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An Algebra Isomorphism for the Landau-Ginzburg Mirror Symmetry Conjecture

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Abstract

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Landau-Ginzburg mirror symmetry takes place in the context of affine singularities in $\mathbb{C}^N$. Given such a singularity defined by a quasihomogeneous polynomial W and an appropriate group of symmetries G, one can construct the FJRW theory (see [3]). This construction fills the role of the A-model in a mirror symmetry proposal of Berglund and Hübsch [1]. The conjecture is that the A-model of W and G should match the B-model of a dual singularity and dual group (which we denote by W^{T} and G^{T}). The B-model construction is based on the Milnor ring, or local algebra, of the singularity. We verify this conjecture for a wide class of singularities on the level of Frobenius algebras, generalizing work of Krawitz [10]. We also review the relevant parts of the constructions.

Keywords: mirror symmetry, Landau-Ginzburg models, FJRW theory, mathematical physics, Frobenius algebra

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Chapter 1. Introduction

1.1 Background

Landau-Ginzburg mirror symmetry was inspired by an early proposal of Berglund and Hübsch [1] in 1993. This version of mirror symmetry involves a duality between constructions based on affine complex singularities with certain groups of symmetries, called *orbifold groups*. For such a pair, there are two constructions which are called the A-model and the B-model. The conjecture is that the A-model of a singularity and symmetry group should match the B-model of an appropriately chosen dual singularity and dual group. We call the dual singularity and dual group the *transpose singularity* and *transpose group*. This is suggestive of the construction of the dual singularity (see Section 2.1). In this paper, we consider the Frobenius algebra structures of the A-model and the B-model, although we should note that the conjecture extends to even larger structures.

For some time, the constructions of the A-model and the B-model were incomplete. In a series of papers [4, 2, 3] Fan, Jarvis, and Ruan resolved a conjecture of Witten and constructed a full cohomological field theory (called the FJRW theory) for a singularity and a group of symmetries. A restriction of this theory (to genus zero with three marked points) gives a Frobenius algebra for the A-model which we call $\mathscr{H}_{W,G}$, where W is the polynomial defining the singularity and G is the orbifold group.

For the B-model, in the case that the orbifold group is trivial, the Frobenius algebra is given by the Milnor ring (or local algebra) of the singularity. However, to formulate the conjecture in more generality, one needs to consider B-models with non-trivial orbifolding. The construction of the orbifold B-model as a vector space was given by Intrilligator and Vafa [5], however, until more recently the orbifold B-model was lacking a product structure. In [10], Krawitz followed a recipe of Kaufmann [8, 7, 9] and wrote down a multiplication for the orbifold Milnor ring, which we call $\mathscr{B}_{W,G}$. He also gave a general formula for the transpose group and resolved the conjecture completely on the level of vector spaces. He also proved the conjecture on the level of algebras for the case of the B-model with trivial orbifold group.

The purpose of this paper is to extend these results. We show that for a wide class of singularities, for any admissible orbifold group the A-model Frobenius algebra matches the dual orbifold B-model Frobenius algebra.

The FJRW theory is a geometric construction based on the moduli space of curves. It is thus interesting to see it match up with the orbifold Milnor ring, which is defined more algebraically. The product structure of the FJRW theory is determined by the genus-zero three-point correlators, which are given by a $\mathbb{C}$-valued function on $\mathscr{H}_{W,G}^{\otimes 3}$. The FJRW theory satisfies the axioms of a cohomological field theory, and these axioms allow us to compute some of these correlators in a straightforward way. When orbifolding by a trivial symmetry group on the B-side (and thus the maximal group on the A-side), the relevant insertions are mostly what we call *narrow* (called *Neveu-Schwarz* in [3] and [11]). In these cases, the axioms we mentioned do provide straightforward ways to compute the correlators. Thus, the case of the trivial group on the B-side is more tractable and was solved in [10].

When we orbifold by non-trivial symmetry groups on the B-side (and thus smaller groups on the A-side) the situation is more difficult, and instead of being narrow, more of the insertions may be *broad* (called *Ramond* in [3]). In many of these cases, the axioms do not give us enough information. The problem under study in such cases is a PDE-problem, and we do not yet have techniques to solve it explicitly in most cases. In a partial solution to this, we use an additional selection rule (see Property 2.20) that seems not to have been fully exploited in some previous papers.

Orbifolding by a smaller group on the A-side also introduces another difficulty. According to the classification in [13], quasihomogeneous polynomials that meet our non-degeneracy criteria are the decoupled sums of polynomials of three "atomic types". When the orbifold group is a product of groups acting on these sums (as in the case of the maximal symmetry group), the FJRW ring is the tensor product of the pieces, and the B-model can similarly be broken up as a tensor product (see Axiom 2.19 and Proposition 2.23). Thus, attention has been focused on these atomic types. However, if we consider more arbitrary groups that do not necessary break into a product of groups acting independently on the atomic pieces, this method does not apply directly. This paper introduces and justifies a new strategy to take advantage of the "breaking up into tensor products" techniques even in the case of more arbitrary orbifold groups. This also allows us to avoid computation of some of the difficult correlators. Essentially, for each product, we take a subalgebra containing the factors that can also be thought of as a subalgebra of a theory with a group that does break up as a direct product in a useful way.

We consider the case of a polynomial

$$W = \sum_{i=1}^{N} W_i,$$

where each W_i is of either Fermat type or loop type (according to the classification of Kreuzer and Skarke [13], see Section 3.1 of this paper), and prove the following:

Theorem 5.1. *If W is polynomial of the type described above, and G is any admissible orbifold group, then there is an isomorphism of Frobenius algebras*

$$\mathscr{H}_{W,G} \cong \mathscr{B}_{W^\intercal,G^\intercal}.$$

where $\mathscr{H}_{W,G}$ is the Frobenius algebra of the FJRW theory, and $\mathscr{B}_{W^\intercal,G^\intercal}$ is the Frobenius algebra of the orbifold Milnor ring for the transpose singularity and transpose group.

We also show explicitly how to make the pairing of the Frobenius algebras match. This was omitted in [10].

1.2 Overview

In Chapter 2, we review the construction of the A-model and the B-model of a quasihomogeneous polynomial, as well as some properties of these polynomials, for the convenience of the reader and to fix our notation. We focus on providing the definitions and facts needed for our computations rather than on completeness. Then in Chapter 3, we recall some facts about loop and Fermat polynomials and prove some new lemmas that we will need. In Chapter 4, we review the construction of the mirror map in [10] and discuss some issues that complicate our proof. In Chapter 5, we define our own variation of the mirror map and prove Theorem 5.1.

Chapter 2. Review of Construction

2.1 Quasihomogeneous Polynomials

We call a polynomial *invertible* if it has the same number of variables as monomials. We start with an quasihomogeneous, invertible polynomial in variables $X_1, \ldots X_N$:

$$W = \sum_{i=1}^{N} c_i \prod_{j=1}^{N} X_j^{a_{ij}} \in \mathbb{C}[X_1, \ldots, X_N].$$

The matrix $A = (a_{ij})$ encodes the exponents of the polynomial. We require the polynomial to have uniquely determined weights which give it weighted degree 1, i.e., that

$$A \begin{bmatrix} q_1 \\ \vdots \\ q_N \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

has a unique solution. The weighted degree of X_i is then q_i. Alternatively, we can clear the denominators of the q_i to get integer weights w_i. We also require that the polynomial defines an isolated singularity, i.e., the system of equations $\left\{\frac{\partial W}{\partial X_i} = 0\right\}$ has a unique solution at the origin. A polynomial satisfying these two conditions we call *non-degenerate*.

Since the c_i can be absorbed by rescaling the variables, in the sequel we take $c_i = 1$ without loss of generality.

One can form the transpose singularity W^{T} by taking the polynomial corresponding to the matrix A^{T}. This will again be a invertible, non-degenerate polynomial.

Example 2.1. If

$$W = x^3 y + y^7 z + z^4,$$

then the exponent matrix is

$$A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 7 & 1 \\ 0 & 0 & 4 \end{bmatrix}.$$

Then

$$A^{\mathsf{T}} = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 7 & 0 \\ 0 & 1 & 4 \end{bmatrix}$$

defines the transpose polynomial

$$W^{\mathsf{T}} = x^3 + xy^7 + yz^4.$$

Notation 2.2. We use boldface type to represent a column vector, and regular italic type to represent entries in the vector. Thus, by $\mathbf{g}$ we mean the vector $[g_1, \ldots, g_N]^{\mathsf{T}}$, where the N must be understood from context. We also write $\mathbf{1}$ for the vector $[1, \ldots, 1]^{\mathsf{T}}$.

There is an action of $(\mathbb{C}^*)^N$ on $\mathbb{C}[X_1, \ldots, X_N]$ where the tuple $(\lambda_1, \ldots, \lambda_N) \in (\mathbb{C}^*)^N$ acts on X_j by multiplication by λ_j.

Definition 2.3. The *maximal group of diagonal symmetries* $G_W^{\max}$ (or simply $G^{\max}$ if W is clear from context) is the maximal subgroup of $(\mathbb{C}^*)^N$ which fixes the polynomial W.

We prefer to think of symmetries $G_W^{\max}$ as a subgroup of $(\mathbb{Q}/\mathbb{Z})^N$, where the element $[\mathbf{g}]$ corresponding to the class of the vector $\mathbf{g} \in \mathbb{Q}^N$ acts on X_j by multiplication by $\exp(2\pi i g_j)$.

Definition 2.4. If $g = [\mathbf{\Theta}]$ with $0 \leq \Theta_j < 1$, then the Θ_j are called the *phases* of the group element g.

We can find a special set of generators for $G_W^{\max}$ as follows.

Definition 2.5. The group element corresponding to the class of the ith column of A^{-1} we call ρ_i.

Proposition 2.6. *The ρ_i generate the maximal symmetry group G^{max}. Similarly, the rows $\bar{\rho}_i$ of A^{-1} generate the maximal symmetry group of $W^\intercal$.*

Proof. If $[\mathbf{g}]$ is a symmetry of W, then $A\mathbf{g} = \mathbf{v} \in \mathbb{Z}^N$, which implies $\mathbf{g} = A^{-1}\mathbf{v}$, so $[\mathbf{g}] = \sum_i v_i \rho_i$. □

Definition 2.7. The symmetry

$$J := \sum_{j=1}^{N} \rho_j = [A^{-1}\mathbf{1}] = [\mathbf{q}]$$

is called the *exponential grading operator.*

Example 2.8. Consider $W = x^2y + xy^3 + z^5$. The exponent matrix is

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

The exponent matrix A for W is symmetric, thus $W^{\mathsf{T}} = W$.

The exponent matrix has inverse

$$A^{-1} = \begin{bmatrix} \frac{3}{5} & -\frac{1}{5} & 0 \\ -\frac{1}{5} & \frac{2}{5} & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix}.$$

The weights are

$$q_x = \frac{2}{5}, \quad q_y = \frac{1}{5}, \quad q_z = \frac{1}{5}.$$

The maximal symmetry group is generated by the columns of A^{-1}:

$$G^{\max} = \left\langle \left(\frac{3}{5}, -\frac{1}{5}, 0\right), \left(-\frac{1}{5}, \frac{2}{5}, 0\right), \left(0, 0, \frac{1}{5}\right) \right\rangle = \left\langle \left(0, 0, \frac{1}{5}\right), \left(\frac{1}{5}, \frac{3}{5}, 0\right) \right\rangle.$$

2.2 Milnor Rings

Definition 2.9. The *Jacobian ideal* $\mathrm{Jac}(W)$ of a polynomial W is the ideal generated by the partial derivatives:

$$\mathrm{Jac}(W) := \left(\frac{\partial W}{\partial X_1}, \ldots, \frac{\partial W}{\partial X_N}\right)$$

and the *Milnor ring* $\mathscr{Q}_W$ (also called the *local algebra*) is

$$\mathscr{Q}_W := \mathbb{C}[X_1, \ldots, X_N] / \mathrm{Jac}(W).$$

The Milnor ring is a finite-dimensional $\mathbb{C}$-vector space. Its dimension is given by the *Milnor number* μ_W. The Milnor number can be computed using the formula

$$\mu_W = \prod_{j=1}^{N} \left(\frac{1}{q_j} - 1\right).$$

The Milnor ring is graded by the weighted degree of monomials. The subspace of highest weighted degree is one-dimensional and has weighted degree equal to the *central charge* $\hat{c}$ of the singularity. The central charge is given by the formula

$$\hat{c} = \sum_{j=1}^{N}(1 - 2q_j).$$

Let $\operatorname{hess} W$ be the determinant of the Hessian matrix $\left(\frac{\partial^2 W}{\partial X_i \partial X_j}\right)_{ij}$. Then $\operatorname{hess} W$ spans the one-dimensional subspace of top weighted degree in the Milnor ring.

There is a pairing on the Milnor ring determined by

$$fg = \frac{\langle f, g\rangle}{\mu_W} \operatorname{hess} W + \text{ lower degree terms.}$$

Thus, if f and g are weighted-homogeneous elements of $\mathscr{Q}_W$, then the pairing is given by

$$\langle f, g\rangle = \begin{cases} \mu_W \frac{fg}{\operatorname{hess} W} & \text{if } fg \text{ has weighted degree } \hat{c}, \\ 0 & \text{otherwise.} \end{cases}$$

The quotient $fg/\operatorname{hess} W$ makes sense since the $\hat{c}$ degree subspace is one-dimensional, and so fg must be a scalar multiple of $\operatorname{hess} W$. This pairing makes the Milnor ring into a Frobenius algebra, i.e., it has the property that $\langle fg, h\rangle = \langle f, gh\rangle$.

Example 2.10. Consider the polynomial from Example 2.8, $W = x^2y + xy^3 + z^5$. The Jacobian ideal is

$$\left(2xy + y^3, x^2 + 3xy^2, 5z^4\right).$$

We can compute the Milnor number

$$\mu_w = (\frac{5}{2} - 1)(5 - 1)(5 - 1) = 24$$

and we can find a basis of monomials

$$\mathscr{Q}_W = \operatorname{span}(1, z, z^2, z^3, y, yz, yz^2, yz^3, y^2, y^2z, y^2z^2, y^2z^3, x, xz, xz^2, xz^3, xy, xyz, xyz^2,$$
$$xyz^3, xy^2, xy^2z, xy^2z^2, xy^2z^3).$$

The central change is

$$\hat{c} = \frac{7}{5}$$

and we see that the monomial xy^2z^3 has weighted degree $\frac{7}{5}$. We can compute the Hessian determinant as

$$\operatorname{hess} W = -80x^2z^3 - 180y^4z^3,$$

but using the Jacobian relations we can write this as an element of $\mathscr{Q}_W$ as

$$\operatorname{hess} W = 600xy^2z^3.$$

We can use this information to compute the pairing. For example,

$$\langle xy, yz^3 \rangle = 24\frac{1}{600} = \frac{1}{25}$$

while

$$\langle z, y \rangle = 0$$

since yz does not have top degree.

2.3 The A-model

The A-model is the so-called FJRW ring. We give a description in terms of Milnor rings which is more elementary than the full definition and sufficient for our computations.

We pick a subgroup of G of $G^{\max}$, and we require that $J \in G$. We call a group containing

J *admissible*. For each $g \in G$, we define W_g to be the restriction of the polynomial W to the coordinates that are fixed by g. The *unprojected* g *sector* $\mathscr{H}_g$ can be described by

$$\mathscr{H}_g \cong \mathscr{Q}_{W_g} \cdot dX_{i_1} \wedge \cdots \wedge dX_{i_{N_g}},$$

where N_g is the number of variables fixed by g, and $i_1, \ldots i_{N_g}$ are the indexes of the fixed coordinates. The sector is called *narrow* if $N_g = 0$, and *broad* otherwise. The group G acts on $\mathscr{H}_g$ by acting on the Milnor ring and the volume form— we note that the addition of the volume form gives the action of G a "determinant twist". The state space of the A-model is then given by taking the direct sum of these sectors and taking G invariants:

$$\mathscr{H}_{W,G} := \left(\bigoplus_{g \in G} \mathscr{H}_g \right)^G. \tag{2.1}$$

The Milnor ring always has a basis of monomials. Thus, we can write a basis for $\mathscr{H}_{W,G}$ of elements of the form

$$\lceil \mathbf{X}^{\mathbf{r}} \,;g \rfloor.$$

Here by $\mathbf{X}^{\mathbf{r}}$ we mean the monomial $\prod_{j=1}^{N_g} X_{i_j}^{r_j}$, which is an element of $\mathscr{Q}_g$. We say that this element is in the g-*sector*. We do not explicitly write the volume form, since it is determined by the group element. We should mention that the notation $\lceil \bullet \,; \bullet \rfloor$ is not standard, but was invented for this paper to avoid various problems with other notations.

We endow $\mathscr{H}_{W,G}$ with a pairing as follows. Since g and $-g$ fix the same coordinates, there is a natural isomorphism

$$I : (\mathscr{H}_g)^G \cong (\mathscr{H}_{-g})^G$$

$$\lceil \mathbf{X}^{\mathbf{r}} \,;g \rfloor \mapsto \lceil \mathbf{X}^{\mathbf{r}} \,;-g \rfloor$$

Then the pairing on the Milnor ring $\mathscr{Q}_g$ induces a pairing

$$\langle\cdot,\cdot\rangle_g : (\mathscr{H}_g)^G \otimes (\mathscr{H}_{-g})^G \to \mathbb{C}$$

by

$$\langle \lceil \mathbf{X}^{\mathbf{r}} \, ; g \rfloor , \lceil \mathbf{X}^{\mathbf{s}} \, ; -g \rfloor \rangle_g = \left\langle \lceil \mathbf{X}^{\mathbf{r}} \, ; g \rfloor , I^{-1}(\lceil \mathbf{X}^{\mathbf{s}} \, ; -g \rfloor) \right\rangle_{\mathscr{Q}_{W_g}} .$$

Then the pairing on $\mathscr{H}_{W,G}$ is the "direct sum" of these pairings— that is, if two basis elements are from sectors g and $-g$, we use the pairing $\langle\cdot,\cdot\rangle_g$ above, and otherwise, the pairing is 0. We write the matrix of this pairing (with respect to a fixed basis) as $\eta_{\alpha,\beta}$ and its inverse as $\eta^{\alpha,\beta}$.

The A-model is a graded Frobenius algebra with a new $\mathbb{Q}$-grading that we call the *W-degree*. For $\alpha \in \mathscr{H}_g$, the W-degree is given by

$$\deg_W \alpha = N_g + 2\sum_{j=1}^{N}(\Theta_j - q_j)$$

where the Θ_j are the phases of g.

The FJRW theory consists of a full cohomological field theory. The Frobenius algebra structure, however, comes from just the genus-zero, three-point correlators. These are a map

$$\langle\cdot,\cdot,\cdot\rangle : \mathscr{H}_{W,G}^{\otimes 3} \to \mathbb{C}.$$

If we wish to emphasize that the correlator is being computed in $\mathscr{H}_{W,G}$, we will use a superscript, as in $\langle\cdot,\cdot,\cdot\rangle^{W,G}$. We discuss the computation of the correlators in Section 2.4.

The product of the FJRW ring is given by

$$\alpha \star \beta = \sum_{\tau,\sigma} \langle \alpha, \beta, \tau \rangle \, \eta^{\tau,\sigma} \sigma \tag{2.2}$$

where the sum is over all pairs of elements σ, τ from a fixed basis.

The product $\star$ on the A-model respects the W-degree. That is, for basis elements α, β, we have

$$\deg_W(\alpha \star \beta) = \deg_W(\alpha) + \deg_W(\beta).$$

Example 2.11. Consider again $W = x^2y + xy^3 + z^5$. Let us pick the group

$$G = \langle J \rangle = \left\langle \left(\frac{2}{5}, \frac{1}{5}, \frac{1}{5} \right) \right\rangle.$$

The restricted polynomial $W_{(0,0,0)}$ is just W, since $(0,0,0)$ fixes all three variables. Thus

$$\mathscr{H}_{(0,0,0)} = \mathbb{Q}_W \cdot dx \wedge dy \wedge dz.$$

Examining the basis for $\mathbb{Q}_W$ from Example 2.10, we see that the only G-invariants are

$$xyz^3 \, dx \wedge dy \wedge dz, \quad xy^2z^2 \, dx \wedge dy \wedge dz, \quad y \, dx \wedge dy \wedge dz, \quad z \, dx \wedge dy \wedge dz.$$

Any other group element g has no fixed variables. Thus, $W_g = 0$ and $\mathscr{Q}_{W_g} \cong \mathbb{C}$. Thus $\mathscr{H}_g \cong \mathbb{C}$ with no volume form. The action of G on $\mathscr{H}_g$ is trivial, so we see that $\mathscr{H}_{W,G}$ has the following basis (sorted by W-degree):

Basis element	Degree
$\lceil 1\,;\left(\frac{2}{5},\frac{1}{5},\frac{1}{5}\right)\rfloor$	0
$\lceil 1\,;\left(\frac{1}{5},\frac{3}{5},\frac{3}{5}\right)\rfloor$	$\frac{6}{4}$
$\lceil xyz^3\,;(0,0,0)\rfloor$	$\frac{7}{5}$
$\lceil xy^2z^2\,;(0,0,0)\rfloor$	$\frac{7}{5}$
$\lceil y\,;(0,0,0)\rfloor$	$\frac{7}{5}$
$\lceil z\,;(0,0,0)\rfloor$	$\frac{7}{5}$
$\lceil 1\,;\left(\frac{4}{5},\frac{2}{5},\frac{2}{5}\right)\rfloor$	$\frac{8}{5}$
$\lceil 1\,;\left(\frac{3}{5},\frac{4}{5},\frac{4}{5}\right)\rfloor$	$\frac{14}{5}$

2.4 A-model axioms

The genus-zero, three-point correlators may be difficult to compute in general. However, they satisfy some axioms and properties that allow us to compute them in a straightforward way in many cases.

For this section, we assume that for $i = 1, 2, 3$, $\gamma_i \in (\mathscr{H}_{g_i})^G \subset \mathscr{H}_{W,G}$ are basis elements of the FJRW ring, with the phases of g_i being Θ^i_j.

Axiom 2.12 (Dimension)**.** *A genus 0 three point correlator $\langle\gamma_1, \gamma_2, \gamma_3\rangle$ vanishes unless*

$$\sum_{i=1}^{3} \deg_W \gamma_i = 2\hat{c}.$$

Axiom 2.13 (Symmetry)**.** *Let $\sigma \in S_3$. Then*

$$\langle\gamma_1, \gamma_2, \gamma_3\rangle = \left\langle\gamma_{\sigma(1)}, \gamma_{\sigma(2)}, \gamma_{\sigma(3)}\right\rangle$$

The next axioms relate to the degree of certain line bundles. For genus zero, these degrees are given by

$$l_j = q_j - \sum_{i=1}^{3} \Theta_j^i$$

for $j = 1, 2, 3$.

Axiom 2.14 (Integer Line Bundle Degrees). *The correlator $\langle \gamma_1, \gamma_2, \gamma_3 \rangle$ vanishes unless $l_j \in \mathbb{Z}$ for $j = 1, \ldots, N$.*

The following observation follows from Axiom 2.14 and is recorded in [10].

Proposition 2.15. *Suppose $\langle \gamma_1, \gamma_2, \gamma_3 \rangle$ does not vanish. Then $g_3 = J - g_1 - g_2$. Thus $\gamma_1 \star \gamma_2 \in \mathscr{H}^G_{g_1+g_2-J}$.*

Proof. Notice that Axiom 2.14 tells us that

$$J - g_1 - g_2 - g_3 = 0$$

so

$$g_3 = J - g_1 - g_2.$$

Then the definition of the A-side multiplication and pairing implies that the product is in the sector $-g_3$, which gives the desired result. □

Remark 2.16. The A-model product is not G graded, but it has a related property. Another way to state the result of Proposition 2.15 is

$$\mathscr{H}^G_{g_1+J} \star \mathscr{H}^G_{g_2+J} \subset \mathscr{H}^G_{g_1+g_2+J}$$

Axiom 2.17 (Concavity). *If the correlator is not required to vanish by Axiom 2.12 or Axiom 2.14, and if $l_j < 0$ for all $j = 1, 2, 3$, then $\langle \gamma_1, \gamma_2, \gamma_3 \rangle = 1$.*

.

Axiom 2.18 (Pairing)**.** *Let* $\mathbb{1} = \lceil 1\,; J \rfloor$*. Then*

$$\langle \gamma_1, \gamma_2, \mathbb{1} \rangle = \langle \gamma_1,\ \gamma_2 \rangle \,.$$

One can check from this axiom that $\mathbb{1}$ is the identity element (with respect to the multiplication (2.2)) in the FJRW ring. Do not confuse the identity element of the ring with the elements in the identity sector (the sector corresponding to the group identity of G).

For the next axiom, notice that for two singularities W_1 and W_2, we have a natural isomorphism $\mathscr{Q}_{W_1+W_2} \cong \mathscr{Q}_{W_1} \otimes \mathscr{Q}_{W_2}$. We state this axiom in somewhat more detail than it appears in, for example, [10] or [11].

Axiom 2.19 (Sums of Singularities)**.** *Suppose* W_1 *and* W_2 *are non-degenerate, quasihomogeneous polynomials with no variables in common. Suppose* G_1 *and* G_2 *are admissible groups of diagonal symmetries for* W_1 *and* W_2 *respectively. Then* $G_1 \oplus G_2$ *is an admissible group of diagonal symmetries for* $W_1 + W_2$*. Suppose*

$$\lceil m_i n_i\,; g_i + h_i \rfloor \in \mathscr{H}_{W_1+W_2, G_1 \oplus G_2}$$

for $i = 1, 2, 3$*, with* $m_i \in \mathscr{Q}_{W_1}$*,* $n_i \in \mathscr{Q}_{W_2}$*,* $g_i \in G_1$*, and* $h_i \in G_2$*. Then the three point correlator*

$$\langle \lceil m_1 n_1\,; g_1 + h_1 \rfloor, \lceil m_2 n_2\,; g_2 + h_2 \rfloor, \lceil m_3 n_3\,; g_3 + h_3 \rfloor \rangle^{W_1+W_2, G_1 \oplus G_2},$$

has the same value as

$$\langle \lceil m_1\,; g_1 \rfloor, \lceil m_2\,; g_2 \rfloor, \lceil m_3\,; g_3 \rfloor \rangle^{W_1, G_1} \cdot \langle \lceil n_1\,; h_1 \rfloor, \lceil n_2\,; h_2 \rfloor, \lceil n_3\,; h_3 \rfloor \rangle^{W_2, G_2}.$$

This gives an isomorphism

$$\mathscr{H}_{W_1,G_1} \otimes \mathscr{H}_{W_2,G_2} \cong \mathscr{H}_{W_1+W_2,G_1\oplus G_2}.$$

The following property is an additional selection rule which is crucial to our proof.

Property 2.20 ($G^{\max}$-invariance of correlators)**.** *The three point correlator is invariant under the action of G^{max}, i.e.*

$$\langle h\gamma_1, h\gamma_2, h\gamma_3\rangle = \langle \gamma_1, \gamma_2, \gamma_3\rangle$$

for all $h \in G^{max}$.

One can verify Property 2.20 by examining the construction of the virtual class. See [3] for details of the construction.

Remark 2.21. Property 2.20 gives us another selection rule as follows. Recall that the correlators are $\mathbb{C}$-multilinear and that the action of h on a monomial is multiplication by a scalar in $\mathbb{C}$. Thus, if we take the product of these actions on each insertion, the correlator vanishes unless this product is 1.

If the monomial of γ_i is $\mathbf{X}^{\mathbf{r}_i}$, and $h = [\mathbf{h}] \in G_W^{\max}$, then this can be reduced to the criteria that

$$\sum_{i=1}^{3} (\mathbf{r}_i + \mathbf{1})^{\mathsf{T}}\mathbf{h} \in \mathbb{Z}.$$

(where in each summand we restrict $\mathbf{h}$ and $\mathbf{r}_i$ to only the fixed coordinates of g_i). The $\mathbf{1}$ comes from the determinant twist (the h-action on the volume form in $\mathscr{H}_g$).

Example 2.22. Continuing with Example 2.11, let us compute the product

$$\left\lceil 1\,; \left(\frac{1}{5}, \frac{3}{5}, \frac{3}{5}\right)\right\rfloor \star \left\lceil 1\,; \left(\frac{4}{5}, \frac{2}{5}, \frac{2}{5}\right)\right\rfloor.$$

To use Proposition 2.15, we compute

$$J - \left(\frac{1}{5}, \frac{3}{5}, \frac{3}{5}\right) - \left(\frac{4}{5}, \frac{2}{5}, \frac{2}{5}\right) = \left(\frac{2}{5}, \frac{1}{5}, \frac{1}{5}\right) - \left(\frac{1}{5}, \frac{3}{5}, \frac{3}{5}\right) - \left(\frac{4}{5}, \frac{2}{5}, \frac{2}{5}\right)$$
$$= \left(\frac{2}{5}, \frac{1}{5}, \frac{1}{5}\right).$$

Thus we need only compute correlators of the form

$$\left\langle \left\lceil 1\,; \left(\frac{1}{5}, \frac{3}{5}, \frac{3}{5}\right)\right\rfloor, \left\lceil 1\,; \left(\frac{4}{5}, \frac{2}{5}, \frac{2}{5}\right)\right\rfloor, \left\lceil m\,; \left(\frac{2}{5}, \frac{1}{5}, \frac{1}{5}\right)\right\rfloor \right\rangle.$$

The only possible monomial for m is 1. The value of this correlator is 1. This can be computed using either Axiom 2.17 or Axiom 2.18. We also need to compute the appropriate entry of the inverse pairing matrix. Since the pairing matrix is a symmetric block matrix with only one non-zero block in each block-row, we only need to compute the inverse of the appropriate block. In this case, we are concerned with the block containing $\lceil 1\,; \frac{2}{5}, \frac{1}{5}, \frac{1}{5}\rfloor$, which is a 1×1-block since the only thing that pairs with it $\lceil 1\,; (\frac{3}{5}, \frac{4}{5}, \frac{4}{5})\rfloor$. The pairing of these two is 1. Thus, the product is

$$\left\lceil 1\,; \left(\frac{1}{5}, \frac{3}{5}, \frac{3}{5}\right)\right\rfloor \star \left\lceil 1\,; \left(\frac{4}{5}, \frac{2}{5}, \frac{2}{5}\right)\right\rfloor = \left\langle \left\lceil 1\,; \left(\frac{1}{5}, \frac{3}{5}, \frac{3}{5}\right)\right\rfloor, \left\lceil 1\,; \left(\frac{4}{5}, \frac{2}{5}, \frac{2}{5}\right)\right\rfloor, \left\lceil 1\,; \frac{2}{5}, \frac{1}{5}, \frac{1}{5}\right\rfloor \right\rangle \times$$
$$\cdots \times \eta^{\lceil 1\,; (\frac{2}{5}, \frac{1}{5}, \frac{1}{5})\rfloor, \lceil 1\,; (\frac{3}{5}, \frac{4}{5}, \frac{4}{5})\rfloor} \left\lceil 1\,; \left(\frac{3}{5}, \frac{4}{5}, \frac{4}{5}\right)\right\rfloor$$
$$= \left\lceil 1\,; \left(\frac{3}{5}, \frac{4}{5}, \frac{4}{5}\right)\right\rfloor.$$

2.5 The B-model

The construction of the state space of the B-model is very similar to the Milnor ring construction of the A-model state space. This time, instead of requiring that the orbifold group G contain J, we require it to be contained in SL_N. Here, we are thinking of the elements of G as linear transformations of $\mathbb{C}^N$. In our notation, $[\mathbf{g}] \in SL_N$ is equivalent to $\sum_i g_i \in \mathbb{Z}$.

We take unprojected sectors

$$\mathscr{B}_g = \mathscr{Q}_{W_g} \cdot dX_{i_1} \wedge \cdots \wedge dX_{i_{N_g}}$$

and then take G-invariants of the direct sum:

$$\mathscr{B}_W = \left(\bigoplus_g \mathscr{B}_g \right)^G .$$

The pairing is defined in the same way as the A-model.

A product was suggested in [10], making the B-model into a Frobenius algebra. This product is a generalization of the product on the Milnor ring (which is just the product on the polynomial quotient ring). Contrast this with the very geometric definition of the A-model product.

To define the B-model product, first note that for all g there is a surjective homomorphism of Milnor rings $\mathscr{Q}_e \to \mathscr{Q}_g$ given by setting all variables not fixed by g equal to 0. Thus, $\mathscr{Q}_g$ may be thought of as a cyclic $\mathscr{Q}_e$-module with generator $\lceil 1\,; g \rfloor$. Thus the multiplication is determined by the products of the module generators $\lceil 1\,; g \rfloor$. The B-model product is G graded. The product of the module generators $\lceil 1\,; g \rfloor$ and $\lceil 1\,; h \rfloor$ is determined by choice of elements $\gamma_{g,h} \in \mathscr{Q}_{g+h}$ via

$$\lceil 1\,; g \rfloor \star \lceil 1\,; h \rfloor = \gamma_{g,h} \lceil 1\,; g+h \rfloor .$$

Let I_g, I_h, I_{g+h} be the sets of indexes fixed by the group elements g, h, and $g+h$, respectively. Then $\gamma_{g,h}$ is given by

$$\gamma_{g,h} = \begin{cases} (\mu_{g\cap h} \operatorname{hess} W_{g+h}) \,/\, (\mu_{g+h} \operatorname{hess} W_{g\cap h}) & \text{if } I_g \cup I_h \cup I_{g+h} = \{1, \dots, N\} \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

where by $W_{g\cap h}$ we mean W restricted to the variables fixed by both g and h, and μ_g and

$\mu_{g\cap h}$ are the Milnor numbers for W_g and $W_{g\cap h}$ respectively. In [10], it is shown that this product is associative. It is also possible to show that the quotient of hessians is always a polynomial.

The orbifold Milnor ring also has a $\mathbb{Q}$-grading that matches the W-degree in our mirror symmetry. For an element $\lceil m\,; g\rfloor$, with the *non-fixed* variables of g being $\{i_1, \ldots, i_{M_g}\}$, the grading can be computed by the formula

$$\deg \lceil m\,; g\rfloor = M_g + 2\deg m - 2\sum_{j=1}^{M_g} q_{i_j},$$

where $\deg m$ is the weighted degree of the monomial m.

The orbifold Milnor ring has the tensor product property analogous to Axiom 2.19.

Proposition 2.23. *Suppose W_1 and W_2 are non-degenerate, quasihomogeneous polynomials with no variables in common. Suppose G_1 and G_2 are groups of diagonal symmetries contained in SL_N. Then $G_1 \oplus G_2$ is contained in SL_N and is a group of diagonal symmetries for $W_1 + W_2$. There is an isomorphism*

$$\mathscr{B}_{W_1,G_1} \otimes \mathscr{B}_{W_2,G_2} \cong \mathscr{B}_{W_1+W_2,G_1\times G_2}.$$

Proof. This is easy to check from the definitions. One uses the facts that $\operatorname{hess}(W_1+W_2) = \operatorname{hess} W_1 + \operatorname{hess} W_2$ and $\mu_W = \mu_{W_1}\mu_{W_2}$. $\square$

The motivation for the orbifold Milnor ring product comes from orbifold cohomology. Suppose g, h, and k are group elements with $g + h + k = 0$. We have a diagram

$$\begin{tikzcd}
\mathscr{B}_g \arrow[dr, "r_g"] & & \mathscr{B}_h \arrow[dl, "r_h"'] & \\
& \mathscr{B}_{(g,h,k)} \arrow[r, bend right, "r_k^*"'] & \mathscr{B}_k \arrow[l, "r_k"'] \arrow[r, "\cong"] & \mathscr{B}_{g+h}
\end{tikzcd}$$

Here $\mathscr{B}_{(g,h,k)}$ is the Milnor ring of W restricted to the common fixed variables of g, h, and k, and the map r_g is the map that sets all variables not fixed by g, h and k to 0. Since we have a non-degenerate pairing, one can get the dual map r_k^*. To multiply an element of $\mathscr{B}_g$ with an element of $\mathscr{B}_h$, one maps them both into $\mathscr{B}_{(g,h,k)}$ and computes their product with some analog of the virtual class of orbifold cohomology. In our case this is either 1 or 0 depending on the rule described above. This is necessary to ensure associativity. Then one maps by r_k^* and then by the natural isomorphism between $\mathscr{B}_k$ and $\mathscr{B}_{-k} = \mathscr{B}_{g+h}$ to land in the correct sector to preserve the G grading. Working out the dual map results in formula (2.3) above. Our paper offers further evidence that this is the correct product for the B-side.

Example 2.24. Let us use the polynomial $W^T = x^2y + xy^3 + z^5$. We pick the symmetry group

$$G^{\mathsf{T}} = \left\langle \left(\frac{1}{5}, \frac{3}{5}, \frac{1}{5} \right) \right\rangle.$$

We will see later the method of constructing the transpose group G^{T}. Notice that the sum of the entries in the group generator is an integer. We can compute a basis for $\mathscr{B}_{W^{\mathsf{T}},G^{\mathsf{T}}}$ as

Basis element	Degree
$\lceil 1\,;(0,0,0)\rfloor$	0
$\lceil yz^2\,;(0,0,0)\rfloor$	$\frac{6}{5}$
$\lceil 1\,;\left(\frac{1}{5},\frac{3}{5},\frac{1}{5}\right)\rfloor$	$\frac{7}{5}$
$\lceil 1\,;\left(\frac{2}{5},\frac{1}{5},\frac{2}{5}\right)\rfloor$	$\frac{7}{5}$
$\lceil 1\,;\left(\frac{3}{5},\frac{4}{5},\frac{3}{5}\right)\rfloor$	$\frac{7}{5}$
$\lceil 1\,;\left(\frac{4}{5},\frac{2}{5},\frac{4}{5}\right)\rfloor$	$\frac{7}{5}$
$\lceil xyz\,;(0,0,0)\rfloor$	$\frac{8}{5}$
$\lceil xy^2z^3\,;(0,0,0)\rfloor$	$\frac{14}{5}$

.

Notice that the dimension of the vector space and the degrees match those of Example 2.11, as guaranteed by the theorem in [10].

We can compute the multiplication. For example,

$$\left\lceil 1\,;\left(\frac{1}{5},\frac{3}{5},\frac{1}{5}\right)\right\rfloor \star \left\lceil 1\,;\left(\frac{4}{5},\frac{2}{5},\frac{4}{5}\right)\right\rfloor = (\operatorname{hess} W/\mu_W)\left\lceil 1\,;0\right\rfloor = 25\left\lceil xy^2z^3\,;(0,0,0)\right\rfloor$$

but

$$\left\lceil 1\,;\left(\frac{1}{5},\frac{3}{5},\frac{1}{5}\right)\right\rfloor \star \left\lceil 1\,;\left(\frac{3}{5},\frac{4}{5},\frac{3}{5}\right)\right\rfloor = 0$$

since $\gamma_{g,h} = 0$.

Chapter 3. Fermat and Loop Polynomials

3.1 Classification and Properties

Invertible non-degenerate quasihomogeneous polynomials are completely classified. The usual reference is [13]. Any such polynomial is the decoupled sum of polynomials of one of three types. There is the *Fermat* type:

$$W = X^a,$$

the *loop* type:

$$W = X_1^{a_1}X_2 + X_2^{a_2}X_3 + \cdots + X_N^{a_N}X_1,$$

and the *chain* type:

$$W = X_1^{a_1}X_2 + X_2^{a_2}X_3 + \cdots + X_N^{a_N}.$$

We also assume that $a_i \geq 2$, so that there are no terms of the form X_iX_j. Notice that taking the transpose of any of these preserves the atomic type. The variables, however, will be in the reverse order.

Example 3.1. The polynomial $W = x^2y + xy^3 + z^5$ that we have been using in our examples is a sum of a two-variable loop and a Fermat.

Remark 3.2. One can easily check that both the loop type and the Fermat type polynomials have the property that an element of $G^{\max}$ either fixes all variables (i.e., is the identity) or has non-trivial action on all of them. This property will be important in our proof.

We recall the following lemma from [12] which we will need.

Lemma 3.3. *The Milnor ring of a loop type polynomial is generated over* $\mathbb{C}$ *by the basis* $\{\prod_{i=1}^{N} X_i^{\alpha_i} : 0 \leq \alpha_i \leq a_i - 1\}$ *and has dimension* $\mu_W = \prod_{i=1}^{N} a_i$.

Notice that the element of top degree is $\mathbf{X}^{\mathbf{a-1}}$, so the Hessian is scalar multiple of $\mathbf{X}^{\mathbf{a}-1}$.

The analogous lemma for Fermat types is obvious.

Lemma 3.4. *The Milnor ring of a Fermat type polynomial is generated over* $\mathbb{C}$ *by the basis* $\{X^\alpha : 0 \leq \alpha \leq a_i - 2\}$ *and has dimension* $\mu_W = a - 1$.

To define the mirror map, we will also need the following lemma, also from [12].

Lemma 3.5. *Let* W *be a loop polynomial. Any symmetry other than* $-J$ *of* W *may be written uniquely as*

$$g = A^{-1}\boldsymbol{\alpha} = \sum_{i=1}^{N} \alpha_i \rho_i$$

with $0 \leq \alpha_i \leq a_i - 1$. *If* N *is even then*

$$-J = \sum_{i \text{ even}} (a_i - 1)\rho_i = \sum_{i \text{ odd}} (a_i - 1)\rho_i.$$

(If N *is odd, then* $-J$ *cannot be written in the form described here.)*

We restate this in a way that is often more useful for us.

Corollary 3.6. *Let* W *be a loop polynomial. Then any non-trivial symmetry of* W *may be written uniquely as*

$$J + \sum_{i=1}^{N} \alpha_i \rho_i$$

with $0 \le \alpha_i \le a_i - 1$.

If N *is even, then the identity can be written as either*

$$0 = J + \sum_{i \text{ even}} \rho_i^{a_i - 1} = J + \sum_{i \text{ odd}} \rho_i^{a_i - 1}.$$

(If N *is odd, then the identity cannot be written in the form described here.)*

The corresponding fact for Fermat is obvious:

Lemma 3.7. *Let* W *be a Fermat polynomial. Then the symmetries of* W *are precisely* $\{[\frac{\alpha}{a}] : 0 \le \alpha \le a - 1\} = \{\alpha\rho : 0 \le \alpha \le a - 1\}$. *Every non-trivial symmetry can be written uniquely as* $\alpha\rho + J$ *with* $0 \le \alpha \le a - 2$.

In the sequel, when we are given elements of the Milnor rings and symmetry groups associated to Fermat and loop type polynomials, we will assume that they are written in the special forms described here.

3.2 Some new loop lemmas

We will need some more lemmas about loops that we prove here. Suppose W is a loop type polynomial

$$W = X_1^{a_1} X_2 + X_2^{a_2} X_3 + \cdots + X_N^{a_N} X_1,$$

with exponent matrix

$$A = \begin{bmatrix} a_1 & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & 1 \\ 1 & 0 & \dots & 0 & a_N \end{bmatrix}$$

and let $\mathbf{a} = [a_1, \ldots, a_N]^\mathsf{T}$. Notice that these lemmas apply also to W^T, since W^T is also a loop polynomial.

Notation 3.8. In the following lemmas, we take the indexes modulo N, i.e., if $i = N$ then $a_{i+1} = a_1$.

Lemma 3.9. *For a loop type polynomial with exponent matrix A, we have*

$$-J = [A^{-1}\mathbf{a}].$$

Proof. Observe that

$$\begin{aligned} -\mathbf{1} &= \mathbf{a} - A\mathbf{1} \\ -A^{-1}\mathbf{1} &= A^{-1}\mathbf{a} - \mathbf{1} \\ -J &= [A^{-1}\mathbf{a}] \end{aligned}$$

□

Lemma 3.10. *If $\mathbf{X}^\mathbf{t} \in \mathscr{Q}_W$ is a scalar multiple of the hessian, then $[(A^\mathsf{T})^{-1}(\mathbf{t} + \mathbf{2})] = 0$.*

Proof. Recall that $\mathbf{X}^{\mathbf{a}-\mathbf{1}}$ is a multiple of the hessian. Notice that if $\mathbf{t} = \mathbf{a} - \mathbf{1}$, then the result follows from Lemma 3.9. For a loop polynomial, the Jacobian relations are $X_i^{a_i} = -a_{i+1}X_{i+1}^{a_{i+1}-1}X_{i+2}$. Using one of these relations corresponds to adding or subtracting the vector $\mathbf{u}_i = [0, \ldots, 0, a_i, -(a_{i+1} - 1), -1, 0, \ldots, 0]^\mathsf{T}$ to $\mathbf{t}$, where the first non-zero entry is in the ith spot and we "wrap around" as necessary. Thus, if $\mathbf{X}^\mathbf{t}$ is a multiple of the hessian, then we will have $\mathbf{a} - \mathbf{1} = \mathbf{t} + \sum k_i\mathbf{u}_i$ for some integers k_i. We see then that it is sufficient to show for all i that $[(A^\mathsf{T})^{-1}\mathbf{u}_i] = 0$, i.e. that $(A^\mathsf{T})^{-1}\mathbf{u}_i = \mathbf{n}$ for some integer vector $\mathbf{n}$. If we let $\mathbf{n} = [0, \ldots, 0, 1, -1, 0, \ldots 0]^\mathsf{T}$, with the 1 in the ith spot, then $A^\mathsf{T}\mathbf{n} = \mathbf{u}_i$. □

Notation 3.11. In the next lemma we use the following variation of the Kronecker δ function:

$$\delta^i_{\text{even}} = \begin{cases} 1 & \text{if } i \text{ is even} \\ 0 & \text{if } i \text{ is odd} \end{cases}$$

with δ^i_{odd} defined similarly.

Lemma 3.12. *Suppose that*

$$\mathbf{v} = A\mathbf{n}$$

where $\mathbf{v}$ *has integer entries* $2 \leq v_i \leq 2a_i$. *Then one of the following is true:*

1. $\mathbf{v} = \mathbf{a} + \mathbf{1}$
2. $v_i = \delta^i_{odd}(2a_i - 2) + 2$
3. $v_i = \delta^i_{even}(2a_i - 2) + 2.$

The latter two cases can occur only if N *is even.*

Proof. We have a set of inequalities

$$a_i n_i + n_{i+1} \leq 2a_i \tag{3.1}$$

and

$$a_i n_i + n_{i+1} \geq 2. \tag{3.2}$$

Now, suppose that $n_j \leq 0$ for some j. Then, using (3.2), we see that $n_{j+1} \geq 2$. Then we use (3.1) to see that

$$a_{j+1}2 + n_{j+2} \leq 2a_{j+1}$$

which implies that $n_{j+2} \leq 0$.

This tells us that if any n_i is not 1, then the entries of the vector $\mathbf{n}$ must alternate being less than or equal to zero and being greater than or equal to 2. This is not possible, of course, if N is odd.

Suppose now that for some j, $n_j \geq m$, where $m \geq 3$ is an integer. Then equation (3.1) gives

$$\begin{aligned} a_j m + n_{j+1} &\leq 2a_i \\ n_{j+1} &\leq (2-m)a_j \end{aligned}$$

Then from (3.2) we have

$$\begin{aligned} a_{j+1}(2-m)a_j + n_{j+2} &\geq 2 \\ n_{j+2} &\geq 2 + a_j a_{j+1}(m-2) \\ &\geq 2 + 4(m-2) \\ &= m + (3m-6) \\ &\geq m+1. \end{aligned}$$

We can continue repeating this argument to show that we can find an entry of $\mathbf{n}$ larger than any natural number, which is of course impossible. So we see that $n_i \leq 2$ for all i. Using (3.2), we then have

$$a_i n_i + 2 \geq 2$$

so we see that $n_i \geq 0$. Thus, we see that either $\mathbf{n} = \mathbf{1}$, $\mathbf{n} = [2, 0, 2, 0, \ldots, 2, 0]^\mathsf{T}$, or $\mathbf{n} = [0, 2, 0, 2, \ldots, 0, 2]^\mathsf{T}$, and the latter two cases can occur only when n is even.

Now if $\mathbf{n} = \mathbf{1}$, then we see that $\mathbf{v} = A\mathbf{1} = \mathbf{a} + \mathbf{1}$, as desired.

Now, suppose that k is even and $\mathbf{n} = [2, 0, 2, 0 \ldots, 2, 0]^{\mathsf{T}}$. We see then that

$$\mathbf{v} = \begin{bmatrix} a_1 & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & 1 \\ 1 & 0 & \ldots & 0 & a_k \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ \vdots \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2a_1 \\ 2 \\ \vdots \\ 2a_{k-1} \\ 2 \end{bmatrix}$$

so for i odd we have $v_i = 2a_i$, and for i even we have $v_i = 2$. A symmetric argument applies when $\mathbf{n} = [0, 2, 0, 2, \ldots, 0, 2]^{\mathsf{T}}$. □

Remark 3.13. Suppose that $0 \leq r_i, s_i \leq a_i - 1$ (as in Lemma 3.5) and $[(A^{\mathsf{T}})^{-1}(\mathbf{r}+\mathbf{s}+\mathbf{2})] = 0$. Then we see that the vector $\mathbf{v} = \mathbf{r} + \mathbf{s} + \mathbf{2}$ satisfies the hypothesis of Lemma 3.12. We can then conclude that one of the following is true:

1. $\mathbf{r} + \mathbf{s} = \mathbf{a} - \mathbf{1}$
2. $r_i = s_i = \delta^i_{\text{odd}}(a_i - 1)$
3. $r_i = s_i = \delta^i_{\text{even}}(a_i - 1)$

Lemma 3.14. *If W has an even number of variables, then in the Milnor ring we have*

$$\prod_{i \text{ odd}} X_i^{2a_i - 2} = \prod_{i \text{ even}} (-a_i) \prod_{i=1}^{N} X_i^{a_i - 1}$$

and

$$\prod_{i \text{ even}} X_i^{2a_i - 2} = \prod_{i \text{ odd}} (-a_i) \prod_{i=1}^{N} X_i^{a_i - 1}.$$

(Thus these are also multiples of $\operatorname{hess} W$*.)*

Proof. By symmetry it suffices to prove the first equality. Notice that the Jacobian relations for a loop polynomial are $X_i^{a_i} = -a_{i+1}X_{i+1}^{a_{i+1}-1}X_{i+2}$. We apply this relation and get

$$\begin{aligned}
\prod_{i \text{ odd}} X_i^{2a_i-2} &= \prod_{i \text{ odd}} (-a_{i+1}) X_i^{a_i-2} X_{i+1}^{a_{i+1}-1} X_{i+2} \\
&= \prod_{i \text{ odd}} X_i^{a_i-2} X_{i+2} \prod_{i \text{ even}} (-a_i) X_i^{a_i-1} \\
&= \prod_{i \text{ odd}} X_i^{a_i-1} \prod_{i \text{ even}} (-a_i) X_i^{a_i-1} \\
&= \prod_{i \text{ even}} (-a_i) \prod_{i=1}^{N} X_i^{a_i-1}
\end{aligned}$$

□

Corollary 3.15. *Suppose now that W is a sum of Fermat type polynomials and loop type polynomials with exponent matrix A. Suppose $0 \le r_i, s_i \le a_i - 1$ for i corresponding to a variable in a loop polynomial, and $0 \le r_i, s_i \le a_i - 2$ for i corresponding to a variable in a Fermat polynomial. Then the following are equivalent:*

1. $[(A^{\mathsf{T}})^{-1}(\mathbf{r} + \mathbf{s} + \mathbf{2})] = 0$
2. $\mathbf{X}^{\mathbf{r}+\mathbf{s}}$ *is a scalar multiple of the hessian.*

Proof. First we notice that it suffices to prove this for an atomic loop type and an atomic Fermat type polynomial. For the Fermat type, it is obvious.

For the loop type, to see that (1) implies (2), first use Remark 3.13. Then apply Lemma 3.14, if necessary, to show that $\mathbf{X}^{\mathbf{r}+\mathbf{s}}$ is a scalar multiple of $\mathbf{X}^{\mathbf{a}-\mathbf{1}}$.

The implication (2) implies (1) is the content of Lemma 3.10. □

Chapter 4. The Mirror Map

4.1 The Transpose Group

We recall here the definition of the transpose group from [10]. If G is a symmetry group for the polynomial W with exponent matrix A, then the transpose group G^{T} is a symmetry group for the transpose polynomial W^{T}. Informally, one could say that the transpose group is the maximal group of symmetries that would fix all elements of G if the elements of G were interpreted as monomials (i.e. if the coefficients of the generators were interpreted as exponents of monomials). This interpretation hints at the mirror map described in the next section. More precisely, we have:

Definition 4.1. The dual group G^{T} is defined as a set to be

$$G^{\mathsf{T}} := \left\{[\mathbf{g}] \mid \mathbf{g}^{\mathsf{T}}\mathbf{a} \in \mathbb{Z} \text{ for all } [A^{-1}\mathbf{a}] \in G\right\} \tag{4.1}$$

$$= \left\{[\mathbf{g}] \mid \mathbf{g}^{\mathsf{T}}A\mathbf{b} \in \mathbb{Z} \text{ for all } [\mathbf{b}] \in G\right\} \tag{4.2}$$

One can check that G^{T} is a group and that the definition is independent of the presentation of the elements of G. Additionally, the transpose group has the following properties, which are verified in [10]:

- If G contains J, then G^{T} is contained in SL_N, and vice-versa.
- $(G^{\mathsf{T}})^{\mathsf{T}} = G$
- If $G' \leq G$, then $G^{\mathsf{T}} \leq (G')^{\mathsf{T}}$.

Example 4.2. Using the groups from our previous examples, one can check that if

$$G = \left\langle \left(\frac{2}{5}, \frac{1}{5}, \frac{1}{5}\right) \right\rangle$$

then

$$G^{\mathsf{T}} = \left\langle \left(\frac{1}{5}, \frac{3}{5}, \frac{1}{5} \right) \right\rangle.$$

For example, we check that

$$\begin{bmatrix} \frac{1}{5} & \frac{3}{5} & \frac{1}{5} \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} \frac{2}{5} \\ \frac{1}{5} \\ \frac{1}{5} \end{bmatrix} = 1.$$

(It is coincidence that these two groups have the same order. In fact, $|G^{\max} : G| = |G^T|$.)

4.2 The Vector Space Mirror Map

The vector space mirror map $\mathscr{B}_{W^{\mathsf{T}},G^{\mathsf{T}}} \to \mathscr{H}_{W,G}$ (inspired at least in part by [12]) is described in [10] as

$$\left\lceil \prod_j Y_j^{\alpha_j}\,; \sum_i (r_i+1)\bar{\rho}_i \right\rfloor \mapsto \left\lceil \prod_i X^{r_i}\,; \sum_j (\alpha_j+1)\rho_j \right\rfloor \tag{4.3}$$

Notice that the roles of the group elements and monomials are interchanged. The $+1$ appearing in the coefficients of the group elements can be thought of as corresponding to the volume form which we have suppressed.

In (4.3) the range of the index i should be the same on both sides, and the range of the index j should be the same on both sides. The index j should range over the indexes of the fixed coordinates of $\sum_i (r_i + 1)\bar{\rho}_i$ and the index i should range over the indexes of the fixed coordinates of $\sum_j (\alpha_j + 1)\rho_j$. This condition ensures that the mirror map is well defined except in the case of an even-variable loop.

There is some subtlety in the case of an even-variable loop polynomial. Suppose we want to map an element from the identity sector and that $\boldsymbol{\alpha}$ is such that $[A^{-1}(\boldsymbol{\alpha} + \mathbf{1})] = 0$. This happens precisely when either $\alpha_i = \delta^i_{\text{odd}}(a_i - 1)$ or $\alpha_i = \delta^i_{\text{even}}(a_i - 1)$ (see Corollary 3.6). In this case the range of i should be $\{1, \ldots, N\}$, thus, we must write the identity as $\sum_{i=1}^{N}(r_i +$

$1)\bar{\rho}_i$. By Corollary 3.6, there are two ways to do this. Thus, the description above leaves an ambiguity about how to map a certain two dimensional subspace of $\mathscr{B}_{W^\intercal,G^\intercal}$ to a two dimensional subspace of $\mathscr{H}_{W,G}$. However, these subspaces are homogeneous and have the right grading, so this is no obstruction to proving the graded vector space isomorphism. For example, one could simply pick one way to write the group identity for the case $\alpha_i = \delta^i_{\text{odd}}(a_i - 1)$ and the other for the case $\alpha_i = \delta^i_{\text{even}}(a_i - 1)$.

Example 4.3. We demonstrate the vector space mirror map for some of the basis elements of the FJRW ring and orbifold Milnor ring in Example 2.11 and Example 2.24.

$$\lceil 1\,;(0,0,0)\rfloor = \lceil x^0y^0z^0\,;0\rfloor \mapsto \left\lceil 1\,;A^{-1}\begin{bmatrix}0+1\\0+1\\0+1\end{bmatrix}\right\rfloor = \left\lceil 1\,;\left(\frac{2}{5},\frac{1}{5},\frac{1}{5}\right)\right\rfloor$$

$$\lceil yz^2\,;(0,0,0)\rfloor = \lceil x^0y^1z^2\,;0\rfloor \mapsto \left\lceil 1\,;A^{-1}\begin{bmatrix}0+1\\1+1\\2+1\end{bmatrix}\right\rfloor = \left\lceil 1\,;\left(\frac{1}{5},\frac{3}{5},\frac{3}{5}\right)\right\rfloor$$

$$\left\lceil 1\,;\left(\frac{1}{5},\frac{3}{5},\frac{1}{5}\right)\right\rfloor = \left\lceil 1\,;\left(A^\intercal\right)^{-1}\begin{bmatrix}0+1\\1+1\\0+1\end{bmatrix}\right\rfloor \mapsto \lceil x^0y^1z^0\,;0\rfloor = \lceil y\,;(0,0,0)\rfloor$$

$$\left\lceil 1\,;\left(\frac{4}{5},\frac{2}{5},\frac{4}{5}\right)\right\rfloor = \left\lceil 1\,;\left(A^\intercal\right)^{-1}\begin{bmatrix}1+1\\1+1\\3+1\end{bmatrix}\right\rfloor \mapsto \lceil x^1y^1z^3\,;0\rfloor = \lceil xyz^3\,;(0,0,0)\rfloor$$

4.3 Algebra Isomorphism

For the algebra isomorphism for unorbifolded Milnor ring, [10] uses the map $\mathscr{B}_{W^\intercal,0} \to \mathscr{H}_{W,G_W^{\max}}$ that acts on generators as follows:

$$Y_i \mapsto \lceil 1\,;\rho_i + J\rfloor\,. \tag{4.4}$$

Lemma 3.2 in [10] asserts that the product of two narrow sectors is given by

$$\lceil 1\,;[A^{-1}(\boldsymbol{\beta}+\mathbf{1})]\rfloor \star \lceil 1\,;[A^{-1}(\boldsymbol{\gamma}+\mathbf{1})]\rfloor = \lceil 1\,;[A^{-1}(\boldsymbol{\beta}+\boldsymbol{\gamma}+\mathbf{1})]\rfloor \tag{4.5}$$

as long as $\boldsymbol{\beta}+\boldsymbol{\gamma} \preceq \mathbf{a}-\mathbf{1}$ (componentwise) and $[A^{-1}(\boldsymbol{\beta}+\boldsymbol{\gamma}+\mathbf{1})] \neq 0$. It then follows that the algebra map (4.4) agrees with the vector space map (4.3) on the basis element $\mathbf{Y}^{\boldsymbol{\alpha}}$ as long as neither $\alpha_i = \delta^i_{\text{odd}}(a_i-1)$ nor $\alpha_i = \delta^i_{\text{even}}(a_i-1)$. In [10] it is shown that images of the generators Y_i satisfy the same relations as the Jacobian relations of the Milnor ring. Since the spaces have the same dimension, it then suffices to show that this map is surjective. As we noted, this map "almost" agrees with the vector space map, which is surjective. In the cases of $\alpha_i = \delta^i_{\text{odd}}(a_i-1)$ and $\alpha_i = \delta^i_{\text{even}}(a_i-1)$, one can check that the image does in fact land in the identity sector, so this map is a graded vector space homomorphism. Although it is omitted in [10], one can generalize a trick used in the computations of [11] to prove that the images of these two are linearly independent. We provide this proof in Appendix A. However, we do not know how to say more than that the images of $\prod_{i \text{ odd}} Y_i^{a_i-1}$ and $\prod_{i \text{ even}} Y_i^{a_i-1}$ under the map (4.4) are linearly independent in the two dimensional identity sector. Thus, in the case of the even-variable loop, we cannot explicitly describe the algebra isomorphism on all basis elements.

Our proof will use both of these maps to define a mirror map for "sum of Fermat and loop" types. We will also rescale these maps so that they give a Frobenius algebra isomorphisms rather than just an algebra isomorphism.

Chapter 5. Frobenius Algebra Isomorphism for Sums of Loops and Fermats

5.1 Set up

We consider the special case of a polynomial

$$W = \sum_i W_i \tag{5.1}$$

with transpose polynomial

$$W^{\mathsf{T}} = \sum_i W_i^{\mathsf{T}}$$

where each W_i is either of a Fermat type or loop type. Let G be an arbitrary admissible group of diagonal symmetries of W and G^{T} the transpose group defined in [10].

Our main result is the following.

Theorem 5.1. *There is an isomorphism of Frobenius algebras*

$$\mathscr{H}_{W,G} \cong \mathscr{B}_{W^{\mathsf{T}},G^{\mathsf{T}}}.$$

Notation 5.2. Notice that for $\lceil m\,;g\rfloor \in \mathscr{B}_{W^{\mathsf{T}},G^{\mathsf{T}}}$ we can write $m = \prod_i m_i$ where m_i is a monomial in $\mathscr{Q}_{W_i}$. We can also write $g = \sum_i g_i$, where g_i acts trivially except on W_i. In the sequel when we write any element of $\mathscr{B}_{W^{\mathsf{T}},G^{\mathsf{T}}}$ in this form, assume that the product and group element are as described above. Similar remarks apply to the A-model.

Notation 5.3. Let I a subset of $\{1, \dots, N\}$. Then we let

- $g_I = \sum_{i \in I} g_i$
- $m_I = \prod_{i \in I} m_i$

- $W_I^{\mathsf{T}} = \sum_{i \in I} W_i^{\mathsf{T}}$.

5.2 Summary of Proof

We would like to utilize the tensor product property of both the FJRW rings and the orbifold Milnor rings, but our group may not break up into a product. Our idea is to take some partitions $\mathcal{I}$ of $\{1, \ldots, N\}$ and for each such partition break up the polynomial as $W^{\mathsf{T}} = \sum_{I \in \mathcal{I}} W_I^{\mathsf{T}}$. We pick a new group $\bigoplus_{I \in \mathcal{I}} G_I^{\mathsf{T}}$ (which is *not* necessarily isomorphic to G^{T}) and take the subalgebra $B_{\mathcal{I}}$ that is essentially the "intersection" of $\mathscr{B}_{W^{\mathsf{T}},G^{\mathsf{T}}}$ with $\mathscr{B}_{\sum W_I^{\mathsf{T}}, \bigoplus G_I^{\mathsf{T}}}$. Then we apply the tensor product property (see Proposition 2.23) to move to $\bigotimes_{I \in \mathcal{I}} \mathscr{B}_{W_I^{\mathsf{T}}, G_I^{\mathsf{T}}}$. Then we apply a suitable modification $\phi_{\mathcal{I}}$ of Krawitz's mirror map to the pieces. The image of $B_{\mathcal{I}}$ under $\phi_{\mathcal{I}}$ will turn out to be the subalgebra that is essentially the intersection of $\mathscr{H}_{W,G}$ and $\mathscr{H}_{\sum W_I, \bigoplus G_I}$ which can also be considered as a subalgebra of the desired FJRW ring $\mathscr{H}_{W,G}$. This is summarized by the following diagram.

$$
\begin{tikzcd}
\mathscr{B}_{W^{\mathsf{T}},G^{\mathsf{T}}} & & & & \mathscr{H}_{W,G} \\
\mathscr{B}_{\sum W_I^{\mathsf{T}}, \bigoplus G_I^{\mathsf{T}}} \arrow[d, "\cong"] & & & & \mathscr{H}_{\sum W_I, \bigoplus G_I} \\
\bigotimes_{I \in \mathcal{I}} \mathscr{B}_{W_I^{\mathsf{T}}, G_I^{\mathsf{T}}} & B_{\mathcal{I}} \arrow[l, hook'] \arrow[ul, hook] \arrow[uul, hook] \arrow[r, "\phi_{\mathcal{I}}"] & \phi_{\mathcal{I}}(B_{\mathcal{I}}) \arrow[r, hook] \arrow[urr, hook] \arrow[uurr, hook] & & \bigotimes_{I \in \mathcal{I}} \mathscr{H}_{W_I, G_I} \arrow[u, "\cong"']
\end{tikzcd}
$$

We will have show why we can consider $B_{\mathcal{I}}$ and $\phi_{\mathcal{I}}(B_{\mathcal{I}})$ as subalgebras as shown above. We will check that each basis element of $\mathscr{B}_{W^{\mathsf{T}},G^{\mathsf{T}}}$ is contained in some $B_{\mathcal{I}}$, and it will be clear that the maps agree on the overlaps for varying choices of $\mathcal{I}$ and G_I, so these maps determine a well defined set map from a basis of $\mathscr{B}_{W^{\mathsf{T}},G^{\mathsf{T}}}$ to a basis of $\mathscr{H}_{W,G}$. Such a map is not automatically a homomorphism, but we will to check that for most pairs of basis elements $\alpha, \beta \in \mathscr{B}_{W^{\mathsf{T}},G^{\mathsf{T}}}$, one can find a $B_{\mathcal{I}}$ containing both. Then we will check that $\phi_{\mathcal{I}}$ respects the product $\alpha \star \beta$. For pairs where this method fails, we will check the homomorphism properties directly.

5.3 Splitting on the B-side

Lemma 5.4. *Suppose P is a quasihomogeneous, non-degenerate, invertible polynomial, and H and H' are symmetry groups of P contained in SL. Suppose that B is a subalgebra of $\mathscr{B}_{P,H}$ that is invariant under H' and which contains only sectors with group elements in H'. Then one can consider B as a subalgebra of $\mathscr{B}_{P,H'}$ in the obvious way.*

Proof. Notice that the definition of the B-model multiplication makes no reference to the orbifold group, but only to the group elements involved. □

Now suppose that $\lceil \prod m_i \, ; \sum g_i \rfloor \in \mathscr{B}_{W^\intercal, G^\intercal}$. Let $\mathcal{I}$ be a partition of $\{1, \ldots, N\}$. Suppose for each $I \in \mathcal{I}$ we have a group of symmetries $G_I^\intercal$ of $W_I^\intercal$ contained in SL.

Definition 5.5. We say the element $\lceil \prod m_i \, ; \sum g_i \rfloor$ *splits nicely* with respect to $\mathcal{I}$ and $\{G_I^\intercal\}$ if it satisfies the following properties (for all $I \in \mathcal{I}$):

1. $\lceil m_I \, ; g_I \rfloor \in \mathscr{B}_{W_I^\intercal, G_I^\intercal}$
2. g_I is either trivial, or acts non-trivially on all W_i for $i \in I$.
3. If $g_I = 0$ and $m_I \neq 1$, then $|I| = 1$, i.e. $W_I = W_j$ for some j.

We say $\bigotimes_{I \in \mathcal{I}} \lceil m_I \, ; g_I \rfloor \in \bigotimes_{I \in \mathcal{I}} \mathscr{B}_{W_I^\intercal, G_I^\intercal}$ is *split nicely* if it satisfies these properties.

Given $\mathcal{I}$ and $\{G_I^\intercal\}$, let $B_\mathcal{I}$ be the subalgebra of $\mathscr{B}_{W^\intercal, G^\intercal}$ generated by basis elements that split nicely. Then by Lemma 5.4, we can consider $B_\mathcal{I}$ as a subalgebra of $\mathscr{B}_{\sum W_I, \bigoplus G_I}$. Thus we may also consider it as a subalgebra of $\bigotimes\limits_{I \in \mathcal{I}} \mathscr{B}_{W_I^\intercal, G_I^\intercal}$ via the tensor product property (Proposition 2.23).

5.4 The New Mirror Map

We next want to define a map

$$\phi_{\mathcal{I}} : B_{\mathcal{I}} \to \bigotimes_{I \in \mathcal{I}} \mathscr{H}_{W_I, G_I}.$$

We consider $B_{\mathcal{I}}$ as a subalgebra of $\bigotimes_{I\in\mathcal{I}} \mathscr{B}_{W_I^\intercal, G_I^\intercal}$ and define our map on generators: consider $\bigotimes_I \lceil \prod m_I \,; \sum g_I \rfloor$. We can give definitions separately for each factor in the tensor product. On the factors with $g_I = (A_I^\intercal)^{-1}(\mathbf{r}+\mathbf{1})$ non-trivial, we take

$$\lceil 1 \,; (A_I^\intercal)^{-1}(\mathbf{r}+\mathbf{1}) \rfloor \mapsto \left(\prod_{i \in I} k_i^{-\mathbf{w}_i \cdot \mathbf{r}_i} \right) \lceil \mathbf{X}^{\mathbf{r}} \,; 0 \rfloor \tag{5.2}$$

where the k_i are complex constants to be determined, $\mathbf{r}$ is as described in Corollary 3.6 and Lemma 3.7, $\mathbf{r}_i$ is the vector containing the entries of $\mathbf{r}$ corresponding to W_i, $\mathbf{w}_i$ is the integer weight vector for the variables of W_i, and $\mathbf{w}_i \cdot \mathbf{r}_i$ is the vector dot product.

On the factors with $g_I = 0$ and $|I| = 1$, the domain of this map is a subalgebra of $\mathscr{B}_{W_i,0}$, which is just the Milnor ring of W_i, an atomic Fermat or loop polynomial. The map is defined on generators of the Milnor ring by

$$\lceil Y_{i,j} \,; 0 \rfloor \mapsto k_i^{w_{i,j}} \lceil 1 \,; \rho_i + J_i \rfloor \tag{5.3}$$

and then restricted to the appropriate domain. Here $Y_{i,j}$ is the jth variable in $W_i^\intercal$, $w_{i,j}$ is the integer weight of the jth variable in W_i, and J_i is the exponential grading operator for W_i.

If $g_I = 0$ and $m_I = 1$, $\phi_{\mathcal{I}}$ takes the identity to the identity:

$$\lceil 1 \,; 0 \rfloor \mapsto \lceil 1 \,; J_I \rfloor .$$

One can easily see that these maps are rescaling of the maps in Chapter 4, so the definition of the transpose group ensures that they land in the specified codomain. We will show that this map respects the product in Section 5.5.

By Axiom 2.19, we know that $\bigotimes_{I\in\mathcal{I}}\mathscr{H}_{W_I,G_I}\cong\mathscr{H}_{\sum W_I,\bigoplus G_I}$, so we can think of $\phi_{\mathcal{I}}(B_{\mathcal{I}})$ as a subalgebra of $\mathscr{H}_{\sum W_I,\bigoplus G_I}$. Since our map is a combination and rescaling of Krawitz's mirror maps, one can see that we the elements of $\phi_{\mathcal{I}}(B_{\mathcal{I}})$ actually look like elements of $\mathscr{H}_{W,G}$. We will prove the following analog of Lemma 5.4 in Section 5.5.

Lemma 5.6. *The subalgebra $\phi_{\mathcal{I}}(B_{\mathcal{I}})$ of*

$$\bigotimes_{I\in\mathcal{I}}\mathscr{H}_{W_I,G_I}\cong\mathscr{H}_{W,\bigoplus_{I\in\mathcal{I}}G_I}$$

can be considered as a subalgebra of $\mathscr{H}_{W,G}$.

We now have a map from a set of generators of $B_{\mathcal{I}}$ to $\mathscr{H}_{W,G}$. We want to get a map from a set of generators for $\mathscr{B}_{W^T,G^T}$ to $\mathscr{H}_{W,G}$ defined as follows. For any basis element $\gamma\in\mathscr{B}_{W^\intercal,G^\intercal}$, pick any $\mathcal{I}$ and groups G^I so that γ splits nicely and define

$$\varphi(\gamma)=\phi_{\mathcal{I}}(\gamma)\tag{5.4}$$

and extend the map linearly. To check that this is well defined, we need the following observation.

Lemma 5.7. *For any standard basis element $\gamma=\lceil\prod m_i\,;\sum g_i\rfloor\in\mathscr{B}_{W^\intercal,G^\intercal}$ there exists a partition $\mathcal{I}$ and groups G_I so that γ splits nicely.*

Furthermore, the image $\phi_{\mathcal{I}}(\gamma)$ is independent of choice of $\mathcal{I}$ and $\{G_I\}$.

Proof. Let I_0 be the set of i so that $g_i=0$. Let I_g be the set of i with $g_i\neq 0$. Then we choose the partition

$$\mathcal{I}=\{I_g\}\cup\{\{i\}\}_{i\in I_0}.$$

We choose the group $G_{I_g}^{\mathsf{T}}$ for $W_{I_g}^{\mathsf{T}}$ to be the group generated by g_{I_g}, and we choose the trivial groups for the others. In light of Remark 3.2, it is clear that these groups satisfy the properties described in Definition 5.5.

The second claim is easy to check from the definitions— equation (5.2) respects the splitting, and both (5.2) and (5.3) make reference only to the group elements, and not to which subgroup they are members of. □

Example 5.8. Suppose W^{T} was the sum of five Fermat types and we had a basis element that looked like

$$\left\lceil X_1^3\,;\left(0,0,\frac{1}{3},\frac{1}{2},\frac{1}{6}\right)\right\rfloor.$$

Then we would take $I_g = \{3,4,5\}$, $I_0 = \{1,2\}$ and let $G_{I_g} = \left\langle\left(\frac{1}{3},\frac{1}{2},\frac{1}{6}\right)\right\rangle$ (where G_{I_g} acts on the variables indexed by I_g) and take G_{I_0} to be the trivial group acting on the first two variables.

Remark 5.9. We see that φ still gives the graded vector space isomorphism. See Chapter 4 for a discussion of this.

5.5 Splitting on the A-side

In this section we prove the A-side analog of Lemma 5.4. The key observation is in the following lemma. The proof is due to Tyler Jarvis.

Lemma 5.10. *Let P be a quasihomogeneous, non-degenerate, invertible polynomial and H and H' be admissible subgroups of G_P^{max}. Suppose we have $\lceil m\,;g\rfloor$, $\lceil n\,;h\rfloor$, and $\lceil p\,;k\rfloor$ which may be thought of as elements of either $\mathscr{H}_{P,H}$ or $\mathscr{H}_{P,H'}$ (i.e., the group elements are in both H and H' and the monomials with volume forms are invariant under both H and H'.)*

Then

$$\langle\lceil m\,;g\rfloor\,,\lceil n\,;h\rfloor\,,\lceil p\,;k\rfloor\rangle^{P,H} = \langle\lceil m\,;g\rfloor\,,\lceil n\,;h\rfloor\,,\lceil p\,;k\rfloor\rangle^{P,H'}\,. \tag{5.5}$$

That is, we may compute the three point correlator in either FJRW ring.

Proof. First we consider the special case when $H \leq H'$.

The correlator $\langle \lceil m\,;g\rfloor\,,\lceil n\,;h\rfloor\,,\lceil p\,;k\rfloor\rangle^{P,H}$ is defined (see [3, Def 4.6.2]) as

$$\langle \lceil m\,;g\rfloor\,,\lceil n\,;h\rfloor\,,\lceil p\,;k\rfloor\rangle^{P,H} := \int_{\overline{\mathscr{M}}_{0,3}} \Lambda_{0,3}^{P,H}(m,n,p),$$

where $\overline{\mathscr{M}}_{0,3}$ is the moduli space of three-pointed, genus-zero stable curves, and $\Lambda_{0,3}^{P,H}(m,n,p)$ is defined (see [3, Def 4.2.1]) to be the Poincaré dual of the pushforward of the virtual cycle, capped with the classes m, n, and p:

$$\Lambda_{0,3}^{P,H}(m,n,p) := \frac{1}{\deg(\mathrm{st}^{P,H})} PD\ \mathrm{st}_*^{P,H}\left(\left[\overline{\mathscr{W}}_{0,3,H}(P;g,h,k)\right]^{vir} \cap (m \cup n \cup p)\right).$$

Here $\mathrm{st}^{P,H} : \overline{\mathscr{W}}_{0,3,H}(P;g,h,k) \to \overline{\mathscr{M}}_{0,3}$ is the forgetful map taking the moduli space of genus-zero, stable P-curves with admissible group H to the moduli space of genus-zero stable curves with three marked points, defined simply by forgetting the P-structure on the curve.

Similarly, the correlator $\langle \lceil m\,;g\rfloor\,,\lceil n\,;h\rfloor\,,\lceil p\,;k\rfloor\rangle^{P,H'}$ is defined (see [3, Def 4.6.2]) as

$$\langle \lceil m\,;g\rfloor\,,\lceil n\,;h\rfloor\,,\lceil p\,;k\rfloor\rangle^{P,H'} := \int_{\overline{\mathscr{M}}_{0,3}} \Lambda_{0,3}^{P,H'}(m_i,n_i,p_i),$$

with

$$\Lambda_{0,3}^{P,H'}(m,n,p) := \frac{1}{\deg(\mathrm{st}^{P,H'})} PD\ \mathrm{st}_*^{P,H'}\left(\left[\overline{\mathscr{W}}_{0,3,H'}(P;g,h,k)\right]^{vir} \cap (m \cup n \cup p)\right).$$

According to [3, 2.3.1], there is a finite morphism of stacks $a : \overline{\mathscr{W}}_{0,3,H}(P;g,h,k) \to \overline{\mathscr{W}}_{0,3,H'}(P;g,h,k)$, surjective onto an open and closed substack of $\overline{\mathscr{W}}_{0,3,H'}(P)$. Moreover, $\overline{\mathscr{W}}_{0,3,H'}(P)$ actually has only a single geometric point, corresponding to the unique genus-zero, three-pointed P-curve with markings g, h, and k, respectively. Therefore, in this case, the morphism a is surjective and finite.

Theorem 6.3.5 of [2] shows that the virtual class $\left[\mathscr{W}_{0,3,H'}(P;g,h,k)\right]^{vir}$ is the pullback along a of the virtual cycle $\left[\overline{\mathscr{W}}_{0,3,H'}(P;g,h,k)\right]^{vir}$ on $\overline{\mathscr{W}}_{0,3,H'}(P)$.

Now since $\lceil m_i\,;g_i\rfloor$, $\lceil n_i\,;h_i\rfloor$, and $\lceil p_i\,;k_i\rfloor$ are in $\mathscr{H}_{P_i,H_i}$, we have

$$
\begin{aligned}
&\langle \lceil m\,;g\rfloor\,,\lceil n\,;h\rfloor\,,\lceil p\,;k\rfloor\rangle^{P,H}\\
&=\int_{\overline{\mathscr{M}}_{0,3}} \Lambda^{P,H}_{0,3}(m,n,p),\\
&=\int_{\overline{\mathscr{M}}_{0,3}} \frac{PD\ \mathrm{st}^{P,H}_*\left(\left[\overline{\mathscr{W}}_{0,3,H}(P;g,h,k)\right]^{vir}\cap(m\cup n\cup p)\right)}{\deg(\mathrm{st}^{P,H})}\\
&=\int_{\overline{\mathscr{M}}_{0,3}} \frac{PD\ \mathrm{st}^{P,H'}_*\,a_*\left(a^*\left[\overline{\mathscr{W}}_{0,3,H'}(P;g,h,k)\right]^{vir}\cap(m\cup n\cup p)\right)}{\deg(\mathrm{st}^{P,H})}\\
&=\int_{\overline{\mathscr{M}}_{0,3}} \frac{\deg(a)PD\ \mathrm{st}^{P,H'}_*\left(\left[\overline{\mathscr{W}}_{0,3,H'}(P;g,h,k)\right]^{vir}\cap(m\cup n\cup p)\right)}{\deg(\mathrm{st}^{P,H})}\\
&=\int_{\overline{\mathscr{M}}_{0,3}} \frac{PD\ \mathrm{st}^{P,H'}_*\left(\left[\overline{\mathscr{W}}_{0,3,H'}(P;g,h,k)\right]^{vir}\cap(m\cup n\cup p)\right)}{\deg(\mathrm{st}^{P,H'})}\\
&=\int_{\overline{\mathscr{M}}_{0,3}} \Lambda^{P,H'}_{0,3}(m,n,p),\\
&=\langle \lceil m\,;g\rfloor\,,\lceil n\,;h\rfloor\,,\lceil p\,;k\rfloor\rangle^{P,H'}.
\end{aligned}
$$

Most of these equalities are immediate consequences of the definitions, and the fourth equality follows from the fact that, in this case, the morphism a is finite and surjective, so the pushforward of the pullback of any class along a is simply $\deg(a)$ times the original class.

For the general case when H is not necessarily a subgroup of H', notice that the special case proved above implies that we could compute both correlators in $\mathscr{H}_{P,H\cap H'}$ and get the same result. $\square$

Lemma 5.10 allows us to prove the following.

Corollary 5.11. *Suppose* $\lceil m\,;g\rfloor$ *and* $\lceil n\,;h\rfloor$ *can be thought of as elements of either* $\mathscr{H}_{P,H}$

or $\mathscr{H}_{P,H'}$. Then the product $\lceil m\,;g\rfloor \star \lceil n\,;h\rfloor$ looks the same whether is it computed in $\mathscr{H}_{P,H}$ or $\mathscr{H}_{P,H'}$.

Proof. By definition of multiplication

$$\lceil m\,;g\rfloor \star_{P,H} \lceil n\,;h\rfloor = \sum_{\sigma,\tau} \langle \lceil m\,;g\rfloor\,, \lceil n\,;h\rfloor\,, \sigma\rangle\, \eta^{\sigma,\tau}\tau$$

where σ and τ range over a basis of $\mathscr{H}_{P,H}$. On the other hand,

$$\lceil m\,;g\rfloor \star_{P,H'} \lceil n\,;h\rfloor = \sum_{\sigma',\tau'} \langle \lceil m\,;g\rfloor\,, \lceil n\,;h\rfloor\,, \sigma'\rangle\, \eta^{\sigma',\tau'}\tau'$$

where σ' and τ' range over a basis of $\mathscr{H}_{P,H'}$.

For basis elements that are in the basis of both $\mathscr{H}_{P,H}$ and $\mathscr{H}_{P,H'}$, Lemma 5.10 tells us that we can compute the correlators (and thus also the pairing) in either place. It suffices then to show that if we have a basis element $\lceil p\,;k\rfloor$ of $\mathscr{H}_{P,H}$ that is not in $\mathscr{H}_{P,H'}$, the correlator

$$\langle \lceil m\,;g\rfloor\,, \lceil n\,;h\rfloor\,, \lceil p\,;k\rfloor\rangle^{P,H} \tag{5.6}$$

vanishes. (This also gives us the symmetric condition with H and H' interchanged.) There are two reasons why $\lceil p\,;k\rfloor$ might not be in $\mathscr{H}_{P,H'}$. It may be that $\lceil p\,;k\rfloor$ is not invariant under H'. However $\lceil m\,;g\rfloor$ and $\lceil n\,;h\rfloor$ *are* invariant under H', thus the three point correlator (5.6) is not invariant and vanishes by Property 2.20. It may be that $k \notin H'$. But g and h are in H', so (5.6) vanishes by Proposition 2.15 and group closure. $\square$

Now the proof promised in Section 5.4 follows easily.

Proof of Lemma 5.6. As we noted, the construction of the mirror maps implies that the elements of $\mathscr{H}_{W,\bigoplus_{I\ in\mathcal{I}} G_I}$ "look like" elements of $\mathscr{H}_{W,G}$, so this is just an application of Corollary 5.11. $\square$

5.6 The Product

We need to check that φ, as defined by (5.4), is a homomorphism. Our strategy is to first show that for most products, we can choose a partition $\mathcal{I}$ and groups $\{G_I\}_{I\in\mathcal{I}}$ so that both factors split nicely. Then we will check that $\phi_{\mathcal{I}}$ respects such products.

The are some cases where we can not chose a partition that splits both factors nicely, but we can handle them in a different way. Before proving the necessary lemma, we isolate the following fact.

Lemma 5.12. *Suppose P is a decoupled sum $P_1 + P_2$, and following Notation 5.2, we have three basis elements $\lceil m_1 \cdot m_2\, ; g_1 + g_2 \rfloor$, $\lceil n_1 \cdot n_2\, ; h_1 + h_2 \rfloor$, and $\lceil p_1 \cdot p_2\, ; k_1 + k_2 \rfloor$ of $\mathscr{H}_{P,H}$. Suppose $\lceil m_1\, ; g_1 \rfloor$ and $\lceil n_1\, ; h_1 \rfloor$ are invariant under $G^{max}_{P_1}$. Then the correlator*

$$\langle \lceil m_1 \cdot m_2\, ; g_1 + g_2 \rfloor , \lceil n_1 \cdot n_2\, ; h_1 + h_2 \rfloor , \lceil p_1 \cdot p_2\, ; k_1 + k_2 \rfloor \rangle \tag{5.7}$$

is vanishes unless both

1. *$\lceil p_1\, ; k_1 \rfloor$ is also invariant under $G^{max}_{P_1}$.*
2. *The correlator*
$$\langle \lceil m_1\, ; g_1 \rfloor , \lceil n_1\, ; h_1 \rfloor , \lceil p_1\, ; k_1 \rfloor \rangle^{P_1, G^{max}_{P_1}} \tag{5.8}$$

is non-vanishing.

Proof. Condition 1 follows from $G^{\max}$-invariance (Property 2.20) since $G^{\max}_{P_1}$ is a subgroup of $G^{\max}_P$. If this condition is satisfied, then we can pick groups $H_1 = G^{\max}_{P_1}$ and $H_2 = \pi_2(H)$, where π_2 is projection onto the second factor of $G^{\max}_P \cong G^{\max}_{P_1} \oplus G^{\max}_{P_2}$. We want to see that the element $\lceil m_2\, ; g_2 \rfloor$ is invariant under $\pi_2(H)$. A group element $l_2 \in \pi_2(H)$ comes from an element $(l_1, l_2) \in H$. The ring element $\lceil m_1 + m_2\, ; g_1 + g_2 \rfloor$ is invariant under (l_1, l_2) and $(-l_1, 0)$ and thus under $(0, l_2)$. Thus $\lceil m_2\, ; g_2 \rfloor$ is invariant under l_2. Similar arguments apply to $\lceil n_1\, ; h_1 \rfloor$ and $\lceil p_1\, ; k_1 \rfloor$.

Thus Lemma 5.10 applies and we can compute (5.7) in the ring $\mathscr{H}_{W_1+W_2,H_1\oplus H_2}$. Thus by Axiom 2.19, the value of the correlator (5.7) is the product of (5.8) and

$$\langle \lceil m_2\,;g_2\rfloor\,,\lceil n_2\,;h_2\rfloor\,,\lceil p_2\,;k_2\rfloor\rangle^{P_2,H_2}\,,$$

from which the result follows. □

For pairs described in the following lemma, we cannot simultaneously split them nicely. However, the lemma shows that both the product and the product of the images vanish.

Notation 5.13. We write the vector of variables of W_j^{T} as $\mathbf{Y}_j$ and similarly the vector of variables of W_j as $\mathbf{X}_j$.

Lemma 5.14. *Suppose we have a pair of B-side elements,* $\lceil\prod m_i\,;\sum g_i\rfloor$ *and* $\lceil\prod n_i\,;\sum h_i\rfloor$*, and suppose that for some* j*, we have* $m_j=1$*,* $g_j=[(A_j^{\mathsf{T}})^{-1}(\mathbf{s}+1)]\neq 0$*, and* $n_j=\mathbf{Y}_j^{\boldsymbol{\beta}}\neq 1$*,* $h_j=0$*. Then the products*

$$\left\lceil\prod m_i\,;\sum g_i\right\rfloor\star\left\lceil\prod n_i\,;\sum h_i\right\rfloor \tag{5.9}$$

and

$$\varphi(\left\lceil\prod m_i\,;\sum g_i\right\rfloor)\star\varphi(\left\lceil\prod n_i\,;\sum h_i\right\rfloor) \tag{5.10}$$

both vanish.

Remark 5.15. We noted in Chapter 4 that there is some subtlety involved in determining the mirror map for even-variable loops. In particular we have not been able to determine in all cases whether the algebra isomorphism should take the a basis element of the B-side to a scalar multiple of a basis element on the A-side or to some linear combination of basis elements. This will complicate the notation of the following proof, but will not disrupt the strategy.

Proof of Lemma 5.14. Since $g_j + h_j \neq 0$ but $\mathbf{Y}_j^{\boldsymbol{\beta}} \neq 1$, it follows from the definition of the B-side multiplication that (5.9) will vanish.

It remains to see that (5.10) vanishes as well. The image of $\lceil \prod m_i \, ; \sum g_i \rfloor$ will be a linear combination

$$\sum_l c_l \left\lceil \prod \hat{m}_i^l \, ; \sum \hat{g}_i^l \right\rfloor ,$$

where $\hat{m}_j^l = \mathbf{X}_j^{\mathbf{s}}$, $\hat{g}_j^l = 0$ for any l.

In the case that W_j is an even variable loop and $\beta_k = \delta_{\mathrm{odd}}^k(a_k - 1)$ or $\beta_k = \delta_{\mathrm{even}}^k(a_k - 1)$, we can see that the image of $\lceil \prod n_i \, ; \sum h_i \rfloor$ will be a linear combination $\sum_l c_l' \left\lceil \prod \hat{n}_i^l \, ; \hat{h}_i^l \right\rfloor$ where $\hat{h}_j^l = 0$ and n_j^l is either $\prod_{k \text{ even}} X_{j,k}^{a_{j,k}-1}$ or $\prod_{k \text{ odd}} X_{j,k}^{a_{j,k}-1}$. For the potentially non-vanishing correlators, we can apply Lemma 5.12) and examine pieces of the form

$$\left\langle \lceil \mathbf{X}_j^{\mathbf{s}} \, ; 0 \rfloor , \left\lceil \prod_{k \text{ odd/even}} X_{j,k}^{a_{j,k}-1} \, ; 0 \right\rfloor , \lceil 1 \, ; J_{W_j} \rfloor \right\rangle \tag{5.11}$$

(where we filled in the third spot using Proposition 2.15). We see that this is non-vanishing only if $\mathbf{X}_j^{\mathbf{s}}$ pairs with $\prod_{k \text{ odd/even}} X_{j,k}^{a_{j,k}-1}$, which will only happen if $\mathbf{s}$ is of the form $s_k = \delta_{\mathrm{odd/even}}^k(a_k - 1)$. But in that case we have $[(A_j^{\mathsf{T}})^{-1}(\mathbf{s} + \mathbf{1})] = 0$, contradicting our assumption.

Suppose now that $\boldsymbol{\beta}$ is not of that special form. Then $\hat{n}_j^l = 1$, $\hat{h}_j^l = [A_j^{-1}(\boldsymbol{\beta} + \mathbf{1})] \neq 0$. Again using Lemma 5.12, we can examine pieces of the form

$$\left\langle \lceil \mathbf{X}_j^{\mathbf{s}} \, ; 0 \rfloor , \lceil 1 \, ; [A_j^{-1}(\boldsymbol{\beta} + \mathbf{1})] \rfloor , \lceil 1 \, ; [-A_j^{-1}\boldsymbol{\beta}] \rfloor \right\rangle \tag{5.12}$$

(where we filled in the third spot using Proposition 2.15). Notice that is follows from Lemma 3.5 that $A_j^{-1}\boldsymbol{\beta} \neq 0$. Now we apply Remark 2.21 to $\rho_{j,k}$ and see that this correlator vanishes unless $\rho_{i,k}^{\mathsf{T}}(\mathbf{s} + \mathbf{1}) \in \mathbb{Z}$. But this implies that $[(A_j^{\mathsf{T}})^{-1}(\mathbf{s} + \mathbf{1})] = 0$, again contradicting the hypothesis. $\square$

We can now show how to pick the partitions and groups promised at the beginning of the chapter.

Definition 5.16. Excluding the cases described in Lemma 5.14, consider a B-side product $\lceil \prod m_i \,; \sum g_i \rfloor \star \lceil \prod n_i \,; \sum h_i \rfloor$. Define the following subsets of indexes as follows:

- Let I_h be the set of indexes such that $g_i = 0$ and $h_i \neq 0$ (then by assumption $m_i = 1$).
- Let I_g be the set of indexes such that $h_i = 0$ and $g_i \neq 0$ (then $n_i = 1$).
- Let $I_{g,h}$ be the set of indexes where $g_i, h_i \neq 0$.
- Let I_0 be the set of indexes such that $g_i = h_i = 0$.

Now we define the partition

$$\mathcal{I} = \{I_g, I_h, I_{g,h}\} \cup \{\{i\}\}_{i \in I_0}.$$

Definition 5.17. For the partition $\mathcal{I}$, define groups as follows:

- Let $G^{\mathsf{T}}_{I_h}$ be the group of symmetries of W_{I_h} generated by h_{I_h}.
- Let $G^{\mathsf{T}}_{I_g}$ be the group of symmetries of W_{I_g} generated by g_{I_g}
- Let $G^{\mathsf{T}}_{I_{g,h}}$ be the group of symmetries of $W_{I_{g,h}}$generated by $h_{I_{g,h}}$ and $g_{I_{g,h}}$.
- Let G^{T}_i be the trivial group of symmetries of W_i for $i \in I_0$

Example 5.18. Suppose we have a sum of six Fermats, and have a pair of basis elements that look like:

$$\lceil x_3^0 x_6^\bullet \,; (\bullet, \bullet, 0, \bullet, \bullet, 0) \rfloor$$

and

$$\lceil x_4^0 x_5^0 x_6^\bullet \,; (\bullet, \bullet, \bullet, 0, 0, 0) \rfloor \,,$$

where the $\bullet$ in the group elements represents a non-zero entry, and the $\bullet$ in the exponent represents a possibly non-zero exponent. Then we would take $I_g = \{4,5\}$, $I_h = \{3\}$, $I_{g,h} = \{1,2\}$, and $I_0 = \{6\}$.

Lemma 5.19. *The groups $\{G_I^{\mathsf{T}}\}$ described in Definition 5.17 are each contained in SL. Both $\lceil \prod m_i \,; \sum g_i \rfloor$ and $\lceil \prod n_i \,; \sum h_i \rfloor$ split nicely (see Definition 5.5) with respect to $\mathcal{I}$ and $\{G_I^{\mathsf{T}}\}$.*

Proof. Conditions 2 and 3 of Definition 5.5 follow directly from the construction. It is also easy to see that g and h are contained in $\bigoplus_{I \in \mathcal{I}} G_I^{\mathsf{T}}$, so to check condition 1 we just need to see that the elements $\lceil \prod m_i \,; \sum g_i \rfloor$ and $\lceil \prod n_i \,; \sum h_i \rfloor$ are invariant under $\bigoplus_{I \in \mathcal{I}} G_I^{\mathsf{T}}$.

We will check that all the generators mentioned in Definition 5.17 fix $\lceil \prod n_i \,; \sum h_i \rfloor$. It is clear that $\lceil \prod n_i \,; \sum h_i \rfloor$ is invariant under h_{I_h} and $h_{I_{g,h}}$, since the indexes in I_h and $I_{g,h}$ have no fixed variables in $\lceil \prod n_i \,; \sum h_i \rfloor$.

Notice that the only non-trivial part of the action of g on $\lceil \prod n_i \,; \sum h_i \rfloor$ is the action of g_{I_g} on $\lceil 1 \,; 0 \rfloor \in \mathscr{B}_{W_{I_g}^{\mathsf{T}}, G_{I_g}^{\mathsf{T}}}$ since all other indexes i have either g_i trivial or represent non-fixed variables. Since $\lceil \prod n_i \,; \sum h_i \rfloor$ is invariant under g, this shows that $\lceil \prod n_i \,; \sum h_i \rfloor$ is invariant under the action of g_{I_g}. Also, the action of g_{I_g} on $\lceil 1 \,; 0 \rfloor$ is precisely the determinant of g_{I_g}, which shows that $g_{I_g} \in SL$.

Notice that $g = g_{I_g} + g_{I_{g,h}}$. But $g \in SL$ and fixes both by hypothesis and we found above that $g_{I_g} \in SL$, so by group closure $g_{I_{g,h}} \in SL$ as well. Also g fixes $\lceil \prod n_i \,; \sum h_i \rfloor$, and we showed above that g_{I_g} fixes this as well, so it follows that $g_{I_{g,h}}$ fixes $\lceil \prod n_i \,; \sum h_i \rfloor$.

A completely symmetric argument shows that $\lceil \prod m_i \,; \sum g_i \rfloor$ is invariant under $\bigoplus_{I \in \mathcal{I}} G_I^{\mathsf{T}}$ and that h_{I_h} and $h_{I_{g,h}}$ are in SL. $\square$

Lemma 5.20. *With $\mathcal{I}$ and $\{G_I^{\mathsf{T}}\}$ as defined above in Definition 5.16 and Definition 5.17, there are choices of $k_i \in \mathbb{C}$ in* (5.2) *and* (5.3) *so that $\phi_{\mathcal{I}}$ is a homomorphism.*

Proof. It suffices to check this for each factor in the tensor product. There are three cases. First, consider one of the factors $\mathscr{B}_{W_i,0}$ for $i \in I_0$. The map (5.3) is the same as the map in

[10] (see (4.4) in this paper) up to a scalar. But notice that if W_i is quasihomogeneous with weights $w_{i,j}$, then the Jacobian relations are also quasihomogeneous with the same weights. Thus, the rescaling preserves the Jacobian relations, and thus the isomorphism in [10]. The isomorphism for the Fermat case was verified in [6].

Next, in the symmetric cases $\mathscr{B}_{W^{\mathsf{T}}_{I_g},G^{\mathsf{T}}_{I_g}}$ and $\mathscr{B}_{W^{\mathsf{T}}_{I_h},G^{\mathsf{T}}_{I_h}}$, notice that the products here are just products with the identity. Since $\phi_{\mathcal{I}}$ preserves the identity, we are good here.

Lastly, we consider the factor $\mathscr{B}_{W_{I_{g,h}},G_{I_{g,h}}}$. To reduce notational clutter, we drop the subscript $I_{g,h}$ in the following computations. The product is of the form

$$\lceil 1\,;(A^{\mathsf{T}})^{-1}(\mathbf{r}+\mathbf{1})\rfloor \star \lceil 1\,;A^{\mathsf{T}}(\mathbf{s}+\mathbf{1})\rfloor\,,$$

with both group elements non-trivial. Notice that by definition the B-side product vanishes precisely when $[(A^{\mathsf{T}})^{-1}(\mathbf{r}+\mathbf{s}+\mathbf{2})]\neq 0$. The corresponding A-side product is

$$\left(\prod_{i\in I_{g,h}} k_i^{-\mathbf{w}_i\cdot(\mathbf{r}_i+\mathbf{s}_i)}\right)\left\lceil \prod \mathbf{X}^{\mathbf{r}}\,;\mathbf{0}\right\rfloor \star \left\lceil \prod \mathbf{X}^{\mathbf{s}}\,;\mathbf{0}\right\rfloor \tag{5.13}$$

We need to show that the B-side product vanishes if and only if the A-side product does. The A-side product in the identity sector is computed using the pairing, and is non-vanishing if and only if $\mathbf{X}^{\mathbf{r}+\mathbf{s}}$ is a scalar multiple of the Hessian. Thus, Corollary 3.15 gives us what we need.

Now, if the products do not vanish, then we see that we must have $\mathbf{r}+\mathbf{s}=\mathbf{a}-\mathbf{1}$. This follows from Remark 3.13, since if either (2) or (3) were true the B-side product would not be a product of non-identity sectors. Then the B-side product is by definition

$$\frac{1}{\mu}\lceil \mathrm{hess}_W\,;0\rfloor \tag{5.14}$$

Notation 5.21. For W_i a loop, let

$$b_i = \mu_i \mathbf{Y}_i^{\mathbf{a}_i - \mathbf{1}} / \operatorname{hess} W_i,$$

and for W_i a Fermat, let

$$b_i = \mu_i Y_i^{a_i - 2} / \operatorname{hess} W_i.$$

Using this notation, (5.14) may be written

$$\left(\prod_i \frac{1}{b_i}\right) \left\lceil \prod_{i \text{ loop}} \mathbf{Y}_i^{\mathbf{a}_i - \mathbf{1}} \prod_{i \text{ Fermat}} Y_i^{a_i - 2} \, ; 0 \right\rfloor .$$

To compute the image of this under $\phi_{\mathcal{I}}$, we can use Lemma 4.2 in [10], quoted in this paper as (4.5). Note the this is valid for any polynomial, not just loops. The image is then

$$\left(\prod_{i \text{ loop}} \frac{1}{b_i} k_i^{\mathbf{w}_i \cdot (\mathbf{a}_i - \mathbf{1})}\right) \left(\prod_{i \text{ Fermat}} \frac{1}{b_i} k_i^{w_i (a_i - 2)}\right) \left\lceil 1 \, ; \sum_{i \text{ loop}} [A_i^{-1} \mathbf{a}_i] + \sum_{i \text{ Fermat}} \left[\frac{1}{a_i}(a_i - 1)\right] \right\rfloor \tag{5.15}$$

Here the exponent of k_i in the first product is the vector dot product. We choose k_i so that $k_i^{\mathbf{w}_i \cdot (\mathbf{a}_i - \mathbf{1})} = b_i$ for a loop, and $k_i^{w_i(a_i-2)} = b_i$ for a Fermat. Simplifying and using Lemma 3.9, (5.15) becomes

$$\lceil 1 \, ; -J \rfloor$$

On the other hand the A-side product (5.13) is

$$\prod_{i \in I_3} \left(k_i^{-\mathbf{w}_i \cdot (\mathbf{r}_i + \mathbf{s}_i)} \left\langle \left\lceil \prod \mathbf{X}_i^{\mathbf{r}_i} \, ; \mathbf{0} \right\rfloor , \left\lceil \prod \mathbf{X}_i^{\mathbf{s}_i} \, ; \mathbf{0} \right\rfloor \right\rangle \right) \lceil 1 \, ; -J \rfloor \tag{5.16}$$

(Here we used the fact that the pairing "breaks up" across decoupled sums.) As we noted, since $\mathbf{X}^{\mathbf{r}+\mathbf{s}}$ is equal to the Hessian, we must have $\mathbf{r}_i + \mathbf{s}_i = \mathbf{a}_i - \mathbf{1}$ for a loop and $r_i + s_i = a_i - 2$

for a Fermat. Also notice that the pairing in (5.16) just b_i. Thus (5.16) is

$$\left(\prod_{i \text{ loop}} b_i k_i^{-\mathbf{w}_i \cdot (\mathbf{a}_i - \mathbf{1})}\right)\left(\prod_{i \text{ Fermat}} b_i k_i^{-w_i(a_i-2)}\right)\lceil 1\,;-J\rfloor\,.$$

By our choice of k_i, this is $\lceil 1\,;-J\rfloor$, as desired. □

5.7 The Pairing

We have now established the algebra isomorphism, and it remains to check that our choices of k_i cause the pairing to be preserved.

Lemma 5.22. *Consider a pair as described in Lemma 5.14. The both the pairing of these elements and the pairing of their images vanish.*

Proof. Clearly the pairing of the elements on the B-side vanishes, since they are not from inverse sectors. The form of their images was computed explicitly in the proof of Lemma 5.14, and we saw in (5.11) that if the pairing were non-trivial, it would violate the hypothesis. In (5.12), we can see that the images do not come from inverse sectors. □

Now excluding that case, we can construct $\mathcal{I}$ and $\{G_I\}$ as we did in Definition 5.16 and Definition 5.17, so we just need the following.

Lemma 5.23. *The map $\phi_{\mathcal{I}}$ respects the pairing. That is,*

$$\langle \alpha, \beta \rangle = \langle \phi_{\mathcal{I}}(\alpha), \phi_{\mathcal{I}}(\beta) \rangle\,.$$

Proof. It suffices to check for each factor in the tensor product.

First we check the pieces corresponding to I_g and I_h. Here the B-side pairing of $\lceil 1\,; g\rfloor$ and $\lceil 1\,; 0\rfloor$ vanishes. The image of $\lceil 1\,; 0\rfloor$ is $\lceil 1\,; J\rfloor$, which pairs non-trivially only with $\lceil 1\,; -J\rfloor$. But the element $\lceil 1\,; g\rfloor$ will map to the A-side identity sector. So the A-side pairing vanishes as well.

We next check the pairing in the I_0 pieces. Let $j \in I_0$, and suppose W_j is a loop. We drop the subscript j and consider

$$\left\langle \left\lceil \mathbf{Y}^{\boldsymbol{\alpha}}\,; 0 \right\rfloor,\ \left\lceil \mathbf{Y}^{\boldsymbol{\beta}}\,; 0 \right\rfloor \right\rangle \tag{5.17}$$

This pairing is non-zero if and only if $\mathbf{Y}^{\boldsymbol{\alpha}+\boldsymbol{\beta}}$ is a multiple of the hessian. The corresponding A-side pairing is

$$k^{\mathbf{w}\cdot(\boldsymbol{\alpha}+\beta)} \left\langle \left\lceil 1\,; A^{-1}(\boldsymbol{\alpha}+\mathbf{1}) \right\rfloor,\ \left\lceil 1\,; A^{-1}(\boldsymbol{\beta}+\mathbf{1}) \right\rfloor \right\rangle = k^{\mathbf{w}\cdot(\boldsymbol{\alpha}+\beta)} \cdot 1 \tag{5.18}$$

To compute the mirror map above, we assumed that both $A^{-1}(\boldsymbol{\alpha}+\mathbf{1})$ and $A^{-1}(\boldsymbol{\beta}+\mathbf{1})$ were non-trivial and used (4.5). We lose no generality by doing this— we already know that φ is a homomorphism, and so we can use the Frobenius property of the pairing to adjust (5.17) to ensure that we can use (4.5). The A-side pairing (5.18) is non-vanishing if and only if $[A^{-1}(\boldsymbol{\alpha}+\boldsymbol{\beta}+\mathbf{2})] = 0$. Corollary 3.15 gives the same vanishing criteria for (5.17), so it only remains to check that (5.17) matches (5.18) when both are non-vanishing.

By Remark 3.13 there are three cases. If $\boldsymbol{\alpha}+\boldsymbol{\beta} = \mathbf{a}-\mathbf{1}$, then $k^{\mathbf{w}\cdot(\boldsymbol{\alpha}+\beta)} = b$ (by choice of k), and (5.17) is equal to b straight from the definitions (b is defined in Notation 5.21, recall that we have dropped the subscript).

If we have $\alpha_i + \beta_i = \delta_{\text{odd}}(2a_i - 2)$, then we have (using Lemma 3.14)

$$\left\langle \left\lceil \mathbf{Y}^{\boldsymbol{\alpha}}\,; 0 \right\rfloor,\ \left\lceil \mathbf{Y}^{\boldsymbol{\beta}}\,; 0 \right\rfloor \right\rangle = b \prod_{\text{even}} (-a_i).$$

On the other hand, using the Frobenius property,

$$\begin{aligned}\left\langle \phi\left(\lceil \mathbf{Y}^{\boldsymbol{\alpha}}\,;0\rfloor\right),\ \phi\left(\lceil \mathbf{Y}^{\boldsymbol{\beta}}\,;0\rfloor\right)\right\rangle &= \left\langle \phi\left(\lceil \mathbf{Y}^{\boldsymbol{\alpha}+\boldsymbol{\beta}}\,;0\rfloor\right),\ \phi\left(\lceil 1\,;0\rfloor\right)\right\rangle \\ &= \left\langle \prod_{\text{even}}(-a_j)\phi\left(\lceil \mathbf{Y}^{\mathbf{a}-\mathbf{1}}\,;0\rfloor\right),\ \lceil 1\,;J\rfloor\right\rangle \\ &= \left(\prod_{\text{even}}(-a_j)\right) b\end{aligned}$$

where the second equality follows from Lemma 3.14, and the last equality is as in the first case. A symmetric argument works for the case of $\alpha_j+\beta_j=\delta_{\text{even}}(2a_j-2)$.

If W_j is of Fermat type, it is easy to perform the check described above, since there is only one case.

Next, we need to check the pairing in the factor corresponding to $I_{g,h}$. We consider

$$\left\langle \lceil 1\,;(A^{\mathsf{T}})^{-1}(\mathbf{r}+\mathbf{1})\rfloor,\ \lceil 1\,;(A^{\mathsf{T}})^{-1}(\mathbf{s}+\mathbf{1})\rfloor\right\rangle$$

which is non vanishing (and equal to 1) if and only if $[(A^{\mathsf{T}})^{-1}(\mathbf{r}+\mathbf{s}+\mathbf{2})]=0$. The corresponding A-side pairing is

$$\left(\prod_{i\in I_0} k_i^{-\mathbf{w}_i\cdot(\mathbf{r}_i+\mathbf{s}_i)}\right)\langle \lceil \mathbf{X}^{\mathbf{r}}\,;0\rfloor,\ \lceil \mathbf{X}^{\mathbf{s}}\,;0\rfloor\rangle \tag{5.19}$$

which is also non-vanishing if and only if $[(A^{\mathsf{T}})^{-1}(\mathbf{r}+\mathbf{s}+\mathbf{2})]=0$ by Corollary 3.15. If it is non-vanishing, then given that neither $[(A^{\mathsf{T}})^{-1}(\mathbf{r}+\mathbf{1})]$ nor $[(A^{\mathsf{T}})^{-1}(\mathbf{s}+\mathbf{1})]$ is the identity, we know by Remark 3.13 that $\mathbf{r}_i+\mathbf{s}_i=\mathbf{a}_i-\mathbf{1}$ for W_i a loop. We also easily see that $r_i+s_i=a_i-2$ for W_i a Fermat. Thus (5.19) is equal to

$$\left(\prod \frac{1}{b_i}\right)\left(\prod b_i\right)=1$$

as desired. $\square$

This completes the proof of Theorem 5.1.

Appendix A. Proof of Surjectivity of the Algebra Isomorphism for Even-Variable Loops

For Fermats, chains, and odd-variable loops, the following result was easy from Lemma 3.2 of [10], but for even variable loops, we need the following.

Lemma A.1. *Suppose W is a loop polynomial with an even number of variables. Then the map $\mathscr{B}_{W,0} \to \mathscr{H}_{W,G_W^{max}}$ from equation* (4.4)*, given on algebra generators by*

$$Y_i \mapsto \lceil 1\,; \rho_i + J \rfloor$$

is surjective.

Proof. For brevity, let Z_i be the image of Y_i under (4.4), so $Z_i = \lceil 1\,; \rho_i + J \rfloor$. Recall from Sections 4.2 and 4.3 that all that remains to check is that the map is onto the two dimensional sector of the identity group element. One can check using Remark 2.16 and Corollary 3.6 that $\prod_{\text{odd}} Z_i^{a_i-1}$ and $\prod_{\text{odd}} Z_i^{a_i-1}$ are in this subspace. Let

$$\alpha = \prod_{\text{odd}} Z_i^{a_i-1} - \prod_{\text{even}} \left(-a_i Z_i^{a_i-1}\right)$$

and

$$\beta = \prod_{\text{even}} Z_i^{a_i-1} - \prod_{\text{odd}} \left(-a_i Z_i^{a_i-1}\right).$$

These are both elements of the two-dimensional identity sector, and are images of (4.4). Thus, in order to check surjectivity, it suffices to show that these two are linearly independent.

It is sufficient to find δ and ϵ such that

$$\alpha \star \delta = 0, \quad \beta \star \delta \neq 0$$
$$\alpha \star \epsilon \neq 0, \quad \beta \star \epsilon = 0$$

For then suppose $c_1\alpha + c_2\beta = 0$. Then $c_1\alpha\star\delta + c_2\beta\star\delta = 0$, so $c_2 = 0$. Similarly, multiplication by ϵ gives $c_1 = 0$.

Lemmas 3.3 and 3.4 of [10] say that

$$\lceil 1\,; \rho_i + J\rfloor^{a_i} = -a_{i-1}\,\lceil 1\,; \rho_{i-2} + (a_{i-1}-1)\rho_{i-1} + J\rfloor \tag{A.1}$$

and combining this with (4.5) we have

$$Z_i^{a_i} = -a_{i-1} Z_{i-1}^{a_{i-1}-1} Z_{i-2}$$

Pick $\delta = \prod_{\text{odd}} Z_i$. Then, using (4.5) and (A.1), we have

$$\begin{aligned}
\alpha \star \delta &= \prod_{\text{odd}} Z_i^{a_i} - \left(\prod_{\text{even}} \left(-a_i Z_i^{a_i-1}\right)\right)\left(\prod_{\text{odd}} Z_i\right) \\
&= \prod_{\text{odd}} \left(-a_{i-1} Z_{i-1}^{a_{i-1}-1} Z_{i-2}\right) - \prod_{\text{odd}} \left(-a_{i-1} Z_{i-1}^{a_{i-1}-1} Z_i\right) \\
&= 0.
\end{aligned}$$

Pick $\epsilon = \prod_{\text{even}} Z_i$. Then

$$\begin{aligned}
\alpha \star \epsilon &= \left(\prod_{\text{odd}} Z_i^{a_i-1}\right)\left(\prod_{\text{even}} Z_i\right) - \prod_{\text{even}} (-a_i Z_i^{a_i}) \\
&= \prod_{\text{odd}} Z_i^{a_i-1} Z_{i-1} - \prod_{\text{even}} (-a_i Z_i^{a_i}) \\
&= \prod_{\text{even}} Z_{i-1}^{a_{i-1}-1} Z_{i-2} - \prod_{\text{even}} \left(a_i a_{i-1} Z_{i-1}^{a_{i-1}-1} Z_{i-2}\right) \\
&= \left(1 - \prod_i a_i\right) \prod_{\text{even}} Z_{i-1}^{a_{i-1}-1} Z_{i-2}.
\end{aligned}$$

The coefficient is non-zero since $a_i \geq 2$, and we can use (4.5) to see that $\prod_{\text{even}} Z_{i-1}^{a_{i-1}-1} Z_{i-2}$ is non-zero. The same computations, replacing *even* with *odd*, give the desired relations for β. □

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