 = A_{12}$	$A_3 \star A_{11} = A_{13}$
$A_4 \star A_4 = A_9$	$A_4 \star A_5 = A_{10}$
$A_4 \star A_6 = A_{11}$	$A_4 \star A_8 = A_{12}$
$A_4 \star A_{10} = A_{13}$	$A_5 \star A_5 =???$
$A_5 \star A_7 = \pm A_{12}$	$A_5 \star A_9 = A_{13}$
$A_6 \star A_6 = A_{12}$	$A_6 \star A_8 = A_{13}$
$A_7 \star A_7 = -\frac{1}{3} A_{13}$	

The primitive elements in this ring are A_1, A_2, and A_4. It turns out that all three-point correlators can be determined by the axioms in Section 2.3.6, except $\langle A_3, A_5, A_5 \rangle$, and we only know that $\langle A_2, A_5, A_7 \rangle$ and $\langle A_3, A_3, A_7 \rangle$ must be plus or minus one. In fact, reconstruction will show us that both of the latter are equal to -1.

Using the reconstruction lemma we find that

$$\langle A_2, A_{10}, A_{11}, A_2 \star A_3 \rangle : \quad \langle A_2, A_5, A_{10}, A_{11} \rangle = \tfrac{1}{5}, \text{ and}$$
$$\langle A_2, A_5, A_{11}, A_3 \star A_6 \rangle : \quad \langle A_2, A_5, A_{10}, A_{11} \rangle = -\tfrac{1}{15} \langle A_3, A_5, A_5 \rangle,$$

which tells us that $\langle A_3, A_5, A_5 \rangle = -3$. We now know our missing products:

$$A_3 \star A_5 = -3A_9, , \text{ and } A_5 \star A_5 = -3A_{11}$$

Lemma 25 and Equation 4.5 tell us the the entire Frobenius manifold structure can be determined from genus-0 k-point correlators for

$$k \leq 2 + \frac{1 + 16/15}{1 - 1/3} \leq 6$$

There are eight basic genus-zero four-point correlators, seven of which are concave. Using the formula for Λ and reconstruction, we find that

$$\begin{array}{cc}
\langle A_2, A_2, A_{10}, A_{13}\rangle = \frac{1}{5} & \langle A_2, A_4, A_9, A_{12}\rangle = -\frac{1}{15} \\
\langle A_2, A_4, A_{11}, A_{11}\rangle = -\frac{1}{15} & \langle A_4, A_4, A_4, A_{13}\rangle = \frac{4}{15} \\
\langle A_4, A_4, A_6, A_{12}\rangle = \frac{1}{5} & \langle A_4, A_4, A_8, A_{11}\rangle = \frac{2}{15} \\
\langle A_4, A_4, A_9, A_{10}\rangle = \frac{1}{15} & \langle A_2, A_4, A_7, A_{13}\rangle = -\frac{1}{15}
\end{array}$$

There are no basic 5 or 6-point genus-zero correlators.

If we let $X = A_2$ and $Y = A_4$ then the we find that $Y^2 = -3x^5$ and degree three part of the Frobenius manifold generating function is

$$\begin{aligned}
\phi_0(X,Y) = \; & \frac{1}{6}\left(\sum_{\sum a_i=3, \sum b_i=2} t_{X^{a_1}Y^{b_1}} t_{X^{a_2}Y^{b_2}} t_{X^{a_3}Y^{b_3}} + \sum_{a_i \leq 4, \sum a_i = 8} t_{X^{a_1}} t_{X^{a_2}} t_{X^{a_3}}\right) \\
& + \text{higher order terms}
\end{aligned}$$

The higher order terms in the function can be found using the pairing, three-point correlators and basic four-point correlators listed above.

5.2.5 $Z_{12} = x^3y + xy^4$.

For this singularity we find that

$$J = \left(\frac{3}{11}, \frac{2}{11}\right), \text{ and } G^{max} = \left\langle \left(\frac{1}{11}, \frac{8}{11}\right)\right\rangle.$$

So, $\langle J\rangle = G^{max}$ is the only admissible group.

The FJRW vector space structure for $G = \langle J\rangle = G^{max}$ is given by

Element	$\mathbb{C}$-Degree
$A_1 = \lceil 1\,; J \rfloor$	0
$A_2 = \lceil 1\,; 8J \rfloor$	2/11
$A_3 = \lceil 1\,; 6J \rfloor$	3/11
$A_4 = \lceil 1\,; 4J \rfloor$	4/11
$A_5 = \lceil 1\,; 2J \rfloor$	5/11
$A_6 = \lceil x^2\,; \mathbf{0} \rfloor$	6/11
$A_7 = \lceil y^3\,; \mathbf{0} \rfloor$	6/11
$A_8 = \lceil 1\,; 9J \rfloor$	7/11
$A_9 = \lceil 1\,; 7J \rfloor$	8/11
$A_{10} = \lceil 1\,; 5J \rfloor$	9/11
$A_{11} = \lceil 1\,; 3J \rfloor$	10/11
$A_{12} = \lceil 1\,; 10J \rfloor$	12/11

Nontrivial Products	
$A_2 \star A_2 = A_4$	$A_2 \star A_3 = A_5$
$A_2 \star A_4 =???$	$A_2 \star A_5 = A_8$
$A_2 \star A_6 =???$	$A_2 \star A_7 =???$
$A_2 \star A_8 = A_{10}$	$A_2 \star A_9 = A_{11}$
$A_2 \star A_{11} = A_{12}$	$A_3 \star A_3 =???$
$A_3 \star A_4 = A_8$	$A_3 \star A_5 = A_9$
$A_3 \star A_6 =???$	$A_3 \star A_7 =???$
$A_3 \star A_8 = A_{11}$	$A_3 \star A_{10} = A_{12}$
$A_4 \star A_4 =???$	$A_4 \star A_5 = A_{10}$
$A_4 \star A_6 =???$	$A_4 \star A_7 =???$
$A_4 \star A_9 = A_{12}$	$A_5 \star A_5 = A_{11}$
$A_5 \star A_8 = A_{12}$	$A_6 \star A_6 = -\frac{4}{11} A_{12}$
$A_6 \star A_7 = \frac{1}{11} A_{12}$	$A_7 \star A_7 = \frac{3}{11} A_{12}$

It turns out that all three-point correlators can be determined by the axioms in Section 2.3.6, except

$$\langle A_2 A_4, A_6\rangle \quad \langle A_2 A_4, A_7\rangle \quad \langle A_3 A_3, A_6\rangle$$
$$\langle A_3 A_3, A_7\rangle \quad \langle A_4 A_4, A_4\rangle$$

Unfortunately, reconstruction doesn't entirely determine whether these are nonzero or not. Although Krawitz proved the Landau Ginzburg conjecture for all invertible cases where $G = G^{max}$, the map he provides is not defined constructively in the case of an even variable loop. He proves that there exists a map that will respect the product structure, but we cannot use it to gain any more information about the A model three-point correlators.

5.2.6 $Z_{13} = x^3 y + y^6$**.** For this singularity we find that

$$J = \left(\frac{5}{18}, \frac{1}{6}\right), \text{ and } G^{max} = \left\langle \left(\frac{1}{18}, \frac{5}{6}\right) \right\rangle.$$

So, $\langle J \rangle = G^{max}$ is the only admissible group.

The FJRW vector space structure for $G = \langle J \rangle = G^{max}$ with is given by

Element	$\mathbb{C}$-Degree
$A_1 = \lceil 1\,;J \rfloor$	0
$A_2 = \lceil 1\,;8J \rfloor$	1/9
$A_3 = \lceil 1\,;15J \rfloor$	2/9
$A_4 = \lceil 1\,;4J \rfloor$	1/3
$A_5 = \lceil 1\,;13J \rfloor$	1/3
$A_6 = \lceil 1\,;11J \rfloor$	4/9
$A_7 = \lceil 1\,;2J \rfloor$	4/9
$A_8 = \lceil 1\,;9J \rfloor$	5/9
$A_9 = \lceil x^2\,;\mathbf{0} \rfloor$	5/9
$A_{10} = \lceil 1\,;7J \rfloor$	2/3
$A_{11} = \lceil 1\,;16J \rfloor$	2/3
$A_{12} = \lceil 1\,;5J \rfloor$	7/9
$A_{13} = \lceil 1\,;14J \rfloor$	7/9
$A_{14} = \lceil 1\,;3J \rfloor$	8/9
$A_{15} = \lceil 1\,;10J \rfloor$	1
$A_{16} = \lceil 1\,;17J \rfloor$	10/9

Nontrivial Products		
$A_2 \star A_2 = A_3$	$A_2 \star A_3 = A_4$	$A_2 \star A_4 = A_6$
$A_2 \star A_5 = A_7$	$A_2 \star A_6 = \pm A_9$	$A_2 \star A_7 = A_8$
$A_2 \star A_8 = A_{11}$	$A_2 \star A_9 = \pm A_{10}$	$A_2 \star A_{10} = A_{13}$
$A_2 \star A_{11} = A_{12}$	$A_2 \star A_{13} = A_{14}$	$A_2 \star A_{14} = A_{15}$
$A_2 \star A_{15} = A_{16}$	$A_3 \star A_3 = A_6$	$A_3 \star A_4 = \pm 3A_9$
$A_3 \star A_5 = A_8$	$A_3 \star A_6 = ???$	$A_3 \star A_7 = A_{11}$
$A_3 \star A_8 = A_{12}$	$A_3 \star A_9 = \pm A_{13}$	$A_3 \star A_{10} = A_{14}$
$A_3 \star A_{13} = A_{15}$	$A_3 \star A_{14} = A_{16}$	$A_4 \star A_4 = ???$
$A_4 \star A_5 = A_{11}$	$A_4 \star A_6 = ???$	$A_4 \star A_7 = A_{12}$
$A_4 \star A_9 = \pm A_{14}$	$A_4 \star A_{10} = \pm A_{15}$	$A_4 \star A_{13} = \pm A_{16}$
$A_5 \star A_5 = A_{10}$	$A_5 \star A_6 = A_{12}$	$A_5 \star A_7 = A_{13}$
$A_5 \star A_8 = A_{14}$	$A_5 \star A_{11} = A_{15}$	$A_5 \star A_{12} = A_{16}$
$A_6 \star A_6 = ???$	$A_6 \star A_9 = A_{15}$	$A_6 \star A_{10} = A_{16}$
$A_7 \star A_7 = A_{14}$	$A_7 \star A_8 = A_{15}$	$A_7 \star A_9 = A_{16}$
$A_8 \star A_8 = A_{16}$	$A_9 \star A_9 = -\frac{1}{3}A_{16}$	

In this case, we can determine the primitive elements without knowing all products. The primitive elements in this ring are A_1, A_2, and A_5. Lemma 25 and Equation 4.5 tell us the the entire Frobenius manifold structure can be determined from genus-0 k-point correlators for

$$k \leq 2 + \frac{1 + 10/9}{1 - 1/3} \leq 6$$

It turns out that all three-point correlators can be determined by the axioms in Section

2.3.6, except the following,

$$\langle A_2A_6, A_9\rangle \quad \langle A_3A_4, A_9\rangle$$
$$\langle A_3A_6, A_6\rangle \quad \langle A_4A_4, A_6\rangle$$

The axioms do tell us that the first two of the above correlators must be equal to plus or minus one, and the reconstruction lemma shows that the last two are both equal to -3. This gives us some formerly unknown products.

$$A_3 \star A_6 = -3A_{12} \quad A_6 \star A_6 = -3A_{14}$$
$$A_4 \star A_4 = -3A_{12} \quad A_4 \star A_6 = -3A_{13}$$

There are nine basic genus-zero four-point correlators, eight of which are concave. Using the formula for Λ and reconstruction, we find that

$$\langle A_2, A_2, A_{12}, A_{16}\rangle = \tfrac{1}{6} \qquad \langle A_2, A_5, A_{10}, A_{15}\rangle = -\tfrac{1}{18}$$
$$\langle A_2, A_5, A_{13}, A_{14}\rangle = -\tfrac{1}{18} \qquad \langle A_5, A_5, A_5, A_{16}\rangle = \tfrac{5}{18}$$
$$\langle A_5, A_5, A_7, A_{15}\rangle = \tfrac{2}{9} \qquad \langle A_5, A_5, A_8, A_{14}\rangle = \tfrac{1}{6}$$
$$\langle A_5, A_5, A_{10}, A_{12}\rangle = \tfrac{1}{18} \qquad \langle A_5, A_5, A_{11}, A_{13}\rangle = \tfrac{1}{9}$$
$$\langle A_2, A_5, A_9, A_{16}\rangle = -\tfrac{1}{18}$$

There are no basic 5 or 6-point genus-zero correlators.

If we let $X = A_2$ and $Y = A_5$ then the degree three part of the Frobenius manifold generating function is

$$\phi_0(X,Y) = \frac{1}{6}\left(\sum_{\sum a_i=4, \sum b_i=2} t_{X^{a_1}Y^{b_1}} t_{X^{a_2}Y^{b_2}} t_{X^{a_3}Y^{b_3}} - 3(t_1 t_{X^5}^2 + t_{X^2} t_{X^4}^2 + t_{X^3}^2 t_{X^4}) \pm \frac{1}{3}(t_X t_{X^4} t_{X^5} + t_{X^2} t_{X^3} t_{X^5})\right) + \text{higher order terms}$$

The higher order terms in the function can be found using the pairing, three-point correlators and basic four-point correlators listed above.

5.2.7 $U_{12} = x^3 + y^3 + z^4$**.** For this singularity we find that

$$J = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{4}\right), \text{ and } G^{max} = \left\langle \left(\frac{1}{3}, 0, 0\right), \left(0, \frac{1}{3}, \frac{1}{4}\right) \right\rangle.$$

So, $\langle J \rangle$ and G^{max} are the only two admissible groups. G^{max} splits using the decomposition, $U_{12} = A_2 + A_2 + A_3$. So the entire A model structure can be determined using the sums of singularities axiom, and the computations in [1].

$\mathbf{W = U_{12}, G = \langle J \rangle}$:

If we use $G = \langle J \rangle$, then the vector space basis is given by

Element	$\mathbb{C}$-Degree
$A_1 = \lceil 1\,; J \rfloor$	0
$A_2 = \lceil 1\,; 10J \rfloor$	1/4
$A_3 = \lceil x\,; 9J \rfloor$	1/3
$A_4 = \lceil y\,; 9J \rfloor$	1/3
$A_5 = \lceil 1\,; 7J \rfloor$	1/2
$A_6 = \lceil x\,; 6J \rfloor$	7/12
$A_7 = \lceil y\,; 6J \rfloor$	7/12
$A_8 = \lceil 1\,; 5J \rfloor$	2/3
$A_9 = \lceil x\,; 3J \rfloor$	5/6
$A_{10} = \lceil y\,; 3J \rfloor$	5/6
$A_{11} = \lceil 1\,; 2J \rfloor$	11/12
$A_{12} = \lceil 1\,; 11J \rfloor$	7/6

Nontrivial Products	
$A_2 \star A_2 = A_5$	$A_2 \star A_3 =???$
$A_2 \star A_4 =???$	$A_2 \star A_6 =???$
$A_2 \star A_7 =???$	$A_2 \star A_8 = A_{11}$
$A_2 \star A_{11} = A_{12}$	$A_3 \star A_4 =???$
$A_3 \star A_5 =???$	$A_3 \star A_7 =???$
$A_3 \star A_{10} = \frac{1}{9}A_{12}$	$A_4 \star A_5 =???$
$A_4 \star A_6 =???$	$A_4 \star A_9 = \frac{1}{9}A_{12}$
$A_5 \star A_8 = A_{12}$	$A_6 \star A_7 = A_{12}$

All three-point correlators can be determined by the axioms in Section 2.3.6 except for

$$\langle A_2, A_3, A_7 \rangle \quad \langle A_2, A_4, A_6 \rangle \quad \langle A_3, A_4, A_5 \rangle$$

The reconstruction lemma tells us that

$$\langle A_2, A_2, A_5, A_3 \star A_{10}\rangle : \quad \tfrac{1}{36} = \langle A_2, A_2, A_5, A_{12}\rangle = 9\langle A_2, A_3, A_7\rangle\langle A_2, A_5, A_6, A_{10}\rangle,$$
$$\langle A_2, A_5, A_2, A_3 \star A_{10}\rangle : \quad \tfrac{1}{36} = \langle A_2, A_2, A_5, A_{12}\rangle = 9\langle A_3, A_4, A_5\rangle\langle A_2, A_2, A_9, A_{10}\rangle$$
$$\langle A_2, A_2, A_5, A_4 \star A_9\rangle : \quad \tfrac{1}{36} = \langle A_2, A_2, A_5, A_{12}\rangle = 9\langle A_2, A_4, A_6\rangle\langle A_2, A_5, A_7, A_9\rangle$$

which tells us that $\langle A_2, A_4, A_6\rangle$, $\langle A_2, A_3, A_7\rangle$, and $\langle A_3, A_4, A_5\rangle$ are nonzero. If we let them equal the complex numbers a, b, and c, respectively, then we can now describe our missing products:

$$A_2 \star A_3 = 9bA_6 \quad A_2 \star A_4 = 9aA_7 \quad A_2 \star A_6 = 9aA_9$$
$$A_2 \star A_7 = 9bA_{10} \quad A_3 \star A_4 = cA_8 \quad A_3 \star A_5 = 9cA_9$$
$$A_3 \star A_7 = bA_{11} \quad A_4 \star A_5 = 9cA_{10} \quad A_4 \star A_6 = aA_{11}$$

It turns out from other reconstruction relations that $c = 9ab$ We know see that the primitive elements in this ring are A_1, A_2, A_3, and A_4.

Lemma 25 and Equation 4.5 tell us the the entire Frobenius manifold structure can be determined from genus-0 k-point correlators for

$$k \leq 2 + \frac{1 + 7/6}{1 - 1/3} \leq 6$$

There are eight basic genus-zero four-point correlators, only one of which is concave. Unfortunately, our current methods for using the reconstruction lemma don't allow us to completely determine all values of the basic four-point correlators. We can determine that for some (possibly zero) complex numbers k_1 and k_2

$$\langle A_2, A_2, A_5, A_{12}\rangle = \tfrac{1}{4} \quad \langle A_2, A_5, A_9, A_{10}\rangle = \tfrac{1}{324c}$$
$$\langle A_3, A_3, A_3, A_{12}\rangle = k_1 \quad \langle A_3, A_3, A_6, A_{11}\rangle \tfrac{k_1}{9b}$$
$$\langle A_3, A_3, A_8, A_9\rangle = \tfrac{k_1}{81ab} \quad \langle A_4, A_4, A_4, A_{12}\rangle = k_2$$
$$\langle A_4, A_4, A_7, A_{11}\rangle = \tfrac{k_2}{9b} \quad \langle A_4, A_4, A_8, A_{10}\rangle = \tfrac{k_2}{81ab}$$

There are no basic 5 or 6-point genus-zero correlators.

If we let $X = A_2$, $Y = A_3$, and $Z = A_4$ then the degree three part of the Frobenius manifold generating function is

$$\phi_0(X,Y,Z) = \frac{1}{6}\left(9ab \sum_{\substack{\sum a_i=2 \\ \sum b_i = \sum c_i = 1 \\ (a_i, b_i c_i) \neq (2,1,1)}} \prod_{i=1}^{3} t_{X^{a_i}Y^{b_i}Z^{c_i}} + 81ab t_1^2 t_{X^2YZ} \right) + \text{higher order terms}$$

The higher order terms in the function can be found using the pairing, three-point correlators and basic four-point correlators listed above.

5.2.8 $W_{12} = x^4 + y^5$. For this singularity we find that $J = \left(\frac{1}{4}, \frac{1}{5}\right)$, and $G^{max} = \left\langle \left(\frac{1}{4}, \frac{1}{5}\right)\right\rangle$. So, $\langle J \rangle = G^{max}$ is the only admissible group. This group splits using the decomposition, $W_{12} = A_3 + A_4$. So the entire A model structure can be determined using the sums of singularities axiom, and the computations in [1].

5.2.9 $W_{13} = x^4 + xy^4$. For this singularity we find that

$$J = \left(\frac{1}{4}, \frac{3}{16}\right), \text{ and } G^{max} = \left\langle \left(\frac{1}{4}, \frac{3}{16}\right) \right\rangle.$$

So, $\langle J \rangle = G^{max}$ is the only admissible group.

The FJRW vector space structure for $G = G^{max} = \langle J \rangle$ with is given by

Element	$\mathbb{C}$-Degree
$A_1 = \lceil 1\,;J\rfloor$	0
$A_2 = \lceil 1\,;17J\rfloor$	3/16
$A_3 = \lceil 1\,;6J\rfloor$	1/4
$A_4 = \lceil 1\,;13J\rfloor$	3/8
$A_5 = \lceil 1\,;2J\rfloor$	7/16
$A_6 = \lceil 1\,;11J\rfloor$	1/2
$A_7 = \lceil 1\,;9J\rfloor$	916
$A_8 = \lceil 1\,;18J\rfloor$	5/8
$A_9 = \lceil 1\,;7J\rfloor$	11/16
$A_{10} = \lceil 1\,;14J\rfloor$	3/4
$A_{11} = \lceil 1\,;3J\rfloor$	7/8
$A_{12} = \lceil 1\,;19J\rfloor$	15/16
$A_{13} = \lceil 1\,;19J\rfloor$	9/8

Nontrivial Products	
$A_2 \star A_2 = A_4$	$A_2 \star A_3 = A_5$
$A_2 \star A_4 = \pm 4A_7$	$A_2 \star A_5 = A_8$
$A_2 \star A_6 = A_9$	$A_2 \star A_8 = \pm A_{10}$
$A_2 \star A_9 = A_{11}$	$A_2 \star A_{10} = A_{12}$
$A_2 \star A_{12} = A_{13}$	$A_3 \star A_3 = A_6$
$A_3 \star A_4 = A_8$	$A_3 \star A_5 = A_9$
$A_3 \star A_6 = A_{10}$	$A_3 \star A_8 = A_{11}$
$A_3 \star A_9 = A_{12}$	$A_3 \star A_{11} = A_{13}$
$A_4 \star A_4 =???$	$A_4 \star A_5 = A_{10}$
$A_4 \star A_6 = A_{11}$	$A_4 \star A_7 = \pm A_{12}$
$A_4 \star A_{10} = A_{13}$	$A_5 \star A_5 = A_{11}$
$A_5 \star A_6 = A_{12}$	$A_5 \star A_9 = A_{13}$
$A_6 \star A_8 = A_{13}$	$A_7 \star A_7 = -\frac{1}{4}A_{13}$

The primitive elements in this ring are A_1, A_2, and A_3. It turns out that all three-point correlators can be determined by the axioms in Section 2.3.6 except $\langle A_2, A_4, A_7\rangle$ which must be equal to ± 1 and $\langle A_4, A_4, A_4\rangle$ which the reconstruction lemma shows must be equal to -4.

Lemma 25 and Equation 4.5 tell us the the entire Frobenius manifold structure can be determined from genus-0 k-point correlators for

$$k \leq 2 + \frac{1 + 11/10}{1 - 1/4} \leq 5$$

There are seven basic genus-zero four-point correlators, six of which are concave. Using the formula for Λ, and the reconstruction lemma, we find that

$$\begin{array}{rr} \langle A_2, A_2, A_8, A_{13}\rangle = \frac{1}{5} & \langle A_2, A_2, A_{11}, A_{11}\rangle = \frac{1}{4} \\ \langle A_3, A_3, A_{10}, A_{12}\rangle = -\frac{1}{16} & \langle A_3, A_3, A_6, A_{13}\rangle = \frac{3}{16} \\ \langle A_3, A_3, A_9, A_{12}\rangle = \frac{1}{8} & \langle A_3, A_3, A_{10}, A_{11}\rangle = \frac{1}{16} \\ \langle A_2, A_3, A_7, A_{13}\rangle = -\frac{1}{16} & \end{array}$$

There are no basic 5-point genus-zero correlators.

If we let $X = A_2$, $Y = A_3$, and $Z = A_4$ then the degree three part of the Frobenius manifold generating function is

$$\phi_0(X, Y, Z) = \frac{1}{6}\left(9ab \sum_{\substack{\sum a_i = 2 \\ \sum b_i = \sum c_i = 1 \\ (a_i, b_i c_i) \neq (2,1,1)}} \prod_{i=1}^{3} t_{X^{a_i}Y^{b_i}Z^{c_i}} + 81abt_1^2 t_{X^2YZ} \right) + \text{higher order terms}$$

The higher order terms in the function can be found using the pairing, three-point correlators and basic four-point correlators listed above.

5.2.10 $Q_{10} = x^2y + y^4 + z^3$**.** For this singularity we find that $J = \left(\frac{3}{8}, \frac{1}{4}, \frac{1}{3}\right)$, and $G^{max} = \left\langle\left(\frac{3}{8}, \frac{1}{4}, \frac{1}{3}\right)\right\rangle$. So, $\langle J\rangle = G^{max}$ is the only admissible group. This group splits using the decomposition, $Q_{10} = D_5 + A_2$. So the entire A model structure can be determined using the sums of singularities axiom, and the computations in [1].

5.2.11 $Q_{11} = x^2y + y^3z + z^3$**.** For this singularity we find that

$$J = \left(\frac{7}{18}, \frac{2}{9}, \frac{1}{3}\right), \text{ and } G^{max} = \left\langle\left(\frac{7}{18}, \frac{2}{9}, \frac{1}{3}\right)\right\rangle.$$

So, $\langle J\rangle = G^{max}$ is the only admissible group.

The FJRW vector space structure for $G = G^{max} = \langle J\rangle$ with is given by

Element	$\mathbb{C}$-Degree
$A_1 = \lceil 1\,; J \rfloor$	0
$A_2 = \lceil 1\,; 16J \rfloor$	1/6
$A_3 = \lceil 1\,; 14J \rfloor$	5/18
$A_4 = \lceil 1\,; 13J \rfloor$	1/3
$A_5 = \lceil 1\,; 11J \rfloor$	4/9
$A_6 = \lceil 1\,; 10J \rfloor$	1/2
$A_7 = \lceil y^2\,; 9J \rfloor$	5/9
$A_8 = \lceil 1\,; 8J \rfloor$	11/18
$A_9 = \lceil 1\,; 7J \rfloor$	2/3
$A_{10} = \lceil 1\,; 5J \rfloor$	7/9
$A_{11} = \lceil 1\,; 4J \rfloor$	5/6
$A_{12} = \lceil 1\,; 2J \rfloor$	17/18
$A_{13} = \lceil 1\,; 17J \rfloor$	10/9

Nontrivial Products	
$A_2 \star A_2 = A_4$	$A_2 \star A_3 = A_5$
$A_2 \star A_4 =???$	$A_2 \star A_5 = A_8$
$A_2 \star A_6 = A_9$	$A_2 \star A_8 =???$
$A_2 \star A_9 = A_{11}$	$A_2 \star A_{10} = A_{12}$
$A_2 \star A_{12} = A_{13}$	$A_3 \star A_3 = \pm 3A_7$
$A_3 \star A_4 = A_8$	$A_3 \star A_6 = A_{10}$
$A_3 \star A_7 = \pm A_{11}$	$A_3 \star A_9 = A_{12}$
$A_3 \star A_{11} = A_{13}$	$A_4 \star A_4 =???$
$A_4 \star A_5 =???$	$A_4 \star A_6 = A_{11}$
$A_4 \star A_8 =???$	$A_4 \star A_{10} = A_{13}$
$A_5 \star A_6 = A_{12}$	$A_5 \star A_9 = A_{13}$
$A_6 \star A_8 = A_{13}$	$A_7 \star A_7 = -\frac{1}{3}A_{13}$

All three-point correlators can be determined by the axioms in Section 2.3.6 except

$$\langle A_2, A_4, A_8\rangle \quad \langle A_3, A_3, A_7\rangle \quad \langle A_4, A_5, A_5\rangle$$

We know that $\langle A_3, A_3, A_7\rangle = \pm 1$. We shall say that the value of $\langle A_3, A_3, A_7\rangle$ is a.

The reconstruction lemma tells us that

$$\begin{aligned} \langle A_4, A_2, A_{11}, A_3 \star A_6\rangle : &\quad \langle A_2, A_4, A_{10}, A_{11}\rangle = \langle A_4, A_5, A_6, A_{11}\rangle, \text{ and} \\ \langle A_{11}, A_2, A_4, A_3 \star A_6\rangle : &\quad \langle A_2, A_4, A_{10}, A_{11}\rangle = \langle A_4, A_5, A_6, A_{11}\rangle - \tfrac{1}{9} - \tfrac{1}{18}\langle A_2, A_4, A_8\rangle \end{aligned}$$

Which means that $\langle A_2, A_4, A_8\rangle = -2$. Similarly, we can use reconstruction on the correlator $\langle A_4, A_4, A_9, A_{10}\rangle$ to show that $\langle A_4, A_5, A_5\rangle = -2$. So,

$$\begin{aligned} &A_2 \star A_4 = -2A_6 \quad A_2 \star A_8 = -2A_{10} \quad A_4 \star A_5 = -2A_9 \\ &A_4 \star A_8 = -2A_{12} \quad A_5 \star A_5 = -2A_{10} \end{aligned}$$

The primitive elements in this ring are A_1, A_2, and A_3. Lemma 25 and Equation 4.5 tell us the the entire Frobenius manifold structure can be determined from genus-0 k-point correlators for

$$k \leq 2 + \frac{1 + 10/9}{1 - 5/18} \leq 5$$

There are seven basic genus-zero four-point correlators, three of which are concave. Using the formula for Λ, and the reconstruction lemma, we find that

$$\begin{array}{ll} \langle A_2, A_2, A_9, A_{13}\rangle = \frac{2}{9} & \langle A_2, A_2, A_{11}, A_{12}\rangle = \frac{1}{9} \\ \langle A_2, A_3, A_{11}, A_{11}\rangle = -\frac{1}{9} & \langle A_2, A_3, A_7, A_{13}\rangle = \frac{a-2}{9} \\ \langle A_3, A_3, A_5, A_{13}\rangle = -\frac{a-2}{9} & \langle A_3, A_3, A_8, A_{12}\rangle = -\frac{a-2}{9} \\ \langle A_3, A_3, A_{10}, A_{10}\rangle = -\frac{1}{6} & \end{array}$$

There are no basic 5-point genus-zero correlators.

5.2.12 $\quad Q_{12} = x^2y + y^5 + z^3$**.** For this singularity we find that $J = \left(\frac{2}{5}, \frac{1}{5}, \frac{1}{3}\right)$, and $G^{max} = \left\langle \left(\frac{7}{10}, \frac{3}{5}, \frac{2}{3}\right) \right\rangle$. So, $\langle J \rangle$, and G^{max} are the only admissible groups. They both split using the decomposition, $Q_{12} = D_4 + A_2$. So in each case the entire A model structure can be determined using the sums of singularities axiom, and the computations in [1].

5.2.13 $\quad S_{11} = x^2y + y^2z + z^4$**.** For this singularity we find that

$$J = \left(\frac{5}{16}, \frac{3}{8}, \frac{1}{4}\right), \text{ and } G^{max} = \left\langle \left(\frac{5}{16}, \frac{3}{8}, \frac{1}{4}\right) \right\rangle.$$

So, $\langle J \rangle = G^{max}$ is the only admissible group.

The FJRW vector space structure for $G = G^{max} = \langle J \rangle$ with is given by

Element	$\mathbb{C}$-Degree
$A_1 = \lceil 1\,;J \rfloor$	0
$A_2 = \lceil 1\,;14J \rfloor$	3/16
$A_3 = \lceil 1\,;13J \rfloor$	1/4
$A_4 = \lceil 1\,;11J \rfloor$	3/8
$A_5 = \lceil 1\,;10J \rfloor$	7/16
$A_6 = \lceil 1\,;9J \rfloor$	1/2
$A_7 = \lceil y\,;8J \rfloor$	9/16
$A_8 = \lceil 1\,;7J \rfloor$	5/8
$A_9 = \lceil 1\,;6J \rfloor$	11/16
$A_{10} = \lceil 1\,;5J \rfloor$	3/4
$A_{11} = \lceil 1\,;3J \rfloor$	7/8
$A_{12} = \lceil 1\,;2J \rfloor$	15/16
$A_{13} = \lceil 1\,;15J \rfloor$	9/8

Nontrivial Products	
$A_2 \star A_2 = A_4$	$A_2 \star A_3 = A_5$
$A_2 \star A_4 = \pm 2A_7$	$A_2 \star A_5 = A_8$
$A_2 \star A_6 = A_9$	$A_2 \star A_7 = \pm A_{10}$
$A_2 \star A_9 = A_{11}$	$A_2 \star A_{10} = A_{12}$
$A_2 \star A_{12} = A_{13}$	$A_3 \star A_3 = ???$
$A_3 \star A_4 = A_8$	$A_3 \star A_5 = ???$
$A_3 \star A_6 = A_{10}$	$A_3 \star A_8 = ???$
$A_3 \star A_9 = A_{12}$	$A_3 \star A_{11} = A_{13}$
$A_4 \star A_4 = ???$	$A_4 \star A_6 = A_{11}$
$A_4 \star A_7 = A_{12}$	$A_4 \star A_{10} = A_{13}$
$A_5 \star A_5 = ???$	$A_5 \star A_6 = A_{12}$
$A_5 \star A_9 = A_{13}$	$A_6 \star A_8 = A_{13}$
$A_7 \star A_7 = -\frac{1}{2}A_{13}$	

All three-point correlators can be determined by the axioms in Section 2.3.6 except

$$\langle A_2, A_4, A_7\rangle \quad \langle A_3, A_3, A_8\rangle \quad \langle A_3, A_5, A_5\rangle \quad \langle A_4, A_4, A_4\rangle$$

We know that $\langle A_2, A_4, A_7\rangle = a = \pm 1$, and the reconstruction lemma can be used to show that $\langle A_3, A_3, A_8\rangle = \langle A_3, A_5, A_5\rangle = \langle A_4, A_4, A_4\rangle = -2$. This gives us the following new product information:

$$A_3 \star A_3 = -2A_6 \quad A_3 \star A_5 = -2A_9 \quad A_3 \star A_8 = -2A_{11}$$
$$A_4 \star A_4 = -2A_{10} \quad A_5 \star A_5 = -2A_{11}$$

So, the primitive elements in this ring are A_1, A_2, and A_3. Lemma 25 and Equation 4.5 tell us the the entire Frobenius manifold structure can be determined from genus-0 k-point correlators for

$$k \leq 2 + \frac{1 + 9/8}{1 - 1/4} \leq 5$$

There are seven basic genus-zero four-point correlators, three of which are concave. Using the formula for Λ, and the reconstruction lemma, we find that

$$\langle A_2, A_2, A_8, A_{13}\rangle = \tfrac{1}{4} \quad \langle A_2, A_3, A_{10}, A_{12}\rangle = -\tfrac{1}{8}$$
$$\langle A_2, A_2, A_{11}, A_{11}\rangle = \tfrac{1}{8} \quad \langle A_2, A_3, A_7, A_{13}\rangle = -\tfrac{a}{8}$$
$$\langle A_3, A_3, A_6, A_{13}\rangle = \tfrac{3}{8} \quad \langle A_3, A_3, A_9, A_{12}\rangle = \tfrac{1}{4}$$
$$\langle A_3, A_3, A_{10}, A_{11}\rangle = \tfrac{1}{8}$$

There are no basic 5-point genus-zero correlators.

5.2.14 $S_{12} = x^2y + y^2z + xz^3$**.** For this singularity we find that

$$J = \left(\frac{4}{13}, \frac{5}{13}, \frac{3}{13}\right), \text{ and } G^{max} = \left\langle \left(\frac{4}{13}, \frac{5}{13}, \frac{3}{13}\right)\right\rangle.$$

So, $\langle J\rangle = G^{max}$ is the only admissible group.

The FJRW vector space structure for $G = G^{max} = \langle J\rangle$ with is given by

Element	$\mathbb{C}$-Degree
$A_1 = \lceil 1\,; J\rfloor$	0
$A_2 = \lceil 1\,; 11J\rfloor$	3/13
$A_3 = \lceil 1\,; 10J\rfloor$	4/13
$A_4 = \lceil 1\,; 9J\rfloor$	5/13
$A_5 = \lceil 1\,; 8J\rfloor$	6/13
$A_6 = \lceil 1\,; 7J\rfloor$	7/13
$A_7 = \lceil 1\,; 6J\rfloor$	8/13
$A_8 = \lceil 1\,; 5J\rfloor$	9/13
$A_9 = \lceil 1\,; 4J\rfloor$	10/13
$A_{10} = \lceil 1\,; 3J\rfloor$	11/13
$A_{11} = \lceil 1\,; 2J\rfloor$	12/13
$A_{12} = \lceil 1\,; 12J\rfloor$	15/13

Nontrivial Products	
$A_2 \star A_2 = A_5$	$A_2 \star A_3 = A_6$
$A_2 \star A_4 = A_7$	$A_2 \star A_5 =???$
$A_2 \star A_6 = A_9$	$A_2 \star A_7 = A_{10}$
$A_2 \star A_8 = A_{11}$	$A_2 \star A_{11} = A_{12}$
$A_3 \star A_3 =???$	$A_3 \star A_4 = A_8$
$A_3 \star A_5 = A_9$	$A_3 \star A_6 =???$
$A_3 \star A_7 = A_{11}$	$A_3 \star A_{10} = A_{12}$
$A_4 \star A_4 =???$	$A_4 \star A_5 = A_{10}$
$A_4 \star A_6 = A_{11}$	$A_4 \star A_9 = A_{12}$
$A_5 \star A_5 =???$	$A_5 \star A_8 = A_{12}$
$A_6 \star A_7 = A_{12}$	

All three-point correlators can be determined by the axioms in Section 2.3.6 except

$$\langle A_2, A_5, A_5\rangle \quad \langle A_3, A_3, A_6\rangle \quad \langle A_4, A_4, A_4\rangle$$

The reconstruction lemma tells us that each of these are nonzero, and in fact $\langle A_2, A_5, A_5\rangle = \langle A_3, A_3, A_6\rangle = -2$, $\langle A_4, A_4, A_4\rangle = -3$. The new product information we gain is:

$$A_2 \star A_5 = -2A_8 \quad A_3 \star A_3 = -2A_7 \quad A_3 \star A_6 = -2A_{10}$$
$$A_4 \star A_4 = -3A_9 \quad A_5 \star A_5 = -2A_{11}$$

The primitive elements in this ring are A_1, A_2, A_3, and A_4. Lemma 25 and Equation 4.5 tell us the the entire Frobenius manifold structure can be determined from genus-0 k-point correlators for

$$k \leq 2 + \frac{1 + 15/13}{1 - 5/13} \leq 6 \tag{5.3}$$

The nonzero basic genus-zero four-point correlators are shown below. Using the formula for Λ, and the reconstruction lemma, we find that

$$\langle A_2, A_2, A_6, A_{12}\rangle = \tfrac{3}{13} \qquad \langle A_2, A_2, A_9, A_{11}\rangle = \tfrac{1}{13}$$
$$\langle A_2, A_3, A_8, A_{11}\rangle = -\tfrac{2}{13} \qquad \langle A_2, A_4, A_7, A_{11}\rangle = \tfrac{1}{13}$$
$$\langle A_2, A_4, A_9, A_9\rangle = -\tfrac{2}{13} \qquad \langle A_3, A_4, A_7, A_{10}\rangle = -\tfrac{3}{13}$$
$$\langle A_2, A_2, A_{10}, A_{10}\rangle = -\tfrac{2}{13} \qquad \langle A_2, A_3, A_5, A_{12}\rangle = \tfrac{3}{13}$$
$$\langle A_2, A_3, A_9, A_{10}\rangle = \tfrac{1}{13} \qquad \langle A_2, A_4, A_4, A_{12}\rangle = \tfrac{4}{13}$$
$$\langle A_2, A_4, A_8, A_{10}\rangle = \tfrac{1}{13} \qquad \langle A_3, A_3, A_4, A_{12}\rangle = \tfrac{5}{13}$$
$$\langle A_3, A_3, A_7, A_{11}\rangle = \tfrac{3}{13} \qquad \langle A_3, A_3, A_8, A_{10}\rangle = \tfrac{1}{13}$$
$$\langle A_3, A_3, A_9, A_9\rangle = -\tfrac{2}{13} \qquad \langle A_3, A_4, A_6, A_{11}\rangle = \tfrac{3}{13}$$
$$\langle A_3, A_4, A_8, A_9\rangle = \tfrac{1}{13} \qquad \langle A_4, A_4, A_5, A_{11}\rangle = \tfrac{5}{13}$$
$$\langle A_4, A_4, A_6, A_{10}\rangle = \tfrac{1}{13} \qquad \langle A_4, A_4, A_7, A_9\rangle = \tfrac{2}{13}$$
$$\langle A_4, A_4, A_8, A_8\rangle = -\tfrac{3}{13}$$

There are no basic 5 or 6-point genus-zero correlators.

5.3 Bimodal Singularities of Corank 2

5.3.1 $W = J_{3,0} = x^3 + y^9, G = \langle J \rangle$. Notice that $J_{3,0} = x^3 + y^9 = A_2 + A_8$, so G^{max} for this singularity will split, and the entire A model structure can be determined using the sums of singularities axiom, and the computations in [1].

Using $G = \langle J \rangle$, we find the vector space basis,

Element	$\mathbb{C}$-Degree
$A_1 = \lceil 1\,; J \rfloor$	0
$A_2 = \lceil 1\,; 4J \rfloor$	1/3
$A_3 = \lceil 1\,; 2J \rfloor$	4/9
$A_4 = \lceil xy^2\,; \mathbf{0} \rfloor$	5/9
$A_5 = \lceil y^5\,; \mathbf{0} \rfloor$	5/9
$A_6 = \lceil 1\,; 7J \rfloor$	2/3
$A_7 = \lceil 1\,; 5J \rfloor$	7/9
$A_8 = \lceil 1\,; 8J \rfloor$	10/9

All three-point correlators can be found using the axioms. The primitive elements are A_1, A_2, A_3, A_4, and A_5, and the Frobenius manifold structure can be determined from the basic genus-zero, k-point correlators for $k \leq 7$.

The nonzero basic genus-zero, four-point correlators are

$$\langle A_2, A_2, A_2, A_8 \rangle = \tfrac{1}{9} \qquad \langle A_2, A_2, A_6, A_7 \rangle = \tfrac{2}{9}$$
$$\langle A_2, A_3, A_6, A_6 \rangle = \tfrac{1}{9} \qquad \langle A_2, A_3, A_3, A_7 \rangle = \tfrac{1}{3}$$
$$\langle A_2, A_4, A_5, A_6 \rangle = -\tfrac{1}{243} \qquad \langle A_3, A_4, A_4, A_4 \rangle = a$$
$$\langle A_3, A_5, A_5, A_5 \rangle = -\tfrac{1}{531{,}441a},$$

for some nonzero constant a.

The nonzero basic genus-zero, five-point correlators are

$$\langle A_2, A_3, A_3, A_7, A_8\rangle = \tfrac{1}{27} \qquad \langle A_2, A_4, A_4, A_4, A_8\rangle = -\tfrac{2a}{9}$$
$$\langle A_2, A_5, A_5, A_5, A_8\rangle = \tfrac{2}{4{,}782{,}969a} \qquad \langle A_3, A_3, A_3, A_6, A_8\rangle = \tfrac{1}{27}$$
$$\langle A_3, A_3, A_4, A_5, A_8\rangle = -\tfrac{1}{729} \qquad \langle A_3, A_4, A_5, A_7, A_7\rangle = -\tfrac{1}{729}$$
$$\langle A_4, A_4, A_4, A_6, A_7\rangle = -\tfrac{2a}{9} \qquad \langle A_5, A_5, A_5, A_6, A_7\rangle = \tfrac{2}{4{,}782{,}969a}$$

The nonzero basic genus-zero, six-point correlators are

$$\langle A_2, A_3, A_4, A_5, A_8, A_8\rangle = \tfrac{1}{6561} \quad \langle A_4, A_4, A_5, A_5, A_7, A_8\rangle = -\tfrac{2}{177147}$$

There are no nonzero basic seven-point correlators.

5.3.2 $W = Z_{1,0} = x^3y + y^7, G = \langle J\rangle$. Unfortunately we do not have enough information to determine the values of the basic higher point correlators in this case.

5.3.3 $W = Z_{1,0} = x^3y + y^7, G = G^{max}$. If we let $\gamma = \left(\frac{1}{21}, \frac{6}{7}\right)$, then the vector space basis for this example is given by

Element	$\mathbb{C}$-Degree	Element	$\mathbb{C}$-Degree
$A_1 = \lceil 1\,; 6\gamma\rfloor$	0	$A_2 = \lceil 1\,; 5\gamma\rfloor$	2/21
$A_3 = \lceil 1\,; 4\gamma\rfloor$	4/21	$A_4 = \lceil 1\,; 3\gamma\rfloor$	2/7
$A_5 = \lceil 1\,; 13\gamma\rfloor$	1/3	$A_6 = \lceil 1\,; 2\gamma\rfloor$	8/21
$A_7 = \lceil 1\,; 12\gamma\rfloor$	3/7	$A_8 = \lceil 1\,; \gamma\rfloor$	10/21
$A_9 = \lceil 1\,; 11\gamma\rfloor$	11/21	$A_{10} = \lceil x^2\,; \mathbf{0}\rfloor$	4/7
$A_{11} = \lceil 1\,; 10\gamma\rfloor$	13/21	$A_{12} = \lceil 1\,; 20\gamma\rfloor$	2/3
$A_{13} = \lceil 1\,; 9\gamma\rfloor$	5/7	$A_{14} = \lceil 1\,; 19\gamma\rfloor$	16/21
$A_{15} = \lceil 1\,; 8\gamma\rfloor$	17/21	$A_{16} = \lceil 1\,; 18\gamma\rfloor$	6/7
$A_{17} = \lceil 1\,; 17\gamma\rfloor$	20/21	$A_{18} = \lceil 1\,; 16\gamma\rfloor$	22/21
$A_{19} = \lceil 1\,; 15\gamma\rfloor$	8/7		

All three-point correlators can be found using the axioms, except for $\langle A_2, A_8, A_{10}\rangle$, $\langle A_3, A_6, A_{10}\rangle$, and $\langle A_4, A_4, A_{10}\rangle$ which must each be equal to either plus or minus 1, and $\langle A_3, A_8, A_8\rangle$, $\langle A_4, A_6, A_8\rangle$, and $\langle A_6, A_6, A_6\rangle$ which the reconstruction lemma shows are all equal to -3. We set the value of $\langle A_2, A_8, A_{10}\rangle = a$. The primitive elements are A_1, A_2, and A_5, and the Frobenius manifold structure can be determined from the basic genus-zero, k-point correlators for $k \leq 6$.

The nonzero basic genus-zero, four-point correlators are

$$\begin{array}{cc}
\langle A_2, A_2, A_{15}, A_{19}\rangle = \frac{1}{7} & \langle A_2, A_5, A_{12}, A_{18}\rangle = -\frac{1}{21} \\
\langle A_2, A_5, A_{14}, A_{17}\rangle = -\frac{1}{21} & \langle A_2, A_5, A_{16}, A_{16}\rangle = -\frac{1}{21} \\
\langle A_5, A_5, A_5, A_{19}\rangle = -\frac{2}{7} & \langle A_5, A_5, A_7, A_{18}\rangle = \frac{5}{21} \\
\langle A_5, A_5, A_9, A_{17}\rangle = \frac{4}{21} & \langle A_5, A_5, A_{11}, A_{16}\rangle = \frac{1}{21} \\
\langle A_5, A_5, A_{12}, A_{15}\rangle = \frac{1}{21} & \langle A_5, A_5, A_{13}, A_{14}\rangle = \frac{2}{21} \\
\langle A_2, A_5, A_{10}, A_{19}\rangle = \frac{1-2a}{21}, &
\end{array}$$

There are no nonzero basic five or six-point correlators.

5.3.4 $W = W_{1,0} = x^4 + y^6, G = \langle J\rangle$.

Notice that $W_{1,0} = x^4 + y^6 = A_3 + A_5$, so G^{max} for this singularity will split, and the entire A model structure can be determined using the sums of singularities axiom, and the computations in [1].

Using $G = \langle J\rangle$, we find the vector space basis,

Element	$\mathbb{C}$-Degree
$A_1 = \lceil 1\,;J \rfloor$	0
$A_2 = \lceil 1\,;9J \rfloor$	1/3
$A_3 = \lceil 1\,;2J \rfloor$	5/12
$A_4 = \lceil 1\,;7J \rfloor$	1/2
$A_5 = \lceil xy^2\,;\mathbf{0} \rfloor$	7/12
$A_6 = \lceil 1\,;5J \rfloor$	2/3
$A_7 = \lceil 1\,;10J \rfloor$	3/4
$A_8 = \lceil 1\,;3J \rfloor$	5/6
$A_9 = \lceil 1\,;11J \rfloor$	7/6

All three-point correlators can be found using the axioms. The primitive elements are A_1, A_2, A_3, A_4, and A_5, and the Frobenius manifold structure can be determined from the basic genus-zero, k-point correlators for $k \leq 8$.

The nonzero basic genus-zero, four-point correlators are

$$\begin{array}{cc}
\langle A_2, A_2, A_2, A_9 \rangle = \frac{1}{6} & \langle A_2, A_2, A_6, A_8 \rangle = \frac{1}{6} \\
\langle A_2, A_2, A_7, A_7 \rangle = \frac{1}{3} & \langle A_2, A_3, A_6, A_7 \rangle = \frac{1}{6} \\
\langle A_3, A_3, A_4, A_8 \rangle = \frac{1}{4} & \langle A_3, A_3, A_6, A_6 \rangle = \frac{1}{6} \\
\langle A_3, A_4, A_4, A_7 \rangle = \frac{1}{4} & \langle A_2, A_5, A_5, A_6 \rangle = -\frac{1}{144} \\
\langle A_4, A_4, A_5, A_5 \rangle = -\frac{1}{96} &
\end{array}$$

The nonzero basic genus-zero, five-point correlators are

$$\langle A_2, A_3, A_3, A_8, A_9\rangle = \tfrac{1}{24} \qquad \langle A_2, A_3, A_4, A_7, A_9\rangle = \tfrac{1}{24}$$
$$\langle A_2, A_4, A_5, A_5, A_9\rangle = \tfrac{1}{576} \qquad \langle A_3, A_3, A_3, A_7, A_9\rangle = \tfrac{1}{24}$$
$$\langle A_3, A_3, A_4, A_6, A_9\rangle = \tfrac{1}{24} \qquad \langle A_3, A_3, A_5, A_5, A_9\rangle = -\tfrac{1}{576}$$
$$\langle A_3, A_5, A_5, A_7, A_8\rangle = -\tfrac{1}{576} \qquad \langle A_4, A_4, A_4, A_4, A_9\rangle = \tfrac{1}{8}$$
$$\langle A_4, A_4, A_4, A_8, A_8\rangle = \tfrac{1}{8} \qquad \langle A_4, A_5, A_5, A_6, A_8\rangle = \tfrac{1}{576}$$
$$\langle A_5, A_5, A_5, A_5, A_8\rangle = \tfrac{1}{13824}$$

The nonzero basic genus-zero, six-point correlators are

$$\langle A_2, A_2, A_5, A_5, A_9, A_9\rangle = -\tfrac{1}{1728} \qquad \langle A_3, A_3, A_4, A_4, A_9, A_9\rangle = \tfrac{1}{48}$$
$$\langle A_4, A_4, A_5, A_5, A_8, A_9\rangle = -\tfrac{1}{1152} \quad \langle A_5, A_5, A_5, A_5, A_6, A_9\rangle = -\tfrac{1}{27{,}648}$$

The only nonzero basic genus-zero, seven-point correlators is

$$\langle A_4, A_5, A_5, A_5, A_5, A_9, A_9\rangle = \tfrac{1}{55{,}296}$$

There are no nonzero basic eight-point correlators.

5.4 Bimodal Singularities of Corank 3

5.4.1 $W = Q_{2,0} = x^3 + xy^4 + yz^2, G = \langle J\rangle$**.** Unfortunately, we do not have enough information to determine the values of all basic correlators for $k \leq 8$.

5.4.2 $W = Q_{2,0} = x^3 + xy^4 + yz^2, G = G^{max}$**.**

Element	$\mathbb{C}$-Degree	Element	$\mathbb{C}$-Degree
$A_1 = \lceil 1\,; 14\gamma \rfloor$	0	$A_2 = \lceil 1\,; 17\gamma \rfloor$	1/8
$A_3 = \lceil 1\,; 20\gamma \rfloor$	1/4	$A_4 = \lceil 1\,; 13\gamma \rfloor$	7/24
$A_5 = \lceil 1\,; 23\gamma \rfloor$	3/8	$A_6 = \lceil 1\,; 16\gamma \rfloor$	5/12
$A_7 = \lceil 1\,; 2\gamma \rfloor$	1/2	$A_8 = \lceil 1\,; 19\gamma \rfloor$	13/24
$A_9 = \lceil y^3\,; 12\gamma \rfloor$	7/12	$A_{10} = \lceil 1\,; 5\gamma \rfloor$	5/8
$A_{11} = \lceil 1\,; 22\gamma \rfloor$	2/3	$A_{12} = \lceil 1\,; 8\gamma \rfloor$	3/4
$A_{13} = \lceil 1\,; \gamma \rfloor$	19/24	$A_{14} = \lceil 1\,; 11\gamma \rfloor$	7/8
$A_{15} = \lceil 1\,; 4\gamma \rfloor$	11/12	$A_{16} = \lceil 1\,; 7\gamma \rfloor$	25/24
$A_{17} = \lceil 1\,; 10\gamma \rfloor$	7/6		

All three-point correlators can be found using the axioms, except $\langle A_4, A_4, A_9 \rangle$ which we know must equal plus or minus one, and $\langle A_2, A_5, A_{11} \rangle$, $\langle A_3, A_3, A_{11} \rangle$, $\langle A_3, A_5, A_8 \rangle$, and $\langle A_5, A_5, A_6 \rangle$ which must each be equal to -2 by the reconstruction lemma. We will say that the value of $\langle A_4, A_4, A_9 \rangle$ is a. The primitive elements are A_1, A_2, and A_3, and the Frobenius manifold structure can be determined from the basic genus-zero, k-point correlators for $k \leq 6$.

The nonzero basic genus-zero, four-point correlators are

$$\langle A_2, A_4, A_9, A_{17} \rangle = -\tfrac{a}{6} \qquad \langle A_4, A_4, A_6, A_{17} \rangle = \tfrac{1}{3}$$
$$\langle A_4, A_4, A_8, A_{16} \rangle = \tfrac{1}{3} \qquad \langle A_4, A_4, A_{11}, A_{15} \rangle = \tfrac{1}{3}$$
$$\langle A_4, A_4, A_{13}, A_{13} \rangle = -\tfrac{1}{6}$$

There are no nonzero basic genus-zero five or six-point correlators.

5.4.3 $W = Q^T_{2,0} = x^3y + y^4z + z^2, G = \langle J \rangle = G^{max}$.

Element	$\mathbb{C}$-Degree	Element	$\mathbb{C}$-Degree
$A_1 = \lceil 1\,;J \rfloor$	0	$A_2 = \lceil 1\,;11J \rfloor$	1/8
$A_3 = \lceil 1\,;21J \rfloor$	1/4	$A_4 = \lceil 1\,;9J \rfloor$	7/24
$A_5 = \lceil y^3\,;8J \rfloor$	3/8	$A_6 = \lceil 1\,;7J \rfloor$	5/12
$A_7 = \lceil 1\,;19J \rfloor$	1/2	$A_8 = \lceil 1\,;5J \rfloor$	13/24
$A_9 = \lceil 1\,;17J \rfloor$	7/12	$A_{10} = \lceil y^3\,;16J \rfloor$	5/8
$A_{11} = \lceil 1\,;15J \rfloor$	2/3	$A_{12} = \lceil 1\,;3J \rfloor$	3/4
$A_{13} = \lceil 1\,;13J \rfloor$	19/24	$A_{14} = \lceil 1\,;23J \rfloor$	7/8

All three-point correlators can be found using the axioms, except $\langle A_4, A_5, A_5 \rangle$ which we can show to be some nonzero complex number a, and $\langle A_2, A_6, A_6 \rangle$, and $\langle A_3, A_3, A_6 \rangle$, which must each be equal to -3 by the reconstruction lemma. The primitive elements are A_1, A_2, A_4, and A_5, and the Frobenius manifold structure can be determined from the basic genus-zero, k-point correlators for $k \leq 6$.

The nonzero basic genus-zero, four-point correlators are

$$
\begin{array}{ll}
\langle A_2, A_2, A_8, A_{14} \rangle = \frac{1}{8} & \langle A_2, A_2, A_{11}, A_{13} \rangle = \frac{1}{8} \\
\langle A_2, A_4, A_9, A_{13} \rangle = -\frac{1}{12} & \langle A_2, A_4, A_{12}, A_{12} \rangle = \frac{-1}{12} \\
\langle A_4, A_4, A_4, A_{14} \rangle = \frac{7}{24} & \langle A_4, A_4, A_7, A_{13} \rangle = \frac{5}{24} \\
\langle A_4, A_4, A_8, A_{12} \rangle = \frac{1}{8} & \langle A_4, A_4, A_9, A_{11} \rangle = \frac{1}{24} \\
\langle A_2, A_4, A_6, A_{14} \rangle = \frac{1}{8} & \langle A_2, A_5, A_5, A_{14} \rangle = -\frac{a}{8} \\
\langle A_2, A_5, A_{10}, A_{11} \rangle = \frac{1}{32} & \langle A_3, A_5, A_5, A_{13} \rangle = -\frac{a}{8} \\
\langle A_4, A_4, A_{10}, A_{10} \rangle = \frac{1}{48a} & \langle A_4, A_5, A_9, A_{10} \rangle = -\frac{1}{96} \\
\langle A_5, A_5, A_6, A_{12} \rangle = -\frac{a}{8} & \langle A_5, A_5, A_7, A_{11} \rangle = -\frac{a}{8} \\
\langle A_5, A_5, A_8, A_8 \rangle = -\frac{a}{8} & \langle A_5, A_5, A_9, A_9 \rangle = \frac{a}{24}
\end{array}
$$

The nonzero basic genus-zero, five-point correlators are

$$\langle A_2, A_4, A_4, A_{14}, A_{14}\rangle = -\tfrac{1}{96} \quad \langle A_4, A_5, A_5, A_{12}, A_{14}\rangle = -\tfrac{a}{96}$$
$$\langle A_4, A_5, A_5, A_{13}, A_{13}\rangle = -\tfrac{a}{96}$$

There are no nonzero basic genus-zero six-point correlators.

5.4.4 $W = S_{1,0} = x^2y + y^2z + z^5, G = \langle J\rangle$. Unfortunately, we do not have enough information to find the value of this correlator nor many of the higher point basic correlators.

5.4.5 $W = S_{1,0} = x^2y + y^2z + z^5, G = G^{max}$. For $\gamma = \left(\frac{1}{20}, \frac{9}{10}, \frac{1}{5}\right)$, the vector space basis is given by,

Element	$\mathbb{C}$-Degree	Element	$\mathbb{C}$-Degree
$A_1 = \lceil 1\,; 6\gamma\rfloor$	0	$A_2 = \lceil 1\,; 7\gamma\rfloor$	3/20
$A_3 = \lceil 1\,; \gamma\rfloor$	1/4	$A_4 = \lceil 1\,; 8\gamma\rfloor$	3/10
$A_5 = \lceil 1\,; 2\gamma\rfloor$	2/5	$A_6 = \lceil 1\,; 9\gamma\rfloor$	9/20
$A_7 = \lceil 1\,; 16\gamma\rfloor$	1/2	$A_8 = \lceil 1\,; 3\gamma\rfloor$	11/20
$A_9 = \lceil y\,; 10\gamma\rfloor$	3/5	$A_{10} = \lceil 1\,; 17\gamma\rfloor$	13/20
$A_{11} = \lceil 1\,; 4\gamma\rfloor$	7/10	$A_{12} = \lceil 1\,; 11\gamma\rfloor$	3/4
$A_{13} = \lceil 1\,; 18\gamma\rfloor$	4/5	$A_{14} = \lceil 1\,; 12\gamma\rfloor$	9/10
$A_{15} = \lceil 1\,; 19\gamma\rfloor$	19/20	$A_{16} = \lceil 1\,; 13\gamma\rfloor$	21/20
$A_{17} = \lceil 1\,; 14\gamma\rfloor$	6/5		

All three-point correlators can be found using the axioms, except $\langle A_2, A_6, A_9\rangle$ and $\langle A_4, A_4, A_9\rangle$ which we can show must be equal to one, and $\langle A_3, A_3, A_{11}\rangle$, $\langle A_3, A_5, A_8\rangle$, $\langle A_4, A_6, A_6\rangle$, and $\langle A_5, A_5, A_5\rangle$ which we can show must all equal -2. The primitive elements are A_1, A_2, and A_3, and the Frobenius manifold structure can be determined from the basic genus-zero, k-point correlators for $k \leq 5$.

The nonzero basic genus-zero, four-point correlators are

$$\begin{array}{cc} \langle A_2, A_2, A_{11}, A_{17}\rangle = \frac{1}{5} & \langle A_2, A_3, A_{12}, A_{16}\rangle = -\frac{1}{10} \\ \langle A_2, A_3, A_{14}, A_{14}\rangle = -\frac{1}{10} & \langle A_2, A_2, A_{15}, A_{15}\rangle = -\frac{1}{10} \\ \langle A_2, A_3, A_9, A_{17}\rangle = -\frac{1}{10} & \langle A_3, A_3, A_7, A_{17}\rangle = \frac{2}{5} \\ \langle A_3, A_3, A_{10}, A_{16}\rangle = \frac{3}{10} & \langle A_3, A_3, A_{12}, A_{15}\rangle = \frac{1}{10} \\ \langle A_3, A_3, A_{13}, A_{14}\rangle = \frac{1}{5} & \end{array}$$

There are no nonzero basic genus-zero five-point correlators.

5.4.6 $W = S_{1,0}^T = x^2 + xy^2 + yz^5, G = \langle J\rangle = G^{max}$**.** In this example there are many unknown three-point and higher-point correlators. We do not have enough information to find their values.

5.5 Bimodal Exceptional Singularities

5.5.1 $W = E_{18} = x^3 + y^{10}, G = \langle J\rangle = G^{max}$**.** For this singularity we find that $J = \left(\frac{1}{3}, \frac{1}{10}\right)$, and $G^{max} = \left\langle\left(\frac{1}{3}, \frac{1}{10}\right)\right\rangle$. So, $\langle J\rangle = G^{max}$ is the only admissible group. This group splits using the decomposition, $E_{18} = A_2 + A_9$. So the entire A model structure can be determined using the sums of singularities axiom, and the computations in [1].

5.5.2 $W = E_{19} = x^3 + xy^7, G = \langle J\rangle = G^{max}$**.**

Element	$\mathbb{C}$-Degree	Element	$\mathbb{C}$-Degree
$A_1 = \lceil 1\,;J\rfloor$	0	$A_2 = \lceil 1\,;13J\rfloor$	1/7
$A_3 = \lceil 1\,;4J\rfloor$	2/7	$A_4 = \lceil 1\,;11J\rfloor$	2/7
$A_5 = \lceil 1\,;16J\rfloor$	3/7	$A_6 = \lceil 1\,;2J\rfloor$	3/7
$A_7 = \lceil 1\,;7J\rfloor$	4/7	$A_8 = \lceil 1\,;14J\rfloor$	4/7
$A_9 = \lceil y^6\,;\mathbf{0}\rfloor$	4/7	$A_{10} = \lceil 1\,;19J\rfloor$	5/7
$A_{11} = \lceil 1\,;5J\rfloor$	5/7	$A_{12} = \lceil 1\,;10J\rfloor$	6/7
$A_{13} = \lceil 1\,;17J\rfloor$	6/7	$A_{14} = \lceil 1\,;8J\rfloor$	1
$A_{15} = \lceil 1\,;20J\rfloor$	8/7		

All three-point correlators can be found using the axioms, except for $\langle A_4, A_4, A_9\rangle$ which we know to be ± 1. This means that the primitive elements are A_1, A_2, and A_4, and the Frobenius manifold structure can be determined from the basic genus-zero, k-point correlators for $k \leq 5$.

The nonzero basic genus-zero, four-point correlators are

$$\langle A_2, A_2, A_{10}, A_{15}\rangle = \tfrac{2}{21} \qquad \langle A_2, A_2, A_{12}, A_{14}\rangle = \tfrac{1}{21}$$
$$\langle A_2, A_4, A_{12}, A_{12}\rangle = -\tfrac{1}{21} \qquad \langle A_2, A_4, A_9, A_{15}\rangle = \tfrac{a}{1029}$$
$$\langle A_4, A_4, A_6, A_{15}\rangle = \tfrac{1}{3} \qquad \langle A_4, A_4, A_8, A_{14}\rangle = \tfrac{1}{3}$$
$$\langle A_4, A_4, A_{11}, A_{13}\rangle = \tfrac{1}{3}$$

There are no nonzero basic five-point correlators.

5.5.3 $W = E_{20} = x^3 + y^{11}, G = \langle J\rangle = G^{max}$.

For this singularity we find that $J = \left(\frac{1}{3}, \frac{1}{11}\right)$, and $G^{max} = \left\langle\left(\frac{1}{3}, \frac{1}{11}\right)\right\rangle$. So, $\langle J\rangle = G^{max}$ is the only admissible group. This group splits using the decomposition, $E_{20} = A_2 + A_{10}$. So the entire A model structure can be determined using the sums of singularities axiom, and the computations in [1].

5.5.4 $W = Z_{17} = x^3 y + y^8, G = \langle J\rangle = G^{max}$.

Element	$\mathbb{C}$-Degree	Element	$\mathbb{C}$-Degree
$A_1 = \lceil 1\,;J \rfloor$	0	$A_2 = \lceil 1\,;18J \rfloor$	1/12
$A_3 = \lceil 1\,;11J \rfloor$	1/6	$A_4 = \lceil 1\,;4J \rfloor$	1/4
$A_5 = \lceil 1\,;21J \rfloor$	1/3	$A_6 = \lceil 1\,;9J \rfloor$	1/3
$A_7 = \lceil 1\,;14J \rfloor$	5/12	$A_8 = \lceil 1\,;2J \rfloor$	5/12
$A_9 = \lceil 1\,;7J \rfloor$	1/2	$A_{10} = \lceil 1\,;19J \rfloor$	1/2
$A_{11} = \lceil 1\,;12J \rfloor$	7/12	$A_{12} = \lceil 1\,;\mathbf{0} \rfloor$	7/12
$A_{13} = \lceil 1\,;5J \rfloor$	2/3	$A_{14} = \lceil 1\,;17J \rfloor$	2/3
$A_{15} = \lceil 1\,;10J \rfloor$	3/4	$A_{16} = \lceil 1\,;22J \rfloor$	3/4
$A_{17} = \lceil 1\,;15J \rfloor$	5/6	$A_{18} = \lceil 1\,;3J \rfloor$	5/6
$A_{19} = \lceil 1\,;20J \rfloor$	11/3	$A_{20} = \lceil 1\,;13J \rfloor$	1
$A_{21} = \lceil 1\,;6J \rfloor$	13/12	$A_{22} = \lceil 1\,;23J \rfloor$	7/6

All three-point correlators can be found using the axioms, except

$$\begin{array}{lll} \langle A_2, A_9, A_{12}\rangle & \langle A_3, A_7, A_{12}\rangle & \langle A_4, A_5, A_{12}\rangle \\ \langle A_3, A_9, A_9\rangle & \langle A_4, A_7, A_9\rangle & \langle A_5, A_5, A_9\rangle \\ \langle A_5, A_7, A_7\rangle, & & \end{array}$$

where the first three must each be equal to ± 1. Using the reconstruction lemma we can show that the last four are all equal to -3.

The primitive elements are A_1, A_2, and A_6, and the Frobenius manifold structure can be determined from the basic genus-zero, k-point correlators for $k \leq 6$.

The nonzero basic genus-zero, four-point correlators are

$$\langle A_2, A_2, A_{17}, A_{22}\rangle = \tfrac{1}{8} \qquad \langle A_2, A_6, A_{14}, A_{21}\rangle = -\tfrac{1}{24}$$
$$\langle A_2, A_6, A_{15}, A_{20}\rangle = -\tfrac{1}{24} \qquad \langle A_2, A_6, A_{18}, A_{19}\rangle = -\tfrac{1}{24}$$
$$\langle A_6, A_6, A_6, A_{22}\rangle = \tfrac{7}{24} \qquad \langle A_6, A_6, A_8, A_{21}\rangle = \tfrac{1}{4}$$
$$\langle A_6, A_6, A_{10}, A_{20}\rangle = \tfrac{5}{24} \qquad \langle A_6, A_6, A_{11}, A_{19}\rangle = \tfrac{1}{6}$$
$$\langle A_6, A_6, A_{13}, A_{18}\rangle = \tfrac{1}{8} \qquad \langle A_6, A_6, A_{14}, A_{17}\rangle = \tfrac{1}{24}$$
$$\langle A_6, A_6, A_{15}, A_{16}\rangle = \tfrac{1}{12} \qquad \langle A_2, A_6, A_{12}, A_{22}\rangle = -\tfrac{1}{24}$$

There are no nonzero basic five or six-point correlators.

5.5.5 $W = Z_{17}^T = x^3 + xy^8, G = G^{max}$**.** If $\gamma = (2/3, 1/24)$, then

Element	$\mathbb{C}$-Degree	Element	$\mathbb{C}$-Degree
$A_1 = \lceil 1\,; 2\gamma \rfloor$	0	$A_2 = \lceil 1\,; 5\gamma \rfloor$	1/8
$A_3 = \lceil 1\,; 8\gamma \rfloor$	1/4	$A_4 = \lceil 1\,; \gamma \rfloor$	7/24
$A_5 = \lceil 1\,; 11\gamma \rfloor$	3/8	$A_6 = \lceil 1\,; 4\gamma \rfloor$	5/12
$A_7 = \lceil 1\,; 14\gamma \rfloor$	1/2	$A_8 = \lceil 1\,; 7\gamma \rfloor$	13/24
$A_9 = \lceil y^7\,; \mathbf{0} \rfloor$	7/12	$A_{10} = \lceil 1\,; 17\gamma \rfloor$	5/8
$A_{11} = \lceil 1\,; 10\gamma \rfloor$	2/3	$A_{12} = \lceil 1\,; 20\gamma \rfloor$	3/4
$A_{13} = \lceil 1\,; 13\gamma \rfloor$	19/24	$A_{14} = \lceil 1\,; 23\gamma \rfloor$	7/8
$A_{15} = \lceil 1\,; 16\gamma \rfloor$	11/12	$A_{16} = \lceil 1\,; 19\gamma \rfloor$	25/24
$A_{17} = \lceil 1\,; 22\gamma \rfloor$	7/6		

All three-point correlators can be found using the axioms. The primitive elements are A_1, A_2, A_3, A_5, and A_6, and the Frobenius manifold structure can be determined from the basic genus-zero, k-point correlators for $k \leq 6$.

The nonzero basic genus-zero, four-point correlators are

$$\langle A_2, A_2, A_{12}, A_{17}\rangle = \tfrac{1}{12} \qquad \langle A_2, A_2, A_{14}, A_{16}\rangle = \tfrac{1}{24}$$
$$\langle A_2, A_4, A_{14}, A_{14}\rangle = -\tfrac{1}{24} \qquad \langle A_2, A_4, A_9, A_{17}\rangle = -\tfrac{a}{24}$$
$$\langle A_4, A_4, A_6, A_{17}\rangle = \tfrac{1}{3} \qquad \langle A_4, A_4, A_8, A_{16}\rangle = \tfrac{1}{3}$$
$$\langle A_4, A_4, A_{11}, A_{15}\rangle = \tfrac{1}{3} \qquad \langle A_4, A_4, A_{13}, A_{13}\rangle = \tfrac{1}{3}$$

There are no nonzero basic five or six-point correlators.

5.5.6 $W = Z_{17}^T = x^3 + xy^8, G = \langle J \rangle$. If $\gamma = (2/3, 1/24)$, then

Element	$\mathbb{C}$-Degree	Element	$\mathbb{C}$-Degree
$A_1 = \lceil 1\,; J \rfloor$	0	$A_2 = \lceil 1\,; 4J \rfloor$	1/8
$A_3 = \lceil 1\,; 8J \rfloor$	1/4	$A_4 = \lceil 1\,; 2J \rfloor$	7/24
$A_5 = \lceil xy^3\,; \mathbf{0} \rfloor$	3/8	$A_6 = \lceil y^7\,; \mathbf{0} \rfloor$	5/12
$A_7 = \lceil 1\,; 5J \rfloor$	1/2	$A_8 = \lceil 1\,; 10J \rfloor$	13/24
$A_9 = \lceil 1\,; 8J \rfloor$	7/12	$A_{10} = \lceil 1\,; 11J \rfloor$	5/8

All three-point correlators can be found using the axioms. The primitive elements are A_1, A_2, A_3, A_5, and A_6, and the Frobenius manifold structure can be determined from the basic genus-zero, k-point correlators for $k \leq 8$.

The nonzero basic genus-zero, four-point correlators are

$$\langle A_2, A_2, A_4, A_{10}\rangle = \tfrac{1}{12} \qquad \langle A_2, A_2, A_8, A_9\rangle = \tfrac{1}{6}$$
$$\langle A_2, A_3, A_8, A_8\rangle = -\tfrac{1}{12} \qquad \langle A_3, A_3, A_3, A_9\rangle = -\tfrac{1}{3}$$
$$\langle A_3, A_3, A_7, A_7\rangle = \tfrac{1}{3} \qquad \langle A_2, A_5, A_5, A_8\rangle = -\tfrac{1}{288}$$
$$\langle A_2 A_6, A_6, A_8\rangle = \tfrac{1}{96} \qquad \langle A_3 A_5, A_5, A_6\rangle = \pm\tfrac{1}{576}$$
$$\langle A_3 A_6, A_6, A_6\rangle = \pm\tfrac{1}{192} \qquad \langle A_4 A_4, A_5, A_5\rangle = -\tfrac{1}{288}$$
$$\langle A_4 A_4, A_6, A_6\rangle = \tfrac{1}{96}$$

Where $\langle A_3 A_5, A_5, A_6\rangle$ and $\langle A_3 A_6, A_6, A_6\rangle$ must both either be positive or negative.

The nonzero basic five-point correlators are:

$$\langle A_2, A_3, A_3, A_9, A_{10}\rangle = \tfrac{1}{36} \qquad \langle A_2, A_5, A_5, A_6, A_{10}\rangle = \mp\tfrac{1}{1152}$$
$$\langle A_2, A_6, A_6, A_6, A_{10}\rangle = \mp\tfrac{1}{3456} \qquad \langle A_3, A_3, A_3, A_8, A_{10}\rangle = \tfrac{1}{36}$$
$$\langle A_3, A_3, A_5, A_5, A_{10}\rangle = -\tfrac{1}{864} \qquad \langle A_3, A_3, A_6, A_6, A_{10}\rangle = \tfrac{1}{288}$$
$$\langle A_3, A_5, A_5, A_7, A_9\rangle = -\tfrac{1}{864} \qquad \langle A_3, A_6, A_6, A_7, A_9\rangle = \tfrac{1}{288}$$
$$\langle A_4, A_5, A_5, A_6, A_9\rangle = \mp\tfrac{1}{1152} \qquad \langle A_4, A_6, A_6, A_6, A_9\rangle = \mp\tfrac{1}{3456}$$
$$\langle A_5, A_6, A_6, A_7, A_8\rangle = \mp\tfrac{1}{1152} \qquad \langle A_6, A_6, A_6, A_7, A_8\rangle = \mp\tfrac{1}{3456}$$

The nonzero basic six-point correlators are:

$$\langle A_2, A_3, A_5, A_5, A_{10}, A_{10}\rangle = \tfrac{1}{10368} \qquad \langle A_2, A_3, A_6, A_6, A_{10}, A_{10}\rangle = -\tfrac{1}{3456}$$
$$\langle A_5, A_5, A_5, A_5, A_7, A_{10}\rangle = \tfrac{1}{82944} \qquad \langle A_5, A_5, A_5, A_5, A_9, A_9\rangle = \tfrac{1}{82944}$$
$$\langle A_5, A_5, A_6, A_6, A_7, A_{10}\rangle = -\tfrac{1}{82944} \qquad \langle A_5, A_5, A_6, A_6, A_9, A_9\rangle = -\tfrac{1}{82944}$$
$$\langle A_6, A_6, A_6, A_6, A_7, A_{10}\rangle = \tfrac{1}{9216} \qquad \langle A_6, A_6, A_6, A_6, A_9, A_9\rangle = \tfrac{1}{9216}$$

There are no nonzero basic seven or eight point correlators.

5.5.7 $W = Z_{18} = x^3y + xy^6, G = \langle J\rangle = G^{max}$**.** The three and higher point correlators for this singularity cannot be determined using the methods we are currently aware of.

5.5.8 $W = Z_{19} = x^3y + y^9, G = \langle J\rangle = G^{max}$**.**

Element	$\mathbb{C}$-Degree	Element	$\mathbb{C}$-Degree
$A_1 = \lceil 1\,; J \rfloor$	0	$A_2 = \lceil 1\,; 11J \rfloor$	2/27
$A_3 = \lceil 1\,; 21J \rfloor$	4/27	$A_4 = \lceil 1\,; 4J \rfloor$	2/9
$A_5 = \lceil 1\,; 14J \rfloor$	8/27	$A_6 = \lceil 1\,; 19J \rfloor$	1/3
$A_7 = \lceil 1\,; 24J \rfloor$	10/27	$A_8 = \lceil 1\,; 2J \rfloor$	11/27
$A_9 = \lceil 1\,; 7J \rfloor$	4/9	$A_{10} = \lceil 1\,; 12J \rfloor$	13/27
$A_{11} = \lceil 1\,; 17J \rfloor$	14/27	$A_{12} = \lceil 1\,; 22J \rfloor$	5/9
$A_{13} = \lceil 1\,; \mathbf{0} \rfloor$	16/27	$A_{14} = \lceil 1\,; 5J \rfloor$	17/27
$A_{15} = \lceil 1\,; 10J \rfloor$	2/3	$A_{16} = \lceil 1\,; 15J \rfloor$	19/27
$A_{17} = \lceil 1\,; 20J \rfloor$	20/27	$A_{18} = \lceil 1\,; 25J \rfloor$	7/9
$A_{19} = \lceil 1\,; 3J \rfloor$	22/27	$A_{20} = \lceil 1\,; 8J \rfloor$	23/27
$A_{21} = \lceil 1\,; 13J \rfloor$	8/9	$A_{22} = \lceil 1\,; 23J \rfloor$	26/27
$A_{23} = \lceil 1\,; 6J \rfloor$	28/27	$A_{24} = \lceil 1\,; 16J \rfloor$	10/9
$A_{25} = \lceil 1\,; 26J \rfloor$	32/27		

All three-point correlators can be found using the axioms except

$$\begin{array}{cc} \langle A_2, A_{11}, A_{13}\rangle & \langle A_3, A_9, A_{13}\rangle \\ \langle A_4, A_7, A_{13}\rangle & \langle A_5, A_5, A_{13}\rangle \\ \langle A_3, A_{11}, A_{11}\rangle & \langle A_4, A_9, A_{11}\rangle \\ \langle A_5, A_7, A_{11}\rangle & \langle A_5, A_9, A_9\rangle \\ \langle A_7, A_7, A_9\rangle, & \end{array}$$

where the first four are known to be equal to ± 1, and we can use the reconstruction lemma to show that the last four are all equal to -3.

This means that the primitive elements are A_1, A_2, and A_6, and the Frobenius manifold structure can be determined from the basic genus-zero, k-point correlators for $k \leq 6$.

The nonzero basic genus-zero, four-point correlators are

$$
\begin{array}{cc}
\langle A_2, A_2, A_{20}, A_{25}\rangle = \frac{1}{9} & \langle A_2, A_6, A_{15}, A_{24}\rangle = -\frac{1}{27} \\
\langle A_2, A_6, A_{17}, A_{23}\rangle = -\frac{1}{27} & \langle A_2, A_6, A_{19}, A_{22}\rangle = -\frac{1}{27} \\
\langle A_2, A_6, A_{21}, A_{21}\rangle = -\frac{1}{27} & \langle A_6, A_6, A_6, A_{25}\rangle = \frac{8}{27} \\
\langle A_6, A_6, A_8, A_{24}\rangle = \frac{7}{27} & \langle A_6, A_6, A_{10}, A_{23}\rangle = \frac{2}{9} \\
\langle A_6, A_6, A_{12}, A_{22}\rangle = \frac{5}{27} & \langle A_6, A_6, A_{14}, A_{21}\rangle = \frac{4}{27} \\
\langle A_6, A_6, A_{15}, A_{20}\rangle = \frac{1}{27} & \langle A_6, A_6, A_{16}, A_{19}\rangle = \frac{1}{9} \\
\langle A_6, A_6, A_{17}, A_{18}\rangle = \frac{2}{27} & \langle A_2, A_6, A_{13}, A_{25}\rangle = -\frac{1}{27}
\end{array}
$$

There are no nonzero basic five or six-point correlators.

5.5.9 $W = Z_{19}^T = x^3 + xy^9, G = \langle J\rangle = G^{max}$.

Element	$\mathbb{C}$-Degree	Element	$\mathbb{C}$-Degree
$A_1 = \lceil 1\,; J\rfloor$	0	$A_2 = \lceil 1\,; 16J\rfloor$	1/9
$A_3 = \lceil 1\,; 4J\rfloor$	2/9	$A_4 = \lceil 1\,; 14J\rfloor$	8/27
$A_5 = \lceil 1\,; 19J\rfloor$	1/3	$A_6 = \lceil 1\,; 2J\rfloor$	11/27
$A_7 = \lceil 1\,; 7J\rfloor$	4/9	$A_8 = \lceil 1\,; 17J\rfloor$	14/27
$A_9 = \lceil 1\,; 22J\rfloor$	5/9	$A_{10} = \lceil 1\,; \mathbf{0}\rfloor$	16/27
$A_{11} = \lceil 1\,; 5J\rfloor$	17/27	$A_{12} = \lceil 1\,; 10J\rfloor$	2/3
$A_{13} = \lceil 1\,; 20J\rfloor$	20/27	$A_{14} = \lceil 1\,; 25J\rfloor$	7/9
$A_{15} = \lceil 1\,; 8J\rfloor$	23/27	$A_{16} = \lceil 1\,; 13J\rfloor$	8/9
$A_{17} = \lceil 1\,; 23J\rfloor$	26/27	$A_{18} = \lceil 1\,; 11J\rfloor$	29/27
$A_{19} = \lceil 1\,; 26J\rfloor$	32/27		

All three-point correlators can be found using the axioms except $\langle A_4, A_4, A_{10}\rangle$, which must be equal to ± 1. The primitive elements are A_1, A_2, and A_4, and the Frobenius manifold structure can be determined from the basic genus-zero, k-point correlators for $k \leq 6$.

The nonzero basic genus-zero, four-point correlators are

$$\begin{array}{ll}
\langle A_2, A_2, A_{14}, A_{19}\rangle = \frac{2}{27} & \langle A_2, A_2, A_{16}, A_{18}\rangle = -\frac{1}{27} \\
\langle A_2, A_4, A_{16}, A_{16}\rangle = -\frac{1}{27} & \langle A_2, A_4, A_{10}, A_{19}\rangle = -\frac{a}{27} \\
\langle A_4, A_4, A_6, A_{19}\rangle = \frac{1}{3} & \langle A_4, A_4, A_8, A_{18}\rangle = \frac{1}{3} \\
\langle A_4, A_4, A_{11}, A_{17}\rangle = \frac{1}{3} & \langle A_4, A_4, A_{13}, A_{15}\rangle = \frac{1}{3}
\end{array}$$

There are no nonzero basic five or six-point correlators.

5.5.10 $W = U_{16} = x^3 + xy^2 + z^5$**.** For this singularity we find that $J = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{5}\right)$, and $G^{max} = \left\langle\left(\frac{1}{3}, \frac{5}{6}, \frac{1}{5}\right)\right\rangle$. So, $\langle J\rangle$ and G^{max} are the only admissible groups. They both split using the decomposition, $E_{16} = D_2 + A_4$. So the entire A model structure can be determined using the sums of singularities axiom, and the computations in [1].

5.5.11 $W = W_{17} = x^3 + xy^4, G = \langle J\rangle$**.**

Element	$\mathbb{C}$-Degree
$A_1 = \lceil 1\,; J\rfloor$	0
$A_2 = \lceil 1\,; 4J\rfloor$	1/2
$A_3 = \lceil 1\,; 2J\rfloor$	1/2
$A_4 = \lceil xy\,; \mathbf{0}\rfloor$	1/2
$A_5 = \lceil y^3\,; \mathbf{0}\rfloor$	1/2
$A_6 = \lceil 1\,; 5J\rfloor$	1

The primitive elements are A_1, A_2, and A_3, and the Frobenius manifold structure can be determined from the basic genus-zero, k-point correlators for $k \leq 6$. Unfortunately, we do not have enough information to find basic correlator values for this example.

5.5.12 $W = W_{17} = x^3 + xy^4, G = G^{max}$**.** Let $\gamma = (2/3, 1/12)$, then

Element	$\mathbb{C}$-Degree
$A_1 = \lceil 1\,; 2\gamma \rfloor$	0
$A_2 = \lceil 1\,; 5\gamma \rfloor$	2/27
$A_3 = \lceil 1\,; \gamma \rfloor$	4/27
$A_4 = \lceil 1\,; 8\gamma \rfloor$	2/9
$A_5 = \lceil 1\,; 4\gamma \rfloor$	8/27
$A_6 = \lceil 1\,; \mathbf{0} \rfloor$	1/3
$A_7 = \lceil 1\,; 11\gamma \rfloor$	10/27
$A_8 = \lceil 1\,; 7\gamma \rfloor$	11/27
$A_9 = \lceil 1\,; 10\gamma \rfloor$	4/9

All three-point correlators can be found using the axioms except $\langle A_3, A_3, A_6 \rangle$ which must be equal to ± 1. In fact, the reconstruction lemma shows us that it is equal to 1.

The primitive elements are A_1, A_2, and A_3, and the Frobenius manifold structure can be determined from the basic genus-zero, k-point correlators for $k \leq 5$.

The nonzero basic genus-zero, four-point correlators are

$$\begin{array}{cc} \langle A_2, A_2, A_4, A_9 \rangle = \frac{1}{6} & \langle A_2, A_2, A_7, A_8 \rangle = \frac{1}{12} \\ \langle A_2, A_3, A_7, A_7 \rangle = -\frac{1}{12} & \langle A_2, A_3, A_6, A_9 \rangle = -\frac{1}{12} \\ \langle A_3, A_3, A_5, A_9 \rangle = \frac{1}{3} & \langle A_3, A_3, A_8, A_8 \rangle = \frac{1}{3} \end{array}$$

There are no nonzero basic five-point correlators.

5.5.13 $W = W_{18} = x^4 + y^7, G = \langle J \rangle = G^{max}$. For this singularity we find that $J = \left(\frac{1}{4}, \frac{1}{7}\right)$, and $G^{max} = \left\langle \left(\frac{1}{4}, \frac{1}{7}\right) \right\rangle$. So, $\langle J \rangle = G^{max}$ is the only admissible group. This group splits using the decomposition, $W_{18} = A_3 + A_6$. So the entire A model structure can be determined using the sums of singularities axiom, and the computations in [1].

5.5.14 $W = Q_{16} = x^3 + y^7 + yz^2$. For this singularity we find that $J = \left(\frac{1}{3}, \frac{1}{7}, \frac{3}{7}\right)$, and $G^{max} = \left\langle \left(\frac{1}{3}, \frac{1}{7}, \frac{13}{14}\right) \right\rangle$. So, $\langle J \rangle$ and G^{max} are the only admissible groups. They both split using the decomposition, $Q_{16} = A_2 + D_6$. So in each case the entire A model structure can be determined using the sums of singularities axiom, and the computations in [1].

5.5.15 $W = Q_{17} = x^3 + xy^5 + yz^2, G = \langle J \rangle = G^{max}$.

Element	$\mathbb{C}$-Degree	Element	$\mathbb{C}$-Degree
$A_1 = \lceil 1\,; J \rfloor$	0	$A_2 = \lceil 1\,; 10J \rfloor$	1/10
$A_3 = \lceil 1\,; 19J \rfloor$	1/5	$A_4 = \lceil 1\,; 28J \rfloor$	3/10
$A_5 = \lceil 1\,; 8J \rfloor$	3/10	$A_6 = \lceil 1\,; 7J \rfloor$	2/5
$A_7 = \lceil 1\,; 17J \rfloor$	2/5	$A_8 = \lceil 1\,; 16J \rfloor$	1/2
$A_9 = \lceil 1\,; 26J \rfloor$	1/2	$A_{10} = \lceil 1\,; 25J \rfloor$	3/5
$A_{11} = \lceil 1\,; 5J \rfloor$	3/5	$A_{12} = \lceil 1\,; 15J \rfloor$	3/5
$A_{13} = \lceil 1\,; 4J \rfloor$	7/10	$A_{14} = \lceil 1\,; 14J \rfloor$	7/10
$A_{15} = \lceil 1\,; 13J \rfloor$	4/5	$A_{16} = \lceil 1\,; 23J \rfloor$	4/5
$A_{17} = \lceil 1\,; 22J \rfloor$	9/10	$A_{18} = \lceil 1\,; 2J \rfloor$	9/10
$A_{19} = \lceil 1\,; 11J \rfloor$	1	$A_{20} = \lceil 1\,; 20J \rfloor$	11/10
$A_{21} = \lceil 1\,; 29J \rfloor$	6/5		

All three-point correlators can be found using the axioms except

$$\begin{array}{ll} \langle A_2, A_6, A_{14} \rangle & \langle A_3, A_4, A_{14} \rangle \\ \langle A_3, A_6, A_{11} \rangle & \langle A_4, A_4, A_{11} \rangle \\ \langle A_4, A_6, A_9 \rangle & \langle A_5, A_5, A_{12} \rangle \\ \langle A_6, A_6, A_7 \rangle & \end{array}$$

The reconstruction lemma shows that all of the above correlators are equal to -2, except for $\langle A_5, A_5, A_{12} \rangle$ which is equal to -1. The primitive elements are A_1, A_2, and A_5, and the Frobenius manifold structure can be determined from the basic genus-zero, k-point correlators for $k \leq 6$.

The nonzero basic genus-zero, four-point correlators are

$$\langle A_2, A_2, A_{15}, A_{21}\rangle = \tfrac{2}{15} \qquad \langle A_2, A_2, A_{17}, A_{20}\rangle = \tfrac{1}{15}$$
$$\langle A_2, A_5, A_{17}, A_{17}\rangle = -\tfrac{1}{15} \qquad \langle A_2, A_5, A_{12}, A_{21}\rangle = -\tfrac{1}{15}$$
$$\langle A_5, A_5, A_7, A_{21}\rangle = \tfrac{1}{3} \qquad \langle A_5, A_5, A_9, A_{20}\rangle = \tfrac{1}{3}$$
$$\langle A_5, A_5, A_{11}, A_{19}\rangle = \tfrac{1}{3} \qquad \langle A_5, A_5, A_{14}, A_{18}\rangle = \tfrac{1}{3}$$
$$\langle A_5, A_5, A_{16}, A_{16}\rangle = \tfrac{1}{3}$$

There are no nonzero basic five or six-point correlators.

5.5.16 $W = Q_{17}^T = x^3y + y^5z + z^2, G = \langle J\rangle$**.** Unfortunately we do not have enough information to find the basic correlator values.

5.5.17 $W = Q_{17}^T = x^3y + y^5z + z^2, G = G^{max}$**.** For $\gamma = \left(\frac{1}{30}, \frac{9}{10}, \frac{1}{2}\right)$, the vector space basis is given by,

Element	$\mathbb{C}$-Degree	Element	$\mathbb{C}$-Degree
$A_1 = \lceil 1\,; 9\gamma\rfloor$	0	$A_2 = \lceil 1\,; 7\gamma\rfloor$	2/15
$A_3 = \lceil 1\,; 5\gamma\rfloor$	4/15	$A_4 = \lceil 1\,; 19\gamma\rfloor$	1/3
$A_5 = \lceil 1\,; 3\gamma\rfloor$	2/5	$A_6 = \lceil y^4\,; 10\gamma\rfloor$	13/30
$A_7 = \lceil 1\,; 17\gamma\rfloor$	7/15	$A_8 = \lceil 1\,; \gamma\rfloor$	8/15
$A_9 = \lceil 1\,; 15\gamma\rfloor$	3/5	$A_{10} = \lceil 1\,; 29\gamma\rfloor$	2/3
$A_{11} = \lceil 1\,; 13\gamma\rfloor$	11/15	$A_{12} = \lceil y^4\,; 20\gamma\rfloor$	23/30
$A_{13} = \lceil 1\,; 27\gamma\rfloor$	4/5	$A_{14} = \lceil 1\,; 11\gamma\rfloor$	13/15
$A_{15} = \lceil 1\,; 25\gamma\rfloor$	14/15	$A_{16} = \lceil 1\,; 23\gamma\rfloor$	16/15
$A_{17} = \lceil 1\,; 21\gamma\rfloor$	6/5		

All three-point correlators can be found using the axioms except

$$\langle A_2, A_8, A_8\rangle \quad \langle A_3, A_5, A_8\rangle$$
$$\langle A_4, A_6, A_6\rangle \quad \langle A_5, A_5, A_5\rangle$$

The reconstruction lemma shows that all of the above correlators are equal to -3, except for $\langle A_4, A_6, A_6\rangle$ which is equal to plus or minus one. The primitive elements are A_1, A_2, A_4 and A_6, and the Frobenius manifold structure can be determined from the basic genus-zero, k-point correlators for $k \le 6$.

The nonzero basic genus-zero, four-point correlators are

$$
\begin{array}{cc}
\langle A_2, A_2, A_{11}, A_{17}\rangle = \frac{1}{10} & \langle A_2, A_2, A_{14}, A_{16}\rangle = \frac{1}{10} \\
\langle A_2, A_4, A_{10}, A_{16}\rangle = -\frac{1}{15} & \langle A_2, A_4, A_{13}, A_{15}\rangle = -\frac{1}{15} \\
\langle A_4, A_4, A_4, A_{17}\rangle = \frac{3}{10} & \langle A_4, A_4, A_7, A_{16}\rangle = \frac{7}{30} \\
\langle A_4, A_4, A_9, A_{15}\rangle = \frac{1}{6} & \langle A_4, A_4, A_{10}, A_{14}\rangle = \frac{1}{30} \\
\langle A_4, A_4, A_{11}, A_{13}\rangle = \frac{1}{10} & \langle A_2, A_4, A_8, A_{17}\rangle = \frac{1}{10} \\
\langle A_2, A_6, A_6, A_{17}\rangle = -\frac{a}{10} & \langle A_2, A_6, A_{12}, A_{14}\rangle = \frac{1}{50} \\
\langle A_3, A_6, A_6, A_{16}\rangle = -\frac{a}{10} & \langle A_4, A_4, A_{12}, A_{12}\rangle = \frac{1}{75a} \\
\langle A_4, A_6, A_{10}, A_{12}\rangle = -\frac{1}{150} & \langle A_5, A_6, A_6, A_{15}\rangle = -\frac{a}{10} \\
\langle A_6, A_6, A_7, A_{14}\rangle = -\frac{a}{10} & \langle A_6, A_6, A_8, A_{13}\rangle = -\frac{a}{10} \\
\langle A_6, A_6, A_9, A_{11}\rangle = -\frac{a}{10} & \langle A_6, A_6, A_{10}, A_{10}\rangle = \frac{a}{30}
\end{array}
$$

The nonzero basic genus-zero, four-point correlators are

$$
\begin{array}{ll}
\langle A_2, A_4, A_4, A_{17}, A_{17}\rangle = -\frac{1}{150} & \langle A_4, A_6, A_6, A_{13}, A_{17}\rangle = -\frac{a}{150} \\
\langle A_4, A_6, A_6, A_{15}, A_{16}\rangle = -\frac{a}{150} &
\end{array}
$$

There are no nonzero basic six-point correlators.

5.5.18 $W = Q_{18} = x^3 + y^8 + yz^2, G = \langle J\rangle = G^{max}$**.** For this singularity we find that $J = \left(\frac{1}{3}, \frac{1}{8}, \frac{7}{16}\right)$, and $G^{max} = \left\langle\left(\frac{1}{3}, \frac{1}{8}, \frac{15}{16}\right)\right\rangle$. So, $\langle J\rangle = G^{max}$ is the only admissible group. This group splits using the decomposition, $Q_{18} = A_2 + D_7$. So the entire A model structure can be determined using the sums of singularities axiom, and the computations in [1].

5.5.19 $W = S_{16} = x^2y + y^2z + xz^2, G = \langle J\rangle = G^{max}$**.**

Element	$\mathbb{C}$-Degree	Element	$\mathbb{C}$-Degree
$A_1 = \lceil 1\,;J \rfloor$	0	$A_2 = \lceil 1\,;8J \rfloor$	2/27
$A_3 = \lceil 1\,;7J \rfloor$	4/27	$A_4 = \lceil 1\,;15J \rfloor$	2/9
$A_5 = \lceil 1\,;6J \rfloor$	8/27	$A_6 = \lceil 1\,;14J \rfloor$	1/3
$A_7 = \lceil 1\,;5J \rfloor$	10/27	$A_8 = \lceil 1\,;13J \rfloor$	11/27
$A_9 = \lceil 1\,;4J \rfloor$	4/9	$A_{10} = \lceil 1\,;12J \rfloor$	13/27
$A_{11} = \lceil 1\,;3J \rfloor$	14/27	$A_{12} = \lceil 1\,;11J \rfloor$	5/9
$A_{13} = \lceil 1\,;2J \rfloor$	16/27	$A_{14} = \lceil 1\,;10J \rfloor$	17/27
$A_{15} = \lceil 1\,;9J \rfloor$	2/3	$A_{16} = \lceil 1\,;16J \rfloor$	19/27

All three-point correlators can be found using the axioms except

$$\begin{array}{ll} \langle A_2, A_7, A_7\rangle & \langle A_3, A_3, A_9\rangle \\ \langle A_3, A_6, A_6\rangle & \langle A_4, A_4, A_7\rangle \\ \langle A_5, A_5, A_5\rangle & \end{array}$$

The reconstruction lemma tells us that all of the above correlators are equal to -2 except for $\langle A_5, A_5, A_5\rangle$ which is -4. The primitive elements are A_1, A_2, A_3, and A_5, and the Frobenius manifold structure can be determined from the basic genus-zero, k-point correlators for $k \leq 6$.

The nonzero basic genus-zero, four-point correlators are

$$\langle A_2, A_2, A_9, A_{16}\rangle = \tfrac{3}{17} \quad \langle A_2, A_2, A_{12}, A_{15}\rangle = \tfrac{1}{17} \quad \langle A_2, A_3, A_{10}, A_{15}\rangle = \tfrac{-2}{17}$$
$$\langle A_2, A_3, A_{13}, A_{13}\rangle = \tfrac{-2}{17} \quad \langle A_2, A_5, A_8, A_{15}\rangle = \tfrac{1}{17} \quad \langle A_2, A_5, A_{11}, A_{13}\rangle = \tfrac{1}{17}$$
$$\langle A_2, A_5, A_{12}, A_{12}\rangle = \tfrac{-2}{17} \quad \langle A_3, A_5, A_8, A_{14}\rangle = \tfrac{-4}{17} \quad \langle A_3, A_5, A_{11}, A_{11}\rangle = \tfrac{-4}{17}$$
$$\langle A_2, A_2, A_{14}, A_{14}\rangle = \tfrac{-2}{17} \quad \langle A_2, A_3, A_7, A_{16}\rangle = \tfrac{3}{17} \quad \langle A_2, A_3, A_{12}, A_{14}\rangle = \tfrac{1}{17}$$
$$\langle A_2, A_5, A_5, A_{16}\rangle = \tfrac{5}{17} \quad \langle A_2, A_5, A_{10}, A_{14}\rangle = \tfrac{1}{17} \quad \langle A_3, A_3, A_5, A_{16}\rangle = \tfrac{7}{17}$$
$$\langle A_3, A_3, A_8, A_{15}\rangle = \tfrac{5}{17} \quad \langle A_3, A_5, A_{10}, A_{14}\rangle = \tfrac{1}{17} \quad \langle A_3, A_3, A_{11}, A_{13}\rangle = \tfrac{3}{17}$$
$$\langle A_3, A_3, A_{12}, A_{12}\rangle = \tfrac{-2}{17} \quad \langle A_3, A_5, A_6, A_{15}\rangle = \tfrac{5}{17} \quad \langle A_3, A_5, A_9, A_{13}\rangle = \tfrac{3}{17}$$
$$\langle A_3, A_5, A_{10}, A_{12}\rangle = \tfrac{1}{17} \quad \langle A_4, A_5, A_5, A_{15}\rangle = \tfrac{6}{17} \quad \langle A_5, A_5, A_6, A_{14}\rangle = \tfrac{1}{17}$$
$$\langle A_5, A_5, A_7, A_{13}\rangle = \tfrac{7}{17} \quad \langle A_5, A_5, A_8, A_{12}\rangle = \tfrac{3}{17} \quad \langle A_5, A_5, A_9, A_{11}\rangle = \tfrac{2}{17}$$
$$\langle A_5, A_5, A_{10}, A_{10}\rangle = \tfrac{-4}{17}$$

The nonzero basic five-point correlators are

$$\langle A_2, A_2, A_5, A_{16}, A_{16}\rangle = -\tfrac{4}{289} \quad \langle A_2, A_3, A_3, A_{16}, A_{16}\rangle = \tfrac{8}{289} \quad \langle A_3, A_5, A_5, A_{13}, A_{16}\rangle = -\tfrac{4}{289}$$
$$\langle A_3, A_5, A_5, A_{15}, A_{15}\rangle = -\tfrac{8}{289} \quad \langle A_5, A_5, A_5, A_{11}, A_{16}\rangle = \tfrac{20}{289} \quad \langle A_5, A_5, A_5, A_{14}, A_{15}\rangle = \tfrac{12}{289}$$

There are no nonzero basic 6-point correlators.

5.5.20 $W = S_{17} = x^2y + y^2z + z^6, G = \langle J\rangle = G^{max}$.

Element	$\mathbb{C}$-Degree	Element	$\mathbb{C}$-Degree
$A_1 = \lceil 1\,;J \rfloor$	0	$A_2 = \lceil 1\,;8J \rfloor$	1/8
$A_3 = \lceil 1\,;7J \rfloor$	1/4	$A_4 = \lceil 1\,;15J \rfloor$	1/4
$A_5 = \lceil 1\,;14J \rfloor$	3/8	$A_6 = \lceil 1\,;22J \rfloor$	3/8
$A_7 = \lceil 1\,;21J \rfloor$	1/2	$A_8 = \lceil 1\,;5J \rfloor$	1/2
$A_9 = \lceil 1\,;13J \rfloor$	1/2	$A_{10} = \lceil 1\,;4J \rfloor$	5/8
$A_{11} = \lceil 1\,;20J \rfloor$	5/8	$A_{12} = \lceil 1\,;12J \rfloor$	5/8
$A_{13} = \lceil 1\,;19J \rfloor$	3/4	$A_{14} = \lceil 1\,;11J \rfloor$	3/4
$A_{15} = \lceil 1\,;3J \rfloor$	3/4	$A_{16} = \lceil 1\,;10J \rfloor$	7/8
$A_{17} = \lceil 1\,;2J \rfloor$	7/8	$A_{18} = \lceil 1\,;17J \rfloor$	1
$A_{19} = \lceil 1\,;9J \rfloor$	1	$A_{20} = \lceil 1\,;16J \rfloor$	9/8
$A_{21} = \lceil 1\,;23J \rfloor$	5/4		

All three-point correlators can be found using the axioms except

$$\begin{array}{cc} \langle A_2, A_8, A_{12}\rangle & \langle A_3, A_3, A_{14}\rangle \\ \langle A_3, A_5, A_{10}\rangle & \langle A_3, A_7, A_7\rangle \\ \langle A_4, A_6, A_{12}\rangle & \langle A_4, A_8, A_8\rangle \\ \langle A_5, A_5, A_7\rangle & \langle A_6, A_6, A_8\rangle \end{array}$$

where all the above correlators are equal to -2 except for $\langle A_2, A_8, A_{12}\rangle$ and $\langle A_4, A_6, A_{12}\rangle$ which are each equal to plus or minus one. The primitive elements are A_1, A_2, and A_3, and the Frobenius manifold structure can be determined from the basic genus-zero, k-point correlators for $k \leq 5$.

The nonzero basic genus-zero, four-point correlators are

$$\begin{array}{cc}
\langle A_2, A_2, A_{14}, A_{21}\rangle = \frac{1}{6} & \langle A_2, A_3, A_{13}, A_{20}\rangle = -\frac{1}{12} \\
\langle A_2, A_3, A_{17}, A_{19}\rangle = -\frac{1}{12} & \langle A_2, A_2, A_{18}, A_{18}\rangle = -\frac{1}{12} \\
\langle A_2, A_3, A_{12}, A_{21}\rangle = -\frac{a}{12} & \langle A_3, A_3, A_9, A_{21}\rangle = -\frac{5}{12} \\
\langle A_3, A_3, A_{11}, A_{20}\rangle = \frac{1}{3} & \langle A_3, A_3, A_{13}, A_{18}\rangle = \frac{1}{12} \\
\langle A_3, A_3, A_{15}, A_{19}\rangle = \frac{1}{4} & \langle A_3, A_3, A_{16}, A_{17}\rangle = \frac{1}{6}
\end{array}$$

There are no nonzero basic five or six-point correlators.

5.5.21 $W = S_{17}^T = x^2 + xy^2 + yz^6, G = \langle J \rangle$**.** Unfortunately we do not have enough information to find the basic correlator values.

5.5.22 $W = S_{17}^T = x^2 + xy^2 + yz^6, G = G^{max}$**.** Unfortunately we do not have enough information to find the basic correlator values.

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