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ABSTRACT

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The total character of a finite group G is the sum of the irreducible characters of G. When the total character of a finite group can be written as a monic polynomial with integer coefficients in an irreducible character of G, we say that G is a total character group. In this thesis we examine the total character of the dicyclic group of order $4n$, the non-abelian groups of order p^3, and the symmetric group on n elements for all $n \geq 1$. The dicyclic group of order $4n$ is a total character group precisely when $n \equiv 2, 3 \bmod 4$, and the associated polynomial is a sum of Chebyshev polynomials of the second kind. The irreducible characters paired with these polynomials are exactly the faithful characters of the dicyclic group. In contrast, the non-abelian groups of order p^3 and the symmetric group on n elements with $n \geq 4$ are not total character groups. Finally, we examine the special case when G is a total character group and the polynomial is of degree 2. In this case, we say that G is a quadratic total character group. We classify groups which are both quadratic total character groups and p-groups.

Keywords: character, total character, symmetric group, p-group

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Contents

List of Tables

Chapter 1. Introduction

The representation theory and character theory of finite groups was introduced by Ferdinand Georg Frobenius (1849-1917) in a series of papers published in 1896 and 1897 [5, p. 1]. Although the representation theory of finite abelian groups had been explored throughout the 19th century, Frobenius presented a way to generalize the theory to any finite group [5, p. 63].

In the following years, many fundamental elements of representation theory were discovered by Frobenius and other eminent mathematicians such as William Burnside (1852-1927), Issai Schur (1875-1941), Heinrich Maschke (1853-1908), and Richard Brauer (1901-1977). Representation theory and character theory as we understand them today are the result of the work of these and many subsequent mathematicians, and continue to be of fundamental importance in current mathematical research.

In this chapter, we present the basics of representation theory and character theory as they pertain to this thesis. We also introduce our research problem, discuss related topics that provide motivation, and examine previous results that have been obtained in this area.

1.1 Representations and characters of finite groups

Representation theory and character theory are closely related. We first introduce representations, then characters of finite groups.

1.1.1 Representations. Throughout Chapter 1, G will be an arbitrary finite group and F will be $\mathbb{R}$ or $\mathbb{C}$. We let $GL(n, F)$ denote the group of invertible $n \times n$ matrices over the field F.

Definition 1.1. [15, p. 30] A *representation* of G over F is a homomorphism ρ from G to $GL(n, F)$, for some n. The *degree* of ρ is the integer n.

One basic example is the homomorphism $\rho : G \to GL(1, F)$ defined by

$$\rho(g) = (1) \text{ for all } g \in G,$$

called the *trivial representation.*

The *regular representation* is another important representation, which in essence represents the action of the group on itself. Let G be a finite group, say $G = \{g_1, g_2, \ldots, g_n\}$. If g_i and g_j are two distinct elements of G, then $gg_i \neq gg_j$, else mutliplying on the left by g^{-1} yields $g_i = g_j$, a contradiction. Therefore gg_i and gg_j are distinct if g_i and g_j are, and so the set $\{gh \mid h \in G\}$ is simply a rearrangement of G. In other words, if we write $gg_i = g_{\sigma_g(i)}$, then σ_g is a permutation. We can represent this rearrangement by a permutation matrix denoted P_g, which has the property that

$$P_g e_i = e_{\sigma_g(i)},$$

where e_i is the usual basis element

$$e_i = [0, \cdots, 0, \underbrace{1}_{i^{th}}, 0, \cdots, 0]^T.$$

The map $\rho_{\text{reg}} : G \to GL(|G|, \mathbb{C})$ defined by

$$g \mapsto P_g \quad \text{for each } g \in G,$$

is indeed a homomorphism and is called the *regular representation* of G.

If $\rho : G \to GL(n, F)$ and $\phi : G \to GL(n, F)$ are two representations, we say that ρ, ϕ are *equivalent* if there exists an invertible $n \times n$ matrix T such that $\rho(g) = T\phi(g)T^{-1}$ for each $g \in G$. We note that given any representation ρ and any $n \times n$ matrix T, the map

$\phi : G \to GL(n, F)$ defined by

$$\phi(g) = T^{-1}\rho(g)T \quad \text{for each } g \in G$$

is also a representation, since for all $g, h \in G$, we have

$$\phi(gh) = T^{-1}\rho(gh)T = T^{-1}\rho(g)\rho(h)T = (T^{-1}\rho(g)T)(T^{-1}\rho(h)T) = \phi(g)\phi(h).$$

This type of equivalence determines an equivalence relation on the set of representations of G. We are only interested in representations up to equivalence.

Representations can sometimes be decomposed into simpler representations. We make the following definition:

Definition 1.2. [17, p. 161] A representation $\rho : G \to GL(n, F)$ is said to be *reducible* if it is equivalent to a representation $\sigma : G \to GL(n, F)$ with the property that $\sigma(g)$ is a block diagonal matrix with blocks of the same sizes for all $g \in G$. If no such representation σ exists, then ρ is said to be *irreducible*.

From this definition it follows that all 1-dimensional representations are irreducible.

A fundamental result known as Maschke's Theorem [15, p. 70] tells us that when F is a field of characteristic zero, any representation is a sum of a finite number of irreducible representations in an essentially unique way. Irreducible representations are the building blocks of representation theory.

The characters of a finite group are intrisically related to the representations of the group. In the next section, we discuss characters of finite groups.

1.1.2 Characters. In his 1896 paper *Über Gruppencharaktere*, Frobenius first defined characters in the context of group determinants [8]. He subsequently discovered that characters also arose from representations and that the two definitions were equivalent [5, p. 50]. In this thesis, we include the definition that arises from representations, as it is more relevant to our methods.

Definition 1.3. Given a representation ρ, the function $\chi_\rho : G \mapsto F$ defined by

$$\chi_\rho(g) = \mathrm{tr}(\rho(g)) \text{ for all } g \in G$$

is called a *character* of G. Here $\mathrm{tr}(\cdot)$ is the trace function. We say that χ_ρ is *the character afforded by* ρ. The *degree* of χ_ρ is the degree of ρ.

Note that if two representations are not equivalent, then they afford different characters. Since a representation is a homomorphism and the trace function is invariant under matrix similarity, if $g, h \in G$ and $\chi = \chi_\rho$ is a character of G, we have

$$\chi_\rho(ghg^{-1}) = \mathrm{tr}(\rho(ghg^{-1})) = \mathrm{tr}(\rho(g)\rho(h)\rho(g)^{-1}) = \mathrm{tr}(\rho(h)) = \chi_\rho(h).$$

Thus any two conjugate elements of G have the same character value, and it follows that characters are constant on conjugacy classes [15, p. 119]. A map from G into F which is constant on conjugacy classes of G is called a *class function* [15, p. 152]. So, in particular, characters are class functions.

An example of a character is the one afforded by the trivial representation, namely the homomorphism $\chi : G \to F$ defined by

$$\chi(g) = 1 \text{ for all } g \in G.$$

We naturally call this the *trivial character*.

Another example is the character χ_{reg} afforded by the regular representation ρ_{reg}, for which we get $\chi_{\mathrm{reg}}(1) = |G|$ and $\chi_{\mathrm{reg}}(g) = 0$ if $g \neq 1$.

Some useful properties of characters rely on inner products. If V is a vector space over $\mathbb{C}$, then an *inner product* is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ that satisfies the following conditions:

(1) $\langle \theta, \psi \rangle = \overline{\langle \psi, \theta \rangle}$ for all $\theta, \psi \in V$;

(2) $\langle \lambda_1\theta_1 + \lambda_2\theta_2, \psi\rangle = \lambda_1\langle\theta_1,\psi\rangle + \lambda_2\langle\theta_2,\psi\rangle$for all $\lambda_1, \lambda_2 \in \mathbb{C}$ and all $\theta_1, \theta_2, \psi \in V$;

(3) $\langle\theta,\theta\rangle > 0$ if $\theta \neq 0$.

Since the set of all functions from G to $\mathbb{C}$ forms a vector space V over $\mathbb{C}$ with operations

$$(\theta+\psi)(g) = \theta(g) + \psi(g) \text{ for } \theta, \psi \in V,$$
$$(\lambda\theta)(g) = \lambda(\theta(g)) \text{ for } \lambda \in \mathbb{C}, \theta \in V,$$

we can define an inner product on two such functions θ, ψ by

$$\langle\theta,\psi\rangle = \frac{1}{|G|}\sum_{g\in G}\theta(g)\overline{\psi(g)},$$

and it is easily shown that this map satisfies the three conditions of an inner product [15, p. 135].

Characters are functions from G to $\mathbb{C}$, so this definition gives an inner product on characters. A basic proposition concerning the inner product of characters is given in [15, p. 135], which we restate here for future use.

Proposition 1.4. *Assume that G has exactly ℓ conjugacy classes, with representatives $g_1, g_2, \ldots, g_\ell$. Let χ and ψ be characters of G.*

(1) $\langle\chi,\psi\rangle = \langle\psi,\chi\rangle = \frac{1}{|G|}\sum_{g\in G}\chi(g)\psi(g^{-1})$, and this is a real number.

(2) $\langle\chi,\psi\rangle = \sum_{i=1}^{\ell}\frac{\chi(g_i)\overline{\psi(g_i)}}{|C_G(g_i)|}$.

Property (2) of Proposition 1.4 will be useful as we discuss the dicyclic group of order $4n$ in Chapter 2 and quadratic total character groups in Chapter 5.

1.1.3 Irreducible characters and the character table.

Irreducible representations yield irreducible characters, which are of critical importance in character theory.

Definition 1.5. Let χ be the character of G afforded by a representation ρ of G. If ρ is reducible, then χ is said to be a *reducible character*. If ρ is irreducible, then χ is *irreducible*.

The number of irreducible characters of a finite group G is the same as the number of conjugacy classes of G [15, p. 152]. The set of irreducible characters of G is denoted $\operatorname{Irr}(G)$. A useful fact proved in [15, p. 153] states that $\operatorname{Irr}(G)$ is a basis for the complex vector space of class functions of G. Since characters are class functions, every character can therefore be written as a unique linear combination of the irreducible characters. In fact, if G is a finite group and $\chi_1, \dots, \chi_k$ are the irreducible characters of G, then any character χ of G can be written as

$$\chi = a_1\chi_1 + a_2\chi_2 + \cdots + a_k\chi_k$$

for some $a_i \in \mathbb{Z}$, where the a_i are uniquely determined by χ. We say that χ_i is a *constituent* of χ if $a_i \neq 0$. In the particular case of the regular representation defined in Section 1.1.1, we find in [15, p. 127] that

$$\chi_{\text{reg}} = \sum_{i=1}^{k} d_i\chi_i = \sum_{i=1}^{k} \chi_i(1) \cdot \chi_i,$$

where $\chi_1, \dots, \chi_k$ are the irreducible characters of G and d_i is the degree of χ_i for each i, $1 \leq i \leq k$. Since $d_i \geq 1$ for all i, every irreducible character is a constituent of χ_{reg}.

Since there are finitely many irreducible characters of G, we can list them in a table, called a character table of G.

Definition 1.6. Let $\chi_1, \chi_2, \dots, \chi_k$ be the irreducible characters of G and let $g_1, g_2, \dots, g_k$ be representatives of the conjugacy classes of G. The $k \times k$ matrix whose ij-entry is $\chi_i(g_j)$ (for all i, j with $1 \leq i \leq k, 1 \leq j \leq k$) is called a *character table* of G [15, p. 159].

For example, the symmetric group on three elements, S_3, has three conjugacy classes and three irreducible characters, which are shown in its character table (see Table 1.1).

The character table of any finite group G satisfies important orthogonality relations, which rely on the definition of inner product given in Section 1.1.2. Before stating these relations, we first need to understand a particular property concerning the inner product of irreducible characters [15, p. 140].

g	e	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

Table 1.1: Character Table of S_3

Proposition 1.7. *Let χ, ψ be two irreducible characters of G. Then*

$$\langle \chi, \psi \rangle = \begin{cases} 1 & \chi = \psi; \\ 0 & \chi \neq \psi. \end{cases}$$

In other words, the inner product function on the set of irreducible characters is the Kronecker delta function δ_{ij} defined by

$$\delta_{ij} = \begin{cases} 1 & i = j; \\ 0 & i \neq j. \end{cases}$$

This fact leads to the orthogonality relations satisfied by the rows and columns of every character table, which are given in [15, p. 161]:

Theorem 1.8 (Orthogonality Relations)**.** *Let $\chi_1, \dots, \chi_k$ be the irreducible characters of G, and let $g_1, \dots, g_k$ be representatives of the conjugacy classes of G. Then the following relations hold for any $r, s \in \{1, \dots, k\}$.*

(1) The row orthogonality relations:

$$\sum_{i=1}^{k} \frac{\chi_r(g_i)\overline{\chi_s(g_i)}}{|C_G(g_i)|} = \delta_{rs}.$$

(2) The column orthogonality relations:

$$\sum_{i=1}^{k} \chi_i(g_r)\overline{\chi_i(g_s)} = \delta_{rs}|C_G(g_r)|.$$

The row and column orthogonality relations can be used to calculate the values of an irreducible character when all the other irreducible characters and other relavent information, such as the degree of the character in question, is known. They are also helpful in determining whether an irreducible character is a constituent of any other character.

Now that we have discussed some preliminary definitions, examples, and properties of representations and characters of finite groups, we are ready to introduce the specific research question of this thesis, which concerns the total character of a finite group.

1.2 The total character

The first paper introducing the total character of a finite group was written by Amy Cottrell and Eirini Poimenidou and was published in 2000 [4]. In this paper, Cottrell and Poimenidou defined the *total character* τ of a finite group G to be the sum of the irreducible characters of G. When the character table of a group G is known, the total character τ can be obtained by summing over the rows of the character table. For example, Table 1.2 lists both the character table and total character τ of the group S_3.

g	e	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1
τ	4	0	1

Table 1.2: Character Table and Total Character of S_3

One interesting property of the total character is that τ_G is always integral-valued. This follows from the fact that if ψ is an irreducible character of G, then all the Galois conjugates of ψ appear in the character table (see [6, p. 558-654] for an introduction to general Galois theory, and [12, p. 152-153] for Galois conjugates of characters). The sum of all these Galois conjugates will give a character having values that are both rational and algebraic integers, and hence integers (Corollary 1 in [16, p. 15]).

After defining the total character, Cottrell and Poimenidou examined a question posed by K.W. Johnson that involves the total character and monic polynomials.

Question. *Let G be a finite group and let τ be the total character of G. Does there exist an irreducible character χ of G and a monic polynomial $f(x) \in \mathbb{Z}[x]$ such that $f(\chi) = \tau$?*

If there is a positive answer to this question for the group G, we say that G is a *total character group*. As an example, note that for the group $G = S_3$ above, the character χ_3 and the monic polynomial $f(x) = x^2 \in \mathbb{Z}[x]$ satisfy $(\chi_3)^2 = \tau$. Here, to find $(\chi_3)^2$, we identify χ_3 with the corresponding row in the character table, $(2, 0, -1)$, and then square the entries in this row to obtain $(\chi_3)^2 = (4, 0, 1) = \tau$. Thus S_3 is a total character group.

A motivating concept behind Johnson's question, as discussed in [4], is that of character sharpness. If G is a finite group and χ is a character of G of degree n, let

$$L = \{\chi(g) \mid g \neq 1\}$$

and

$$f_L(x) = \prod_{l \in L} (x - \ell).$$

Cameron and Kiyota showed in [2] that $f_L(x) \in \mathbb{Z}[x]$ and that $|G|$ divides $f_L(n)$. When $f_L(n) = |G|$, the character χ is said to be *sharp*. Define

$$f_L(\chi) = \prod_{\ell \in L} (\chi - \ell).$$

If χ is sharp, then $f_L(\chi) = \chi_{\text{reg}}$, the character afforded by the regular representation ρ. This is a consequence of the fact that if 1_G is the identity of G, we have

$$f_L(\chi)(1_G) = f_L(\chi(1_G)) = f_L(n) = |G| = \chi_{\text{reg}}(1_G),$$

and if $g \neq 1$, then $\chi(g) = l'$ for some $l' \in L$, so

$$f_L(\chi(g)) = \prod_{\ell \in L}(\chi(g) - \ell) = (\ell' - \ell') \prod_{\ell \in L,\, \ell \neq \ell'} (\chi(g) - \ell) = 0 = \chi_{\text{reg}}(g).$$

Since every irreducible character is a constituent of the regular representation, we see that if χ is sharp, every irreducible character is a constituent of $f_L(\chi)$. Since it can be shown that $f_L(\chi)$ is a monic polynomial in χ with integer coefficients [2], Johnson's question arose naturally in this context.

In [4, p. 13], Cottrell and Poimenidou noted that all irreducible characters χ satisfying $f(\chi) = \tau$ for the dihedral group D_{2n}, n odd, were sharp characters. The natural question of whether a character must be sharp in order to satisfy this property was suggested. However, in a follow up paper [18], sharp characters were mentioned purely for contextualization purposes, and no answer to the sharp character question was given. In this thesis, we present a counterexample to this conjecture and show that T_{12}, the dicyclic group of order 12, has a non-sharp character χ satisfying $f(\chi) = \tau$ (see Section 2.6). We therefore know that a character does not have to be sharp in order to yield a positive answer to Johnson's question.

In the light of character sharpness, Cottrell and Poimenidou gave a partial answer to Johnson's question for the dihedral group D_{2n}. They proved the following (see Theorem 3.7 in [4, p. 11]):

Theorem 1.9. *Let G be a dihedral group of order $2n$ with n odd. Let τ be the total character of G. There exists a monic polynomial $g(x) \in \mathbb{Z}[x]$ exists such that $g(\chi) = \tau$ for every faithful irreducible character χ. Furthermore, if $g'(\chi) = \tau$ for some faithful irreducible character χ, then* $\deg g'(x) \geq \deg g(x)$.

In their proof, Cottrell and Poimenidou defined $x_r = \epsilon^r + \epsilon^{-r} = 2\cos\left(\frac{2\pi r}{n}\right)$, where

$\epsilon = e^{2\pi i/n}$ and $1 \leq r \leq \frac{n-1}{2}$. They then showed that the polynomial

$$g(x) = (x - x_1)(x - x_2)\cdots(x - x_t),$$

where $t = \frac{n-1}{2}$, satisfied $g(\chi) = \tau$ for all faithful irreducible characters χ.

Poimenidou expanded upon this work in the paper "Total characters and Chebyshev polynomials," published with Homer Wolfe in 2003 [18]. This paper gives necessary and sufficient conditions for the dihedral group D_{2n} to be a total character group, and proves that the associated monic polynomial $g(x)$ is, in fact, a sum of Chebyshev polynomials of the first kind. Poimenidou and Wolfe proved the following (Theorem 1.2 in [18, p. 2248]):

Theorem 1.10. *Let $G \cong D_{2n}$ and let τ be the total character of G.*

(1) If n is odd or if $n = 2m$, where $m \equiv 2, 3 \bmod 4$, then there exists a unique monic polynomial $P(x) \in \mathbb{Z}[x]$ such that $P(\chi) = \tau$ for any faithful $\chi \in Irr(G)$. Moreover,

$$P(x) = \begin{cases} 2\sum_{k=0}^{m} T_k^*\left(\frac{x}{2}\right) - 2T_{m-1}^*\left(\frac{x}{2}\right), & \text{if } n = 2m-1,\ n \equiv 1, 3 \bmod 8; \\ 2\sum_{k=0}^{m-1} T_k^*\left(\frac{x}{2}\right), & \text{if } n = 2m-1,\ n \equiv 5, 7 \bmod 8; \\ 2\sum_{k=0}^{m} T_k^*\left(\frac{x}{2}\right), & \text{if } n = 2m,\ m \equiv 2, 3 \bmod 4; \end{cases}$$

where $T_k^*(x)$ is the k^{th} Chebyshev polynomial of the first kind.

(2) If $n = 2m$ and $m \equiv 1 \bmod 4$, then there exists a unique monic polynomial $P(x) \in \mathbb{Z}[x]$ such that $P(x) = 2\tau$ for any faithful $\chi \in Irr(G)$, where

$$P(x) = 2\left[T_{m+1}^*\left(\frac{x}{2}\right) + 2\sum_{k=0}^{m} T_k^*\left(\frac{x}{2}\right) - T_{m-1}^*\left(\frac{x}{2}\right)\right].$$

Moreover, there does not exist a polynomial $P(x) \in \mathbb{Z}[x]$ such that $P(\chi) = \tau$ for any $\chi \in Irr(G)$.

(3) If $G \cong D_{2n}$, where $n \equiv 0 \bmod 8$, then there does not exist a polynomial $P(x) \in \mathbb{C}[x]$ such that $P(\chi) = \tau$ for any $\chi \in Irr(G)$.

The proof of this theorem constitutes the remainder of [18], which builds upon the preliminary results proved in [4]. To our knowledge, these two papers are the only papers that have been published specifically on the subject of the total character.

In this thesis, we answer Johnson's question for three more families of groups, the first of which is the dicyclic group of order $4n$, denoted T_{4n}. The dicyclic group result is similar to Poimenidou and Wolfe's [18] in that T_{4n} is a total character group precisely when $n \equiv 2, 3 \bmod 4$. Furthermore, the necessary monic polynomial is a sum of Chebyshev polynomials of the second kind, and the irreducible characters that yield the total character are exactly the irreducible faithful characters.

Following the dicyclic groups, we examine the non-abelian groups of order p^3, with p an odd prime. Unlike the dicyclic and dihedral groups, these groups are never total character groups. However, the proof has a similar computational style as that of the dicyclic groups because the character table is known.

The last family of groups we examine are the symmetric groups. The symmetric group on n elements, S_n, is a group of fundamental importance in group theory, representation theory and character theory. The S_n result is similar to the p^3 result in that for all values of n excepting $n = 1, 2$, and 3, the symmetric group on n elements is not a total character group. Additionally, the technique used in proving the negative result differs greatly from the proofs of the T_{4n} and p^3 results.

In the final chapter of this thesis, we consider the special case when the associated polynomial is of degree two. If a group G is a total character group and the polynomial is of degree two, we say that G is a quadratic total character group. We prove that if G is both a quadratic total character group and a p-group, then $p = 2$ and G is extraspecial.

Chapter 2. The Dicyclic Group of Order $4n$

2.1 Background

The dicyclic group of order $4n$ has presentation

$$T_{4n} = \langle a, b \mid a^{2n} = 1, a^n = b^2, b^{-1}ab = a^{-1} \rangle.$$

From these relations, we see that every element of the dicyclic group can be written uniquely as $a^k b^j$, where $0 \leq k < 2n$ and $j = 0$ or 1. Throughout Chapter 2, let $\tau = \tau_{4n}$ be the total character of T_{4n}. Furthermore, since T_4 is isomorphic to the cyclic group of order 4 and is not considered dicyclic, we assume throughout this chapter that $n \geq 2$. Our main result is:

Theorem 2.1. *Let $G = T_{4n}$, the dicyclic group of order $4n$.*

(1) If $n \equiv 2, 3$ mod 4, then T_{4n} is a total character group: there exists a monic polynomial $F_n(x) \in \mathbb{Z}[x]$ such that $F_n(\chi) = \tau$ for any faithful irreducible character χ of G. Moreover,

$$F_n(x) = U_n\left(\frac{x}{2}\right) + U_{n-1}\left(\frac{x}{2}\right) + 1,$$

where $U_n(x)$ is the n^{th} Chebyshev polynomial of the second kind.

(2) If $n \equiv 0, 1$ mod 4, then T_{4n} is not a total character group.

A proof of this theorem requires that we understand the character table and the total character of T_{4n}, as well as Chebyshev polynomials of the second kind. For $n \equiv 2, 3$ mod 4, we identify all characters χ of T_{4n} such that $U_n(\frac{\chi}{2}) + U_{n-1}(\frac{\chi}{2}) + 1 = \tau$. When $n \equiv 0, 1$ mod 4, we show that no irreducible character when substituted in a monic polynomial with integer coefficients gives the total character of T_{4n}.

2.2 Character table and total character of T_{4n}

The character table of T_{4n} is well known and can be found in [15, p. 420]. The dicyclic group of order $4n$ has $n+3$ conjugacy classes:

$$\{1\},\ \{a^n\},\ \{a^r, a^{-r}\}\ (1 \le r \le n-1),\ \{a^{2j}b \mid 0 \le j \le n-1\},$$
$$\{a^{2j+1}b \mid 0 \le j \le n-1\}.$$

There are four irreducible linear characters, χ_1, χ_2, χ_3, and χ_4, which behave differently when n is odd or n is even. There are $n-1$ irreducible characters of degree two represented by $\psi_j, 1 \le j \le n-1$, which are given by the same formula for any $n \ge 2$. The character table of T_{4n} for n odd is given as Table 2.1, where $\omega = e^{2\pi i/2n}$ is a primitive $2n^{th}$ root of unity. Table 2.2 shows the character table of T_{4n} for n even.

g	1	a^n	$a^r(1 \le r \le n-1)$	b	ab
$\lvert C_G(g)\rvert$	$4n$	$4n$	$2n$	4	4
χ_1	1	1	1	1	1
χ_2	1	-1	$(-1)^r$	i	$-i$
χ_3	1	1	1	-1	-1
χ_4	1	-1	$(-1)^r$	$-i$	i
$\psi_j,\ 1 \le j \le n-1$	2	$2(-1)^j$	$\omega^{rj} + \omega^{-rj}$	0	0

Table 2.1: Character Table of T_{4n}, n odd

g	1	a^n	$a^r(1 \le r \le n-1)$	b	ab
$\lvert C_G(g)\rvert$	$4n$	$4n$	$2n$	4	4
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	$(-1)^r$	1	-1
χ_4	1	1	$(-1)^r$	-1	1
$\psi_j,\ 1 \le j \le n-1$	2	$2(-1)^j$	$\omega^{rj} + \omega^{-rj}$	0	0

Table 2.2: Character Table of T_{4n}, n even

Using the character tables above, we can compute the value of $\tau(g)$ for each conjugacy class representative g by summing over rows.

We first determine a simple expression for $\sum_{j=1}^{n-1} \omega^{rj} + \omega^{-rj}$.

Lemma 2.2. *Let $\omega = e^{2\pi i/2n}$ be a primitive $2n^{th}$ root of unity. Then*

$$\sum_{j=1}^{n-1} \omega^{rj} + \omega^{-rj} = -1 - (-1)^r,$$

for any $1 \leq r \leq n-1$.

Proof. As $\omega = e^{2\pi i/2n}$, we compute

$$0 = (\omega^r)^{2n} - 1 = (\omega^r - 1)((\omega^r)^{2n-1} + (\omega^r)^{2n-2} + ... + (\omega^r) + 1).$$

Now the order of ω is $2n$, but $1 \leq r \leq n-1$, so $\omega^r \neq 1$. This implies $\sum_{j=0}^{2n-1} \omega^{rj} = 0$. If we multiply by ω^{-rn}, we get $\sum_{j=0}^{2n-1} \omega^{r(j-n)} = 0$, and re-indexing yields $\sum_{j=-n}^{n-1} \omega^{rj} = 0$. Then we have:

$$\begin{aligned} 0 = \sum_{j=-n}^{n-1} \omega^{rj} &= (\omega^r)^{-n} + \sum_{j=-n+1}^{-1} \omega^{rj} + (\omega^r)^0 + \sum_{j=1}^{n-1} \omega^{rj} \\ &= (\omega^n)^{-r} + \sum_{j=1}^{n-1} \omega^{-rj} + 1 + \sum_{j=1}^{n-1} \omega^{rj} \\ &= 1 + (-1)^{-r} + \sum_{j=1}^{n-1} (\omega^{rj} + \omega^{-rj}). \end{aligned}$$

Thus,

$$\sum_{j=1}^{n-1} (\omega^{rj} + \omega^{-rj}) = -1 - (-1)^{-r} = 1 - (-1)^r.$$

□

Since $\psi_j(a^r) = \omega^{rj} + \omega^{-rj}$, the value of $\tau(a^r)$ is an easy consequence of Lemma 2.2.

Proposition 2.3. *For τ the total character of T_{4n},*

$$\tau(a^r) = \begin{cases} 0 & r \text{ odd;} \\ 2 & r \text{ even.} \end{cases}$$

Proof. By the definition of τ, we have

$$\tau(a^r) = \sum_{i=1}^{4} \chi_i(a^r) + \sum_{j=1}^{n-1} \psi_j(a^r).$$

From Table 2.1, the sum of the four linear characters $\sum_{i=1}^{4} \chi_i(a^r)$ is 0 when r is odd and 4 when r is even. By Lemma 2.2,

$$\sum_{j=1}^{n-1} \psi_j(a^r) = \sum_{j=1}^{n-1} (\omega^{rj} + \omega^{-rj}) = -1 - (-1)^r = \begin{cases} 0 & r \text{ odd;} \\ -2 & r \text{ even.} \end{cases}$$

This gives the result. □

We now easily obtain the other values of τ_{4n}, beginning with the case when n is odd.

Proposition 2.4. *Let τ be the total character of T_{4n}, n odd. Then the following table gives the value of $\tau(g)$ for every conjugacy class representative g of T_{4n}.*

g	1	a^n	$a^r (1 \leq r \leq n-1)$	b	ab
τ	$2n+2$	0	$\begin{cases} 0 & r \text{ odd} \\ 2 & r \text{ even} \end{cases}$	0	0

Proof. Using Table 2.1, we see that $\tau(1) = 4 + 2(n-1) = 2n+2$ and $\tau(b) = \tau(ab) = 0$. For the element $g = a^n$, we see from Table 2.1 again that

$$\tau(a^n) = 1 - 1 + 1 - 1 + \sum_{j=1}^{n-1} 2(-1)^j = \sum_{j=1}^{n-1} 2(-1)^j,$$

which equals 0 since $n-1$ is even. By Proposition 2.3, the value of $\tau(a^r)$ is 0 when r is odd and 2 when r is even. This finishes the table. □

Using Table 2.2, we next calculate τ_{4n} when n is even.

Proposition 2.5. *Let τ be the total character of T_{4n}, n even. Then the following table gives the value of $\tau(g)$ for every conjugacy class representative g of T_{4n}.*

g	1	a^n	$a^r(1 \leq r \leq n-1)$	b	ab
τ	$2n+2$	2	$\begin{cases} 0 & r \text{ odd} \\ 2 & r \text{ even} \end{cases}$	0	0

Proof. Summing values found in Table 2.2 shows that $\tau(1) = 4 + 2(n-2) = 2n+2$ and $\tau(b) = \tau(ab) = 0$. The values of $\tau(a_r)$ follow from Proposition 2.3. Finally, in considering $\tau(a^n)$ for n even, we have

$$\tau(a^n) = 1 + 1 + 1 + 1 + \sum_{j=1}^{n-1} 2(-1)^j = 4 + (-2) = 2.$$

This finishes the table. □

We have now calculated τ_{4n} for both n odd and n even. The total character values are given below for future reference as Tables 2.3 and 2.4.

g	1	a^n	$a^r(1 \leq r \leq n-1)$	b	ab
τ	$2n+2$	0	$\begin{cases} 0 & r \text{ odd} \\ 2 & r \text{ even} \end{cases}$	0	0

Table 2.3: Total Character of T_{4n}, n odd

g	1	a^n	$a^r(1 \leq r \leq n-1)$	b	ab
τ	$2n+2$	2	$\begin{cases} 0 & r \text{ odd} \\ 2 & r \text{ even} \end{cases}$	0	0

Table 2.4: Total Character of T_{4n}, n even

2.3 Faithful characters

Determining when T_{4n} is a total character group involves a certain type of character called a faithful character. In this section, we first define what a faithful character is and then discuss the relationship between faithful characters and total character groups. Finally, we identify the faithful characters of the dicyclic group of order $4n$.

Definition 2.6. For any group G, a *faithful character* is a character χ such that $\chi(1) \neq \chi(g)$ for each nontrivial $g \in G$.

One example of a faithful character is the regular character χ_{reg}, since $\chi_{\text{reg}}(1) = |G|$ and $\chi_{\text{reg}}(g) = 0$ if $g \neq 1$. The total character is also faithful, as shown in Proposition 2.7.

Proposition 2.7. *Let G be a finite group. Then the total character τ is faithful.*

Proof. Suppose by way of contradiction that τ is not faithful, so there exists $g \neq 1$ such that $\tau(1) = \tau(g)$. Let $\chi_1, \dots, \chi_k$ be the irreducible characters of G, and let d_i be the degree of χ_i for each i. Then since $\tau(g) = \tau(1)$, we have

$$\sum_{i=1}^{k} \chi_i(g) = \sum_{i=1}^{k} \chi_i(1).$$

Since $|\chi_i(g)| \leq \chi_i(1)$ for all i, $1 \leq i \leq k$, this implies $\chi_i(g) = \chi_i(1)$ for all irreducible characters χ_i of G. Therefore if χ_{reg} is the regular character of G, we have

$$\chi_{\text{reg}}(g) = \sum_{i=1}^{k} d_i \chi_i(g) = \sum_{i=1}^{k} d_i \chi_i(1) = \chi_{\text{reg}}(1),$$

a contradiction since the regular character is faithful and $g \neq 1$. Therefore the total character is faithful. $\square$

The next proposition exhibits the relationship between faithful characters and total character groups.

Proposition 2.8. *Let χ be an irreducible character of G and let τ be the total character of G. If χ is not faithful, then there does not exist a polynomial $f(x) \in \mathbb{C}[x]$ such that $f(\chi) = \tau$.*

Proof. Suppose such a polynomial exists, call it $f(x) \in \mathbb{C}[x]$. Since χ is not faithful, there exists $g_0 \in G$, $g_0 \neq 1$, such that $\chi(1) = \chi(g_0)$. Since $f(\chi) = \tau$, we have $f(\chi(g)) = \tau(g)$ for all $g \in G$, which implies that

$$\tau(1) = f(\chi(1)) = f(\chi(g_0)) = \tau(g_0).$$

This is a contradiction, since τ is faithful. Thus there does not exist $f(x) \in \mathbb{C}[x]$ such that $f(\chi) = \tau$. $\square$

By Proposition 2.8, we need only consider the faithful irreducible characters of G in determining whether or not G is a total character group.

In order to utilize Proposition 2.8 when $G = T_{4n}$, we want to know which irreducible characters of the dicyclic group of order $4n$ are faithful. We see from Tables 2.1 and 2.2 that no irreducible linear character is faithful. For the remaining characters, we have $\psi_j(1) = \psi_j(a^n) = 2$ when j is even, hence the only potentially faithful irreducible characters are the degree two characters ψ_j with j odd. In order for ψ_j to be faithful when j is odd, we must have $\psi_j(a^r) \neq 2$ for all r, $1 \leq r \leq n-1$. The following lemma gives necessary and sufficient conditions for ψ_j to be a faithful character.

Lemma 2.9. *The irreducible character ψ_j is faithful if and only if j is odd and n does not divide rj for all r, $1 \leq r \leq n-1$.*

Proof. Let ψ_j be a faithful character for some j. Then since $2 = \psi_j(1) \neq \psi_j(a^n) = 2(-1)^j$, we see that j is odd.

Now from the definition of $\omega = e^{2\pi i/2n}$,

$$\psi_j(a^r) = \omega^{rj} + \omega^{-rj} = e^{i(rj/n)\pi} + e^{i(-rj/n)\pi} = 2\cos\left(\frac{rj}{n}\pi\right).$$

Now suppose that n divides rj. Then since $\frac{rj}{n} \in \mathbb{Z}$, $\psi_j(a^{2r}) = 2\cos\left(\frac{2rj}{n}\pi\right) = 2$. But ψ_j is a faithful character, hence $a^{2r} = 1$. We now have a contradiction, since $1 \leq r \leq n-1$ and $|a| = 2n$. Thus n does not divide rj for all r, $1 \leq r \leq n-1$.

Conversely, let j be an odd integer and assume n does not divide rj for all r, $1 \leq r \leq n-1$. Then $\frac{rj}{n}$ is not an integer, so $\psi_j(a^r) = 2\cos(\frac{rj}{n}\pi) \neq 2 = \psi_j(1)$. We also see by inspection that none of $\psi_j(a^n)$, $\psi_j(b)$, or $\psi_j(ab)$ are equal to $\psi_j(1) = 2$ when j is odd. Thus $\psi_j(g) \neq \psi_j(1)$ for all conjugacy class representatives $g \neq 1$ of T_{4n}, so ψ_j is faithful. $\square$

Next, we show that for certain values of n, every faithful character χ does satisfy $f(\chi) = \tau$ for some monic polynomial $f(x) \in \mathbb{Z}[x]$. In this case, the correct polynomial is a sum of Chebyshev polynomials of the second kind.

2.4 Chebyshev polynomials of the second kind

Chebyshev polynomials of the second kind are defined recursively in [7, p. 2] by

$$U_0(x) = 1; \quad U_1(x) = 2x; \quad U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x).$$

There is also a useful equivalent trigonometric definition given in [3], namely

$$U_n(\cos(\theta)) = \frac{\sin((n+1)\theta)}{\sin(\theta)}.$$

For our proof of Theorem 2.1, we need to know the values $U_n(1), U_n(-1)$, and $U_n(0)$, which we calculate using the recursive definition. We will first show by induction on n that $U_n(1) = n+1$. From the definition,

$$U_0(1) = 1; \quad U_1(1) = 2;$$

so our base case holds. Next, assume $U_{n-1}(1) = n$ and $U_n(1) = n+1$. Then we have

$$U_{n+1}(1) = 2U_n(1) - U_{n-1}(1) = 2(n+1) - n = n+2 = (n+1)+1.$$

Therefore, by induction, we have $U_n(1) = n + 1$ for all n.

Next, we will show again by induction on n that $U_n(-1) = (n+1)(-1)^n$. For the base case, note that

$$U_0(-1) = 1 = (0+1)(-1)^0; \quad U_1(-1) = -2 = (1+1)(-1)^1.$$

Assume that $U_{n-1}(-1) = (n)(-1)^{n-1}$ and $U_n(-1) = (n+1)(-1)^n$. Then

$$\begin{aligned} U_{n+1}(-1) &= -2U_n(-1) - U_{n-1}(-1) \\ &= -2(n+1)(-1)^n - n(-1)^{n-1} \\ &= 2(n+1)(-1)^{n+1} - n(-1)^{n+1} \\ &= (n+2)(-1)^{n+1}. \end{aligned}$$

Thus, by induction, $U_n(-1) = (n+1)(-1)^n$ for all n.

The value of $U_n(0)$ depends on n, as follows. From the trigonometric definition and the identity $\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$, we have

$$U_n(0) = U_n\left(\cos\left(\frac{\pi}{2}\right)\right) = \frac{\sin\left((n+1)\frac{\pi}{2}\right)}{\sin\left(\frac{\pi}{2}\right)} = \frac{\sin\left(\frac{\pi}{2}n\right)\cos\left(\frac{\pi}{2}\right) + \cos\left(\frac{\pi}{2}n\right)\sin\left(\frac{\pi}{2}\right)}{\sin\left(\frac{\pi}{2}\right)} = \cos\left(\frac{\pi}{2}n\right).$$

Thus,

$$U_n(0) = \begin{cases} 1 & n \equiv 0 \bmod 4; \\ 0 & n \equiv 1, 3 \bmod 4; \\ -1 & n \equiv 2 \bmod 4. \end{cases}$$

We finish our discussion of Chebyshev polynomials by showing that the polynomial $F_n(x) = U_n(\frac{x}{2}) + U_{n-1}(\frac{x}{2}) + 1$ is a monic polynomial with integer coefficients. From the recursive definition of $U_n(x)$, we see that $U_n(x)$ has degree n and that the leading coefficient is 2^n. This implies that the leading coefficient of $F_n(x)$ is the leading coefficient of $U_n(\frac{x}{2})$, which is $2^n(\frac{x}{2})^n = x^n$. Therefore $F_n(x)$ is monic. Also, a simple induction argument on n

shows that $F_n(x)$ does in fact have integer coefficients for each value of n. First, note that

$$F_1(x) = U_1\left(\frac{x}{2}\right) + U_0\left(\frac{x}{2}\right) + 1 = 2\frac{x}{2} + 1 + 1 = x + 2;$$

$$F_2(x) = U_2\left(\frac{x}{2}\right) + U_1\left(\frac{x}{2}\right) + 1 = 4\left(\frac{x}{x}\right)^2 - 1 + x + 1 = x^2 + x;$$

so our base case holds. Next, assume that $F_n(x)$ and $F_{n-1}(x)$ have integer coefficients. We see that

$$\begin{aligned} F_{n+1}(x) &= U_{n+1}\left(\frac{x}{2}\right) + U_n\left(\frac{x}{2}\right) + 1 \\ &= 2\left(\frac{x}{2}\right)U_n\left(\frac{x}{2}\right) - U_{n-1}\left(\frac{x}{2}\right) + 2\left(\frac{x}{2}\right)U_{n-1}\left(\frac{x}{2}\right) - U_{n-2}\left(\frac{x}{2}\right) + 1 \\ &= x\left(U_n\left(\frac{x}{2}\right) + U_{n-1}\left(\frac{x}{2}\right)\right) - \left(U_{n-1}\left(\frac{x}{2}\right) + U_{n-2}\left(\frac{x}{2}\right)\right) + 1 \\ &= xF_n(x) - F_{n-1}(x) - x + 2, \end{aligned}$$

which has integer coefficients by the inductive hypothesis. Thus $F_n(x)$ has integer coefficients for all n, and is therefore a legitimate candidate for showing that T_{4n} is a total character group when $n \equiv 2, 3 \bmod 4$.

2.5 Proof of main result

We now have the preliminaries necessary to prove the main result of this chapter.

2.5.1 T_{4n} is a total character group for $n \equiv 2, 3 \bmod 4$.

Proof of Theorem 2.1(1). Let χ be a faithful character of T_{4n}. By Lemma 2.7, $\chi = \psi_j$ for some odd j, where n does not divide rj for $1 \leq r \leq n-1$. We summarize the values of ψ_j (j odd) and the total character τ in Tables 2.5 and 2.6.

g	1	a^n	$a^r (1 \leq r \leq n-1)$	b	ab
ψ_j, j odd	2	-2	$\omega^{rj} + \omega^{-rj}$	0	0
τ (n odd)	$2n+2$	0	$\begin{cases} 0 & r \text{ odd} \\ 2 & r \text{ even} \end{cases}$	0	0

Table 2.5: Faithful Characters and Total Character of T_{4n}, n odd

g	1	a^n	$a^r (1 \leq r \leq n-1)$	b	ab
ψ_j, j odd	2	-2	$\omega^{rj} + \omega^{-rj}$	0	0
τ (n even)	$2n+2$	2	$\begin{cases} 0 & r \text{ odd} \\ 2 & r \text{ even} \end{cases}$	0	0

Table 2.6: Faithful Characters and Total Character of T_{4n}, n even

We need to show that for $F_n(x) = U_n(\frac{x}{2}) + U_{n-1}(\frac{x}{2}) + 1$, we have $F_n(\psi_j(g)) = \tau(g)$ for each conjugacy class representative g. We begin with the case $n \equiv 3 \bmod 4$ and use the values in Table 2.5. First, consider the representative $g = 1$. Since $U_n(1) = n+1$,

$$F_n(\psi_j(1)) = F_n(2) = U_n(1) + U_{n-1}(1) + 1 = (n+1) + ((n-1)+1) + 1 = 2n+2 = \tau(1).$$

We next consider the representative $g = a^n$. From Section 2.3, $U_n(-1) = (n+1)(-1)^n$, so

$$F_n(\psi_j(a^n)) = F_n(-2) = U_n(-1) + U_{n-1}(-1) + 1 = (n+1)(-1)^n + (n)(-1)^{n-1} + 1.$$

Since n is odd, we have $F_n(\psi_j(a^n)) = -(n+1) + n + 1 = 0$. We see that $\tau(a^n) = 0$ when n is odd, so $F_n(\psi_j(a^n)) = \tau(a^n)$.

Next, we need to show that $F_n(\psi_j(a^r)) = \tau(a^r)$ for each r in the range $1 \leq r \leq n-1$. From the proof of Lemma 2.3, $\psi_j(a^r) = \omega^{rj} + \omega^{-rj} = 2\cos\left(\frac{rj}{n}\pi\right)$, so

$$F_n(\psi_j(a^r)) = F_n\left(2\cos\left(\frac{rj}{n}\pi\right)\right) = U_n\left(\cos\left(\frac{rj}{n}\pi\right)\right) + U_{n-1}\left(\cos\left(\frac{rj}{n}\pi\right)\right) + 1.$$

Using the trigonometric definition of $U_n(x)$, we have

$$F_n(\psi_j(a^r)) = \frac{\sin((n+1)\frac{rj}{n}\pi)}{\sin(\frac{rj}{n}\pi)} + \frac{\sin((n)\frac{rj}{n}\pi)}{\sin(\frac{rj}{n}\pi)} + 1,$$

and applying the law of addition of the sine function to $\frac{\sin((n+1)\frac{rj}{n}\pi)}{\sin(\frac{rj}{n}\pi)} = \frac{\sin(rj\pi+\frac{rj}{n}\pi)}{\sin(\frac{rj}{n}\pi)}$ shows that

$$F_n(\psi_j(a^r)) = \frac{\sin(rj\pi)\cos(\frac{rj}{n}\pi) + \cos(rj\pi)\sin(\frac{rj}{n}\pi)}{\sin(\frac{rj}{n}\pi)} + \frac{\sin(rj\pi)}{\sin(\frac{rj}{n}\pi)} + 1.$$

Also, we know $\frac{rj}{n} \notin \mathbb{Z}$, so $\sin(\frac{rj}{n}\pi) \neq 0$. Since $rj \in \mathbb{Z}$, we have $\sin(rj\pi) = 0$, therefore

$$F_n(\psi_j(a^r)) = \frac{\cos(rj\pi)\sin(\frac{rj}{n}\pi)}{\sin(\frac{rj}{n}\pi)} + 1,$$

and after cancellation we get $F_n(\psi_j(a^r)) = \cos(rj\pi) + 1$. Since j is odd, rj is odd when r is odd and even when r is even. We thus obtain the desired value of $F_n(\psi_j(a^r))$:

$$F_n(\psi_j(a^r)) = \begin{cases} 0 & r \quad \text{odd;} \\ 2 & r \quad \text{even;} \end{cases}$$

which is the same as $\tau(a^r)$. Thus, $F_n(\psi_j(a^r)) = \tau(a^r)$ for all r, $1 \leq r \leq n-1$.

Finally, we need to show that $F_n(\psi_j(b)) = \tau(b)$ and $F_n(\psi_j(ab)) = \tau(ab)$. We first recall the values of $U_n(0)$ that were calculated in Section 2.3:

$$U_n(0) = \begin{cases} 1 & n \equiv 0 \text{ mod } 4; \\ 0 & n \equiv 1, 3 \text{ mod } 4; \\ -1 & n \equiv 2 \text{ mod } 4. \end{cases}$$

When $n \equiv 3$ mod 4, we have $n - 1 \equiv 2$ mod 4, so

$$F_n(\psi_j(b)) = F_n(0) = U_n(0) + U_{n-1}(0) + 1 = 0 + (-1) + 1 = 0 = \tau(b).$$

Similarly, $F_n(\psi_j(ab)) = F_n(0) = 0 = \tau(ab)$. We have now shown that $F_n(\psi_j) = \tau$ for $n \equiv 3 \pmod 4$, where ψ_j is an arbitrary faithful irreducible character of T_{4n}, and $F_n(x) = U_n(\frac{x}{2}) + U_{n-1}(\frac{x}{2}) + 1$. Notice there is such a character (take $j = 1$ for example).

We next consider the case $n \equiv 2$ mod 4, which is similar to the first case. We now use the values of ψ_j and τ found in Table 2.6. For $g = 1$, we again have

$$F_n(\psi_j(1)) = U_n(1) + U_{n-1}(1) + 1 = 2n + 2 = \tau(1).$$

Next, let $g = a^n$. Since $n \equiv 2$ mod 4 instead of $n \equiv 3$ mod 4, the signs on the first two terms of the sum $F_n(\psi_j(a^n))$ switch, and we get

$$F_n(\psi_j(a^n)) = (n + 1) + (-n) + 1 = 2 = \tau(a^n).$$

We next consider $g = a^r$, $1 \le r \le n - 1$. Since the value of $\psi_j(a^r)$ is the same for n odd or n even, as is $U_n(\psi_j(a^r)) = \cos(rj\pi) + 1$, we have $F_n(\psi_j(a^r)) = \tau(a^r)$. Finally, we notice that when $n \equiv 2$ mod 4, $n - 1 \equiv 1$ mod 4, so using the values of $U_n(0)$ above we have

$$F_n(\psi_j(b)) = F_n(0) = U_n(0) + U_{n-1}(0) + 1 = -1 + 0 + 1 = 0 = \tau(b),$$

and similarly, $F_n(\psi_j(ab)) = F_n(0) = 0 = \tau(ab)$.

We have just proved that $F_n(\psi_j) = \tau$ for $n \equiv 2$ mod 4, where ψ_j is an arbitrary faithful irreducible character of T_{4n}. Since ψ_j was arbitrary in both cases, we have that $F_n(\chi) = \tau$ for all faithful irreducible characters χ of T_{4n}, $n \equiv 2, 3$ mod 4. This concludes the proof of Theorem 2.1(1). □

2.5.2 $\mathbf{T_{4n}}$ is not a total character group for $\mathbf{n \equiv 0, 1}$ mod 4. We now prove Theorem 2.1(2) in separate cases. First, we consider the case $n \equiv 0$ mod 4, and afterward the case $n \equiv 1$ mod 4. The first case can be proved using a contradiction that arises directly from the character table and total character values. However, in the proof of the case $n \equiv 1$ mod 4, we must introduce a specific character and then demonstrate a contradiction in the parity of certain inner products. Before beginning the case $n \equiv 0 \bmod 4$, we relate a pertinent result.

Proposition 2.10. *Let G be any group. Let χ be an irreducible character of G and let τ be the total character of G. If there exist $g_1, g_2 \in G$ such that $\chi(g_1) = \chi(g_2)$ but $\tau(g_1) \neq \tau(g_2)$, then there does not exist $f(x) \in \mathbb{C}[x]$ such that $f(\chi) = \tau$.*

Proof. Suppose by way of contradiction there exists $f(x) \in \mathbb{C}[x]$ such that $f(\chi) = \tau$. Then

$$\tau(g_1) = f(\chi(g_1)) = f(\chi(g_2)) = \tau(g_2),$$

a contradiction. Therefore there does not exist $f(x) \in \mathbb{C}[x]$ such that $f(\chi) = \tau$. $\square$

Now consider the case when $n \equiv 0 \bmod 4$.

Proof of Theorem 2.1(2) for $n \equiv 0$ mod 4. Let χ be an irreducible character of T_{4n}. If χ is not faithful, then by Proposition 2.8, there does not exist $f(x) \in \mathbb{Z}[x]$ such that $f(\chi) = \tau$. Therefore if we want to find χ satisfying $f(\chi) = \tau$, we must have $\chi = \psi_j$ for some odd j. To utilize Proposition 2.10, we need to find $g_1, g_2 \in T_{4n}$ with $\psi_j(g_1) = \psi_j(g_2)$ and $\tau(g_1) \neq \tau(g_2)$.

Let $g_1 = a^{\frac{n}{2}}$ and let $g_2 = b$. Since j is odd,

$$\psi_j(a^{\frac{n}{2}}) = 2\cos\left(\frac{\frac{n}{2} \cdot j}{n}\pi\right) = 2\cos\left(\frac{j}{2}\pi\right) = 0.$$

From the character table, $\psi_j(b) = 0$, so $\psi_j(a^{\frac{n}{2}}) = 0 = \psi_j(b)$. Next, note that since $n \equiv 0 \bmod 4$, $\frac{n}{2}$ is even, hence $\tau(a^{\frac{n}{2}}) = 2$. However, $\tau(b) = 0$, therefore $\tau(a^{\frac{n}{2}}) \neq \tau(b)$. By Proposition 2.8, there does not exist $f(x) \in \mathbb{Z}[x]$ with $f(\psi_j) = \tau$. Since ψ_j was an arbitrary faithful irreducible character, and non-faithful irreducible characters do not satisfy $f(\chi) = \tau$, there does not exist an irreducible character χ of T_{4n} and a polynomial $f(x) \in \mathbb{Z}[x]$ such that $f(\chi) = \tau$. Thus T_{4n} is not a total character group when $n \equiv 0 \bmod 4$. $\square$

Next, we turn to the case when $n \equiv 1 \bmod 4$. Throughout this section, we freely use the properties of the character inner product given in Sections 1.1.2 and 1.1.3.

Proof of Theorem 2.1(2) for $n \equiv 1$ mod 4. Let χ be a faithful irreducible character of T_{4n}, and suppose, by way of contradiction, that there exists $f(x) \in \mathbb{Z}[x]$ such that $f(\chi) = \tau$. Since χ is faithful, $\chi = \psi_j$ for some odd j. To obtain a contradiction, we define a special character θ and show that for any odd j, $\langle \theta, f(\psi_j)\rangle \equiv 0 \pmod 2$ but $\langle \theta, \tau\rangle \equiv 1 \pmod 2$, where $\langle \cdot, \cdot \rangle$ is the inner product defined in Section 1.1.2.

Let $\theta = \sum_{\chi \in \mathcal{S}} \chi$, where

$$\mathcal{S} = \{\chi \in \operatorname{Irr}(T_{4n}) \mid \chi \neq \chi_1 \text{ and } \chi(1) = \chi(a^n)\}.$$

From the character table of T_{4n} for n odd, we see that $\theta = \chi_3 + \sum_{j \text{ even}} \psi_j$. Then

$$\left\langle \theta, \sum_{i=1}^{4} \chi_i \right\rangle = \left\langle \chi_3 + \sum_{j \text{ even}} \psi_j, \sum_{i=1}^{4} \chi_i \right\rangle = \langle \chi_3, \chi_3 \rangle = 1$$

and

$$\left\langle \theta, \sum_{j=1}^{n-1} \psi_j \right\rangle = \left\langle \chi_3 + \sum_{j \text{ even}} \psi_j, \sum_{j=1}^{n-1} \psi_j \right\rangle = \sum_{j \text{ even}} \langle \psi_j, \psi_j \rangle.$$

Thus

$$\langle \theta, \tau \rangle = \left\langle \left(\chi_3 + \sum_{j \text{ even}} \psi_j \right), \left(\sum_{i=1}^{4} \chi_i + \sum_{j=1}^{n-1} \psi_j \right) \right\rangle = 1 + \sum_{j \text{ even}} \langle \psi_j, \psi_j \rangle.$$

Since $n \equiv 1$ mod 4 and j ranges from 1 to $n-1$, the number of characters ψ_j with j even is $\frac{n-1}{2}$, which is even. Thus $\sum_{j \text{ even}} \langle \psi_j, \psi_j \rangle = \sum_{j \text{ even}} 1 = \frac{n-1}{2} \equiv 0$ mod 2, so $\langle \theta, \tau \rangle \equiv 1$ mod 2.

We now show that $\langle \theta, f(\psi_j)\rangle \equiv 0$ mod 2. We may assume $f(x) = a_0 + a_1 x + \cdots + a_m x^m$ for some $a_i \in \mathbb{Z}$. To prove $\langle \theta, f(\psi_j)\rangle \equiv 0$ mod 2, we will show that $\langle \theta, (\psi_j)^k \rangle \equiv 0$ mod 2 for any $k \in \mathbb{Z}^+$, where we still have j odd. In order to do so, we first find the character values $\theta(g)$ for every conjugacy class representative g of T_{4n}.

From Table 2.1, we see that $\theta(1) = \theta(a^n) = 1 + 2\left(\frac{n-1}{2}\right) = n$. Similarly, $\theta(b) = \theta(ab) = -1$.

To calculate $\theta(a^r)$ for $1 \leq r \leq n-1$, we note that

$$\begin{aligned}
\theta(a^r) &= 1 + (\omega^{2r} + \omega^{-2r}) + (\omega^{4r} + \omega^{-4r}) + \cdots + (\omega^{(n-1)r} + \omega^{-(n-1)r}) \\
&= 1 + (\omega^{2r(1)} + \omega^{2r(n-1)}) + (\omega^{2r(2)} + \omega^{2r(n-2)}) + \cdots + (\omega^{2r(\frac{n-1}{2})} + \omega^{2r(n-\frac{n-1}{2})}) \\
&= \sum_{i=0}^{n-1} \omega^{2ri}.
\end{aligned}$$

We know $\omega^{2nr} - 1 = 0$, so factoring gives

$$(\omega^{2r} - 1)((\omega^{2r})^{(n-1)} + (\omega^{2r})^{(n-2)} + \cdots + \omega^{2r} + 1) = 0.$$

Since $\omega^{2r} \neq 1$, $\theta(a^r) = \sum_{i=0}^{n-1} \omega^{2ri} = 0$. The character values for θ are listed in Table 2.7, as well as the values of ψ_j when j is odd, for reference as we calculate the inner product:

g	1	a^n	$a^r(1 \leq r \leq n-1)$	b	ab
$\lvert C_G(g)\rvert$	$4n$	$4n$	$2n$	4	4
θ	n	n	0	-1	-1
ψ_j (j odd)	2	-2	$\omega^{rj} + \omega^{-rj}$	0	0

Table 2.7: Character Values for θ and ψ_j, j odd

Now we show that $\langle \theta, (\psi_j)^k \rangle \equiv 0 \pmod 2$ for any $k \in \mathbb{Z}^+$ and any j odd. If k is odd, then from the inner product formula we have

$$\langle \theta, (\psi_j)^k \rangle = \frac{n(2^k)}{4n} + \frac{n(-2)^k}{4n} + \sum_{j=1}^{n-1} \frac{0 \cdot (\omega^{rj} + \omega^{-rj})^k}{2n} + \frac{-1(0)^k}{4} + \frac{-1(0)^k}{4}.$$

Since k is odd,

$$\langle \theta, (\psi_j)^k \rangle = \frac{n(2^k)}{4n} + \frac{-n(2^k)}{4n} = 0.$$

When k is even and $k > 0$, we have

$$\langle \theta, (\psi_j)^k \rangle = \frac{n(2^k)}{4n} + \frac{n(2^k)}{4n} + 0 + 0 + 0 = \frac{2n(2^k)}{4n} = 2^{k-1} \equiv 0 \bmod 2.$$

For the case $k = 0$ we see that

$$\langle \theta, (\psi_j)^k \rangle = \langle \theta, \chi_1 \rangle = \frac{n(1)}{4n} + \frac{n(1)}{4n} + 0 + \frac{-1(1)}{4} + \frac{-1(1)}{4} = \frac{2n}{4n} - \frac{1}{2} = 0.$$

Thus $\langle \theta, (\psi_j)^k \rangle \equiv 0 \bmod 2$ for all $k \in \mathbb{Z}^+$. But we have

$$\langle \theta, f(\psi_j) \rangle = \left\langle \theta, \sum_{k=0}^{m} a_k (\psi_j)^k \right\rangle = \sum_{k=0}^{m} \langle \theta, a_k (\psi_j)^k \rangle = \sum_{k=0}^{m} \overline{a_k} \langle \theta, (\psi_j)^k \rangle \equiv \sum_{k=0}^{m} \overline{a_k}(0) \equiv 0 \bmod 2.$$

However, $\langle \theta, \tau \rangle \equiv 1 \pmod 2$. Since $f(\chi) = \tau$, this gives the required contradiction. Thus T_{4n} is not a total character group when $n \equiv 1 \bmod 4$. □

In [4], Cottrell and Poimenidou noted that all of the characters that worked for the dihedral group D_{2n}, n odd, were indeed sharp characters, and posed the following question (Question 4.1 in [4, p. 13]):

Question. *If χ is an irreducible character of a finite group G such that for some monic polynomial $g(x) \in \mathbb{Z}[x]$ we have $g(x) = \tau$, is χ necessarily sharp?*

We provide a negative answer to this question. Consider T_{12}, the dicyclic group of order 12. The character table and the total character of T_{12} are shown in Table 2.8.

g_i	1	a^3	a	a^2	b	ab
$\lvert C_G(g_i)\rvert$	12	12	6	6	4	4
χ_1	1	1	1	1	1	1
χ_2	1	-1	-1	1	i	$-i$
χ_3	1	1	1	1	-1	-1
χ_4	1	-1	-1	1	$-i$	i
ψ_1	2	-2	1	-1	0	0
ψ_2	2	2	-1	-1	0	0
τ	8	0	0	2	0	0

Table 2.8: Character Table and Total Character of T_{12}

The polynomial $F_3(x) = x^3 + x^2 - 2x = x(x+2)(x-1)$ does satisfy $F_3(\psi_1) = \tau$ for ψ_1, the only faithful irreducible character of T_{12}. In the notation of character sharpness given in

Section 1.2, we have $L = \{\psi_1(g) \mid g \neq 1\} = \{-2, 1, -1, 0\}$, so

$$f_L(x) = (x+2)(x-1)(x+1)x.$$

However, since ψ_1 is degree 2, we have $f_L(2) = 4(1)(3)(2) = 24 \neq 12$, and therefore, ψ_1 is not sharp. Thus a character must not necessarily be sharp in order to satisfy $f(\chi) = \tau$.

Chapter 3. Non-Abelian Groups of Order p^3

3.1 Background

There are only two isomorphism types for non-abelian groups of order p^3, with p an odd prime. They have the following presentations [15, p. 304]:

$$H_1 = \langle a, b \mid a^{p^2} = b^p = 1, b^{-1}ab = a^{p+1} \rangle, \text{ and}$$

$$H_2 = \langle a, b, z \mid a^p = b^p = z^p = 1, az = za, bz = zb, b^{-1}ab = az \rangle.$$

Throughout this chapter, let G be a non-abelian group of order p^3 for some odd prime p, and let $\tau = \tau_G$ be the total character of G. We prove that G is not a total character group.

3.2 Irreducible characters and the total character

We first give formulas for the irreducible characters of G, which can be found in [15, p. 302]. Every element of G (in both cases) is of the form

$$a^r b^s z^t,$$

for some $z \in Z(G)$ and r, s, t with $0 \leq r, s, t \leq p-1$. Let $\epsilon = e^{2\pi i/p}$ be a primitive p^{th} root of unity. Then the irreducible characters of G are

$$\chi_{u,v} \quad (0 \leq u \leq p-1, 0 \leq v \leq p-1),$$

$$\phi_u \quad (1 \leq u \leq p-1),$$

where for all r, s, t,

$$\chi_{u,v}(a^r b^s z^t) = \epsilon^{ru+sv},$$

$$\phi_u(a^r b^s z^t) = \begin{cases} p\epsilon^{ut}, & \text{if } r = s = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Next, we calculate τ. By summing over rows, we have

$$\tau(a^r b^s z^t) = \sum_{u=0}^{p-1} \sum_{v=0}^{p-1} \chi_{u,v}(a^r b^s z^t) + \sum_{u=1}^{p-1} \phi_u(a^r b^s z^t).$$

When $r = s = 0$,

$$\begin{aligned} \tau(a^r b^s z^t) &= \sum_{u=0}^{p-1} \sum_{v=0}^{p-1} \epsilon^{ru+sv} + \sum_{u=1}^{p-1} p\epsilon^{ut} \\ &= \sum_{u=0}^{p-1} \sum_{v=0}^{p-1} 1 + p \sum_{u=1}^{p-1} \epsilon^{ut} \\ &= p^2 + p \sum_{u=1}^{p-1} \epsilon^{ut}. \end{aligned}$$

When in addition $t = 0$, then $a^r b^s z^t = 1_G$ is the identity of G, and

$$\tau(a^r b^s z^t) = \tau(1_G) = p^2 + p \sum_{u=1}^{p-1} 1 = p^2 + p(p-1) = 2p^2 - p.$$

If $0 < t \leq p - 1$, then ϵ^t is a primitive p^{th} root of unity, hence

$$\tau(a^r b^s z^t) = \tau(z^t) = p^2 + p \sum_{u=1}^{p-1} (\epsilon^t)^u = p^2 + p(-1) = p^2 - p.$$

Finally, when $r \neq 0$ or $s \neq 0$, then either ϵ^r or ϵ^s is a primitive p^{th} root of unity, so

$$\begin{aligned} \tau(a^r b^s z^t) &= \sum_{u=0}^{p-1} \sum_{v=0}^{p-1} \epsilon^{ru+sv} + \sum_{u=1}^{p-1} 0 \\ &= \sum_{u=0}^{p-1} (\epsilon^r)^u \sum_{v=0}^{p-1} (\epsilon^s)^v \\ &= 0. \end{aligned}$$

We now have the total character of G:

$$\tau(a^r b^s z^t) = \begin{cases} 2p^2 - p, & \text{if } r = s = t = 0, \\ p^2 - p, & \text{if } r = s = 0,\ t \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

We now present our main result concerning groups of order p^3 with p an odd prime.

3.3 Proof of main result

Theorem 3.1. *Let G be a non-abelian group of order p^3 for some odd prime p. Then G is not a total character group.*

Proof. If χ is an irreducible character satisfying $f(\chi) = \tau$ for some $f(x) \in \mathbb{Z}[x]$, we know χ must be faithful (Proposition 2.8). Since $\chi_{u,v}$ is linear, it is not faithful. This follows from the fact that linear characters are homomorphisms $\lambda : G \to \mathbb{C}^\times$, and since $\lambda(G)$ is a subgroup of the abelian group $\mathbb{C}^\times$, it is abelian. Since G is non-abelian, λ is not injective, in other words, not faithful. Thus the only characters that can satisfy $f(\chi) = \tau$ for some $f(x) \in \mathbb{Z}[x]$ are the characters $\chi = \phi_u$. We show this is not the case, which implies that G is not a total character group.

Let ϕ_u be such a character. Suppose by way of contradiction there exists $f(x) \in \mathbb{Z}[x]$ with $f(\phi_u) = \tau$. Note that for any $\alpha \in G$, the value $\phi_u(\alpha)$ is a root of $\prod_{t=0}^{p-1}(x - p\epsilon^{ut})$. Consider the polynomial $g(x) \in \mathbb{Z}[x]$ defined by

$$g(x) = f(x) - \left(\prod_{t=0}^{p-1} (x - p\epsilon^{ut}) \right) k(x),$$

where $k(x) \in \mathbb{Z}[x]$ is chosen so that $\deg(g(x)) \leq p - 1$, using the Euclidean algorithm and noting that $\prod_{t=0}^{p-1}(x - p\epsilon^{ut})$ is a polynomial of degree p. Note that since $\prod_{t=0}^{p-1}(x - p\epsilon^{ut})$ is monic and has integer coefficients, $g(x) \in \mathbb{Z}[x]$ as well. Now since $f(p\epsilon^{ut_0}) = f(\phi_u(z^{t_0})) = \tau(z^{t_0})$, for any $0 \leq t_0 \leq p-1$, we have

$$g(p\epsilon^{ut_0}) = f(p\epsilon^{ut_0}) - \left(\prod_{t=0}^{p-1}(p\epsilon^{ut_0} - p\epsilon^{ut})\right) k(p\epsilon^{ut_0}) = f(p\epsilon^{ut_0}) = \tau(z^{t_0}).$$

Now let $h(x) = g(x) - (p^2 - p)$. Since $g(x) \in \mathbb{Z}[x]$, we have $h(x) \in \mathbb{Z}[x]$. Then $\deg(h(x)) = \deg(g(x)) \leq p-1$ and for each t in the range $1 \leq t \leq p-1$ we have

$$h(p\epsilon^{ut}) = g(p\epsilon^{ut}) - (p^2 - p) = \tau(z^t) - (p^2 - p) = 0.$$

This implies that $p\epsilon^{ut}$ is a root of $h(x)$ for $1 \leq t \leq p-1$, therefore $\prod_{t=1}^{p-1}(x - p\epsilon^{ut})$ divides $h(x)$. Since the degree of $h(x)$ is at most $p-1$ and each $p\epsilon^{ut}$ is a distinct root for $1 \leq t \leq p-1$, this implies that the $\deg(h(x)) = p-1$, and

$$h(x) = c\prod_{t=1}^{p-1}(x - p\epsilon^{ut})$$

for some constant c. Since $h(x) \in \mathbb{Z}[x]$, we must have $c \in \mathbb{Z}$. Now consider that

$$h(p) = h(p\epsilon^0) = g(p\epsilon^0) - (p^2 - p) = \tau(z^0) - (p^2 - p) = (2p^2 - p) - (p^2 - p) = p^2,$$

but we also have

$$h(p) = c\prod_{t=1}^{p-1}(p - p\epsilon^{ut}) = cp^{p-1}\prod_{t=1}^{p-1}(1 - (\epsilon^u)^t).$$

By [16, p. 9], $\prod_{t=1}^{p-1}(1 - (\epsilon^u)^t) = p$, hence

$$h(p) = cp^{p-1}\prod_{t=1}^{p-1}(1 - (\epsilon^u)^t) = cp^{p-1}p = cp^p.$$

Thus $c = \frac{p^2}{p^p} \in \mathbb{Z}$ for some odd prime p, which is a contradiction, so there does not exist $f(x) \in \mathbb{Z}[x]$ such that $f(\phi_u) = \tau$. Since u was arbitrary, for all $\chi \in \mathrm{Irr}(G)$ there does not exist $f(x) \in \mathbb{Z}[x]$ with $f(\chi) = \tau$. Therefore G is not a total character group. $\square$

Chapter 4. The Symmetric Group on n Elements

4.1 Background

The next group we examine is the symmetric group on n elements, S_n. The symmetric group is intriguing to many mathematicians not only for its structural beauty, but also because it appears in many facets of algebra.

We first recall some basic facts about S_n. From [6, p. 29] we learn that every element g of S_n can be written uniquely as a product of disjoint cycles of lengths $n_1, n_2, \ldots, n_r$, and without loss of generality we may assume $n_1 \geq n_2 \geq \cdots \geq n_r \geq 1$. Since the cycles are disjoint, $\sum_{i=1}^{r} n_i = n$. The r-tuple $(n_1, n_2, \ldots, n_r)$ is called the *cycle type* of g. The conjugacy classes of S_n are determined by cycle type: for all $g, h \in S_n$, g and h are conjugate if and only if they have the same cycle type [6, p. 126].

One final preliminary from the representation theory of the symmetric group we take note of is that all the irreducible characters of S_n are integer-valued [19, p. 87]. As a result, we can consider the character values modulo 2. Throughout this chapter, let $\tau = \tau_n$ be the total character of the symmetric group on n elements.

4.2 Symmetric group result

In this chapter, we prove that excepting $n = 1, 2, 3$, the symmetric group S_n is not a total character group. Furthermore, we can exclude the monic condition on $f(x)$ and still retain a negative answer for $n \geq 4$. This result is stated in the following theorem.

Theorem 4.1. *If $n \geq 4$, S_n is not a total character group.*

The proof consists of looking at the character table of S_n modulo 2 and showing that $\tau \not\equiv f(\chi) \bmod 2$ for any polynomial $f(x) \in \mathbb{Z}[x]$ and $\chi \in \mathrm{Irr}(S_n)$. To do this we make use of the Murnaghan-Nakayama Rule.

4.3 The character table of S_n

The complexity and beauty of the character table of S_n were first unfolded by the mathematician Ferdinand Georg Frobenius, who discovered many of the fundamental properties of the representation theory of the symmetric group. As one of his many results in this area, Frobenius produced a complex formula that can be used to calculate individual values within the character table. Both Frobenius' formula and an equivalent formula that we will use, called the Murnaghan-Nakayama Rule, will be discussed more fully in later sections. Before considering individual character values, we relate some facts pertaining to the character table in general.

4.3.1 Conjugacy classes and irreducible characters associated with partitions.

In this section, we highlight relevant aspects of the conjugacy classes and irreducible characters of S_n. From the work of Frobenius and many subsequent mathematicians, we know that the character table of S_n is closely related to partitions of n.

Definition 4.2. Let n be a positive integer. A *partition* λ of n, notated $\lambda \vdash n$, is an r-tuple $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ of integers with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0$ such that $\sum_{i=1}^{r} \lambda_i = n$.

We purposefully use the same notation for both a partition of n and the cycle type of an element g in S_n, namely the parenthetical notation $(\lambda_1, \lambda_2, \ldots, \lambda_r)$ or $(n_1, n_2, \ldots, n_r)$. This is a natural consequence of the fact that cycle types of elements of S_n are in a one-to-one correspondence with partitions of n. Since conjugacy classes are determined by cycle type, we can index the conjugacy classes of S_n by partitions of n.

The character table of the symmetric group on n elements has the surprising property that not only the conjugacy classes, but also the irreducible characters can be indexed by partitions of n in a natural way. This correspondence was first established by Frobenius, and is obtained in the following way [9, p. 45-46]:

We first represent a partition of n visually by a diagram of boxes called a *Young diagram.* If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)$ is a partition of n, then the Young diagram associated with λ has

λ_i boxes in the i^{th} row, with the rows of boxes aligned on the left. We denote the Young diagram associated with the partition λ by $[\lambda]$. The Young diagram $[\lambda]$ is said to be of *shape* λ. Since a partition is nonincreasing by definition, no row or column of $[\lambda]$ has more boxes than the preceding row or column. For example, if $n = 8$ and $\lambda = (3, 2, 2, 1)$, then the associated Young diagram is

$$[\lambda] = \begin{array}{|c|c|c|} \hline & & \\ \hline & & \multicolumn{1}{c}{} \\ \cline{1-2} & & \multicolumn{1}{c}{} \\ \cline{1-2} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

where $[\lambda]$ has shape $\lambda = (3, 2, 2, 1)$. A *Young tableaux* is a Young diagram that has the numbers $1, 2, \ldots, n$ placed in the n boxes, with the standard tableaux having the numbers in an increasing sequence in each row and column, as follows:

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & \multicolumn{1}{c}{} \\ \cline{1-2} 6 & 7 & \multicolumn{1}{c}{} \\ \cline{1-2} 8 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

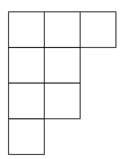
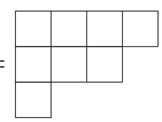
For each Young tableaux, there exists a subgroup of S_n that permutes the elements in each row of the tableaux among themselves. In our example above, it would be the subgroup that permutes $\{1, 2, 3\}$, $\{4, 5\}$, $\{6, 7\}$, and $\{8\}$ among themselves, which would be isomorphic to $S_3 \times S_2 \times S_2 \times S_1$. It is outside of the scope of this thesis, but this subgroup and a related subgroup obtained from the columns can be manipulated to obtain a representation ρ_λ of S_n, from which the irreducible character χ_λ associated with λ can be determined [9, p. 45-46].

Since χ_λ can be constructed from the Young tableaux $[\lambda]$, the character values of χ_λ and the shape of $[\lambda]$ have a strong correlation. As a result, there are formulae that use properties of $[\lambda]$ to get information about the character values of χ_λ. The Frobenius formula and Murnaghan-Nakayama rule mentioned previously are two such formulae that calculate $\chi_\lambda(g)$ for $g \in S_n$ using the shape of $[\lambda]$ (see Section 4.5).

4.3.2 Young diagrams. In this section, we introduce some terminology associated with Young diagrams and their corresponding partitions. The specific types of Young diagrams that we mention are particularly useful because of the relationship between χ_λ and $[\lambda]$ outlined in the previous section.

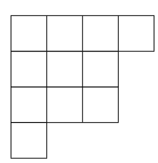
If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t)$ is a partition of n, we define $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_t)$ to be the *conjugate partition* to λ, where λ'_i is the number of λ_j's in $(\lambda_1, \lambda_2, \ldots, \lambda_t)$ such that $\lambda_j \geq i$. The conjugate partition of our example $\lambda = (3, 2, 2, 1)$ above is $\lambda' = (4, 3, 1)$.

Conjugate partitions correspond to conjugate Young diagrams $[\lambda]$ and $[\lambda']$, where $[\lambda']$ can be produced by interchanging the rows and columns of $[\lambda]$. Equivalently, $[\lambda']$ is the Young diagram that results from reflection across the main diagonal of $[\lambda]$. Below are the Young diagrams $[\lambda]$ and $[\lambda']$ associated with $\lambda = (3, 2, 2, 1)$ and $\lambda' = (4, 3, 1)$:

$[\lambda] =$  $[\lambda'] =$ 

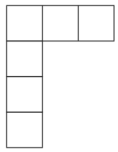
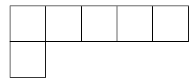
The irreducible characters χ_λ and $\chi_{\lambda'}$ associated with conjugate partitions have an important relationship that will be explored in our proof of Theorem 4.1.

Conjugate partitions need not be distinct from one another; for example, the conjugate partition of $\lambda = (4, 3, 3, 1)$ is $\lambda' = (4, 3, 3, 1)$:

$[\lambda] = [\lambda'] =$ 

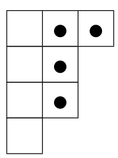
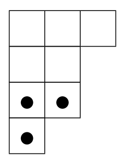
When λ has the property that $\lambda = \lambda'$, then λ is called a *symmetric partition.* If λ is a symmetric partition, we say that the corresponding Young diagram $[\lambda]$ is a *symmetric Young diagram.* When $[\lambda]$ is a symmetric Young diagram, then $[\lambda] = [\lambda']$, so $[\lambda]$ has reflectional symmetry along the main diagonal. This symmetry can be seen in the example $\lambda = (4, 3, 3, 1)$ above.

In addition to Young diagrams, we are also interested in hooks. A *hook* is a Young diagram of the form $\lambda = (n - \ell, 1, 1, \ldots, 1)$, where there are ℓ 1's. For example, if $n = 6$, then $\lambda_1 = (3, 1, 1, 1)$ and $\lambda_2 = (5, 1)$ are both hooks. The corresponding Young diagrams $[\lambda_1]$ and $[\lambda_2]$ are pictured below.

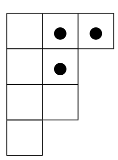
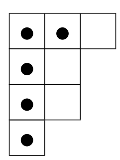
$[\lambda_1] =$  $[\lambda_2] =$ 

The box in the upper left-most corner is called the box where the hook *originates*. The *length* of the hook is the number of boxes in the hook.

The term "hook" is also frequently used to describe a collection of boxes within a Young diagram that would form a hook Young diagram if taken by themselves; in addition, the hook within the diagram must extend to the right and bottom edge of the Young diagram it is within. For instance, the bulleted boxes in the diagrams below are hooks:

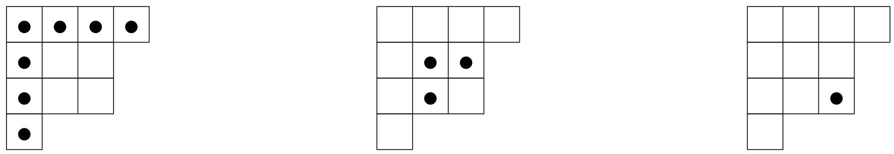
 

whereas the bulleted boxes in the following diagrams are not hooks:

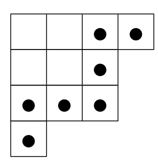
Whether the term "hook" references a Young diagram that is a hook, or simply a hook within a Young diagram, should be clear from context. If there is danger of confusion, we will refer to a Young diagram that is a hook as a "hook diagram," and to the other as a "hook within the diagram."

A *diagonal hook* is a hook within a symmetric Young diagram that originates at one of the boxes on the main diagonal. The bulleted boxes below are the diagonal hooks within the symmetric Young diagram $[\lambda]$ that has shape $\lambda = (4, 3, 3, 1)$:

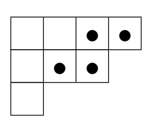


Note that a single box may form a hook, as in the example above to the far right. Unlike the bulleted boxes directly above, the bulleted boxes in the diagram below are not considered to be diagonal hooks because (1) they are within a non-symmetric Young diagram, and (2) they are not centered on the main diagonal:

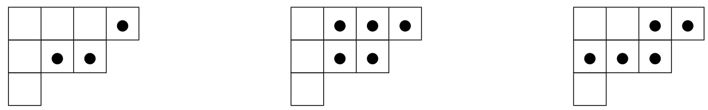
For our next definition, we first note that the *boundary boxes* of the Young diagram $[\lambda]$ are the boxes with an edge or corner exposed on the right side of the diagram, as shown below:



A *skew hook* within a Young diagram $[\lambda]$ is a connected subset of boundary boxes with the property that removing them leaves a smaller Young diagram. The *length* of a skew hook is the number of boxes in the skew hook. For example, the bulleted boxes in the diagram below form a skew hook:



On the other hand, the bulleted boxes in the following diagrams do not form a skew hook. On the left, the bulleted boxes are not connected; in the center, the bulleted boxes are not all boundary boxes; on the right, removing the bulleted boxes does not leave a Young diagram.



There is a one-to-one correspondance between skew hooks and ordinary hooks of the same size within the diagram: a skew hook corresponds to the regular hook with the same end boxes, and vice versa, as can be seen in the bulleted boxes below.

Skew Hook Corresponding Ordinary Hook

We are interested in skew hooks because they are a key component in the application of the Murnaghan-Nakayama rule.

If h is a skew hook or an ordinary hook within $[\lambda]$, a symmetric Young diagram, then the *conjugate of* h is h', the skew hook or ordinary hook produced by reflecting the boxes of h across a line drawn down the main diagonal of $[\lambda]$. The following is an example of a hook h and its conjugate hook h':

Hooks, skew hooks, diagonal hooks, and conjugate hooks each play an important role in the proof of Theorem 4.1.

We need one final definition. The *hook length* of a box within a Young diagram is the number of boxes in the hook originating at that box. In other words, the hook length of the box is the number of boxes to the right and below the given box, including the box itself. The Young diagram $[\lambda]$ having shape $\lambda = (3, 2, 2, 1)$ is pictured below with each hook length written in its respective box:

6	4	1
4	2	
3	1	
1		

In our proof, we will focus particularly on the hook lengths of the diagonal hooks within a symmetric Young diagram. For simplicity we will refer to the set of hook lengths of the diagonal hooks as the *diagonal hook lengths*. The diagonal hook lengths of a symmetric Young diagram form a special kind of partition of n that will be used to determine the value of the total character on elements of S_n.

4.4 Formulae for calculating irreducible characters

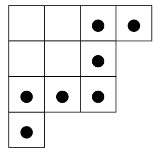
We will use the components of Young diagrams defined in the previous section as we next examine individual values of the irreducible character χ_λ. Frobenius produced a classic formula for calculating the value of any irreducible character on any specific elements of S_n. An alternate formula called the Murnaghan-Nakayama rule is more useful for our purposes and will therefore be utilized in lieu of Frobenius' formula. The Murnaghan-Nakayama rule is given in [9, p. 59], and we restate it below as Theorem 4.3.

Theorem 4.3. (Murnaghan-Nakayama Rule) *If λ is a partition of n and $g \in S_n$ is written as a product of an m-cycle and a disjoint permutation $h \in S_{n-m}$, then*

$$\chi_\lambda(g) = \sum_{\mu} (-1)^{r(\mu)} \chi_\mu(h),$$

where the sum is over all partitions μ that are obtained from $[\lambda]$ by removing a skew hook of length m, and $r(\mu)$ is the number of vertical steps in the skew hook.

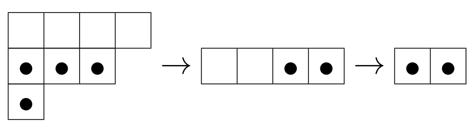
Here, *vertical step* is taken to mean the number of movements from one box to the box directly above it within the skew hook. For example, the following skew hook has three vertical steps:

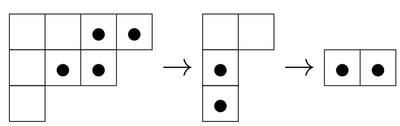


The Murnaghan-Nakayama rule can be restated inductively as follows: If g is written as a product of disjoint cycles of lengths $m_1, m_2, \ldots, m_t$, with the lengths m_i written in any order, then

$$\chi_\lambda(g) = \sum_s (-1)^{r(s)}, \tag{4.1}$$

where the summation is taken over all ways s to decompose $[\lambda]$ by successively removing t skew hooks of lengths $m_1, m_2, \ldots, m_t$ in the given order, and $r(s)$ is the total number of vertical steps in the hooks of s. For example, in S_8 let $\lambda = (4, 3, 1)$ and let $g = (1234)(56)(78)$, and fix the cycle lengths in the order $m_1 = 4, m_2 = 2, m_3 = 2$. Then we have the following decompositions of $[\lambda]$, where the bulleted boxes show the skew hook that is removed to produce the diagram in the next step:

s_1 : 

s_2 : 

These are the only two decompositions of $[\lambda]$ by successively removing three skew hooks of lengths 4, 2, and 2, in that order. There is one vertical step in the first skew hook of s_1 and no vertical step in each of the following skew hooks, which implies $r(s_1) = 1 + 0 + 0 = 1$. There is one vertical step in each of the first two skew hooks of s_2 and none in the third, giving $r(s_2) = 1 + 1 + 0 = 2$. Therefore by (4.1),

$$\chi_\lambda(g) = \sum_s (-1)^{r(s)} = (-1)^{r(s_1)} + (-1)^{r(s_2)} = (-1)^1 + (-1)^2 = 0.$$

Throughout this chapter, we will use the inductive statement of the Murnaghan-Nakayama rule, namely (4.1), instead of the formal statement given in Theorem 4.3. In addition to calculating actual character values, we frequently employ (4.1) to calculate parity.

Corollary 4.4. *Let χ_λ be the irreducible character of S_n associated to the partition λ, and let $g \in S_n$ have cycle type $(m_1, m_2, \ldots, m_t)$. Then*

$$\chi_\lambda(g) \equiv |s| \ mod\ 2,$$

where $|s|$ is the number of ways to decompose $[\lambda]$ by successively removing skew hooks of lengths $m_1, m_2, \ldots, m_t$.

Proof. By the Murnaghan-Nakayama rule,

$$\chi_\lambda(g) = \sum_s (-1)^{r(s)},$$

where the sum is taken over all ways s of decomposing $[\lambda]$ by successively removing skew hooks of lengths $m_1, m_2, \ldots, m_t$, and $r(s)$ is the number of vertical steps in the hooks of s. If there are an even number of decompositions, then we are summing an even number of 1's and -1's, so

$$\chi_\lambda(g) \equiv 0 \text{ mod } 2 \equiv |s| \text{ mod } 2.$$

Similar reasoning holds if $|s|$ is odd. Therefore $\chi_\lambda(g) \equiv |s|$ mod 2 for all χ_λ. □

Corollary 4.4 is particularly useful because of the simple way in which it can be applied when $[\lambda]$ is symmetric. Symmetric Young diagrams are fundamental to understanding the total character of S_n, as will be seen in the next section.

4.5 The total character of S_n

In this section, we calculate the parity of τ, or τ mod 2. For notational simplicity, we will refer to τ mod 2 as $\hat{\tau}$. We will find $\hat{\tau}$ using the Murnaghan-Nakayama rule introduced in the previous section.

4.5.1 Conjugate partitions and the total character. We first explore the relationship between $\hat{\tau}$ and the irreducible characters of S_n corresponding to symmetric partitions. If χ_λ is an irreducible character of S_n and χ_a is the alternating, or sign, character defined by $\chi_a(g) = \operatorname{sgn}(g)$, then $\chi_\lambda \cdot \chi_a$ is also an irreducible character. This can be shown easily using the inner product formula (see Section 1.1.2). Additionally, if $[\lambda]$ is the Young diagram associated with some irreducible character χ_λ, then the Young diagram associated with the irreducible character $\chi_\lambda \cdot \chi_a$ is $[\lambda']$, the conjugate partition [9, p. 47]. In other words, $\chi_{\lambda'} = \chi_\lambda \cdot \chi_a$. This implies that for each $g \in S_n$, $\chi_\lambda(g) + \chi_{\lambda'}(g)$ is either $2\chi_\lambda(g)$ or 0.

When λ is not symmetric, $\lambda \neq \lambda'$, so χ_λ and $\chi_{\lambda'}$ are distinct. In this case, $\chi_\lambda + \chi_{\lambda'} \equiv 0 \bmod 2$ and the characters χ_λ, $\chi_{\lambda'}$ do not contribute to $\hat{\tau}$. This implies that $\hat{\tau}$ depends entirely on the parity of the irreducible characters associated with symmetric partitions. If λ is a symmetric partition, then $\lambda = \lambda'$, so $\chi_\lambda = \chi'_\lambda$ and this cancellation does not happen.

We summarize this information in the following lemma:

Lemma 4.5. *Let τ be the total character of S_n. Then*

$$\tau(g) \equiv \sum_{\lambda:\ \lambda=\lambda'} \chi_\lambda(g) \ \mathit{mod}\ 2.$$

Here the sum is over all partitions λ where $\lambda = \lambda'$.

Proof. Let $\mathcal{S}$ be the set of all partitions of n. Let $\mathcal{T} = \{\lambda : \lambda \neq \lambda'\}$, the set of all non-symmetric partitions of n. Finally, let $\mathcal{U}$ be a subset of T containing exactly one element of each pair $\{\lambda, \lambda'\}$. The total character can be broken up as follows:

$$\begin{aligned}
\tau(g) &= \sum_{\lambda \in \mathcal{S}} \chi_\lambda(g) \\
&= \sum_{\lambda=\lambda'} \chi_\lambda(g) + \sum_{\lambda \in \mathcal{U}} (\chi_\lambda(g) + \chi_{\lambda'}(g)) \\
&= \sum_{\lambda=\lambda'} \chi_\lambda(g) + \sum_{\lambda \in \mathcal{U}} (1 + \chi_a(g))(\chi_\lambda(g)).
\end{aligned}$$

Since $\chi_a(g) = \pm 1$ for all $g \in S_n$, we have $(1 + \chi_a(g))(\chi_\lambda(g)) \equiv 0 \bmod 2$. This implies that $\tau(g) \equiv \sum_{\lambda=\lambda'} \chi_\lambda(g) \bmod 2$, as desired. □

By Lemma 4.5, we only need to consider the irreducible characters associated with symmetric partitions when calculating $\hat{\tau}$.

4.5.2 Formula for $\hat{\tau}$. In order to calculate $\hat{\tau}$, we need to find the value of $\chi_\lambda(g) \bmod 2$ for λ a symmetric partition and $g \in S_n$. The value of $\chi_\lambda(g) \bmod 2$ depends on not only the cycle type of g, but also on the diagonal hook lengths of $[\lambda]$. Note that if $d_1, d_2, \ldots, d_r$ are the diagonal hook lengths of a symmetric Young diagram $[\lambda]$, then without loss of generality we may assume $d_1 \geq d_2 \geq \cdots \geq d_r$. Since each box of a symmetric diagram $[\lambda]$ is contained in exactly one diagonal hook, the union of the diagonal hooks is the entire Young diagram, with no overlapping. Therefore the diagonal hook lengths $(d_1, d_2, \ldots, d_r)$ form a partition of n, and hence are the cycle type for some element $g \in S_n$.

Lemma 4.6. *Let λ be a symmetric partition of n, and let $(d_1, d_2, \ldots, d_r)$ be the diagonal hook lengths of $[\lambda]$. Let $g \in S_n$ have cycle type $(m_1, m_2, \ldots, m_t)$. Then $\chi_\lambda(g) \equiv 1 \bmod 2$ if and only if $t = r$ and $(m_1, m_2, \ldots, m_r) = (d_1, d_2, \ldots, d_r)$.*

Proof. First suppose $g \in S_n$ has cycle type $(d_1, d_2, \ldots, d_r)$. In preparation to apply Corollary 4.4, we fix the cycle lengths of g in the order $d_1, d_2, \ldots, d_r$. We will show that there is only one way to decompose $[\lambda]$ by successively removing r skew hooks of lengths $d_1, d_2, \ldots, d_r$, then apply Corollary 4.4 to get the parity of $\chi_\lambda(g)$.

Our proof is by induction on r. If $r = 1$, then $[\lambda]$ is a hook, so we can indeed remove a skew hook of length d_1, namely $[\lambda]$ itself. Since one step decomposes the entire Young diagram, there is only one way to decompose $[\lambda]$ by removing a skew hook of length d_1.

Now assume that when $k < r$ and $c_1 \geq c_2 \geq \cdots \geq c_k$, there is a unique way to decompose $[\lambda_c]$, the symmetric Young diagram with diagonal hook lengths $c = (c_1, c_2, \ldots, c_k)$, by successively removing k skew hooks of lengths $c_1, c_2, \ldots, c_k$. Let $[\lambda]$ be a symmetric diagram with diagonal hook lengths $(d_1, \ldots, d_r)$, and let $g \in S_n$ have cycle type $(d_1, \ldots, d_r)$. Since d_1 is the length of the hook originating at the box in the upper left corner of the Young diagram, the skew hook corresponding to the diagonal hook of length d_1 is precisely the skew hook consisting of all of the boundary boxes of $[\lambda]$. See an example below for the symmetric partition $\lambda = (4, 4, 3, 2)$:

This is the only skew hook of length d_1, therefore there is only one way to remove a skew hook of length d_1 from $[\lambda]$. After removing that skew hook, we are left with a Young diagram with diagonal hook lengths $(d_2, d_3, \ldots, d_r)$. Since there are $r - 1 < r$ diagonal hooks in this smaller Young diagram, by induction there is only one way to remove skew hooks of lengths $d_2, d_3, \ldots, d_r$ from the remaining boxes of the Young diagram. Thus there is only one way to successively remove r skew hooks of lengths $d_1, d_2, \ldots, d_r$ to decompose $[\lambda]$, so $s = 1$. By Corollary 4.4, $\chi_\lambda(g) \equiv 1 \bmod 2$. It now remains to prove the converse.

Let λ be a symmetric partition with diagonal hook lengths $(d_1, \ldots, d_r)$, and let $g \in S_n$ have cycle type $(m_1, m_2, \ldots m_t)$ with $(m_1, \ldots, m_t) \neq (d_1, \ldots, d_r)$. We will show that there are an even number of ways to decompose $[\lambda]$ by successively removing skew hooks of the appropriate lengths, and again the result will follow by Corollary 4.4.

If s is the decomposition of $[\lambda]$ given by successively removing skew hooks $h_1, h_2, \ldots, h_t$, then s', the decomposition given by removing the conjugate skew hooks $h'_1, h'_2, \ldots, h'_t$, is also a complete decomposition of $[\lambda]$. Clearly, $s = s'$ if and only if $h_1 = h'_1, h_2 = h'_2, \ldots, h_t = h'_t$. But since $[\lambda]$ is symmetric, the only hooks h with $h = h'$ are the diagonal hooks of $[\lambda]$. Therefore a complete decomposition of $[\lambda]$ that uses only self-conjugate skew-hooks would require the removal of only skew hooks corresponding to diagonal hooks. Since $(m_1, m_2, \ldots, m_t) \neq (d_1, d_2, \ldots, d_r)$, at least one skew hook h in every decomposition of $[\lambda]$ given by removing skew hooks of lengths $m_1, m_2, \ldots, m_t$ has the property that $h \neq h'$. This implies that $s \neq s'$ for all decompositions s of $[\lambda]$, so the decompositions s and s' pair up and there are an even number of ways to decompose $[\lambda]$ by successively removing skew hooks of lengths $m_1, \ldots, m_t$. By Corollary 4.4, $\chi_\lambda(g) \equiv 0 \bmod 2$. This completes our proof, therefore $\chi_\lambda(g) \equiv 1 \bmod 2$ if and only if $t = r$ and $(m_1, m_2, \ldots, m_r) = (d_1, d_2, \ldots, d_r)$. □

The diagonal hook lengths $(d_1, d_2, \ldots, d_r)$ in Lemma 4.6 form a specific type of partition called a *distinct odd partition*. For any $n \in \mathbb{Z}^+$, a distinct odd partition of n is a partition $(n_1, n_2, \ldots, n_t)$ such that $n_i \neq n_j$ for $i \neq j$ and n_i is an odd integer for all i. As a consequence of Lemma 4.6, the value of $\hat{\tau}(g)$ is determined by whether or not the cycle type of g is a distinct odd partition, as detailed in the following proposition:

Proposition 4.7. *Let $g \in S_n$ have cycle type $(m_1, m_2, \ldots, m_t)$. Then*

$$\tau(g) \equiv \begin{cases} 1 \ \textit{mod}\ 2 & \textit{if } (m_1, m_2, \ldots, m_t) \textit{ is a distinct odd partition of } n; \\ 0 \ \textit{mod}\ 2 & \textit{otherwise.} \end{cases}$$

Proof. First, suppose that $(m_1, m_2, \ldots, m_t)$ is a distinct odd partition of n. Let λ_g be the partition of n such that $[\lambda_g]$ has diagonal hook lengths $(m_1, \ldots, m_t)$. By Lemma 4.6, $\chi_\lambda(g) \equiv 1 \bmod 2$ if $\lambda = \lambda_g$ and 0 otherwise. Then, since $\tau(g) \equiv \sum_{\lambda=\lambda'} \chi_\lambda(g) \bmod 2$ by

Lemma 4.5, we see that

$$\begin{aligned}\tau(g) \equiv \sum_{\lambda=\lambda'} \chi_\lambda(g) &\equiv \chi_{\lambda_g}(g) + \sum_{\lambda=\lambda',\lambda\neq\lambda_g} \chi_\lambda(g) \\ &\equiv \chi_{\lambda_g}(g) + \sum_{\lambda=\lambda',\lambda\neq\lambda_g} 0 \equiv \chi_{\lambda_g}(g) \equiv 1 \bmod 2.\end{aligned}$$

Next, suppose that $(m_1, \ldots, m_t)$ is not a distinct odd partition of n. Then since the diagonal hook lengths of every symmetric Young diagram form a distinct odd partition of n, there does not exist a symmetric Young diagram $[\lambda]$ such that $[\lambda]$ has diagonal hook lengths $(m_1, \ldots, m_t)$. Thus by Lemma 4.6, $\chi_\lambda(g) \equiv 0 \bmod 2$ for all symmetric partitions λ of n, hence

$$\tau(g) \equiv \sum_{\lambda=\lambda'} \chi_\lambda(g) \equiv \sum_{\lambda=\lambda'} 0 \equiv 0 \bmod 2.$$

This completes our proof of Proposition 4.7. □

4.6 Proof of main result

We are now ready to prove our main result, that S_n is not a total character group for $n \geq 4$. As mentioned previously, we proceed by contradiction. We first clarify the types of contradictions we will look for. We then show that one of these contradictions occurs for any $\chi \in \mathrm{Irr}(S_n)$, $n \geq 4$.

4.6.1 Parity conditions. In this section, we prove that if either of two parity conditions (see Corollary 4.9) arises in $\hat{\tau}$ and the parity of χ, then χ cannot be a character such that $f(\chi) = \tau$ for some $f(x) \in \mathbb{Z}[x]$. We will have need of the following lemma:

Lemma 4.8. *Let χ be an irreducible character of S_n, $n \geq 4$. If there exists a polynomial $f(x) \in \mathbb{Z}[x]$ such that $f(\chi) = \tau$, where τ is the total character of S_n, then $f(\chi)$ is congruent to either χ or $\chi + 1$ modulo 2.*

Proof. Let $f(x) = a_0 + a_1 x + \cdots + a_m x^m$ for some $a_i \in \mathbb{Z}$. Since $\chi(g)^k \equiv \chi(g) \bmod 2$ for all

$g \in S_n$ and for all k, we have

$$\begin{aligned} f(\chi(g)) &= a_0 + a_1\chi(g) + \cdots + a_m\chi(g)^m \\ &\equiv a_0 + (a_1 + \cdots + a_n) \cdot \chi(g) \bmod 2. \end{aligned}$$

Since a_0 and $a_1 + \cdots + a_n$ are constants and this last equation holds for all $g \in S_n$,

$$f(\chi) \equiv a_0 + (a_1 + \cdots + a_n) \cdot \chi \bmod 2,$$

where here a_0 denotes is the constant a_0 function. Thus, either $f(\chi) \equiv 0 \bmod 2$, $f(\chi) \equiv 1 \bmod 2$, $f(\chi) \equiv \chi \bmod 2$, or $f(\chi) \equiv \chi + 1 \bmod 2$.

Since $n \geq 4$, there exist $g_1, g_2 \in S_n$ such that the cycle type of g_1 is a distinct odd partition and the cycle type of g_2 is not. By Proposition 4.7, $\tau(g)$ has odd parity when g has cycle type a distinct odd partition of n, and even parity otherwise, so $\tau(g_1) \equiv 1 \bmod 2$ and $\tau(g_2) \equiv 0 \bmod 2$. This implies $\tau \not\equiv 0, 1 \bmod 2$. By hypothesis, $f(\chi) \equiv \tau \bmod 2$, therefore $f(\chi) \not\equiv 0, 1 \bmod 2$. Thus $f(\chi) \equiv \chi \bmod 2$ or $f(\chi) \equiv \chi + 1 \bmod 2$. □

Our parity conditions follow as a corollary to this characterization of $f(\chi)$.

Corollary 4.9 (Parity Conditions)**.** *Let χ be an irreducible character of S_n, $n \geq 4$, such that either of the following conditions hold:*

(1) There exist $g_1, g_2 \in S_n$ such that $\chi(g_1) \equiv \chi(g_2)$ mod 2 and $\tau(g_1) \not\equiv \tau(g_2)$ mod 2.

(2) There exist $g_1, g_2 \in S_n$ such that $\chi(g_1) \not\equiv \chi(g_2)$ mod 2 and $\tau(g_1) \equiv \tau(g_2)$ mod 2.

Then there does not exist a polynomial $f(x) \in \mathbb{Z}[x]$ such that $f(\chi) = \tau$.

Proof. Let $\chi \in \mathrm{Irr}(S_n)$, and first assume that condition (1) holds. Suppose by way of contradiction that there exists a polynomial $f(x) \in \mathbb{Z}[x]$ such that $f(\chi) = \tau$. By Lemma 4.8, either $f(\chi) \equiv \chi \bmod 2$ or $f(\chi) \equiv \chi + 1 \bmod 2$. If $f(\chi) \equiv \chi \bmod 2$, then

$$\tau(g_1) = f(\chi(g_1)) \equiv \chi(g_1) \equiv \chi(g_2) \equiv f(\chi(g_2)) = \tau(g_2) \bmod 2,$$

which contradicts the fact that $\tau(g_1) \not\equiv \tau(g_2)$ mod 2. If, on the other hand, $f(\chi) \equiv \chi + 1$ mod 2, then since $\chi(g_1) \equiv \chi(g_2)$ mod 2, we have $\chi(g_1) + 1 \equiv \chi(g_2) + 1$ mod 2, so

$$\tau(g_1) = f(\chi(g_1)) \equiv \chi(g_1) + 1 \equiv \chi(g_2) + 1 \equiv f(\chi(g_2)) = \tau(g_2) \text{ mod } 2.$$

This is a contradiction, therefore if (1) holds, there does not exist $f(x) \in \mathbb{Z}[x]$ such that $f(\chi) = \tau$.

Now assume that (2) holds, and suppose that there exists a polynomial $f(x) \in \mathbb{Z}[x]$ such that $f(\chi) = \tau$. If $f(\chi) \equiv \chi$ mod 2, then

$$\chi(g_1) \equiv f(\chi(g_1)) = \tau(g_1) \equiv \tau(g_2) = f(\chi(g_2)) \equiv \chi(g_2) \text{ mod } 2,$$

a contradiction. On the other hand, if $f(\chi) \equiv \chi + 1$ mod 2, then

$$\chi(g_1) + 1 \equiv f(\chi(g_1)) = \tau(g_1) \equiv \tau(g_2) = f(\chi(g_2)) \equiv \chi(g_2) + 1 \text{ mod } 2,$$

which implies that $\chi(g_1) \equiv \chi(g_2)$ mod 2, a contradiction. Thus, if (2) holds, there does not exist $f(x) \in \mathbb{Z}[x]$ such that $f(\chi) = \tau$. □

It remains to show that one of these conditions (which we refer to as *Parity Conditions (1) and (2)*) holds for each irreducible character χ of S_n. We will first consider characters associated with symmetric partitions, and then characters associated with non-symmetric partitions.

4.6.2 Characters associated with symmetric partitions.

All characters associated with symmetric partitions can be shown to exhibit Parity Condition (1) when there are at least two distinct odd partitions of n. The existence of two such partitions is guaranteed when $n \geq 8$, as will be seen shortly. We therefore assume $n \geq 8$ and prove Theorem 4.1 for $n = 4, 5, 6$, and 7 on a case by case basis (see "Proof of Theorem 4.1" and Appendix A).

Proposition 4.10. *Let λ be a symmetric partition of S_n, $n \geq 8$. Then there does not exist a polynomial $f(x) \in \mathbb{Z}[x]$ such that $f(\chi_\lambda) = \tau$.*

Proof. If $n \geq 8$ is even, then $(n-1,1)$ and $(n-3,3)$ are distinct odd partitions of n. If $n \geq 8$ is odd, then (n) and $(n-4,3,1)$ are distinct odd partitions of n. Therefore for all $n \geq 8$, there are at least two distinct odd partitions of n.

Since λ is symmetric, the Young diagram $[\lambda]$ is a symmetric diagram, so let $(d_1, d_2, \ldots, d_r)$ be the diagonal hook lengths of $[\lambda]$. Because there are at least two elements of S_n with cycle type a distinct odd partition, we can choose g_1 to be an element with cycle type $(m_1, m_2, \ldots, m_t)$, where $(m_1, m_2, \ldots, m_t)$ is a distinct odd partition of n and $(m_1, m_2, \ldots, m_t) \neq (d_1, d_2, \ldots, d_r)$. Let g_2 be an element of S_n whose cycle lengths do not form a distinct odd partition. By Lemma 4.6, $\chi_\lambda(g_1) \equiv \chi_\lambda(g_2) \equiv 0 \bmod 2$, and by Proposition 4.7, $\tau(g_1) \equiv 1 \bmod 2$ and $\tau(g_2) \equiv 0 \bmod 2$. Since g_1 and g_2 satisfy Parity Condition (1), there does not exist a polynomial $f(x) \in \mathbb{Z}[x]$ such that $f(\chi_\lambda) = \tau$. □

4.6.3 Characters associated with non-symmetric partitions.

In order to get the parity contradictions needed to apply Corollary 4.9 to characters associated to non-symmetric partitions, we will use the n-cycle $\sigma = (123 \ldots n)$. We therefore need to know the value of $\chi_\lambda(\sigma)$ for χ_λ an irreducible character of S_n.

Lemma 4.11. *If $\sigma = (123 \cdots n)$ is the cycle of length n in S_n and $\chi_\lambda \in Irr(S_n)$, then*

$$\chi_\lambda(\sigma) \equiv \begin{cases} 1 \textit{ mod } 2 & \textit{if } \lambda = (n-\ell, 1, 1, \ldots, 1) \textit{ for some } \ell, 0 \leq \ell \leq n; \\ 0 \textit{ mod } 2 & \textit{otherwise.} \end{cases}$$

Proof. To use Corollary 4.4, we want to find the number of ways of decomposing $[\lambda]$ by removing a single skew hook of length n, the length of the only cycle of σ. If λ has shape $(n-\ell, 1, 1, \ldots, 1)$, there is exactly one hook (and therefore one corresponding skew hook) within $[\lambda]$ of length n. Therefore removing a skew hook of length n can be done in exactly one way. This implies that $s = 1$, so by Corollary 4.4, $\chi_\lambda(\sigma) \equiv 1 \bmod 2$. If $[\lambda]$ is not a hook diagram, then the hook length of every box in $[\lambda]$ has length strictly less than n, thus there is no hook and therefore no corresponding skew hook of length n. Therefore we cannot decompose $[\lambda]$ by removing a skew hook of length n, so $s = 0$ and by Corollary 4.4, $\chi_\lambda(\sigma) \equiv 0 \bmod 2$. $\square$

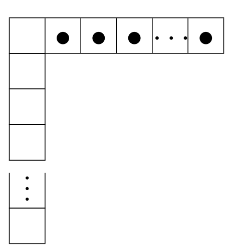
Now that we have the value of $\chi_\lambda(\sigma)$, we will calculate the parity of $\chi_\lambda(g)$ for other specific elements $g \in S_n$ as needed on a case-by-case basis depending on χ_λ, and then use Corollary 4.9. We consider the cases n odd and n even separately. Within each of these cases, we further consider subcases based on the *maximum hook length* of $[\lambda]$, which is the maximum of the hook lengths for all of the hooks within the Young diagram $[\lambda]$. We first consider the case when n is odd.

Proposition 4.12. *Let λ be a non-symmetric partition of n, where $n \geq 9$ is an odd integer. Then there does not exist a polynomial $f(x) \in \mathbb{Z}[x]$ such that $f(\chi_\lambda) = \tau$.*

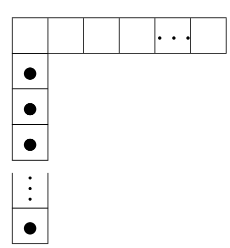
Proof. We will show that Parity Condition (1) holds for χ_λ. Let k be the maximum hook length of $[\lambda]$, which is also the hook length of the upper left box of the diagram. We consider three cases. If $k = n$, then $[\lambda]$ has only one hook and is therefore a hook diagram. If $k = n - 1$, then $[\lambda]$ is almost a hook diagram; it has one main hook with one extra box in the corner of the hook. Our third case will be when $k < n - 1$.

Case 1: $k = n$.

If $k = n$, then $[\lambda]$ is a hook diagram: $\lambda = (n - \ell, 1, 1, \ldots, 1)$ for some $0 \leq \ell \leq n$. Note that $n - \ell - 1$ is the length of the skew hook in the first row, and ℓ is the length of the skew hook corresponding to the first column, as pictured below:



Skew hook of length $n - \ell - 1$



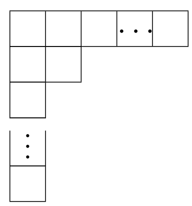
Skew hook of length ℓ

Let $m = \max\{n-\ell-1, \ell\}$, and let $g_1 \in S_n$ be an element of cycle type $(m, n-m)$, with the cycle lengths fixed in the order $m, n-m$. Since λ is not symmetric, $n-\ell-1 \neq \ell$, so we can only remove a skew hook of length m in one way. Since the next step must remove the remaining $n-m$ boxes, which all live in one row or column, there is exactly one way to take away the remaining boxes, so $s = 1$. By Corollary 4.4, $\chi_\lambda(g_1) \equiv 1 \bmod 2$. Since n is odd, no two integers form a distinct odd partition of n, so the cycle lengths of g_1 do not form a distinct odd partition of n. This implies that $\tau(g_1) \equiv 0 \bmod 2$, by Proposition 4.7.

Now let σ be a cycle of length n, so that σ has cycle type (n). Since $[\lambda]$ is a hook diagram, by Lemma 4.11 we have $\chi_\lambda(\sigma) \equiv 1 \equiv \chi_\lambda(g_1) \bmod 2$. Since n is odd, (n) is a distinct odd partition of n, so $\tau(\sigma) \equiv 1 \not\equiv \tau(g_1) \bmod 2$ by Proposition 4.7. Thus g_1 and $g_2 = \sigma$ satisfy Parity Condition (1).

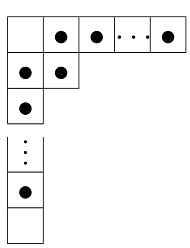
Case 2: $k = n-1$**.**

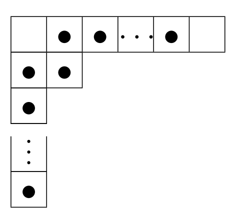
If $k = n-1$, then $\lambda = (n-\ell, 2, 1, \ldots, 1)$ for some $2 \leq \ell \leq n-2$, where there are $\ell - 2$ 1's:



We consider two cases, depending on the value of ℓ.

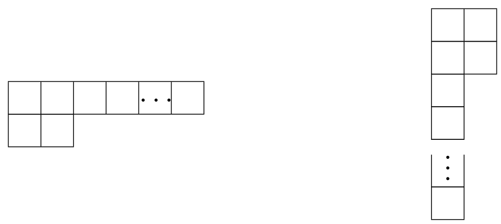
If $2 < l < n-2$, then $[\lambda]$ has at least three rows and columns. Let $g_1 \in S_n$ be the element with cycle lengths $n-2$ and 2, fixed in that order. The only connected regions of boundary boxes of length $n-2$ are those which include all boundary boxes except for one:



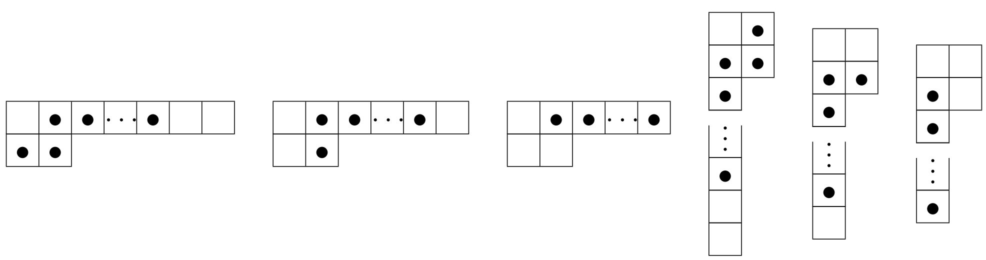


Since removing either of these sets of boxes does not leave a Young diagram, by definition there is no skew hook of length $n-2$. Therefore we cannot decompose $[\lambda]$ by removing skew hooks of lengths $n-2, 2$, so by Corollay 4.4, $\chi_\lambda(g_1) \equiv 0 \bmod 2$. Since $(n-2,2)$ is not a distinct odd partition of n, $\tau(g_1) \equiv 0 \bmod 2$. Now consider the element σ of cycle type (n). Since λ is not a hook, $\chi_\lambda(\sigma) \equiv 0 \equiv \chi_\lambda(g_1) \bmod 2$, and since n is odd, $\tau(\sigma) \equiv 1 \not\equiv \tau(g_1) \bmod 2$. Thus g_1 and $g_2 = \sigma$ satisfy Parity Condition (1).

Now suppose that $l = 2$ or $l = n-2$. Then $[\lambda]$ has one of two shapes pictured below:



Let $g_1 \in S_n$ be an element with cycle lengths $n-3$ and 3, fixed in that order. Then there are only three possible connected regions of boundary boxes of length $n-3$ for each of the possible shapes for $[\lambda]$, as pictured below:



In every case, removing these sets of boundary boxes does not leave a smaller Young diagram, so there is no skew hook of length $n-3$. Therefore we cannot decompose $[\lambda]$ by removing skew hooks of lengths $n-3, 3$, hence $\chi_\lambda(g_1) \equiv 0 \equiv \chi_\lambda(\sigma) \bmod 2$ by Corollary 4.4. Since n is odd, $n-3$ is even, so $(n-3,3)$ is not a distinct odd partition of n. By Proposition 4.7, $\tau(g_1) \equiv 0 \not\equiv \tau(\sigma) \bmod 2$. Thus g_1 and $g_2 = \sigma$ satisfy Parity Condition (1).

Case 3: $k < n-1$.

If $k < n-1$, then no hook has length n or $n-1$, so no skew hook has length n or $n-1$. Let $g_1 \in S_n$ be an element with cycle lengths $n-1$ and 1, fixed in that order. Since there is no skew hook of length $n-1$, there is no way to decompose $[\lambda]$ by successively removing skew hooks of lengths $n-1$ and 1. By Corollary 4.4, this tells us that $\chi_\lambda(g_1) \equiv 0 \equiv \chi_\lambda(\sigma) \bmod 2$. Since $(n-1,1)$ is not a distinct odd partition of n, $\tau(g_1) \equiv 0 \not\equiv \tau(\sigma) \bmod 2$. Thus g_1 and $g_2 = \sigma$ satisfy Parity Condition (1).

In all cases we were able to find $g_1, g_2 \in S_n$ satisfying Parity Condition (1). Thus, when $n \geq 9$ is odd and $\chi_\lambda \in \mathrm{Irr}(S_n)$ is associated with a non-symmetric partition λ, there does not exist a polynomial $f(x) \in \mathbb{Z}[x]$ such that $f(\chi_\lambda) = \tau$. □

We next consider the case when n is even.

Proposition 4.13. *Let λ be a non-symmetric partition of n, where n is an even integer and $n \geq 8$. Then there does not exist a polynomial $f(x) \in \mathbb{Z}[x]$ such that $f(\chi_\lambda) = \tau$.*

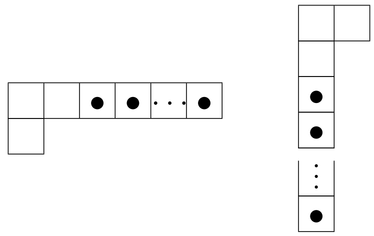
Proof. Throughout this proof, we will rely on the conclusion of Lemma 4.11, that for $\sigma = (123\cdots n)$, $\chi_\lambda(\sigma) \equiv 1 \bmod 2$ if $[\lambda]$ is a hook diagram and 0 otherwise. Since (n) is not a distinct odd partition of n when n is even, we also know from Proposition 4.7 that $\tau(\sigma) \equiv 0 \bmod 2$. Our proof is again split into cases based on k, the maximum hook length of $[\lambda]$.

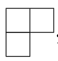
Case 1: $k = n$.

If the maximum hook length of the hooks of $[\lambda]$ is n, then $[\lambda]$ is a hook diagram associated with a partition $\lambda = (n-\ell, 1, \ldots, 1)$ for some ℓ. Note that since $[\lambda]$ is a hook diagram, $\chi_\lambda(\sigma) \equiv 1 \bmod 2$. Within this case, we have three subcases based on the value of ℓ.

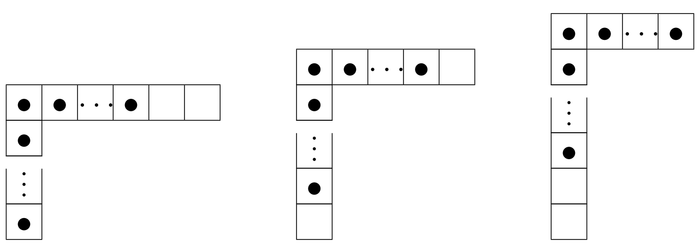
If $\ell = 0$ or $\ell = n-1.$, we have $\lambda = (n)$ or $\lambda = (1, 1, 1, \ldots, 1)$. These two partitions correspond to the trivial character and the sign (alternating) character [9, p. 47]. In both of these cases we have $\chi_\lambda(g) \equiv 1 \bmod 2$ for all $g \in S_n$. Since there exist $g_1, g_2 \in S_n$ such that $\tau(g_1) \not\equiv \tau(g_2) \bmod 2$, Parity Condition (1) holds.

If $\ell = 1$ or $\ell = n-2$, we have $\lambda = (n-1, 1)$ or $\lambda = (2, 1, 1, \ldots, 1)$. Let $g_1 \in S_n$ be an element with cycle lengths $n-3$, 2, and 1. Since $n-3 > 1$, there is only one skew hook of length $n-3$, namely the one beginning at the far right of the first row when $\lambda = (n-1, 1)$, or the one beginning at the bottom of the first column when $\lambda = (2, 1, 1, \ldots, 1)$:



Since there is only one skew hook of length $n-3$, every decomposition must begin by removing that skew hook. However, there is no skew hook of length 2 in the remaining Young diagram , so there is no way to decompose $[\lambda]$ by successively removing skew hooks of lengths $n-3, 2, 1$. By Corollary 4.4, $\chi_\lambda(g_1) \equiv 0 \not\equiv \chi_\lambda(\sigma) \bmod 2$. Since g_1 is not a distinct odd partition of n, $\tau(g_1) \equiv 0 \equiv \tau(\sigma) \bmod 2$. Thus g_1 and $g_2 = \sigma$ satisfy Parity Condition (2).

Finally, consider when $1 < \ell < n-2$. Let $g_1 \in S_n$ have cycle lengths $n-2$ and 2, fixed in that order, and consider the number of ways of decomposing $[\lambda]$ by successively removing skew hooks of lengths $n-2, 2$. Note that when $n > 4$, we have $n-2 > 2$, so the only connected regions of boundary boxes of lengths $n-2$ do not leave a smaller Young diagram, as demonstrated below:

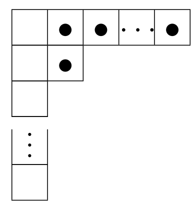


This means that there is no skew hook of length $n-2$, so there is no way to decompose $[\lambda]$ by successively removing skew hooks of lengths $n-2, 2$. Then $s = 0$, so by Corollary 4.4, $\chi_\lambda(g_1) \equiv 0 \not\equiv \chi_\lambda(\sigma) \bmod 2$. Since the cycle lengths do not form a distinct odd partition, we know that $\tau(\alpha) \equiv 0 \equiv \tau(\sigma) \bmod 2$. Then g_1 and $g_2 = \sigma$ satisfy Parity Condition (2).

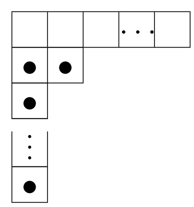
Case 2: $k = n - 1$**.**

When $k = n-1$, we have $\lambda = (n-\ell, 2, 1, \ldots, 1)$ for some ℓ with $2 \le \ell \le n-2$. In this case, $[\lambda]$ is not a hook and λ is not a distinct odd partition, so $\chi_\lambda(\sigma) \equiv 0 \bmod 2$ and $\tau(\sigma) \equiv 0 \bmod 2$. We want to find g_1 with $\tau(g_1) \equiv 0 \equiv \tau(\sigma) \bmod 2$ and $\chi_\lambda(g_1) \equiv 1 \not\equiv \chi_\lambda(\sigma) \bmod 2$. Within Case 2, we have two subcases, depending on the value of ℓ.

First, suppose $2 < \ell < n-2$. Notice that $n - \ell$ is the number of boxes that if removed would leave the first column:



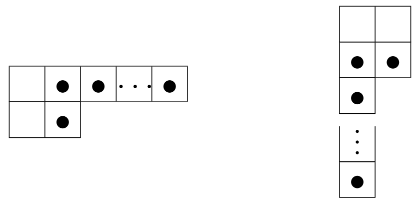
whereas ℓ is the number of boxes that if removed would leave only the first row:



Let $m = \max\{n-\ell, \ell\}$. Since n is even, it is possible that $(m, n-m)$ is a distinct odd partition, so to guarantee that the cycle type of g_1 is not a distinct odd partition we let g_1 be an element with cycle lengths $m, n-m-2, 2$, fixed in that order. Thus $\tau(g_1) \equiv 0 \equiv \tau(\sigma) \bmod 2$. Since $[\lambda]$ is not symmetric, we know that $n - \ell \neq \ell$, hence there really is only one maximum. Then there is only one skew hook of length m in $[\lambda]$, namely the skew hook starting from whichever end will leave one row or column after removal. After removing that skew hook, there is only one skew hook of length $n-m-2$ in the remaining row or column: the skew hook starting at either the far right box of the row or the bottom box of the column. Finally, after removing that skew hook, there is only one skew hook of length 2 left, which finishes the decomposition. Since only one skew hook was possible at each step of the decomposition, we know that $s = 1$, hence $\chi_\lambda(g_1) \equiv 1 \not\equiv \chi_\lambda(\sigma) \bmod 2$ by Corollary

4.4. This is our desired g_1, so we see that g_1 and $g_2 = \sigma$ meet Parity Condition (2).

Next, suppose that $\ell = 2$ or $\ell = n - 2$. In both cases there is only one skew hook of length $n - 2$:



Let $g_1 \in S_n$ be an element with cycle lengths $n - 2, 2$, fixed in that order. After removing the only skew hook of length $n - 2$, there is only one skew hook of length 2, so $s = 1$ and $\chi_\lambda(g_1) \equiv 1 \not\equiv \chi_\lambda(\sigma) \bmod 2$. Since the cycle type of g_1 is not a distinct odd partition of n, we have $\tau(g_1) \equiv 0 \equiv \tau(\sigma) \bmod 2$. Then g_1 and $g_2 = \sigma$ satisfy Parity Condition (2).

Case 3: $k < n - 1$**.**

For this case, consider an element $g_1 \in S_n$ having cycle type $(n - 1, 1)$. Since n is even, the cycle type of g_1 is a distinct odd partition of n, so $\tau(g_1) \equiv 1 \bmod 2$. Since there is no skew hook of length $n - 1$ in $[\lambda]$, it is impossible to remove a skew hook of length $n - 1$, hence $\chi_\lambda(g_1) \equiv 0 \bmod 2$. We know that $[\lambda]$ is not a hook, so $\chi_\lambda(\sigma) \equiv 0 \equiv \chi_\lambda(g_1) \bmod 2$. Since n is even, $\tau(\sigma) \equiv 0 \not\equiv \tau(g_1) \bmod 2$. Then the elements g_1 and $g_2 = \sigma$ satisfy Parity Condition (1).

This concludes consideration of all cases. In each case we found $g_1, g_2 \in S_n$ satisfying either Parity Condition (1) or (2), so there does not exist a polynomial $f(x) \in \mathbb{Z}[x]$ such that $f(\chi_\lambda) = \tau$. $\square$

We are now ready to piece together our proof of Theorem 4.1.

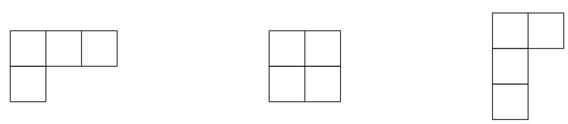
Proof of Theorem 4.1. We first prove this theorem for $n = 4, 5, 6$, and 7 on a case by case basis. The character tables for S_4, S_5, S_6, and S_7 are well known and can be obtained in many ways. We calculate the character table of S_n for these small values of n from the computer program MAGMA [1]. First, consider $n = 4$. The group S_4 has five conjugacy classes with representatives $g_1 = 1, g_2 = (12)(34), g_3 = (12), g_4 = (123)$, and $g_5 = (1234)$. We present both the character table and the total character of S_4 modulo 2 in Table 4.1.

	g_1	g_2	g_3	g_4	g_5
χ_1	1	1	$\mathbf{1}$	$\mathbf{1}$	1
χ_2	1	1	$\mathbf{1}$	$\mathbf{1}$	1
χ_3	0	0	0	1	0
χ_4	1	1	1	0	1
χ_5	1	1	1	0	1
τ	0	0	0	1	0

Table 4.1: Character Table and Total Character of S_4 modulo 2

We see that parity contradictions occur for the characters χ_1 and χ_2. For χ_i, $i = 1, 2$, $\chi_i(g_3) \equiv \chi_i(g_4) \equiv 1 \bmod 2$, but $\tau(g_3) \equiv 0 \bmod 2$ and $\tau(g_4) \equiv 1 \bmod 2$. The corresponding values of χ_1 and χ_2 are bolded in the table. This is Parity Condition (2), so no polynomial with integer coefficients in χ_1 and χ_2 will yield τ.

As we examine the other three irreducible characters of S_n, we see that similar parity contradictions do not arise. These three characters correspond with the following Young diagrams:



For the symmetric case, there is only one symmetric partition $\lambda = (2, 2)$ of $n = 4$, so we have no element $g \in S_4$ with the cycle type a distinct odd partition such that $\chi_\lambda(g) = 0$. Our other parity arguments do not hold in the non-symmetric cases because of similar exceptions that only occur when n is small. We therefore use the regular character table of S_4 to finish proving Theorem 4.1 for $n = 4$. Both the character table and the total character of S_4 are given in Table 4.2.

	g_1	g_2	g_3	g_4	g_5
χ_1	1	1	1	1	1
χ_2	1	1	-1	1	-1
χ_3	$\mathbf{2}$	$\mathbf{2}$	0	-1	0
χ_4	3	$\mathbf{-1}$	$\mathbf{-1}$	0	1
χ_5	3	$\mathbf{-1}$	1	0	$\mathbf{-1}$
τ	10	2	0	1	0

Table 4.2: Character Table and Total Character of S_4

To show that there does not exist $f(x) \in \mathbb{Z}[x]$ such that $f(\chi_i) = \tau$ for $i = 3, 4, 5$, we use Proposition 2.10. Recall this proposition:

Proposition 2.10. *Let G be any group. Let χ be an irreducible character of G and let τ be the total character of G. If there exist $g_1, g_2 \in G$ such that $\chi(g_1) = \chi(g_2)$ but $\tau(g_1) \neq \tau(g_2)$, then there does not exist $f(x) \in \mathbb{C}[x]$ such that $f(\chi) = \tau$.*

From Table 4.2, we see that Proposition 2.10 applies to χ_3, χ_4, and χ_5, using the representatives g_1 and g_2 for χ_3, g_2 and g_3 for χ_4, and g_2 and g_5 for χ_5. The appropriate values are bolded in the table.

We have now verified that for all $\chi \in \mathrm{Irr}(S_4)$, there does not exist $f(x) \in \mathbb{Z}[x]$ with $f(\chi) = \tau$, hence S_4 is not a total character group. The cases $n = 5, 6$, and 7 are proved similarly (see Appendix A).

If $n \geq 8$ is odd, by Proposition 4.12 there does not exist $\chi \in \mathrm{Irr}(S_n)$ and a polynomial $f(x) \in \mathbb{Z}[x]$ such that $f(\chi) = \tau$. By Proposition 4.13, the same result holds when $n \geq 8$ is even. Therefore for all $n \geq 4$, there does not exist an irreducible character χ of S_n and a polynomial $f(x) \in \mathbb{Z}[x]$ such that $f(\chi) = \tau$. This concludes the proof that S_n is not a total character group for $n \geq 4$. □

Chapter 5. Quadratic Total Character Groups

5.1 Preliminaries

We say that a finite group G is a *quadratic total character group* if there is an irreducible character χ and a polynomial $f(x) = f_{G,\chi}(x) \in \mathbb{Z}[x]$ of degree two such that $f(\chi) = \tau_G$. In this chapter, we show that if G is a quadratic total character group and a p-group, then $p = 2$ and G is extraspecial. Recall that a p-group G is said to be *extraspecial* if $G' = Z(G)$ and $|G'| = |Z(G)| = p$ [13, p. 123].

When G is a quadratic total character group, we can write $f_{G,\chi}(x) = ax^2 + bx + c$ for some $a, b, c \in \mathbb{Z}$. We begin by proving some basic facts about the polynomial $f_{G,\chi}(x)$.

Lemma 5.1. *Let G be a quadratic total character group with total character $\tau_G = \tau$. Let $\tau = a\chi^2 + b\chi + c$, with $a, b, c \in \mathbb{Z}$. Then:*

(1) $a = 1$ and $\langle \chi^2, \psi \rangle = 1$ for any $\psi \in Irr(G) \setminus \{1, \chi\}$.

(2) $b = 1 - \langle \chi^2, \chi \rangle \leq 1$.

(3) $c = 1 - \langle \chi^2, 1 \rangle \leq 1$.

(4) $\chi^2 + b\chi = \overline{\chi}^2 + b\overline{\chi}$.

(5) For any $g \in G$ the complex number $\chi(g)$ is an algebraic integer of degree at most 2.

Proof. (1) Let ψ be any irreducible character of G not equal to 1 or χ. Then we have

$$1 = \langle \tau, \psi \rangle = \langle a\chi^2 + b\chi + c, \psi \rangle = a\langle \chi^2, \psi \rangle,$$

and since $a, \langle \chi^2, \psi \rangle \in \mathbb{Z}$ and $\langle \chi^2, \psi \rangle \geq 0$ we see that $a = 1$ and $\langle \chi^2, \psi \rangle = 1$.

(2) We have $1 = \langle \tau, \chi \rangle = \langle \chi^2 + b\chi + c, \chi \rangle = \langle \chi^2, \chi \rangle + b$.

(3) Similarly, $1 = \langle \tau, 1 \rangle = \langle \chi^2, 1 \rangle + c$.

(4) Since $\tau(g) \in \mathbb{Z}$ for all $g \in G$ (see Section 1.2, pg. 8), $\overline{\tau} = \tau$. Then since $\tau = \chi^2 + b\chi + c$, taking complex conjugates of both sides yields

$$\overline{\chi}^2 + b\overline{\chi} + c = \overline{\tau} = \tau = \chi^2 + b\chi + c.$$

The result follows.

(5) For $g \in G$, we have $(\chi^2 + b\chi + c)(g) = \tau(g) \in \mathbb{Z}$. Thus $[\chi(g)]^2 + b[\chi(g)] + (c - \tau(g)) = 0$, so $\chi(g)$ is a root of the quadratic integral polynomial $x^2 + bx + (c - \tau(g))$, and hence an algebraic integer of degree at most 2. $\square$

If we have $\chi^2 = \tau_G$, and G is solvable, then G is isomorphic to S_3^k for some $k \in \mathbb{N}$ by Theorem 1.1 in [10, p. 655]. Thus we assume that b and c are not both 0 in what follows.

Lemma 5.2. *Let G be a non-abelian total character group. Let $z \in Z(G)$. Then z has order* 1, 2, *or* 4.

Proof. Let z have order k, and let $d = \chi(1)$. Assume that $\tau_G = a\chi^2 + b\chi + c$. Let ρ_χ be the representation affording χ. Since z is a central element, $\rho_\chi(z)$ is in the center of $GL_d(\mathbb{C})$, hence is a scalar matrix. Since z has order k, the entry on the main diagonal must be ζ where $\zeta^k = 1$. Then $\chi(z) = d\zeta$, where $d = \chi(1)$. In fact ζ has order k since χ is faithful (see Proposition 2.8).

If $\zeta = \overline{\zeta}$, then $\zeta^2 = 1$ and k divides 2.

So now assume that $\zeta \neq \overline{\zeta}$. Then Lemma 5.1(4) gives $\chi(z)^2 + b\chi(z) = \overline{\chi}(z)^2 + b\overline{\chi}(z)$, hence $\chi(z)^2 - \overline{\chi}(z)^2 = b[\overline{\chi}(z) - \chi(z)]$. Factoring yields

$$(d\zeta + d\overline{\zeta})(d\zeta - d\overline{\zeta}) = b(d\overline{\zeta} - d\zeta).$$

Since $\zeta \neq \overline{\zeta}$, we can divide both sides by $d(\overline{\zeta} - \zeta)$ so that we have $\zeta + \overline{\zeta} = -b/d$. Since $\zeta + \overline{\zeta}$ is an algebraic integer and $b/d \in \mathbb{Q}$ we must have $b/d \in \mathbb{Z}$. It follows that $\zeta + \overline{\zeta} = -b/d \in \{0, 1, -1, 2, -2\}$, where the cases $-b/d = \pm 2$ imply that $\zeta = \pm 1$. Thus we can assume that

$\zeta + \overline{\zeta} = -b/d \in \{0, 1, -1\}$.

Let $\zeta = x + iy$, $x, y \in \mathbb{R}$, so that $\zeta + \overline{\zeta} = 2x \in \{0, 1, -1\}$. Since ζ is a root of unity, we know that $\sqrt{x^2 + y^2} = 1$. If $2x = 0$, then a simple calculation shows that $\zeta = \pm i$. In this case, $|z| = k = 4$. If $2x = \pm 1$, then $\zeta \in \left\{\frac{\pm 1 \pm \sqrt{3}}{2}\right\}$ is a 3^{rd} or a 6^{th} root of unity. Thus z has order 1, 2, 3, 4, or 6.

Now, if z has order 3, then $-1 = \zeta + \bar{\zeta} = -b/d$, which shows that $b = d$. Since χ is faithful and G is non-abelian we have $b = d > 1$, contradicting Lemma 5.1(2). This shows that G does not have a central element of order 3 and thus $k \neq 3, 6$. □

Corollary 5.3. *If G is a quadratic total character group and a non-abelian p-group, then $p = 2$.*

Proof. If G is a p-group, then the center of G is nontrivial, so $|Z(G)| > 1$. Since G is a p-group, $|Z(G)| = p^k$ for some $k > 0$, hence by Cauchy's Theorem, $Z(G)$ has an element of order p. Since G is a non-abelian quadratic total character group, we must have $p = 2$ by Lemma 5.2. □

Now let G' be the commutator subgroup of G. The following result is true for any group.

Lemma 5.4. *Let $\tau = \tau_G$ and let $g \notin G'$. Then $\tau(g) = 0$.*

Proof. Since $g \notin G'$ there is a linear character λ for G such that $\lambda(g) \neq 1$. Now multiplication by λ permutes the elements of $\mathrm{Irr}(G)$, and so $\tau\lambda = \tau$, or $\tau(1 - \lambda) = 0$. Evaluating this equation at g gives $\tau(g)(1 - \lambda(g)) = 0$, showing that $\tau(g) = 0$. □

5.2 p-Groups

For the rest of the chapter, we assume that G is a non-abelian p-group and a quadratic total character group with $\tau = \chi^2 + b\chi + c$ for some irreducible character χ. Then G is a 2-group by Corollary 5.3, say $|G| = 2^n$. Let $z \in Z(G)$ have order 2. Then $\rho_\chi(z)$ is either the identity or the negative identity matrix, hence $\chi(z) = \chi(1)$ or $\chi(z) = -\chi(1)$. By Proposition 2.8, χ must be a faithful character, so $\chi(z) = -\chi(1)$. Let $\chi(1) = d$, so that $\chi(z) = -d$.

Let $\mathrm{Irr}(G)$ denote the set of irreducible characters of G, and let $\mathrm{Irr}(G)^* = \mathrm{Irr}(G) \setminus \{\chi\}$. If $\psi \in \mathrm{Irr}(G)^*$, then by a similar reasoning we either have $\psi(z) = \psi(1)$ or $\psi(z) = -\psi(1)$. For each $k \geq 0$ we let N_{2^k} denote the number of $\psi \in \mathrm{Irr}(G)^*$ such that $\psi(z) = \psi(1) = 2^k$, and let N_{-2^k} denote the number of $\psi \in \mathrm{Irr}(G)^*$ such that $\psi(z) = -\psi(1) = -2^k$.

Lemma 5.5. *Let G be a quadratic total character group and a non-abelian 2-group, with $\tau_G = \chi^2 + b\chi + c$. Then with the notation as above, $N_{-2^k} = 0$ for all $k \geq 0$. In particular, there is only one irreducible character $\psi \in Irr(G)$ such that $\psi(z) = -\psi(1)$, namely χ.*

Proof. Let $d = \chi(1)$. By the definition of N_{2^k} and N_{-2^k},

$$\tau(1) = d + \sum_{k=0}^{\infty}(N_{2^k} + N_{-2^k})2^k = d^2 + bd + c. \tag{5.1}$$

Here the sum in the middle is finite. We also have

$$\tau(z) = -d + \sum_{k=0}^{\infty}(N_{2^k} - N_{-2^k})2^k = d^2 - bd + c. \tag{5.2}$$

Subtracting equation (5.2) from equation (5.1) we get

$$2d + 2\sum_{k=0}^{\infty} N_{-2^k}2^k = 2bd \leq 2d, \tag{5.3}$$

where the last inequality comes from Lemma 5.1(2). Since $N_{-2^k} \geq 0$, it follows that $N_{-2^k} = 0$ for all $k \geq 0$. □

Using Lemma 5.5 and the column orthogonality relations, we calculate $d = \chi(1)$, b, and $Z(G)$.

Lemma 5.6. *Let G be a non-abelian 2-group of order 2^n. Assume that G is a quadratic total character group with $\tau = \chi^2 + b\chi + c$, $b, c \in \mathbb{Z}$, and let $d = \chi(1)$. Then n is odd and $d = 2^{(n-1)/2}$. Furthermore, $b = 1$ and $Z(G) \cong C_2$.*

Proof. First, recall the Column Orthogonality Relations (Theorem 1.8(2)): If $\chi_1, \ldots, \chi_\ell$ are the irreducible characters of G, then for any $g_r, g_s \in G$,

$$\sum_{i=1}^{\ell} \chi_i(g_r)\overline{\chi_i(g_s)} = \delta_{rs}|C_G(g_r)|.$$

Letting $g_r = 1$ and $g_s = z$, we obtain

$$\chi(1)\overline{\chi}(z) + \sum_{k=0}^{\infty} N_{2^k} 2^{2k} = -d^2 + \sum_{k=0}^{\infty} N_{2^k} 2^{2k} = 0.$$

Similarly, letting $g_r = g_s = 1$, we get

$$\chi(1)\overline{\chi}(1) + \sum_{k=0}^{\infty} N_{2^k} 2^{2k} = d^2 + \sum_{k=0}^{\infty} N_{2^k} 2^{2k} = |G| = 2^n.$$

Subtracting we see that $2d^2 = 2^n$, showing that n is odd and that $d = 2^{(n-1)/2}$.

Now note that in the first equality of (5.3) we cannot have $b \leq 0$, since $2d > 0$ and the left-hand side is greater than 0. Since $b \leq 1$ and $b \in \mathbb{Z}$, we see that $b = 1$.

Finally, we show that $Z(G) \cong C_2$. By Theorem 2.32 in [12, p. 29], since χ is faithful, the center of G is cyclic. Let $Z(G) = \langle w \rangle$. By Lemma 5.2, w has order 2 or 4. We show that w has order 2.

So suppose that w has order 4. Then since χ is faithful, we must have $\chi(w) = d \cdot i$, where $\chi(1) = d$ and $i^2 = -1$. Now we get

$$-d^2 + b(d \cdot i) + c = \chi(w)^2 + b\chi(w) + c = \tau(w) \in \mathbb{Z}.$$

Since $d, b, c, \tau(w) \in \mathbb{Z}$ and $d \neq 0$, this forces $b = 0$, a contradiction. Thus $Z(G) \cong C_2$. $\square$

Next, we prove an important fact concerning $\chi(g)$ for $g \in G$.

Lemma 5.7. *If G is a quadratic total character group and a non-abelian 2-group of order 2^n, with $\tau = \chi^2 + \chi + c$, then $\chi(g) = 0$ for all $g \in G \setminus Z(G)$.*

Proof. First, we show that $\chi(g) \in \mathbb{Z}$ for all $g \in G$. Suppose by way of contradiction that there exists $g \in G$ such that $\alpha = \chi(g) \notin \mathbb{Q}$. By Lemma 5.1(5) we see that α is in a quadratic extension of $\mathbb{Q}$: $\mathbb{Q}(\alpha) = \mathbb{Q}(\sqrt{m})$ for some squarefree $m \in \mathbb{Z}$. Let $\overline{\alpha}$ be the conjugate of α. Then from Lemma 5.1(4), we have

$$\chi(g)^2 + b\chi(g) = \overline{\chi(g)}^2 + b\overline{\chi(g)},$$

hence

$$\alpha^2 + \alpha = \overline{\alpha}^2 + \overline{\alpha},$$

from which we get $\alpha^2 - \overline{\alpha}^2 = \overline{\alpha} - \alpha$ and, since $\alpha \neq \overline{\alpha}$, we get $\alpha + \overline{\alpha} = -1$.

Write $\alpha = x + y\sqrt{m}, x, y \in \mathbb{Q}$. Then the above equation gives $x = -1/2$. Since $\tau(g) \in \mathbb{Z}$ (Lemma 1.9) and $\tau(g) = \chi^2(g) + \chi(g) + c$, and $c \in \mathbb{Z}$, we know that $\chi^2(g) + \chi(g) \in \mathbb{Z}$. Thus

$$(1/2 + y\sqrt{a})^2 + (1/2 + y\sqrt{a}) = 1/4 + y\sqrt{a} + y^2a + 1/2 + y\sqrt{a} \in \mathbb{Z},$$

from which it follows that $y = 0$. This contradicts the assumption that $\alpha \notin \mathbb{Q}$, thus we must have $\alpha = \chi(g) \in \mathbb{Q}$. Since $\chi(g)$ is an algebraic integer and a rational number, $\chi(g) \in \mathbb{Z}$.

Now assume that the entries in the row of the character table corresponding to χ are

$$d, -d, \alpha_1, \alpha_2, \ldots, \alpha_s. \tag{5.4}$$

Here $\chi(1) = d, \chi(z) = -d$.

Recall the Row Orthogonality Relations (Theorem 1.8(1)): If $\chi_r, \chi_s \in \mathrm{Irr}(G)$ and $g_1, \ldots, g_k$ are representatives of the conjugacy classes of G, then

$$\sum_{i=1}^{k} \frac{\chi_r(g_i)\overline{\chi_s(g_i)}}{|C_G(g_i)|} = \delta_{rs}.$$

Let $\chi_r = \chi_s = \chi$, and choose $g_i \in G$ such that $\chi(g_i) = \alpha_i$ from (5.4). Then since $|C_G(1)| =$

$|C_G(z)| = |G|$, we have

$$\frac{d \cdot \overline{d}}{|G|} + \frac{-d \cdot \overline{-d}}{|G|} + \sum_{i=1}^{s} \frac{\chi(g_i)\overline{\chi(g_i)}}{|C_G(g_i)|} = 1.$$

Let $\mu_i = |C_i|$, where C_i is the conjugacy class containing g_i. Then multiplying both sides of the equation above by $|G|$ yields

$$2d^2 + \sum_{i=1}^{s} \frac{|G|}{|C_G(g_i)|}\alpha_i\overline{\alpha_i} = 2d^2 + \sum_{i=1}^{s} \mu_i\alpha_i^2 = |G| = 2^n.$$

Since $d = 2^{(n-1)/2}$, we know $2d^2 = 2^n = |G|$, hence

$$2d^2 + \sum_{i=1}^{s} \mu_i\alpha_i^2 = |G| = 2d^2.$$

Then $\mu_i > 0$ and $\alpha_i \in \mathbb{Q}$ for all $1 \leq i \leq s$ forces $\alpha_i = 0$ for $1 \leq i \leq s$. Thus $\chi(g) = 0$ for $g \in G \setminus Z(G)$. □

Finally, we are ready to prove our main result.

5.3 Proof of main result

Theorem 5.8. *Let G be a non-abelian quadratic total character group and a p-group, with $|G| = p^n$. Then $p = 2$ and G is extraspecial.*

Proof. By Corollary 5.3, we know that $p = 2$. It remains to show that G is an extraspecial 2-group, or in other words, that $G' = Z(G)$ and that $|G'| = |Z(G)| = 2$. To do this, we introduce some new notation. From Lemma 5.6 we have $\tau = \chi^2 + \chi + c$, with $c \in \mathbb{Z}$. Let

$$\Sigma_0 = \sum_{k=0}^{\infty} N_{2^k}; \quad \Sigma_1 = \sum_{k=0}^{\infty} N_{2^k}2^k; \quad \Sigma_2 = \sum_{k=0}^{\infty} N_{2^k}2^{2k}.$$

From the Column Orthogonality Relations, if $\chi_1, \ldots, \chi_\ell$ are the irreducible characters of G, then

$$\sum_{k=1}^{\ell} \chi_i(1)^2 = \sum_{k=1}^{\ell} \chi_i(1)\overline{\chi_i(1)} = |C_G(1)| = |G| = 2^n,$$

hence

$$\sum_{k=0}^{\infty} N_{2^k} 2^{2k} + d^2 = 2^n.$$

From this we obtain $\Sigma_2 = 2^n - d^2 = 2^n - 2^{n-1} = 2^{n-1}$. Then from (5.1) and (5.2) we see that $\tau(1) = d + \Sigma_1$ and $\tau(z) = -d + \Sigma_1$. Since $\chi(g) = 0$ for $g \neq 1, z$ by Lemma 5.7, we have for $g \neq 1, z$ that $\tau(g) = \chi(g)^2 + \chi(g) + c = c$. Thus the values of τ (in the same order as in (5.4)) are

$$d + \Sigma_1, -d + \Sigma_1, c, c, \ldots, c.$$

We want to show that $c = 0$. Let

$$r = 1 + \sum_{k=0}^{\infty} N_{2^k} = 1 + \Sigma_0 \geq 3,$$

be the number of conjugacy classes of G. Let the classes be $C_1 = \{1\}, C_2 = \{z\}, C_3, \ldots, C_r$, with representatives $g_1, \ldots, g_r$, and let $\mu_i = |C_i|, i \leq r$.

By Proposition 1.4, the inner product of τ with the principal character χ_0 is

$$\langle \tau, \chi_0 \rangle = \frac{\tau(1)\overline{\chi_0(1)}}{|G|} + \frac{\tau(z)\overline{\chi_0(z)}}{|G|} + \sum_{i=3}^{r} \frac{\tau(g_i)\overline{\chi_0(g_i)}}{|C_G(g_i)|} = \frac{d + \Sigma_1}{|G|} + \frac{-d + \Sigma_1}{|G|} + \sum_{i=3}^{r} \frac{c}{|C_G(g_i)|}.$$

Since $\frac{|G|}{|C_G(g_i)|} = |C_i| = \mu_i$, and $\langle \tau, \chi_0 \rangle = 1$, multiplying by $|G|$ gives

$$2^n = |G| = (d + \Sigma_1) + (-d + \Sigma_1) + \sum_{i=3}^{r} \mu_i c = 2\Sigma_1 + (2^n - 2)c. \tag{5.5}$$

By a similar reasoning, since $\langle \tau, \tau \rangle = r$, taking the inner product of τ with itself yields

$$r \cdot 2^n = (d + \Sigma_1)^2 + (-d + \Sigma_1)^2 + \sum_{i=3}^{r} \mu_i c^2 = 2d^2 + 2\Sigma_1^2 + (2^n - 2)c^2. \tag{5.6}$$

Since $r = 1 + \Sigma_0$ and $2d^2 = 2^n$, Equation (5.6) gives $(1+\Sigma_0)2^n = 2^n + 2\Sigma_1^2 + (2^n-2)c^2$, and so we get

$$2^n\Sigma_0 = 2\Sigma_1^2 + (2^n-2)c^2. \tag{5.7}$$

From (5.5) we get $2^n = 2\Sigma_1 + (2^n-2)c$. Multiplying this last equation by Σ_1 and subtracting the resulting equation from (5.7) we get

$$2^n(\Sigma_0 - \Sigma_1) = (2^n-2)c^2 - (2^n-2)c\Sigma_1 = (2^n-2)(c^2 - c\Sigma_1). \tag{5.8}$$

Equation (5.8) shows that 2^{n-1} divides $c^2 - c\Sigma_1$.

Now $\Sigma_1 \leq \Sigma_2 = 2^{n-1}$, so (5.5) shows that $c \geq 0$. From Lemma 5.1(3) we have $c \leq 1$ and so we must have $c = 0$ or 1.

If $c = 1$, then (5.8) shows that 2^{n-1} divides $1 - \Sigma_1$, which is a contradiction since $1 \leq \Sigma_1 \leq 2^{n-1}$. Thus we must have $c = 0$.

From (5.5) we see that $\Sigma_1 = 2^{n-1}$, and then from (5.7) we see that $\Sigma_0 = 2^{n-1} = \Sigma_1$.

It follows that $\Sigma_0 = \Sigma_1 = \Sigma_2$. From this we see that $N_{2^k} = 0$ for $k > 0$ and $N_1 = 2^{n-1}$. Since N_1 is the number of linear characters of G, we see that G has 2^{n-1} linear characters. By [15, p. 174], the number of distinct linear characters of G is equal to $|G/G'|$, hence $|G/G'| = 2^{n-1}$. Since $|G| = 2^n$, we have $|G'| = |Z(G)| = 2$.

Finally, it remains to show that $G' = Z(G)$. By Theorem 1(2) of [6, p. 188], if H is a nontrivial normal subgroup of a p-group P, then H intersects the center $Z(P)$ nontrivially. Since G' is always a normal subgroup of G and we have shown that G' is nontrivial, we have $G' \cap Z(G) \neq 1$. But $|G'| = |Z(G)| = 2$, so $G' = Z(G)$. Thus G is extraspecial. $\square$

Chapter 6. Questions for Further Study

A natural topic of study following our investigation into the symmetric group is to determine if the alternating group A_n, the subgroup of S_n consisting of all even permutations, is a total character group for any values of n. Because the alternating group A_n is a subgroup of index 2 in S_n [6, p. 109], its character table is closely related to that of S_n. For example, any irreducible character of S_n restricted to A_n is either an irreducible character of A_n or is the sum of two irreducible characters of A_n [20, p. 48]. This fact and other connections between the two character tables may be utilized to determine if A_n is a total character group.

In addition to examining the total characters of other finite groups, another worthwhile pursuit is to more fully understand the significance of sharp characters and faithful characters to this research question. As shown in Section 2.6, a character need not be sharp in order to satisfy $f(\chi) = \tau$, but we think that some connection to sharp characters nonetheless exists, and this requires further investigation. Similarly, we observed that whenever one character χ of a group G satisfies $f(\chi) = \tau$, then every faithful character of G also satisfies the same property, and with the same monic polynomial $f(x) \in \mathbb{Z}[x]$. Whether this situation happens only in the groups previously discussed, in any group that meets certain conditions, or in all finite groups is a motivating topic for future study.

One final area of future study is that of rational total character groups. We say that a p-group G is a *rational total character group* if there is a polynomial $f_G(x) \in \mathbb{Q}[x]$ of degree p and an irreducible character $\chi \in \mathrm{Irr}(G)$ such that $f_G(\chi) = \tau_G$, where τ_G is the total character of G. Rational total character groups are a generalization of quadratic total character groups, and therefore our initial attempts at understanding them relied on methods similar to those used in Chapter 5. Our current conjecture concerning rational total character groups is:

Conjecture. *Let G be a p-group of order p^k. Then G is a rational total character group with $deg(f_G) = p$ if and only if $Z(G)$ is cyclic of order p and $G/Z(G)$ is elementary abelian.*

The following list summarizes the aforementioned questions for future study:

1. Is A_n, the alternating group, a total character group for $n \geq 4$?

2. What conditions can be placed on a finite group G to guarantee that G is a total character group?

3. If a finite group G is a total character group, do all of its faithful characters χ satisfy $f(\chi) = \tau$ for the same monic polynomial $f(x) \in \mathbb{Z}[x]$?

4. What is the relationship between sharp characters and total character groups?

5. Which finite simple groups are total character groups? We conjecture that no non-abelian finite simple groups are total character groups.

6. What are necessary and sufficient conditions for a p-group G to be a rational total character group?

   In investigating this question we were led to the following conjecture:

7. If $|G| = p^n$, p odd, and χ is a non-principal irreducible character of G, then $\langle \chi, \chi^k \rangle = 0$ for $1 < k < p$.

As in most areas of mathematical research, the total character remains an object not yet fully understood. This thesis expands upon and adds to to the current body of information concerning the total character. We hope that future research pertaining to the questions posed above will also provide a valuable contribution toward our understanding of total character groups and character theory in general.

The results of this thesis have been submitted for publication [11].

Appendix A. Proof of Theorem 4.1 for Small Values of n

Proof of Theorem 4.1 for $n = 5, 6$, and 7. Suppose that for $\chi_\lambda \in \mathrm{Irr}(S_n)$, $n = 5, 6, 7$, there exist $g, h \in S_n$ such that $\chi_\lambda(g) = \chi_\lambda(h)$, but $\tau(g) \neq \tau(h)$. Then if we suppose that $f(\chi_\lambda) = \tau$ for some $f(x) \in \mathbb{Z}[x]$, we see

$$\tau(g) = f(\chi_\lambda(g)) = f(\chi_\lambda(h)) = \tau(h),$$

which contradicts the fact that $\tau(g) \neq \tau(h)$. Therefore if this condition holds, there does not exist $f(x) \in \mathbb{Z}[x]$ such that $f(\chi_\lambda) = \tau$. Using the character tables for S_5, S_6, and S_7 reproduced from MAGMA, we can find such elements g, h for every irreducible character, so our theorem is true for $n = 5, 6, 7$. The tables are given on the following pages as Tables A.1, A.2, and A.3. The corresponding contradicting values are bolded, as well as listed next to the table.

	g_1	g_2	g_3	g_4	g_5	g_6	g_7
χ_1	**1**	**1**	1	1	1	1	1
χ_2	**1**	-1	**1**	1	-1	1	-1
χ_3	4	2	**0**	1	**0**	-1	-1
χ_4	4	-2	**0**	1	**0**	-1	1
χ_5	5	1	1	$\mathbf{-1}$	$\mathbf{-1}$	0	1
χ_6	5	-1	**1**	-1	**1**	0	-1
χ_7	6	0	-2	**0**	**0**	1	0
τ	26	0	2	2	0	1	0

χ	g	h
χ_1 :	g_1	g_2
χ_2 :	g_1	g_3
χ_3 :	g_3	g_5
χ_4 :	g_3	g_5
χ_5 :	g_4	g_5
χ_6 :	g_3	g_5
χ_7 :	g_4	g_5

Table A.1: Character Table and Total Character of S_5

	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}	g_{11}
χ_1	**1**	**1**	1	1	1	1	1	1	1	1	1
χ_2	1	−1	−1	1	1	**1**	**1**	−1	1	−1	−1
χ_3	5	−3	1	1	2	**−1**	**−1**	−1	0	0	1
χ_4	5	**−1**	3	1	**−1**	2	−1	1	0	−1	0
χ_5	5	3	**−1**	1	2	**−1**	−1	1	0	0	−1
χ_6	5	1	−3	1	**−1**	2	**−1**	−1	0	1	0
χ_7	9	3	3	**1**	0	0	**1**	−1	−1	0	0
χ_8	9	−3	−3	**1**	0	0	**1**	1	−1	0	0
χ_9	10	2	2	−2	1	**1**	0	0	0	**1**	−1
χ_{10}	10	−2	−2	−2	1	**1**	0	0	0	−1	**1**
χ_{11}	16	**0**	1	**0**	−2	−2	0	0	1	0	0
τ	76	0	0	4	4	4	0	0	1	0	0

χ	g	h
χ_1 :	g_1	g_2
χ_2 :	g_6	g_7
χ_3 :	g_6	g_7
χ_4 :	g_2	g_5
χ_5 :	g_3	g_6
χ_6 :	g_5	g_7
χ_7 :	g_4	g_7
χ_8 :	g_4	g_7
χ_9 :	g_6	g_{10}
χ_{10} :	g_6	g_{11}
χ_{11} :	g_2	g_4

Table A.2: Character Table and Total Character of S_6

	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}	g_{11}	g_{12}	g_{13}	g_{14}	g_{15}
χ_1	**1**	**1**	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	−1	−1	1	1	**1**	−1	**1**	1	1	−1	−1	1	−1	−1
χ_3	6	4	0	2	3	**0**	2	**0**	1	−1	1	0	−1	−1	−1
χ_4	6	−4	0	2	3	**0**	−2	**0**	1	−1	−1	0	−1	1	1
χ_5	14	4	0	**2**	−1	**2**	−2	0	−1	−1	1	0	0	−1	1
χ_6	14	−4	0	**2**	−1	**2**	2	0	−1	−1	−1	0	0	1	−1
χ_7	14	−6	−2	**2**	**2**	−1	0	0	−1	2	0	1	0	−1	0
χ_8	14	6	2	**2**	**2**	−1	0	0	−1	2	0	−1	0	1	0
χ_9	15	5	−3	−1	3	**0**	1	−1	**0**	−1	−1	0	1	0	1
χ_{10}	15	−5	3	−1	3	**0**	−1	−1	**0**	−1	1	0	1	0	−1
χ_{11}	20	0	0	−4	**2**	**2**	0	0	0	2	0	0	−1	0	0
χ_{12}	21	−1	3	**1**	−3	0	**1**	−1	1	1	−1	0	0	−1	1
χ_{13}	21	**1**	−3	**1**	−3	0	−1	−1	1	1	1	0	0	1	−1
χ_{14}	35	5	1	**−1**	**−1**	−1	−1	1	0	−1	−1	1	0	0	−1
χ_{15}	35	−5	−1	**−1**	**−1**	−1	1	1	0	−1	1	−1	0	0	1
τ	232	0	0	8	10	4	0	0	2	2	0	0	1	0	0

χ	g	h
χ_1 :	g_1	g_2
χ_2 :	g_6	g_8
χ_3 :	g_6	g_8
χ_4 :	g_6	g_8
χ_5 :	g_4	g_6
χ_6 :	g_4	g_6
χ_7 :	g_4	g_5
χ_8 :	g_4	g_5
χ_9 :	g_6	g_9
χ_{10} :	g_6	g_9
χ_{11} :	g_5	g_6
χ_{12} :	g_4	g_7
χ_{13} :	g_2	g_4
χ_{14} :	g_4	g_5
χ_{15} :	g_4	g_5

Table A.3: Character Table and Total Character of S_7

This completes our proof of Theorem 4.1 for $n = 5, 6$, and 7. $\square$

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