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Topics in Analytic Number Theory

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Topics in Analytic Number Theory

by

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ABSTRACT

Topics in Analytic Number Theory

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The thesis is in two parts. The first part is the paper "The Distribution of k-free integers" that my advisor, Dr. Roger Baker, and I submitted in February 2009. The reader will note that I have inserted additional commentary and explanations which appear in smaller text. Dr. Baker and I improved the asymptotic formula for the number of k-free integers less than x by taking advantage of exponential sum techniques developed since the 1980's. Both of us made substantial contributions to the paper. I discovered the exponent in the error term for the cases $k = 3, 4$, and worked the case $k = 3$ completely. Dr. Baker corrected my work for $k = 4$ and proved the result for $k = 5$. He then generalized our work into the paper as it now stands. We also discussed and both contributed to parts of section 3 on bounds for exponential sums.

The second part represents my own work guided by my advisor. I study the zeros of derivatives of Dirichlet L-functions. The first theorem gives an analog for a result of Speiser on the zeros of $\zeta'(s)$. He proved that RH is equivalent to the hypothesis that $\zeta'(s)$ has no zeros with real part strictly between 0 and $\frac{1}{2}$. The last two theorems discuss zero-free regions to the left and right for $L^{(k)}(s, \chi)$.

ACKNOWLEDGMENTS

I owe and attribute a lot to my advisor for guiding and teaching me how to effectively research.

The proofs of the zero-free regions follow a line of reasoning similar to Robert Spira's [26]. I generalize his arguments so that I may apply them to Dirichlet L-functions.

My main lines of reasoning in the analog of Speiser's result are derived from papers of Montgomery and Levinson [19], Titchmarsh [28], and Berndt [7].

Contents

1 The Distribution of k-free integers

Abstract

Let $k \in \{3, 4, 5\}$. Let

$$R_k(x) = \sum_{\substack{n \le x \\ n \text{ is } k\text{-free}}} 1 - \frac{x}{\zeta(k)}.$$

We give new upper bounds for $R_k(x)$ conditional on the Riemann hypothesis, improving work of S.W. Graham and J. Pintz. The method stays close to that devised by H.L. Montgomery and R.C. Vaughan, with the improvement depending on exponential sum results.

1.1 Introduction

Let $k \ge 2$. A positive integer n is said to be k-free if n is not divisible by the k-th power of a prime. Let

$$R_k(x) = \sum_{\substack{n \le x \\ n \text{ is } k\text{-free}}} 1 - \frac{x}{\zeta(k)}.$$

An elementary argument yields

$$R_k(x) \ll x^{1/k}.$$

In the opposite direction,

$$R_k(x) = \Omega(x^{1/2k}).$$

(See [10] or [22]).

Assuming the Riemann hypothesis (RH), Montgomery and Vaughan [20] obtained

$$R_k(x) \ll x^{1/(k+1)+\epsilon}. \tag{1.1}$$

Here ϵ is an arbitrarily small positive number. The exponent in (1.1) has been improved for every value of k. For $k \geq 3$, see Graham and Pintz [14]. For $k = 2$, there have been papers by Graham [12], Baker and Pintz [6] and Jia [18]; the exponent in [18] is $17/54 + \epsilon$.

No-one has yet been able to improve the approximation in Theorem 1 of [20] under RH: for $N \geq 1$,

$$R_k(x) = -\sum_{n \leq y_k} \mu(n)\psi(xn^{-k}) + \mathrm{O}(x^{1/2+\epsilon} y_k^{(1-k)/2} + y_k^{1/2+\epsilon}). \tag{1.2}$$

Here $\psi(w) = w - [w] - 1/2$. Once (1.2) is applied, exponential sums dominate the discussion.

Montgomery and Vaughan's result

Let

$$a_n = \sum_{d^k | n} \mu(n).$$

Then a_n is the indicator function of whether or not the k-th power part of n is 1 or greater than 1. That is $a_n = \begin{cases} 1, & \text{n is k-free;} \\ 0, & \text{n is not k-free.} \end{cases}$

It follows that a_n is the n-th coefficient of the series $\frac{\zeta(s)}{\zeta(ks)}$. Then, $Q_k(x) = \sum_{n \leq x} a_n$. The standard estimate for such a coefficient sum of a Dirichlet series may be obtained using an inverse Mellin transform, i.e. Perron's formula

$$\sum_{n \leq x}{}' a_n = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{\zeta(s)}{\zeta(ks)} \frac{x^s}{s} ds + R$$

where $R \to \infty$ as $T \to \infty$, $c > 1$, and the dash signifies that if x is an integer, it only has weight $\frac{1}{2}$ in the sum (See [21, Theorem 5.2]).

Instead of using Perron's formula directly, Montgomery and Vaughan used a different approach [20] dependent on RH. They break $Q_k(x)$ into two sums depending on a variable y_k which they introduce.

$$Q_k(x) = \sum_{n\le x}\sum_{d^k|n}\mu(n) = \sum_{n\le x}\sum_{\substack{d^k|n\\ d>y_k}}\mu(n) + \sum_{n\le x}\sum_{\substack{d^k|n\\ d\le y_k}}\mu(n).$$

In this first sum, they interpret $b_n = \sum_{\substack{d^k|n\\ d>y_k}}\mu(n)$ as the nth coefficient of the Dirichlet series $\zeta(s)\sum_{d>y_k}\mu(d)d^{-ks}$ and apply a version of Perron's formula using a contour integral and residue calculus. They use an estimate for $\zeta(s)$ and $\zeta(s)^{-1}$ dependent on RH. The other sum is manipulated using the estimate $\sum_{d\le y_k}\mu(d) = O(y_k^{1/2+\epsilon})$ dependent on RH. Putting the pieces together, Montgomery and Vaughan proved (1.2). In section 4 below, we choose the value of y_k.

Jia's paper, which appeared after [14], contains an estimate for exponential sums which has found several applications. Abstractions of the estimate are given by Wu [30]. The exponential sum estimate devised by Heath-Brown [17] is crucial in [6], [14], [18]; it is abstracted in Baker [2]. In the present paper we combine these results with theorems from Robert and Sargos [23] and Baker [4]. The former paper is a natural culmination of the work of Fouvry and Iwaniec [11]; the starting point is the double large sieve. The latter paper is related to earlier work of Baker and Kolesnik [5]. Actually, we shall adapt the results from [4], [30] a little.

Theorem. *Assume RH. We have*

$$R_k(x) \ll x^{\theta_k+\epsilon} \quad (k = 3,\ 4,\ 5)$$

where

$$\theta_3 = \frac{17}{74} = 0.2297...,\ \theta_4 = \frac{17}{94} = 0.1808...,\ \theta_5 = 0.15.$$

For comparison, θ_k replaces $7/(6k+8)$ in [9], and we have

$$\frac{7}{30} = 0.2333...,\ \frac{7}{38} = 0.1842...,\ \frac{7}{46} = 0.1521....$$

In contrast with [20], [12], [6], [14], [18], we apply Heath-Brown's "generalized Vaughan identity" [17], or more precisely, the variant provided in [4]. We add a new result on the application of this decomposition (the case $\frac{1}{6} < a \leq \frac{1}{5}$ of Lemma 3(iii) below) which may be of independent interest.

Overview of Method

We concentrate most of our efforts on the sum $S(x) = \sum_{n \leq y_k} \mu(n)\psi(xn^{-k})$ in (1.2). In section 4, we indicate that

$$\left| \sum_{D<n\leq 2D} \mu(n)\psi(xn^{-k}) \right| \leq \left| \sum_{0<|h|\leq H} a_h \sum_{D<n\leq 2D} \mu(n)e(hxn^{-k}) \right| + \sum_{|h|\leq H} b_h \sum_{n\sim D} e(hxn^{-k})$$

where a_h, b_h are explicitly determined and H is an introduced parameter (for details see section 4). Thus $S(x)$ is approximated by exponential sums of the type

$$\sum_{n\sim D} \mu(n)e(hxn^{-k}).$$

These are broken down into subsums by the Heath-Brown Decomposition below. Combinatorially, all these subsums can be classified as having a certain trait or not, that is being Type I and summed over a certain range or Type II and summed over another (See lemma 3). In our work, we determine cases when exponential sum estimates, stated in section 3, give the bound x^{θ_k} for both traits, and therefore for the whole case. These cases piece together so that every subsum has the bound x^{θ_k}. The decomposition only introduces a log factor and so we arrive at the bound $x^{\theta_k+\epsilon}$ for $S(x)$. For the remaining terms in $R_k(x)$, we choose y_k so that they are $\ll x^{\theta_k+\epsilon}$

Note: We arrived at the bound $x^{\theta_k+\epsilon}$ by finding which cases the exponential sum results we examined gave a bound of $x^{\theta-\delta}$ where $\delta > 0$ and θ was the previous record. Then we pieced the puzzle together, optimizing our value of δ. We modified some of the exponential sum results when needed (see Lemmas 6 and 7).

We close this section with a few remarks on notation. Implied constants depend at most on k and ϵ, except in section 3 where they may depend on u, v, α, β, γ, κ

and λ. We write "$A \asymp B$" if $A \ll B \ll A$. We write "$n \sim N$" for $N < n \le 2N$. As usual, $e(z) = e^{2\pi iz}$. The cardinality of a finite set E is written $|E|$.

1.2 Decomposition of sums involving the Möbius function

Let $2 \le D < D' \le 2D$ and let f be a complex function on $[D, D']$. The sum

$$\sum_{D<n\le D'} \mu(n) f(n)$$

can be decomposed into $O((logD)^{2l-1})$ sums of the form

$$\sum_{\substack{n_i \sim N_i \\ D < n_1 \ldots n_{2l-1} \le D'}} \mu(n_l) \ldots \mu(n_{2l-1}) f(n_1 \ldots n_{2l-1}). \tag{2.1}$$

Here $N_i > \frac{1}{2}$, $\prod_{i=1}^{2l-1} N_i \asymp D$ and

$$2N_i \le (2D)^{1/l} \text{ if } i \ge l. \tag{2.2}$$

This is the variant of the Heath-Brown Decomposition given by Baker [4, Section 2]. I discuss it below:

The Decomposition

Let $g(s) = \sum_{n \le Y} \mu(n) n^{-s}$ where Y is chosen later. First, we derive an identity for $\mu(n)$. Trivially,

$$\zeta(s)^{-1} = \zeta(s)^{-1} - \zeta(s)^{-1}(1 - \zeta(s)g(s))^l + \zeta(s)^{-1}(1 - \zeta(s)g(s))^l.$$

Expanding the first product,

$$-\zeta(s)^{-1} \sum_{j=0}^{l} \binom{l}{j} (-1)^j \zeta(s)^j g(s)^j = \sum_{j=0}^{l} \binom{l}{j} (-1)^{j-1} \zeta(s)^{j-1} g(s)^j.$$

This becomes:

$$-\zeta(s)^{-1} + \sum_{j=1}^{l} \binom{l}{j} (-1)^{j-1}\zeta(s)^{j-1}g(s)^j.$$

Therefore,

$$\zeta(s)^{-1} = \sum_{j=1}^{l} \binom{l}{j} (-1)^{j-1}\zeta(s)^{j-1}g(s)^j + \zeta(s)^{-1}(1-\zeta(s)g(s))^l. \qquad (*)$$

Since we may equate the coefficients of n^{-s} the two Dirichlet series on either side, we have an identity for $\mu(n)$. Choose $Y = (2D)^{1/l}$. Note that

$$\zeta(s)g(s) = \sum_{j\geq 1} \Big(\sum_{\substack{n|j \\ n\leq Y}} \mu(n)\Big) j^{-s} = 1 + \sum_{j>Y} \Big(\sum_{\substack{n|j \\ n\leq Y}} \mu(n)\Big) j^{-s}.$$

Then the last term of (*) makes no contribution to the coefficient of n^{-s} on the right side if $n \leq 2D$.

Putting this identity for $\mu(n)$ into the sum $\sum_{D<n\leq D'} f(n)\mu(n)$ we obtain the above decomposition.

To apply the decomposition, we need results of the following kind. The numbers α_i $(1 \leq i \leq r)$ in Lemma 1 arise as exponents for which $N_i \asymp D^{\alpha_i}$. (To be precise, let $r = 2l-1$, $N_0 = 2^{2l-1}N_1...N_{2l-1}$ and define α_i by

$$2N_i = N_0^{\alpha_i} \text{ for } 1 \leq i \leq 2l-1.)$$

Lemma 1. *Let* $0 \leq \alpha_1 \leq ... \leq \alpha_r$, $\alpha_1 + ... + \alpha_r = 1$. *For* $S \subseteq \{1, ..., r\}$, *we write* $\sigma_s = \sum_{i\in S} \alpha_i$.

(i) Let h *be an integer,* $h \geq 3$. *Suppose that* $\alpha_r \leq \frac{2}{h+1}$. *Then some* $\sigma_s \in [\frac{1}{h}, \frac{2}{h+1}]$.

(ii) Let $\lambda \geq \frac{2}{3}$. *Suppose that* $\alpha_r \leq \lambda$. *Then some* $\sigma_s \in [1-\lambda, \lambda]$.

Proof. See [4, Lemma 1].

The following lemma and Lemma 3(iii) are based on a result in Heath-Brown [4], which it strengthens in the case $\frac{1}{6} < a \leq \frac{1}{5}$.

Lemma 2. *Make the hypotheses of Lemma 1.*

(i) Let $\frac{1}{5} < a \leq \frac{1}{3}$. Suppose that $\alpha_r \leq \frac{1-a}{2}$. Then some $\sigma_s \in [a, 2a]$.

(ii) Let $0 < a \leq \frac{1}{5}$. Suppose that $\alpha_r \leq \frac{1-a}{2}$. Then some $\sigma_s \in [a, \frac{1}{3}]$.

Proof. (i) We have $\frac{1}{h+1} < a \leq \frac{1}{h}$ for some natural number h, which must be 3 or 4. Now

$$\alpha_r \leq \frac{1-a}{2} < \frac{2}{5} \leq \frac{2}{h+1}.$$

By Lemma 1(i), some $\sigma_s \in [\frac{1}{h}, \frac{2}{h+1}] \subseteq [a, 2a]$.

(ii) Suppose that no $\sigma_s \in [a, \frac{1}{3}]$. Let

$$T = \{j : \alpha_j \leq \frac{1}{3} - a\} \quad , \quad U = \{j : \frac{1}{3} - a < \alpha_j < a\}$$

(U is empty if $a \leq \frac{1}{6}$), and

$$V = \{j : \alpha_j > \frac{1}{3}\}.$$

Thus

$$\sigma_T + \sigma_U + \sigma_V = 1. \tag{2.3}$$

Clearly

$$|V| \leq 2 \quad , \quad \sigma_V \leq |V| \left(\frac{1-a}{2}\right) \leq 1 - a. \tag{2.4}$$

Suppose for a moment that U is nonempty. Then for any $j \in U$,

$$\sigma_T + \alpha_j < a. \tag{2.5}$$

To see this suppose the contrary, and take the smallest $\sigma_W \geq a - \alpha_j$ with $W \subseteq T$. Then

$$\sigma_W - \alpha_k < a - \alpha_j$$

for any $k \in W$. Hence $a \leq \sigma_W + \alpha_j < a + \alpha_k \leq \frac{1}{3}$, contrary to our hypothesis. A similar argument gives $\sigma_T < a$ if U is empty, and it follows that

$$\sigma_T + \sigma_U < a + \max(0, |U| - 1)a = \max(1, |U|)a. \tag{2.6}$$

Suppose for a moment that $|U| \geq 2$. Take α_i, α_j in U, $i \neq j$. Then

$$\alpha_i + \alpha_j > \frac{2}{3} - 2a > a.$$

Consequently, $\alpha_i + \alpha_j > \frac{1}{3}$. This yields

$$\text{If } |U| \geq 2, \text{ then } \sigma_U > \frac{|U|}{2}\frac{1}{3} = \frac{|U|}{6}. \tag{2.7}$$

In particular, $|U| \leq 5$.

We now consider all possibilities for $|U|$.

Suppose $|U| = 0$ or 1. From (2.6),

$$\sigma_T + \sigma_U < a,$$

and from (2.4),

$$\sigma_T + \sigma_U + \sigma_V < 1,$$

which is absurd.

Suppose $|U| = 2$ or 3. Then $\sigma_U > \frac{1}{3}$ from (2.7), and so $\sigma_V < \frac{2}{3}$ and $|V| \le 1$,

$$\sigma_V \le \frac{1-a}{2}.$$

In conjunction with (2.6), this yields

$$\sigma_T + \sigma_U + \sigma_V < 3a + \frac{1-a}{2} \le 1,$$

which is absurd.

Suppose finally that $|U| = 4$ or 5. From (2.6) and (2.7),

$$\frac{2}{3} < \sigma_T + \sigma_U < 5a \le 1,$$

so that $\sigma_V \in (0, \frac{1}{3})$. This is absurd and the lemma is proved.

Lemma 3. *Let f be a complex function on $(D, D']$ where $2 \le D < D' \le 2D$.*

(i) Suppose that

$$\sum_{\substack{m\sim M \\ D<mn\le D'}} \sum_{n\sim N} a_m b_n f(mn) \ll B \tag{2.8}$$

whenever $|a_m| \le 1$, $|b_n| \le 1$ and

$$D^{1/h} \ll N \ll D^{2/(h+1)}$$

where h is a natural number, $h \ge 3$. Suppose further that

$$\sum_{\substack{m\sim M \\ D<mn\le D'}} a_m \sum_{n\sim N} f(mn) \ll B \tag{2.9}$$

whenever $|a_m| \le 1$ and

$$N \gg D^{2/(h+1)}.$$

Then

$$\sum_{D<d\leq D'} \mu(d)f(d) \ll BD^{\epsilon}. \tag{2.10}$$

(ii) Let $\lambda \geq \frac{2}{3}$. *Suppose that (2.8) holds whenever* $|a_m| \leq 1$, $|b_n| \leq 1$ *and*

$$D^{1-\lambda} \ll N \ll D^{1/2},$$

while (2.9) holds whenever $|a_m| \leq 1$ *and*

$$N \gg D^{\lambda}.$$

Then (2.10) holds.

(iii) Let $0 < a \leq \frac{1}{3}$. *Let* $b = 2a$ *if* $a > \frac{1}{5}$ *and* $b = \frac{1}{3}$ *if* $a \leq \frac{1}{5}$. *Suppose that (2.8) holds whenever* $|a_m| \leq 1$, $|b_n| \leq 1$ *and*

$$D^a \ll N \ll D^b,$$

while (2.9) holds whenever $|a_m| \leq 1$ *and*

$$N \gg D^{(1-a)/2}.$$

Then (2.10) holds.

Proof. We prove (iii); the other proofs are similar. Take $l = 4$ in the decomposition into sums (2.1). We may suppose that D is large. Given one of the sums (2.1), let N_0 and $\alpha_1, ..., \alpha_{2l-1}$ have the values assigned before Lemma 1. We claim that the sum (2.1) is $O(BN^{\epsilon/2})$. If we can group the variables as

$$n = \prod_{i \in S} \quad , \quad m = \prod_{i \notin S} n_i$$

with $\sigma_S \in [a, b]$, then from (2.8) the sum in (2.1) is $O(BN^{\epsilon/2})$, since the corresponding coefficients a_m, b_n are $O(N^{\epsilon/2})$. If this grouping is not possible, then by Lemma 2 there is an $\alpha_j > \frac{1-a}{2}$. Now we group the variables as

$$m = \prod_{i \neq j} n_i \quad , \quad n = n_j.$$

Note that $j < l$ from (2.2). It follows from (2.9) that the sum in (2.1) is $O(BN^{\epsilon/2})$. Now (iii) follows at once.

1.3 Lemmas on Exponential Sums

We quote two preliminary lemmas from Graham and Kolesnik [13].

Lemma 4. *(Kusmin-Landau) If the real function f is continuously differentiable, and f' is monotonic with*

$$0 < \lambda \leq |f'| \leq \frac{1}{2}$$

on the interval I, then

$$\sum_{n \in I} e\Big(f(n)\Big) \ll \lambda^{-1}.$$

Proof. See [13, Theorem 2.1].

Lemma 5. *Let*

$$E(H) = \sum_{i=1}^{u} A_i H^{a_i} + \sum_{j=1}^{v} B_j H^{-b_j}$$

where A_i, B_j, a_i and b_j are positive. Let $0 \leq H_1 \leq H_2$. Then there is some $H \in (H_1, H_2]$ with

$$E(H) \ll \sum_{i=1}^{u} \sum_{j=1}^{v} ({A_i}^{b_j} {B_j}^{a_i})^{1/(a_i+b_j)} + \sum_{i=1}^{u} A_i H_1^{a_i} + \sum_{j=1}^{v} B_j {H_2}^{-b_j}.$$

Proof. This follows at once from [13, Lemma 2.4].

A brief, non-rigorous synopsis of the theory of Exponent Pairs and the A- and B-processes

An exponent pair is a pair (λ, κ) such that if f meets certain requirements, then $S = \sum_{n \in I} e(f(n)) \ll L^k N^l$ where $I = (a, b] \subset [N, 2N]$ and $f' \asymp L$. Applying the A-process or the B-process yields estimates for S which can be expressed as $L^\kappa N^\lambda$. In particular, these processes yield a method for obtaining new exponent pairs. Suppose that (k, l) is an exponent pair. Note that in the following couple paragraphs, the estimates mentioned for the A- and B-processes contain an inner exponential sum. The exponent pair will apply to that inner sum and then simplifying, we may write (λ, κ) in terms of (l, k). Specifically, if we denote $(\lambda, \kappa) = A(k, l)$ or $B(k, l)$,

$$A(k,l) = \left(\frac{k}{2k+2}, \frac{k+l+1}{2k+2}\right)$$

$$B(k,l) = \left(l - \frac{1}{2}, k + \frac{1}{2}\right)$$

For more details see [13, Sections 3.1, 3.3-3.5]

A-Process (Weyl Shift, or enriched Cauchy's inequality)

This arises as Cauchy's inequality applied to the sum $\sum_{a<n\le b} e(f(n))$ [13, Lemma 2.5]. For $H > 0$,

$$H \sum_{a<n\le b} e(f(n)) = \sum_{k=1}^{H} \sum_{a<n\le b} e(f(n)) = \sum_{a-H<n\le b-1} \sum_{k=1}^{H} e(f(n+k)).$$

Cauchy's inequality $|v \cdot u|^2 \le |v|^2 |u|^2$ is now applied:

$$H^2 |\sum_{a<n\le b} e(f(n))|^2 \le (H + |b-a|) \sum_{a-H<n\le b-1} |\sum_{k=1}^{H} e(f(n+k))|^2.$$

After some manipulating and a change of variable, we arrive at

$$|S|^2 \le \frac{2|I|^2}{H} + \frac{4|I|}{H} \sum_{1\le h\le H} |S_1(h)|$$

where

$$S_1(h) = \sum_{a<n\le b-h} e(f(n+h) - f(n)).$$

The estimates for the A-process are based off of this inequality and [13, Theorem 2.2].

B-Process (Poisson Summation)

Let $F(x) = e(f(x))$ for $a < x \leq b$. Otherwise, set $F(x) = 0$. Then

$$\hat{F}(\nu) = \int_a^b e(f(t) - \nu t)dt.$$

By Poisson summation [27, 6.3 Theorem 4],

$$\sum_{a<n\leq b} e(f(n)) = \sum_{\nu\in\mathbb{Z}} \int_a^b e(f(t) - \nu t)dt.$$

We assume that $f^{(2)} < 0$ on I and set $\alpha = f'(b)$, $\beta = f'(a)$. Graham and Kolesnik in [13, Lemma 3.4] give an estimate for the integral when $\alpha \leq \nu \leq \beta$ as

$$\frac{e(-\frac{1}{8} - f(x_\nu) - \nu x_\nu)}{|f^{(2)}(x_\nu)|^{1/2}} + O(E)$$

where x_ν is chosen so that $f'(x_\nu) = \nu$ where $\nu \in [\alpha, \beta]$. (Note that x_ν exists by choice of α, β). This gives the summands in the main term of the B-process technique plus an error. The other terms can be bounded using [13, Lemma 3.1] and interpreting $\hat{F}(\nu)$ as $\frac{1}{2\pi i\nu}\hat{F}'(\nu)$. Graham and Kolesnik actually don't directly apply Poisson summation as above. Through the use of Riemann-Stieljes integration and the Fourier expansion of $\psi(x)$, they achieve:

$$\sum_{n\in I} e(f(n)) = \sum_{\alpha\leq\nu\leq\beta} \frac{e(-\frac{1}{8} - f(x_\nu) - \nu x_\nu)}{|f^{(2)}(x_\nu)|^{1/2}} + O(\log(BN^{-2} + 2) + B^{-1/2}N)$$

where B is related to the sizes of the second, third and fourth derivatives of f [13, Lemmas 3.5, 3.6].

In the remainder of this section, we write $L = log(D+2)$. Our first lemma is a variant of [4, Theorem 5].

Lemma 6. *Let (κ, λ) be an exponent pair. Let α, β be real constants, $\beta \neq 0$, $\beta < 1$,*

$\alpha < 0$. *Suppose that*

$$X > 0, \frac{1}{2} \leq N \ll M, MN \asymp D, D < D' \leq 2D. \tag{3.1}$$

Let

$$S_1 = \sum_{\substack{m \sim M \\ D \leq mn < D'}} a_m \sum_{n \sim N} b_n e\left(\frac{X m^\alpha n^\beta}{M^\alpha N^\beta}\right) \tag{3.2}$$

where $|a_m| \leq 1$, $|b_n| \leq 1$. *Then*

$$S_1 \ll L^{7/4}\Big(DN^{-1/2} + DM^{-1/4} + D^{5/4}X^{-1/4}N^{-3/4} + D^{5/4}X^{-1/4}M^{-3/8} \tag{3.3}$$

$$+(D^{11+10\kappa}X^{1+2\kappa}N^{2(\lambda-\kappa)})^{1/(14+12\kappa)} + (D^{5+4\kappa}N^{\lambda-\kappa})^{1/(6+4\kappa)}\Big).$$

Proof. In [4], proof of Theorem 5, it is shown that

$$L^{-7}S_1^4 \ll \frac{D^4}{Q^2} + D^3\left(\left(\frac{XQ^3}{D}\right)^{1/2+\kappa} N^{\lambda-\kappa} + N + D^2X^{-1}Q^{-3}\right) \tag{3.4}$$

for any natural number $Q \leq \min(c^2N, cM^{-1/2})$. Here and in the remainder of the paper, c is a small positive constant. With a little thought we see that (3.4) is true for any real Q with $0 < Q \leq \min(c^2N, cM^{-1/2})$. Now Lemma 5 yields

$$L^{-7}S_1^4 \ll D^4N^{-2} + D^4M^{-1} + (D^4)^{(3+6\kappa)/(7+6\kappa)}(D^{5/2-\kappa}X^{1/2+\kappa}N^{\lambda-\kappa})^{4/(7+6\kappa)}$$

$$+(D^5X^{-1})^{(3+6\kappa)/(9+6\kappa)}(D^{5/2-\kappa}X^{1/2+\kappa}N^{\lambda-\kappa})^{6/(9+6\kappa)}$$

$$+D^5X^{-1}M^{-3/2} + D^5X^{-1}N^{-3}.$$

After some simplification, we obtain (3.3).

Our next lemma improves (2.2) of [30, Theorem 2]; the term $(X^{-1}M^{14}N^{23})^{1/22}$ has been dropped.

Lemma 7. *Let α, β be real constants, $\alpha\beta(\alpha-1)(\beta-1) \neq 0$. Suppose that (3.1) holds and $|a_m| \le 1$, $|b_n| \le 1$. The sum S_1 in (3.2) satisfies*

$$S_1 \ll L^3\Big((XM^3N^4)^{1/5} + (X^4M^{10}N^{11})^{1/16} + (XM^7N^{10})^{1/11} + MN^{1/2} + X^{-1/2}MN\Big). \tag{3.5}$$

Proof. Define S_0 in the same way as S_1 but without the condition $D < mn \le D'$. It suffices to prove (3.5) with S_1, L^3 replaced by S_0, L^2, as explained in Harman [15, pp. 49-50].

If $X \le M$, the double large sieve [11, Theorem 2] yields

$$S_0 \ll L\Big((XMN)^{1/2} + MN^{1/2} + X^{-1/2}MN\Big)$$

which is satisfactory since $(XMN)^{1/2} \ll MN^{1/2}$.

Suppose now that $X > M$. We follow the proof of [30, Theorem 2] to save space. Obviously we may assume that $N > L^3$, since $S_0 \ll MN \ll MN^{1/2}L^{3/2}$ otherwise. The first step (a Weyl shift) gives

$$S_0^2 \ll \frac{(MN)^2}{Q} + LM^{3/2}NQ^{-1}|S(Q_1)| \tag{3.6}$$

where $Q \in [L, \frac{N}{L}]$ is a parameter at our disposal, and $1 \le Q_1 \le Q$. Here

$$S(Q_1) = \sum_{q_1 \sim Q_1} \sum_{n+q_1, n \sim N} b_{n+q_1}\bar{b}_n \sum_{m \sim M} m^{-1/2} e\Big(Am^\alpha t(n, q_1)\Big),$$

$$A = \frac{X}{M^\alpha N^\beta} \quad , \quad t(n, q_1) = (n+q_1)^\beta - n^\beta.$$

Suppose initially that

$$X(MN)^{-1}Q \geq c. \tag{3.7}$$

The next step (a B process) yields

$$L^{-1}S(Q_1) \ll (XM^{-1}N^{-1}Q_1)^{1/2}S^*(Q_1) \tag{3.8}$$

$$+(XM^{-1}N^{-1}Q_1^3)^{1/2} + M^{-1/2}NQ_1 + (X^{-1}MNQ_1)^{1/2} + (X^{-2}MN^4)^{1/2}.$$

Here

$$S^*(Q_1) = \sum_{n\sim N}\left|\sum_{q_1\sim Q_1} b_{n+q_1}e\Big(\theta q_1 + \tilde{\alpha}(At)^{\gamma}l^{1-\gamma}\Big)\right|,$$

θ is a real constant, $l \asymp X(MN)^{-1}Q_1$, $\gamma = \frac{1}{1-\alpha}$ and $\tilde{\alpha} = |1-\alpha||\alpha|^{\alpha/(1-\alpha)}$.

For the next step, suppose for the moment that $Q_1 \geq L$. A Weyl shift yields

$$S^*(Q_1)^2 \ll (NQ_1)^2Q_2^{-1} + NQ_1Q_2^{-1}\sum_{1\leq q_2\leq Q_2}|S_2(q_2)| \tag{3.9}$$

where Q_2 is chosen below, $Q_2 \leq c\sqrt{Q_1}$, and

$$S_2(q_2) = \sum_{n\sim N}\sum_{q_1+q_2,q_1\sim Q_1}\bar{b}_{n+q_1}b_{n+q_1+q_2}e\Big(t_1(n,q_1,q_2)\Big)$$

with

$$t_1(n,q_1,q_2) = \tilde{\alpha}A^{\gamma}l^{1-\gamma}(t(n,q_1+q_2)^{\gamma} - t(n,q_1)^{\gamma}).$$

Note that Wu uses the incorrect expression $S_2(q_1,q_2)$ in place of $S_2(q_2)$.

The final step in [30], another Weyl shift, yields

$$\left(\frac{S_2(q_2)}{L}\right)^2 \ll (NQ_1)^2Q_2^{-2} + NQ_1Q_2^{-2}\sum_{1\leq q_3\leq Q_2^2}\sum_{q_1\sim Q_1}|S_3(q_1,q_2,q_3)| \tag{3.10}$$

with

$$S_3(q_1, q_2, q_3) = \sum_{n' \sim N} e\Big(f(n')\Big)$$

for an f (depending on the q_i) with

$$f'(n') \asymp X N^{-2} Q_1^{-1} q_2 q_3 \quad (n' \sim N).$$

We choose

$$Q_2 = c \min(Q_1^{1/2}, (X^{-1} N^2 Q_1)^{1/3}),$$

so that

$$|f'(n')| \leq \frac{1}{2} \quad \text{for } n' \sim N$$

and

$$S_3(q_1, q_2, q_3) \ll (X N^{-2} Q_1^{-1} q_2 q_3)^{-1}$$

(Lemma 4). From (3.10),

$$\left(\frac{S_2(q_2)}{L}\right)^2 \ll (NQ_1)^2 Q_2^{-2} + N Q_1^2 Q_2^{-2} (X N^{-2} Q_1^{-1} q_2)^{-1} \sum_{1 \leq q_3 \leq Q_2^2} q_3^{-1},$$

$$L^{-3} S_2(q_2)^2 \ll N^2 Q_1^2 Q_2^{-2} + X^{-1} N^3 Q_1^3 Q_2^{-2} q_2^{-1}.$$

Now (3.9) yields

$$L^{-3/2} S^*(Q_1)^2 \ll N^2 Q_1^2 Q_2^{-1} + N Q_1 Q_2^{-1} \sum_{q_2 \leq Q_2} (N Q_1 Q_2^{-1} + X^{-1/2} N^{3/2} Q_1^{3/2} Q_2^{-1} q_2^{-1/2})$$

$$\ll N^2 Q_1^2 Q_2^{-1} + N^{5/2} Q_1^{5/2} Q_2^{-3/2} X^{-1/2}$$

$$\ll N^2 Q_1^{3/2} + N^{4/3} Q_1^{5/3} X^{1/3} + N^{5/2} Q_1^{7/4} X^{-1/2},$$

where we used the value of Q_2 in the last step.

Recalling (3.8),

$$L^{-7/4}S(Q_1) \ll \tag{3.11}$$

$$(XM^{-1}N^{-1}Q_1)^{1/2}(NQ_1^{3/4} + N^{2/3}Q_1^{5/6}X^{1/6} + N^{5/4}Q_1^{7/8}X^{-1/4})$$

$$+X^{1/2}M^{-1/2}N^{-1/2}Q_1^{3/2} + M^{-1/2}NQ_1 + X^{-1/2}M^{1/2}N^{1/2}Q_1^{1/2} + X^{-1}M^{1/2}N^2$$

$$\ll X^{1/2}M^{-1/2}N^{1/2}Q_1^{5/4} + X^{2/3}M^{-1/2}N^{1/6}Q_1^{4/3} + X^{1/4}M^{-1/2}N^{3/4}Q_1^{11/8} + X^{-1}M^{1/2}N^2.$$

We were able to discard three of the first last four terms in the first bound in (3.11):

$$X^{1/2}M^{-1/2}N^{-1/2}Q_1^{3/2} \ll X^{1/2}M^{-1/2}N^{1/2}Q_1^{5/4}, \text{ since } Q_1 < N;$$

$$M^{-1/2}NQ_1 \ll X^{1/2}M^{-1/2}N^{1/2}Q_1^{5/4}, \text{ since } X^{1/2}N^{1/2} \gg N;$$

$$X^{-1/2}M^{1/2}N^{1/2}Q_1^{1/2} < X^{1/2}M^{-1/2}N^{1/2}Q_1^{5/4}, \text{ since } M < X.$$

Recalling (3.6),

$$L^{-7/4}S_0^2 \ll \frac{(MN)^2}{Q} + LM^{3/2}NQ^{-1}\Big(X^{1/2}M^{-1/2}N^{1/2}Q_1^{5/4} +$$

$$X^{2/3}M^{-1/2}N^{1/6}Q_1^{4/3} + X^{1/4}M^{-1/2}N^{3/4}Q_1^{11/8} + X^{-1}M^{1/2}N^2\Big)$$

Thus,

$$L^{-7/4}S_0^2 \ll \tag{3.12}$$

$$L\frac{(MN)^2}{Q} + LMN^{3/2}X^{1/2}Q^{1/4} + LMN^{7/6}X^{2/3}Q^{1/3} + LMN^{7/4}X^{1/4}Q^{3/8}.$$

Here we used

$$LM^{3/2}NQ^{-1}(X^{-1}M^{1/2}N^2) = LM^2N^3X^{-1}Q^{-1} < LMN^3Q^{-1} \ll LM^2N^2Q^{-1}.$$

The inequality (3.12) remains valid if $Q_1 < L$. For then (3.6) yields trivially

$$S_0^2 \ll \frac{(MN)^2}{Q} + L^2M^2N^2Q^{-1} \ll L^2(MN)^2Q^{-1}.$$

Suppose now that $X(MN)^{-1}Q < c$. We remove $m^{-1/2}$ from $S(Q_1)$ by partial summation and apply Lemma 4, since

$$\frac{d}{dm}\Big(Am^\alpha t(n,q_1)\Big) < \frac{1}{2} \quad (m \sim M)$$

for all relevant n, q_1. Now (3.6) yields

$$S_0^2 \ll \frac{(MN)^2}{Q} + \frac{MN}{Q}\sum_{1\le q_1\le Q}\Big(X(MN)^{-1}q_1\Big)^{-1}$$

$$\ll \frac{(MN)^2}{Q} + LM^2N^2Q^{-1}X^{-1} \ll \frac{L(MN)^2}{Q}$$

since $M < X$. Thus (3.12) always holds, and indeed remains valid for $Q \in (0, L]$. An application of Lemma 4 with $H_1 = 0$, $H_2 = \frac{N}{L}$ yields

$$L^{-11/4}S_0^2 \ll M^2N + (MN^{3/2}X^{1/2})^{4/5}(M^2N^2)^{1/5} + (MN^{7/6}X^{2/3})^{3/4}(M^2N^2)^{1/4}$$

$$+(MN^{7/4}X^{1/4})^{8/11}(M^2N^2)^{3/11}.$$

This gives the required variant of (3.5) for S_0 and completes the proof.

Lemma 8. *Let α, β, γ be real constants with $\alpha < 1$, $\alpha\beta\gamma \neq 0$. Let (κ, λ) be an* *exponent pair. Let M, M_1, M_2 be real numbers $\geq \frac{1}{2}$ and $X \geq M_1 M_2$. Let*

$$S_2 = \sum_{\substack{m \sim M \\ D \leq m m_1 < D'}} \sum_{m_1 \sim M_1} \sum_{m_2 \sim M_2} a_m b_{m_1, m_2} e\left(\frac{X m^\alpha m_1^\beta m_2^\gamma}{M^\alpha M_1^\beta M_2^\gamma}\right)$$

where $|a_n| \leq 1$, $|b_{m_1,m_2}| \leq 1$. Then

$$S_2 \ll M M_1 M_2 (\log 12 M M_1 M_2)^2 \{(M_1 M_2)^{-1/2} + \left(\frac{X}{M_1 M_2}\right)^{\kappa/(2+2\kappa)} M^{-(1+\kappa-\lambda)/(2+2\kappa)}\}.$$

Proof. By a similar remark to that at the beginning of the last proof, this follows from [2, Theorem 2].

Lemma 9. *Let α, β, γ be real constants with $\alpha(\alpha-1)\beta\gamma \neq 0$. Let H, M, N be real numbers $\geq \frac{1}{2}$, let $X > 0$ and*

$$S = \sum_{h \sim H} \sum_{m \sim M} \max \left| \sum_{n=N_1}^{N_2} e\left(\frac{X m^\alpha h^\beta n^\gamma}{M^\alpha H^\beta N^\gamma}\right) \right|$$

where the maximum is taken over $1 \leq N_1 \leq N_2 \leq N$. Then

$$S \ll (HNM)^{1+\epsilon} \left(\left(\frac{X}{HMN^2}\right)^{1/4} + \frac{1}{N^{1/2}} + \frac{1}{X} \right).$$

Proof. See [23, Theorem 3].

1.4 Proof of Theorem

Throughout this section, $k \in \{3, 4, 5\}$. We write

$$y_k = x^{(1-2\theta_k)/(k-1)},$$

so that (1.2) yields

$$R_k(x) = -\sum_{n \le y_k} \mu(n)\psi(xn^{-k}) + O(x^{\theta_k+\epsilon}).$$

Accordingly it suffices to prove that

$$\sum_{n \sim D} \mu(n)\psi(xn^{-k}) \ll x^{\theta_k + 3\epsilon/4} \tag{4.1}$$

for

$$x^{\theta_k} < D < x^{(1-2\theta_k)/(k-1)}. \tag{4.2}$$

((4.1) is trivial for smaller D.)

Let $H \ge 1$. Vaaler [29] (see also [13, Appendix]) gave the approximation

$$\left| \psi(w) - \sum_{0<|h|\le H} a_h e(hw) \right| \le B(w),$$

where $B(w) = \sum_{|h|\le H} b_h e(hw)$ is non-negative; the a_h, b_h are given explicitly and satisfy

$$a_h \ll \frac{1}{h} \quad , \quad b_h \ll \frac{1}{H}. \tag{4.3}$$

It follows that

$$\left| \sum_{n\sim D} \mu(n)\psi(xn^{-k}) - \sum_{0<|h|\le H} a_h \sum_{n\sim D} \mu(n) e(hxn^{-k}) \right| \le \sum_{|h|\le H} b_h \sum_{n\sim D} e(hxn^{-k}).$$

We select $H = [Dx^{-\theta_k}]$. The contribution to the right-hand side from $h = 0$ is

$$\ll DH^{-1} \ll x^{\theta_k}$$

from (4.3). Accordingly, after a splitting-up argument, we need to show that

$$\sum_{h\sim K} c_h \sum_{n\sim D} \mu(n) e(hxn^{-k}) \ll K x^{\theta_k + 2\epsilon/3} \tag{4.4}$$

whenever (4.2) holds, $|c_h| \le 1$ and

$$\frac{1}{2} \le K \le D x^{-\theta_k}; \tag{4.5}$$

for our proof will show that (4.4) remains valid with 1 in place of $\mu(n)$.

In the rest of this section, we write

$$S_I(D,K,N) = \sum_{h\sim K} c_h \sum_{\substack{m\sim M \\ mn\sim D}} a_m \sum_{n\sim N} e\left(\frac{hx}{(mn^k)}\right),$$

$$S_{II}(D,K,N) = \sum_{h\sim K} \sum_{\substack{m\sim M \\ mn\sim D}} a_m \sum_{n\sim N} b_n e\left(\frac{hx}{(mn^k)}\right),$$

where c_h, a_m, b_n are unspecified numbers of absolute value ≤ 1, and D, K are assumed to satisfy (4.2), (4.5).

Lemma 10. *(i) Suppose that*

$$S_{II}(D,K,N) \ll K x^{\theta_k + \epsilon/3} \tag{4.6}$$

whenever

$$D^2 x^{-2\theta_k} \ll N \ll D^{1/2}. \tag{4.7}$$

Then (4.4) holds.

(ii) Suppose that

$$K < x^{5\theta_k - 1} D^{k-7/2}. \tag{4.8}$$

Then (4.4) holds.

(iii) Suppose that

$$KD^{8-k} < x^{10\theta_k - 1}. \tag{4.9}$$

Then (4.4) holds.

Proof. (i) Since $\theta_k > 5/(6k+4)$, we deduce from (4.2) that

$$D^2 x^{-2\theta_k} \ll D^{1/3-c}. \tag{4.10}$$

We can thus apply Lemma 3(ii) with the choice

$$D^\lambda = x^{2\theta_k} D^{-1} \quad , \quad D^{1-\lambda} = D^2 x^{-2\theta_k}.$$

Our hypothesis gives (4.6) for the range

$$D^{1-\lambda} \ll N \ll D^{1/2}.$$

We claim that

$$S_I(D, K, N) \ll K x^{\theta_k + \epsilon/3} \tag{4.11}$$

for

$$N \gg x^{1-4\theta_k} D^{3-k}. \tag{4.12}$$

This is a consequence of Lemma 9; we must show that

$$KD\left(\frac{x}{N^{1/4} D^{k+1}}\right)^{1/4} + KDN^{-1/2} + KD(KxD^{-k})^{-1} \ll Kx^{\theta_k}.$$

The first term on the left gives rise to the condition (4.12). The second term produces the condition

$$N \gg D^2 x^{-2\theta_k}. \tag{4.13}$$

Moreover, (4.2) gives

$$D^2 x^{-2\theta_k} < x^{1-4\theta_k} D^{3-k},$$

so that (4.13) follows from (4.12). Finally, it is easy to see that

$$x^{-1} D^{1+k} \ll x^{\theta_k},$$

so that the third term gives no difficulty, proving our claim.

Now

$$D^{\lambda} > x^{1-4\theta_k} D^{3-k},$$

or equivalently,

$$D^{4-k} < x^{6\theta_k - 1}.$$

For $k = 3$, this follows from $\theta_3 > 3/14$. For $k = 4$, we use $\theta_4 > 1/6$, and for $k = 5$, we require $D > x^{1/10}$, which is a consequence of (4.2). We conclude that (4.11) holds for $N \gg D^{\lambda}$, and (i) follows at once.

(ii) We apply Lemma 7 to show that (4.6) holds in the range (4.7) under the hypothesis (4.8). Treating the variable h trivially, this reduces to verifying that

$$(KxD^{-k}M^3N^4)^{1/5} + (KxD^{-k}M^{10}N^{11})^{1/16} + (KxD^{-k}M^7N^{10})^{1/11}$$

$$+MN^{1/2} + (KxD^{-k})^{-1/2}D \ll x^{\theta_k}.$$

The term $MN^{1/2}$ gives rise to the lower bound for N in (4.7). Next,

$$(KxD^{-k})^{-1/2}D \ll D^{k/2+1}x^{-1/2} \ll x^{\theta_k}$$

follows from (4.2), because

$$\theta_k > 3/(4k+2).$$

The condition

$$(KxD^{-k}M^3N^4)^{1/5} \ll x^{\theta_k}$$

may be rewritten

$$KN \ll x^{5\theta_k - 1}D^{k-3}. \tag{4.14}$$

In this form, it follows from (4.8) for $N \ll D^{1/2}$.

The condition

$$(KxD^{-k}M^{10}N^{11})^{1/16} \ll x^{\theta_k}$$

may be rewritten

$$KN \ll x^{16\theta_k - 1}D^{k-10}.$$

Now

$$x^{5\theta_k - 1}D^{k-3} < x^{16\theta_k - 1}D^{k-10}$$

from (4.2), because

$$\theta_k > 7/(11k+3).$$

Finally, the condition

$$(KxD^{-k}M^7N^{10})^{1/11} \ll x^{\theta_k}$$

may be rewritten

$$K^{1/3}N \ll x^{(11\theta_k - 1)/3}D^{(k-7)/3}.$$

To show that this follows from (4.14), it suffices to verify that

$$x^{5\theta_k - 1}D^{k-3} < x^{(11\theta_k - 1)/3}D^{(k-7)/3},$$

and this in turn follows from (4.2). This completes the proof of (ii).

(iii) Similarly, on applying Lemma 6 with $\kappa = \lambda = 1/2$, we must show that (4.7) and (4.9) together imply

$$DN^{-1/2} + DM^{-1/4} + D^{5/4}(KxD^{-k})^{-1/4}N^{-3/4}$$

$$+D^{5/4}(KxD^{-k})^{-1/4}M^{-3/8} + D^{4/5}(KxD^{-k})^{1/10} + D^{7/8} \ll x^{\theta_k}.$$

To begin with, we observe that (4.9) implies

$$D \ll x^{(10\theta_k - 1)/(8-k)}. \tag{4.15}$$

Thus

$$D^{7/8} \ll x^{(70\theta_k - 7)/(64-8k)} < x^{\theta_k}$$

because

$$\theta_k < 7/(6+8k). \tag{4.16}$$

Since $M \gg D^{1/2}$, it follows also that

$$DM^{-1/4} \ll x^{\theta_k}.$$

As before, the term $DN^{-1/2}$ gives rise to the lower bound on N in (4.7). Next,

$$D^{4/5}(KxD^{-k})^{1/10} = (KD^{8-k})^{1/10}x^{1/10} < x^{\theta_k}$$

from (4.9). Next,

$$D^{5/4}(KxD^{-k})^{-1/4}N^{-3/4} \ll (D^{5+k}x^{-1})^{1/4}(D^2x^{-2\theta_k})^{-3/4}$$

$$= (D^{k-1}x^{-1+6\theta_k})^{1/4} < x^{\theta_k}$$

from (4.2).

Finally, since $M \gg D^{1/2}$,

$$D^{5/4}(KxD^{-k})^{-1/4}M^{-3/8} \ll (D^{5+k}x^{-1})^{1/4}D^{-3/16} = (D^{17+4k})^{1/16}x^{-1/4} \ll x^{\theta_k}.$$

To see this, we use (4.15):

$$D^{17+4k} \ll x^q \quad , \quad \text{where}$$

$$q = (17+4k)(10\theta_k - 1)/(8-k) < 16\theta_k + 4,$$

as a consequence of (4.16). This completes the proof of (iii).

We have yet to use Lemma 8, which is capable of extending the range (4.7) on the left.

Lemma 11. *Suppose that*

$$K^{-1}D^2x^{-2\theta_k} \ll N \ll \min(D^{k-4}x^{6\theta_k-1}, D^{1/2}). \tag{4.17}$$

Then (4.6) holds.

Proof. We apply Lemma 8 with N, K in place of M_1, M_2, and $\kappa = \lambda = 1/2$. The condition $X \gg M_1M_2$ reduces to

$$xD^{-k} \gg N.$$

Since $N \ll D^{1/2}$, this reduces in turn to $\theta_k \geq 3/(4k+2)$, which was used earlier.

Now we need to show that (4.17) implies

$$KD(NK)^{-1/2} + KD(xD^{-k}N^{-1})^{1/6}M^{-1/3} \ll Kx^{\theta_k}.$$

The term $KD(NK)^{-1/2}$ gives rise to the lower bound for N in (4.17). The condition

$$KD(xD^{-k}N^{-1})^{1/6}M^{-1/3} \ll Kx^{\theta_k}$$

is equivalent to

$$D^6\left(\frac{x}{D^{k+2}}\right)N \ll x^{6\theta_k},$$

which reduces to the upper bound for N in (4.17).

Guide to the proof: intervals with the bound $x^{\theta_k+\epsilon}$

(Below: $\lambda \geq \frac{2}{3}$. This may be different for different cases.)

Let $K < x^{5\theta_k-1}D^{k-7/2}$.

For k=3,4,5:

(Type I) $N \gg D^{\lambda}$ where $\lambda \geq \frac{2}{3}$ (Lemma 9).

(Type II) $D^{1-\lambda} \ll N \ll D^{1/2}$ (Lemma 7).

Let $K \geq x^{5\theta_k-1}D^{k-7/2}$.

For k=3:

(Type I) $N \gg D^{2/5}$ (Lemma 9).

(Type II) $D^{1/4} \ll N \ll D^{2/5}$ (Lemma 8).

For k=4, $D \geq x^{9/47}$:

(Type I) $N \gg D^{2/5}$ (Lemma 9).

(Type II) $D^{1/4} \ll N \ll D^{2/5}$ (Lemma 8).

For k=4, $D < x^{9/47}$:

(Type I) $N \gg D^{\lambda}$ where $\lambda \geq \frac{2}{3}$ (Lemma 9).

(Type II) $D^{1-\lambda} \ll N \ll D^{1/2}$ (Lemma 6).

For $k = 5$, $kD^3 < x^{1/2}$

(Type I) $N \gg D^{\lambda}$ where $\lambda \geq \frac{2}{3}$ (Lemma 9).

(Type II) $D^{1-\lambda} \ll N \ll D^{1/2}$ (Lemma 6).

For $k = 5$, $kD^3 \geq x^{1/2}$

(Type I) $N \gg D^{(1-a)/2}$ where $\lambda \geq \frac{2}{3}$ (Lemma 9).

(Type II) $D^{a} \ll N \ll D^{b}$ where a, b are as in Lemma 3(iii) (Lemma 8).

Proof of the Theorem. Let D, K satisfy (4.2), (4.5). We must show that (4.4) holds.

Suppose first that $k = 3$. If

$$K < x^{11/74} D^{-1/2}$$

then we are done, by Lemma 10(ii). Suppose now that

$$K \geq x^{11/74} D^{-1/2}. \tag{4.18}$$

We apply Lemma 3(i). Thus it suffices to prove (4.6) for

$$D^{1/4} \ll N \ll D^{2/5} \tag{4.19}$$

and (4.11) for

$$N \gg D^{2/5}. \tag{4.20}$$

Lemma 11 gives (4.6) in the range

$$K^{-1} D^2 x^{-17/37} \ll N \ll \min(D^{-1} x^{14/37}, D^{1/2}).$$

From (4.18), (4.2),

$$K^{-1}D^2x^{-17/37} \leq D^{5/2}x^{-45/74} < D^{1/4},$$

$$D^{-1}x^{14/37} > D^{2/5}.$$

Thus (4.6) holds in the required range.

As shown in the proof of Lemma 10, (4.11) holds in the range (4.12), which becomes

$$N \gg x^{3/37}$$

for $k = 3$. Since $x^{3/37} < D^{2/5}$ from (4.2), the discussion of the case $k = 3$ is complete.

Now let $k = 4$. From Lemma 10(ii), we may suppose that

$$K \geq x^{-9/94}D^{1/2}. \tag{4.21}$$

As above, it suffices to prove (4.6) in the range (4.19), and (4.11) in the range (4.20).

Lemma 11 gives (4.6) in the range

$$K^{-1}D^2x^{-17/47} \ll N \ll x^{4/47}, \tag{4.22}$$

since $D^{1/2} > x^{17/188} > x^{4/47}$. Now

$$K^{-1}D^2x^{-17/47} < x^{-25/94}D^{3/2} < D^{1/4},$$

since $D < x^{10/47}$ from (4.2). Moreover,

$$x^{4/47} > D^{2/5},$$

also from (4.2). This gives (4.6) in the desired range.

As for (4.11), the range (4.12) becomes

$$N \gg x^{13/47} D^{-1}. \tag{4.23}$$

If $D \geq x^{9/47}$, then we obtain (4.11) in the desired range by combining the ranges (4.22) and (4.23).

It remains to prove (4.4) in the case $D < x^{9/47}$. In this case, (4.5) yields

$$KD^4 \leq D^5 x^{-17/94} < x^{73/94},$$

and (4.4) follows from Lemma 10(iii).

Finally, let $k = 5$. In view of Lemma 10(ii), (iii), we may suppose that

$$K \geq x^{-1/4} D^{3/2} \tag{4.24}$$

and

$$KD^3 \geq x^{1/2}. \tag{4.25}$$

We may apply Lemma 3(iii) with

$$D^a = K^{-1} D^2 x^{-3/10},$$

which was shown in (4.10) to be smaller than $D^{1/3}$. We must show that (4.6) holds for

$$K^{-1} D^2 x^{-3/10} \ll N \ll \max(D^{1/3}, K^{-2} D^4 x^{-3/5}),$$

and that (4.11) holds for

$$N \gg K^{1/2} D^{-1/2} x^{3/20}.$$

We apply Lemma 11. We have to show that

$$\max(D^{1/3}, K^{-2}D^4x^{-3/5}) \ll Dx^{-1/10}$$

(since $Dx^{-1/10} < D^{1/2}$ from (4.2)). The bound

$$D^{1/3} < Dx^{-1/10}$$

follows from (4.2). Also,

$$K^{-2}D^4x^{-3/5}(Dx^{-1/10})^{-1} = K^{-2}D^3x^{-1/2} \leq 1$$

from (4.24). This gives (4.6) in the required range.

As for (4.11), the range (4.12) satisfies our requirements, because we obtain

$$x^{2/5}D^{-2} \leq K^{1/2}D^{-1/2}x^{3/20}$$

on rearranging (4.25). This finishes the discussion for $k = 5$, and completes the proof of the Theorem.

2 Zeros of $L^{(k)}(s,\chi)$

This part of the thesis discusses the zeros of certain Dirichlet series that have an analytic continuation over the whole complex plane. A Dirichlet Series is a series of the type

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

The function $\zeta(s)$ provides a common example. Let σ denote the real part of s. These series absolutely converge for $\sigma > \sigma_0$ and diverge for $\sigma < \sigma_0$ for some σ_0 called the abscissa of absolute convergence.

They are useful for multiplicative counting problems. Here is an example: The n-th coefficient of $(\zeta(s))^2$ is equal to $\sum_{ab=n} 1$ which is the number of divisors of n. The question then that comes into play is one of analysis–determining the coefficients of the series. (In a similar manner, power series may be used for additive problems, such as in the number of representations a number has as a sum of two squares). See [21, Chp. 1].

A Dirichlet L-function $L(s,\chi)$ is a Dirichlet series whose coefficients are given by $\chi(n)$ where χ is a multiplicative character to some modulus q. In other words, χ is a homomorphism $\chi : \mathbb{Z}_q^\times \to S^1 \subset \mathbb{C}$ where $\chi(n) = \chi(\bar{n})$. Further, for $(n,q) \neq 1$, we define $\chi(n) = 0$. Thus, χ is a function from $\mathbb{Z}$ to $S^1 \bigcup \{0\}$.

A character to the modulus q is primitive if it does not have a period less than q. A character χ to the modulus q is induced by χ_1 to the modulus q_1 if

$$\chi(n) = \begin{cases} \chi_1(n), & \text{if (n,q)=1;} \\ 0, & \text{otherwise.} \end{cases}$$

Every character χ is induced by one and only one character χ_1 which is primitive [21, Theorem 9.2].

Dirichlet L-functions, in particular are useful for multiplicative counting problems in arithmetic progressions. An example is counting the number of primes in the arithmetic progression $\{a + qk : k \in \mathbb{Z}\}$ that are $\leq x$. Let $\psi(x, \chi)$ represent the sum of the coefficients $n \leq x$ of $-\frac{L'}{L}(s, \chi)$. Then by orthogonal relations of characters we can relate $\psi(x, \chi)$ to $\sum_{\substack{p^m \leq x \\ p^m \equiv d(q)}} \log(p)$ which then by partial summation yields the estimate. See [21, Sections 6.2, 11.3].

In the case of counting primes in arithmetic progressions, greater knowledge of the zeros of $L(s, \chi)$ can help us in the analysis of $\psi(x, \chi)$. Perron's formula, used for partial sums of coefficients, yields:

$$\psi(x, \chi) = \frac{-1}{2\pi i} \int_{c-iT}^{c+iT} \frac{L'}{L}(s, \chi) \frac{x^s}{s} ds + R$$

where $R \to \infty$ as $T \to \infty$ where $c > 1$ [21, Theorem 11.16]. The integral is evaluated by means of a contour integral. It is not surprising then that knowledge of the zeros gives improved estimates. In particular, due to the analytic continuation of $L(s, \chi)$, knowledge of the zeros in the critical strip $0 \leq \sigma \leq 1$ is particularly helpful.

The Generalized Riemann Hypothesis (GRH) states: $L(s, \chi)$ only has nontrivial zeros with real part $= \frac{1}{2}$. For counting primes in arithmetic progressions, the estimate for $\sum_{\substack{p^m \leq x \\ p^m \equiv d(q)}} \log(p)$ is improved from $\frac{x}{\phi(q)} + O_A(xe^{-c_1\sqrt{\log x}})$ to $\frac{x}{\phi(q)} + O(x^{1/2}(\log x)^2)$ on GRH (the former estimate assumes $q \leq (\log x)^A$) [21, Cor. 11.19, 13.8].

In the results that follow, I provide a partial equivalence to GRH and I also discuss zero-free regions to the left and right for derivatives of Dirichlet L-functions.

The first theorem generalizes Speiser's result (1934) [24] that the Riemann hypothesis is equivalent to $\zeta'(s)$ having no zeros in the left half of the critical strip.

Theorem 1. *For primitive characters χ, $L(s, \chi)$ only has finitely many zeros in the critical strip with $\sigma \neq \frac{1}{2}$ if and only if $L'(s, \chi)$ has finitely many with $0 < \sigma < \frac{1}{2}$. In particular, if $\chi(-1) = -1$ and $q > 2\pi e^{17/6}$, then $L(s, \chi)$ is zero-free in the critical* *strip with* $\sigma \neq \frac{1}{2}$ *if and only if* $L'(s,\chi)$ *has no zeros for* $0 < \sigma < \frac{1}{2}$*.*

Theorem 2. *For each* $\epsilon > 0$*,* $L^{(k)}(s,\chi) \neq 0$ *in the region determined by* $\sigma < -\epsilon$*,* $|t| > \epsilon$*, and* $|s| > l(k,q)$ *where* $l(k,q)$ *is determined completely by* k *and* q *where* q *is the modulus of* χ*. This result holds for primitive and imprimitive characters* χ*.*

A zero-free region to the right is immediate from general results on Dirichlet series [1, Theorems 11.3, 11.4]. Here, I give an explicit zero-free region on the right for $L^{(k)}(s,\chi)$:

Theorem 3. *Let* $l = \min\{n : n > 1, (q,n) = 1\}$*. Then* $L^{(k)}(s,\chi) \neq 0$ *in the region* $\sigma \geq l + 1 + \frac{kl^{1/2}}{\log^{1/2} l}$*.*

Note that the right side of the inequality in Theorem 3 is an increasing function of k. For odd q, $L'(s,\chi)$ is zero-free in the region $\sigma \geq 4.6986....$

2.1 Theorem 1: Zeros in the critical strip of $L'(s,\chi)$

Assume throughout the following, unless otherwise stated that χ is a primitive character. I do not classify the principal character as primitive or imprimitive.

The goal of the first lemma is to find an explicit constant for the asymptotic formula of $\frac{\Gamma'}{\Gamma}$ in the region $\sigma > 0$ and $s \neq 0$.

Let $|\arg s| < \pi$. N.G. de Bruijn gives an estimate for $\log\Gamma(s)$ based on the Euler-Maclaurin summation formula applied to $S_n(z) = \sum_{k=1}^{n} \log(z+k-1)$ [8, p. 47]. From Euler's product formula for $\Gamma(s)$, he gives that $\log\Gamma(z) = \lim_{n\to\infty}[(z-1)\log n + S_n(1) - S_n(z)]$. The estimate obtained is:

$$\log\Gamma(s) = (s - \frac{1}{2})\log(s) - s + \omega_1(s) + 1 - \omega_1(1) \tag{1}$$

where

$$\omega_1(s) = \frac{1}{12s} - \frac{1}{2}\int_0^\infty (s+x)^{-2} B_2(x-[x])dx$$

($B_2(x)$ is the second Bernoulli polynomial).

Define $\omega(s) = -\omega_1(s) + \frac{1}{12s}$.

We also have from [8, p. 48] that $1 - \omega_1(1) = \frac{1}{2}\log(2\pi)$.

Thus we have

$$\Gamma(s) = \sqrt{2\pi}e^{-s}s^{s-1/2}e^{1/(12s)-\omega(s)}. \tag{2}$$

Let Ω be the region in $\mathbb{C}$ such that $s \neq 0$ and $\sigma \geq 0$.

Lemma 1. *For $j \geq 0$ and $s \in \Omega$, $\omega^{(j)}(s) = O\left(\frac{1}{|s|^{j+1}}\right)$. In particular, $|\omega^{(j)}(s)| \leq \frac{B_2}{2}(j+1)!(1+\frac{1}{j+1})|s|^{-1-j}$. ($B_2 = \frac{1}{6}$ is the second Bernoulli number).*

Proof. I follow the same reasoning as in [25]. Consider first the integral

$$\int_0^\infty |s+x|^{-k}dx$$

where $k > 1$. Note that if $|\arg s| < \pi$ ($s \neq 0$), this integral converges.

We have

$$\int_0^\infty |s+x|^{-k}dx = \int_0^\infty ((\sigma+x)^2+t^2)^{-k/2}dx$$

$$= \int_0^\infty (\sigma^2+t^2+x^2+2x\sigma)^{-k/2}dx \leq \int_0^\infty (|s|^2+x^2)^{-k/2}dx.$$

Note that this last inequality depends on $\sigma \geq 0$.

Since $\min(|s|^{-k}, x^{-k}) \geq (|s|^2+x^2)^{-k/2}$, we have:

$$\int_0^\infty (|s|^2+x^2)^{-k}dx \leq \int_0^{|s|} |s|^{-k}dx + \int_{|s|}^\infty x^{-k}dx = (1+\frac{1}{k-1})|s|^{1-k}.$$

Note that $B_2(x-[x]) \leq B_2$. The above calculations show that if $F(s,x) = (s+x)^{-k}B_2(x-[x])$ for $k > 1$, then $\int_0^\infty |F(s,x)|dx \leq B_2(1+\frac{1}{k-1})|s|^{-1}$ for $s \in \Omega$. Note then that in compact subsets of Ω, $\int_0^\infty |F(s,x)|dx$ is bounded. We may then apply [3, Lemma 4.1], an easy application of Fubini's, Morera's and Cauchy's theorems, so that $f(s) = \int_0^\infty F(s,x)dx$ is holomorphic in Ω and $f'(s) = \int_0^\infty F_s(s,x)dx$.

By a simple induction,

$$|\omega^{(j)}(s)| = |\frac{1}{2}\int_0^\infty (s+x)^{-2-j}(-1)^j(j+1)!B_2(x-[x])dx| \leq$$

$$\frac{B_2}{2}(j+1)!(1+\frac{1}{j+1})|s|^{-1-j} = O(\frac{1}{|s|^{j+1}})$$

□

From this lemma, if $s \in \Omega$, then $\frac{\Gamma'}{\Gamma}(s) = \log(s) - \frac{1}{2s} - \frac{1}{12s^2} + A$ where $|A| \leq \frac{1}{4}|s|^{-2}$. Thus, by the triangle inequality:

$$\frac{\Gamma'}{\Gamma}(s) = \log(s) - \frac{1}{2s} + R \quad \text{where } |R| \leq \frac{1}{3}|s|^{-2} \tag{3}$$

Lemma 2. *A list of facts about $L(s,\chi)$ for nonprincipal, primitive characters χ:*

(i) *$L(s,\chi)$ has an analytic continuation over the entire plane.*

(ii) *If ρ is a zero of $L(s,\chi)$ and not a pole of $\Gamma(\frac{1}{2}(s+\tilde{a}))$, or if it is, has multiplicity at least two, then $L(1-\bar{\rho}) = 0$.*

(iii) *$L(s,\chi)$ is zero-free for $\sigma = 1$.*

(iv) *If $\chi(-1) = -1$, $L(s,\chi)$ has zeros at $s = -1, -3, -5, ...$ corresponding to the simple poles of $\Gamma(\frac{1}{2}(s+\tilde{a}))$.*

(v) *If $\chi(1) = 1$, $L(s,\chi)$ has zeros at $s = -2, -4, -6, ...$ corresponding to the simple poles of $\Gamma(\frac{1}{2}(s+\tilde{a}))$.*

(vi) *All zeros that correspond to the poles of $\Gamma(\frac{1}{2}(s+\tilde{a}))$ are simple. All other zeros, the zeros of $\xi(s,\chi)$, are called nontrivial and satisfy $0 < \Re\rho < 1$.*

Proof. Statements (i),(iv), and (v) follow from the fact that

$$\xi(s,\chi) = (\frac{\pi}{q})^{-(1/2)(s+\tilde{a})}\Gamma(\frac{1}{2}(s+\tilde{a}))L(s,\chi)$$

is an integral function [9, pgs. 69-71].

The hypothesis in statement (ii) is equivalent to saying $\xi(\rho,\chi) = 0$. From the functional equation, $\xi(1-\bar{\rho},\chi) = \frac{i^{\tilde{a}}q^{1/2}}{\tau(\chi)}\xi(\bar{\rho},\bar{\chi})$ [9, p. 71]. By the equations at the top of [9, p. 69] and at the foot of [9, p. 70], it is clear that $\xi(\bar{\rho},\bar{\chi}) = \overline{\xi(\rho,\chi)}$. It therefore follows that $\xi(1-\bar{\rho},\chi) = 0$ and therefore $L(1-\bar{\rho},\chi) = 0$.

Statement (iii) follows from the fact that $L(1,\chi) \neq 0$ [9, pgs. 31-34] and a zero-free region on the right in the critical strip [9, p. 93]. Note that [9, p. 93] states that for real χ, there may be a possible zero in part of the region given. Thus, I include the result that $L(1,\chi) \neq 0$.

From the Euler product formula of $L(s,\chi)$, $L(s,\chi)$ has no zeros for $\sigma > 1$ It follows that $\xi(s,\chi)$ has no zeros in this region since $\Gamma(s)$ has no zeros. Then, by the contrapositive of (ii), $L(s,\chi)$ has only simple zeros for $\sigma < 0$ and $\xi(s,\chi)$ has no zeros for $\sigma < 0$. From (iii) and the contrapositive of (ii), we therefore obtain that all the zeros of $\xi(s,\chi)$, the nontrivial zeros, are in the region $0 < \sigma < 1$. □

For a primitive character χ,

$$\Re\frac{L'(s,\chi)}{L(s,\chi)} = -\frac{1}{2}\log(\frac{q}{\pi}) - \frac{1}{2}\Re\frac{\Gamma'(\frac{1}{2}(s+\tilde{a}))}{\Gamma(\frac{1}{2}(s+\tilde{a}))} + \Re\sum_{\rho}\frac{1}{s-\rho} \tag{4}$$

where ρ runs over all the nontrivial zeros of $L(s,\chi)$ [9, p. 83]. Let

$$I = \Re\sum_{\rho}\frac{1}{s-\rho}.$$

From lemma 2(ii), if $\rho = \beta + i\gamma$ and $\beta < \frac{1}{2}$, then $1-\bar{\rho} = 1-\beta+i\gamma$ is also a zero.

Then, we have

$$\Re \sum_{\beta \neq 1/2} \frac{1}{s-\rho} = \Re \sum_{\beta < 1/2} \left(\frac{1}{s-\rho} + \frac{1}{s-1+\bar{\rho}} \right).$$

Note that:

$$\Re\left(\frac{1}{s-\rho} + \frac{1}{s-1+\bar{\rho}} \right) = \Re \frac{(s-1+\bar{\rho}+s-\rho)(\bar{s}-\bar{\rho})(\bar{s}-1+\rho)}{|s-\rho|^2|s-1+\bar{\rho}|^2}$$

$$= \Re \frac{[(2\sigma-1)+2(t-\gamma)i][(\sigma-\beta)+(\gamma-t)i][(\sigma+\beta-1)+(\gamma-t)i]}{|s-\rho|^2|s-1+\bar{\rho}|^2}$$

$$= \frac{(2\sigma-1)(\sigma-\beta)(\sigma+\beta-1)-(2\sigma-1)(\gamma-t)^2-2(\sigma-\beta)(t-\gamma)^2+2(\sigma+\beta-1)(t-\gamma)^2}{|s-p|^2|s-1+\bar{\rho}|^2}$$

$$= (2\sigma-1)\left(\frac{(\sigma-\beta)(\sigma+\beta-1)-(\gamma-t)^2+2(\gamma-t)^2}{|s-\rho|^2|s-1+\bar{\rho}|^2} \right)$$

$$= (2\sigma-1)\left(\frac{(\sigma^2-\sigma+\frac{1}{4})-\frac{1}{4}-(\beta^2-\beta+\frac{1}{4})+\frac{1}{4}+(\gamma-t)^2}{|s-\rho|^2|s-1+\bar{\rho}|^2} \right).$$

Thus,

$$\Re\left(\frac{1}{s-\rho} + \frac{1}{s-1+\bar{\rho}} \right) = -2(\frac{1}{2}-\sigma)\frac{(t-\gamma)^2+(\sigma-\frac{1}{2})^2-(\frac{1}{2}-\beta)^2}{|s-\rho|^2|s-1+\bar{\rho}|^2}.$$

And we may write:

$$\Re \sum_{\beta \neq 1/2} \frac{1}{s-\rho} = -2(\frac{1}{2}-\sigma) \sum_{\beta < 1/2} \frac{(t-\gamma)^2+(\sigma-\frac{1}{2})^2-(\frac{1}{2}-\beta)^2}{|s-\rho|^2|s-1+\bar{\rho}|^2}$$

Further note that $\Re(\bar{s}-\bar{\rho}) = \sigma - \frac{1}{2}$ so that

$$\Re \sum_{\beta = 1/2} \frac{1}{s-\rho} = -(\frac{1}{2}-\sigma) \sum_{\beta = 1/2} \frac{1}{|s-\rho|^2}.$$

Thus, factoring out $-(\frac{1}{2}-\sigma)$ and labeling

$$I_1 = 2\sum_{\beta<1/2} \frac{(t-\gamma)^2 + (\sigma-\frac{1}{2})^2 - (\frac{1}{2}-\beta)^2}{|s-\rho|^2|s-1+\bar{\rho}|^2} + \sum_{\beta=1/2} \frac{1}{|s-\rho|^2}, \tag{5}$$

we obtain that

$$I = -(\frac{1}{2}-\sigma)I_1. \tag{6}$$

Note that (5) and (6) are identical to [19, (2.2) and (2.3)].

The following lemma then trivially follows:

Lemma 3. *Suppose* $L(s,\chi)\neq 0$. *If* $\sigma = \frac{1}{2}$, $I=0$. *If* $\sigma \leq 0$, $I<0$.

Proof. For the first part, see (6). For the second part see both (5) and (6) noting that $(\sigma-\frac{1}{2})^2 > (\frac{1}{2}-\beta)^2$ in this case. □

Lemma 4. *Suppose* $L(s,\chi)\neq 0$ *and* $0\leq\sigma\leq\frac{1}{2}$. *If*

(i) $|t| > \frac{2\pi}{q}e^{17/6}$ *or*

(ii) $\chi(-1) = -1$ *and* $q > 2\pi e^{17/6}$, *then*

$$-\frac{1}{2}\log\left(\frac{q}{\pi}\right) - \frac{1}{2}\Re\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}(s+\tilde{a})\right) < 0.$$

Proof. From (3) noting that $\Re\frac{1}{2(\frac{1}{2})(s+\tilde{a})} = \frac{\sigma+\tilde{a}}{|s+\tilde{a}|^2}$,

$$\Re\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}(s+\tilde{a})\right) = \log|\frac{1}{2}(s+\tilde{a})| - \frac{\sigma+\tilde{a}}{|s+\tilde{a}|^2} + \Re R$$

where

$$|R| \leq \frac{1}{3|\frac{1}{2}(s+\tilde{a})|^2} = \frac{4}{3|s+\tilde{a}|^2}.$$

So,

$$\log\left(\frac{q}{\pi}\right) + \Re\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}(s+\tilde{a})\right) \geq \log(\frac{q}{\pi}) + \left(\log|\frac{1}{2}(s+\tilde{a})| - \frac{\sigma+\tilde{a}}{|s+\tilde{a}|^2} - \frac{4}{3|s+\tilde{a}|^2}\right).$$

Thus, it suffices if

$$\log|\frac{q}{2\pi}(s+\tilde{a})| > \frac{\sigma+\tilde{a}}{|s+\tilde{a}|^2} + \frac{4}{3|s+\tilde{a}|^2}$$

or

$$\frac{q}{2\pi}|(s+\tilde{a})| > e^{(3\sigma+3\tilde{a}+4)/(3|s+\tilde{a}|^2)}.$$

Suppose that $|t|>1$. Then, it suffices if $\frac{q}{2\pi}|t| > e^{17/6}$ or $|t| > \frac{2\pi}{q}e^{17/6}$.

Suppose that $\chi(-1) = -1$ so that $\tilde{a} = 1$. Then it suffices if $\frac{q}{2\pi} > e^{17/6}$ or $q > 2\pi e^{17/6}$.

□

Let $H_1, H_2 \in \mathbb{R}$. Also, suppose $H_1 < H_2$. Define

$$D_\chi(H_1,H_2) = \{\rho\}^c \bigcap \{s : 0 \le \sigma \le \frac{1}{2} \text{ and } H_1 \le t \le H_2\}$$

where $\{\rho\}^c$ is the complement of the set of nontrivial zeros.

Lemma 5. *Let H_1 and H_2 be such that no zero of $L(s,\chi)$ has imaginary part H_1 or H_2. Suppose*

$$-\frac{1}{2}\log\left(\frac{q}{\pi}\right) - \frac{1}{2}\mathfrak{R}\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}(s+\tilde{a})\right) < 0$$

everywhere in $D_\chi(H_1,H_2)$. Given $\epsilon > 0$, there exists a simple contour $\eta : [0,1] \to \mathbb{C}$ such that

$$\eta([0,1]) \subset D_\chi(H_1,H_2),$$

$$\mathfrak{R}\eta(s) > \frac{1}{2} - \epsilon,$$

$$\eta(0) = \frac{1}{2} + iH_1, \quad \eta(1) = \frac{1}{2} + iH_2 \text{ and}$$

$$\mathfrak{R}\frac{L'}{L}(\eta(u),\chi) < 0 \quad (u \in [0,1]).$$

Proof. For $s \in \{\rho\}^c$ such that $\sigma = \frac{1}{2}$, $I = 0$ from (6).

Let $a_1, ...a_m$ denote the zeros of $L(s,\chi)$ with real part $\frac{1}{2}$ in $D_\chi(H_1, H_2)$. The term $|s-a_i|^{-2}$ in (5) can be made arbitrarily large for s close to a_i. So there exists a small semicircle of radius ϵ_i about a_i in $D_\chi(H_1, H_2)$ such that $I_1 > 0$ which implies $I < 0$. We may choose $\epsilon_i < \epsilon$, $\epsilon_i < \frac{1}{2}\min\{|\Im a_i - \Im a_j| : i \neq j\}$ and $\epsilon_i < \min\{|\Im a_i - H_j| : j = 1, 2\}$. Thus, the desired contour is on $\sigma = \frac{1}{2}$ indented by the semicircles of radius ϵ_i about each a_i.

□

Lemma 6. *Assume all the hypotheses of lemma 5. Also suppose if* $\chi(-1) = 1$ *that either* $H_1 < H_2 < 0$ *or* $0 < H_1 < H_2$. *Then given* $\epsilon > 0$, *there exists a simple closed contour* $\lambda : [0,1] \to \mathbb{C}$ *such that*

$$\lambda([0,1]) \subset D_\chi(H_1, H_2),$$

$$L(s,\chi) \textit{ and } L'(s,\chi) \textit{ are both nonzero on } \lambda,$$

$$\lambda(u) = 1 - 2u + iH_2 \quad u \in [\frac{1}{4}, \frac{1}{2}],$$

$$\lambda(u) = 2u - \frac{3}{2} + iH_1 \quad u \in [\frac{3}{4}, 1],$$

λ *encloses all the zeros of* $L(s,\chi)$ *and* $L'(s,\chi)$ *in the region* $H_1 < t < H_2$, $0 < \sigma < \frac{1}{2}$, *and*

$$\Re\frac{L'}{L}(\lambda(u),\chi) < 0 \quad u \in [0, \frac{1}{4}] \bigcup [\frac{1}{2}, \frac{3}{4}].$$

Proof. We form a contour traversed in the counter clockwise direction. From the hypotheses of lemma 5, we may choose the lines $t = H_2$ and $t = H_1$ as the top ($u \in [\frac{1}{4}, \frac{1}{2}]$) and bottom ($u \in [\frac{3}{4}, 1]$) pieces of the contour as described in the statement of the lemma. The sides we construct will begin and end at corners of the rectangle. If we choose $\delta > 0$ so that there are no zeros of $L(s,\chi)$ or $L'(s,\chi)$ in the region $\frac{1}{2} - \delta < \sigma < \frac{1}{2}$, then by lemma 5, there exists a contour η in $D_\chi(H_1, H_2)$ with real part greater than $\frac{1}{2} - \delta$ for which $\Re \frac{L'}{L}(s, \chi) < 0$ for $s \in \eta$. We may choose η as the right piece of the contour.

Now $L(s, \chi) \neq 0$ on the line $\sigma = 0$, $H_1 \leq t \leq H_2$ by lemma 2 and the hypothesis if $\chi(-1) = 1$. So by lemma 3, $\Re \frac{L'}{L}(it, \chi) < 0$ $(H_1 \leq t \leq H_2)$. We choose this as the left piece of the contour. Note that since $L(s, \chi)$ is zero-free on the boundary of the closure of $D_\chi(H_1, H_2)$ by lemma 2 and the hypothesis if $\chi(-1) = -1$. Since $\Re \frac{L'}{L}(s, \chi) < 0$ on the contour we have chosen and all zeros of $L(s, \chi)$ and $L'(s, \chi)$ are in the interior of the contour, $L'(s, \chi)$ and $L(s, \chi)$ are both nonzero on the contour.

□

The following lemma is of independent interest. The main ideas for its statement and proof are due to Roger Baker:

Lemma 7. *Suppose h is analytic inside and on a closed contour $\eta : [0, 1] \to \mathbb{C}$ and that $h(\eta(u)) \neq 0$ and $h'(\eta(u)) \neq 0$ for $u \in [0, 1]$. Also suppose $\Re \frac{h'}{h}(\eta(u)) < 0$ for $u \in [0, c]$ $(0 \leq c \leq 1)$. Let N be the number of zeros of h inside η and let N' be the number of zeros of h' inside η. Then if the length of the curve $\eta|_{[c,1]}$ is L, we have:*

$$|N - N'| \leq \frac{1}{2} + \frac{1}{2\pi} CL \tag{7}$$

where $C = \max_{u \in [c,1]} \frac{h'}{h}(\eta(u))$.

Proof. From the argument principle,

$$N = \frac{1}{2\pi}(\arg h(\eta(1)) - \arg h(\eta(0)))$$

and

$$N' = \frac{1}{2\pi}(\arg h'(\eta(1)) - \arg h'(\eta(0))).$$

Note that

$$\arg h(\eta(u)) = \arg h(\eta(0)) + \Im \int_0^u \frac{h'(\eta(v))}{h(\eta(v))} d\eta(v)$$

is a continuous function of u. Similarly, $\arg h'(\eta(u))$ is a continuous function of u.

Thus, the function

$$\arg \frac{h'}{h}(\eta(u)) = \arg h'(\eta(u)) - \arg(h(\eta(u))$$

is continuous. Since $\frac{h'}{h}$ is in the left half plane for $u \in [0, c]$,

$$\arg \frac{h'}{h}(\eta(u)) \in (\frac{\pi}{2}, \frac{3\pi}{2}) \pmod{2\pi} \quad \text{for } u \in [0, c].$$

By continuity,

$$|\arg \frac{h'(\eta(u_2))}{h(\eta(u_2))} - \arg \frac{h'(\eta(u_1))}{h(\eta(u_1))}| < \pi$$

for all $u_1, u_2 \in [0, c]$. So this in particular holds when $u_1 = 0$ and $u_2 = c$ as will be used below.

So,

$$2\pi(N - N') = [\arg h(\eta(1)) - \arg h(\eta(0))] - [\arg h'(\eta(1)) - \arg h'(\eta(0))]$$

$$= [\arg h'(\eta(0)) - \arg h(\eta(0))] - [\arg h'(\eta(1)) - \arg h(\eta(1))]$$

And this is:

$$= \Big([\arg h'(\eta(0)) - \arg h(\eta(0))] - [\arg h'(\eta(c)) - \arg h(\eta(c))]\Big)$$

$$-\Big([\arg h'(\eta(1)) - \arg h(\eta(1))] - [\arg h'(\eta(c)) - \arg h(\eta(c))]\Big).$$

This in turn is

$$= K - \Im \int_c^1 \frac{h'}{h}(\eta(v))\eta'(v)dv$$

where $|K| < \pi$ from above. Since $\frac{h'(\eta(v))}{h(\eta(v))}$ is continuous on $[c, 1]$, it achieves a maximum C. so that

$$|\int_c^1 \frac{h'}{h}(\eta(v))\eta'(v)dv| \leq C \int_c^1 |\eta'(v)|dv = CL$$

where L is the length of the curve η from $u = c$ to $u = 1$.

□

Lemma 8. *Let $\sigma \geq \alpha$ for some fixed α. Then, as $t \to \infty$, $L(s, \chi) = O_q(|t|^A)$ and $L'(s, \chi) = O_q(|t|^B)$.*

Proof. This follows the same line of reasoning as given in [28, Sections 4.12, 5.1] for the Riemann Zeta function. For $\sigma \geq \frac{1}{2}$,

$$L(s, \chi) = s \int_1^{\infty} \sum_{n \leq x} \chi(n) x^{-s-1} dx.$$

The character sum is trivially bounded by $\phi(q)$. Thus,

$$|L(s, \chi)| \leq |s| \int_1^{\infty} \phi(q) x^{-\sigma-1} dx = |s| \frac{\phi(q)}{\sigma}$$

$$= \phi(q)\sqrt{1 + \frac{t^2}{\sigma^2}} \leq \phi(q)\sqrt{1 + 4t^2} \leq (1 + 2|t|)\phi(q).$$

Thus, for $\sigma \geq \frac{1}{2}$, we have that $L(s, \chi) = O_q(|t|)$.

Without loss of generality, we may assume that $\alpha < \frac{1}{2}$ and so consider the region $\alpha \leq \sigma < \frac{1}{2}$. We use the functional equation in the form

$$L(s, \chi) = \frac{\tau(\chi)}{i^{\tilde{a}}} (\frac{2\pi}{q})^s \sec(\frac{\pi}{2}(s - \tilde{a})) \frac{1}{\Gamma(s)} L(1 - s, \chi).$$

From (1) and lemma 1,

$$\log \Gamma(s) = (s - \frac{1}{2}) \log s - s + \frac{1}{2} \log 2\pi + O(\frac{1}{|s|}).$$

Note that

$$\log(\sigma + it) = \log(it(\frac{\sigma}{it} + 1)) = \log(it) + \log(1 - \frac{i\sigma}{t}).$$

And

$$|\log(1 - \frac{i\sigma}{t})| = |\sum_{n \geq 1} \frac{1}{n} \left(\frac{i\sigma}{t}\right)^n | \leq \sum_{n \geq 1} \left|\frac{\sigma}{t}\right|^n = \left|\frac{\sigma}{t}\right| \frac{1}{1 - |\frac{\sigma}{t}|} = O(\frac{1}{|t|})$$

for large t since σ is restricted to a finite interval. So,

$$\log(\sigma + it) = \log(it) + O(\frac{1}{|t|}).$$

Trivially, $-s + O(\frac{1}{|s|}) = -it - \sigma + O(\frac{1}{|t|})$. Thus,

$$\log \Gamma(s) = (\sigma + it - \frac{1}{2})(\log it) - it + (\frac{1}{2} \log 2\pi - \sigma) + O(\frac{1}{|t|})$$

$$= (\sigma + it - \frac{1}{2})(\log i + \log t) - it + (\frac{1}{2} \log 2\pi - \sigma) + O(\frac{1}{|t|})$$

$$= (\sigma + it - \frac{1}{2})(\frac{\pi}{2} i) + (\sigma + it - \frac{1}{2}) \log t - it + (\frac{1}{2} \log 2\pi - \sigma) + O(\frac{1}{|t|})$$

So:

$$\Gamma(s) = t^{\sigma + it - \frac{1}{2}} e^{-(\pi/2)t - i(\pi/2)(\sigma - 1/2) - it} e^{\frac{1}{2} \log 2\pi - \sigma} (1 + O(\frac{1}{|t|}))$$

where $e^{O(1/|t|)} = 1 + O(\frac{1}{|t|})$ by the Maclaurin expansion of e^z.

Now consider

$$\cos(\frac{\pi}{2}(s - \tilde{a}))\Gamma(s) = \frac{1}{2}\left(e^{-(\pi/2)t + i(\pi/2)(\sigma - \tilde{a})}\Gamma(s) + e^{(\pi/2)t - i(\pi/2)(\sigma - \tilde{a})}\Gamma(s)\right).$$

Note that:

$$e^{-(\pi/2)t+i(\pi/2)(\sigma-\tilde{a})}\Gamma(s) = t^{\sigma+it-\frac{1}{2}}e^{-\pi t+i(1/2-\tilde{a}-t)}e^{\frac{1}{2}\log 2\pi-\sigma}(1+O(\frac{1}{|t|}))$$

and

$$e^{(\pi/2)t-i(\pi/2)(\sigma-\tilde{a})}\Gamma(s) = t^{\sigma+it-\frac{1}{2}}e^{i(1/2+(\pi/2)\tilde{a}-\pi\sigma-t)}e^{\frac{1}{2}\log 2\pi-\sigma}(1+O(\frac{1}{|t|}))$$

Note that

$$\left|\frac{1}{e^{\frac{1}{2}\log 2\pi-\sigma}(1+O(\frac{1}{|t|}))e^{-\pi t+i(1/2-\tilde{a}-t)} + e^{\frac{1}{2}\log 2\pi-\sigma}(1+O(\frac{1}{|t|}))e^{i(1/2+(\pi/2)\tilde{a}-\pi\sigma-t)}}\right|$$

$$\leq \frac{1}{e^{\frac{1}{2}\log 2\pi-\sigma}\left||(1+O(\frac{1}{|t|}))| - |(1+O(\frac{1}{|t|}))|e^{-\pi t}\right|} = O(e^{-\frac{1}{2}\log 2\pi+\sigma})$$

for large $|t|$ (for large $t<0$ this still holds). So,

$$\sec(\frac{\pi}{2}(s-\tilde{a}))\frac{1}{\Gamma(s)} = 2t^{-\sigma+1/2-it}O(1) = O(|t|^{-\sigma+1/2}).$$

Trivially, $\frac{\tau(\chi)}{i^{\tilde{a}}}(\frac{2\pi}{q})^s = O_q(1)$ for $\alpha \leq \sigma < \frac{1}{2}$. Since $L(1-s,\chi) = O_q(|t|)$, $L(s,\chi) = O_q(|t|^{3/2-\sigma}) = O_q(|t|^{3/2-\alpha})$.

Last, we show $L'(s,\chi) = O_q(|t|^B)$. From above, for $\sigma > \alpha - 2$, $L(s,\chi) = O_q(|t|^{7/2-\alpha})$. Let $\sigma > \alpha$ and t be large. Then, on a circle of radius 1 about s, $L(s,\chi) = O(|t \pm 1|^{7/2-\alpha}) = O_q(|t|^{7/2-\alpha})$ ($\pm$ depending on whether $\frac{7}{2}-\alpha$ is >0 or <0). By Cauchy's estimates, we have that $L'(s,\chi) = O_q(|t|^{7/2-\alpha})$ for large t. □

Lemma 9. *The change in* $\arg L(\sigma+it,\chi)$ *and in* $\arg L'(\sigma+it,\chi)$ *from* $\sigma=0$ *to* $\sigma=\frac{1}{2}$ *for large* $|t|$ *is* $O_q(\log|t|)$ *where* $L(\sigma+it,\chi)\neq 0$ *and* $L(\sigma+it)\neq 0$ *for* $0\leq\sigma\leq\frac{1}{2}$.

Proof. This proof follows a similar construction that can be found in [7] or in [28, Section 9.4]. Consider $L(s,\chi)$. The proof for $L'(s,\chi)$ is identical. Let l_T be the line segment along $t = T$ for $0 \leq \sigma \leq \frac{1}{2}$. Suppose that $L(s, \chi)$ has no zeros on l_T. Let m denote the number of zeros of $\Re L(s, \chi)$ on l_T. These zeros subdivide l_T into at most $m + 1$ intervals. The sign of $\Re L(s, \chi)$ on each of these is constant. Therefore, the change of argument on each subinterval is at most π. Thus the total change in argument on l_T is bounded by $(m + 1)\pi$.

Let

$$g(z) = \frac{1}{2}(L(z + iT, \chi) + \overline{L(\bar{z} + iT, \chi)}).$$

Note that this is an analytic function in a radius where $L(z + iT, \chi)$ is analytic. To see this, suppose $L(z + iT, \chi) = \sum_{n=0}^{\infty} c_n (z - a)^n$ is a power series representation centered at $a \in \mathbb{R}$. Then,

$$\overline{L(\bar{z} + iT, \chi)} = \overline{\sum_{n=0}^{\infty} c_n (\bar{z} - a)^n} = \sum_{n=0}^{\infty} \bar{c_n} (z - a)^n.$$

Note for $z = \sigma$ real, $g(\sigma) = \Re L(\sigma + iT, \chi)$.

Thus, m denotes the number of zeros of $g(\sigma)$ for $0 \leq \sigma \leq \frac{1}{2}$.

We apply Jensen's formula, noting that $g(z)$ is analytic on the entire plane. Denote by C the circle of radius 1 about $\frac{1}{2}$. Let $n(r)$ be the number of zeros of $g(z)$ in and on a circle of radius r centered at $\frac{1}{2}$. Then

$$\int_0^1 \frac{n(r)}{r} dr \geq n(\frac{1}{2}) \int_{1/2}^1 \frac{dr}{r} = n(\frac{1}{2}) \log 2.$$

Thus,

$$m \leq n(\frac{1}{2}) \leq \frac{1}{2\pi \log 2} \int_{-\pi}^{\pi} \log |g(\frac{1}{2} + \frac{1}{2} e^{i\theta})| d\theta - \frac{1}{\log 2} \log |g(\frac{1}{2})| \tag{8}$$

using Jensen's formula.

From the fact that $L(s, \chi)$ is $O_q(|t|^A)$ $(\sigma \geq -\frac{1}{2})$, we obtain that on the circle of radius 1 about $\frac{1}{2} + iT$, $L(s, \chi) = O_q(|T \pm 2|^A) = O_q(|T|^A)$. Thus, $g(z) = O_q(|T|^A)$

on C. Thus the integral in (8) is bounded by $\frac{1}{\log 2}\log|MT^A| = O_q(\log|T|)$ for some constant M depending on q. Similarly, $\log|g(\frac{1}{2})| = O_q(\log|T|)$.

Thus, $m = O_q(\log|T|)$ so that the variation of the argument of $L(s,\chi)$ on l_T is $O_q(\log|T|)$.

□

Notation:

Let $N^-(a,b,\chi)$ and $N_1^-(a,b,\chi)$ denote the number of zeros of $L(s,\chi)$ and $L'(s,\chi)$ respectively contained in the interior of the rectangle $a \le t \le b$ and $0 \le \sigma \le \frac{1}{2}$.

Put together, the following two lemmas are an analog to [19, Theorem 1].

Lemma 10. *Suppose there exists a set of positive, increasing real numbers* $\{T_j\}_{j=1}^{\infty}$ *and a set of negative, decreasing real numbers* $\{P_j\}_{j=1}^{\infty}$ *such that* $\lim_{j\to\infty} T_j = \infty$ *and* $\lim_{j\to\infty} P_j = -\infty$. *Also suppose that* $\Re\frac{L'}{L}(\sigma + iT_j) < 0$ *and* $\Re\frac{L'}{L}(\sigma + iP_j) < 0$ *for* $0 \le \sigma \le \frac{1}{2}$ *and each* j. *Then*

$$N_1^-(P_i, T_j, \chi) = N^-(P_i, T_j, \chi) + O_\chi(1)$$

for $i, j \ge 1$. *If in particular,* $\chi(-1) = -1$, $q > 2\pi e^{17/6}$, *then*

$$N_1^-(P_i, T_j, \chi) = N^-(P_i, T_j, \chi)$$

for $i, j \ge 1$.

Proof. Let $K = \min_i\{T_i : T_i > \frac{2\pi}{q}e^{17/6}\}$. Also, let $K' = \max_i\{P_i : P_i > \frac{2\pi}{q}e^{17/6}\}$. Note that K and K' depend on χ. Choose $T_j > K$. Then applying lemma 6 with $H_1 = K$ and $H_2 = T_j$, there exists a contour λ enclosing all the zeros of $L(s,\chi)$ and $L'(s,\chi)$ in the interior of the rectangle $0 \le \sigma \le \frac{1}{2}$ and $K \le t \le T_j$. Also, on λ, $\Re\frac{L'(s,\chi)}{L(s,\chi)} < 0$. This also implies $L(s,\chi) \ne 0$ and $L'(s,\chi) \ne 0$ on λ. So, since the absolute value of the change of argument of $\frac{L'}{L}(s,\chi)$ is less than π, the change of argument must be 0. Therefore $N_1^-(K,T_j,\chi) = N^-(K,T_j,\chi)$. Similarly, $N_1^-(P_j,K',\chi) = N^-(P_j,K',\chi)$.

Since there are only finitely many zeros in the interior of the rectangle $K' \le \sigma \le \frac{1}{2}$, $0 \le t \le K$, we have the first statement of the lemma.

For the second statement, we use the hypothesis $\chi(-1) = -1$, $q > 2\pi e^{17/6}$ and set $H_1 = K'$ and $H_2 = K$ in lemma 6 to obtain a contour with the same properties we mentioned for λ. It follows that $N_1^-(K',K,\chi) = N^-(K',K,\chi)$.

□

Lemma 11. *If neither a sequence like $\{T_j\}$ nor $\{P_j\}$ in lemma 10 exists, then $N^-(-T,T,\chi) \ge T + O_\chi(1)$ for sufficiently large $T > 0$.*

Proof. Without loss of generality, suppose that a set $\{T_j\}$ as in lemma 10 does not exist. Then:

(i) for all sufficiently large $t > 0$, there exists $0 \le \sigma \le 1$ such that $\Re\frac{L'}{L}(\sigma + it,\chi)$ is nonnegative.

(ii) From Lemma 4 we have, for sufficiently large $t > 0$ that

$$-\frac{1}{2}\log\left(\frac{q}{\pi}\right) - \frac{1}{2}\Re\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}(s+\tilde{a})\right) < 0$$

so that we must have $I > 0$ or $I_1 < 0$. At least one term in the expression for I_1 must then be negative. This only happens when for some $\beta < \frac{1}{2}$, we have $(\frac{1}{2}-\beta)^2 > (t-\gamma)^2 + (\sigma - \frac{1}{2})^2$. Note that $(\frac{1}{2}-\beta) \le \frac{1}{2}$ so that $|t-\gamma| < \frac{1}{2}$. Thus, if t is an integer n, $|t-n| < \frac{1}{2}$ so that for each sufficiently large positive integer, there is at least one zero with imaginary part differing by less than $\frac{1}{2}$. So, if $t > c$ implies that conditions (i) and (ii) hold, we have $N^-(-T,T,\chi) \ge T - c$ or $N^-(T,\chi) \ge T + O_\chi(1)$ since c depends on χ. □

Lemma 12. $N_1^-(-T,T,\chi) = N^-(-T,T,\chi) + O_\chi(\log T)$.

Proof. We use the notation in the proof of Lemma 9, of l_T to denote the line segment at $t = T$ with $0 \leq \sigma \leq \frac{1}{2}$. By lemma 9, there exists $T_0' > 0$ dependent on χ such that for $t > T_0'$

$$|\Delta_{l_t} \arg \frac{L'}{L}(s)| < K_q \log |t|$$

for some constant $K_q > 0$. Thus, there Let T_0 be such that $0 < T_0 - \max\{T_0', \frac{2\pi}{q} e^{17/6}\} < 1$, and $L(s, \chi)$ is zero-free on l_{T_0} and l_{-T_0}. From dependence on χ,

$$N_1^-(-T_0, T_0, \chi) = O_\chi(1).$$

Let $T > T_0$. By lemma 4,

$$-\frac{1}{2} \log \left(\frac{q}{\pi}\right) - \frac{1}{2} \Re \frac{\Gamma'}{\Gamma} \left(\frac{1}{2}(s + \tilde{a})\right) < 0$$

in $D_\chi(T_0, T)$. By lemma 6, there exists a closed contour $\lambda \subset D_\chi(T_1, T)$ enclosing all the zeros of $L(s, \chi)$ and $L'(s, \chi)$ in the region $T_0 < t < T$, $0 < \sigma < \frac{1}{2}$. This contour has l_{T_0} as its bottom piece and l_T as the top piece. Elsewhere on λ, $\Re \frac{L'(s,\chi)}{L(s,\chi)} < 0$ on two disjoint intervals. Thus the total change of argument on these two disjoint intervals together is $< 2\pi$. So

$$\Delta_\lambda \arg \frac{L'}{L}(s, \chi) = O_q(\log T)$$

on λ. So, $N_1^-(T_0, T, \chi) = N^-(T_0, T, \chi) + O_\chi(\log T)$. Similarly, $N_1^-(-T, -T_0, \chi) = N^-(-T, -T_0, \chi) + O_\chi(\log T)$. □

Proof of Theorem 1.

Proof. Suppose first that there are only finitely many zeroes in the critical strip with $\sigma \neq \frac{1}{2}$. Then, by the contrapositive to lemma 11, there exist sets $\{T_j\}$ and $\{P_j\}$ with properties as stated in Lemma 10. Then applying Lemma 10, $N_1^-(P_i, T_j, \chi) =$

$N^-(P_i,T_j,\chi)+O_\chi(1)$ for $i,j\geq 1$. Consequently, $L'(s,\chi)$ has $O_\chi(1)$ zeros with $0<\sigma<\frac{1}{2}$.

Now if $\chi(-1)=-1$ and $q>2\pi e^{17/6}$, assuming there are no zeros in in the critical strip with $\sigma\neq\frac{1}{2}$, lemma 11 again gives sequences $\{T_j\}$ and $\{P_j\}$ with properties stated as in lemma 10. Lemma 10 then gives $N_1^-(P_i,T_j,\chi)=N^-(P_i,T_j,\chi)$ for $i,j\geq 1$. So $L'(s,\chi)$ has no zeros for $0<\sigma<\frac{1}{2}$.

Now suppose conversely that $L'(s,\chi)$ has only finitely many zeros with $0<\sigma<\frac{1}{2}$. Then by lemma 12 we have that $N^-(-T,T,\chi)=O_\chi(\log T)$. Lemma 11 yields sets $\{T_j\}$ and $\{P_j\}$ as given in lemma 10. Lemma 10 then gives $N_1^-(P_i,T_j,\chi)=N^-(P_i,T_j,\chi)+O\chi(1)$ for $i,j\geq 1$. So $L(s,\chi)$ has only finitely many zeros with $\sigma\neq\frac{1}{2}$ by symmetry.

Similarly, if $\chi(-1)=-1$ and $q>2\pi e^{17/6}$, the same reasoning applies as in the preceding paragraph only lemma 10 gives $N_1^-(P_i,T_j,\chi)=N^-(P_i,T_j,\chi)$ for $i,j\geq 1$. Thus, if $L'(s,\chi)$ has no zeros in the region $0<\sigma<\frac{1}{2}$, by symmetry $L(s,\chi)$ has no zeros in the critical strip with $\sigma\neq\frac{1}{2}$.

□

2.2 Theorem 2: A zero-free region to the left for $L^{(k)}(s,\chi)$

The approach used here to prove theorem 2, follows the same flavor and general idea of [26, Section 2]. Suppose χ is imprimitive. Then, there is a unique primitive character χ_1 that induces it. We have from the Euler product that $L(s,\chi)=P(s)L(s,\chi_1)$ where $P(s)=\prod_{p|q}(1-\chi_1(p)p^{-s})$ [9, p. 37]. If χ is primitive, we may define $\chi_1=\chi$ and $P(s)=1$. In either case χ_1 is primitive and so we may consider the functional equation associated to $L(s,\chi_1)$.

Lemma 13. $\frac{P^{(n)}(s)}{P(s)}=O_{q,\epsilon,n}(1)$ *for* $n\geq 0$ *in the region* $\sigma<-\epsilon$.

Proof. We proceed by induction on n and we only consider the case when $P(s)\neq 1$.

In the case $n = 0$, the result is trivial. For the case $n = 1$, we have the identity

$$P'(s) = P(s) \sum_{p|q} \frac{\chi_1(p)\log(p)p^{-s}}{1 - \chi_1(p)p^{-s}}$$

obtained by logarithmic differentiation.

Multiplying by each term in the sum respectively by $\frac{\bar{\chi}_1(p)p^s}{\bar{\chi}_1(p)p^s}$ we obtain:

$$P'(s) = P(s) \sum_{p|q} \frac{\log(p)}{\bar{\chi}_1(p)p^s - 1} \tag{9}$$

Thus,

$$|\frac{P'(s)}{P(s)}| \leq \sum_{p|q} \frac{\log(p)}{1 - p^{\sigma}} \leq \sum_{p|q} \frac{\log(p)}{1 - p^{-\epsilon}}.$$

So, we have, $\frac{P'(s)}{P(s)} = O_{q,\epsilon}(1)$.

Suppose now that the result holds for $k < n$ and that $n > 1$. Then, by Leibniz's rule using (9),

$$P^{(n)}(s) = \sum_{j=0}^{n-1} \binom{n-1}{j} P^{(j)}(s)G^{(k-j)}(s)$$

where $G(s) = \sum_{p|q} \frac{\log(p)}{\bar{\chi}_1(p)p^s - 1}$. By the inductive hypothesis, dividing by $P(s)$, it suffices to show that $G^{(j)}(s) = O_{q,\epsilon,j}(1)$ for $j \leq n-1$.

As seen above, $G(s) = \frac{P'(s)}{P(s)} = O_{q,\epsilon}(1)$ in the region $\sigma < -\frac{\epsilon}{4}$. Then, let C_s be a circle of radius $\frac{\epsilon}{2}$ about s. (Note that the only poles of $G(s)$ occur when $\sigma = 0$.) There exists a constant K depending on ϵ and q such that $|G(s)| < K$ in the region $\sigma < -\frac{\epsilon}{4}$. By Cauchy's integral formula,

$$|G^{(j)}(s)| = |\frac{j!}{2\pi i} \int_{C_s} \frac{G(\zeta)}{(\zeta - s)^{j+1}} d\zeta| \leq \frac{j!}{2\pi}(2\pi\frac{\epsilon}{2})\frac{K}{(\frac{\epsilon}{2})^{j+1}} = \frac{2^j K j!}{\epsilon^j}.$$

That proves the lemma. $\square$

We use the following notation:

$$\begin{pmatrix} n \\ n_1, j_2, ..., n_r \end{pmatrix} = \frac{n!}{n_1!n_2!...n_r!}.$$

Recall from (2), $\Gamma(s) = \sqrt{2\pi}e^{-s}s^{s-1/2}e^{1/(12s)-\omega(s)}$ where by Lemma 1, $\omega^{(j)}(s) = O(\frac{1}{|s|^{j+1}})$.

Lemma 14. *For $j \geq 0$, we have:*

$$\frac{\Gamma^{(j)}(s)}{\Gamma(s)} = \sum_{j_1+j_2+j_3=j} \begin{pmatrix} j \\ j_1, j_2, j_3 \end{pmatrix} (-1)^{j_1}(\log^{j_2} s + \sum_{m=0}^{j_2-1} E_{j_2m} \log^m s)F_{j_3}$$

where F_j is a polynomial in $1/s$ and the first j derivatives of $\omega(s)$ and E_{jm} is a polynomial in $1/s$.

Proof. We consider $\Gamma(s) = \sqrt{2\pi}e^{-s}s^{s-1/2}e^{1/(12s)-\omega(s)}$.

By the multinomial generalization of Leibniz's derivative rule, we have:

$$\Gamma^{(j)}(s) = \sqrt{2\pi} \sum_{j_1+j_2+j_3=j} \begin{pmatrix} j \\ j_1, j_2, j_3 \end{pmatrix} (-1)^{j_1}e^{-s}(s^{s-1/2})^{(j_2)}(e^{1/(12s)-\omega(s)})^{(j_3)}$$

Dividing by $\Gamma(s)$,

$$\frac{\Gamma^{(j)}(s)}{\Gamma(s)} = \sum_{j_1+j_2+j_3=j} \begin{pmatrix} j \\ j_1, j_2, j_3 \end{pmatrix} (-1)^{j_1} \frac{(s^{s-1/2})^{(j_2)}}{(s^{s-1/2})} \frac{(e^{1/(12s)-\omega(s)})^{(j_3)}}{(e^{1/(12s)-\omega(s)})}$$

We claim that $(s^{s-1/2})^{(j)} = (h_j(s) + \log^j s)s^{s-1/2}$ where $h_j(s) = \sum_{m=0}^{j-1} E_{jm} \log^m s$ and E_{jm} is a polynomial in $1/s$. The 0th case is trivial. For the rest, we proceed by induction. Rewrite $s^{s-1/2}$ as $e^{\log(s)(s-1/2)}$. Then if the $j-1$st derivative is $(h_{j-1}(s) + \log^{j-1} s)e^{\log(s)(s-1/2)}$, the jth derivative is $(h'_{j-1}(s) + \frac{j-1}{s}\log^{j-2} s + (1 - \frac{1}{2s} +$

$\log s)(h_{j-1}(s) + \log^{j-1} s))e^{\log(s)(s-1/2)}$. The claim is evident.

Further note that $(e^{1/(12s)-\omega(s)})^{(j)} = F_j e^{1/(12s)-\omega(s)}$ where F_j is a polynomial in $1/s$ and the first j derivatives of $\omega(s)$ which is also seen by induction: $F_j = F'_{j-1} + (-1/(12s^2) - \omega'(s))F_{j-1}(s)$. Note that $F_0 = 1$.

Combining these two facts we have the lemma. $\square$

Lemma 15. *If* $|t| > \epsilon$,

$$\frac{(\cos(\frac{s\pi}{2} - \frac{\pi\tilde{a}}{2}))^{(j)}}{\cos(\frac{s\pi}{2} - \frac{\pi\tilde{a}}{2})} = O_\epsilon(1)$$

Proof. Note that

$$\frac{(\cos(\frac{s\pi}{2} - \frac{\pi\tilde{a}}{2}))^{(j)}}{\cos(\frac{s\pi}{2} - \frac{\pi\tilde{a}}{2})} = \begin{cases} (-1)^{j/2}(\frac{\pi}{2})^j, & \text{j is even;} \\ (-1)^{(j+1)/2}(\frac{\pi}{2})^j(\tan(\frac{s\pi}{2} - \frac{\pi\tilde{a}}{2})), & \text{j is odd.} \end{cases}$$

$$|\tan(\frac{s\pi}{2} - \frac{\pi\tilde{a}}{2})| = \left|\frac{e^{t\pi/2+(\sigma-\tilde{a})\pi i/2} - e^{-t\pi/2-(\sigma-\tilde{a})\pi i/2}}{e^{t\pi/2+(\sigma-\tilde{a})\pi i/2} + e^{-t\pi/2-(\sigma-\tilde{a})\pi i/2}}\right| \leq \frac{e^{t\pi/2} + e^{-t\pi/2}}{e^{t\pi/2} - e^{-t\pi/2}} =$$

$$\frac{e^{t\pi} + 1}{e^{t\pi} - 1} = \frac{2}{e^{t\pi} - 1} + 1 = O_\epsilon(1)$$

in the region $|t| > \epsilon$. $\square$

The following lemma stated without proof follows from repeated integration by parts (a less generalized version was used by [26, Section 1]):

Lemma 16.

$$\int_c^\infty \frac{\log^j x}{x^\sigma} dx = \frac{c^{1-\sigma} j!}{(\sigma-1)^{j+1}} \sum_{m=0}^{j} \frac{(\log^m c)(\sigma-1)^m}{m!}.$$

Lemma 17.

$$\frac{L^{(j)}(s, \bar{\chi}_1)}{L(s, \bar{\chi}_1)} = O_{q,\epsilon,j}(1)$$

in the region $\sigma > 1 + \epsilon$.

Proof. Note that $\frac{\log^j x}{x^\sigma}$ is monotonically decreasing, (by differentiation) when $j - \sigma \log x$ is negative or $x > e^{j/\sigma}$. Since $e^{j/(1+\epsilon)} > e^{j/\sigma}$, it suffices if $x > e^{j/(1+\epsilon)}$ for $\frac{\log^j x}{x^\sigma}$ to be decreasing. Let $c = [e^{j/(1+\epsilon)} + 1]$. Then:

$$|L^{(j)}(s, \bar{\chi}_1)| = |\sum_{n=2}^{\infty} \frac{\bar{\chi}_1(n) \log^j n}{n^s}| \leq \sum_{n=2}^{c} \frac{\log^j n}{n^\sigma} + \sum_{n=c+1}^{\infty} \frac{\log^j n}{n^\sigma} \leq O_{\epsilon,j}(1) + \int_c^\infty \frac{\log^j x}{x^\sigma} dx$$

$$= O_{\epsilon,j}(1) + \frac{c^{1-\sigma} j!}{(\sigma-1)^{j+1}} \sum_{m=0}^{j} \frac{(\log^m c)(\sigma-1)^m}{m!} = O_{\epsilon,j}(1).$$

We also have that

$$|\frac{1}{L(s, \bar{\chi}_1)}| = |\sum_{n=1}^{\infty} \frac{\mu(n)\bar{\chi}_1(n)}{n^s}| < \zeta(\sigma) \leq 1 + \int_1^\infty \frac{dx}{x^\sigma}$$

which is bounded in the region $\sigma > 1 + \epsilon$. □

Proof of Theorem 2.

Proof. Using the unsymmetric form of the functional equation for $L(s, \bar{\chi}_1)$, we have:

$L(1-s, \chi) = P(1-s)L(1-s, \chi_1)$

$L(1-s, \chi) = P(1-s)\frac{2i^{\tilde{a}}}{\tau(\bar{\chi}_1)}(\frac{q}{2\pi})^s \cos(\frac{\pi}{2}(s-\tilde{a}))\Gamma(s)L(s, \bar{\chi}_1).$

Before continuing, we note that $P(1-s)(\frac{q}{2\pi})^s \cos(\frac{\pi}{2}(s-\tilde{a}))\Gamma(s)L(s, \bar{\chi}_1)$ is nonzero for $\sigma > 1 + \epsilon$.

Now, by the multinomial version of Leibniz's rule, we obtain:

$$\frac{(-1)^k L^{(k)}(1-s, \chi)}{P(1-s)\frac{2i^{\tilde{a}}}{\tau(\bar{\chi}_1)}(\frac{q}{2\pi})^s \cos(\frac{\pi}{2}(s-\tilde{a}))\Gamma(s)L(s, \bar{\chi}_1)} = \tag{10}$$

$$\sum_{\sum_{i=1}^5 r_i = k} \binom{k}{r_1, r_2, r_3, r_4, r_5} \frac{(-1)^{r_1} P^{(r_1)}(1-s)}{P(1-s)} \frac{((\frac{q}{2\pi})^s)^{(r_2)}}{(\frac{q}{2\pi})^s} \frac{(\cos(\frac{\pi}{2}(s-\tilde{a})))^{(r_3)}}{\cos(\frac{\pi}{2}(s-\tilde{a}))} \frac{\Gamma^{(r_4)}(s)}{\Gamma(s)} \frac{L^{(r_5)}(s, \bar{\chi}_1)}{L(s, \bar{\chi}_1)}$$

$$= \sum_{r=1}^{k} \begin{pmatrix} k \\ r \end{pmatrix} \frac{\Gamma^{(r)}(s)}{\Gamma(s)} H_r(s)$$

where $H_r(s)$ is a linear combination (the number of terms is bounded by the number of partitions of $k-r$). By lemmas 13, 15, 17, and since

$$\frac{((\frac{q}{2\pi})^s)^{(j)}}{(\frac{q}{2\pi})^s} = \log^j(\frac{q}{2\pi}) = O_{q,k}(1) \quad \text{for } j \geq k,$$

each term in $H_r(s)$ is $O_{\epsilon,q,k}(1)$. So, $H_r(s) = O_{\epsilon,q,k}(1)$.

Using lemma 14, write the right side of (10) in the form:

$$\sum_{r=1}^{k} \begin{pmatrix} k \\ r \end{pmatrix} H_r(s) \sum_{j_1+j_2+j_3=r} \begin{pmatrix} r \\ j_1, j_2, j_3 \end{pmatrix} (-1)^{j_1} (\log^{j_2} s + \sum_{m=0}^{j_2-1} E_{j_2 m} \log^m s) F_{j_3}.$$

Dividing by $\log^{k-1}(s)$, we obtain:

$$\sum_{r=1}^{k} \begin{pmatrix} k \\ r \end{pmatrix} H_r(s) \sum_{j_1+j_2+j_3=r} \begin{pmatrix} j \\ j_1, j_2, j_3 \end{pmatrix} (-1)^{j_1} (\log^{j_2-k+1} s + \sum_{m=0}^{j_2-1} E_{j_2 m} \log^{m-k+1} s) F_{j_3}$$

All powers of $\log s$ in this sum are at most 0 except for when $j_2 = r = k$ and $j_1 = j_3 = 0$. Hence $F_{j_3} = 1$ from the proof of lemma 14. Note further that $H_k(s) = 1$ so that the $(0,0,0,k,0)$ term in the right side of (10) divided by $\log^{k-1} s$ is $\log s + \sum_{m=0}^{k-1} E_{km} \log^{m-k+1} s$. This being so, we obtain that

$$\frac{(-1)^k L^{(k)}(1-s,\chi)}{\log^{k-1}(s) P(1-s) \frac{2i^{\tilde{a}}}{\tau(\bar{\chi}_1)} (\frac{q}{2\pi})^s \cos(\frac{\pi}{2}(s-\tilde{a})) \Gamma(s) L(s,\bar{\chi}_1)} = \log s + O_{q,\epsilon,k}(1)$$

in the region $\sigma > 1+\epsilon$.

Then, by the triangle inequality, there exists $l(k,q)$ such that $|s| > l(k,q)$ implies that $L^{(k)}(s,\chi) \neq 0$ if $\sigma < -\epsilon$ and $|t| > \epsilon$. □

2.3 Theorem 3: A zero-free region to the right for $L^{(k)}(s,\chi)$

The proof here is almost identical to the proof of [26, Theorem 2] only with a few adjustments.

Proof of Theorem 3.

Proof. Note that $l = \min\{n : n > 1, (q,n) = 1\} = \min\{n : n > 1, \chi(n) \neq 0\}$. Also note that $\frac{k}{\log l} < l + 1 + \frac{kl^{1/2}}{\log^{1/2} l}$. Suppose $\sigma \geq \frac{k}{\log l}$.

Differentiating k times,

$$L^{(k)}(s,\chi) = \sum_{n=2}^{\infty} \frac{\chi(n)\log^k n}{n^s} = \frac{\chi(l)\log^k l}{l^s} + \sum_{n=l+1}^{\infty} \frac{\chi(n)\log^k n}{n^s}.$$

Thus,

$$|L^{(k)}(s,\chi)| \geq \frac{\log^k l}{l^\sigma} - \sum_{n=l+1}^{\infty} \frac{\log^k n}{n^\sigma}.$$

Note that

$$\frac{d}{dx}\frac{\log^k x}{x^\sigma} = \frac{(x^{\sigma-1}\log^{k-1} x)(k - \sigma\log x)}{x^{2\sigma}}.$$

Since $k - \sigma\log l < 0$,

$$|L^{(k)}(s,\chi)| > \frac{\log^k l}{l^\sigma} - \int_l^\infty \frac{\log^k x}{x^\sigma}$$

$$= \frac{\log^k l}{l^\sigma} - \frac{l^{1-\sigma}k!}{(\sigma-1)^{k+1}} \sum_{j=0}^{k} \frac{(\log^j l)(\sigma-1)^j}{j!}$$

by the application of Lemma 16. The right side of the inequality is:

$$\frac{l^{1-\sigma}k!}{(\sigma-1)^{k+1}}\left(\frac{\log^{k+1} l(\sigma-1)^{k+1}}{(l\log l)k!} - \sum_{j=0}^{k} \frac{(\log^j l)(\sigma-1)^j}{j!}\right).$$

Let $z = (\log l)(\sigma - 1)$. Then we require that

$$\frac{z^{k+1}}{(l \log l)k!} - \sum_{j=0}^{k} \frac{z^j}{j!} \geq 0. \tag{11}$$

Note that

$$\sum_{j=0}^{k} \frac{z^j}{j!} \leq \frac{z^k}{k!} + \frac{kz^{k-1}}{(k-1)!} = \frac{z^{k-1}}{k!}(z + k^2).$$

Thus, (11) is true if

$$\frac{z^{k-1}}{k!}\left(\frac{z^2}{l \log l} - (z + k^2)\right) \geq 0.$$

So, we require that

$$\frac{z^2}{l \log l} - z - k^2 \geq 0. \tag{12}$$

From the quadratic formula, equality in 12 holds when

$$z = \frac{1 \pm \sqrt{1 + \frac{4k^2}{l \log l}}}{\frac{2}{l \log l}}.$$

Since (12) represents a parabola opening upward, (12) will hold true for

$$(\log l)(\sigma - 1) = z \geq \frac{1 + \sqrt{1 + \frac{4k^2}{l \log l}}}{\frac{2}{l \log l}}.$$

Simplifying,

$$\sigma - 1 \geq \frac{\left(1 + \sqrt{1 + \frac{4k^2}{l \log l}}\right) l}{2}.$$

From the inequality $\sqrt{a^2 + b^2} < a + b$ for $a, b > 0$,

$$\sqrt{1 + \frac{4k^2}{l \log l}} \leq 1 + \frac{2k}{l^{1/2} \log^{1/2} l}.$$

Thus, it suffices to consider

$$\sigma - 1 \geq \frac{\left(2 + \frac{2k}{l^{1/2} \log^{1/2} 1}\right) l}{2}$$

or

$$\sigma \geq l + 1 + \frac{k l^{1/2}}{\log^{1/2} l}.$$

□

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