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The Minimum Rank Problem Over Finite Fields

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THE MINIMUM RANK PROBLEM OVER FINITE FIELDS

by

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of a dissertation submitted by

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This dissertation has been read by each member of the following graduate committee and by majority vote has been found to be satisfactory.

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As chair of the candidate's graduate committee, I have read the dissertation of Jason Nicholas Grout in its final form and have found that (1) its format, citations, and bibliographical style are consistent and acceptable and fulfill university and department style requirements; (2) its illustrative materials including figures, tables, and charts are in place; and (3) the final manuscript is satisfactory to the graduate committee and is ready for submission to the university library.

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ABSTRACT

THE MINIMUM RANK PROBLEM OVER FINITE FIELDS

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We have two main results. Our first main result is a sharp bound for the number of vertices in a minimal forbidden subgraph for the graphs having minimum rank at most 3 over the finite field of order 2. We also list all 62 such minimal forbidden subgraphs and show that many of these are minimal forbidden subgraphs for any field. Our second main result is a structural characterization of all graphs having minimum rank at most k for any k over any finite field. This characterization leads to a very strong connection to projective geometry and we apply projective geometry results to the minimum rank problem.

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- Raphael Loewy for Corollary 6, a crucial part of our approach in Chapter 2, as well as an upper bound of 14 analogous to the result in Theorem 31.
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- Willem Haemers for recognizing that the graph $F2R3$ in Section 3.2.4 is related to the Fano projective plane and suggesting that we look into connections with projective geometry, thereby leading to the results in Chapter 3.
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Contents

Chapter 1

Introduction

Given a field F and a simple undirected graph G (i.e., an undirected graph without loops or multiple edges) on n vertices, let $S(F, G)$ be the set of symmetric $n \times n$ matrices A with entries in F satisfying $a_{ij} \neq 0$, $i \neq j$, if and only if ij is an edge in G. There is no restriction on the diagonal entries of the matrices in $S(F, G)$. Let

$$\mathrm{mr}(F, G) = \min\{\mathrm{rank}\ A \mid A \in S(F, G)\}.$$

Let $\mathcal{G}_k(F) = \{G \mid \mathrm{mr}(F, G) \leq k\}$, the set of simple graphs with minimum rank at most k.

The problem of finding $\mathrm{mr}(F, G)$ and describing $\mathcal{G}_k(F)$ has recently attracted considerable attention, particularly for the case in which $F = \mathbb{R}$ (see [Nyl96, CdV98, JD99, Hsi01, JS02, CHLW03, vdH03, BFH04, BvdHL04, HLR04, AHK$^+$05, BD05, BFH05a, BFH05b, BvdHL05, DK06, BF07]). The minimum rank problem over $\mathbb{R}$ is a sub-problem of a much more general problem, the inverse eigenvalue problem for symmetric matrices: given a family of real numbers, find every symmetric matrix that has the family as its eigenvalues. More particularly, the minimum rank problem is a sub-problem of the inverse eigenvalue problem for graphs, which fixes a zero/nonzero pattern for the symmetric matrices considered in the inverse eigenvalue problem. The minimum rank problem can also be thought of in this way: given a fixed pattern of off-diagonal zeros, what is the smallest rank that a symmetric matrix having that pattern can achieve?

One of the oldest results implies that a graph G on n vertices has $\mathrm{mr}(F, G) \geq n-1$ if and only if G is the path (see [BD05] for a field-independent proof). In [BvdHL04] and [BvdHL05], the graphs in $\mathcal{G}_2(F)$ were characterized for any field F via their structure and also via forbidden induced subgraphs. Recently, Ding and Kotlov [DK06] independently used structures similar to those introduced in Chapter 3 to obtain a bound on the sizes of minimal forbidden induced subgraphs characterizing $\mathcal{G}_k(F)$ for any finite field F and positive integer k.

We adopt the following notation dealing with fields, vector spaces, and matrices. Given a field F, the group of nonzero elements under multiplication is denoted $F^\times$ and the vector space of dimension k over F is denoted F^k. The finite field of order q is denoted $\mathbb{F}_q$. Given a matrix M, the principal submatrix of columns and rows $x_1, x_2, \ldots, x_m$ is denoted $M[x_1, x_2, \ldots, x_m]$.

As an example of how one might approach the problem of finding the minimum rank of a simple graph, we recall from [BvdHL05] the fullhouse graph in Figure 1.1 (there called $(P_3 \cup 2K_1)^c$), which is the only graph on 5 or fewer vertices for which the minimum rank is field-dependent.

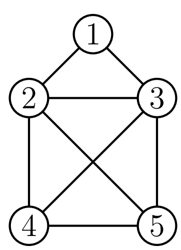


Figure 1.1: A labeled fullhouse graph

If $F \neq \mathbb{F}_2$, there are elements $a, b \neq 0$ in F such that $a + b \neq 0$. Then

$$\begin{bmatrix} a & a & a & 0 & 0 \\ a & a+b & a+b & b & b \\ a & a+b & a+b & b & b \\ 0 & b & b & b & b \\ 0 & b & b & b & b \end{bmatrix} \in S(F, \text{fullhouse})$$

which shows that $\mathrm{mr}(F, \text{fullhouse}) = 2$. The case $F = \mathbb{F}_2$ gives a different result. Let A be any matrix in $S(\mathbb{F}_2, \text{fullhouse})$. Then for some $d_1, d_2, \ldots, d_5 \in \mathbb{F}_2$,

$$A = \begin{bmatrix} d_1 & 1 & 1 & 0 & 0 \\ 1 & d_2 & 1 & 1 & 1 \\ 1 & 1 & d_3 & 1 & 1 \\ 0 & 1 & 1 & d_4 & 1 \\ 0 & 1 & 1 & 1 & d_5 \end{bmatrix} \quad \text{and} \quad \det(A[\{1,2,5\},\{1,3,4\}]) = \begin{vmatrix} d_1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 1,$$

where $A[\{1,2,5\},\{1,3,4\}]$ is the submatrix of A of rows $\{1,2,5\}$ and columns $\{1,3,4\}$. Thus $\mathrm{mr}(\mathbb{F}_2, \text{fullhouse}) \geq 3$. Setting all $d_i = 1$ verifies that $\mathrm{mr}(\mathbb{F}_2, \text{fullhouse}) = 3$.

In spite of this dependence on the field, there are a number of results about minimum rank that are field independent. The minimum rank of a tree, for example, is field independent. It has become clear that results obtained over finite fields will provide important insights for other fields. In this spirit, we will explore field-independent extensions of our Chapter 2 results in Section 2.10.

The presentation of material in this dissertation is oriented towards one familiar with terminology and concepts from linear algebra and graph theory. While terminology and concepts are fairly standardized in linear algebra, such is not necessarily the case in graph theory. In the rest of this section, we will review some of the main concepts and terminology from graph theory that we will be using.

We recall some notation from graph theory.

Definition 1. Given a graph G, $V(G)$ denotes the set of vertices in G and $E(G)$ denotes the set of edges in G. The *order* of a graph is $|G| = |V(G)|$. The complement, G^c, is the graph with vertices $V(G)$ and edges $E(G)^c$ (the set complement of the edges of G). Given two graphs G and H, with $V(G)$ and $V(H)$ disjoint, the *union* of G and H is $G \cup H = (V(G) \cup V(H), E(G) \cup E(H))$. The *join*, $G \vee H$, is the graph obtained from $G \cup H$ by adding the edges $\{xy \mid x \in V(G), y \in V(H)\}$ from all vertices of G to all vertices of H. If $S \subset V(G)$, $G[S]$ denotes the subgraph of G induced by S. If H is an induced subgraph of G, $G - H$ denotes the subgraph induced by $V(G) \setminus V(H)$.

Definition 2. We denote the path on n vertices by P_n. The complete graph on n vertices will be denoted by K_n and has vertices $\{1, 2, \ldots, n\}$ and edges $\{xy \mid x, y \in V(K_n)\}$. We abbreviate $K_n \cup \cdots \cup K_n$ (m times) to mK_n.

Definition 3. Two vertices in a graph are *adjacent* if an edge connects them. A *clique* in a graph is a set of pairwise adjacent vertices. An *independent set* in a graph is a set of pairwise nonadjacent vertices.

The main results of this dissertation characterize $\mathcal{G}_k(F)$, F finite, in two different ways. In Chapter 2, we characterize the special case of $\mathcal{G}_3(\mathbb{F}_2)$ using minimal forbidden subgraphs. In Chapter 3, we directly characterize the structure of graphs in $\mathcal{G}_k(F)$ for any positive integer k and any finite field F. In the next two sections, we will briefly review the concepts behind these characterizations for a set of graphs.

1.1 Forbidden subgraph characterization

Let P be a graph property and let $\mathcal{P}$ be the set of graphs satisfying P. We will assume that P is preserved under taking induced subgraphs (i.e., if a graph has property P, then so does every induced subgraph), or equivalently, the set $\mathcal{P}$ is closed under taking induced subgraphs. If $G \notin \mathcal{P}$, then any graph containing G as an induced subgraph is not in $\mathcal{P}$. Each graph not in $\mathcal{P}$ is called a *forbidden subgraph* for $\mathcal{P}$, since it is not induced in any graph in $\mathcal{P}$.

There is a lot of duplicate information in the set of forbidden subgraphs, though, since any graph that has an induced forbidden subgraph is itself a forbidden subgraph. A graph G is a *minimal forbidden subgraph* if $G \notin \mathcal{P}$, but every induced subgraph of G is in $\mathcal{P}$. The class $\mathcal{P}$ is characterized by the set of minimal forbidden subgraphs in the sense that G is in $\mathcal{P}$ if and only if G does not contain an induced minimal forbidden subgraph. We can think of a minimal forbidden subgraph characterization as giving the structures that graphs in $\mathcal{P}$ do not contain. While the set of forbidden subgraphs is always infinite, the set of minimal forbidden subgraphs may be finite or may have other nice properties.

Forbidden subgraph characterizations, and a generalization, forbidden minor characterizations, play an important role in graph theory. The recently proven Strong Perfect Graph Theorem is a significant example of a forbidden subgraph characterization. The chromatic number of a graph is the smallest number of colors needed to color the vertices of a graph such that adjacent vertices do not have the same color. Clearly the size of the largest clique in a graph is a lower bound for the chromatic number. A perfect graph is a graph in which, for every induced subgraph, the size of the largest induced clique is equal to the chromatic number (e.g., a cycle of even order is perfect, but a cycle of odd order is not since the chromatic number is 3). The Strong Perfect Graph Theorem asserts that a graph is perfect if and only if it does not contain either a cycle of odd order at least 5 or the complement of a cycle of odd order at least 5 as an induced subgraph. An example of a famous forbidden minor characterization is Kuratowski's theorem: a graph is planar if and only if it does not contain K_5 or $K_{3,3}$ as a minor.

Relating this material to the minimum rank problem, since the rank of any matrix is bounded below by the rank of any principal submatrix, we have the following observation.

Observation 1 ([BvdHL04, Observation 5])**.** If H is an induced subgraph of G, then for any field F, $\operatorname{mr}(F, H) \leq \operatorname{mr}(F, G)$.

Example 1. It is well known that $\operatorname{mr}(F, P_{k+2}) = k + 1$ for any field F. Therefore P_{k+2} cannot be an induced subgraph of any graph in $\mathcal{G}_k(F)$.

From the observation, the set $\mathcal{G}_k(F)$ can be characterized by a set of minimal forbidden subgraphs. In [BvdHL04] and [BvdHL05], the authors gave complete lists of minimal forbidden subgraphs for $\mathcal{G}_2(F)$ for any field F. While these lists were finite, Hall [Hal] has recently shown that there are infinitely many minimal forbidden subgraphs characterizing $\mathcal{G}_3(\mathbb{R})$ and $\mathcal{G}_3(\mathbb{C})$. Things are more manageable with finite fields, however. Ding and Kotlov [DK06] have given bounds for the numbers of vertices in the minimal forbidden subgraphs characterizing $\mathcal{G}_k(F)$ for any rank k and any finite field F; these imply that there are only a finite number of minimal forbidden subgraphs when the field F is finite. However, we will see that we still must do much work to get a complete list of minimal forbidden subgraphs in even the simplest unknown case because the bound is far beyond what current computational techniques can manage.

In Chapter 2, we will find a minimal forbidden subgraph characterization for $\mathcal{G}_3(\mathbb{F}_2)$, the simplest unknown case. The proof of this characterization, though, will not be so simple.

1.2 Direct structural characterizations

Forbidden subgraphs characterize a class of graphs by enumerating what the graphs do *not* look like. We can also characterize the class directly by what structure the graphs *do* have. In Chapter 3, we will directly characterize the structure of graphs in $\mathcal{G}_k(F)$ for any positive integer k and any finite field F. The characterization is simply stated and has very strong connections with projective geometries over finite fields. While the proofs in Chapter 2 are technical, specific, and rely on the hypothesis that $F = \mathbb{F}_2$, the proofs and connections in Chapter 3 are much more general and probably will generate more fruitful areas to study in the future. Indeed, we list only a few of the ramifications of the characterization in this dissertation.

Chapter 2

The minimum rank problem over the finite field of order 2: minimum rank 3

In this chapter, we will characterize $\mathcal{G}_3(\mathbb{F}_2)$ using minimal forbidden subgraphs. We will also show how some of the minimal forbidden subgraphs for $\mathcal{G}_3(\mathbb{F}_2)$ are actually minimal forbidden subgraphs for $\mathcal{G}_3(F)$ for any field F (infinite or finite).

Definition 4. Let F be any field. The graph H is a *minimal forbidden subgraph* for the class of graphs $\mathcal{G}_k(F) = \{G \mid \operatorname{mr}(F, G) \leq k\}$ if

(a) $\operatorname{mr}(F, H) \geq k + 1$ and

(b) $\operatorname{mr}(F, H - v) \leq k$ for every vertex $v \in V(H)$.

Let $\mathcal{F}_{k+1}(F)$ be the set of all minimal forbidden subgraphs for $\mathcal{G}_k(F)$.

Observation 2. $G \in \mathcal{G}_k(F) \iff$ no graph in $\mathcal{F}_{k+1}(F)$ is induced in G.

Theorem 6 ($a \iff c$) of [BvdHL04] and Theorem 16 of [BvdHL05] can be restated in terms of our notation:

Theorem 3 ([BvdHL04, Theorem 6])**.** $\mathcal{F}_3(\mathbb{R}) = \{P_4, \ltimes, \text{dart}, P_3 \cup K_2, 3K_2, K_{3,3,3}\}$.

Theorem 4 ([BvdHL05, Theorem 16])**.**

$$\mathcal{F}_3(\mathbb{F}_2) = \{P_4, \ltimes, \text{dart}, P_3 \cup K_2, 3K_2, \text{fullhouse}, P_3 \vee P_3\}.$$

Ding and Kotlov have given bounds for the number of vertices in graphs in $\mathcal{F}_{k+1}(F)$ for any k and any finite field F [DK06]; these imply a bound on the number of graphs in each $\mathcal{F}_{k+1}(F)$. In the case $k = 3$ and $F = \mathbb{F}_2$, Ding and Kotlov have proven that each graph in $\mathcal{F}_4(\mathbb{F}_2)$ has 25 or fewer vertices; this implies that $|\mathcal{F}_4(\mathbb{F}_2)| \leq 1.32 \times 10^{65}$. By explicitly using the fact that $k = 3$ and $F = \mathbb{F}_2$, we will show that every graph in $\mathcal{F}_4(\mathbb{F}_2)$ has 8 or fewer vertices, which will imply that $|\mathcal{F}_4(\mathbb{F}_2)| \leq 13\,598$. We will then find by an exhaustive search all the graphs in $\mathcal{F}_4(\mathbb{F}_2)$, which will prove that $|\mathcal{F}_4(\mathbb{F}_2)| = 62$ and show that our bound on the number of vertices is sharp. Of the 29 graphs in $\mathcal{F}_4(\mathbb{F}_2)$ having vertex connectivity at most one, we will prove that 21 graphs are in $\mathcal{F}_4(F)$ for every field F, while none of the remaining 8 graphs are in $\mathcal{F}_4(F)$ for any field $F \neq \mathbb{F}_2$.

Our approach relies on the following generalization of $\mathcal{F}_{k+1}(F)$.

Definition 5. Given a field F and a graph H, let $\mathcal{F}_{k+1}(F, H)$ be the set of graphs G containing H as an induced subgraph and satisfying

(a) $\operatorname{mr}(F, G) \geq k + 1$ and

(b) for some H induced in G, $\operatorname{mr}(F, G - v) \leq k$ for every $v \in V(G - H)$.

Example 2. Let F be any field, let G be the graph labeled in Figure 2.1, let $H = P_4$,

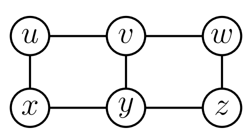
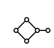


Figure 2.1: G in Example 2.

and let $k = 3$. Since P_5 is induced in G, $\operatorname{mr}(F, G) \geq 3+1$, so condition (a) is satisfied.

Six copies of $H = P_4$ are induced in G. For $H = G[\{u, v, w, x\}]$, we have $\operatorname{mr}(F, G - y) = 4$, so condition (b) is not satisfied for this copy of P_4. However, if $H = G[\{u, v, y, z\}]$, both $G - w$ and $G - x$ are isomorphic to , which has minimum rank 3 by Theorem 2.3 in [BFH04] (see Theorem 39 in this paper). Therefore condition (b) is satisfied for this induced P_4, so $G \in \mathcal{F}_{k+1}(F, P_4)$.

In the notation of Definition 5, $\mathcal{F}_{k+1}(F) = \mathcal{F}_{k+1}(F, \emptyset)$, where $\emptyset$ is the empty graph.

Theorem 5.

$$\mathcal{F}_{k+1}(F) \subseteq \bigcup_{H \in \mathcal{F}_k(F)} \mathcal{F}_{k+1}(F, H).$$

Proof. Let $G \in \mathcal{F}_{k+1}(F)$. Since $\operatorname{mr}(F, G) \geq k+1 > k-1$, $G \notin \mathcal{G}_{k-1}(F)$. Therefore some graph $H \in \mathcal{F}_k(F)$ is induced in G. By definition, $\operatorname{mr}(F, G-v) \leq k$ for every vertex v of G, so $\operatorname{mr}(F, G-v) \leq k$ for every vertex v of $G - H$. By definition, $G \in \mathcal{F}_{k+1}(F, H)$. □

Combining Theorems 4 and 5, we have the following result.

Corollary 6.

$$\begin{aligned}\mathcal{F}_4(\mathbb{F}_2) \subseteq & \bigcup_{H \in \mathcal{F}_3(\mathbb{F}_2)} \mathcal{F}_4(\mathbb{F}_2, H) \\ &= \mathcal{F}_4(\mathbb{F}_2, 3K_2) \cup \mathcal{F}_4(\mathbb{F}_2, P_3 \vee P_3) \cup \mathcal{F}_4(\mathbb{F}_2, \text{dart}) \cup \mathcal{F}_4(\mathbb{F}_2, \ltimes) \\ &\quad \cup \mathcal{F}_4(\mathbb{F}_2, P_3 \cup K_2) \cup \mathcal{F}_4(\mathbb{F}_2, \text{fullhouse}) \cup \mathcal{F}_4(\mathbb{F}_2, P_4).\end{aligned}$$

Sections 2.1–2.9 are devoted to explicitly determining $\mathcal{F}_4(\mathbb{F}_2)$.

2.1 Matrices which attain a minimum rank for $\mathcal{F}_3(\mathbb{F}_2)$

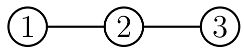
Given a field F and a graph G, it is natural to seek to determine all matrices in $S(F, G)$ which attain the minimum rank of G. Determining these matrices plays a critical role in determining $\mathcal{F}_4(\mathbb{F}_2)$.

Definition 6. Let G be a graph. Let $\mathcal{M}(F, G) = \{A \in S(F, G) \mid \operatorname{rank} A = \operatorname{mr}(F, G)\}$, the set of matrices in $S(F, G)$ that attain the minimum rank of G. Call two matrices in $\mathcal{M}(F, G)$ *equivalent* if and only if they have the same column space. Let $\mathcal{C}(F, G)$ be the resulting set of equivalence classes.

Let G be a graph. In the remainder of this section and in Sections 2.2–2.8, we will assume that $F = \mathbb{F}_2$ and abbreviate our notation as follows: $S(\mathbb{F}_2, G)$ is shortened to

$S(G)$, $\mathrm{mr}(\mathbb{F}_2, G)$ is shortened to $\mathrm{mr}(G)$, $\mathcal{F}_{k+1}(\mathbb{F}_2)$ is shortened to $\mathcal{F}_{k+1}$, $\mathcal{F}_{k+1}(\mathbb{F}_2, G)$ is shortened to $\mathcal{F}_{k+1}(G)$, $\mathcal{M}(\mathbb{F}_2, G)$ is shortened to $\mathcal{M}(G)$, and $\mathcal{C}(\mathbb{F}_2, G)$ is shortened to $\mathcal{C}(G)$.

In the remainder of this section, we determine $\mathcal{M}(G)$ for all of the graphs in $\mathcal{F}_3$.

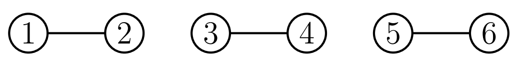
Lemma 7. *With P_3 labeled as*  ,

$$\mathcal{M}(P_3) = \left\{ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \right\}.$$

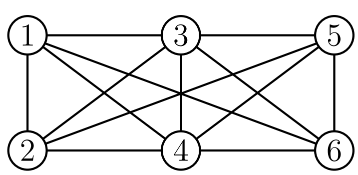
Proof. Since $\mathrm{mr}(P_3) = 2$,

$$A = \begin{bmatrix} x & 1 & 0 \\ 1 & y & 1 \\ 0 & 1 & z \end{bmatrix} \in \mathcal{M}(P_3) \iff \det A = xyz + x + z = 0 \text{ in } \mathbb{F}_2.$$

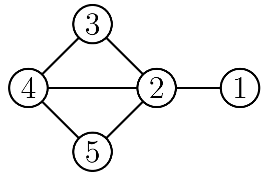
If $x \neq z$, then $\det A = 1$, so $x = z$. Then $\det A = xy$, so $A \in \mathcal{M}(P_3)$ if and only if either $x = y = z = 0$, $x = z = 0$ and $y = 1$, or $x = z = 1$ and $y = 0$. $\square$

Proposition 8. *(a) With $3K_2$ labeled as*  ,

$$\mathcal{M}(3K_2) = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \oplus \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \oplus \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}.$$

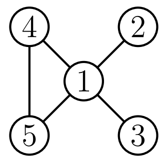
(b) With $P_3 \vee P_3$ labeled as  ,

$$\mathcal{M}(P_3 \vee P_3) = \left\{ \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 \end{bmatrix} \right\}.$$

(c) *With the dart labeled as*  ,

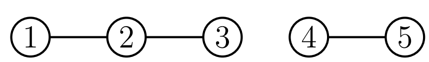
$$\mathcal{M}(\text{dart}) = \left\{ M_1 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix}, M_2 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{bmatrix} \right\}$$

and $\mathcal{C}(\text{dart}) = \{C_1 = \{M_1\}, C_2 = \{M_2\}\}$.

(d) *With* $\ltimes$ *labeled as*  ,

$$\mathcal{M}(\ltimes) = \left\{ M_1 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{bmatrix}, M_2 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{bmatrix}, M_3 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{bmatrix} \right\}$$

and $\mathcal{C}(\ltimes) = \{C_1 = \{M_1, M_2\}, C_2 = \{M_3\}\}$.

(e) *With* $P_3 \cup K_2$ *labeled as*  ,

$$\mathcal{M}(P_3 \cup K_2) = \left\{ M_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, M_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, M_3 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \oplus \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$$

and $\mathcal{C}(P_3 \cup K_2) = \{C_1 = \{M_1, M_2\}, C_2 = \{M_3\}\}$.

(f) With the fullhouse labeled as in Figure 1.1,

$$\mathcal{M}(\text{fullhouse}) = \left\{ M_1 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}, M_2 = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}, \right.$$

$$\left. M_3 = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}, M_4 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix} \right\}$$

and $\mathcal{C}(\text{fullhouse}) = \{C_1 = \{M_1, M_2\}, C_2 = \{M_3\}, C_3 = \{M_4\}\}$.

(g) With P_4 labeled as $\textcircled{1} - \textcircled{2} - \textcircled{3} - \textcircled{4}$,

$$\mathcal{M}(P_4) = \left\{ M_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, M_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \right.$$

$$\left. M_3 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, M_4 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, M_5 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \right\}$$

and $\mathcal{C}(P_4) = \{C_1 = \{M_1, M_2\}, C_2 = \{M_3, M_4\}, C_3 = \{M_5\}\}$.

Proof. It is known [BvdHL04, BvdHL05] that each of the graphs in (a) through (g) has minimum rank 3.

Part (a) follows immediately and (e) follows from Lemma 7. We prove (f) and (g). The proofs of (b), (c), and (d) are similar.

(f) Let

$$A = \begin{bmatrix} v & 1 & 1 & 0 & 0 \\ 1 & w & 1 & 1 & 1 \\ 1 & 1 & x & 1 & 1 \\ 0 & 1 & 1 & y & 1 \\ 0 & 1 & 1 & 1 & z \end{bmatrix} \in \mathcal{M}(\text{fullhouse}).$$

I. $v = 0$. Elementary row and column operations reduce A to

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & w & 1 & 0 & 0 \\ 1 & 1 & x & 0 & 0 \\ 0 & 0 & 0 & y & 1 \\ 0 & 0 & 0 & 1 & z \end{bmatrix}.$$

Then we must have $y = z = 1$. Since $\begin{vmatrix} 0 & 1 & 1 \\ 1 & w & 1 \\ 1 & 1 & x \end{vmatrix} = w + x$, we must have $w = x$, so $w = x = 1$ or $w = x = 0$. This yields the matrices M_2 and M_3 in (f).

II. $v = 1$. Row and column reductions yield

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & w+1 & 0 & 1 & 0 \\ 0 & 0 & x+1 & 1 & 0 \\ 0 & 1 & 1 & y & y+1 \\ 0 & 0 & 0 & y+1 & y+z \end{bmatrix}.$$

If $w = 0$, then B can be further reduced to

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & x+1 & 1 & 0 \\ 0 & 0 & 1 & y+1 & y+1 \\ 0 & 0 & 0 & y+1 & y+z \end{bmatrix},$$

which has rank at least 4, so we must have $w = 1$. Then

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & x+1 & 1 & 0 \\ 0 & 1 & 1 & y & y+1 \\ 0 & 0 & 0 & y+1 & y+z \end{bmatrix},$$

which reduces to

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & x+1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & y+z \end{bmatrix}.$$

In order for rank $C = 3$, we require that $x = 1$ and $y = z$. This yields matrices M_1 and M_4 in (f).

(g) Let

$$A = \begin{bmatrix} w & 1 & 0 & 0 \\ 1 & x & 1 & 0 \\ 0 & 1 & y & 1 \\ 0 & 0 & 1 & z \end{bmatrix} \in \mathcal{M}(P_4).$$

If $w = 0$, by elementary row and column operations the matrix reduces to

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & y & 1 \\ 0 & 0 & 1 & z \end{bmatrix}.$$

In order for rank $B = 3$, we must have $y = z = 1$, but x can be 0 or 1. This yields matrices M_1 and M_2 in (g). If $w = 1$, one row and column operation gives

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & x+1 & 1 & 0 \\ 0 & 1 & y & 1 \\ 0 & 0 & 1 & z \end{bmatrix}.$$

In order for C to have rank 3,

$$\begin{bmatrix} x+1 & 1 & 0 \\ 1 & y & 1 \\ 0 & 1 & z \end{bmatrix} \text{ must be in } \mathcal{M}(P_3),$$

which by Lemma 7 gives three possibilities for x, y, and z, giving matrices M_3, M_4, and M_5 in (g). $\square$

Alternatively, Proposition 8 can be proved by exhaustively calculating the rank of each matrix in $S(G)$ for each $G \in \mathcal{F}_3$. Appendix A contains a collection of SAGE [SAG07] functions and Appendix B contains a similar collection of Magma [BCP97] functions to implement this approach.

2.2 General theorems

Throughout this section, let G be a graph with an induced subgraph H such that $\operatorname{mr}(H) = k$.

For convenience, in sections 2.2–2.8, we will consider G as a complete graph with weighted edges. The weight of an edge, $\operatorname{wt}(ij)$, is the matrix entry corresponding to the edge. Edges with zero weight correspond to nonedges in the original graph. The vertices in $G - H$ will also have weights. Let the vertices of H be labeled $h_1, h_2, \ldots, h_\ell$. The weight $\operatorname{wt}(v)$ of a vertex $v \in V(G - H)$ is the vector $(\operatorname{wt}(vh_1), \operatorname{wt}(vh_2), \ldots, \operatorname{wt}(vh_\ell))^T$ of edge weights between the vertex v and the vertices of H.

2.2.1 Definitions

Definition 7. Let $M \in \mathcal{M}(H)$. We say the vertex v in $G - H$ is *rank-preserving* with respect to M if

$$\operatorname{rank} \begin{bmatrix} M & \operatorname{wt}(v) \end{bmatrix} = \operatorname{rank} M.$$

If v is rank-preserving with respect to M, then M can be augmented by a row and column to obtain a matrix in $S(G[V(H) \cup \{v\}])$ of rank k, so $\operatorname{mr}(G[V(H) \cup \{v\}]) = \operatorname{mr}(H)$. If v is not rank-preserving with respect to M, we say v is *rank-increasing* with respect to M. We say that a set of vertices is *rank-preserving* with respect to M if each vertex is rank-preserving with respect to M, and a set is *rank-increasing* with respect to M if some vertex is rank-increasing with respect to M.

Definition 8. Let $M \in \mathcal{M}(H)$. We say the edge $uv \in G - H$, $u \neq v$, is *rank-preserving* with respect to M if u and v are rank-preserving with respect to M and $\mathrm{wt}(uv)$ is the unique number that satisfies the equality

$$\operatorname{rank} \begin{bmatrix} M & \mathrm{wt}(u) \\ \mathrm{wt}(v)^T & \mathrm{wt}(uv) \end{bmatrix} = \operatorname{rank} M.$$

(If $\mathrm{wt}(u) = Mp$ and $\mathrm{wt}(v) = Mq$, then uv is rank-preserving if and only if $\mathrm{wt}(uv) = q^T Mp$.) If uv is not rank-preserving with respect to M, we say uv is *rank-increasing* with respect to M. Notice that uv is rank-preserving with respect to M if and only if $\mathrm{mr}(G[V(H) \cup \{u, v\}]) = \mathrm{mr}(H)$. We say that a set of edges is *rank-preserving* with respect to M if each edge is rank-preserving with respect to M and is *rank-increasing* with respect to M if some edge is rank-increasing with respect to M.

We emphasize one part of this definition as:

Observation 9. If a vertex v is rank-increasing with respect to M, then each edge incident to v is also rank-increasing with respect to M.

Definition 9. Let $M \in \mathcal{M}(H)$. Given an ordered set of vertex weights $[Mv_1, \ldots, Mv_\ell]$ in $\mathrm{col}(M)$, the column space of M, let A be the matrix with the ith column equal to v_i. Then we say that the $\ell \times \ell$ matrix $P = A^T MA$ is the *rank-preserving table* for the ordered set $[Mv_1, \ldots, Mv_\ell]$ with respect to M. Note that the ij entry of P is the edge weight needed to make the edge between two vertices with weights Mv_i and Mv_j a rank-preserving edge with respect to M.

Example 3. Let $H = P_4$, labeled as in Proposition 8, with corresponding $\mathcal{M}(P_4)$ and $\mathcal{C}(P_4)$. Let G be a graph containing vertices $\{1, 2, 3, 4, u, v\}$ such that $H = G[\{1, 2, 3, 4\}]$ and $G[\{1, 2, 3, 4, u, v\}]$ is one of the graphs in Figure 2.2. Then u and

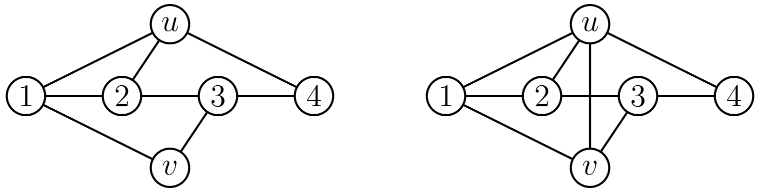


Figure 2.2: Graphs in Example 3.

v have weights $\mathrm{wt}(u) = (1,1,0,1)^T$ and $\mathrm{wt}(v) = (1,0,1,0)^T$. The vertex u is rank-preserving with respect to M_1 and M_2 since $\mathrm{wt}(u) \in \mathrm{col}(M_1) = \mathrm{col}(M_2)$ and is rank-increasing with respect to M_3, M_4, and M_5 since $\mathrm{wt}(u) \notin \mathrm{col}(M_3) = \mathrm{col}(M_4)$ and $\mathrm{wt}(u) \notin \mathrm{col}(M_5)$. Also, v is rank-preserving with respect to M_1, M_2, and M_5 and is rank-increasing with respect to M_3 and M_4. The set of vertices $\{u, v\}$ is rank-preserving with respect to M_1 and M_2 and is rank-increasing with respect to M_3, M_4, and M_5.

The edge uv is rank-increasing with respect to M_3, M_4, and M_5 because the set $\{u, v\}$ is rank-increasing with respect to each of those matrices. If $G[\{1,2,3,4,u,v\}]$ is the graph in Figure 2.2(a), then $\mathrm{wt}(uv) = 0$ and uv is rank-preserving with respect to M_2 and rank-increasing with respect to M_1. If $G[\{1,2,3,4,u,v\}]$ is the graph in Figure 2.2(b), then $\mathrm{wt}(uv) = 1$ and uv is rank-preserving with respect to M_1 and rank-increasing with respect to M_2. Rank-preserving tables with respect to M_1 and M_2 for $[\mathrm{wt}(u), \mathrm{wt}(v)]$ are, respectively,

$$P_1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad P_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Note that $P_1 + P_2 = J$, the all-ones matrix. This property will be important later, so we give it a name now.

Definition 10. Two matrices A and B with entries in $\mathbb{F}_2$ are *complementary* if $A + B = J$, the all-ones matrix.

Definition 11. Let v be a vertex in $G-H$ and V be a set of vertices in $G-H$. Let

$$\mathcal{I}_v = \{M \in \mathcal{M}(H) \mid v \text{ is rank-increasing with respect to } M\}$$

and $\mathcal{I}_V = \cup_{v\in V}\mathcal{I}_v$, the set of matrices for which V is rank-increasing. Let

$$\bar{\mathcal{I}}_V = \{C \in \mathcal{C} \mid V \text{ is rank-increasing with respect to every } M \in C\}.$$

Let uv be an edge in $G-H$ and E be a set of edges in $G-H$. Let

$$\mathcal{I}_{uv} = \{M \in \mathcal{M}(H) \mid uv \text{ is rank-increasing with respect to } M\}$$

and $\mathcal{I}_E = \cup_{uv\in E}\mathcal{I}_{uv}$, the set of matrices for which E is rank-increasing.

Example 4. We will continue from Example 3. We have $\mathcal{I}_u = \{M_3, M_4, M_5\}$ and $\bar{\mathcal{I}}_u = \{C_2, C_3\}$. We also have $\mathcal{I}_v = \{M_3, M_4\}$ and $\bar{\mathcal{I}}_v = \{C_2\}$.

If $\mathrm{wt}(uv) = 0$, as is pictured in Figure 2.2(a), then $\mathcal{I}_{uv} = \{M_1, M_3, M_4, M_5\}$. If $\mathrm{wt}(uv) = 1$, as is pictured in Figure 2.2(b), then $\mathcal{I}_{uv} = \{M_2, M_3, M_4, M_5\}$.

Observation 10. Let V' be a set of vertices in $G-H$ such that $\mathcal{I}_{V'} \neq \mathcal{M}(H)$. Then for every $v \in V'$,

$$\mathrm{wt}(v) \in \bigcap_{M_i \in \mathcal{M}(H)\setminus\mathcal{I}_{V'}} \mathrm{col}(M_i).$$

2.2.2 Theorems

Observation 11. We have $\mathrm{mr}(G) = k$ if and only if there is some $M \in \mathcal{M}(H)$ such that every edge and vertex in $G-H$ is rank-preserving with respect to M. Conversely, $\mathrm{mr}(G) > k$ if and only if there is some set of edges $E' \subseteq E(G-H)$ and vertices $V' \subseteq V(G-H)$ such that $\mathcal{I}_{E'} \cup \mathcal{I}_{V'} = \mathcal{M}(H)$.

Corollary 12. *Assume that* $\mathrm{mr}(G) > k$*. If there are sets* $E' \subset E(G-H)$ *and* $V' \subset V(G-H)$ *such that* $\mathcal{I}_{E'} \cup \mathcal{I}_{V'} = \mathcal{M}(H)$ *and* $(\cup_{xy \in E'}\{x,y\}) \cup V' \subset V(G-H)$ *is a proper subset of* $V(G-H)$*, then* $G \notin \mathcal{F}_{k+1}(H)$*.*

Proof. Let $v \in V(G-H) \setminus ((\cup_{xy\in E'}\{x,y\}) \cup V')$. Then $E' \subset E((G-v)-H)$ and $V' \subset V((G-v)-H)$, so $\mathrm{mr}(G-v) > k$ and $G \notin \mathcal{F}_{k+1}(H)$. □

Proposition 13. *Let* $G \in \mathcal{F}_{k+1}(H)$*. If* $|G-H| \geq 2$*, then for every vertex* v *in* $G-H$*,* $\mathcal{I}_v \neq \mathcal{M}(H)$*. If* $|G-H| \geq 3$*, then for every edge* uv *in* $G-H$*,* $\mathcal{I}_{uv} \neq \mathcal{M}(H)$*.*

Proof. Suppose that $G \in \mathcal{F}_{k+1}(H)$. Suppose there is some vertex $v \in G-H$ which is rank-increasing with respect to every $M \in \mathcal{M}(H)$. Let w be a vertex in $G-H$ other than v. Then $\mathrm{mr}(G-w) > k$, which is a contradiction.

Similarly, suppose that $G \in \mathcal{F}_{k+1}(H)$. Suppose there is some edge uv in $G-H$ which is rank-increasing with respect to every $M \in \mathcal{M}(H)$. Let w be a vertex in $G-H$ other than u or v. Then $\mathrm{mr}(G-w) > k$, which is a contradiction. □

Corollary 14. *Let* $G \in \mathcal{F}_{k+1}(H)$ *and suppose that* $|\mathcal{M}(H)| = 1$*. If* $|G-H| \geq 2$*, then* $\mathcal{I}_v = \emptyset$ *for every vertex* v *in* $G-H$*. If* $|G-H| \geq 3$*, then* $\mathcal{I}_{uv} = \emptyset$ *for every edge* uv *in* $G-H$*.*

Corollary 15. *Suppose that* $|\mathcal{M}(H)| = 1$*. If* $G \in \mathcal{F}_{k+1}(H)$*, then* $|G-H| \leq 2$*.*

Proof. Suppose that $|\mathcal{M}(H)| = 1$ and $\mathcal{M}(H) = \{M\}$. Then if $|G-H| \geq 3$, $\mathcal{I}_{uv} = \emptyset$ for every edge uv in $G-H$ and $\mathcal{I}_v = \emptyset$ for every vertex v in $G-H$. Since every edge and vertex in $G-H$ is rank-preserving with respect to M, $\mathrm{mr}(G) = \mathrm{mr}(H) = k$ and $G \notin \mathcal{F}_{k+1}(H)$. □

Corollary 16. *Let* $G \in \mathcal{F}_{k+1}(H)$*. If* $|G-H| \geq 3$*, then* $G-H$ *contains no vertex* v *with* $\mathrm{wt}(v) = \vec{0}$*.*

Proof. Let $|G-H| \geq 3$ and let v be a vertex of $G-H$ with $\mathrm{wt}(v) = \vec{0}$, the zero vector. Suppose that there is some vertex w of $G-H$ distinct from v such that the edge

vw has nonzero weight. Then the edge vw is rank-increasing for each $M \in \mathcal{M}(H)$, so $\mathcal{I}_{vw} = \mathcal{M}(H)$. This contradicts Proposition 13. Therefore, $\mathrm{wt}(vw) = 0$ for every $w \in V(G - H)$ and v is an isolated vertex in G. Therefore $\mathrm{mr}(G) = \mathrm{mr}(G - v) = k$, so $G \notin \mathcal{F}_{k+1}(H)$, a contradiction. □

Lemma 17. *Let $G \in \mathcal{F}_{k+1}(H)$. If $|G - H| \geq |\mathcal{C}(H)| + 1$, then $\mathcal{I}_{V(G-H)} \neq \mathcal{M}(H)$ (i.e., there exists some $C \in \mathcal{C}$ for which $V(G - H)$ is rank-preserving with respect to each $M \in C$).*

Proof. Suppose that $\mathcal{I}_{V(G-H)} = \mathcal{M}(H)$. Choose vertices $t_1, \ldots, t_{|\mathcal{C}(H)|}$ from $V(G-H)$ such that $C_i \subseteq \mathcal{I}_{t_i}$ for $i = 1, \ldots, |\mathcal{C}(H)|$. Let T be the set containing $t_1, \ldots, t_{|\mathcal{C}(H)|}$. Then $|T| \leq |\mathcal{C}(H)|$ and $\mathcal{I}_T = \mathcal{M}(H)$. Let $v \in V(G - H) \setminus T$. Then $\mathcal{I}_{V(G-H)\setminus\{v\}} = \mathcal{M}(H)$ and $\mathrm{mr}(G - v) > k$, which is a contradiction. Thus there is some $M \in \mathcal{M}(H)$ and corresponding $C \in \mathcal{C}(H)$ for which $V(G - H)$ is rank-preserving. □

By Observation 11, $\mathrm{mr}(G) > k$ if and only if there exist subsets $E' \subseteq E(G - H)$ and $V' \subseteq V(G-H)$ such that $\mathcal{I}_{E'} \cup \mathcal{I}_{V'} = \mathcal{M}(H)$. We will be interested in "minimal" subsets $R \subseteq E(G - H)$ and $T \subseteq V(G - H)$ such that $\mathcal{I}_R \cup \mathcal{I}_T = \mathcal{M}(H)$ because R and T provide an upper bound for $|G - H|$, as the following theorem shows.

Theorem 18. *Assume that $\mathrm{mr}(G) > k$. Let R be a set of edges in $G-H$ and T be a set of vertices in $G-H$ such that $\mathcal{I}_R \cup \mathcal{I}_T = \mathcal{M}(H)$. Let $S = \cup_{ij \in R}\{i, j\}$, the set of vertices incident to the edges in R. If $G \in \mathcal{F}_{k+1}(H)$, then $|G - H| \leq |S| + |T| \leq 2|R| + |T|$.*

Proof. We prove the contrapositive. Suppose that $|G-H| > |S|+|T|$ for some R, S, and T satisfying the hypotheses. Let $v \in V(G-H) \setminus (S \cup T)$ be a vertex in $G-H$ that is different from the vertices in S or T. Then $\mathrm{mr}(G - v) > k$ and $G \notin \mathcal{F}_{k+1}(H)$. □

The basic idea behind our strategy is to minimize the size of $|S| + |T|$ to get an upper bound on the number of vertices in $G - H$ for which $G \in \mathcal{F}_{k+1}(H)$.

In our proofs in Sections 2.3–2.8, we will examine possible cases for $\mathcal{I}_S$, $\mathcal{I}_R$, and $\mathcal{I}_T$. The following four properties will significantly reduce the number of cases we will need to consider.

Assume that G is a graph such that $\operatorname{mr}(G) > k$. Let $R \subseteq E(G-H)$ and $T \subseteq V(G-H)$. Let $S = \cup_{ij \in R}\{i,j\}$, the set of vertices incident to the edges in R. Then the following properties are a direct consequence of the definition of rank-increasing vertices and edges.

P1. $\mathcal{I}_S$ and $\mathcal{I}_T$ are each the union of equivalence classes in $\mathcal{C}(H)$.

P2. $\mathcal{I}_S \subseteq \mathcal{I}_R$ since if $v \in S$ is rank-increasing for a matrix $M \in \mathcal{M}(H)$, then any edge incident to v is also rank-increasing for M (Observation 9).

In addition, if $G \in \mathcal{F}_{k+1}(H)$, $|G-H| \geq |\mathcal{C}(H)|+1$, and $\mathcal{I}_R \cup \mathcal{I}_T = \mathcal{M}(H)$, the following properties are consequences of Lemma 17.

P3. $\bar{\mathcal{I}}_S \cup \bar{\mathcal{I}}_T \neq \mathcal{C}(H)$. This implies that $\mathcal{I}_S \neq \mathcal{M}(H)$ and $\mathcal{I}_T \neq \mathcal{M}(H)$.

P4. There exists a $C \in \mathcal{C}(H)$ such that $C \subseteq \mathcal{I}_R \setminus \mathcal{I}_S$. This implies that $\mathcal{I}_R \neq \emptyset$.

Property 4 is a consequence of $\mathcal{I}_R \cup \mathcal{I}_T = \mathcal{M}(H)$ and properties P1 and P3.

Definition 12. Assume that $\operatorname{mr}(G) > k$. Let $\mathcal{A}$ be the set of triples (R,S,T) such that

(a) $R \subseteq E(G-H)$, $S = \cup_{ij \in R}\{i,j\}$, and $T \subseteq V(G-H)$;

(b) $\mathcal{I}_R \cup \mathcal{I}_T = \mathcal{M}(H)$; and

(c) $2|R| + |T|$ is minimized.

From the triples in $\mathcal{A}$, select those that minimize $|R|$, and from these triples, choose the triples (R,S,T) that minimize $|S|$. We call such an (R,S,T) an *optimal triple* for G and H.

Theorem 19. *Assume that* $\mathrm{mr}(G) > k$. *Let* (R, S, T) *be an optimal triple for* G *and* H. *Then*

(a) For every $v \in T$, $\mathcal{I}_v \not\subseteq (\mathcal{I}_{T\setminus\{v\}} \cup \mathcal{I}_S)$, *and*

(b) For every $uv \in R$, $\mathcal{I}_{uv} \not\subseteq (\mathcal{I}_{R\setminus\{uv\}} \cup \mathcal{I}_S \cup \mathcal{I}_T)$.

Proof. Suppose that S and T do not satisfy (a). Let v be a vertex for which the property does not hold. Then $\mathcal{I}_R \cup \mathcal{I}_{T\setminus\{v\}} = \mathcal{M}(H)$, but $2|R| + |T \setminus \{v\}| < 2|R| + |T|$. This is a contradiction since $(R, S, T) \in \mathcal{A}$.

Suppose that R, S, and T do not satisfy (b). Let uv be an edge for which the property does not hold. Let $R' = R \setminus \{uv\}$, $S' = \cup_{xy \in R'}\{x, y\}$, and $T' = T \cup \{u, v\}$. Then $\mathcal{I}_{R'} \cup \mathcal{I}_{T'} = \mathcal{M}(H)$ and $2|R'| + |T'| \leq 2|R| + |T|$, so $(R', S', T') \in \mathcal{A}$. However, $|R'| < |R|$. This is a contradiction since (R, S, T) is an optimal triple. □

The minimality of $|S|$ was not used in the proof of Theorem 19, but will be used later.

Let (R, S, T) be an optimal triple for G and H. Theorem 19(a) implies that for every vertex $v \in T$, there is class of matrices $C \in \mathcal{C}(H)$ such that v is rank-increasing with respect to every matrix in C, while every other vertex in T and every vertex in S is rank-preserving with respect to every matrix in C. Consequently, there are at most $|\bar{\mathcal{I}}_T \setminus \bar{\mathcal{I}}_S|$ vertices in T. Theorem 19(b) implies that for every edge $uv \in R$, there is some matrix $M \in \mathcal{M}(H)$ such that uv is rank-increasing with respect to M, while every other edge in R and every vertex in $S \cup T$ is rank-preserving with respect to M. Consequently, there are at most $|\mathcal{I}_R \setminus \mathcal{I}_{S\cup T}| = |\mathcal{I}_R \setminus (\mathcal{I}_S \cup \mathcal{I}_T)|$ edges in R.

Corollary 20. *Assume that* $\mathrm{mr}(G) > k$. *If* (R, S, T) *is an optimal triple for* G *and* H, *then*

(a) $|T| \leq |\bar{\mathcal{I}}_T \setminus \bar{\mathcal{I}}_S| = |\{C \in \mathcal{C}(H) \mid C \subseteq (\mathcal{I}_T \setminus \mathcal{I}_S)\}|$, *and*

(b) $|R| \leq |\mathcal{I}_R \setminus (\mathcal{I}_S \cup \mathcal{I}_T)|$.

This corollary gives one upper bound for $|R|$. There will be times that we can prove that an edge in R is rank-increasing for one matrix $M_i \in \mathcal{M}(H)$ if and only if it is also rank-increasing for another matrix $M_j \in \mathcal{M}(H)$. In these cases, we can get a smaller upper bound for $|R|$.

Corollary 21. *Assume that* $\mathrm{mr}(G) > k$ *and let* (R, S, T) *be an optimal triple for* G *and* H*. Then* $S \cap T = \emptyset$*.*

Corollary 22. *Let* $G \in \mathcal{F}_{k+1}(H)$ *and let* (R, S, T) *be an optimal triple for* G *and* H*. If* $\mathcal{I}_R = \mathcal{M}(H)$*, then* $T = \emptyset$ *and* $|G - H| \leq |S| \leq 2|R|$*.*

Proof. Since $\mathcal{I}_R = \mathcal{M}(H)$, $\mathcal{I}_R \cup \mathcal{I}_\emptyset = \mathcal{M}(H)$. Since for any $T \subseteq V(G - H)$, $2|R| + |\emptyset| \leq 2|R| + |T|$, we have $T = \emptyset$ by the minimality of $2|R| + |T|$. By Theorem 18, $|G - H| \leq |S|$. □

The following lemma and corollary give conditions sufficient to reduce the size of the upper bound for $|S|$.

Lemma 23. *Assume that* $\mathrm{mr}(G) > k$*. Let* (R, S, T) *be an optimal triple for* G *and* H*. Suppose that*

(a) $|R| = 2$*,*

(b) If uv *and* wx *are any two edges between vertices in* S*, then either* $\mathcal{I}_{wx} \setminus \mathcal{I}_S = \mathcal{I}_{uv} \setminus \mathcal{I}_S$ *or* $\mathcal{I}_{wx} \setminus \mathcal{I}_S = (\mathcal{I}_R \setminus \mathcal{I}_{uv}) \setminus \mathcal{I}_S$*, and*

(c) there are two (not necessarily distinct) vertices v *and* w*, one incident to each edge of* R*, such that* $\mathcal{I}_{\{v,w\}} = \mathcal{I}_S$*.*

Then $|S| = 3$*.*

Proof. Since $|R| = 2$, we have $3 \leq |S| \leq 4$. Suppose that $|S| = 4$. Let $R = \{uv, wx\}$ and $S = \{u, v, w, x\}$, where $\mathcal{I}_{\{v,w\}} = \mathcal{I}_S$. Let $A = \mathcal{I}_{uv} \setminus \mathcal{I}_S$ and $B = (\mathcal{I}_R \setminus \mathcal{I}_{uv}) \setminus \mathcal{I}_S$. We have $\mathcal{I}_{wx} \setminus \mathcal{I}_S \neq A$ by Theorem 19(b), so $\mathcal{I}_{wx} \setminus \mathcal{I}_S = B$ by hypothesis (b). By hypothesis (b), $\mathcal{I}_{vw} \setminus \mathcal{I}_S = A$ or $\mathcal{I}_{vw} \setminus \mathcal{I}_S = B$.

I. $\mathcal{I}_{vw} \setminus \mathcal{I}_S = A$. Let $R' = \{vw, wx\}$ and $S' = \{v, w, x\}$.

II. $\mathcal{I}_{vw} \setminus \mathcal{I}_S = B$. Let $R' = \{uv, vw\}$ and $S' = \{u, v, w\}$.

Since $\{v, w\} \subseteq S'$, $\mathcal{I}_{S'} = \mathcal{I}_S$. Also $\mathcal{I}_{R'} = A \cup B \cup \mathcal{I}_{S'} = A \cup B \cup \mathcal{I}_S = \mathcal{I}_R$. Therefore, (R', S', T) is a triple such that $\mathcal{I}_{R'} \cup \mathcal{I}_T = \mathcal{M}(H)$, $2|R'| + |T| = 2|R| + |T|$, and $|R'| = |R|$, but $|S'| < |S|$, which contradicts the optimality of (R, S, T). Thus $|S| = 3$. □

Corollary 24. *Assume that* $\mathrm{mr}(G) > k$. *Let* (R, S, T) *be an optimal triple for* G *and* H. *Suppose that*

(a) $|R| = 2$,

(b) If uv *and* wx *are any two edges between vertices in* S, *then either* $\mathcal{I}_{wx} \setminus \mathcal{I}_S = \mathcal{I}_{uv} \setminus \mathcal{I}_S$ *or* $\mathcal{I}_{wx} \setminus \mathcal{I}_S = (\mathcal{I}_R \setminus \mathcal{I}_{uv}) \setminus \mathcal{I}_S$, *and*

(c) $|\bar{\mathcal{I}}_S| \leq 1$.

Then $|S| = 3$.

Proof. Since $|\bar{\mathcal{I}}_S| \leq 1$, there is some vertex $y \in S$ such that $\mathcal{I}_y = \mathcal{I}_S$. Therefore $\mathcal{I}_{\{y,z\}} = \mathcal{I}_S$ for any vertex $z \in S$. Applying Lemma 23 then gives the result. □

In Sections 2.3–2.8, we will determine an upper bound for the number of vertices in graphs in $\mathcal{F}_4(H)$ for each graph H in

$$\mathcal{F}_3 = \{3K_2, P_3 \vee P_3, \text{dart}, \ltimes, P_3 \cup K_2, \text{fullhouse}, P_4\}.$$

We will then apply Corollary 6 to determine the maximum number of vertices in a graph in $\mathcal{F}_4$.

2.3 $H = 3K_2$ or $H = P_3 \vee P_3$

By Proposition 8, $|\mathcal{M}(3K_2)| = 1$ and $|\mathcal{M}(P_3 \vee P_3)| = 1$, so applying Corollary 15 gives the following lemma.

Lemma 25. *If $G \in \mathcal{F}_4(3K_2)$ or $G \in \mathcal{F}_4(P_3 \vee P_3)$, then $|G| \leq 8$.*

2.4 $H =$ Dart

Lemma 26. *If $G \in \mathcal{F}_4(\text{dart})$, then $|G| \leq 7$.*

Proof. Suppose that $G \in \mathcal{F}_4(\text{dart})$ and $|G| \geq 8$ (i.e., $|G - H| \geq 3$). Then $G - H$ has no vertices with zero weight by Corollary 16. Assume that (R, S, T) is an optimal triple for G and the dart. Let $\mathcal{M}(\text{dart}) = \{M_1, M_2\}$ and $\mathcal{C}(\text{dart}) = \{C_1 = \{M_1\}, C_2 = \{M_2\}\}$ be as in Proposition 8(c). By property P1, $\mathcal{I}_S \in \{\emptyset, C_1, C_2, C_1 \cup C_2\}$. By property P3, $\mathcal{I}_S \neq C_1 \cup C_2$. Thus $\mathcal{I}_S \in \{\emptyset, C_1, C_2\}$ and we have the following cases.

Case 1: $\mathcal{I}_S = \emptyset$. By Observation 10, if $v \in S$, then

$$\begin{aligned}\text{wt}(v) \in \text{col}(M_1) \cap \text{col}(M_2) =\{&\vec{0},\ v_1 = (0,1,0,1,0)^T,\ v_2 = (1,0,0,1,0)^T,\\ &v_3 = (1,1,0,0,0)^T\}.\end{aligned}$$

The rank-preserving tables for $[v_1, v_2, v_3]$ with respect to M_1 and M_2 are, respectively,

$$P_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad P_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Since $P_1 = P_2$, an edge in R is rank-preserving for M_1 if and only if it is also rank-preserving for M_2. This combined with property P4 implies that $\mathcal{I}_R = \{M_1, M_2\}$. Since $M_1 \in \mathcal{I}_{uv}$ if and only if $M_2 \in \mathcal{I}_{uv}$ for any edge $uv \in R$, Theorem 19(b) implies that $|R| = 1$ and $|S| = 2$. Since $\mathcal{I}_R = \mathcal{M}(\text{dart})$, $T = \emptyset$ and $|G - H| \leq 2$ by

Corollary 22. This contradicts our assumption that $|G| \geq 8$, so this case cannot occur.

Case 2: $\mathcal{I}_S = \{M_1\}$ **or** $\mathcal{I}_S = \{M_2\}$**.** In each of these cases, by property P4, $\mathcal{I}_R = \{M_1, M_2\}$. By Corollary 20, $|R| \leq 1$, so $|R| = 1$. Again, since $\mathcal{I}_R = \mathcal{M}(\text{dart})$, $T = \emptyset$ and $|G - H| \leq 2$ by Corollary 22. This contradicts our assumption that $|G| \geq 8$, so neither of these cases can occur.

Thus $|G - H| \geq 3$ is impossible, so $|G - H| \leq 2$ and $|G| \leq 7$. $\square$

2.5 $H = \ltimes$

Lemma 27. *If* $G \in \mathcal{F}_4(\ltimes)$*, then* $|G| \leq 8$.

Proof. Suppose that $G \in \mathcal{F}_4(\ltimes)$ and $|G| \geq 8$ (i.e., $|G - H| \geq 3$). Then $G - H$ has no vertices with zero weight by Corollary 16. Assume that (R, S, T) is an optimal triple for G and $\ltimes$. Let $\mathcal{M}(\ltimes) = \{M_1, M_2, M_3\}$ and $\mathcal{C}(\ltimes) = \{C_1 = \{M_1, M_2\}, C_2 = \{M_3\}\}$ be as in Proposition 8(d). By property P1, $\mathcal{I}_S \in \{\emptyset, C_1, C_2, C_1 \cup C_2\}$. By property P3, $\mathcal{I}_S \neq C_1 \cup C_2$. Thus $\mathcal{I}_S \in \{\emptyset, C_1, C_2\}$ and we have the following cases.

Case 1: $\mathcal{I}_S = \emptyset$**.** By Observation 10, if $v \in S$, then

$$\text{wt}(v) \in \bigcap_{i=1}^{3} \text{col}(M_i) = \{\vec{0},\ v_1 = (0,1,1,0,0)^T,\ v_2 = (1,0,0,1,1)^T,\ v_3 = (1,1,1,1,1)^T\}.$$

The rank-preserving tables for $[v_1, v_2, v_3]$ with respect to M_1, M_2, and M_3 are, respectively,

$$P_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad \text{and } P_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Since $P_2 = P_3$, an edge in R is rank-preserving for M_2 if and only if it is also rank-preserving for M_3. Thus, we must either have both M_2 and M_3 in $\mathcal{I}_R$ or have neither in the set. Thus $\mathcal{I}_R \in \{\{M_1, M_2, M_3\}, \{M_2, M_3\}, \{M_1\}\}$. By property P4, $\mathcal{I}_R \neq \{M_1\}$. Therefore we have the following cases.

Subcase 1.1: $\mathcal{I}_R = \{M_1, M_2, M_3\}$**.** Since $\mathcal{I}_R = \mathcal{M}(\ltimes)$, Corollary 22 implies that $T = \emptyset$ and $|G - H| \leq |S| \leq 2|R|$. Since $M_2 \in \mathcal{I}_{uv}$ if and only if $M_3 \in \mathcal{I}_{uv}$ for each edge $uv \in R$, $|R| \leq 2$ by Theorem 19(b). If $|R| \leq 1$, then $|G - H| \leq 2$, a contradiction. If $|R| = 2$, then Theorem 19(b) implies that R consists of an edge uv such that $\mathcal{I}_{uv} = \{M_1\}$ and another edge wx such that $\mathcal{I}_{wx} = \{M_2, M_3\}$. Since the second row and column of P_1, P_2, and P_3 are identical, we see that if any vertex in S, say u, has weight v_2, then the edge in R incident to the vertex must have either $\mathcal{I}_{uv} = \mathcal{I}_R$ or $\mathcal{I}_{uv} = \emptyset$. Neither of these cases occur, so u, v, w, and x each must have weight v_1 or v_3. Note that since the principal submatrices $P_1[1,3]$ and $P_2[1,3] = P_3[1,3]$ are complementary, any edge between vertices with weights v_1 or v_3 must be either rank-increasing for M_1 and rank-preserving for M_2 and M_3, or rank-increasing for M_2 and M_3 and rank-preserving for M_1. This fact combined with the facts that $|R| = 2$ and $|\bar{\mathcal{I}}_S| = 0$ allow us to apply Corollary 24 to conclude that $|S| = 3$, $|G - H| \leq 3$, and $|G| \leq 8$.

Subcase 1.2: $\mathcal{I}_R = \{M_2, M_3\}$**.** Since $\mathcal{I}_R \cup \mathcal{I}_T = \mathcal{M}(H)$, properties P1 and P3 imply that $\mathcal{I}_T = \{M_1, M_2\}$. By Corollary 20, $|R| \leq 1$ and $|T| \leq 1$. By Theorem 18, $|G - H| \leq 3$, so $|G| \leq 8$.

Case 2: $\mathcal{I}_S = \{M_1, M_2\}$**.** By property P4, $\mathcal{I}_R = \{M_1, M_2, M_3\}$, so $T = \emptyset$ and $|G - H| \leq 2|R|$ by Corollary 22. By Corollary 20, $|R| \leq 1$, so $|G - H| \leq 2$. This contradicts the assumption that $|G - H| \geq 3$, so this case cannot occur.

Case 3: $\mathcal{I}_S = \{M_3\}$**.** By Observation 10, if $v \in S$, then

$$\begin{aligned}\mathrm{wt}(v) \in \mathrm{col}(M_1) = \mathrm{col}(M_2) = \{\vec{0},\, v_1 = (0,0,0,1,1)^T,\, v_2 = (0,1,1,0,0)^T,\\ v_3 = (0,1,1,1,1)^T,\, v_4 = (1,0,0,0,0)^T,\, v_5 = (1,0,0,1,1)^T,\\ v_6 = (1,1,1,0,0)^T,\, v_7 = (1,1,1,1,1)^T\}.\end{aligned}$$

The rank-preserving tables for $[v_1, v_2, v_3, v_4, v_5, v_6, v_7]$ with respect to M_1 and M_2 are, respectively,

$$P_1 = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad P_2 = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

By property P4, $\mathcal{I}_R = \{M_1, M_2, M_3\}$, so $T = \emptyset$ and $|G - H| \leq |S|$ by Corollary 22. By Corollary 20, $|R| \leq 2$. If $|R| = 1$, then $|G - H| \leq 2$, which is a contradiction. If $|R| = 2$, then Theorem 19(b) implies that R consists of an edge uv such that $\mathcal{I}_{uv} \setminus \mathcal{I}_S = \{M_1\}$ and another edge wx such that $\mathcal{I}_{wx} \setminus \mathcal{I}_S = \{M_2\}$. Since the first, fourth, and fifth rows and columns of P_1 and P_2 are identical, we see that if any vertex, say u, has weight v_1, v_4, or v_5, then the edge in R incident to the vertex must have either $\mathcal{I}_{uv} \setminus \mathcal{I}_S = \mathcal{I}_R \setminus \mathcal{I}_S$ or $\mathcal{I}_{uv} \setminus \mathcal{I}_S = \emptyset$. Neither of these cases occur, so u, v, w, and x each must have weight v_2, v_3, v_6, or v_7. As in Subcase 1.1, since $P_1[2,3,6,7]$ and $P_2[2,3,6,7]$ are complementary, $|R| = 2$, and $|\bar{\mathcal{I}}_S| = 1$, we can apply Corollary 24 to conclude that $|S| = 3$, $|G - H| \leq 3$, and $|G| \leq 8$. □

2.6 $H = P_3 \cup K_2$

Lemma 28. *If $G \in \mathcal{F}_4(P_3 \cup K_2)$, then $|G| \leq 8$.*

Proof. Suppose that $G \in \mathcal{F}_4(P_3 \cup K_2)$ and $|G| \geq 8$ (i.e., $|G - H| \geq 3$). Then $G - H$

has no vertices with zero weight by Corollary 16. Assume that (R, S, T) is an optimal triple for G and $P_3 \cup K_2$. Let $\mathcal{M}(P_3 \cup K_2) = \{M_1, M_2, M_3\}$ and $\mathcal{C}(P_3 \cup K_2) = \{C_1 = \{M_1, M_2\}, C_2 = \{M_3\}\}$ be as in Proposition 8(e). By properties P1 and P3, $\mathcal{I}_S \in \{\emptyset, C_1, C_2\}$, so we have the following cases.

Case 1: $\mathcal{I}_S = \emptyset$. By Observation 10, if $v \in S$, then

$$\mathrm{wt}(v) \in \bigcap_{i=1}^{3} \mathrm{col}(M_i) = \{\vec{0},\ v_1 = (0,0,0,1,1)^T,\ v_2 = (1,0,1,0,0)^T,\ v_3 = (1,0,1,1,1)^T\}.$$

The rank-preserving tables for $[v_1, v_2, v_3]$ with respect to M_1, M_2, and M_3 are, respectively,

$$P_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad \text{and } P_3 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Since $P_1 = P_3$, an edge in R is rank-preserving for M_1 if and only if it is also rank-preserving for M_3. Thus $\mathcal{I}_R \in \{\{M_1, M_2, M_3\}, \{M_1, M_3\}, \{M_2\}\}$. By property P4, $\mathcal{I}_R \neq \{M_2\}$. Therefore we have the following cases.

Subcase 1.1: $\mathcal{I}_R = \{M_1, M_2, M_3\}$. Since $\mathcal{I}_R = \mathcal{M}(P_3 \cup K_2)$, Corollary 22 implies that $T = \emptyset$ and $|G - H| \le |S| \le 2|R|$. We reason as in Subcase 1.1 in Section 2.5. Since $M_1 \in \mathcal{I}_{uv}$ if and only if $M_3 \in \mathcal{I}_{uv}$ for each edge $uv \in R$, $|R| \le 2$ by Theorem 19(b). If $|R| = 1$, then $|G - H| \le 2$, a contradiction. If $|R| = 2$, then Theorem 19(b) implies that R consists of an edge uv such that $\mathcal{I}_{uv} = \{M_1, M_3\}$ and another edge wx such that $\mathcal{I}_{wx} = \{M_2\}$. Since the first row and column of P_1, P_2, and P_3 are identical, we see that if any vertex, say u, has weight v_1, then the edge in R incident to the vertex must have either $\mathcal{I}_{uv} = \mathcal{I}_R$ or $\mathcal{I}_{uv} = \emptyset$. Neither of these cases occur, so u, v, w, and x each must have weight v_2 or v_3. As in Subcase 1.1 in Section 2.5, since $P_1[2,3]$ and $P_2[2,3]$ are complementary, $|R| = 2$, and $|\bar{\mathcal{I}}_S| = 0$, we can apply Corollary 24 to conclude that $|S| = 3$, $|G - H| \le 3$, and $|G| \le 8$.

Subcase 1.2: $\mathcal{I}_R = \{M_1, M_3\}$**.** Since $\mathcal{I}_R \cup \mathcal{I}_T = \mathcal{M}(H)$, properties P1 and P3 imply that $\mathcal{I}_T = \{M_1, M_2\}$. By Corollary 20, $|R| \leq 1$ and $|T| \leq 1$. By Theorem 18, $|G - H| \leq 3$, so $|G| \leq 8$.

Case 2: $\mathcal{I}_S = \{M_1, M_2\}$**.** By property P4, $\mathcal{I}_R = \{M_1, M_2, M_3\}$, so $T = \emptyset$ and $|G - H| \leq 2|R|$ by Corollary 22. By Corollary 20, $|R| \leq 1$, so $|G - H| \leq 2$ and $|G| \leq 7$. This contradicts the assumption that $|G| \geq 8$, so this case cannot occur.

Case 3: $\mathcal{I}_S = \{M_3\}$**.** By Observation 10, if $v \in S$, then

$$\begin{aligned}\mathrm{wt}(v) \in \mathrm{col}(M_1) = \mathrm{col}(M_2) = \{\vec{0},\, v_1 = (0,0,0,1,1)^T,\, v_2 = (0,1,0,0,0)^T,\\ v_3 = (0,1,0,1,1)^T,\, v_4 = (1,0,1,0,0)^T,\, v_5 = (1,0,1,1,1)^T,\\ v_6 = (1,1,1,0,0)^T,\, v_7 = (1,1,1,1,1)^T\}.\end{aligned}$$

The rank-preserving tables for $[v_1, v_2, v_3, v_4, v_5, v_6, v_7]$ with respect to M_1 and M_2 are, respectively,

$$P_1 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad P_2 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

By property P4, $\mathcal{I}_R = \{M_1, M_2, M_3\}$, so $T = \emptyset$ and $|G - H| \leq |S|$ by Corollary 22. By Corollary 20, $|R| \leq 2$. If $|R| = 1$, then $|G - H| \leq 2$, which is a contradiction. If $|R| = 2$, then Theorem 19(b) implies that R consists of an edge uv such that $\mathcal{I}_{uv} \setminus \mathcal{I}_S = \{M_1\}$ and another edge wx such that $\mathcal{I}_{wx} \setminus \mathcal{I}_S = \{M_2\}$. Since the first three rows and columns of P_1 and P_2 are identical, we see that if any vertex, say u, has weight v_1, v_2, or v_3, then the edge in R incident to the vertex must have either $\mathcal{I}_{uv} \setminus \mathcal{I}_S = \mathcal{I}_R \setminus \mathcal{I}_S$ or $\mathcal{I}_{uv} \setminus \mathcal{I}_S = \emptyset$. Neither of these cases occur, so u, v, w, and x each must have weight v_4, v_5, v_6, or v_7. As in Subcase 1.1 in Section 2.5, since $P_1[4,5,6,7]$

and $P_2[4,5,6,7]$ are complementary, $|R| = 2$, and $|\bar{\mathcal{I}}_S| = 1$, we can apply Corollary 24 to conclude that $|S| = 3$, $|G - H| \leq 3$, and $|G| \leq 8$. □

2.7 $H =$ fullhouse

Lemma 29. *If $G \in \mathcal{F}_4(\text{fullhouse})$, then $|G| \leq 8$.*

Proof. Suppose that $G \in \mathcal{F}_4(\text{fullhouse})$ and $|G| \geq 9$ (i.e., $|G - H| \geq 4$). Then $G - H$ has no vertices with zero weight by Corollary 16. Assume that (R, S, T) is an optimal triple for G and the fullhouse. Let $\mathcal{M}(\text{fullhouse}) = \{M_1, M_2, M_3, M_4\}$ and $\mathcal{C}(\text{fullhouse}) = \{C_1 = \{M_1, M_2\}, C_2 = \{M_3\}, C_3 = \{M_4\}\}$ be as in Proposition 8(f). By properties P1 and P3, $\mathcal{I}_S \in \{\emptyset, C_1, C_2, C_3, C_1 \cup C_2, C_1 \cup C_3, C_2 \cup C_3\}$, so we have the following cases.

Case 1: $\mathcal{I}_S = \emptyset$. By Observation 10, if $v \in S$, then

$$\text{wt}(v) \in \bigcap_{i=1}^{4} \text{col}(M_i) = \{\vec{0},\ v_1 = (0,0,0,1,1)^T\}.$$

The rank-preserving tables for $[v_1]$ with respect to M_1, M_2, M_3, and M_4 are, respectively,

$$P_1 = \begin{bmatrix} 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 1 \end{bmatrix}, \quad \text{and } P_4 = \begin{bmatrix} 0 \end{bmatrix}.$$

Since $P_1 = P_4$ and $P_2 = P_3$ are complementary, an edge uv in R has either $\mathcal{I}_{uv} = \{M_1, M_4\}$ or $\mathcal{I}_{uv} = \{M_2, M_3\}$. Thus $\mathcal{I}_R \in \{\{M_1, M_2, M_3, M_4\}, \{M_1, M_4\}, \{M_2, M_3\}\}$. Therefore we have the following cases.

Subcase 1.1: $\mathcal{I}_R = \{M_1, M_2, M_3, M_4\}$. Since $\mathcal{I}_R = \mathcal{M}(\text{fullhouse})$, Corollary 22 implies that $T = \emptyset$ and $|G - H| \leq |S| \leq 2|R|$. We reason as in Subcase 1.1 in Section 2.5. Since $M_1 \in \mathcal{I}_{uv}$ if and only if $M_4 \in \mathcal{I}_{uv}$ and $M_2 \in \mathcal{I}_{uv}$ if and only if $M_3 \in \mathcal{I}_{uv}$ for each edge $uv \in R$, $|R| \leq 2$ by Theorem 19(b). If $|R| = 1$, then

$|G - H| \le 2$, a contradiction. If $|R| = 2$, then Theorem 19(b) implies that R consists of an edge uv such that $\mathcal{I}_{uv} = \{M_1, M_4\}$ and another edge wx such that $\mathcal{I}_{wx} = \{M_2, M_3\}$. As in Subcase 1.1 in Section 2.5, since $P_1 = P_4$ and $P_2 = P_3$ are complementary, $|R| = 2$, and $|\bar{\mathcal{I}}_S| = 0$, we can apply Corollary 24 to conclude that $|S| = 3$, $|G - H| \le 3$, and $|G| \le 8$. This contradicts the assumption that $|G| \ge 9$, so this case does not occur.

Subcase 1.2: $\mathcal{I}_R = \{M_1, M_4\}$**.** Since $\mathcal{I}_R \cup \mathcal{I}_T = \mathcal{M}(H)$, properties P1 and P3 imply that $\mathcal{I}_T = \{M_1, M_2, M_3\}$. By Corollary 20, $|R| \le 1$ and $|T| \le 2$. If $|T| = 2$, then by Theorem 19(a), T consists of a vertex v such that $\mathcal{I}_v = \{M_1, M_2\}$ and another vertex w such that $\mathcal{I}_w = \{M_3\}$. Since v is rank-preserving with respect to M_3 and M_4, $\mathrm{wt}(v) \in \mathrm{col}(M_3) \cap \mathrm{col}(M_4) = \{\vec{0},\ (0,0,0,1,1)^T\}$, so $\mathrm{wt}(v) = (0,0,0,1,1)^T$. But then $\mathcal{I}_v = \emptyset$, a contradiction, so this case does not occur.

Subcase 1.3: $\mathcal{I}_R = \{M_2, M_3\}$**.** Since $\mathcal{I}_R \cup \mathcal{I}_T = \mathcal{M}(H)$, properties P1 and P3 imply that $\mathcal{I}_T = \{M_1, M_2, M_4\}$. By Corollary 20, $|R| \le 1$ and $|T| \le 2$. Again, if $|T| = 2$, then by Theorem 19(a), T consists of a vertex v such that $\mathcal{I}_v = \{M_1, M_2\}$ and another vertex w such that $\mathcal{I}_w = \{M_4\}$. Proceeding as in Subcase 1.2, $\mathrm{wt}(v) = (0,0,0,1,1)^T$ and $\mathcal{I}_v = \emptyset$, a contradiction, so this case does not occur.

Case 2: $\mathcal{I}_S = \{M_1, M_2\}$**.** By Observation 10, if $v \in S$, then $\mathrm{wt}(v) \in \mathrm{col}(M_3) \cap \mathrm{col}(M_4) = \{\vec{0},\ v_1 = (0,0,0,1,1)^T\}$. But then $\mathcal{I}_S = \emptyset$, a contradiction, so this case does not occur.

Case 3: $\mathcal{I}_S = \{M_3\}$**.** By Observation 10, if $v \in S$, then

$$\mathrm{wt}(v) \in \mathrm{col}(M_1) \cap \mathrm{col}(M_2) \cap \mathrm{col}(M_4) = \{\vec{0},\ v_1 = (0,0,0,1,1)^T,\ v_2 = (1,1,1,0,0)^T,\ v_3 = (1,1,1,1,1)^T\}.$$

The rank-preserving tables for $[v_1, v_2, v_3]$ with respect to M_1, M_2, and M_4 are, respectively,

$$P_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad \text{and } P_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Since $P_1 = P_4$, an edge in R is rank-preserving for M_1 if and only if it is also rank-preserving for M_4. Thus $\mathcal{I}_R \in \{\{M_1, M_2, M_3, M_4\}, \{M_1, M_3, M_4\}, \{M_2, M_3\}\}$. By property P4, $\mathcal{I}_R \neq \{M_2, M_3\}$. Therefore we have the following cases.

Subcase 3.1: $\mathcal{I}_R = \{M_1, M_2, M_3, M_4\}$**.** Since $\mathcal{I}_R = \mathcal{M}(\text{fullhouse})$, Corollary 22 implies that $T = \emptyset$ and $|G - H| \leq |S|$. We reason as in Subcase 1.1 in Section 2.5. Since $M_1 \in \mathcal{I}_{uv}$ if and only if $M_4 \in \mathcal{I}_{uv}$ for each edge $uv \in R$, $|R| \leq 2$ by Theorem 19(b). If $|R| = 1$, then $|G - H| \leq 2$, which is a contradiction. If $|R| = 2$, then Theorem 19(b) implies that R consists of an edge uv such that $\mathcal{I}_{uv} \setminus \mathcal{I}_S = \{M_1, M_4\}$ and another edge wx such that $\mathcal{I}_{wx} \setminus \mathcal{I}_S = \{M_2\}$. Since the third row and column of P_1, P_2, and P_4 are identical, we see that if any vertex, say u, has weight v_3, then the edge in R incident to the vertex must have either $\mathcal{I}_{uv} \setminus \mathcal{I}_S = \mathcal{I}_R \setminus \mathcal{I}_S$ or $\mathcal{I}_{uv} \setminus \mathcal{I}_S = \emptyset$. Neither of these cases occur, so u, v, w, and x each must have weight v_1 or v_2. As in Subcase 1.1 in Section 2.5, since $P_1[1,2] = P_4[1,2]$ and $P_2[1,2]$ are complementary, $|R| = 2$, and $|\bar{\mathcal{I}}_S| = 1$, we can apply Corollary 24 to conclude that $|S| = 3$, $|G - H| \leq 3$, and $|G| \leq 8$. This contradicts the assumption that $|G| \geq 9$, so this case does not occur.

Subcase 3.2: $\mathcal{I}_R = \{M_1, M_3, M_4\}$**.** Since $\mathcal{I}_R \cup \mathcal{I}_T = \mathcal{M}(H)$, properties P1 and P3 imply that $\mathcal{I}_T = \{M_1, M_2\}$ or $\mathcal{I}_T = \{M_1, M_2, M_3\}$. In each of these cases, $|R| \leq 1$ and $|T| \leq 1$ by Theorem 19(b) and Corollary 20, implying that $|G - H| \leq 3$ and $|G| \leq 8$. This contradicts the assumption that $|G| \geq 9$, so these cases do not occur.

Case 4: $\mathcal{I}_S = \{M_4\}$**.** By Observation 10, if $v \in S$, then

$$\text{wt}(v) \in \text{col}(M_1) \cap \text{col}(M_2) \cap \text{col}(M_3) = \{\vec{0},\, v_1 = (0,0,0,1,1)^T,\, v_2 = (0,1,1,0,0)^T,$$
$$v_3 = (0,1,1,1,1)^T\}.$$

The rank-preserving tables for $[v_1, v_2, v_3]$ with respect to M_1, M_2, and M_3 are, respectively,

$$P_1 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad \text{and } P_3 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Since $P_2 = P_3$, an edge in R is rank-preserving for M_2 if and only if it is also rank-preserving for M_3. By properties P2 and P4, $\mathcal{I}_R \in \{\{M_1, M_2, M_3, M_4\}, \{M_2, M_3, M_4\}\}$, so we have the following cases.

Subcase 4.1: $\mathcal{I}_R = \{M_1, M_2, M_3, M_4\}$. Since $\mathcal{I}_R = \mathcal{M}(\text{fullhouse})$, Corollary 22 implies that $T = \emptyset$ and $|G - H| \leq |S|$. We again reason as in Subcase 1.1 in Section 2.5. Since $M_2 \in \mathcal{I}_{uv}$ if and only if $M_3 \in \mathcal{I}_{uv}$ for each edge $uv \in R$, $|R| \leq 2$ by Theorem 19(b). If $|R| = 1$, then $|G-H| \leq 2$, which is a contradiction. If $|R| = 2$, then Theorem 19(b) implies that R consists of an edge uv such that $\mathcal{I}_{uv} \setminus \mathcal{I}_S = \{M_2, M_3\}$ and another edge wx such that $\mathcal{I}_{wx} \setminus \mathcal{I}_S = \{M_1\}$. Since the third row and column of P_1, P_2, and P_3 are identical, we see that if any vertex, say u, has weight v_3, then the edge in R incident to the vertex must have either $\mathcal{I}_{uv} \setminus \mathcal{I}_S = \mathcal{I}_R \setminus \mathcal{I}_S$ or $\mathcal{I}_{uv} \setminus \mathcal{I}_S = \emptyset$. Neither of these cases occur, so u, v, w, and x each must have weight v_1 or v_2. As in Subcase 1.1 in Section 2.5, since $P_1[1,2]$ and $P_2[1,2] = P_3[1,2]$ are complementary, $|R| = 2$, and $|\bar{\mathcal{I}}_S| = 1$, we can apply Corollary 24 to conclude that $|S| = 3$, $|G - H| \leq 3$, and $|G| \leq 8$. This contradicts the assumption that $|G| \geq 9$, so this case does not occur.

Subcase 4.2: $\mathcal{I}_R = \{M_2, M_3, M_4\}$. Since $\mathcal{I}_R \cup \mathcal{I}_T = \mathcal{M}(H)$, properties P1 and P3 imply that $\mathcal{I}_T = \{M_1, M_2\}$ or $\mathcal{I}_T = \{M_1, M_2, M_4\}$. In each of these cases, $|R| \leq 1$ and $|T| \leq 1$ by Theorem 19(b) and Corollary 20, implying that $|G - H| \leq 3$ and $|G| \leq 8$. This contradicts the assumption that $|G| \geq 9$, so these cases do not occur.

Case 5: $\mathcal{I}_S = \{M_1, M_2, M_3\}$ or $\mathcal{I}_S = \{M_1, M_2, M_4\}$. In each of these cases, by property P4, $\mathcal{I}_R = \{M_1, M_2, M_3, M_4\}$, so $T = \emptyset$ and $|G - H| \leq 2|R|$ by Corollary 22.

In each of these cases, $|R| \leq 1$ by Corollary 20, so $|G - H| \leq 2$ and $|G| \leq 7$. This contradicts the assumption that $|G| \geq 9$, so these cases do not occur.

Case 6: $\mathcal{I}_S = \{M_3, M_4\}$. By Observation 10, if $v \in S$, then

$$\begin{aligned} \mathrm{wt}(v) \in \mathrm{col}(M_1) = \mathrm{col}(M_2) = \{\vec{0},\ v_1 = (0,0,0,1,1),\ v_2 = (0,1,1,0,0), \\ v_3 = (0,1,1,1,1),\ v_4 = (1,0,0,0,0),\ v_5 = (1,0,0,1,1), \\ v_6 = (1,1,1,0,0),\ v_7 = (1,1,1,1,1)\}. \end{aligned}$$

The rank-preserving tables for $[v_1, v_2, v_3, v_4, v_5, v_6, v_7]$ with respect to M_1 and M_2 are, respectively,

$$P_1 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad P_2 = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

By property P4, $\mathcal{I}_R = \{M_1, M_2, M_3, M_4\}$, so $T = \emptyset$ and $|G - H| \leq |S| \leq 2|R|$ by Corollary 22. By Corollary 20, $|R| \leq 2$. If $|R| = 1$, then $|G - H| \leq 2$ and $|G| \leq 7$, a contradiction.

Suppose that $|R| = 2$. Let $R = \{uv, wx\}$. Theorem 19(b) implies that R consists of an edge e_1 such that $\mathcal{I}_{e_1} \backslash \mathcal{I}_S = \{M_1\}$ and another edge e_2 such that $\mathcal{I}_{e_2} \backslash \mathcal{I}_S = \{M_2\}$. Since the third, fourth, and seventh rows and columns of P_1 and P_2 are identical, we see that if any vertex, say a vertex in e_1, has weight v_3, v_4, or v_7, then the edge in R incident to the vertex must have either $\mathcal{I}_{e_1} \setminus \mathcal{I}_S = \mathcal{I}_R \setminus \mathcal{I}_S$ or $\mathcal{I}_{e_1} \setminus \mathcal{I}_S = \emptyset$. Neither of these cases occur, so each of the vertices in S must have weight v_1, v_2, v_5, or v_6. Note also that $P_1[1,2,5,6]$ and $P_2[1,2,5,6]$ are complementary. However, we cannot proceed as before and apply Corollary 24 since $|\bar{\mathcal{I}}_S| = 2$.

If there are vertices a and b, one incident to each edge of R, such that $\mathcal{I}_{\{a,b\}} = \mathcal{I}_S$, then we can apply Lemma 23 and conclude that $|S| = 3$, $|G - H| \leq 3$, and $|G| \leq 8$, a contradiction.

Suppose that $|S| = 4$ and there are not two vertices a and b in R such that a is incident to one edge, b is incident to the other edge, and $\mathcal{I}_{\{a,b\}} = \mathcal{I}_S = \{M_3, M_4\}$. By relabeling, if necessary, we then have $\mathcal{I}_u = \{M_3\}$, $\mathcal{I}_v = \{M_4\}$, $\mathcal{I}_w = \emptyset$, and $\mathcal{I}_x = \emptyset$. Recall also that for any vertex $a \in S$, $\mathrm{wt}(a) \in \{v_1, v_2, v_5, v_6\}$. Notice that if a vertex a has weight $\mathrm{wt}(a) = v_1$, then $\mathcal{I}_a = \emptyset$, so $\mathrm{wt}(u) \neq v_1$ and $\mathrm{wt}(v) \neq v_1$. Moreover, $\mathrm{wt}(u) \in \mathrm{col}(M_4)$ while $v_2, v_5 \notin \mathrm{col}(M_4)$. Thus $\mathrm{wt}(u) = v_6$. Also $\mathrm{wt}(v) \in \mathrm{col}(M_3)$ and $v_5, v_6 \notin \mathrm{col}(M_3)$, so $\mathrm{wt}(v) = v_2$. Since $\mathrm{wt}(w), \mathrm{wt}(x) \in \mathrm{col}(M_i)$ for all i, $\mathrm{wt}(w) = \mathrm{wt}(x) = v_1$.

Since $|R| = 2$, either $\mathcal{I}_{uv} \setminus \mathcal{I}_S = \{M_1\}$ and $\mathcal{I}_{wx} \setminus \mathcal{I}_S = \{M_2\}$, or $\mathcal{I}_{uv} \setminus \mathcal{I}_S = \{M_2\}$ and $\mathcal{I}_{wx} \setminus \mathcal{I}_S = \{M_1\}$.

Suppose that $\mathcal{I}_{uv} \setminus \mathcal{I}_S = \{M_1\}$ and $\mathcal{I}_{wx} \setminus \mathcal{I}_S = \{M_2\}$. Since $M_2 \in \mathcal{I}_{wx}$, $\mathrm{wt}(wx) = 0$, which implies that $M_3 \in \mathcal{I}_{wx}$. Either $M_2 \in \mathcal{I}_{vw}$ or $M_2 \notin \mathcal{I}_{vw}$.

I. $M_2 \in \mathcal{I}_{vw}$. Let $R' = \{uv, vw\}$.

II. $M_2 \notin \mathcal{I}_{vw}$. Then $\mathrm{wt}(vw) = 0$, so $M_1 \in \mathcal{I}_{vw}$. Let $R' = \{vw, wx\}$.

In either case, $\mathcal{I}_{R'} = \mathcal{M}(H)$, so $G \notin \mathcal{F}_4(\text{fullhouse})$ by Corollary 12. This is a contradiction.

Suppose that $\mathcal{I}_{uv} \setminus \mathcal{I}_S = \{M_2\}$ and $\mathcal{I}_{wx} \setminus \mathcal{I}_S = \{M_1\}$. Either $M_1 \in \mathcal{I}_{vw}$ or $M_1 \notin \mathcal{I}_{vw}$.

I. $M_1 \in \mathcal{I}_{vw}$. Let $R' = \{uv, vw\}$.

II. $M_1 \notin \mathcal{I}_{vw}$. Then $\mathrm{wt}(vw) = 1$, so $M_2 \in \mathcal{I}_{vw}$. Also, as can easily be checked, $M_3 \in \mathcal{I}_{vw}$. Let $R' = \{vw, wx\}$.

In either case, $\mathcal{I}_{R'} = \mathcal{M}(H)$, so $G \notin \mathcal{F}_4(\text{fullhouse})$ by Corollary 12. This is a contradiction.

Therefore $|S| \neq 4$, so $|G - H| \leq |S| \leq 3$ and $|G| \leq 8$. This contradicts the assumption that $|G| \geq 9$, so this case does not occur.

For every possible value of $\mathcal{I}_S$, we have reached a contradiction. Thus $|G-H| \geq 4$ is impossible, so $|G-H| \leq 3$ and $|G| \leq 8$. $\square$

2.8 $H = P_4$

Lemma 30. *If $G \in \mathcal{F}_4(P_4)$, then $|G| \leq 8$.*

Proof. Suppose that $G \in \mathcal{F}_4(P_4)$ and $|G| \geq 8$ (i.e., $|G-H| \geq 4$). Then $G-H$ has no vertices with zero weight by Corollary 16. Assume that (R, S, T) is an optimal triple for G and P_4. Let $\mathcal{M}(P_4) = \{M_1, M_2, M_3, M_4, M_5\}$ and $\mathcal{C}(P_4) = \{C_1 = \{M_1, M_2\}, C_2 = \{M_3, M_4\}, C_3 = \{M_5\}\}$ be as in Proposition 8(g). By properties P1 and P3, $\mathcal{I}_S \in \{\emptyset, C_1, C_2, C_3, C_1 \cup C_2, C_1 \cup C_3, C_2 \cup C_3\}$, so we have the following cases.

Case 1: $\mathcal{I}_S = \emptyset$. By Observation 10, if $v \in S$, then

$$\mathrm{wt}(v) \in \bigcap_{i=1}^{5} \mathrm{col}(M_i) = \{\vec{0},\, v_1 = (1,0,0,1)^T\}.$$

The rank-preserving tables for $[v_1]$ with respect to M_1, M_2, M_3, M_4, and M_5 are, respectively,

$$P_1 = \begin{bmatrix} 1 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 1 \end{bmatrix}, \quad P_4 = \begin{bmatrix} 0 \end{bmatrix}, \quad \text{and } P_5 = \begin{bmatrix} 1 \end{bmatrix}.$$

Since $P_1 = P_3 = P_5$ and $P_2 = P_4$ are complementary, an edge uv in R has either $\mathcal{I}_{uv} = \{M_1, M_3, M_5\}$ or $\mathcal{I}_{uv} = \{M_2, M_4\}$. Also, by property P4, $\mathcal{I}_R \neq \{M_2, M_4\}$. Thus $\mathcal{I}_R \in \{\{M_1, M_2, M_3, M_4, M_5\}, \{M_1, M_3, M_5\}\}$. Therefore we have the following cases.

Subcase 1.1: $\mathcal{I}_R = \{M_1, M_2, M_3, M_4, M_5\}$. Since $\mathcal{I}_R = \mathcal{M}(P_4)$, Corollary 22 implies that $T = \emptyset$ and $|G-H| \leq 2|R|$. By Theorem 19(b), $|R| \leq 2$, so $|G-H| \leq 4$ and $|G| \leq 8$.

Subcase 1.2: $\mathcal{I}_R = \{M_1, M_3, M_5\}$**.** Since $\mathcal{I}_R \cup \mathcal{I}_T = \mathcal{M}(H)$, properties P1 and P3 imply that $\mathcal{I}_T = \{M_1, M_2, M_3, M_4\}$. By Corollary 20, $|R| \leq 1$ and $|T| \leq 2$. By Theorem 18, $|G - H| \leq 4$, so $|G| \leq 8$.

Case 2: $\mathcal{I}_S = \{M_1, M_2\}$**.** By Observation 10, if $v \in S$, then

$$\mathrm{wt}(v) \in \mathrm{col}(M_3) \cap \mathrm{col}(M_4) \cap \mathrm{col}(M_5) = \{\vec{0},\ v_1 = (0,1,0,1)^T,\ v_2 = (1,0,0,1)^T,\ v_3 = (1,1,0,0)^T\}.$$

The rank-preserving tables for $[v_1, v_2, v_3]$ with respect to M_3, M_4, and M_5 are, respectively,

$$P_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad P_4 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad \text{and } P_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Since $P_3 = P_5$, an edge in R is rank-preserving for M_3 if and only if it is also rank-preserving for M_5. Also, by property P4, $\mathcal{I}_R \neq \{M_1, M_2, M_4\}$. Thus $\mathcal{I}_R \in \{\{M_1, M_2, M_3, M_4, M_5\}, \{M_1, M_2, M_3, M_5\}\}$. Therefore we have the following cases.

Subcase 2.1: $\mathcal{I}_R = \{M_1, M_2, M_3, M_4, M_5\}$**.** Since $\mathcal{I}_R = \mathcal{M}(P_4)$, Corollary 22 implies that $T = \emptyset$ and $|G - H| \leq 2|R|$. By Theorem 19(b), $|R| \leq 2$, so $|G - H| \leq 4$ and $|G| \leq 8$.

Subcase 2.2: $\mathcal{I}_R = \{M_1, M_2, M_3, M_5\}$**.** Since $\mathcal{I}_R \cup \mathcal{I}_T = \mathcal{M}(H)$, properties P1 and P3 imply that $\mathcal{I}_T = \{M_1, M_2, M_3, M_4\}$ or $\mathcal{I}_T = \{M_3, M_4\}$. In each of these cases, $|R| \leq 1$ and $|T| \leq 1$ by Theorem 19(b) and Corollary 20, implying that $|G - H| \leq 3$ and $|G| \leq 7$. This contradicts the assumption that $|G| \geq 8$, so these cases do not occur.

Case 3: $\mathcal{I}_S = \{M_3, M_4\}$**.** By Observation 10, if $v \in S$, then

$$\mathrm{wt}(v) \in \mathrm{col}(M_1) \cap \mathrm{col}(M_2) \cap \mathrm{col}(M_5) = \{\vec{0},\ v_1 = (0,0,1,1)^T,\ v_2 = (1,0,0,1)^T,\ v_3 = (1,0,1,0)^T\}.$$

The rank-preserving tables for $[v_1, v_2, v_3]$ with respect to M_1, M_2, and M_5 are, respectively,

$$P_1 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad \text{and } P_5 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since $P_1 = P_5$, an edge in R is rank-preserving for M_1 if and only if it is also rank-preserving for M_5. Also, by property P4, $\mathcal{I}_R \neq \{M_2, M_3, M_4\}$. Thus $\mathcal{I}_R \in \{\{M_1, M_2, M_3, M_4, M_5\}, \{M_1, M_3, M_4, M_5\}\}$. Therefore we have the following cases.

Subcase 3.1: $\mathcal{I}_R = \{M_1, M_2, M_3, M_4, M_5\}$**.** Since $\mathcal{I}_R = \mathcal{M}(P_4)$, Corollary 22 implies that $T = \emptyset$ and $|G - H| \leq 2|R|$. By Theorem 19(b), $|R| \leq 2$, so $|G - H| \leq 4$ and $|G| \leq 8$.

Subcase 3.2: $\mathcal{I}_R = \{M_1, M_3, M_4, M_5\}$**.** Since $\mathcal{I}_R \cup \mathcal{I}_T = \mathcal{M}(H)$, properties P1 and P3 imply that $\mathcal{I}_T = \{M_1, M_2, M_3, M_4\}$ or $\mathcal{I}_T = \{M_1, M_2\}$. In each of these cases, $|R| \leq 1$ and $|T| \leq 1$ by Theorem 19(b) and Corollary 20, implying that $|G - H| \leq 3$ and $|G| \leq 7$. This contradicts the assumption that $|G| \geq 8$, so these cases do not occur.

Case 4: $\mathcal{I}_S = \{M_5\}$**.** By Observation 10, if $v \in S$, then

$$\mathrm{wt}(v) \in \mathrm{col}(M_1) \cap \mathrm{col}(M_2) \cap \mathrm{col}(M_3) \cap \mathrm{col}(M_4) = \{\vec{0},\ v_1 = (0,1,1,1)^T,\ v_2 = (1,0,0,1)^T,\ v_3 = (1,1,1,0)^T\}.$$

The rank-preserving tables for $[v_1, v_2, v_3]$ with respect to M_1, M_2, M_3, and M_4 are, respectively,

$$P_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad \text{and } P_4 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Since $P_2 = P_4$, an edge in R is rank-preserving for M_2 if and only if it is also rank-preserving for M_4. Thus

$$\mathcal{I}_R \in \{\{M_1, M_2, M_3, M_4, M_5\}, \{M_1, M_2, M_4, M_5\}, \{M_2, M_3, M_4, M_5\}, \{M_2, M_4, M_5\}, \\ \{M_1, M_3, M_5\}, \{M_1, M_5\}, \{M_3, M_5\}\}.$$

By property P4, $\mathcal{I}_R \notin \{\{M_2, M_4, M_5\}, \{M_1, M_3, M_5\}, \{M_1, M_5\}, \{M_3, M_5\}\}$. Therefore we have the following cases.

Subcase 4.1: $\mathcal{I}_R = \{M_1, M_2, M_3, M_4, M_5\}$. Since $\mathcal{I}_R = \mathcal{M}(P_4)$, Corollary 22 implies that $T = \emptyset$ and $|G - H| \leq |S| \leq 2|R|$. By Theorem 19(b), $|R| \leq 3$. If $|R| \leq 2$, then $|G - H| \leq 4$ and $|G| \leq 8$.

Suppose that $|R| = 3$. Then Theorem 19(b) implies that R consists of three edges e_1, e_2, and e_3 such that $\mathcal{I}_{e_1} \setminus \mathcal{I}_S = \{M_1\}$, $\mathcal{I}_{e_2} \setminus \mathcal{I}_S = \{M_2, M_4\}$, and $\mathcal{I}_{e_3} \setminus \mathcal{I}_S = \{M_3\}$.

Since the first row and column of P_1 and P_2 are the same, if an edge $e \in R$ is incident to a vertex of weight v_1, then either $\{M_1, M_2\} \subseteq \mathcal{I}_e$ or $\{M_1, M_2\} \subseteq \mathcal{M}(H) \setminus \mathcal{I}_e$. Therefore e_1 and e_2 are not incident to vertices with weight v_1. Since the third row and column of P_2 and P_3 are the same, if an edge $e \in R$ is incident to a vertex of weight v_3, then either $\{M_2, M_3\} \subseteq \mathcal{I}_e$ or $\{M_2, M_3\} \subseteq \mathcal{M}(H) \setminus \mathcal{I}_e$. Therefore e_2 and e_3 are not incident to vertices with weight v_3. Since $P_1[2] = P_3[2]$, if both vertices incident to an edge $e \in R$ have weight v_2, then $\{M_1, M_3\} \subseteq \mathcal{I}_e$ or $\{M_1, M_3\} \subseteq \mathcal{M}(H) \setminus \mathcal{I}_e$. Therefore e_1 and e_3 each are incident to at least one vertex that does not have weight v_2.

Therefore e_1 must be incident to vertices with weights v_2 and v_3 (implying that $\mathrm{wt}(e_1) = 0$ since $\mathcal{I}_{e_1} = \{M_1\}$) or incident to vertices with weights v_3 and v_3 (implying that $\mathrm{wt}(e_1) = 1$). Each vertex incident to e_2 must have weight v_2, which implies that $\mathrm{wt}(e_2) = 1$. The edge e_3 must be incident to vertices with weights v_1 and v_1 (implying that $\mathrm{wt}(e_3) = 1$) or incident to vertices with weights v_1 and v_2 (implying that $\mathrm{wt}(e_3) = 0$).

Therefore there are at least three vertices u, v, and w in S such that u is incident to e_3, v is incident to e_2, w is incident to e_1, $\mathrm{wt}(u) = v_1$, $\mathrm{wt}(v) = v_2$, and $\mathrm{wt}(w) = v_3$. Let $R' = \{uv, vw\}$. Note that since $v_1, v_3 \notin \mathrm{col}(M_5)$, $M_5 \in \mathcal{I}_{R'}$. Suppose that $|S| \geq 5$. Then the vertices in $R' \cup e$ for any edge $e \in R$ form a proper subset of S. We now have the following possibilities for $\mathcal{I}_{R'}$.

I. $M_1 \notin \mathcal{I}_{R'}$. Then $\mathrm{wt}(vw) = 1$, which implies that $M_2, M_3, M_4 \in \mathcal{I}_{R'}$. Since $\mathcal{I}_{R' \cup e_1} = \mathcal{M}(H)$, $G \notin \mathcal{F}_4(P_4)$ by Corollary 12, which is a contradiction.

II. $M_2, M_4 \notin \mathcal{I}_{R'}$. Then $\mathrm{wt}(uv) = 0$, which implies that $M_3 \in \mathcal{I}_{R'}$. Also $\mathrm{wt}(vw) = 0$, which implies that $M_1 \in \mathcal{I}_{R'}$. Since $\mathcal{I}_{R' \cup e_2} = \mathcal{M}(H)$, $G \notin \mathcal{F}_4(P_4)$ by Corollary 12, which is a contradiction.

III. $M_3 \notin \mathcal{I}_{R'}$. Then $\mathrm{wt}(uv) = 1$, which implies that $M_1, M_2, M_4 \in \mathcal{I}_{R'}$. Since $\mathcal{I}_{R' \cup e_3} = \mathcal{M}(H)$, $G \notin \mathcal{F}_4(P_4)$ by Corollary 12, which is a contradiction.

IV. $\mathcal{I}_{R'} = \mathcal{M}(P_4)$. Since the vertices in R' are a proper subset of the vertices in S, $G \notin \mathcal{F}_4(P_4)$ by Corollary 12, which is a contradiction.

Since each case leads to a contradiction, our assumption that $|S| \geq 5$ must be false. Therefore $|S| \leq 4$, so $|G - H| \leq 4$ and $|G| \leq 8$.

Subcase 4.2: $\mathcal{I}_R = \{M_1, M_2, M_4, M_5\}$**.** Since $\mathcal{I}_R \cup \mathcal{I}_T = \mathcal{M}(H)$, properties P1 and P3 imply that $\mathcal{I}_T = \{M_3, M_4, M_5\}$ or $\mathcal{I}_T = \{M_3, M_4\}$. In each of these cases, $|T| \leq 1$ by Corollary 20. Since $M_2 \in \mathcal{I}_{uv}$ if and only if $M_4 \in \mathcal{I}_{uv}$ for each edge $uv \in R$, $|R| \leq 2$ by Theorem 19(b). If $|R| = 1$, then $|G - H| \leq 3$, a contradiction. If $|R| = 2$, then Theorem 19(b) implies that R consists of an edge uv such that $\mathcal{I}_{uv} \setminus \mathcal{I}_S = \{M_2, M_4\}$

and another edge wx such that $\mathcal{I}_{wx} \setminus \mathcal{I}_S = \{M_1\}$. Since the first row and column of P_1, P_2, and P_4 are identical, we see that if any vertex, say u, has weight v_1, then the edge in R incident to the vertex must have either $\mathcal{I}_{uv} \setminus \mathcal{I}_S = \mathcal{I}_R \setminus \mathcal{I}_S$ or $\mathcal{I}_{uv} \setminus \mathcal{I}_S = \emptyset$. Neither of these cases occur, so u, v, w, and x each must have weight v_2 or v_3. As in Subcase 1.1 in Section 2.5, since $P_1[2,3]$ and $P_2[2,3] = P_4[2,3]$ are complementary, $|R| = 2$, and $|\bar{\mathcal{I}}_S| = 1$, we can apply Corollary 24 to conclude that $|S| = 3$, implying that $|G - H| \leq 4$ and $|G| \leq 8$.

Subcase 4.3: $\mathcal{I}_R = \{M_2, M_3, M_4, M_5\}$**.** Since $\mathcal{I}_R \cup \mathcal{I}_T = \mathcal{M}(H)$, properties P1 and P3 imply that $\mathcal{I}_T = \{M_1, M_2\}$ or $\mathcal{I}_T = \{M_1, M_2, M_5\}$. In each of these cases, $|T| \leq 1$ by Corollary 20. Since $M_2 \in \mathcal{I}_{uv}$ if and only if $M_4 \in \mathcal{I}_{uv}$ for each edge $uv \in R$, $|R| \leq 2$ by Theorem 19(b). If $|R| = 1$, then $|G - H| \leq 3$, a contradiction. If $|R| = 2$, then Theorem 19(b) implies that R consists of an edge uv such that $\mathcal{I}_{uv} \setminus \mathcal{I}_S = \{M_2, M_4\}$ and another edge wx such that $\mathcal{I}_{wx} \setminus \mathcal{I}_S = \{M_3\}$. Note that the third row and column of P_2, P_3, and P_4 are identical; as in the previous case, none of u, v, x, w can have weight v_3, so each must have weight v_1 or v_2. Since $P_3[1,2]$ and $P_2[1,2] = P_4[1,2]$ are complementary, $|R| = 2$, and $|\bar{\mathcal{I}}_S| = 1$, we can apply Corollary 24 to conclude that $|S| = 3$, implying that $|G - H| \leq 4$ and $|G| \leq 8$.

Case 5: $\mathcal{I}_S = \{M_1, M_2, M_3, M_4\}$**,** $\mathcal{I}_S = \{M_1, M_2, M_5\}$**, or** $\mathcal{I}_S = \{M_3, M_4, M_5\}$**.** In each of these cases, by property P4, $\mathcal{I}_R = \{M_1, M_2, M_3, M_4, M_5\}$, so $T = \emptyset$ and $|G - H| \leq 2|R|$ by Corollary 22. In each of these cases, $|R| \leq 2$ by Corollary 20, so $|G - H| \leq 4$ and $|G| \leq 8$. □

2.9 All graphs in $\mathcal{F}_4(\mathbb{F}_2)$

Combining Lemmas 25 through 30 with Corollary 6, we have:

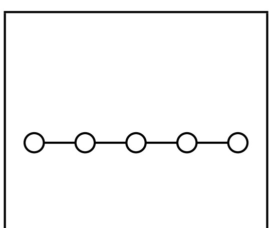
Theorem 31. *All graphs in $\mathcal{F}_4(\mathbb{F}_2)$ have 8 or fewer vertices.*

Theorem 3.1 in [DK06] implies that all graphs in $\mathcal{F}_4(\mathbb{F}_2)$ have 25 or fewer vertices. Because we have made a much more detailed analysis for the field $\mathbb{F}_2$, we have been able to greatly improve their bound in this single case. Since all graphs in $\mathcal{F}_4(\mathbb{F}_2)$ have 8 or fewer vertices, we can do an exhaustive search for all the graphs. Appendix A contains a collection of SAGE [SAG07] functions and Appendix B contains a similar collection of Magma [BCP97] functions to implement this approach. Both of these appendices use the graph generation program "geng" distributed with Brendan McKay's Nauty program [McK90, Version 2.2]. This exhaustive search results in the 62 graphs displayed at the end of this section. Thus, recalling Observation 2, we have:

Theorem 32. $\mathcal{F}_4(\mathbb{F}_2)$ *consists of the 62 graphs listed at the end of this section. For any graph* G, $\operatorname{mr}(\mathbb{F}_2, G) \leq 3$ *if and only if no graph in* $\mathcal{F}_4(\mathbb{F}_2)$ *is induced in* G.

In the listing of the graphs in $\mathcal{F}_4(\mathbb{F}_2)$ that follows, the graphs are sorted by number of vertices. We have also tried to group similarly structured graphs together. Each graph is identified with a number and a graph6 code. The graph6 code is a compact representation of the adjacency matrix (and thus the zero/nonzero pattern of the matrices associated with the graph). The specification of the graph6 code is distributed with Nauty and can also be found on the Nauty website.

We now proceed with the listing of all 62 graphs in $\mathcal{F}_4(\mathbb{F}_2)$.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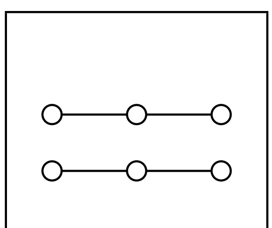
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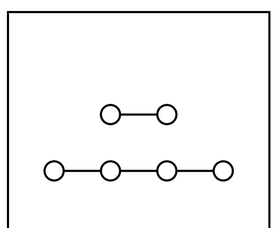
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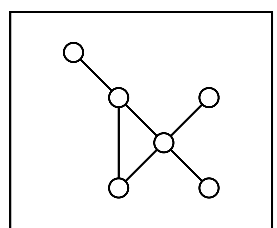
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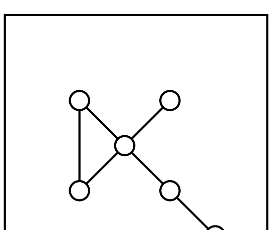
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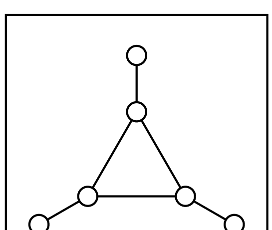
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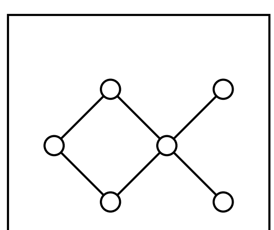
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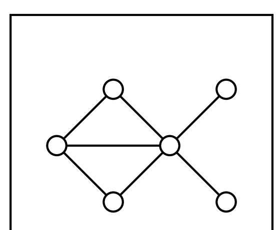
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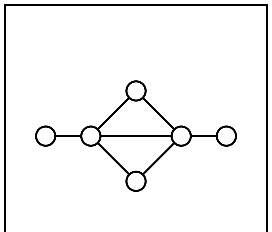

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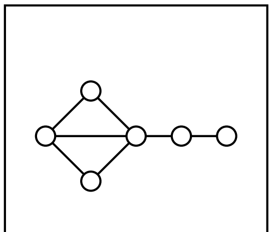

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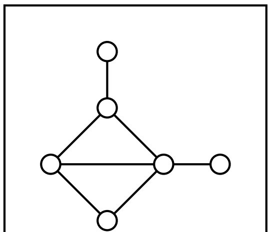

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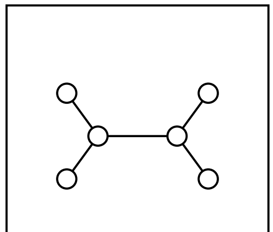

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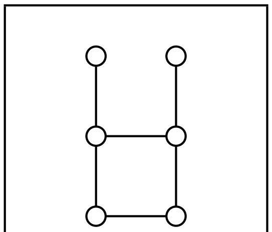

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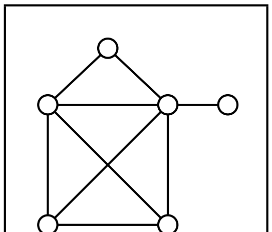

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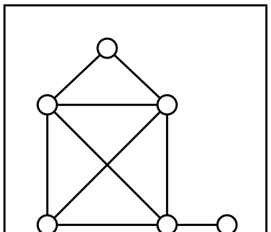

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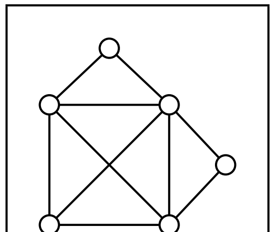

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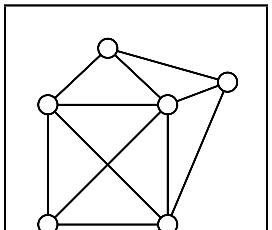

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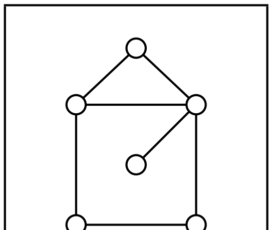

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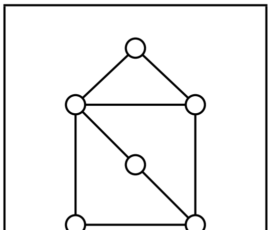

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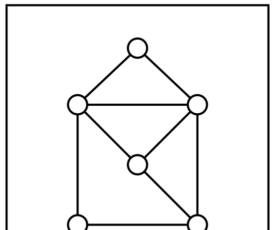

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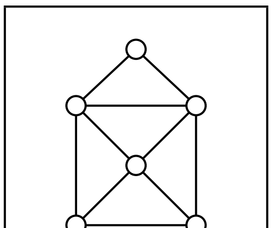

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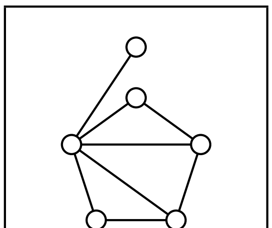

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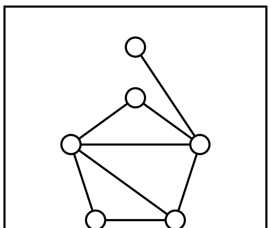

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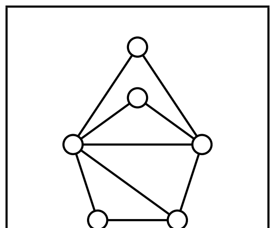

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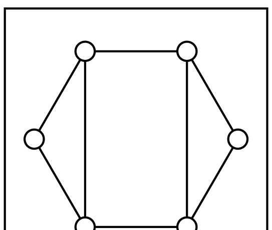

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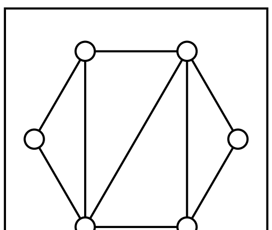

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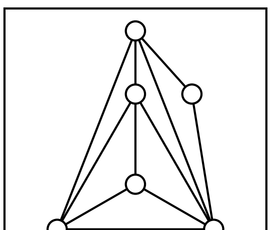

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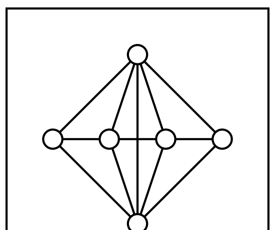

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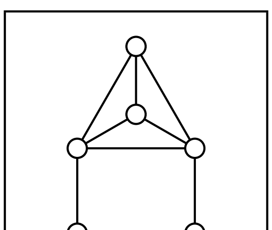

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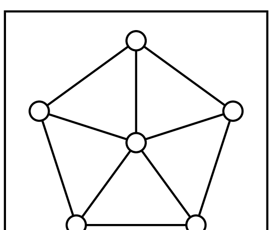

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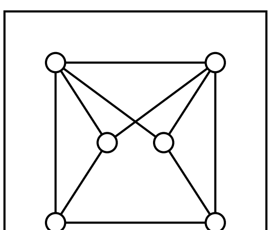

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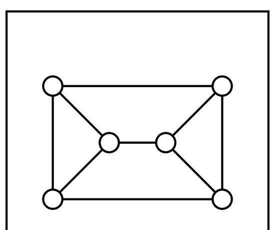

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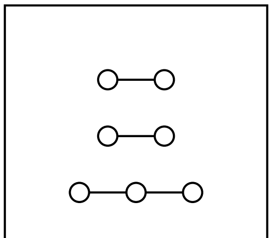
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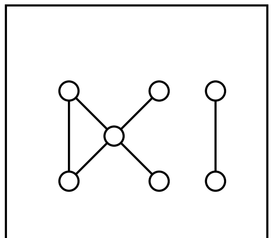
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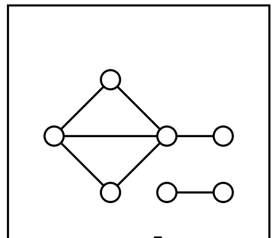
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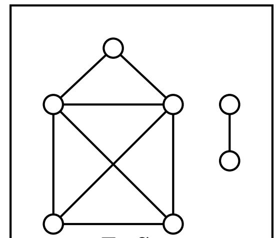
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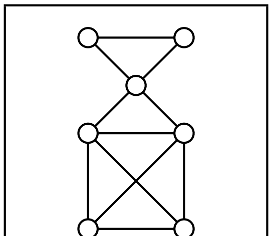
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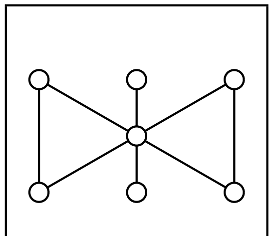
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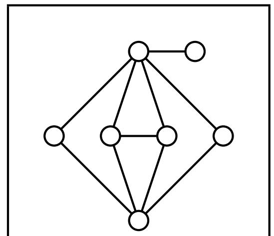
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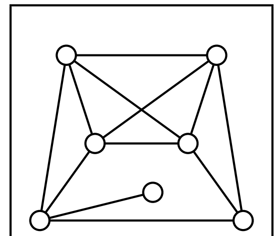
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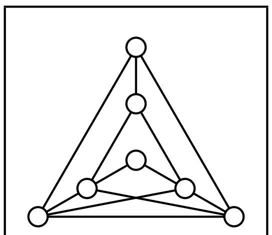
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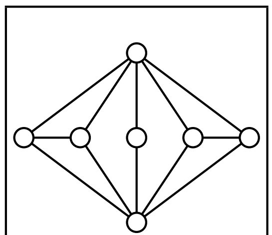
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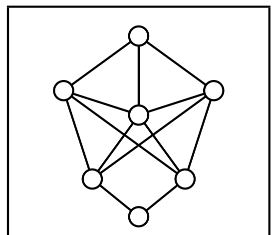
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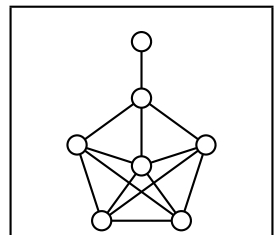
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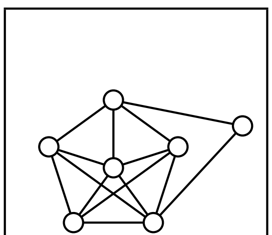
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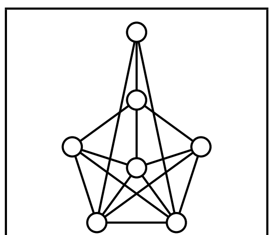
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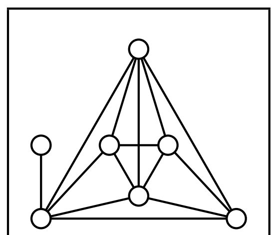
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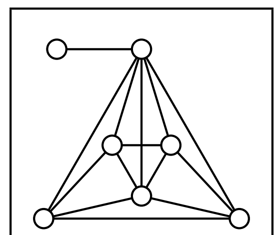
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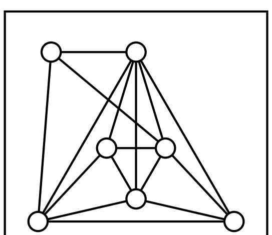
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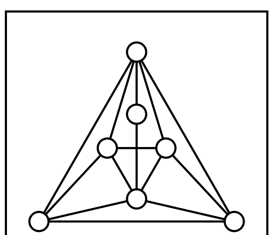
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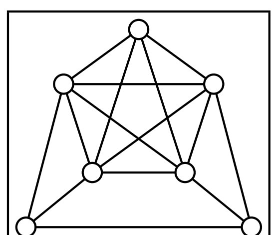
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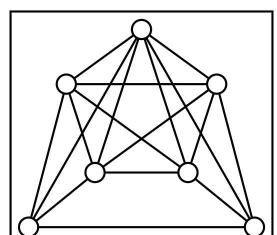
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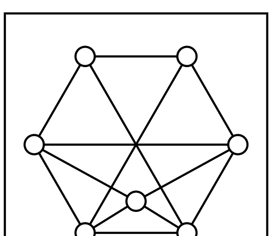
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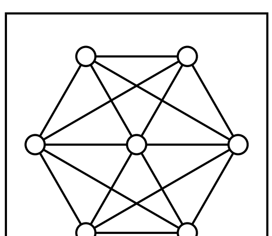
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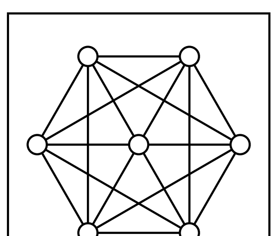
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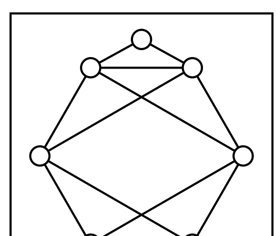
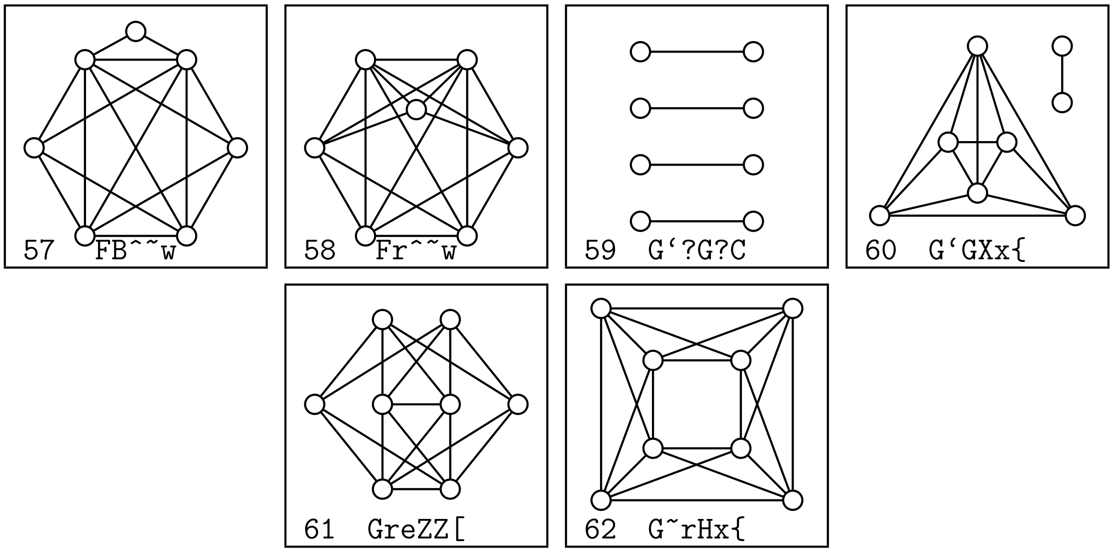

56 FKdzo

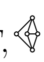


2.10 Graphs in $\mathcal{F}_4(F)$ for other fields

Many of the graphs in $\mathcal{F}_4(\mathbb{F}_2)$ are also in $\mathcal{F}_4(F)$ for any field F. This is the case with most of the disconnected graphs and the connected graphs with a cut vertex in the table.

We need the following elementary facts [BvdHL04].

Observation 33. For any field F

(a) $\mathrm{mr}(F, K_n) = 1$ for $n \geq 2$; $\mathrm{mr}(F, K_{2,3}) = \mathrm{mr}(F,$  $) = 2$; $\mathrm{mr}(F, \ltimes) = \mathrm{mr}(F, \mathrm{dart}) = 3$.

(b) $K_2 \in \mathcal{F}_1(F)$; $\ltimes, \mathrm{dart} \in \mathcal{F}_3(F)$.

(c) If $G = \cup_{i=1}^{k} G_i$, then $\mathrm{mr}(F, G) = \sum_{i=1}^{k} \mathrm{mr}(F, G_i)$.

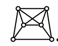
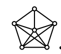
We will also need

Theorem 34 ([Fie69, BD05])**.** *Let F be any field and let G be a graph on n vertices. Then* $\mathrm{mr}(F, G) = n - 1$ *if and only if* $G = P_n$.

A stronger result was proved by Fiedler over $\mathbb{R}$ [Fie69] and his result was extended to any field, with some exceptions for $\mathbb{F}_3$, by Bento and Duarte [BD05].

Corollary 35. *For any field F, $\operatorname{mr}(F, P_n) = n - 1$ and $P_n \in \mathcal{F}_{n-1}(F)$.*

We will also utilize the following

Proposition 36. *Let $\mathcal{E} = \{\text{fullhouse}, G_1 =$ $, G_2 =$ $, P_3 \vee P_3\}$ (G_1 is graph 40 minus the pendant vertex and G_2 is graph 44 minus the pendant vertex). Then for each $G \in \mathcal{E}$, $\operatorname{mr}(\mathbb{F}_2, G) = 3$ and $\operatorname{mr}(F, G) = 2$ for any $F \neq \mathbb{F}_2$. Moreover,* fullhouse, $P_3 \vee P_3 \in \mathcal{F}_3(\mathbb{F}_2)$.

Proof. We already verified the first claim for the fullhouse in the introduction. Taking complements of the others we find that $G_1^c = 2P_3$, $G_2^c = P_3 \cup K_2 \cup K_1$, and $(P_3 \vee P_3)^c = 2K_2 \cup 2K_1$. By Theorems 6 and 7 in [BvdHL04] and Theorems 11 and 15 in [BvdHL05], $\operatorname{mr}(F, G_1) = \operatorname{mr}(F, G_2) = \operatorname{mr}(F, P_3 \vee P_3) = 2$ for $F \neq \mathbb{F}_2$, while $\operatorname{mr}(\mathbb{F}_2, G_1) = \operatorname{mr}(\mathbb{F}_2, G_2) = \operatorname{mr}(\mathbb{F}_2, P_3 \vee P_3) = 3$. The final claim follows from Theorem 16 in [BvdHL05]; it can also be easily checked directly. □

2.10.1 Disconnected graphs

Proposition 37. *If F is any field and $S_i \in \mathcal{F}_{\operatorname{mr}(S_i)}(F)$, $i = 1, \ldots, m$, then*

$$\bigcup_{i=1}^{m} S_i \in \mathcal{F}_{\operatorname{mr}(S_1)+\cdots+\operatorname{mr}(S_m)}(F).$$

Proof. This follows immediately from Observation 33(c) and the definition of $\mathcal{F}_{k+1}(F)$. □

Applying the last four results to the disconnected graphs 2, 3, 33, 34, 35, and 59 in Section 2.9, we have

Theorem 38. *For any field F,*

$$\{2P_3, P_4 \cup K_2, P_3 \cup 2K_2, \ltimes \cup K_2, \text{dart} \cup K_2, 4K_2\} \subseteq \mathcal{F}_4(F).$$

Graphs 36 and 60 are fullhouse $\cup K_2$ and $(P_3 \vee P_3) \cup K_2$. Since fullhouse, $P_3 \vee P_3 \in \mathcal{F}_3(F)$ if and only if $F = \mathbb{F}_2$, graphs 36 and 60 are not in $\mathcal{F}_4(F)$ for any $F \neq \mathbb{F}_2$.

2.10.2 Connected graphs with a cut vertex

We begin by recalling a definition and a known result

Definition 13. Let G and H be graphs, each having a vertex labeled v. Then $G \oplus_v H$ is the graph obtained from $G \cup H$ by identifying the two vertices labeled v. Note that v is necessarily a cut vertex of $G \oplus_v H$ and any graph with a cut vertex can be thought of as a sum $G \oplus_v H$.

Theorem 39 ([Hsi01, BFH04])**.** *Let F be any field and let G and H be graphs, each having a vertex labeled v. Then*

$$\operatorname{mr}(F, G \underset{v}{\oplus} H) = \min\{\operatorname{mr}(F,G) + \operatorname{mr}(F,H),\ \operatorname{mr}(F, G-v) + \operatorname{mr}(F, H-v) + 2\}.$$

This result reduces the calculation of the minimum rank of any graph with a cut vertex to a calculation for smaller graphs. The proofs of Theorem 39 contained in [Hsi01] and [BFH04] are over $\mathbb{R}$, but with slight modifications they can be seen to hold for any field. For completeness, we include a proof in Appendix C.

Corollary 40. $\operatorname{mr}(F, G \oplus_v H) \leq \operatorname{mr}(F,G) + \operatorname{mr}(F,H)$.

We noted in the introduction the following fact about the fullhouse graph.

Proposition 41. *Let G be a graph on 5 or fewer vertices and suppose that $G \neq$* fullhouse*. Then* $\operatorname{mr}(F,G)$ *is independent of the field F.*

We can now establish one criterion for membership in $\mathcal{F}_4(F)$ for any field.

Theorem 42. *Let F be any field and let G be a graph satisfying all of the following*

(a) $|G| = 6$,

(b) $\mathrm{mr}(F,G) = 4$, *and*

(c) P_5 *is not induced in* G.

Then $G \in \mathcal{F}_4(F)$.

Proof. For each vertex v of G, $G - v$ is a graph on 5 vertices distinct from P_5. By Theorem 34, $\mathrm{mr}(F,G) < 5 - 1 = 4$. By Definition 4, $G \in \mathcal{F}_4(F)$. $\square$

Proposition 43. *Graphs 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 18, 22, and 23 are all in* $\mathcal{F}_4(F)$ *for any field* F.

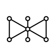
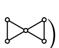
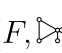
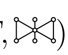
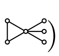
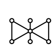
Proof. Each of these graphs has 6 vertices and P_5 is induced in none of them. Moreover, each graph is of the form $G \oplus_v K_2$, where $G \neq$ fullhouse is a graph on 5 vertices. Let $G \oplus_v K_2$ be any of these graphs. By Theorem 39, Proposition 41, and Theorem 32,

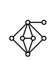
$$\begin{aligned}
\mathrm{mr}(F, G \underset{v}{\oplus} K_2) &= \min\{\mathrm{mr}(F,G) + \mathrm{mr}(F,K_2), \mathrm{mr}(F, G-v) + \mathrm{mr}(F, K_2 - v) + 2\} \\
&= \min\{\mathrm{mr}(\mathbb{F}_2,G) + \mathrm{mr}(\mathbb{F}_2,K_2),\ \mathrm{mr}(\mathbb{F}_2, G-v) + \mathrm{mr}(\mathbb{F}_2, K_2 - v) + 2\} \\
&= \mathrm{mr}(\mathbb{F}_2, G \underset{v}{\oplus} K_2) = 4.
\end{aligned}$$

By Theorem 42, $G \oplus_v K_2 \in \mathcal{F}_4(F)$. $\square$

We note that graphs 14 and 15, which contain the fullhouse, have minimum rank 3 over any field $F \neq \mathbb{F}_2$, so are not in $\mathcal{F}_4(F)$ for $F \neq \mathbb{F}_2$.

We now consider in turn graphs 38 and 39.

Graph 38 (): Applying Theorem 39 three times we see that for any field F, $\mathrm{mr}(F,$$) = 2$, $\mathrm{mr}(F,$$) = 3$, and $\mathrm{mr}(F,$$) = 4$. Theorem 39 also implies that $\mathrm{mr}(F,$$) = 3$. By definition,  $\in \mathcal{F}_4(F)$.

Graph 39 (): Let F be any field. Let H be the graph obtained by deleting the pendant vertex in graph 39, labeled as in Figure 2.3. Since $\ltimes$ is induced in H,

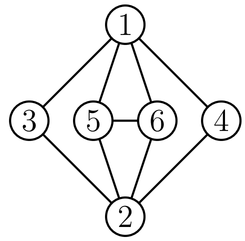


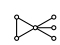
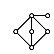
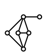
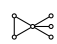
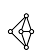
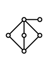
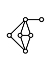
Figure 2.3: $H =$ graph 39 minus the pendant vertex.

$\operatorname{mr}(F, H) \geq \operatorname{mr}(F, \ltimes) = 3$ by Observation 33. Moreover,

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix} \in \mathcal{S}(F, H)$$

and rank $A = 3$. Therefore $\operatorname{mr}(F, H) = 3$. By Theorem 39,

$$\begin{aligned} \operatorname{mr}(F, \text{graph } 39) &= \min\{\operatorname{mr}(H) + \operatorname{mr}(K_2), \operatorname{mr}(\ltimes) + \operatorname{mr}(K_1) + 2\} \\ &= \min\{3 + 1, 3 + 0 + 2\} = 4. \end{aligned}$$

Any graph obtained by deleting a vertex from graph 39 is one of H, $\ltimes \cup K_1$, , , or . By Observation 33, $\operatorname{mr}(F, \ltimes \cup K_1) = 3$. We just saw that  has minimum rank 3. Since $K_{2,3}$ and  have minimum rank 2 over any field by Observation 33, the graphs  and  each have minimum rank at most 3 by Corollary 40. By definition, graph 39 $\in \mathcal{F}_4(F)$ for every field F.

Summarizing,

Proposition 44. *Graphs 38 and 39 are in $\mathcal{F}_4(F)$ for any field F.*

The four remaining connected graphs with cut vertices, graphs 40, 44, 47, and 48, in the table do not belong to $\mathcal{F}_4(F)$ for $F \neq \mathbb{F}_2$. Let G be any of these graphs. Deleting the pendant vertex in G yields one of the last three graphs in Proposition 36, so by that result and Corollary 40, $\operatorname{mr}(F, G) \leq 2 + 1 = 3$ for $F \neq \mathbb{F}_2$.

2.10.3 Summary

We have seen that 6 of the 8 disconnected graphs in Section 2.9 are in $\mathcal{F}_4(F)$ for all fields F, while 15 of 21 of the connected graphs with a cut vertex are in $\mathcal{F}_4(F)$ for all F.

We stated in the introduction that even if one is only interested in the minimum rank problem over $\mathbb{R}$, results obtained over $\mathbb{F}_2$ yield important insights. We have just observed that of the 29 graphs with vertex connectivity at most one in the list of 62 graphs in $\mathcal{F}_4(\mathbb{F}_2)$, 21 of these are also in $\mathcal{F}_4(F)$ for *any* field. While the discrepancy is significant, it is also the case that the amount of overlap is surprising. The analysis of the 2-connected graphs in Section 2.9 is much more complicated and will be pursued in subsequent work.

We have not found all graphs with vertex connectivity less than 2 in $\mathcal{F}_4(F)$, $F \neq \mathbb{F}_2$, by this method. For example, let F be any field with $\operatorname{char} F \neq 2$. Then $\operatorname{mr}(F, K_{3,3,3}) = 3$ and $\operatorname{mr}(F, K_{3,3,2}) = 2$ [BvdHL04]. Let $G = K_{3,3,3} \oplus_v K_2$. By Theorem 39,

$$\begin{aligned}\operatorname{mr}(G) &= \min\{\operatorname{mr}(F, K_{3,3,3}) + \operatorname{mr}(F, K_2), \operatorname{mr}(F, K_{3,3,2}) + \operatorname{mr}(F, K_1) + 2\} \\ &= \min\{3 + 1, 2 + 0 + 2\} = 4.\end{aligned}$$

But since for either of the nonisomorphic graphs $K_{3,3,2} \oplus_v K_2$, we have

$$\operatorname{mr}(F, K_{3,3,2} \bigoplus_v K_2) \leq \operatorname{mr}(F, K_{3,3,2}) + \operatorname{mr}(F, K_2) = 2 + 1 = 3$$

by Corollary 40, it follows that $K_{3,3,3} \oplus_v K_2 \in \mathcal{F}_4(F)$. This graph did not occur in the table $\mathcal{F}_4(\mathbb{F}_2)$ because $\operatorname{mr}(\mathbb{F}_2, K_{3,3,3}) = 2$ [BvdHL04]. It is also easy to see that $K_{3,3,3} \cup K_2 \in \mathcal{F}_4(F)$ if $\operatorname{char} F \neq 2$. We do not know how many other graphs are in

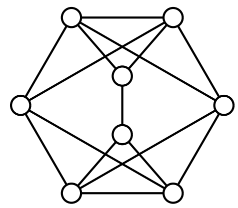


Figure 2.4: An 8 vertex graph in $\mathcal{F}_4(\mathbb{F}_2, P_4) \setminus \mathcal{F}_4(\mathbb{F}_2)$.

$\mathcal{F}_4(F)$ for an arbitrary infinite field F. In at least two cases ($\mathcal{F}_4(\mathbb{R})$ and $\mathcal{F}_4(\mathbb{C})$), the number is infinite [Hal]. It is even difficult to understand the structure of just those graphs in $\mathcal{F}_4(F)$ of the form $G \oplus_v K_2$. Sometimes $G \in \mathcal{F}_3(F)$, but frequently it is not. The analysis of this issue, the question of the number of graphs in $\mathcal{F}_4(F)$, and other related issues are worthwhile topics for further investigation.

In examining the list of graphs in Section 2.9, we see that some of the bounds obtained in Sections 2.3–2.8 do not appear to be sharp. For example, there is no graph in Section 2.9 with 8 vertices that has an induced P_4, even though the bound in Lemma 30 is 8 vertices. This is because there are graphs in $\mathcal{F}_4(\mathbb{F}_2, P_4)$ that are not in $\mathcal{F}_4(\mathbb{F}_2)$. For instance, Figure 2.4 shows a graph on 8 vertices which is in $\mathcal{F}_4(\mathbb{F}_2, P_4)$ (when the induced P_4 contains both center vertices), as can be checked by hand or by using the SAGE or Magma functions in Appendices A or B, respectively. However, the graph is not in $\mathcal{F}_4(\mathbb{F}_2)$, since deleting one of the center vertices yields graph 56 in Section 2.9. This shows that Lemma 30 does indeed provide a sharp bound for the number of vertices in a graph in $\mathcal{F}_4(\mathbb{F}_2, P_4)$.

We have succeeded in obtaining a sharp bound on the number of vertices in a minimal forbidden subgraph for the class of graphs $\mathcal{G}_3(\mathbb{F}_2)$. We have also generated a complete list of these minimal forbidden subgraphs, thereby giving a structural characterization for the graphs having minimum rank 4 or more over $\mathbb{F}_2$. Since this gives a method for generating or recognizing all such graphs, it leads to a theoretical procedure for determining whether a given graph has minimum rank at most 3 over $\mathbb{F}_2$. In the next chapter of this dissertation, a structural characterization will be given for all graphs having minimum rank at most k over any finite field F, leading to a straightforward method for generating or recognizing all such graphs. This structural characterization will relate the graphs in $\mathcal{G}_3(\mathbb{F}_2)$ to the Fano projective plane.

Chapter 3

Structural Characterization

The purpose of this chapter is to characterize the structure of all graphs in $\mathcal{G}_k(\mathbb{F}_q)$ for any q and k (recall that $\mathbb{F}_q$ denotes the finite field with q elements). This characterization leads to some very strong connections with projective geometry over finite fields.

In this chapter, graphs are undirected, may have loops, but will not have multiple edges. To simplify our drawings, a vertex with a loop will be filled (black) and a vertex without a loop will be empty (white). A *simple graph* is a graph without loops. Let G be a graph with loops and G' be the simple version of G without loops. Then the matrix $A \in S(F, G')$ corresponds to G if a_{ii} is nonzero exactly when the vertex i has a loop in G.

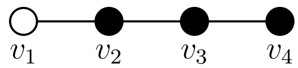
We recall some terminology from graph theory.

Definition 14. Two vertices are *adjacent* if an edge connects them. A *clique* in a graph is a set of pairwise adjacent vertices. An *independent set* in a graph is a set of pairwise nonadjacent vertices. In either of these cases, loops do not matter.

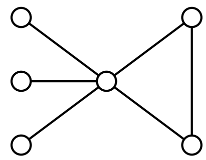
The next definition extends a standard definition introduced in [KSS97] and is particularly used in random graph theory in connection with the regularity lemma.

Definition 15. A *blowup* of a graph G is a new simple graph H constructed by replacing each nonlooped vertex v_i in G with a (possibly empty) independent set V_i, each looped vertex v_i with a (possibly empty) clique V_i, and each edge $v_iv_j \in G$ $(i \neq j)$ with the edges $\{xy \mid x \in V_i, y \in V_j\}$.

Example 5. Let G be the graph



Let $|V_1| = 3$, $|V_2| = 1$, $|V_3| = 2$, and $|V_4| = 0$. Then we obtain the simple blowup graph H:



It is useful to see how matrices corresponding to a graph and a blowup are related. In this example, a matrix M over $\mathbb{F}_3$ corresponding to G and a matrix $N \in S(\mathbb{F}_3, H)$ are, respectively,

$$M = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad N = \left[\begin{array}{ccc|c|cc} 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 2 & 1 & 1 & 0 & 1 & 1 \\ \hline 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right].$$

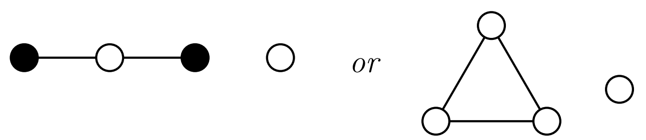
Note that, for example, the entry m_{11} was replaced with a 3×3 zero block in N, the entry m_{12} was replaced with a 3×1 nonzero block in N, the entries in the last row and column of M were replaced with empty blocks (i.e., erased), and the diagonal entries of N were changed to whatever was desired. These substitutions of block matrices correspond to the vertex substitutions used to construct H.

We will introduce our method by proving a special case of a characterization theorem from [BvdHL05]. We will then generalize our method to characterize all simple graphs in $\mathcal{G}_k(\mathbb{F}_q)$ for any q and k. After giving examples for some specific q and k, we will describe the strong connection to projective geometries and list some consequences of this connection. We will finish with some data obtained from using this characterization to computationally determine minimal forbidden subgraphs for $\mathcal{G}_k(\mathbb{F}_q)$.

3.1 A new approach to a recent result

We will introduce our method by giving a proof of a special case of Theorems 5 and 6 of [BvdHL05]. First, we give a statement of the theorem using the concept of blowup graphs.

Theorem 45 ([BvdHL05])**.** *Let G be a simple graph on n vertices. Then* $\operatorname{mr}(\mathbb{F}_2, G) \leq 2$ *if and only if G is a blowup of either*



In our proof of this result, we will need the following lemma and corollary, which hold in any field. We will then give our proof of Theorem 45.

Lemma 46 ([CG01, Theorem 8.9.1])**.** *Let A be an $n \times n$ symmetric matrix of rank k. Then there is an invertible principal $k \times k$ submatrix B of A and a $k \times n$ matrix U such that*

$$A = U^t B U.$$

Corollary 47. *Let A be a symmetric matrix. Then* $\operatorname{rank} A \leq k$ *if and only if there is some invertible $k \times k$ matrix B and $k \times n$ matrix U such that $A = U^t B U$.*

Proof. Let A have rank $r \leq k$. Then by Lemma 46, there is an invertible $r \times r$ matrix B_1 and an $r \times n$ matrix U_1 such that $A = U_1^t B_1 U_1$. Let $B_2 = \begin{bmatrix} B_1 & O \\ O & I_{k-r} \end{bmatrix}$ and $U_2 = \begin{bmatrix} U_1 \\ O \end{bmatrix}$ (where O represents a zero matrix of the appropriate size). Then $A = U_2^t B_2 U_2$.

The reverse implication follows from the rank inequality $\operatorname{rank}(U^t BU) \leq \operatorname{rank} B$.

$\square$

Recall that two square matrices A and B are congruent if there exists some invertible matrix C such that $A = C^t BC$. It is straightforward to show that congruence is an equivalence relation. We will restrict our attention to invertible symmetric matrices. Let $\mathcal{B}$ consist of one representative from each congruence equivalence class. By Corollary 47, if A is a symmetric $n \times n$ matrix with $\operatorname{rank} A \leq k$, then $A \in \{U^t BU \mid B \in \mathcal{B}, U \text{ a } k \times n \text{ matrix}\}$.

We now proceed with our proof of Theorem 45.

Proof of Theorem 45. First, we compute a suitable $\mathcal{B}$, one of the possible sets of representatives from the congruence classes. If B has a nonzero diagonal entry, then $B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$, or $B = I_2$. In any of these three cases, $B^t BB = I_2$, so B is congruent to the identity matrix I_2. If an invertible symmetric matrix B of order 2 over $\mathbb{F}_2$ has all zeros on the diagonal, then the off-diagonal entries must be nonzero, so $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. In this case,

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ac + ac & ad + bc \\ ad + bc & bd + bd \end{bmatrix} = \begin{bmatrix} 0 & ad + bc \\ ad + bc & 0 \end{bmatrix},$$

so any conjugate of B will have a zero diagonal. Therefore, a suitable $\mathcal{B}$ is

$$\mathcal{B} = \left\{ I_2, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}.$$

Because U is over $\mathbb{F}_2$, the columns of U are members of the finite set

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}.$$

Let A be a symmetric $k \times k$ matrix. For any $n \times n$ permutation matrix P, the graphs of A and P^tAP are isomorphic. Therefore we may assume that identical columns of U are contiguous and write $U = \begin{bmatrix} E_1 & E_2 & J & O \end{bmatrix}$ where E_1 is $2 \times p$ matrix with each column equal to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, E_2 is $2 \times q$ matrix with each column equal to $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, J is a $2 \times r$ matrix with each entry equal to 1, and O is a $2 \times t$ zero matrix. Then either

$$A = \begin{bmatrix} E_1^{\mathrm{T}} \\ E_2^{\mathrm{T}} \\ J^{\mathrm{T}} \\ O^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} E_1 & E_2 & J & O \end{bmatrix} = \begin{bmatrix} J_p & O & J_{p,r} & O \\ O & J_q & J_{q,r} & O \\ J_{r,p} & J_{r,q} & O_r & O \\ O & O & O & O_t \end{bmatrix}$$

or else

$$A = \begin{bmatrix} E_1^{\mathrm{T}} \\ E_2^{\mathrm{T}} \\ J^{\mathrm{T}} \\ O^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} E_1 & E_2 & J & O \end{bmatrix} = \begin{bmatrix} O_p & J_{p,q} & J_{p,r} & O \\ J_{q,p} & O_q & J_{q,r} & O \\ J_{r,p} & J_{r,q} & O_r & O \\ O & O & O & O_t \end{bmatrix}.$$

Any graph corresponding to the first matrix is a blowup of the first graph in our statement of Theorem 45, while any graph corresponding to the second matrix is a blowup of the second graph. Thus we have established Theorem 45. □

Observation 48. Note that every block in the above matrices is either a O matrix or a J matrix. Consequently, we could have obtained the zero/nonzero form of the matrices with rank ≤ 2 by only considering $U = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$ and computing

$$A = U^tU = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$A = U^tB_2U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The nonzero diagonal entries correspond to loops in our graphs. This simplified procedure again yields the graphs in Theorem 45.

In the proof of Theorem 45, we noted that any U could be written in a standard form. In Observation 48, we saw how the standard form of U could be simplified to take advantage of the theorem being about blowup graphs. We will now discuss the reasoning behind these constructions and show that an analogous standard form of U exists for any finite field and any k.

In Observation 48, note that the columns of our simplified U consisted of every vector in $\mathbb{F}_2^2$. The following lemma shows that if we take such a matrix U and if B and $\hat{B}$ are congruent, then U^tBU and $U^t\hat{B}U$ have isomorphic graphs.

Lemma 49. *Let U be the matrix with columns $\{v \mid v \in \mathbb{F}_q^k\}$. Let B and C be invertible $k \times k$ matrices with B symmetric. Then the graphs corresponding to U^tBU and $U^t(C^tBC)U$ are isomorphic.*

Proof. Since every vector in $\mathbb{F}_q^k$ appears as a column of U and the mapping $x \mapsto Cx$ is one-to-one, CU is just a column permutation of U. This permutation induces a relabeling of the graph U^tBU to give the graph of $(CU)^tB(CU) = U^t(C^tBC)U$. $\square$

Though this result does not hold for an arbitrary U, there is another smaller U which gives the same result. We first need a definition. Then we will show this extension in Lemma 50

Definition 16. Let F be a field. Two nonzero vectors $v_1, v_2 \in F^k$ are *projectively equivalent* if there exists some nonzero $c \in F$ such that $v_1 = cv_2$.

It is easy to check that projective equivalence is in fact an equivalence relation on the vectors in V.

Let u_i be a column of U. Let $\hat{U}$ be the matrix obtained from U by replacing the column u_i with cu_i for some nonzero $c \in F$. Then the i, j entry of $\hat{U}^tB\hat{U}$, $(cu_i)^tBu_j$, is zero if and only if the i, j entry of U^tBU, $u_i^tBu_j$, is zero. Thus the graphs associated with U^tBU and $\hat{U}^tB\hat{U}$ are isomorphic.

Lemma 50. *Let F be any field, let $x \in F^k$, let $\bar{x}$ denote the projective equivalence class of x, and let $P = \cup_{x \in F^k - \vec{0}} \{\bar{x}\}$, the set of projective equivalence classes in F^k. Let C be an invertible matrix. Then the map $f \colon P \to P$ defined by $f \colon \bar{x} \mapsto \overline{Cx}$ is a bijection.*

Proof. The function f is well-defined since if $Cx = y$, then for any nonzero $k \in F$, $\overline{C(kx)} = \overline{kCx} = \overline{ky} = \bar{y}$. If $\overline{Cx_1} = \overline{Cx_2}$, then for some nonzero $k \in F$, $kCx_1 = Cx_2$, which implies $C(kx_1 - x_2) = 0$, giving $kx_1 = x_2$ since C is invertible. Therefore $\overline{x_1} = \overline{x_2}$ and f is injective. Surjectivity of f also follows from the hypothesis that C is invertible. □

Lemma 51. *Let U be the matrix in which the columns are formed by choosing one representative from each projective equivalence class of vectors in $\mathbb{F}_q^k - \vec{0}$. Let B and C be invertible $k \times k$ matrices with B symmetric. Then the graphs corresponding to $U^t BU$ and $U^t(C^t BC)U$ are isomorphic.*

Proof. Let $T = CU$. Denote the ith column of U by u_i and the ith column of T by t_i. By Lemma 50, the sequence of projective equivalence classes $\overline{t_1}, \overline{t_2}, \ldots, \overline{t_n}$ is just a permutation of the sequence $\overline{u_1}, \overline{u_2}, \ldots, \overline{u_n}$. Form the matrix S in which the ith column is u_j if $\overline{t_i} = \overline{u_j}$, so that S is a column permutation of U. Then the graph of $U^t(C^t BC)U = (CU)^t B(CU) = T^t BT$ is isomorphic to the graph of $S^t BS$ by the reasoning preceding Lemma 50, which is in turn just a relabeling of the graph of $U^t BU$. □

We now find a standard form for any matrix U, as in our proof of Theorem 45. Let U be a $k \times n$ matrix over $\mathbb{F}_q$ and let B be a $k \times k$ invertible matrix over $\mathbb{F}_q$. Let $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_m$ be the projective equivalence classes of $\mathbb{F}_q^k - \vec{0}$, with each x_i as a chosen representative from its class. For each nonzero column u_i, replace u_i with the chosen representative of $\bar{u}_i$. Then permute the columns of U so that the matrix is of the form $\hat{U} = [X_1\ X_2\ \cdots\ X_m\ O]$, where each X_i is a block matrix of columns equal to x_i

and O is a zero block matrix. Note that some of these blocks may be empty. From our results above, $\hat{U}^t B \hat{U}$ has the same graph as $U^t BU$.

As observed in Observation 48, we can obtain the zero/nonzero structure of the block matrix $\hat{U}^t B \hat{U}$ by simply deleting all duplicate columns of $\hat{U}$. Deleting these duplicate columns of $\hat{U}$ leaves us with a matrix that can be obtained from $\tilde{U} = [x_1\ x_2\ \cdots\ x_m\ 0]$ by deleting columns. Then the (simple) graph of $U^t BU$ is a blowup of the (looped) graph of $\tilde{U}^t B \tilde{U}$ since the (simple) graph of $\hat{U}^t B \hat{U}$ is such a blowup graph.

Furthermore, let $\mathcal{B}$ be a set consisting of one representative from each congruence class of invertible symmetric matrices and let $\hat{B}$ be the representative that is congruent to B. Then from our results above, the graphs of $\tilde{U}^t B \tilde{U}$ and $\tilde{U}^t \hat{B} \tilde{U}$ are isomorphic.

There is another simplification we can make. Notice that both graphs displayed in Theorem 45 have an isolated nonlooped vertex. This vertex came from the zero column vector in U and corresponds to the fact that adding any number of isolated vertices to a graph does not change its minimum rank. In any theorem like Theorem 45, the graphs from which we blowup will always have this isolated nonlooped vertex and so will be of the form $G \cup K_1$ (where K_1 does not have a loop). Note that in constructing such a graph G, it is enough to assume that $\tilde{U}$ in the above paragraphs does not have a zero column vector.

Definition 17. Let $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_m$ be the projective equivalence classes of $\mathbb{F}_q^k - \vec{0}$, with each x_i as a chosen representative from its class. Let $\mathcal{B}$ be a set consisting of one representative from each congruence class of invertible symmetric $k \times k$ matrices. Let $U = [x_1\ x_2\ \cdots\ x_m]$, the matrix with column vectors $x_1, x_2, \ldots, x_m$. We define the set of graphs $\mathfrak{g}_k(\mathbb{F}_q)$ as the set of graphs of the matrices in $\{U^t BU \mid B \in \mathcal{B}\}$.

The paragraphs immediately preceding the definition show that the following is true.

Theorem 52. *Let G be a simple graph in $\mathcal{G}_k(\mathbb{F}_q)$. Then G is a blowup of some graph in $\{H \cup K_1 \mid H \in \mathfrak{g}_k(\mathbb{F}_q), K_1$ does not have a loop$\}$.*

The key to characterizing $\mathcal{G}_k(\mathbb{F}_q)$ then depends on finding $\mathcal{B}$ for any k and any finite field. This will be done in the next section.

3.2 Congruence classes of symmetric matrices

Symmetric matrices represent symmetric bilinear forms, and as such, play an important role in projective geometry. Two matrices that are congruent represent the same bilinear form with respect to different bases. Thus, congruence classes for symmetric matrices over finite fields have been studied and characterized for a long time in projective geometry. In this section, we have distilled the pertinent proofs of these characterizations from [Alb38], [Hir98], and [Coh03] to give a suitable $\mathcal{B}$ for invertible symmetric matrices of order k over $\mathbb{F}_q$ for any q and k. In the next section, we will expound more on the connection between the minimum rank problem and projective geometry.

We need the following elementary lemma.

Lemma 53. *If a symmetric matrix $B = \begin{bmatrix} C & D \\ D^t & E \end{bmatrix}$ and C is a square invertible matrix, then B is congruent to $\begin{bmatrix} C & O \\ O & E' \end{bmatrix}$, where E' is a square symmetric matrix of the same order as E and O is a zero matrix.*

Proof. Let $R = C^{-1}D$ so that $CR = D$. Then

$$\begin{aligned}
\begin{bmatrix} I & O \\ -R^t & I \end{bmatrix} \begin{bmatrix} C & D \\ D^t & E \end{bmatrix} \begin{bmatrix} I & -R \\ O & I \end{bmatrix} &= \begin{bmatrix} C & D \\ -R^tC + D^t & -R^tD + E \end{bmatrix} \begin{bmatrix} I & -R \\ O & I \end{bmatrix} \\
&= \begin{bmatrix} C & -CR + D \\ -R^tC + D^t & R^tCR - D^tR - R^tD + E \end{bmatrix} \\
&= \begin{bmatrix} C & O \\ O & E - D^tR \end{bmatrix},
\end{aligned}$$

since $-CR + D = O = (-CR + D)^t = -R^tC + D^t$. $\square$

Lemma 54. *Every symmetric matrix over $\mathbb{F}_q$ is congruent to a matrix of the form* $\operatorname{diag}(a_1, a_2, \ldots, a_s, b_1H_1, b_2H_2, \ldots, b_tH_t)$*, where* $a_i, b_i \in \mathbb{F}_q$*,* $H_i = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$*, and s and t are nonnegative integers.*

Proof. If B is the zero matrix, then the result is true.

If B is not the zero matrix, then the diagonal of B has a nonzero entry or there is some $a_{ij} \neq 0$, $i \neq j$, so that B has a principal submatrix of the form $\begin{bmatrix} 0 & a_{ij} \\ a_{ij} & 0 \end{bmatrix} = a_{ij}H$, where $H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

In the first case, by using a suitable permutation, we may assume that $b_{11} \neq 0$. By Lemma 53, B is congruent to $\operatorname{diag}(b_{11}, B')$.

In the second case, again by using a suitable permutation, we may assume that the upper left 2×2 principal submatrix is $a_{ij}H$. By Lemma 53, B is congruent to $\operatorname{diag}(a_{ij}H, B')$.

Continue this process inductively with B'. Then, again using a suitable permutation, B is congruent to $\operatorname{diag}(a_1, a_2, \ldots, a_s, b_1H, b_2H, \ldots, b_tH)$. □

3.2.1 Odd characteristic

We now consider the case when $\mathbb{F}_q$ has odd characteristic. We first need a well-known result.

Lemma 55. *If $\mathbb{F}_q$ has odd characteristic and $\nu \in \mathbb{F}_q$, then there exists $c, d \in \mathbb{F}_q$ such that $c^2 + d^2 = \nu$.*

Proof. Let $A = \{c^2 \mid c \in \mathbb{F}_q\}$ and $B = \{\nu - d^2 \mid d \in \mathbb{F}_q\}$. Since the map $\sigma \colon \mathbb{F}_q^\times \to \mathbb{F}_q^\times$ given by $\sigma \colon x \mapsto x^2$ has kernel $\{1, -1\}$, there are $(q-1)/2$ squares in $\mathbb{F}_q \backslash \{0\}$. Including zero, there are then $(q+1)/2$ squares in $\mathbb{F}_q$. Thus $|A| = |B| = (q+1)/2$, so $A \cap B \neq \emptyset$, and $c^2 = \nu - d^2$ for some $c, d \in \mathbb{F}_q$. □

Since there are $(q-1)/2$ nonzero squares in $\mathbb{F}_q$, given a nonsquare $\nu \in \mathbb{F}_q$, the set $\{\nu b^2 \mid b \in \mathbb{F}_q, b \neq 0\}$ is a set of $(q-1)/2$ nonsquares in $\mathbb{F}_q$. Consequently, every nonsquare is equal to νb^2 for some $b \in \mathbb{F}_q$.

The matrix aH for any $a \in \mathbb{F}_q$ is congruent to a diagonal matrix:

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} a & a \\ a & -a \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2a & 0 \\ 0 & -2a \end{bmatrix}.$$

Therefore every symmetric matrix is congruent to a diagonal matrix.

Lemma 56. *Every invertible symmetric matrix B of order k over $\mathbb{F}_q$ is congruent to either I_k or $\operatorname{diag}(I_{k-1}, \nu)$, where ν is any nonsquare in $\mathbb{F}_q$.*

Proof. Let C be an invertible diagonal matrix congruent to B, with $C = N^tBN$, and let ν be any nonsquare in $\mathbb{F}_q$.

By a permutation matrix P, let $D = P^tCP = \operatorname{diag}(b_1^2, b_2^2, \ldots, b_s^2, \nu c_1^2, \nu c_2^2, \ldots, \nu c_t^2)$. The first s elements of the diagonal of D are squares in $\mathbb{F}_q$ and the last t elements are nonsquares in $\mathbb{F}_q$.

Let $Q = \operatorname{diag}(b_1^{-1}, b_2^{-1}, \ldots, b_s^{-1}, c_1^{-1}, c_2^{-1}, \ldots, c_t^{-1})$. Let $E = Q^tDQ = \operatorname{diag}(I_s, \nu I_t)$.

Let $c, d \in \mathbb{F}_q$ such that $c^2 + d^2 = \nu$. Let

$$R = \nu^{-1} \begin{bmatrix} c & d \\ -d & c \end{bmatrix}.$$

Since $\det R = \nu^{-2}(c^2 + d^2) = \nu^{-1} \neq 0$, R is invertible. Note that

$$R^t(\nu I_2)R = \nu R^tR = \nu\nu^{-2}(c^2 + d^2)I_2 = I_2.$$

If t is even, let $S = \operatorname{diag}(I_s, R_1, R_2, \ldots, R_{t/2})$, where $R_i = R$. Then $S^tES = I_k$. If t is odd, let $S = \operatorname{diag}(I_s, R_1, R_2, \ldots, R_{(t-1)/2}, 1)$. Then $S^tES = \operatorname{diag}(I_{k-1}, \nu)$. $\square$

The next lemma shows that these two cases are in fact different and gives a simple criteria to determine which congruence class any symmetric matrix is in.

Lemma 57. *If* $\det B$ *is a square (nonsquare) and* $\hat{B}$ *is congruent to* B*, then* $\det \hat{B}$ *is a square (nonsquare).*

Proof. Let $\hat{B} = C^t BC$. Then $\det \hat{B} = (\det C)^2(\det B)$. Thus if $\det B$ is a square, $\det \hat{B}$ is a square. If $\det B$ is a nonsquare, then $\det \hat{B}$ is a nonsquare. $\square$

Since $\det I_k = 1$ is a square and $\det(\mathrm{diag}(I_{k-1}, \nu)) = \nu$ is a nonsquare, we can determine if a matrix is congruent to I_k or $\mathrm{diag}(I_{k-1}, \nu)$ by whether the determinant is a square or not.

It appears then that $|\mathcal{B}| = 2$. However, we can do better in one case since we only are concerned with whether an entry of $U^t BU$ is zero or nonzero.

Definition 18. Let B and $\hat{B}$ be matrices. If $\hat{B} = dC^t BC$ for some invertible matrix C and some nonzero constant d, then B and $\hat{B}$ are *projectively congruent*.

Since multiplying by a nonzero constant preserves the zero/nonzero pattern in a matrix over a field, if B and $\hat{B}$ are projectively congruent, then $U^t BU$ and $U^t \hat{B} U$ give isomorphic graphs.

Lemma 58. *If* k *is odd, then a* $k \times k$ *invertible symmetric matrix is projectively congruent to* I_k*.*

Proof. Let $k = 2\ell - 1$. We can see that $\det(\nu \,\mathrm{diag}(I_{k-1}, \nu)) = \nu^{2\ell-1}\nu = \nu^{2\ell}$ is a square. Thus $\mathrm{diag}(I_{k-1}, \nu)$ is projectively congruent to I_k. $\square$

The results in this subsection give us the following lemma.

Lemma 59. *Let* q *be odd. To determine* $\mathfrak{g}_k(\mathbb{F}_q)$*, we may take* $\mathcal{B}$ *as follows: if* k *is odd, then* $\mathcal{B} = \{I_k\}$*; if* k *is even, then* $\mathcal{B} = \{I_k, \mathrm{diag}(I_{k-1}, \nu)\}$*, where* ν *is any nonsquare in* $\mathbb{F}_q$

3.2.2 Even characteristic

We now consider the case when $\mathbb{F}_q$ has even characteristic. First, we need a well-known result.

Lemma 60. *Every element in a field of characteristic 2 is a square.*

Corollary 61. *Every symmetric matrix is congruent to* $\mathrm{diag}(I_s, H_1, H_2, \ldots, H_t)$.

Proof. By Lemma 54, a symmetric matrix A is congruent to a matrix

$$B = \mathrm{diag}(a_1, a_2, \ldots, a_s, b_1H_1, b_2H_2, \ldots, b_tH_t).$$

Let

$$C = \mathrm{diag}(\frac{1}{\sqrt{a_1}}, \frac{1}{\sqrt{a_2}}, \ldots, \frac{1}{\sqrt{a_s}}, \frac{1}{\sqrt{b_1}}I_2, \frac{1}{\sqrt{b_2}}I_2, \ldots, \frac{1}{\sqrt{b_t}}I_2).$$

Then $C^tBC = \mathrm{diag}(I_s, H_1, H_2, \ldots, H_t)$. $\square$

In Corollary 61, either $s = 0$ or $s > 0$. If $s > 0$, then $\mathrm{diag}(I_s, H_1, H_2, \ldots, H_t)$ is congruent to I_k. Indeed, let

$$A = \mathrm{diag}(1, H) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Then, since $\mathrm{char}\,\mathbb{F}_q = 2$,

$$P^t(AP) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = I_3.$$

If $s = 0$, then B has even order and is congruent to $\mathrm{diag}(H_1, \ldots, H_{k/2})$.

The next lemma shows that these two cases are different.

Lemma 62. *If B has a zero diagonal, then every matrix congruent to B has a zero diagonal.*

Proof. Let $v \in \mathbb{F}_q^k$. Then

$$v^t B v = \sum_{i,j} b_{ij} v_i v_j = \sum_i b_{ii} v_i^2 + \sum_{i<j} b_{ij}(v_i v_j + v_i v_j) = \sum_i b_{ii} v_i^2 = 0.$$

Thus $v^t B v = 0$ for any vector $v \in \mathbb{F}_q^k$. $\square$

The results in this subsection give us the following lemma.

Lemma 63. *Let q be even. To determine $\mathfrak{g}_k(\mathbb{F}_q)$, we may take $\mathcal{B}$ as follows: if k is odd, then $\mathcal{B} = \{I_k\}$; if k is even, then $\mathcal{B} = \{I_k, \operatorname{diag}(H_1, H_2, \dots, H_{k/2})\}$, where $H_i = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.*

3.2.3 Summary

Combining Lemmas 59 and 63, the results of this section can be summarized as the following theorem.

Theorem 64. *The set $\mathfrak{g}_k(\mathbb{F}_q)$ is the set of graphs of the matrices in $\{U^t B U \mid B \in \mathcal{B}\}$, where the columns of U are a maximal set of nonzero vectors in $\mathbb{F}_q^k$ such that no vector is a multiple of another and $\mathcal{B}$ is given by:*

(a) if k is odd, $\mathcal{B} = \{I_k\}$.

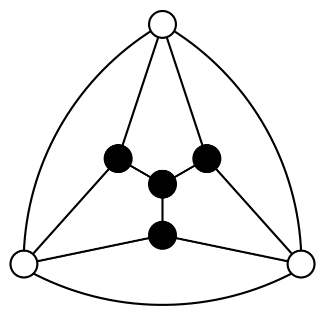
(b) if k is even and $\operatorname{char} \mathbb{F}_q = 2$, $\mathcal{B} = \{I_k, \operatorname{diag}(H_1, H_2, \dots, H_{k/2})\}$, where $H_i = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$.

(c) if k is even and $\operatorname{char} \mathbb{F}_q \neq 2$, $\mathcal{B} = \{I_k, \operatorname{diag}(I_{k-1}, \nu)\}$, where ν is any non-square in $\mathbb{F}_q$.

3.2.4 Examples of characterizations

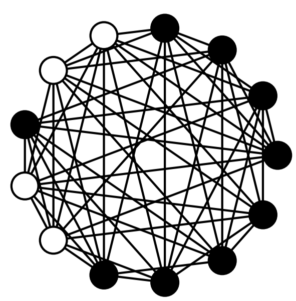
As special cases of Theorem 64, we present the following corollaries which calculate $\mathfrak{g}_k(\mathbb{F}_q)$ for several $\mathbb{F}_q$ and k. In the corollaries, we label a graph in $\mathfrak{g}_k(\mathbb{F}_q)$ using the pattern $FqRk$, signifying that it is a graph for the $\operatorname{mr}(\mathbb{F}_q, G) \leq k$ corollary. To compute these graphs, we used the SAGE functions listed in Appendix D.

Corollary 65. *Let G be any simple graph. Let $F2R3$ be the graph*



Then $\operatorname{mr}(\mathbb{F}_2, G) \leq 3$ *if and only if G is a blowup graph of $F2R3 \cup K_1$.*

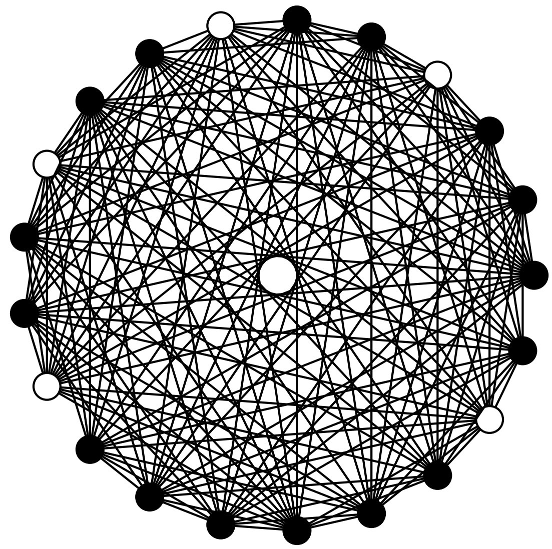
Corollary 66. *Let G be any simple graph. Let $F3R3$ be the graph*



Then $\operatorname{mr}(\mathbb{F}_3, G) \leq 3$ *if and only if G is a blowup graph of $F3R3 \cup K_1$.*

The graphs quickly become more complicated, as the next corollary shows.

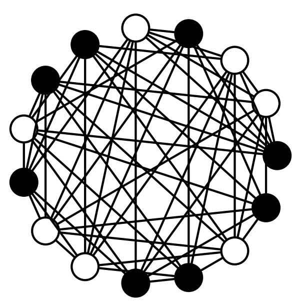
Corollary 67. *Let G be any simple graph. Let $F4R3$ be the graph*



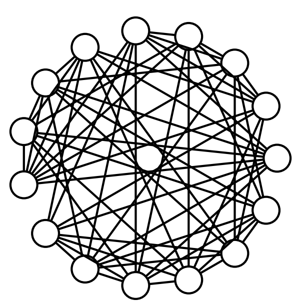
Then $\mathrm{mr}(\mathbb{F}_4, G) \leq 3$ *if and only if* G *is a blowup graph of F4R3* $\cup K_1$*.*

The next corollary gives the simplest previously-unknown result for which $\mathfrak{g}_k(\mathbb{F}_q)$ contains two graphs.

Corollary 68. *Let G be any simple graph. Let F2R4A be the graph*



and let F2R4B be the graph



Then $\mathrm{mr}(\mathbb{F}_2, G) \le 4$ *if and only if* G *is a blowup graph of either* $F2R4A \cup K_1$ *or* $F2R4B \cup K_1$*.*

3.3 Connection to projective geometries

As mentioned previously, the classifications of symmetric matrices in Section 3.2 are standard classification results in projective geometry. In this section, we first introduce appropriate terminology and highlight this connection to projective geometry. We then give some examples of how results in projective geometry can help us understand $\mathfrak{g}_k(\mathbb{F}_q)$ better. For further material, a definitive treatise on projective geometry is contained in the series [Hir98] and [HT91].

3.3.1 Definitions and the connection

We start with basic definitions from projective geometry.

Definition 19. Let $V = \mathbb{F}_q^{n+1}$, the vector space of dimension $n+1$ over $\mathbb{F}_q$. For $x, y \in V - \vec{0}$, we define an equivalence relation by

$$x \sim y \iff x = cy, \qquad \text{where } c \neq 0 \text{ and } c \in \mathbb{F}_q.$$

Denote the equivalence class containing $x \in V - \vec{0}$ as $\bar{x} = \{cx \mid c \neq 0 \text{ and } c \in \mathbb{F}_q\}$. Geometrically, we can think of the class $\bar{x}$ as the set of non-origin points on a line passing through x and the origin in V. These equivalence classes form the *projective geometry* $PG(n, q)$ of dimension n and order q. The equivalence classes are called the *points* of $PG(n, q)$. Each subspace of dimension $m + 1$ in V corresponds to a subspace of dimension m in $PG(n, q)$. If a projective geometry has dimension 2, then it is called a *projective plane*.

Note that there is a shift by one in dimension between a vector space V and its subspaces and the projective geometry associated with V and its subspaces. To help the reader, we will use the term *projective dimension* (or "pdim") when dealing with the dimension of a projective geometry. This is not standard, however.

Definition 20. Let $PG(n,q)$ be the projective geometry of projective dimension n over $\mathbb{F}_q$ and let $\mathcal{S}$ be the set of subspaces of $PG(n,q)$. A *correlation* $\sigma : \mathcal{S} \to \mathcal{S}$ is a bijective map such that for any subspaces $R, T \in \mathcal{S}$, $R \subseteq T \implies \sigma(T) \subseteq \sigma(R)$ and $\operatorname{pdim} \sigma(R) = n - 1 - \operatorname{pdim} R$. A *polarity* is a correlation σ of order 2 (i.e., $\sigma^2 = 1$, the identity map).

Note that any polarity σ maps points in S to hyperplanes (subspaces of projective dimension $n-1$ in S) and hyperplanes to points. If $Y = \sigma(\bar{x})$, then $\sigma(Y) = \bar{x}$ since $\sigma^2 = 1$, so σ induces a bijection between points and hyperplanes. This bijection leads to the next definition.

Definition 21. Let σ be a polarity on $PG(n,q)$. Let $\bar{x}, \bar{y}$ be points in $PG(n,q)$. We say that $\sigma(\bar{x})$ is the *polar* (hyperplane) of $\bar{x}$ and $\bar{x}$ is the *pole* of $\sigma(\bar{x})$. If $\bar{y} \in \sigma(\bar{x})$, then $\bar{x} \in \sigma(\bar{y})$ and we say that $\bar{x}$ and $\bar{y}$ are *conjugate* points. If $\bar{x} \in \sigma(\bar{x})$, then we say that $\bar{x}$ is *self-conjugate* or *absolute*. A subspace of $PG(n,q)$ consisting of absolute points is called *isotropic*.

The next definition gives the connection with symmetric matrices.

Definition 22. Let B be an $(n+1) \times (n+1)$ invertible symmetric matrix over $\mathbb{F}_q$. Define $\sigma : \mathcal{S} \to \mathcal{S}$ by $\sigma : R \mapsto R^{\perp}$, where the orthogonality relation is defined by the symmetric bilinear form B (i.e., $R^{\perp} = \{\bar{y} \mid x^t B y = 0 \text{ for all } \bar{x} \in R\}$). We call σ the polarity associated with B.

The fact that the σ in the previous definition is a polarity is easy to check.

Let M_1 and M_2 be symmetric matrices. Let σ_1 and σ_2 be the associated polarities, respectively. Two polarities are equivalent if the matrices are projectively congruent, i.e., σ_1 is equivalent to σ_2 if $M_1 = dC^t M_2 C$ for some nonzero d and invertible matrix C.

We now summarize from [Hir98, Section 2.1.5] the classification of polarities that are associated with symmetric matrices. Let B be an invertible symmetric matrix over $\mathbb{F}_q$. Let σ be the polarity associated with B.

- If q is odd, then σ is called an *ordinary polarity.*

  When B has even order, there are two associated polarities: the *hyperbolic* and *elliptic* polarities. The correspondence between these types of polarities and the matrices in $\mathcal{B}$ from Theorem 64(c) is slightly nontrivial and is summarized in [Hir98, Corollary 5.19].

  When B has odd order, there is only one associated polarity, the *parabolic* polarity, which corresponds to $\mathcal{B}$ in Theorem 64(a).

- If q is even and $b_{ii} = 0$ for all i, then there is one polarity associated with B, the *null* or *symplectic* polarity. Note that this only occurs when B has even order since otherwise B is not invertible. This case corresponds to the non-identity matrix in the $\mathcal{B}$ in Theorem 64(b).

- If q is even and there is some $b_{ii} \neq 0$, then there is one associated polarity, the *pseudo-polarity.* This case corresponds to the identity matrix in $\mathcal{B}$ in Theorem 64(a) or (b).

We pause to note that there are polarities that are not associated with symmetric matrices. However, since we are only concerned about symmetric matrices, we will restrict ourselves to this case. Information about polarities not associated with symmetric matrices may also be found in [Hir98].

We now examine the connection to graphs.

Definition 23. Let B be an invertible symmetric matrix over $\mathbb{F}_q$ and σ be the associated polarity. The *polarity graph* of σ has as its vertices the points of $PG(n, q)$ and as its edges $\{\bar{x}\bar{y} \mid x^t B y = 0\}$. Thus $\bar{x}$ is adjacent to $\bar{y}$ exactly when $\bar{x}$ and $\bar{y}$ are conjugate (i.e., x and y are orthogonal with respect to B).

In standard literature, loops are not allowed in polarity graphs. However, for our purposes, loops convey needed information, so a vertex $\bar{x}$ in a polarity graph has a loop if and only if $\bar{x}$ is absolute.

In Theorem 64, the vertices of a graph in $\mathfrak{g}_k(\mathbb{F}_q)$ represent the points of the projective geometry $PG(k-1,q)$ and an edge is drawn if the points are *not* conjugate (i.e., $x^tBy \neq 0$). Thus, the graphs in Theorem 64 are exactly the complements of polarity graphs. Recall that, when dealing with looped graphs, a vertex is looped in the complement of a graph if and only if it is nonlooped in the original graph.

Using this connection, we can restate Theorem 64:

Theorem 69. *The set $\mathfrak{g}_k(\mathbb{F}_q)$ is the set of complements of the (looped) polarity graphs of polarities on $PG(k-1,q)$ that are associated with symmetric matrices.*

We challenge the reader to quickly recite the last theorem out loud ten times!

3.3.2 Consequences of the connection

With the main theorem stated as in Theorem 69, we can use a variety of known results about polarity graphs to derive results about graphs in $\mathfrak{g}_k(\mathbb{F}_q)$. For example, an elementary result in projective geometry gives us the size of the graphs in $\mathfrak{g}_k(\mathbb{F}_q)$.

Theorem 70. *Every graph in $\mathfrak{g}_k(\mathbb{F}_q)$ has $\frac{q^k-1}{q-1}$ vertices.*

Proof. There are q^k-1 vectors in $\mathbb{F}_q^k - \vec{0}$. Since there are $q-1$ nonzero constants in $\mathbb{F}_q$, there are $q-1$ elements in each equivalence class $\bar{x}$, so there are $\frac{q^k-1}{q-1}$ points in $PG(k-1,q)$. Thus the graphs derived from U^tBU will have $\frac{q^k-1}{q-1}$ vertices. $\square$

Analyzing the polarities of $PG(k-1,q)$ gives us further information about the looped versus nonlooped vertices in graphs in $\mathfrak{g}_k(\mathbb{F}_q)$. Recall that a vertex $\bar{x}$ is nonlooped in $G \in \mathfrak{g}_k(\mathbb{F}_q)$ if and only if $\bar{x}$ is absolute with respect to the corresponding polarity. Therefore determining the numbers of looped and nonlooped vertices in G is equivalent to finding the numbers of absolute points of the polarities of $PG(k-1,q)$.

In the polarity associated with a symmetric matrix B, a point $\bar{x}$ is conjugate with a point $\bar{y}$ exactly when $x^t By = 0$. Thus $\bar{x}$ is absolute exactly when $x^t Bx = 0$.

Theorem 71. *Let $\mathbb{F}_q$ be a finite field having characteristic 2. One graph in $\mathfrak{g}_k(\mathbb{F}_q)$ will have $\frac{q^{k-1}-1}{q-1}$ nonlooped vertices. If k is even, then the additional graph in $\mathfrak{g}_k(\mathbb{F}_q)$ will have all nonlooped vertices.*

Proof. In a field of characteristic 2, since

$$x^t Bx = \sum_{i,j} b_{ij} x_i x_j = \sum_i b_{ii} x_i^2 + \sum_{i<j} b_{ij}(x_i x_j + x_i x_j) = \sum_i b_{ii} x_i^2 = \left(\sum_i \sqrt{b_{ii}} x_i \right)^2,$$

a point $\bar{x}$ is absolute exactly when $\sum_i \sqrt{b_{ii}} x_i = 0$.

For a symplectic polarity, $b_{ii} = 0$ for all i. Therefore every vertex is nonlooped (i.e., there are $\frac{q^k-1}{q-1}$ nonlooped vertices). A symplectic polarity occurs when k is even.

For a pseudo-polarity, the set of absolute points forms the hyperplane $\sum_i \sqrt{b_{ii}} x_i = 0$. Since a hyperplane of $PG(k-1,q)$ is a projective geometry of projective dimension $k-2$, there are $\frac{q^{k-1}-1}{q-1}$ nonlooped vertices in this graph. □

For the odd characteristic case, we will directly apply a standard result in projective geometry about the number of self-conjugate points in ordinary polarities.

Theorem 72 ([HT91, Theorem 22.5.1(b)])**.** *Let q be odd. Then the number of absolute points in a polarity in $PG(k-1,q)$ is given by:*

$$\begin{cases} \frac{(q^m-1)(q^{m-1}+1)}{q-1} \text{ or } \frac{(q^m+1)(q^{m-1}-1)}{q-1} & \text{if } k = 2m \text{ is even} \\ \frac{q^{2m}-1}{q-1} & \text{if } k = 2m+1 \text{ is odd} \end{cases}$$

Corollary 73. *Let q be odd. If $k = 2m$ is even, then the two graphs in $\mathfrak{g}_k(\mathbb{F}_q)$ will have $\frac{(q^m-1)(q^{m-1}+1)}{q-1}$ and $\frac{(q^m+1)(q^{m-1}-1)}{q-1}$ nonlooped vertices, respectively. If $k = 2m+1$ is odd, then the graph in $\mathfrak{g}_k(\mathbb{F}_q)$ will have $\frac{q^{2m}-1}{q-1}$ nonlooped vertices.*

Field / Vertices:	5	6	7	8	9	10	11	12	Total
$\mathbb{F}_2$	1	31	26	4	0	0	0	0	62
$\mathbb{F}_3$	1	17	6	119	162	53	23	4	385
$\mathbb{F}_5$	1	17	6	1	0	101	?	?	> 126

Table 3.1: Minimal forbidden subgraphs for minimum rank ≤ 3

As another example of how the graphs in $\mathfrak{g}_k(\mathbb{F}_q)$ are related to other known graphs, consider the Erdős-Rényi graphs. For a given finite field $\mathbb{F}_q$, the Erdős-Rényi graph is the polarity graph of the polarity associated with identity matrix of order 3 (i.e., the polarity is over a projective plane) [ERS66]. The Erdős-Rényi graphs play an important role in extremal graph theory. From our results above, the graph in $\mathfrak{g}_3(\mathbb{F}_q)$ is the complement of the (looped) Erdős-Rényi graph associated with the field $\mathbb{F}_q$. Thus the graphs in Corollaries 65, 66, and 67 are complements of (looped) Erdős-Rényi graphs.

Other results can be applied to graphs in $\mathfrak{g}_k(\mathbb{F}_q)$. For example, see [CG01, Section 10.12] for results symplectic polarity graphs over $\mathbb{F}_2$ (for example, the complement of F2R4B in Corollary 68). Another interesting reference is [Par76], in which various automorphism groups and other properties of polarity graphs are worked out.

3.4 Forbidden subgraphs

Using the characterization of $\mathcal{G}_k(\mathbb{F}_q)$ developed in this chapter, we can find minimal forbidden subgraphs (as in Chapter 2) by a computer search. Our preliminary search takes advantage of the fact that a minimal forbidden subgraph for $\mathcal{G}_k(\mathbb{F}_q)$ is a graph G such that G is not a blowup of a graph in $\mathfrak{g}_k(\mathbb{F}_q)$, but every subgraph of G is a blowup of a graph in $\mathfrak{g}_k(\mathbb{F}_q)$. The results from our preliminary search for the case $k = 3$ are summarized in Table 3.1. We also note the following interesting observations from the results of our preliminary search. (As in Section 2.9, we identify a particular graph by its graph6 code, a compact representation of the adjacency matrix.)

- The only graph on 5 vertices that has minimum rank greater than three is P_5.
- On 6 vertices, the forbidden subgraphs for $\mathbb{F}_3$ and $\mathbb{F}_5$ are the same and these are a subset of the forbidden subgraphs for $\mathbb{F}_2$.
- On 7 vertices, the forbidden subgraphs for $\mathbb{F}_3$ and $\mathbb{F}_5$ are the same. These are a subset of the forbidden subgraphs for $\mathbb{F}_2$ except for `F?D~w`. However, this graph has minimum rank 4 or more in $\mathbb{F}_2$ since it contains `E@^w`, one of the (extra) forbidden subgraphs on 6 vertices in $\mathbb{F}_2$.
- On 8 vertices, $4K_2$ is the only graph in common (pairwise) between each of $\mathbb{F}_2$, $\mathbb{F}_3$, and $\mathbb{F}_5$. The field $\mathbb{F}_5$ has only this forbidden graph. Also, $\mathbb{F}_3$ has a much larger number of forbidden graphs compared to $\mathbb{F}_2$ or $\mathbb{F}_5$.
- For 9 vertices, $\mathbb{F}_5$ has no forbidden subgraphs. However, $\mathbb{F}_5$ has 101 forbidden subgraphs for 10 vertices.
- On 10 vertices, $\mathbb{F}_5$ and $\mathbb{F}_3$ share the forbidden subgraphs `IJXX~NZnW`, `IsXayw~No`, and `IqOxr|}~_`. However, every forbidden graph for $\mathbb{F}_5$ except these three and the $\mathbb{F}_5$-forbidden graph `IrfMP{}Xw` were already forbidden in $\mathbb{F}_3$ by lower order graphs.

Chapter 4

Conclusion

In this dissertation, we have had two major results and many more minor ones. Our first main result was a sharp bound on the number of vertices in forbidden subgraphs characterizing graphs that have minimum rank at most 3 over $\mathbb{F}_2$. Our second main result was a structural characterization for graphs having minimum rank at most k for any positive integer k and over any finite field. This structural characterization exposed a strong connection to projective geometry and we used some results from projective geometry to derive information about our characterization.

Of the two main results, the second shows much more promise for further research. There are many more results in projective geometry that can be applied to give information about the minimum rank problem.

We conclude with a short list of interesting open questions.

(a) Can the methods in Chapter 2 be extended to answer questions about the minimum rank problem restricted to adjacency matrices?

(b) How do properties of a graph G affect the properties of any blowup graph of G?

(c) What other results from projective geometry can be used to describe the graphs in $\mathfrak{g}_k(\mathbb{F}_q)$?

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Appendix A

SAGE code to generate forbidden graphs

```
# Return all matrices in S(F,G).  Since we are only changing
# the diagonal entries, we are assuming that F=F_2, the finite
# field with two elements.
def matrices_in_S(F,graph):
    if F != FiniteField(2):
        raise NotImplementedError
    order=len(graph.vertices())
    adj=coerce(MatrixSpace(F,order,order),
               graph.adjacency_matrix())
    results=[]
    for v in VectorSpace(F,order):
        b=adj.copy()
        for i in xrange(order):
            b[i,i]=v[i]
        results.append(b)
    return results


# Return the minimum rank of graph over the field F
def minrank(F,graph):
    return min([rank(m) for m in matrices_in_S(F,graph)])

# Return all matrices in S(F,G) that acheive the minimum rank
def minrank_matrices(F,graph):
    mr=minrank(F,graph)
    return [m for m in matrices_in_S(F,graph) if rank(m)==mr]
```

```
# Return the subgraphs of graph with order size
def subgraphs(graph, size):
    return [graph.subgraph(s)
            for s in combinations_iterator(graph.vertices(), size)]


# Return True if a subgraph of graph is isomorphic to one of the graphs
# in canonicalsubgraphs_by_size
def isomorphic_subgraph(graph, canonicalsubgraphs_by_size):
    for size, subgraphs in canonicalsubgraphs_by_size.items():
        for vertices in combinations_iterator(graph.vertices(), size):
            if graph.subgraph(vertices).canonical_label() in subgraphs:
                return True
    return False



# A utility function that splits a list into a dictionary,
# based on a criteria function.  The critera function takes
# an element of the list and returns a value.  A key in the
# dictionary is an element of the image of criteria, and the
# associated value is a list of elements of the list mapping
# to that image.
def split_list(list, criteria):
    d=dict()
    for i in list:
        j=(criteria)(i)
        if j in d:
            d[j].append(i)
        else:
            d[j]=[i]
    return d

# Need the subprocess module to access McKay's geng program
import subprocess
```

```
def generate_forbidden_graphs(F,mr,numvertices,forbiddengraphs):
    sub_canonical=split_list([s.canonical_label()
                              for s in forbiddengraphs],
                             lambda x: len(x.vertices()))
    allgraphs=graphs_list.from_graph6( \
        subprocess.Popen(["geng",str(numvertices)],
                         stdout=subprocess.PIPE,
                         stderr=subprocess.PIPE)\
        .communicate()[0].splitlines())
    newforbidden=[]
    for g in allgraphs:
        if isomorphic_subgraph(g,sub_canonical):
            continue
        if(minrank(F,g) > mr):
            newforbidden.append(g)
    return newforbidden


# Call forbidden_graphs_F2_mr3 for the 62 graphs
def forbidden_graphs_F2_mr3():
    f=[]
    for i in xrange(1,9):
        f+=generate_forbidden_graphs(FiniteField(2),3,i,f)
    return f
```

Appendix B

Magma programs

```
// We are working in F_2.
F:=FiniteField(2);

// This function returns all matrices in S(F_2,G) by adding
// all possible diagonal matrices to the adjacency matrix of G.
matrices_in_S:=function(graph)
    return {DiagonalMatrix(F,x)+AdjacencyMatrix(graph):
            x in Subsequences({x: x in F}, #Vertices(graph))};
end function;

// This function returns the minimum rank of a matrix by brute
// force computation.
minrank:=function(graph)
    return Min({Rank(m): m in matrices_in_S(graph)});
end function;

// This function returns the matrices in S(F_2,G) that attain
// the minimum rank.
minrank_matrices:=function(graph)
    return {m: m in matrices_in_S(graph) | Rank(m) eq minrank(graph)};
end function;

// This function returns true if and only if a subgraph of graph is
// isomorphic to a graph in graphlist
// (i.e., if graph is forbidden by graphlist).
isomorphic_subgraph:=function(graph,graphlist)
    if exists(t){<subgraph,fgraph>:
            subgraph in {sub<graph|s>: s in Subsets(Set(VertexSet(graph)))},
```

```
        fgraph in graphlist
        | IsIsomorphic(subgraph,fgraph)} then
       return true;
    else
        return false;
    end if;
end function;

// This is another version of the isomorphic_subgraph function.
isomorphic_subgraph:=function(graph,graphlist)
    for subgraph in {sub<graph|s>: s in Subsets(Set(VertexSet(graph)))} do
        if exists(t){ fgraph: fgraph in graphlist |
                      IsIsomorphic(subgraph,fgraph)} then
             return true;
        end if;
    end for;
    return false;
end function;

// This function appends a list of forbidden subgraphs with
// numvertices vertices to forbiddengraphs.  The geng program
// must be in the current directory.
generate_forbidden_graphs:=function(numvertices,forbiddengraphs)
    allgraphs:=OpenGraphFile("cmd geng "
        *IntegerToString(numvertices), 0, 0);
    while true do
        more, graph:=NextGraph(allgraphs);
        if more then
            if minrank(graph) ge 4
            and not isomorphic_subgraph(graph,forbiddengraphs) then
                Include(~forbiddengraphs,graph);
            end if;
        else
            break;
        end if;
    end while;
```

```
    return forbiddengraphs;
end function;

// Initialize the forbiddengraphs set and generate the forbidden
// subgraphs with 8 or fewer vertices.
forbiddengraphs:={};
for i in [1..8] do
    forbiddengraphs:=generate_forbidden_graphs(i,forbiddengraphs);
end for;

// Now forbiddengraphs contains all graphs in \mathcal{F}_4(F_2) as
// Magma graphs.
```

Appendix C

Field independent proof of Theorem 39

First recall a definition, a well-known fact, and the statement of the theorem.

Definition. Let G and H be graphs, each having a vertex labeled v. Then $G \oplus_v H$ is the graph obtained from $G \cup H$ by identifying the two vertices labeled v.

Lemma ([Nyl96])**.** *If F is any field and G is a graph with a vertex v, then* $\operatorname{mr}(F, G - v) \le \operatorname{mr}(F, G) \le \operatorname{mr}(F, G - v) + 2$.

Theorem ([Hsi01, BFH04])**.** *Let F be any field and let G and H be graphs, each having a vertex labeled v. Then*

$$\operatorname{mr}(F, G \underset{v}{\oplus} H) = \min\{\operatorname{mr}(F,G)+\operatorname{mr}(F,H),\ \operatorname{mr}(F,G-v)+\operatorname{mr}(F,H-v)+2\}. \quad \text{(C.1)}$$

Proof. Since v is a cut vertex of the connected graph $G \oplus_v H$, $(G \oplus_v H) - v = (G - v) \cup (H - v)$. By the lemma and Observation 33,

$$\operatorname{mr}(F, G \underset{v}{\oplus} H) \le \operatorname{mr}(F, G - v) + \operatorname{mr}(F, H - v) + 2.$$

Let v be the last vertex of G and the first vertex of H. Let

$$M = \begin{bmatrix} A & b \\ b^T & c_1 \end{bmatrix} \in S(F, G) \quad \text{and} \quad N = \begin{bmatrix} c_2 & d^T \\ d & E \end{bmatrix} \in S(F, H),$$

such that $\operatorname{rank} M = \operatorname{mr}(F, G)$ and $\operatorname{rank} N = \operatorname{mr}(F, H)$. Let

$$\hat{M} = \begin{bmatrix} A & b & 0 \\ b^T & c_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \hat{N} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & c_2 & d^T \\ 0 & d & E \end{bmatrix}.$$

Then $\hat{M} + \hat{N} \in S(F, G \oplus_v H)$ so

$$\begin{aligned} \text{mr}(F, G \underset{v}{\oplus} H) &\leq \text{rank}\,(\hat{M} + \hat{N}) \\ &\leq \text{rank}\ \hat{M} + \text{rank}\ \hat{N} = \text{rank}\ M + \text{rank}\ N \\ &= \text{mr}(F, G) + \text{mr}(F, H). \end{aligned}$$

This proves the $\leq$ in (C.1).

Now let $M \in S(F, G \oplus_v H)$ with rank $M = \text{mr}(F, G \oplus_v H)$. Write

$$M = \begin{bmatrix} A & b & 0 \\ b^T & c & d^T \\ 0 & d & E \end{bmatrix}.$$

Now

$$\begin{aligned} \text{rank}\ A + \text{rank}\ E &\leq \text{rank} \begin{bmatrix} A & b & 0 \\ 0 & d & E \end{bmatrix} && \text{(C.2)} \\ &\leq \text{rank}\ M && \text{(C.3)} \\ &\leq \text{rank}\ A + \text{rank}\ E + 2. && \text{(C.4)} \end{aligned}$$

It follows that one of the three inequalities (C.2), (C.3), or (C.4) is an equality.

I. Suppose that (C.2) and (C.4) are strict inequalities. Then

$$\text{rank}\ M = \text{rank} \begin{bmatrix} A & b & 0 \\ 0 & d & E \end{bmatrix} = \text{rank}\ A + \text{rank}\ E + 1.$$

Consequently $\begin{bmatrix} b \\ d \end{bmatrix} \notin \text{col} \begin{bmatrix} A & 0 \\ 0 & E \end{bmatrix}$, so either $b \notin \text{col}(A)$ or $d \notin \text{col}(E)$. Assume $b \notin \text{col}(A)$. Then $b^T \notin \text{row}(A)$, so

$$\text{rank}\ M = \text{rank} \begin{bmatrix} A & b & 0 \\ b^T & c & d^T \\ 0 & d & E \end{bmatrix} > \text{rank} \begin{bmatrix} A & b & 0 \\ 0 & d & E \end{bmatrix},$$

a contradiction. Therefore, this case does not occur. So either (C.2) or (C.4) is an equality.

II. Suppose (C.2) is an equality. Then

$$\text{rank}\begin{bmatrix} A & 0 \\ 0 & E \end{bmatrix} = \text{rank}\begin{bmatrix} A & 0 & b \\ 0 & E & d \end{bmatrix}.$$

Thus $\begin{bmatrix} b \\ d \end{bmatrix} \in \text{col}\begin{bmatrix} A & 0 \\ 0 & E \end{bmatrix}$, which implies that $b = Au$, $d = Ev$ for some vectors u and v. Then

$$\hat{A} = \begin{bmatrix} A & Au \\ u^T A & u^T Au \end{bmatrix} = \begin{bmatrix} A & b \\ b^T & u^T Au \end{bmatrix} \in S(F, G)$$

and rank $\hat{A} =$ rank A. Similarly,

$$\hat{E} = \begin{bmatrix} v^T Ev & v^T E \\ Ev & E \end{bmatrix} = \begin{bmatrix} v^T Ev & d^T \\ d & E \end{bmatrix} \in S(F, H)$$

and rank $\hat{E} =$ rank E. It follows that

$$\begin{aligned} \text{mr}(F, G \underset{v}{\oplus} H) &= \text{rank } M \\ &\geq \text{rank}\begin{bmatrix} A & 0 \\ 0 & E \end{bmatrix} = \text{rank } A + \text{rank } E = \text{rank } \hat{A} + \text{rank } \hat{E} \\ &\geq \text{mr}(F, G) + \text{mr}(F, H). \end{aligned}$$

III. Suppose that (C.4) is an equality. Since $A \in S(F, G-v)$ and $B \in S(F, H-v)$, rank $A \geq \text{mr}(F, G-v)$ and rank $B \geq \text{mr}(F, H-v)$. Then

$$\text{mr}(F, G \underset{v}{\oplus} H) = \text{rank } M \geq \text{mr}(F, G-v) + \text{mr}(F, H-v) + 2.$$

Combining cases I, II, and III, we have proven the $\geq$ in (C.1). $\square$

Appendix D

SAGE code to generate graphs

```
def bilinearforms(F,mr):
    # Construct a matrix space for our bilinear forms
    MSpace=MatrixSpace(F,mr)
    # The identity matrix is always
    # a congruence class representative
    forms=[MSpace.identity_matrix()]
    # Add the extra matrices in even rank cases
    if(mod(mr,2)==0): # even rank
        if(mod(F.characteristic(),2)==0): # characteristic 2
            # Add diag(H_1, H_2, ..., H_{mr/2})
            hyperbolic=matrix(F,[[0,1],[1,0]])
            forms.append(reduce(lambda x,y:x.block_sum(y), \
                                    [hyperbolic]*Integer(mr/2)))
        else: # odd characteristic
            # Add diag(I_{n-1}, ν), where ν is a non-square
            identity=MSpace.identity_matrix()
            # Find a non-square
            for a in F:
                if not a.is_square():
                    break
            if a.is_square():
                return "error"
            identity[mr-1,mr-1]=a
            forms.append(identity)
    return forms

def markedgraphs(F,mr):
    # U has one vector for every equivalence class in PG(mr-1,q)
```

```
    U=matrix(F,[list(v)
                for v in ProjectiveSpace(mr-1,F)]).transpose()
    forms=bilinearforms(F,mr)
    grammatrices=[U.transpose()*m*U for m in forms]
    graphs=[Graph(m) for m in grammatrices]
    for i in range(len(graphs)):
        graphs[i].loops(true);
        graphs[i].add_edges([[j,j] for j in range(len(graphs[i])) \
                            if grammatrices[i][j,j] != 0])
    return graphs

def showmarkedgraphs(F,mr):
    for g in markedgraphs(F,mr):
        # Vertices with loops are black, others are white
        colorsdict={}
        blackvertices=g.loop_vertices()
        whitevertices=[i for i in range(len(g)) \
                        if i not in g.loop_vertices()]
        if(len(blackvertices)>0):
            colorsdict['black']=blackvertices
        if(len(whitevertices)>0):
            colorsdict['white']=whitevertices
        g.show(layout='circular', color_dict=colorsdict, \
               vertex_labels=false)
```