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The Minimum Rank, Inverse Inertia, and Inverse Eigenvalue Problems for Graphs

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Abstract

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For a graph G we define $\mathcal{S}(G)$ to be the set of all real symmetric $n \times n$ matrices whose off-diagonal zero/nonzero pattern is described by G. We show how to compute the minimum rank of all matrices in $\mathcal{S}(G)$ for a class of graphs called outerplanar graphs. In addition, we obtain results on the possible eigenvalues and possible inertias of matrices in $\mathcal{S}(G)$ for certain classes of graph G. We also obtain results concerning the relationship between two graph parameters, the zero forcing number and the path cover number, related to the minimum rank problem.

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Contents

List of Figures

Chapter 1. Introduction

Given a graph $G = (V, E)$ on n vertices we define $\mathcal{S}(G)$ to be the set of all real symmetric $n \times n$ matrices whose off-diagonal zero/nonzero pattern is given by the graph (and no condition is put on the diagonal entries). That is

$$\mathcal{S}(G) = \{A \in M_{n\times n} \mid A \text{ is symmetric, and } a_{ij} \neq 0, i \neq j \text{ if and only if } ij \in E(G)\}.$$

The Inverse Eigenvalue Problem asks the following question: given n real numbers and a graph G on n vertices, is there a matrix in $\mathcal{S}(G)$ with those numbers as its eigenvalues? This question has proven to be extremely difficult to answer. Some results in connection to this problem can be found in [1]. A simpler, related question that has received a great deal of attention recently is the Minimum Rank Problem: what is the smallest possible rank among matrices in $\mathcal{S}(G)$ for a given graph G? Knowing the minimum rank gives information on the possible multiplicities of eigenvalues. A good introduction to the minimum rank problem can be found in [2].

The *inertia* of a matrix A is the triple $(\pi(A), \nu(A), \delta(A))$ where $\pi(A)$, $\nu(A)$, $\delta(A)$ denote the number of positive eigenvalues, number of negative eigenvalues, and the multiplicity of 0 as an eigenvalue of A respectively. Another problem, whose difficulty lies between that of the minimum rank and inverse eigenvalue problems, is the Inverse Inertia Problem: what are the possible inertias of matrices in $\mathcal{S}(G)$ for a given graph G? Good introductions to the inverse inertia problem can be found in [3] and [4]. In this thesis, we obtain results for all three of these problems, and investigate some graph parameters related to the minimum rank problem.

1.1 Preliminaries

In this section we introduce some of the basic definitions and important results and ideas related to the minimum rank problem.

Definition 1.1. Let F be any field. Given a graph $G = (V, E)$ on n vertices, define

$$\mathcal{S}^F(G) = \{F\text{-valued symmetric } n \times n \text{ matrices } A \mid a_{ij} \neq 0, i \neq j \text{ if and only if } ij \in E(G)\}.$$

The *minimum rank* over F of G, denoted $\operatorname{mr}^F(G)$, is the smallest rank among matrices in $\mathcal{S}^F(G)$. That is

$$\operatorname{mr}^F(G) = \min\{\operatorname{rank} A \mid A \in \mathcal{S}^F(G)\}.$$

Likewise, we define the *maximum nullity* over F of G, $M^F(G)$, by

$$M^F(G) = \max\left\{ \text{nullity } A \,\middle|\, A \in \mathcal{S}^F(G) \right\}.$$

Note that reordering the vertices of the graph equates to a similarity transformation by a permutation matrix for all the matrices in $\mathcal{S}^F(G)$. Thus the minimum rank is invariant under reordering of the vertices. The same is true for all questions of eigenvalues and inertia that we will address later on.

We note the following:

Observation 1.2. *Given a graph G on n vertices, for any field F,*

$$\operatorname{mr}^F(G) + M^F(G) = n$$

so finding the minimum rank and the maximum nullity are equivalent problems.

We will be primarily concerned with the minimum rank over the real field. Whenever we do not specify the field we are working over, it will be assumed that we are working over the real field. In particular, we will typically write just $\mathrm{mr}(G)$ in place of $\mathrm{mr}^{\mathbb{R}}(G)$.

We give as a simple illustration a well known example.

Example 1.3. We will compute the minimum rank for the complete graph on n vertices, $K_n, n > 1$, over any field. Define J_n to be the $n \times n$ matrix all of whose entries are 1. Notice that $\mathrm{rank}(J_n) = 1$ and $J_n \in \mathcal{S}^F(K_n)$. Therefore $\mathrm{mr}^F(K_n) \leq 1$. For any graph with an edge, every corresponding matrix has a nonzero entry, so of course $\mathrm{mr}^F(K_n) > 0$. Hence $\mathrm{mr}^F(K_n) = 1$, and so by Observation 1.2, $\mathrm{M}^F(K_n) = n - 1$. Of course, for $n = 1$, the zero matrix is in $\mathcal{S}^F(K_1)$ so $\mathrm{mr}^F(K_1) = 0$.

It is well known that $K_n, n \geq 2$, are in fact the only connected graphs whose minimum rank is 1.

Definition 1.4. Given a vertex v of the graph G we define the *rank spread*, $r_v(G)$, of v in G by

$$r_v(G) = \mathrm{mr}(G) - \mathrm{mr}(G - v).$$

Likewise, the *edge spread* of an edge e of G is

$$\mathrm{mr}(G) - \mathrm{mr}(G - e).$$

Rank spread and edge spread can be thought of as how much a vertex or edge increases the minimum rank when added to a graph. We have the following well-known fact concerning rank spreads and edge spreads.

Lemma 1.5. *[5, Proposition 2.1] Let F be any field. For any vertex v in $V(G)$,*

$$\mathrm{mr}^F(G - v) + 2 \geq \mathrm{mr}^F(G) \geq \mathrm{mr}^F(G - v).$$

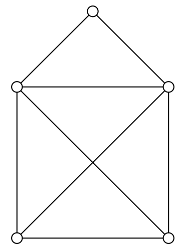

Figure 1.1: The Full House Graph

For any edge e in $E(G)$,

$$\mathrm{mr}^F(G-e)+1 \geq \mathrm{mr}^F(G) \geq \mathrm{mr}^F(G-e)-1.$$

In other words, the rank spread of any vertex is between 0 and 2, and the edge spread of any edge between -1 and 1.

Sometimes the field that we are working over does make a difference in questions of minimum rank. However, there are graphs whose minimum rank is the same no matter what field we are working over. We say that such graphs have *field independent* minimum rank. Trees are an example of a class of graphs whose minimum rank is known to be field independent. In Example 1.3 we saw that K_n is another example whose minimum rank is field independent. In chapter 3, we will prove field independence of the minimum rank of a class of graphs known as *outerplanar* graphs. Knowing that a graph's minimum rank is field independent can be significant computationally. While computing the minimum rank over the real field is an infinite problem, to find the minimum rank over a finite field, say $\mathbb{F}_2$ for example, requires the computation of the rank of only finitely many matrices.

The smallest graph whose minimum rank is not field independent is the graph on five vertices known as the *full house*, pictured in Figure 1.1.

It is known (see [6, Proposition 2]) that if G is the full house, then $\mathrm{mr}^F(G) = 2$ for any $F \neq \mathbb{F}_2$, but $\mathrm{mr}^{\mathbb{F}_2}(G) = 3$.

We include one result that we will need later on.

Theorem 1.6. *[7, Theorem 2.5] Let F be a field, let G be a graph, let e be an edge adjacent to a vertex of degree at most 2, and let G_e be the graph obtained by subdividing e once. Then* $\mathrm{mr}^F(G_e) = \mathrm{mr}^F(G) + 1$.

We will include a few simple examples that we will need later on.

Example 1.7. The *star* on $n \geq 3$ vertices, denoted S_n, is the graph with one central vertex, and $n-1$ pendant vertices adjacent to that vertex. The star S_5 is shown in Figure 1.2. Label the vertices of S_n so the central vertex is first. Then the $n \times n$ matrix

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \\ 1 & 0 & \cdots & 0 \end{bmatrix} \in \mathcal{S}(S_n)$$

and has rank 2 over any field. Since $n \geq 3$, S_n is not a complete graph, so its minimum rank is not 1. Thus $\mathrm{mr}^F(S_n) = 2$ for any field F.

Example 1.8. The *cycle* on n vertices, denoted C_n is the graph obtained by subdividing an edge of K_3 $n-3$ times. The cycle C_5 is shown in Figure 1.2. By Example 1.3 and Theorem 1.6, $\mathrm{mr}^F(C_n) = n - 2$ over any field F.

Example 1.9. A *double cycle* is a graph obtained by successive edge subdivisions of exterior edges of the *diamond* graph, shown in Figure 1.2. A double cycle is also shown in Figure 1.2. The matrix

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

is a rank 2 matrix over any field corresponding to the diamond. It is not a complete graph,

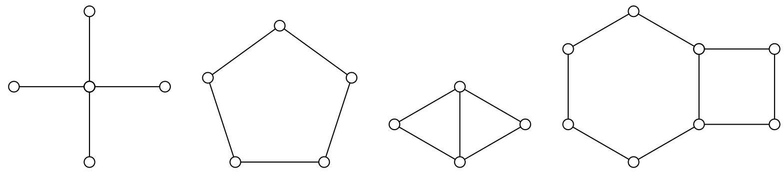

Figure 1.2: A Star, A Cycle, The Diamond, and A Double Cycle

so its minimum rank over any field is 2. Then by Theorem 1.6, the minimum rank of a double cycle on n vertices over any field is $n-2$ (we subdivide $n-4$ times).

A table containing the minimum rank of all graphs on seven or fewer vertices can be found at www.aimath.org/pastworkshops/matrixspectrum.html. It also gives the minimum rank of some basic families of graphs. We will reference this table from time to time when we need to use the minimum rank of a graph that is already known.

1.2 Separations of Graphs

A *separation* (G_1, G_2) of a graph G is a pair of subgraphs $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ such that $V_1 \cup V_2 = V$, $E_1 \cup E_2 = E$, and $E_1 \cap E_2 = \emptyset$. The *order* of a separation is $|V_1 \cap V_2|$. A *k-separation* is a separation of order k. If (G_1, G_2) is a separation of G, with $R = V_1 \cap V_2$, then we will sometimes write

$$G = G_1 \underset{R}{\oplus} G_2.$$

If $R = \{v\}$ (so we have a 1-separation) then we call G the *vertex sum* of G_1 and G_2 and write

$$G = G_1 \underset{v}{\oplus} G_2.$$

The vertex v in a vertex sum is called a *cut vertex*.

An important formula for the minimum rank of the vertex sum of two graphs is given in the following theorem.

Theorem 1.10. *[8, Theorem 2.3] Let G_1 and G_2 be graphs on at least 2 vertices, each with* *a vertex labelled v, and let F be any field. Then*

$$\mathrm{mr}^F(G_1 \underset{v}{\oplus} G_2) = \min\left\{\, \mathrm{mr}^F(G_1) + \mathrm{mr}^F(G_2), \mathrm{mr}^F(G_1 - v) + \mathrm{mr}^F(G_2 - v) + 2 \,\right\}.$$

It was verified in [6] that this formula holds over any field.

This theorem is extremely useful in that, for graphs with a cut vertex, the computation of minimum rank is reduced to the computation of the minimum ranks of smaller graphs.

There is also a similar (but more complicated) formula for a 2-separation. This formula involves adding edges and contracting edges, which may give rise to a multigraph, that is, a graph with parallel edges. In [9], $\mathcal{S}^F(G)$ is extended to allow for graphs with parallel edges.

Definition 1.11. Let $G = (V, E)$ be a multigraph graph with $V = \{1, 2, \ldots, n\}$. If F is a field unequal to $\mathbb{F}_2$, we define $\mathcal{S}^F(G)$ as the set of all F-valued symmetric $n \times n$ matrices $A = [a_{ij}]$ with

(i) $a_{ij} = 0$ if $i \neq j$ and i and j are not adjacent,

(ii) $a_{ij} \neq 0$ if $i \neq j$ and i and j are connected by exactly one edge,

(iii) $a_{ij} \in F$ if $i \neq j$ and i and j are connected by multiple edges, and

(iv) $a_{ii} \in F$ for all $i \in V$.

If $F = \mathbb{F}_2$, we define $\mathcal{S}^{\mathbb{F}_2}(G)$ as the set of all $\mathbb{F}_2$-valued symmetric $n \times n$ matrices $A = [a_{ij}]$ with

(i) $a_{ij} \neq 0$ if $i \neq j$ and i and j are connected by an odd number of edges,

(ii) $a_{ij} = 0$ if $i \neq j$ and i and j are connected by an even number of edges, and

(iii) $a_{ii} \in \mathbb{F}_2$ for all $i \in V$.

For a graph G with parallel edges, let $\mathcal{G}$ be the set of all simple graphs obtained from G by deleting all or all but one of the parallel edges. Then $\mathrm{mr}^F(G) = \min\{\mathrm{mr}^F(H) | H \in \mathcal{G}\}$. Thus finding the minimum rank of a graph with parallel edges is reduced to finding the minimum rank of simple graphs.

Theorem 1.12. *[9, Corollary 15] Let (G_1, G_2) be a 2-separation of G, let H_1 and H_2 be obtained from G_1 and G_2, respectively, by adding an edge between the vertices of $R = \{r_1, r_2\} = V(G_1) \cap V(G_2)$, and let $\overline{G_1}$ and $\overline{G_2}$ be obtained from G_1 and G_2, respectively, by identifying r_1 and r_2. Then*

$$\begin{aligned}\mathrm{mr}^F(G) = \min\{&\mathrm{mr}^F(G_1) + \mathrm{mr}^F(G_2),\\ &\mathrm{mr}^F(G_1 - r_1) + \mathrm{mr}^F(G_2 - r_1) + 2,\\ &\mathrm{mr}^F(G_1 - r_2) + \mathrm{mr}^F(G_2 - r_2) + 2,\\ &\mathrm{mr}^F(G_1 - R) + \mathrm{mr}^F(G_2 - R) + 4,\\ &\mathrm{mr}^F(H_1) + \mathrm{mr}^F(H_2),\\ &\mathrm{mr}^F(\overline{G_1}) + \mathrm{mr}^F(\overline{G_2}) + 2\}.\end{aligned}$$

Any graph which has a vertex of degree 2 has a 2-separation of the form $G = (G_1, P_3)$ where the endpoints of P_3 are the vertices in common. The following result gives a simplification of Theorem 1.12 in this situation.

Theorem 1.13. *[9, Corollary 18] Let G be a graph and let u be a vertex of degree two in G with neighbors v and w. Let H_1 be the graph obtained from G by deleting u and adding an edge between v and w. Let $\overline{G_1}$ be the graph obtained from G by deleting u and identifying v and w. Then*

$$\mathrm{mr}^F(G) = \min\{\mathrm{mr}^F(H_1) + 1, \mathrm{mr}^F(\overline{G_1}) + 2\}.$$

Lemma 1.14. *[9, Lemma 10] Let G be a graph and let v_1, v_2 be vertices of G. Let $\overline{G}$ be obtained from G by identifying v_1 and v_2. Then $\mathrm{mr}^F(\overline{G}) \geq \mathrm{mr}^F(G) - 2$.*

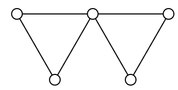

Figure 1.3: The Bowtie

We include one consequence of the 2-separation formula, due to John Sinkovic, that we will need later on. It involves a graph known as the *bowtie* shown in Figure 1.3.

Lemma 1.15. *Let (G_1, G_2) be a 2-separation of G such that G_2 is the bowtie and the common vertices are two vertices of degree 2 which are not adjacent in G_2. Then*

$$\mathrm{mr}^F(G) = \mathrm{mr}^F(G_1) + 2.$$

Proof. Since G has a 2-separation we apply Theorem 1.12 to $G = (G_1, G_2)$. For the values of the minimum ranks of the graphs associated with G_2, and to verify that they are indeed field independent, we simply cite the online database of graphs up through 7 vertices (see the remark at the end of Section 1.1). Let $R = \{r_1, r_2\}$, where r_1 and r_2 are non-adjacent vertices of degree two in G_2. Now $\mathrm{mr}^F(G_2) = \mathrm{mr}^F(G_2 - r_i) = \mathrm{mr}^F(G_2 - R) = 2$.

Let H_2 be the graph obtained from adding the edge r_1r_2 to G_2. Then $\mathrm{mr}^F(H_2) = 3$. The graph $\overline{G_2}$, obtained by identifying r_1 and r_2 together, is the diamond with an extra edge between the vertices of degree 3. In order to compute $\mathrm{mr}^F(\overline{G_2})$ when $F \neq \mathbb{F}_2$, we must consider two graphs, the diamond, and C_4. The minimum rank of either graph over any field F is two. When $F = \mathbb{F}_2$, we delete the pair of edges obtaining C_4. Thus $\mathrm{mr}^F(\overline{G_2}) = 2$ for any field F.

Filling in the appropriate values we obtain,

$$\begin{aligned}\mathrm{mr}^F(G) = \min\{\, &\mathrm{mr}^F(G_1) + 2, \mathrm{mr}^F(G_1 - r_1) + 4, \mathrm{mr}^F(G_1 - r_2) + 4,\\ &\mathrm{mr}^F(G_1 - R) + 6, \mathrm{mr}^F(H_1) + 3, \mathrm{mr}^F(\overline{G_1}) + 4\}.\end{aligned}$$

We will now show that each of the last five terms is greater then or equal to $\mathrm{mr}^F(G_1)+2$. By the first part of Lemma 1.5, $\mathrm{mr}^F(G_1 - r_i) \geq \mathrm{mr}^F(G_1) - 2$. Thus $\mathrm{mr}^F(G_1 - r_i) + 4 \geq \mathrm{mr}^F(G_1) + 2$ for $i = 1, 2$. Applying the same lemma twice, $\mathrm{mr}^F(G_1 - R) \geq \mathrm{mr}^F(G_1) - 4$. Thus $\mathrm{mr}^F(G_1 - R) + 6 \geq \mathrm{mr}^F(G_1) + 2$.

If $r_1r_2 \notin E(G_1)$. Then H_1 and G_1 differ by an edge. Applying the second part of Lemma 1.5, $\mathrm{mr}^F(H_1) \geq \mathrm{mr}^F(G_1) - 1$. Thus $\mathrm{mr}^F(H_1) + 3 \geq \mathrm{mr}^F(G_1) + 2$. In the case where $r_1r_2 \in E(G_1)$, H_1 has two edges between r_1 and r_2. Thus $\mathrm{mr}^F(H_1) = \mathrm{mr}^F(G_1)$ or $\mathrm{mr}^F(H_1) = \mathrm{mr}^F(G_1 - r_1r_2)$. In the latter case, if we apply the same lemma, $\mathrm{mr}^F(G_1 - r_1r_2) \geq \mathrm{mr}^F(G_1) - 1$. Thus in both cases $\mathrm{mr}^F(H_1) + 3 \geq \mathrm{mr}^F(G_1) + 2$.

By Lemma 1.14, $\mathrm{mr}^F(\overline{G_1}) \geq \mathrm{mr}^F(G_1) - 2$. Thus $\mathrm{mr}^F(\overline{G_1}) + 4 \geq \mathrm{mr}^F(G_1) + 2$. □

1.3 Minimum Positive Semidefinite Rank

A question related to the minimum rank question that has also received a great deal of attention is that of finding the minimum rank among positive semidefinite matrices corresponding to a graph.

Definition 1.16. For a graph G, the *minimum positive semidefinite rank*, denoted $\mathrm{mr}_+(G)$, is defined as

$$\mathrm{mr}_+(G) = \min\{\mathrm{rank}\, A \mid A \in \mathcal{S}(G) \text{ and } A \text{ is positive semidefinite}\}.$$

The *maximum positive semidefinite nullity*, $M_+(G)$, is defined analogously.

When computing mr_+ or M_+ for a graph, we are taking the minimum or maximum over a smaller set of matrices than when simply computing the minimum rank or maximum nullity. Thus we have the following observation.

Observation 1.17. *For any graph G,* $\mathrm{mr}_+(G) \geq \mathrm{mr}(G)$ *and* $\mathrm{M}_+(G) \leq \mathrm{M}(G)$.

In some cases, computing the minimum positive semidefinite rank of a graph is easier than computing the minimum rank, which can sometimes give a useful bound on the minimum rank.

We have formulas for mr_+ for graphs with 1-separations and 2-separations, similar to the formulas for minimum rank, but simpler.

Theorem 1.18. *[10, Corollary 2.4] Let (G_1, G_2) be a 1-separation of a graph G. Then*

$$\mathrm{mr}_+(G) = \mathrm{mr}_+(G_1) + \mathrm{mr}_+(G_2).$$

Theorem 1.19. *[10, Corollary 2.9] Let (G_1, G_2) be a 2-separation of a graph G, and let H_1 and H_2 be obtained from G_1 and G_2 respectively by adding an edge between the vertices of the separation. Then*

$$\mathrm{mr}_+(G) = \min\{\mathrm{mr}_+(G_1) + \mathrm{mr}_+(G_2), \mathrm{mr}_+(H_1) + \mathrm{mr}_+(H_2)\}.$$

Theorem 1.18 has the following consequence.

Theorem 1.20. *[10, Theorem 2.6] If T is a tree on n vertices, $M_+(G) = 1$ and $\mathrm{mr}_+(G) = n - 1$.*

Chapter 2. Zero Forcing and Path Covers

2.1 Basic Facts and Definitions

We now describe an important tool in determining the maximum nullity of a graph called the zero forcing number. The idea of zero forcing and its importance is introduced in [11].

Definition 2.1. Let G be a graph in which every vertex is colored either black or white. We will describe a procedure for changing the color of vertices in G.

- *Color Change Rule*: If the vertex u of G is black, and has only one white neighbor v, then color v black. We say u *forces* v and we write $u \rightarrow v$.

- Given a coloring of G, the *derived coloring* is the coloring of G obtained by repeatedly applying the color change rule until no more forces are possible.

- A *zero forcing set* is a subset Z of $V(G)$ such that if every vertex of Z is colored black and each vertex not in Z is white, then the derived coloring of G is all black.

- The *zero forcing number* of G, denoted $Z(G)$ is the size of the smallest zero forcing set. Any zero forcing set of size $Z(G)$ is called a *minimal zero forcing set.*

The significance of the zero forcing number to the minimum rank problem is shown in the following result.

Proposition 2.2. *[11, Proposition 2.4] For any graph G and any field F, $M^F(G) \leq \mathrm{Z}(G)$.*

Thus, the zero forcing number gives an upper bound on the maximum nullity. This is especially significant since the bound is valid over any field.

Another graph parameter that has proven useful in the minimum rank problem is the path cover number. A *path* is a graph that can be obtained by repeated edge subdivision of the complete graph K_2. An *induced subgraph* of a graph G is a subgraph of G that can be obtained by repeated deletion of vertices of G. In other words, an induced subgraph of G contains a subset of the vertex set of G and every edge that occurs in G between those vertices.

Definition 2.3. A *path cover* of a graph G is a set of disjoint paths that occur as induced subgraphs of G whose union includes all the vertices of G. The *path cover number* of G, denoted $P(G)$, is the smallest number of paths in a path cover.

The following is a well-known and important result. In particular, it establishes the field independence of the minimum rank of a tree.

Theorem 2.4. *[11, Proposition 4.2] [12] If T is a tree and F any field, then*

$$Z(T) = P(T) = M^F(T).$$

While the path cover number is helpful in many cases (as in when we are working with trees), there is no general theorem analogous to Proposition 2.2 comparing the path cover number and the maximum nullity (see [13] and [14] for some results on this). In this chapter, we investigate the relationship between the path cover number and the zero forcing number, showing equality $P(G) = Z(G)$ for a certain class of graphs called *cactuses*. We will also investigate examples where equality does not occur.

Definition 2.5. Let Z be a zero forcing set of a graph G. The *forcing sequence* is the list of forces in the order they occur when constructing the derived coloring of G. A *forcing chain* is a sequence $(v_1, v_2, \cdots, v_k)$ of vertices such that v_i forces v_{i+1} for $i = 1, \cdots k-1$. A *maximal zero forcing chain* is a forcing chain that is not a proper subset of another zero forcing chain.

The following gives an important relationship between the zero forcing number and the path cover number.

Lemma 2.6. *[15, Lemma 4.2] If $(v_1, v_2, \cdots, v_k)$ is a forcing chain corresponding to some zero forcing set of the graph G, then the subgraph induced by the vertices $v_1, \cdots, v_k$ of G is a path.*

Corollary 2.7. *For any graph G, $P(G) \leq Z(G)$.*

Proof. Let $Z \subseteq V(G)$ be a zero forcing set of size $Z(G)$. Then each vertex of Z is the first vertex of a forcing chain, and by Lemma 2.6, each of these chains is an induced path. Since Z is a zero forcing set, these paths cover all the vertices of G, and are clearly disjoint since any vertex can only force once in the zero forcing sequence, so this gives a path cover of G of size $Z(G)$. Thus $P(G) \leq Z(G)$. □

Definition 2.8. A path cover $\mathcal{P}$ of a graph G is called *Z-induced* if there is some zero forcing set that induces it as in Corollary 2.7.

Definition 2.9. Let Z be a zero forcing set for the graph G. The *reversal* of G is the set of last vertices in the maximal zero forcing chains of a forcing sequence for Z. Notice that the number of vertices in the reversal of Z is the same as that in Z.

Proposition 2.10. *[16, Theorem 2.6][15, Theorem 3.3] If Z is a zero forcing set for a graph, then so is the reversal of G.*

Proposition 2.11. *[16, Theorem 2.9][15, Theorem 3.4] Given any vertex v of a connected graph G on more than one vertex, there is a minimal zero forcing set Z of G that does not contain v. In other words, no vertex is required to be in a minimal zero forcing set for such a graph.*

The following lemma will be used several times in this section and the next.

Lemma 2.12. *If G is a graph with the property that every path cover is Z-induced, then $Z(G) = P(G)$.*

Proof. We have $P(G) \leq Z(G)$ by Corollary 2.7. For the reverse inequality, let $\mathcal{P}$ be a path cover of size $P(G)$. Then since $\mathcal{P}$ is Z-induced, there is a zero forcing set Z of size $P(G)$, hence $Z(G) \leq P(G)$ and the result follows. $\square$

The proof of the following result comes from [11, Proposition 4.2]. We reproduce their proof here, with slight modifications, to point out a slightly stronger result than is mentioned there.

Theorem 2.13. *Let T be a tree. Let $\mathcal{P} = \{P_1, P_2, \cdots, P_k\}$ be any path cover of of T, and for $i = 1, 2, \cdots, k$, let u_i be either endpoint of P_i. Then $\{u_1, u_2, \cdots, u_k\}$ is a zero forcing set for T. In particular, any path cover of T is Z-induced.*

Proof. We proceed by induction on $k = |\mathcal{P}|$. For $k = 1$, the path cover has a single path, which implies that T is a path, thus coloring either endpoint black is a zero forcing set.

Now assume this is true for path covers of size less than k and suppose $|\mathcal{P}| = k$. Note that the number of edges in any path P_i is $|V(P_i)| - 1$, so since the paths are disjoint, induced, and cover T, the total number of edges in the path cover is

$$\sum_{i=1}^{k}(|V(P_i)| - 1) = n - k$$

where $n = |V(T)|$.

Define $\widetilde{T}$ to be that graph with $\mathcal{P}$ as its vertex set and an edge between P_i and P_j if and only if there is an edge of T with one vertex in P_i and the other in P_j. Clearly $\widetilde{T}$ has k vertices and is connected since T is connected. Notice that the edges of $\widetilde{T}$ are in one to one correspondence with the edges not in $\mathcal{P}$ since T is a tree. So since T has $n-1$ edges, and the path cover has a total of $n-k$ edges, $\widetilde{T}$ has $n-1-(n-k) = k-1$ edges. Thus $\widetilde{T}$ is a tree. Therefore it has a pendant vertex, say P_j. Call the path P_j in T a *pendant path.* Notice that there is only one edge in T connecting P_j to the rest of the graph.

The graph $T - P_j$ is a tree with path cover $\mathcal{P} - \{P_j\}$ of size less than k, so by the induction hypothesis, it is Z-induced, with $Z = \{u_1, \cdots, u_{j-1}, u_{j+1}, \cdots, u_k\}$. Then in T, color each of the vertices in Z and u_j black. Now, there is only one vertex in P_j incident to an edge not in the path; call it v_j. Then since this is the only edge coming out of P_j, the vertex u_j in P_j can force the vertices of this path until we get to v_j (including v_j). Now, since this is the only edge connecting P_j to the rest of the graph, and v_j is now black, then Z can force $T - P_j$ since it is a zero forcing set. Now since all of $T - P_j$ is black, v_j can force the rest of the path P_j to be black. Thus, $\{u_1, u_2, \cdots, u_k\}$ is a zero forcing set for T that induces the path cover $\mathcal{P}$. $\square$

Lemma 2.12 and Theorem 2.13 have the immediate consequence, already stated in Theorem 2.4:

Corollary 2.14. *If T is a tree, then $Z(T) = P(T)$.*

2.2 Vertex Sums and Cactuses

The result in Corollary 2.14 has been well-known since the publication of [11]. In this section we will generalize this result to a larger class of graphs known as cactuses. We develop first an important tool.

Lemma 2.15. *Let G_1 and G_2 be two graphs for which every path cover is Z-induced. Let*

$$G = G_1 \underset{v}{\oplus} G_2$$

be the vertex sum at some vertex v of G_1 and G_2. Then every path cover of G is Z-induced, and hence $Z(G) = P(G)$.

Proof. Let $\mathcal{P}$ be a path cover of G. Let P_k be the path containing the vertex v.

Case 1: Suppose P_k lies entirely within G_1 or G_2. Without loss of generality, suppose P_k lies entirely in G_1. Then define $\mathcal{P}_1$ to contain the paths of $\mathcal{P}$ that are in G_1, and $\mathcal{P}_2$ the paths of $\mathcal{P}$ that are in G_2 along with the path that is the single vertex v. Then $\mathcal{P}_1$ is a path cover of G_1 and $\mathcal{P}_2$ is a path cover of G_2. Then both $\mathcal{P}_1$ and $\mathcal{P}_2$ are Z-induced, say by Z_1 and Z_2 respectively. We claim that

$$Z_1 \cup (Z_2 - \{v\})$$

is a zero forcing set for G. This is because the forcing sequence of Z_1 is unaffected by G_2 until we have forced v. Once v is black, because it is a single vertex path, and no vertex of $G_2 - v$ is adjacent to a vertex of $G_1 - v$, $Z_2 - \{v\}$ can force the vertices of $G_2 - v$ to be black. (If $v \in Z_1$, so v is black to begin with, then $Z_2 - \{v\}$ can force $G_2 - v$ at the outset.) Then,

Z_1 can finish forcing G_1. So this is a zero forcing set, and clearly induces the path cover $\mathcal{P}$.

Case 2: Suppose P_k intersects both G_1 and G_2 in more than just v. Let P_1^* be the sub-path of P_k in G_1 and P_2^* the sub-path of P_k in G_2. Define $\mathcal{P}_1$ to be the paths from $\mathcal{P}$ that are entirely in G_1 along with P_1^*, and $\mathcal{P}_2$ to be the paths from $\mathcal{P}$ that are entirely in G_2 along with P_2^*. Then $\mathcal{P}_1$ and $\mathcal{P}_2$ are path covers of G_1 and G_2 respectively, and are thus Z-induced. Since v is an endpoint of a path in both these path covers, then by Proposition 2.10, taking reversals if necessary, we can choose a zero forcing set Z_1 of G_1 such that $v \notin Z_1$, and a zero forcing set Z_2 of G_2 such that $v \in Z_2$, where Z_1 and Z_2 induce $\mathcal{P}_1$ and $\mathcal{P}_2$ respectively. (Note that v is not a path by itself, since this puts us in Case 1.) Then

$$Z_1 \cup (Z_2 - \{v\})$$

is a zero forcing set for G, since, as in Case 1, Z_1 can force in G_1 until we get to v, and then since v was the end of a path in G_1, it does not need to force anything else in G_1, it just needs to be black, so Z_1 can force all of G_1. Then since v is black, now Z_2 can force G_2. Again, this clearly induces the path cover $\mathcal{P}$. Lemma 2.12 then gives the final statement of the lemma. □

Definition 2.16. A *block* of a graph is a maximal subgraph with no cut vertex. A *cactus* is a graph in which every block is either a cycle or K_2. Equivalently, a cactus is a graph in which any edge is in at most one cycle.

A cactus has also been referred to as a *block-cycle graph*, and some results concerning the maximum nullity of such graphs can be found in [13].

Lemma 2.17. *Every path cover of a cycle C_n is Z-induced.*

Proof. Let $\{P_1, P_2, \cdots, P_k\}$ be any path cover of C_n. Then we must have two paths with adjacent endpoints. Color those two adjacent endpoints black, and then these can force along the cycle to the ends of the paths. Then color the endpoints of the paths following those black, and those can force the rest of their paths. Continue in this way, and the zero forcing set consisting of the endpoints of the paths we colored black induces the path cover. □

Theorem 2.18. *If G is a cactus, then $Z(G) = P(G)$.*

Proof. It is known that a cactus can be built by vertex summing trees and cycles, so by induction and Theorem 2.13, Lemma 2.17, and Lemma 2.15, we see that any path cover of a cactus is Z-induced and thus $P(G) = Z(G)$. □

2.3 Consequences, Counterexamples, and Open Questions

In this section, we discuss some of the consequences of the above results, and look into possible generalizations of them.

The following theorem is known.

Theorem 2.19. *[13, Corollary 5.3] If G is a unicyclic graph, then $0 \leq P(G) - M(G) \leq 1$.*

In other words, the maximum nullity and path cover number differ by at most one for a unicyclic graph (the path cover number never being smaller). In fact, [13] gives a complete characterization of which unicyclic graphs attain equality, and which have $M(G) = P(G) - 1$. The latter case occurs only if the *trimmed form* is an odd n-sun, $n > 3$ (see [13] for a description of trimming a graph and the definition of an n-sun).

Since any unicyclic graph is clearly a cactus, then this result and Theorem 2.18 give us the same result for the zero forcing number:

Corollary 2.20. *If G is a unicyclic graph, then $0 \leq Z(G) - M(G) \leq 1$.*

It was the investigation of the relationship between the zero forcing number and maximum nullity for unicyclic graphs that led us to the result in Theorem 2.18.

There is in fact a generalization of Theorem 2.19 for cactus graphs in [13] (although there they are referred to as *block-cycle* graphs).

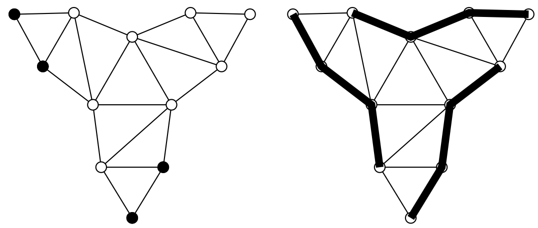

Figure 2.1: The Pinwheel

Theorem 2.21. *[13, Corollary 6.4] If G is a cactus, then $0 \leq P(G) - M(G) \leq co(G)$ where $co(G)$ denotes the number of odd cycles of length greater than 3 in G.*

Hence, Theorem 2.18 implies the same relationship between Z and M for cactuses.

We note in passing a general result concerning the relationship between maximum nullity and path cover number for outerplanar graphs. (See Definition 3.9 for the definition of an outerplanar graph.)

Theorem 2.22. *[14, Theorem 2.8] If G is an outerplanar graph, then $M(G) \leq P(G)$.*

Theorem 2.18 generalizes the equality $Z(G) = P(G)$ from the class of trees (for which this equality was already known) to the class of cactuses. This leads us to ask how far this generalization can be taken. In light of Theorem 2.22, a natural question to ask is whether $P(G) = Z(G)$ for any outerplanar graph. The answer to this is no, as seen in the following counterexample, due to Leslie Hogben.

Example 2.23. Consider the graph G in Figure 2.1 known as the pinwheel. Here $P(G) \leq 3$, but the zero forcing number $Z(G) = 4$ (see [16, Example 2.11]). The first graph in the figure indicates a minimal zero forcing set, while the second indicates a minimal path cover.

In all of the proofs from the preceding section, we showed the equality $P(G) = Z(G)$ by means of Lemma 2.12 by showing that every path cover is Z-induced. This leads us to ask if this is the case for all graphs where the path cover number is equal to the zero forcing number. The answer is, again, no.

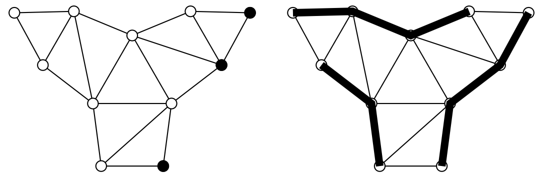

Figure 2.2: A Graph with a Non-Z-induced Minimal Path Cover

Example 2.24. Let G be the graph in Figure 2.2. We will argue that $P(G) = 3$. The second picture in the figure shows a path cover of size 3, so $P(G) \leq 3$. If there were a path cover of size 2, then one of the paths must have at least 6 vertices. However, any choice of 6 successive vertices around the exterior of the graph would contain a triangle, and would thus not be an induced path. But any path of 6 or more vertices that uses an interior edge and that does not contain a triangle cuts the graph into two nontrivial pieces, so they cannot be covered with a single path that is disjoint with the first path. Thus $P(G) = 3$. The first picture in the figure indicates a zero forcing set of size 3, and since $Z(G) \geq P(G)$, $Z(G) = 3$ as well. However, the path cover indicated is not induced by any zero forcing set, as can be verified by checking all possible combinations of coloring one endpoint of each path black.

We have, thus, some open questions:

Question. Is there an outerplanar graph on fewer than 12 vertices with $Z(G) > P(G)$?

Question. Can we characterize all graphs G for which $Z(G) = P(G)$?

We suspect the answer to the first question to be no. The second could be very difficult to answer.

Chapter 3. Minimum Rank of Outerplanar Graphs

In this chapter, we will give a result that solves the minimum rank problem for a certain class of graphs called outerplanar graphs. This will be done by way of a generalization of a well-known tool in the minimum rank problem known as the clique-cover number. We will also look at some results specifically related to the maximum nullity of an outerplanar graph. The results of this chapter were all done in joint work with John Sinkovic.

3.1 Clique-Cover Number and Other Covers

Definition 3.1. Let G be a graph. A *clique* in G is a complete graph occurring as a subgraph of G. The *clique-cover number* of G is the minimum number of cliques needed to cover every edge of G.

It is well known (see [11, Observation 1.1]) that, for any graph G, $\operatorname{mr}(G) \leq \operatorname{cc}(G)$, making the clique cover number a useful tool in the minimum rank problem. A graph is called *chordal* if it has no cycles of length more than 3 occurring as induced subgraphs. We also have the following theorem, making the clique cover number especially useful for the minimum semidefinite rank problem.

Theorem 3.2. *[17, Theorem 3.6] If G is a connected chordal graph, then* $\operatorname{mr}_+(G) = \operatorname{cc}(G)$.

We will generalize the idea of the clique cover number to covers of graphs using other graphs whose minimum rank is known.

Definition 3.3. Let G be a graph on n vertices. For a collection $\mathcal{C}$ of subgraphs of G, we say that $\mathcal{C}$ *covers* G, or that $\mathcal{C}$ is a cover of G, if every vertex and every edge of G is in at least one graph in $\mathcal{C}$. A cover $\mathcal{C}$ is *non-overlapping* if every edge of G is in exactly one of the subgraphs of $\mathcal{C}$.

We define the *rank sum* of a cover, denoted $\operatorname{rs}(\mathcal{C})$, to be the sum of all the minimum ranks of the graphs in the cover.

Definition 3.4. Let T be a collection of graphs. A cover $\mathcal{C}$ is of *type* T if all subgraphs of $\mathcal{C}$ belong to T.

Lemma 3.5. *If G is a graph then* $\operatorname{mr}(G) \leq \operatorname{rs}(\mathcal{C})$ *for any cover $\mathcal{C}$ of G.*

Proof. Let $G_1, \cdots, G_m$ be the graphs in $\mathcal{C}$. Choose $A_k \in \mathcal{S}(G_k)$ such that $\operatorname{rank}(A_k) = \operatorname{mr}(G_k)$, $k = 1, \cdots, m$. Define $\hat{A}_k$ to be the $|G| \times |G|$ matrices $\hat{A}_k = [\hat{a}_{ij}^{(k)}]$ where $\hat{a}_{ij}^{(k)}$ is $a_{ij}^{(k)}$ if $i, j \in V(G_k)$ and is 0 otherwise. Let $A = c_1\hat{A}_1 + \cdots + c_m\hat{A}_m$ where $c_1, \cdots, c_m$ are nonzero constants chosen so that no nonzero off-diagonal entries of the $\hat{A}_k$'s cancel in the sum. Thus $A \in \mathcal{S}(G)$ and

$$\operatorname{mr}(G) \leq \operatorname{rank}(A) \leq \operatorname{rank}(\hat{A}_1) + \cdots + \operatorname{rank}(\hat{A}_m) = \operatorname{mr}(G_1) + \cdots + \operatorname{mr}(G_m) = \operatorname{rs}(\mathcal{C}).$$

□

Note that the proof above may require us to work in an infinite field in order to choose the constants c_k so that no off-diagonal entry cancels in the sum to guarantee $A \in \mathcal{S}(G)$. Such c_k's may not exist in a finite field. However, if the cover is non-overlapping, then in the above proof, we can simply take $A = A_1 + \cdots + A_m$, and no off-diagonal entries will cancel since they will be non-zero in only one of the A_k's. Hence, we obtain the following result.

Lemma 3.6. *If G is any graph, F any field, and $\mathcal{C}$ any non-overlapping cover, then* $\operatorname{mr}^F(G) \leq \operatorname{rs}(\mathcal{C})$.

Our goal will be to prove equality for certain classes of graphs and certain types of covers. Whenever we have a cover $\mathcal{C}$ for a graph G with $\operatorname{mr}(G) = \operatorname{rs}(\mathcal{C})$, we will say that $\mathcal{C}$ is a *minimal cover* of G. We will begin with some preliminary lemmas.

Lemma 3.7. *Let F be a field, let G be the vertex sum at v of G_1, G_2, and T a cover type that includes all stars. If G_i and $G_i - v$ all have non-overlapping covers of type T whose rank sum is the minimum rank, then so does G.*

Proof. By Theorem 1.10

$$\operatorname{mr}^F(G) = \min\{\operatorname{mr}^F(G_1) + \operatorname{mr}^F(G_2), \operatorname{mr}^F(G_1 - v) + \operatorname{mr}^F(G_2 - v) + 2\}.$$

For $i = 1, 2$, let $\mathcal{C}_i$ be a non-overlapping cover of G_i with $\mathrm{rs}(\mathcal{C}_i) = \mathrm{mr}^F(G_i)$ and $\mathcal{C}'_i$ a non-overlapping cover of $G_i - v$ with $\mathrm{rs}(\mathcal{C}'_i) = \mathrm{mr}^F(G_i - v)$. Then

$$\mathrm{mr}^F(G) = \min\{\mathrm{rs}(\mathcal{C}_1) + \mathrm{rs}(\mathcal{C}_2), \mathrm{rs}(\mathcal{C}'_1) + \mathrm{rs}(\mathcal{C}'_2) + 2\}.$$

If $\mathrm{mr}^F(G) = \mathrm{rs}(\mathcal{C}_1) + \mathrm{rs}(\mathcal{C}_2)$, then define $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$. This is a non-overlapping cover of G since G_1 and G_2 do not share any edges, and

$$\mathrm{rs}(\mathcal{C}) = \mathrm{rs}(\mathcal{C}_1) + \mathrm{rs}(\mathcal{C}_2) = \mathrm{mr}^F(G).$$

If $\mathrm{mr}^F(G) = \mathrm{rs}(\mathcal{C}'_1) + \mathrm{rs}(\mathcal{C}'_2) + 2$, let $\mathcal{C}$ be a cover of G consisting of all the graphs in $\mathcal{C}'_1$ and $\mathcal{C}'_2$ and the star induced in G by v and all its neighbors. Then $\mathcal{C}$ is non-overlapping since $G_1 - v$ and $G_2 - v$ do not share any edges with the star. Then

$$\mathrm{rs}(\mathcal{C}) = \mathrm{rs}(\mathcal{C}'_1) + \mathrm{rs}(\mathcal{C}'_2) + 2 = \mathrm{mr}^F(G).$$

□

Lemma 3.8. *Let F be a field. Let G be a graph with a 2-separation (G_1, G_2) such that G_2 is the bowtie, and the common vertices are two vertices of degree 2 which are not adjacent in G_2 (as in Lemma 1.15). Let T be any cover type including cliques. Then if there is a non-overlapping cover of type T for G_1 whose rank sum is equal to the minimum rank, then the same is true for G.*

Proof. By Lemma 1.15, $\mathrm{mr}^F(G) = \mathrm{mr}^F(G_1) + 2$. Let $\mathcal{C}_1$ be a non-overlapping cover of G_1. Define $\mathcal{C}$ to contain all the graphs of $\mathcal{C}_1$ as well as the two cliques of the bowtie. Then $\mathcal{C}$ is a non-overlapping cover of type T for G and

$$\mathrm{rs}(\mathcal{C}) = \mathrm{rs}(\mathcal{C}_1) + 2 = \mathrm{mr}^F(G_1) + 2 = \mathrm{mr}^F(G).$$

□

3.2 Outerplanar Graphs

Definition 3.9. A graph G is *outerplanar* if G has a planar embedding such that every vertex is adjacent to the unbounded face.

A *minor* of a graph G is a graph obtained from G by successive deletion of vertices or edges, and contraction of edges. It is known that a graph is outerplanar if and only if it has no K_4 or $K_{2,3}$ minor. In particular, the class of outerplanar graphs is closed under the operations of deleting vertices and edges, and contracting edges.

Definition 3.10. A graph G is said to be *k-connected* if there is no set of $k-1$ vertices in G whose removal gives a disconnected graph. By convention, K_n is $(n-1)$-connected.

It is well known that a nontrivial outerplanar graph always has a vertex of degree one or two. Thus an outerplanar graph is never 3-connected.

Definition 3.11. In a 2-connected outerplanar graph, a *terminal cycle* is a cycle in which only one edge is contained in any other cycle. A *partially terminal cycle* is a non-terminal cycle adjacent to at most one other non-terminal cycle.

Recall that a double cycle with n vertices has minimum rank $n-2$ (see Example 1.9).

Lemma 3.12. *Any 2-connected outerplanar graph G is either a cycle, a double cycle, or has a terminal cycle and a partially terminal cycle.*

Proof. Let H be the "interior tree" corresponding to G, that is, the graph obtained by taking a vertex for each interior face of the graph, and an edge between two vertices if the corresponding faces share an edge. Since G is outerplanar, it can be drawn with every vertex adjacent to the exterior face so that interior faces share only a single edge. If H were to contain a cycle, then there would be a vertex in G corresponding to the interior of that cycle of H that would not be adjacent to the exterior face, contradicting that G is outerplanar. So H is a tree. If H is K_1 or K_2, then G is a cycle or a double cycle. It is known (see [18, Lemma 13]) that a tree has a pendant vertex and a vertex adjacent to at most one non-pendant neighbor. These correspond to a terminal cycle and a partially terminal cycle in G respectively. □

Lemma 3.13. *Let C be a cycle in a 2-connected outerplanar graph, and $\mathcal{C}$ be a minimal cover of G consisting of stars, cliques, cycles, and double cycles, such that two edges of C are covered by graphs in $\mathcal{C}$ other than C. Then C is not needed in the minimal cover of G. That is, there is a minimal cover $\mathcal{C}'$ of the graph that does not contain C. Similarly, C does not need to be covered by a double cycle.*

Proof. Suppose $C \in \mathcal{C}$. Then C contributes $k-2$ to the rank sum. Now, two of the edges of C are already covered by some other graphs of $\mathcal{C}$, so let $\mathcal{C}'$ be the collection of subgraphs of G consisting of the graphs in $\mathcal{C}$ except for C, and the $k-2$ K_2's for the rest of the edges of C not already covered. Each of these contributes 1 to the rank sum of $\mathcal{C}'$, so that $\mathrm{rs}(\mathcal{C}') \le \mathrm{rs}(\mathcal{C})$. Since $\mathcal{C}$ was a minimal cover, so is $\mathcal{C}'$. Similarly, suppose a double cycle on r vertices covers C in $\mathcal{C}$. This contributes $r-2$ to the rank sum. Create $\mathcal{C}'$ by replacing the double cycle with the other cycle in the double cycle (this has $r-k+2$ vertices, and so contributes $r-k$ to the rank sum), and the $k-2$ edges not covered by the other graphs. Then $\mathrm{rs}(\mathcal{C}') \le \mathrm{rs}(\mathcal{C})$ as above. □

Proposition 3.14. *Let G be an outerplanar graph, and Θ the cover type consisting of cliques, stars, cycles, and double cycles. Then there is a non-overlapping cover $\mathcal{C}$ of G of type Θ such that* $\mathrm{mr}(G) = \mathrm{rs}(\mathcal{C})$.

Proof. We proceed by induction on $|G|$. For our base cases, note that the result is clear for any clique, star, cycle, or double cycle. So assume this to be true for all outerplanar graphs of order less than $|G|$.

If G is disconnected, we simply take a minimal cover for each component. It is clear that the union is a minimal cover of G. If G has a 1-separation (G_1, G_2), it is clear that $G_1, G_2, G_1 - v$, and $G_2 - v$ are still outerplanar. So by the induction hypothesis and Lemma 3.7, the result follows. So suppose G is 2-connected. We will look at several different cases.

I. Suppose G has a terminal cycle C_k of length $k \geq 4$. Let G' be a graph with a terminal cycle C of length 3 such that G can be obtained from G' by $k-3$ edge subdivisions of edges in the terminal cycle. Let e and f be the edges of C contained in no other cycle of G, and u the degree 2 vertex. By Theorem 1.6, $\mathrm{mr}(G) = \mathrm{mr}(G') + k - 3$. By the induction hypothesis, we can find a non-overlapping cover $\mathcal{C}'$ of G' with $\mathrm{rs}(\mathcal{C}') = \mathrm{mr}(G')$. We can assume that e and f are not covered by a star centered at u, since a star at a degree two vertex can simply be replaced by two K_2's. We will thus naturally identify the edges e and f of G' with edges of C_k in G that are incident to vertices not of degree two.

Case 1. If e and f are not covered by a cycle or double cycle in $\mathcal{C}'$, then $\mathcal{C}'$ covers all of G (including the copies of e and f in G) except for $k-3$ edges in the terminal cycle. Let $\mathcal{C}$ be $\mathcal{C}'$ along with the $k-3$ copies of K_2. Then $\mathcal{C}$ is a non-overlapping cover of G whose rank sum is $\mathrm{mr}(G') + k - 3$, hence $\mathrm{mr}(G) = \mathrm{rs}(\mathcal{C})$.

Case 2. If $C \in \mathcal{C}'$, then let $\mathcal{C} = \mathcal{C}' - \{C\} \cup \{C_k\}$. Then $\mathcal{C}$ is a non-overlapping cover of G with

$$\mathrm{rs}(\mathcal{C}) = \mathrm{rs}(\mathcal{C}') - 1 + (k-2) = \mathrm{mr}(G') + k - 3 = \mathrm{mr}(G).$$

Case 3. If C is covered by a double cycle, B' in $\mathcal{C}'$, let B be the double cycle in G obtained by replacing C in B' with C_k. Then let $\mathcal{C} = \mathcal{C}' - \{B'\} \cup \{B\}$ so $\mathcal{C}$ is a non-overlapping cover of G with

$$\mathrm{rs}(\mathcal{C}) = \mathrm{rs}(\mathcal{C}') + k - 3 = \mathrm{mr}(G') + k - 3 = \mathrm{mr}(G).$$

II. Suppose every terminal cycle in G is of length 3. Let C_k be a partially terminal cycle. Consider G as the union of P_3 and G_1 where the P_3 is taken from a terminal triangle that is next to C_k. Let u be the degree two vertex in P_3, and v, w its neighbors. Then by Theorem

1.13, $\mathrm{mr}(G) = \min\{\mathrm{mr}(H_1) + 1, \mathrm{mr}(\overline{G_1}) + 2\}$ where H_1 and $\overline{G_1}$ are defined in Theorem 1.13. Notice that H_1 has a multiple edge between v and w so let H_1' be the graph not including the edge vw and H_1'' the graph including vw. Each of H_1', H_1'', and $\overline{G_1}$ are outerplanar graphs on fewer vertices, so by the induction hypothesis, we can obtain a non-overlapping cover of type Θ for each whose rank sum equals the minimum rank.

Case 1. $\mathrm{mr}(G) = \mathrm{mr}(H_1') + 1$. Notice that G can be thought of as the union of H_1' and the K_3 induced by u, w, v. These do not share any edges. Then let $\mathcal{C}'$ be a minimal non-overlapping cover for H_1' and let $\mathcal{C} = \mathcal{C}' \cup \{K_3\}$, where the K_3 covers u, v, and w. Then $\mathcal{C}$ is a non-overlapping cover of G, with

$$\mathrm{rs}(\mathcal{C}) = \mathrm{rs}(\mathcal{C}') + 1 = \mathrm{mr}(H_1') + 1 = \mathrm{mr}(G)$$

and the result follows.

Case 2. $\mathrm{mr}(G) = \mathrm{mr}(H_1'') + 1$. Again, G can be thought of as the union of H_1'' and the K_3 induced by u, v, w. These do share edges, so we will need to work a little harder to obtain a non-overlapping cover. Let $\mathcal{C}''$ be a minimal non-overlapping cover for H_1''. We will look at what covers the edge wv in $\mathcal{C}''$.

If $wv \in \mathcal{C}''$, define $\mathcal{C} = \mathcal{C}'' - \{wv\} \cup \{K_3\}$. Then $\mathcal{C}$ is a non-overlapping cover of G with

$$\mathrm{mr}(G) \leq \mathrm{rs}(\mathcal{C}) = \mathrm{rs}(\mathcal{C}'') - 1 + 1 = \mathrm{mr}(H_1'') < \mathrm{mr}(G)$$

a contradiction. So this case does not occur. This is expected since if that single edge needs to belong to a minimal cover, then H_1' will have smaller minimum rank than H_1''.

If wv is covered by a star S at v in H_1'', then define S' to be the star in G at v that includes all the edges in S and the edge uv. Define $\mathcal{C} = \mathcal{C}'' - \{S\} \cup \{S', uw\}$. Then $\mathcal{C}$ is a non-overlapping cover of G with

$$\mathrm{rs}(\mathcal{C}) = \mathrm{rs}(\mathcal{C}'') - 2 + 2 + 1 = \mathrm{mr}(H_1'') + 1 = \mathrm{mr}(G).$$

If wv is covered by the partially terminal C_k, then let G_2 be the double cycle induced by C_k and u, v, w. Let $\mathcal{C} = \mathcal{C}'' - \{C_k\} \cup \{G_2\}$. Then $\mathcal{C}$ is a non-overlapping cover of G with

$$\mathrm{rs}(\mathcal{C}) = \mathrm{rs}(\mathcal{C}'') - (k-2) + (k+1-2) = \mathrm{mr}(H_1'') + 1 = \mathrm{mr}(G).$$

If wv is covered by some double cycle in H_1'', let B be that double cycle. Then B consists of the cycle C_k and some other cycle C. Let C have r vertices, so B has $r+k-2$ vertices. Define $\mathcal{C} = \mathcal{C}'' - \{B\} \cup \{C, uvw, \text{the } k-2 \text{ other edges of } C_k\}$. Then $\mathcal{C}$ is a non-overlapping cover of G with

$$\mathrm{rs}(\mathcal{C}) = \mathrm{rs}(\mathcal{C}'') - ((r+k-2)-2) + (r-2) + 1 + (k-2) = \mathrm{mr}(H_1'') + 1 = \mathrm{mr}(G).$$

Thus, no matter what covers wv in the minimal cover, we can construct a minimal non-overlapping cover of G, and the result follows for this case.

Case 3. $\mathrm{mr}(G) = \mathrm{mr}(\overline{G_1}) + 2$. Depending on how many vertices are in the partially terminal cycle, we may or may not get multiple edges in $\overline{G_1}$, so we will look at two cases.

Subcase 1. Suppose the partially terminal cycle has length 3. We will distinguish the cases where the partially terminal cycle has only one terminal triangle next to it, or where it has more than one.

If there is more than one, then we actually have a 2-separation with a bowtie. By the induction hypothesis and Lemma 3.8, the result follows.

If there is only one terminal triangle, it is the triangle uvw. Let x be the other vertex of the partially terminal triangle. When we identify v and w to obtain $\overline{G_1}$, we get a double edge between w and x. Let $\overline{G_1}'$ be the graph without this edge, and $\overline{G_1}''$ the graph with this edge. Then we get two more cases within this case.

a. $\mathrm{mr}(G) = \mathrm{mr}(\overline{G_1}') + 2$. Think of G as the union of $\overline{G_1}'$ and the diamond induced by u, v, w, x. These do not share any edges, so let $\mathcal{C}'$ be a minimal non-overlapping cover of $\overline{G_1}'$ and let $\mathcal{C} = \mathcal{C}' \cup \{diamond\}$. Then $\mathcal{C}$ is a non-overlapping cover of G with

$$\mathrm{rs}(\mathcal{C}) = \mathrm{rs}(\mathcal{C}') + 2 = \mathrm{mr}(\overline{G_1}') + 2 = \mathrm{mr}(G).$$

b. $\mathrm{mr}(G) = \mathrm{mr}(\overline{G_1}'') + 2$. Think of G as the union of $\overline{G_1}''$, the K_2 induced by vx, and the K_3 induced by u, v, w. None of these share an edge, so let $\mathcal{C}''$ be a minimal non-overlapping cover of $\overline{G_1}''$ and let $\mathcal{C} = \mathcal{C}'' \cup \{K_2, K_3\}$.Then $\mathcal{C}$ is a non-overlapping cover of G, with

$$\mathrm{rs}(\mathcal{C}) = \mathrm{rs}(\mathcal{C}'') + 2 = \mathrm{mr}(\overline{G_1}'') + 2 = \mathrm{mr}(G).$$

Subcase 2. Now assume that the partially terminal C_k has 4 or more vertices, so $\overline{G_1}$ does not have any double edges. Let z be the vertex of $\overline{G_1}$ obtained by the identification of v and w. Notice that $\overline{G_1}$ has a terminal or partially terminal C_{k-1}. Let $\overline{\mathcal{C}}$ be a minimal non-overlapping cover for $\overline{G_1}$. We will look at what possibly covers z in this cover.

If $\overline{\mathcal{C}}$ contains a star S with central vertex z, then we may assume that star will cover two edges of C_{k-1} in $\overline{G_1}$, so by Lemma 3.13 we can assume $C_{k-1} \notin \overline{\mathcal{C}}$ and that C_{k-1} is not covered by a double cycle in $\overline{\mathcal{C}}$. Let S' be the star of G consisting of w and all its neighbors, and S'' the star consisting of v and all its neighbors except the edge wv. Let $\mathcal{C} = \overline{\mathcal{C}} - \{S\} \cup \{S', S''\}$. This then covers G, since it covers everything from G_1, as well as the terminal triangle, and is non-overlapping. Also,

$$\mathrm{rs}(\mathcal{C}) = \mathrm{rs}(\overline{\mathcal{C}}) + 2 = \mathrm{mr}(\overline{G_1}) + 2 = \mathrm{mr}(G).$$

If the cycle $C_{k-1} \in \overline{\mathcal{C}}$, then let B be the double cycle in G consisting of C_k and the terminal triangle. Then $\mathrm{mr}(B) = \mathrm{mr}(C_{k-1}) + 2$. Let $\mathcal{C} = \overline{\mathcal{C}} - \{C_{k-1}\} \cup \{B\}$. Then $\mathcal{C}$ is a non-overlapping cover of G with

$$\operatorname{rs}(\mathcal{C}) = \operatorname{rs}(\overline{\mathcal{C}}) + 2 = \operatorname{mr}(\overline{G_1}) + 2 = \operatorname{mr}(G).$$

If C_{k-1} is part of a double cycle B in $\overline{\mathcal{C}}$, then let C_r be the other cycle in that double cycle, and consider the K_3 that is the terminal triangle. Define $\mathcal{C} = \overline{\mathcal{C}} - \{B\} \cup \{C_r, K_3, (k-2)K_2\}$ where the $k-2$ copies of K_2 cover the remaining edges of C_k. Notice that B has $(k-1)+r-2$ vertices, so $\operatorname{mr}(B) = k + r - 5$. Then $\mathcal{C}$ is a non-overlapping cover of G with

$$\operatorname{rs}(\mathcal{C}) = \operatorname{rs}(\overline{\mathcal{C}}) - (k + r - 5) + (r - 2) + 1 + (k - 2) = \operatorname{rs}(\overline{\mathcal{C}}) + 2 = \operatorname{mr}(\overline{G_1}) + 2 = \operatorname{mr}(G).$$

Suppose $\overline{\mathcal{C}}$ does not contain a star centered at z, C_{k-1}, or a double cycle that includes C_{k-1}. Then z is covered only by cliques or stars not centered at z in $\overline{\mathcal{C}}$. Then just take $\mathcal{C}$ to be everything in $\overline{\mathcal{C}}$ along with the terminal triangle. This will be a non-overlapping cover of G, and

$$\operatorname{mr}(G) \leq \operatorname{rs}(\mathcal{C}) < \operatorname{rs}(\overline{\mathcal{C}}) + 2 = \operatorname{mr}(\overline{G_1}) + 2 = \operatorname{mr}(G),$$

a contradiction. So this is another case that does not actually occur. This is expected, since if a star at z, C_{k-1}, or a double cycle is not needed in a minimal cover of $\overline{G_1}$, then the cover we do have of $\overline{G_1}$ can also cover H_1', so $\operatorname{mr}(H_1') + 1$ will be smaller than $\operatorname{mr}(\overline{G_1}) + 2$.

□

Note that the double cycles can simply be covered with two cycles to achieve the same rank sum, so they are not really necessary in computing the minimum rank. We include them to obtain a non-overlapping cover which will help us prove a field independence result later. But for the real field we have the following immediate consequence.

Corollary 3.15. *If G is an outerplanar graph, and C the cover type consisting of stars, cliques, and cycles, then there is a (not necessarily non-overlapping) cover of G of type C*

whose rank sum is the minimum rank of G.

This result allows us to compute the minimum rank of any outerplanar graph by finding a minimal cover. We will now use this to obtain a field independence result for outerplanar graphs.

Theorem 3.16. *[9, Proposition 16] Let G be a graph with no subgraph homeomorphic to K_4. Then $\mathrm{mr}_F(G)$ is the same over any field $F \neq \mathbb{F}_2$.*

Lemma 3.17. *If G is outerplanar, then $\mathrm{mr}^{\mathbb{F}_2}(G) \leq \mathrm{mr}(G)$.*

Proof. By Proposition 3.14 we can obtain a non-overlapping cover $\mathcal{C}$ of G with $\mathrm{mr}(G) = \mathrm{rs}(\mathcal{C})$ and by Lemma 3.6 $\mathrm{mr}^{\mathbb{F}_2}(G) \leq \mathrm{rs}(\mathcal{C}) = \mathrm{mr}(G)$. □

Lemma 3.18. *If G is outerplanar, then $\mathrm{mr}^{\mathbb{F}_2}(G) \geq \mathrm{mr}(G)$.*

Proof. We proceed by induction on $|G|$. For the base cases, the result is clear for K_1, $2K_1$, or K_2 (see Example 1.3). Assume the result for all graphs of order less than $|G|$. If G is disconnected, then the result is clear. If G has a cut vertex, then since the formula in Theorem 1.10 works over any field, the result follows. So assume G is 2-connected. Since G is outerplanar, it has a 2-separation. Notice that, when we start with a simple graph, performing the operations of adding an edge or contracting an edge, the only parallel edges that can result are double edges (no triple or higher edges). Then notice in Definition 1.11, when we compute the minimum rank over $\mathbb{F}_2$ of a graph with a double edge, we simply compute the minimum rank of the corresponding graph with no edge in that position. Thus, in the formula from Theorem 1.12, when we compute the minimum rank over $\mathbb{F}_2$, we take the minimum over fewer graphs in the last two terms when dealing with double edges than when working over other fields. Each summand of the other terms has minimum rank over $\mathbb{F}_2$ larger than or equal to the minimum rank over $\mathbb{R}$ by the induction hypothesis. Thus $\mathrm{mr}^{\mathbb{F}_2}(G) \geq \mathrm{mr}(G)$. □

Theorem 3.19. *If G is an outerplanar graph, F is any field, and C is the cover type consisting of cycles, stars, and cliques, then there is a cover of G of type C whose rank sum is* $\mathrm{mr}^F(G)$.

Proof. By Corollary 3.15, the result holds over the real field. It is well known that K_4 is not a minor of an outerplanar graph, thus by Theorem 3.16, the result follows for any field not equal to $\mathbb{F}_2$. Then Lemma 3.17 and Lemma 3.18 give the result over any field. □

Corollary 3.20. *If G is a tree, F is any field, and S is the cover type consisting of stars and cliques, then there is a cover of G of type S whose rank sum is* $\mathrm{mr}^F(G)$.

This corollary also follows from results in [3], but our proof is different.

3.3 Minimum Positive Semidefinite Rank of Outerplanar Graphs

In this section, we will prove a result similar to that of the previous section, but for minimum positive semidefinite rank. Because the formulas for 1-separations and 2-separations for minimum positive semidefinite rank are so much simpler than those for minimum rank, the result and proof are correspondingly nicer.

Definition 3.21. If $\mathcal{C}$ is a cover of a graph G, define the *positive semidefinite rank sum* of $\mathcal{C}$, $\mathrm{rs}_+(\mathcal{C})$, to be the sum of the minimum positive semi-definite ranks of the graphs in the cover.

Lemma 3.22. *If G is any graph and $\mathcal{C}$ is any cover of G, then* $\mathrm{mr}_+(G) \leq \mathrm{rs}_+(\mathcal{C})$.

Proof. We proceed exactly as in the proof of Lemma 3.5, choosing positive semi-definite A_k's and positive constants c_k. Then the same proof works since the sum of positive semidefinite matrices is positive semidefinite. □

Theorem 3.23. *Let G be outerplanar, and let O be the cover type consisting of cliques and cycles (alternatively think of these as just edges and cycles since G is outerplanar). Then there is a cover of type O of G with* $\mathrm{mr}_+(G) = \mathrm{rs}_+(\mathcal{C})$.

Proof. We will proceed by induction on the order of G. For our base cases, note that the result is clear for cliques and cycles. If G is disconnected, simply take a minimal cover for each component, then it is clear that the union is a minimal cover for G. So assume G is connected.

If G has a 1-separation, $G = (G_1, G_2)$, by Theorem 1.18 we have $\mathrm{mr}_+(G) = \mathrm{mr}_+(G_1) + \mathrm{mr}_+(G_2)$. For $i = 1, 2$ each G_i is still outerplanar, so by the induction hypothesis, there is a cover $\mathcal{C}_i$ of type O for G_i, with $\mathrm{mr}_+(G_i) = \mathrm{rs}_+(\mathcal{C}_i)$. Let $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$. Then $\mathcal{C}$ is a cover for G with

$$\mathrm{rs}_+(\mathcal{C}) = \mathrm{rs}_+(\mathcal{C}_1) + \mathrm{rs}_+(\mathcal{C}_2) = \mathrm{mr}_+(G_1) + \mathrm{mr}_+(G_2) = \mathrm{mr}_+(G).$$

If G does not have a cut vertex, then G has a 2-separation (G_1, G_2) where G_2 is a terminal cycle C_k. By Theorem 1.19, $\mathrm{mr}_+(G) = \min\{\mathrm{mr}_+(G_1) + \mathrm{mr}_+(G_2), \mathrm{mr}_+(H_1) + \mathrm{mr}_+(H_2)\}$. If $\mathrm{mr}_+(G) = \mathrm{mr}_+(G_1) + \mathrm{mr}_+(G_2)$ then proceed exactly as in the case where G has a cut vertex. So suppose $\mathrm{mr}_+(G) = \mathrm{mr}_+(H_1) + \mathrm{mr}_+(H_2) = \mathrm{mr}_+(H_1) + \mathrm{mr}_+(C_k) = \mathrm{mr}_+(H_1) + (k-2)$. Let H_1' be H_1 with the edge between the vertices of the 2-separation, and H_1'' the graph without. Then $\mathrm{mr}_+(G) = \min\{\mathrm{mr}_+(H_1') + k - 2, \mathrm{mr}_+(H_1'') + k - 2\}$. Let $\mathcal{C}'$ be a minimal cover of H_1' and $\mathcal{C}''$ a minimal cover for H_1''. Then, for either case, either $\mathcal{C} = \mathcal{C}' \cup \{C_k\}$ or $\mathcal{C} = \mathcal{C}'' \cup \{C_k\}$, is a cover for G which adds $k - 2$ to the rank sum. So

$$\mathrm{rs}(\mathcal{C}) = \mathrm{rs}(\mathcal{C}') + k - 2 = \mathrm{mr}_+(H_1') + k - 2 = \mathrm{mr}_+(G)$$

or

$$\mathrm{rs}(\mathcal{C}) = \mathrm{rs}(\mathcal{C}'') + k - 2 = \mathrm{mr}_+(H_1'') + k - 2 = \mathrm{mr}_+(G).$$

□

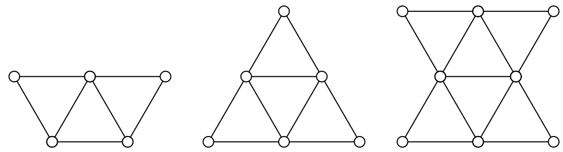

Figure 3.1: The Gem, the Supertriangle, and the 2-gem

3.4 Path Covers and Maximum Nullity of Outerplanar 2-Trees

The following theorem of John Sinkovic gives an important relationship between the maximum nullity and the path cover number of an outerplanar graph.

Theorem 3.24. *[14, Theorem 2.8] If G is an outerplanar graph, then $M(G) \leq P(G)$.*

He also proves equality for a class of graphs known as *partial 2-paths*.

Definition 3.25. A *2-tree* is a graph defined recursively as follows: a triangle is a 2-tree, and a new 2-tree can be obtained from an old one by attaching a vertex to two adjacent vertices. A *partial 2-tree* is a subgraph of a 2-tree. A *2-path* is a 2-tree with exactly two vertices of degree two. A *partial 2-path* is a subgraph of a 2-path.

A straightforward induction argument shows that 2-trees are chordal.

Theorem 3.26. *[14, Theorem 3.17] If G is a partial 2-path, then $M(G) = P(G)$.*

In this same paper, it is conjectured that this equality can be generalized to the class of outerplanar 2-trees. In this section, we prove a special case of this conjecture, namely when the outerplanar 2-tree does not contain a subgraph we call the *2-gem*.

Definition 3.27. The *gem* is the graph on five vertices in Figure 3.1, the *supertriangle* is the graph on six vertices the same figure, and the *2-gem* is the graph on eight vertices in the same figure.

Note that all these graphs are outerplanar 2-trees. The 2-gem can be thought of as two gems connected along an edge, or alternatively, as two supertriangles that overlap.

The idea here is that we can obtain a clique cover of an outerplanar 2-tree by including each triangle except for the central triangles of supertriangles as long as the supertriangles do not overlap so that these central triangles share an edge. Thus we have an observation:

Observation 3.28. *Let G be a 2-gem free outerplanar 2-tree, let t be the number of triangles that are in G, and let s be the number of supertriangles that occur as subgraphs of G. Then* $\mathrm{cc}(G) \leq t - s$.

Definition 3.29. A *terminal gem* in a 2-tree is a subgraph isomorphic to the gem that is connected to the rest of the graph along the single exterior edge not incident to a degree two vertex. A *terminal diamond* in a 2-tree is a subgraph isomorphic to a diamond that is connected to the rest of the graph along a single exterior edge.

Lemma 3.30. *Any outerplanar 2-tree on more than four vertices has either a terminal gem or a terminal diamond.*

Proof. We proceed by induction on the order of G. The only 2-tree on 5 vertices is the gem which has a terminal diamond. Since G is an outerplanar 2-tree, it has a degree two vertex v. By the induction hypothesis, $G - v$ has either a terminal diamond or a terminal gem, since $G - v$ is still an outerplanar 2-tree. If v is not adjacent to this terminal diamond or gem, then the diamond or gem is still terminal in G. If v is adjacent to one of the degree two vertices of the terminal diamond or gem in $G - v$, then v along with the triangle of that degree two vertex gives a terminal diamond. Otherwise, v is adjacent to only non-degree two vertices in the terminal diamond. Then adding v back in turns this terminal diamond into a terminal gem. □

Lemma 3.31. *If G is an outerplanar 2-tree on $n \geq 4$ vertices, then the number of degree two vertices in G is equal to the number of supertriangles in G plus two.*

Proof. We proceed by induction on the order of G. If G is an outerplanar 2-tree on four vertices, then it is the diamond, which has no supertriangle and two degree two vertices.

So suppose G has more than 4 vertices. Then we can find a terminal gem or a terminal diamond. If it is a gem, then it is part of a supertriangle, so deleting a degree two vertex decreases the number of degree two vertices by one, and the number of supertriangles by one. If it is a diamond, then the degree two vertex is not part of a supertriangle, and deleting the degree two vertex does not decrease the number of degree two vertices or the number of supertriangles. Thus by induction, the result follows. $\square$

Lemma 3.32. *Let G be an outerplanar 2-tree and let d be the number of degree two vertices in G. Then $P(G) \leq d$.*

Proof. Since G is outerplanar, we can draw G in the plane so that each vertex is adjacent to the unbounded face. We can then construct a path cover of G by choosing a degree two vertex, and following a path along the exterior of the graph that terminates at the vertex just before the next degree two vertex. Continuing in this manner, we obtain d paths. These paths are disjoint since a 2-tree is 2-connected. Note that any triangle in G either has two interior edges, or a vertex that has degree two in G. Since these paths stopped at the degree two vertices and only include exterior edges, we do not induce any triangle. Since G is chordal, we do not induce any larger cycle either. Thus each is an induced path. Thus we have a path cover of d paths, so $P(G) \leq d$. $\square$

Theorem 3.33. *If G is a 2-gem free outerplanar 2-tree, then*

$$M(G) = P(G) \text{ and } \operatorname{mr}(G) = \operatorname{cc}(G).$$

Proof. Let $n = |G|$. If $n = 3$, $G = K_3$ and the result is clear. So let $n \geq 4$. Let d be the number of degree two vertices, t the number of triangles in G, and s the number of supertriangles in G. Observe that $n = t + 2$ (this is a straightforward induction argument). By Observation 3.28, $\operatorname{mr}(G) \leq \operatorname{cc}(G) \leq t - s$, and by Theorem 3.24, Lemma 3.32, and

Lemma 3.31, $M(G) \leq P(G) \leq d = s + 2$. Then

$$n = \mathrm{mr}(G) + M(G) \leq (t - s) + (s + 2) = t + 2 = n.$$

Thus we get equality in each case, so in particular $M(G) = P(G)$ and $\mathrm{mr}(G) = \mathrm{cc}(G)$. $\square$

Corollary 3.34. *If G is a 2-gem free outerplanar 2-tree on $n \geq 4$ vertices, then $P(G) = d$ and $\mathrm{cc}(G) = t - s$. In particular, there is a minimal path cover of G consisting of only exterior paths, and a minimal clique cover of G that includes each triangle of G except for the central triangle of each supertriangle.*

Chapter 4. Inverse Inertia for Certain Graphs

4.1 Facts About Inertia

In this chapter we address the inverse inertia problem as mentioned in the introduction. Our results from Chapter 3 will lead to some consequences about this problem for outerplanar graphs. We will also solve this problem for any k-connected partial k-path, and any graph whose minimum rank is $n - 2$. We will begin with some preliminary definitions and known results. Most of these can be found in [4].

Definition 4.1. Given a matrix A we define the *inertia* of A as the triple $(\pi(A), \nu(A), \delta(A))$ where $\pi(A)$ denotes the number of positive eigenvalues of A, $\nu(A)$ the number of negative eigenvalues of A, and $\delta(A)$ the multiplicity of 0 as an eigenvalue of A.

Notice that $\pi(A) + \nu(A) + \delta(A) = n$, where n is the order of the matrix. Thus, if we know the size of the matrix that we are dealing with, then knowing any two entries of the inertia determines the third. This motivates the following definition.

Definition 4.2. The *partial inertia* of a matrix A, denoted $\operatorname{pin}(A)$, is the ordered pair $(\pi(A), \nu(A))$ where π and ν are as in Definition 4.1.

Definition 4.3. Given a graph G, the *inertia set* of G, denoted $\mathcal{I}(G)$, is the set of all possible partial inertias that can be obtained by matrices in $\mathcal{S}(G)$. That is

$$\mathcal{I}(G) = \{(r,s) \in \mathbb{N} \times \mathbb{N} \mid \operatorname{pin}(A) = (r,s) \text{ for some } A \in \mathcal{S}(G)\}.$$

(Here we include the number 0 in $\mathbb{N}$.)

We note that for any matrix A, $\pi(A) + \nu(A) = \operatorname{rank} A$, and thus if G is a graph on n vertices and $(r,s) \in \mathcal{I}(G)$, then $\operatorname{mr}(G) \leq r+s \leq n$. With this in mind, we give the following definition.

Definition 4.4. The *k-line* is the subset of $\mathbb{N} \times \mathbb{N}$ whose coordinates add up to k, i.e. $\{(r,s) \in \mathbb{N} \times \mathbb{N} \mid r+s=k\}$. The *minimum rank line* for a graph G is the k-line where $k = \operatorname{mr}(G)$. The *trapezoid* from the l-line to the k-line, denoted $T[l,k]$, is the set

$$T[l,k] = \{(r,s) \in \mathbb{N} \times \mathbb{N} \mid l \leq r+s \leq k\}.$$

A few simple observations are in order.

Observation 4.5. *Let G be a graph on n vertices.*

(i) $\mathcal{I}(G) \subseteq T[\operatorname{mr}(G), n]$.

(ii) There is at least one point in the inertia set on the minimum rank line.

(iii) The point $(\operatorname{mr}_+(G), 0)$ *is the first point of* $\mathcal{I}(G)$ *on the x-axis, and* $(0, \operatorname{mr}_+(G))$ *is the first point on the y-axis.*

(iv) If $(r,s) \in \mathcal{I}(G)$*, then* $(s,r) \in \mathcal{I}(G)$*. Thus the inertia set of G is symmetric over the line $y = x$.*

Definition 4.6. A graph G on n vertices is called *trapezoidal* if $\mathcal{I}(G) = T[\mathrm{mr}(G), n]$. In other word, a graph is trapezoidal if it can have every possible partial inertia not forbidden by the minimum rank or number of vertices.

Definition 4.7. Let $\mathbb{N}^2$ denote $\mathbb{N} \times \mathbb{N}$, and let $Q, R \subseteq \mathbb{N}^2$. We define the following:

- $\mathbb{N}^2_{\leq k} = \{(r,s) \in \mathbb{N}^2 \mid r + s \leq k\}$.
- $Q + R = \{(r_1 + r_2, s_1 + s_2) \mid (r_1, s_1) \in Q, (r_2, s_2) \in R\}$.
- $Q^{\rightarrow} = Q + \{(1,0)\}$.
- $Q^{\uparrow} = Q + \{(0,1)\}$.
- For n a positive integer, $[Q]_n = Q \cap \mathbb{N}^2_{\leq n}$.
- The *northeast expansion* of Q, $Q^{\nearrow} = Q + \mathbb{N}^2$.

The following lemma, known as the Northeast Lemma, is one of the most useful in determining inertia sets.

Lemma 4.8. *[3, Lemma 1.1](Northeast Lemma) Let G be a graph on n vertices and suppose $A \in \mathcal{S}(G)$ with $\mathrm{pin}(A) = (\pi, \nu)$. Then for every pair of integers (r,s) with $r \geq \pi$ and $s \geq \nu$, $r + s \leq n$, there exists a matrix $B \in \mathcal{S}(G)$ with $\mathrm{pin}(B) = (r,s)$.*

Another way of stating the Northeast Lemma is to say that the northeast expansion of any point in the inertia set, up to the n-line, is also in the inertia set. That is, if $Q \subseteq \mathcal{I}(G)$, then $[Q^{\nearrow}]_n \subseteq \mathcal{I}(G)$.

Proposition 4.9. *[3, Proposition 2.3] If G is any graph on n vertices, $T[n-1,n] \subseteq \mathcal{I}(G)$. In other words, the inertia set of any graph contains the $(n-1)$-line and the n-line.*

Definition 4.10. A matrix A is *inertially balanced* if $|\pi(A) - \nu(A)| \leq 1$. A graph G is *inertially balanced* is there is an inertially balanced $A \in \mathcal{S}(G)$ with $\mathrm{rank}(A) = \mathrm{mr}(G)$.

An example of a graph that is not inertially balanced is given in [3].

At this point, a simple example is in order to illustrate some of these ideas.

Example 4.11. We will determine the inertia set for any complete graph $K_n, n > 1$. We saw in Example 1.3 that $\mathrm{mr}(K_n) = 1$. Thus, by Observation 4.5, there is some point along the 1-line in the inertia set for K_n. Since $(1, 0)$ and $(0, 1)$ are the only points on the 1-line, then the symmetry part of this same observation gives us that $\mathcal{I}(K_n)$ contains the entire 1-line. Indeed, the all ones matrix J_n and $-J_n$ are matrices in $\mathcal{S}(K_n)$ that attain these partial inertias. Then by the Northeast Lemma, we find that $T[1, n] \subseteq \mathcal{I}(K_n)$, so in fact, $\mathcal{I}(K_n) = T[1, n]$ for $n > 1$. Of course, for $n = 1$ we have $\mathcal{I}(K_1) = T[0, 1]$. In particular, K_n is trapezoidal and inertially balanced for all n.

We now include some known results that will be useful to us later on.

Lemma 4.12. *[4, Proposition 3.9] (Subadditivity of Inertia) Let A, B, and C be real symmetric $n \times n$ matrices with $A + B = C$. Then*

$$\pi(C) \leq \pi(A) + \pi(B) \text{ and } \nu(C) \leq \nu(A) + \nu(B).$$

Theorem 4.13. *[4, Theorem 6.1] Let G be a graph and e and edge of G. Let G_e be the graph obtained from G by subdividing the edge e. If $(r, s) \in \mathcal{I}(G)$, then $(r + 1, s)$ and $(r, s + 1)$ are contained in $\mathcal{I}(G_e)$.*

Corollary 4.14. *[4, Corollary 6.3] Let G be a graph with edge e and G_e the graph that results from subdivision of e. Then $\mathcal{I}(G)^{\uparrow} \cup \mathcal{I}(G)^{\rightarrow} \subseteq \mathcal{I}(G_e)$.*

Theorem 4.15. *[4, Theorem 8.1] Let G and H be connected, trapezoidal graphs, so $\mathcal{I}(G) = T[\mathrm{mr}(G), |G|]$ and $\mathcal{I}(H) = T[\mathrm{mr}(H), |H|]$. If $\mathrm{mr}(G \cup H) = \mathrm{mr}(G) + \mathrm{mr}(H)$, then*

$$\mathcal{I}(G \cup H) = T[\mathrm{mr}(G) + \mathrm{mr}(H), |G \cup H|].$$

These results are useful in that they give the inertia sets for graphs in terms of smaller graphs.

The paper [4] computes the inertia set for every graph on six or fewer vertices, and for several families of graphs. We include some examples here that we will need later.

Example 4.16. The cycle C_n can be obtained from edge subdivisions of K_3, thus, since $\mathrm{mr}(C_n) = n-2$, by Theorem 4.13, $\mathcal{I}(C_n) = T[n-2, n]$. In particular, C_n is trapezoidal and inertially balanced.

The star S_n has $\mathrm{mr}(S_n) = 2$ and $\mathrm{mr}_+(S_n) = n-1$ (see Example 1.7 and Theorem 1.20). Thus $\mathcal{I}(S_n) = \{(1,1)\}^{\nearrow} \cup T[n-1, n]$. So for $n \geq 4$, S_n is not trapezoidal, but S_n is inertailly balanced for all n.

4.2 Inertia of Outerplanar Graphs

We will now use our results on the minimum rank of an outerplanar graph in terms of the subgraph covers from Chapter 3 to obtain some results about the inertia sets of outerplanar graphs.

Corollary 4.17. *(to Theorem 3.19) If G is an outerplanar graph, then G is inertially balanced.*

Proof. Let $\mathcal{C}$ be a cover of G with $\mathrm{mr}(G) = \mathrm{rs}(\mathcal{C})$ consisting of m cliques $G_1, \cdots, G_m$, p stars $H_1, \cdots, H_p$, and q cycles $F_1, \cdots, F_q$. Each of these graphs is inertially balanced, so we can choose $A_i \in \mathcal{S}(G_i)$ such that $\mathrm{pin}(A_i) = (1,0)$ or $(0,1)$, $B_i \in \mathcal{S}(H_i)$ such that $\mathrm{pin}(B_i) = (1,1)$ and $C_i \in \mathcal{S}(F_i)$ such that $\mathrm{pin}(C_i)$ is balanced. Then we can choose the A_i's, B_i's, and C_i's so that

$$\begin{aligned} 0 \leq & (\pi(A_1) - \nu(A_1)) + \cdots + (\pi(A_m) - \nu(A_m)) \\ & + (\pi(B_1) - \nu(B_1)) + \cdots + (\pi(B_p) - \nu(B_p)) \\ & + (\pi(C_1) - \nu(C_1)) + \cdots + (\pi(C_q) - \nu(C_q)) \leq 1. \end{aligned}$$

Then pad these matrices with zeros in the appropriate way (see Lemma 3.5), and let $A \in \mathcal{S}(G)$ be the sum of all of them. By Lemma 4.12,

$$\pi(A) \leq \sum_{i=1}^{m} \pi(A_i) + \sum_{i=1}^{p} \pi(B_i) + \sum_{i=1}^{q} \pi(C_i)$$

and

$$\nu(A) \leq \sum_{i=1}^{m} \nu(A_i) + \sum_{i=1}^{p} \nu(B_i) + \sum_{i=1}^{q} \nu(C_i).$$

Adding these inequalities, we have

$$\begin{aligned}
\mathrm{mr}(G) &\leq \pi(A) + \nu(A) \\
&\leq \sum_{i=1}^{m} (\pi(A_i) + \nu(A_i)) + \sum_{i=1}^{p} (\pi(B_i) + \nu(B_i)) + \sum_{i=1}^{q} (\pi(C_i) + \nu(C_i)) \\
&= \sum_{i=1}^{m} \mathrm{mr}(G_i) + \sum_{i=1}^{p} \mathrm{mr}(H_i) + \sum_{i=1}^{q} \mathrm{mr}(F_i) \\
&= \mathrm{rs}(\mathcal{C}) = \mathrm{mr}(G)
\end{aligned}$$

so we get equality in each case. Thus $0 \leq \pi(A) - \nu(A) \leq 1$, so G is inertially balanced. $\square$

Theorem 4.18. *If G is any graph such that the minimum rank is equal to the rank sum of some cover consisting only of graphs whose inertia sets are trapezoids, then $\mathcal{I}(G)$ is a trapezoid.*

Proof. Let $\mathrm{mr}(G) = r$ and let $\mathcal{C} = \{G_1, \cdots, G_k\}$ be such a cover of G with $\mathrm{mr}(G_i) = r_i$ so $\sum_{i=1}^{k} r_i = r$. We will show that the point $(m, r-m) \in \mathcal{I}(G)$ for $m \leq r$. Each $\mathcal{I}(G_i)$ is trapezoid, thus we have the points $(j, r_i - j) \in \mathcal{I}(G_i)$ for all i, where j ranges from 0 to r_i. Then since $r_1 + \cdots + r_k = r$ and $0 \leq m \leq r$ and the j's range from 0 to r_i, then for $i = 1, \cdots, k$ choose $j_1, \cdots, j_k$ such that $j_1 + \cdots + j_k = m$. Then let $A_i \in \mathcal{S}(G_i)$ with $\mathrm{pin}(A_i) = (j_i, r_i - j_i)$. Pad these matrices with zeroes as above, and let $A \in \mathcal{S}(G)$ be their sum. Then

$$\pi(A) \leq \sum_{i=1}^{k} \pi(A_i) = \sum_{i=1}^{k} j_i = m$$

and

$$\nu(A) \leq \sum_{i=1}^{k} \nu(A_i) = \sum_{i=1}^{k} (r_i - j_i) = r - m.$$

Then since $\operatorname{rank} A = \pi(A) + \nu(A) \leq (r-m) + m = r = \operatorname{mr}(G)$ we have equality in both cases, so $\operatorname{pin}(A) = (m, r-m)$. So $(m, k-m) \in \mathcal{I}(G)$. Then by the Northeast Lemma, the full trapezoid is in $\mathcal{I}(G)$. $\square$

Corollary 4.19. *If G is an outerplanar graph, then $\mathcal{I}(G)$ is a trapezoid if and only if there is a cover of G consisting of only cliques and cycles whose rank sum is the minimum rank (so a star is not necessary to achieve the minimum rank).*

Proof. ($\Rightarrow$) If $\mathcal{I}(G)$ is a trapezoid, then $\operatorname{mr}(G) = \operatorname{mr}_+(G)$. Then by Theorem 3.23, there is a cover $\mathcal{C}$ consisting of only cliques and cycles with $\operatorname{rs}_+(\mathcal{C}) = \operatorname{mr}_+(G)$. Then

$$\operatorname{mr}_+(G) = \operatorname{mr}(G) \leq \operatorname{rs}(\mathcal{C}) \leq \operatorname{rs}_+(\mathcal{C}) = \operatorname{mr}_+(G)$$

so we get equality. Thus the minimum rank is attained by a minimal cover of only cliques and cycles.

($\Leftarrow$) By Examples 4.11 and 4.16, the inertia sets of complete graphs and cycles are trapezoids. Thus by Theorem 4.18, $\mathcal{I}(G)$ is a trapezoid. $\square$

4.3 Partial k-Paths

In this section, we determine the inertia sets for partial k-paths that are k-connected. Our computations will rely heavily on known results concerning the maximum nullity of such graphs.

Definition 4.20. A *k-tree* is a graph that can be described recursively as follows. K_k is a k-tree, and new k-trees can be built from old ones by adding a new vertex attaching it to a K_k in the old graph. A *partial k-tree* is a subgraph of a k-tree.

Notice in particular that a tree is a 1-tree. A k-tree can be thought of a a tree of K_{k+1}'s. Also notice that each new vertex added has degree k, motivating our next definition.

Definition 4.21. A *k-path* is a k-tree that has exactly two vertices of degree k. K_{k+1} is also considered a (degenerate) k-path. A *partial k-path* is a subgraph of a k-path.

So a 1-path is just a path, and a k-path can be thought of as a path of K_{k+1}'s.

We have the following important result on the maximum nullity of a partial k-path.

Theorem 4.22. *[19, Theorem 12] If G is a k-connected partial k-path, then $M(G) = k$.*

In particular, this result implies that the minimum rank of a k-connected partial k-path is $n - k$. This will allow us to determine the inertia sets of such graphs.

Theorem 4.23. *If G is a k-connected partial k-path on n vertices, then*

$$\mathcal{I}(G) = T[n - k, n].$$

In particular, a k-connected partial k-path is trapezoidal.

Proof. We proceed by induction on $n = |G|$. The smallest k-connected partial k-path is the complete graph K_{k+1}. Example 4.11 gives us the result for this base case. So assume G has more than $k+1$ vertices. Notice that since G is a partial k-path, it is not $(k+1)$-connected, so since it is k-connected, G has a k-separation. So we can write

$$G = G_1 \cup G_2$$

where G_1 and G_2 are both k-connected partial k-paths on strictly fewer vertices than G, and

$$|G_1 \cap G_2| = k.$$

Let $l = |G_1|$ and $m = |G_2|$ so that $n = l + m - k$. By induction, $\mathcal{I}(G_1) = T[l-k, l]$ and $\mathcal{I}(G_2) = T[m-k, m]$. Then

$$\mathrm{mr}(G_1) + \mathrm{mr}(G_2) = (l-k) + (m-k) = (l+m-k) - k = n - k = \mathrm{mr}(G).$$

So by Theorem 4.15, $\mathcal{I}(G) = T[n-k, n]$. $\square$

4.4 Graphs of Minimum Rank $n-2$

In this section, we will use the result from the previous section, in addition to other tools we have discussed, to determine the inertia sets for all graphs whose minimum rank is $n-2$. This will be possible because of an important result characterizing such graphs found in [20]. First we need a definition.

Definition 4.24. A graph $G \neq P_n$ is a *graph of two parallel paths* if there exist two disjoint induced paths of G that cover all the vertices of G and such that any edges between the two paths can be drawn without crossing.

Graphs of two parallel paths are precisely those graphs whose zero forcing number is 2.

Theorem 4.25. *[20, Theorem 5.1] For a graph G, $M(G) = 2$ if and only if G is a graph of two parallel paths or is one of a list of a few exceptional graphs.*

The exceptional graphs that are not graphs on two parallel paths that have maximum nullity two are shown in Figure 4.1, where the dashed line indicates an edge that may or may not be there, and the bold line indicates an edge that may be subdivided arbitrarily many times.

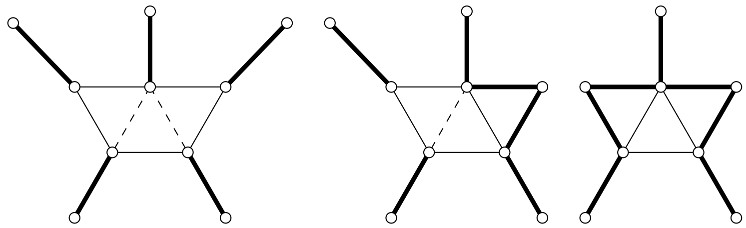

Figure 4.1: Exceptional Maximum Nullity 2 Graphs

Corollary 4.26. *(to Theorem 4.23) If G is a 2-connected partial 2-path, then* $\mathcal{I}(G) = T[n-2, n]$. *In particular, a 2-connected partial 2-path is trapezoidal.*

Before we can determine the inertia sets of all graphs of two parallel paths, we need the following result from [3].

Observation 4.27. *Let P_n denote the path on n vertices. Then $\mathcal{I}(P_n) = T[n-1, n]$.*

This also follows readily from Theorem 1.6, Example 4.11, and Theorem 4.13 applied to $K_2 (= P_2)$.

Theorem 4.28. *If G is any graph of two parallel paths on n vertices that is not a tree, then* $\mathcal{I}(G) = T[n-2, n]$.

Proof. We proceed by induction on n. The base case K_3 follows from Corollary 4.26, so assume the result for graphs on less than n vertices. If G is 2-connected we are done again by Corollary 4.26. If G is not 2-connected, then notice that we can write

$$G = P_k \bigoplus_v G'$$

where G' is a graph of two parallel paths that is not a tree. Let $l = |G'|$, so $l < n$, and $\mathcal{I}(G') = T[l-2, l]$ by the induction hypothesis. Also, we know that $\mathcal{I}(P_k) = T[k-1, k]$ by Observation 4.27. Notice that

$$\operatorname{mr}(P_k) + \operatorname{mr}(G') = (k-1) + (l-2) = (k+l-1) - 2 = n-2 = \operatorname{mr}(G)$$

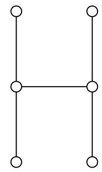

Figure 4.2: The H-Graph

so by Theorem 4.15, $\mathcal{I}(G) = T[n-2, n]$. $\square$

Theorem 4.29. *If G is any of the exceptional graphs of maximum multiplicity 2, then*

$$\mathcal{I}(G) = T[n-2, n]$$

where $n = |G|$.

Proof. Notice that all of the exceptional graphs (see Figure 4.1) can be written as

$$G = G' \bigoplus_{v} P_k$$

where P_k is a path on k vertices and G' is a graph satisfying the hypotheses of Theorem 4.28. Let $n = |G|, m = |G'|$ and notice $n = m + k - 1$. We have $\mathrm{mr}(G) = n - 2, \mathrm{mr}(P_k) = k - 1$, and $\mathrm{mr}(G') = m - 2$. Also, $\mathcal{I}(P_k) = T[k-1, k]$ and by Theorem 4.28, $\mathcal{I}(G') = T[m-2, m]$. Thus,

$$\mathrm{mr}(G') + \mathrm{mr}(P_k) = (m-2) + (k-1) = (m+k-1) - 2 = n - 2 = \mathrm{mr}(G)$$

so by Theorem 4.15, $\mathcal{I}(G) = T[n-2, n]$. $\square$

Notice that a tree T is a graph on two parallel paths if and only if $P(T) = 2$.

Theorem 4.30. *Let T be a tree on two parallel paths (i.e. any tree for which $M(T) = 2$)*

with n vertices. Then

$$\mathcal{I}(T) = T[n-1,n] \cup \{(n-3,1), \cdots, (1,n-3)\}.$$

Proof. By Proposition 4.9 we have $T[n-1,n] \subseteq \mathcal{I}(T)$ and since $M(T) = 2$, $\text{mr}(T) = n-2$ so we do not have any inertias below the $(n-2)$-line. So it suffices to show that the minimum rank line consists of exactly $\{(n-3,1), \cdots, (1,n-3)\}$.

In [4] this is confirmed for such trees on up to six vertices. Any tree with path cover number two on more than six vertices can be obtained by successive edge subdivisions of either the star S_4, or the H-graph (Figure 4.2). Now, again by [4], we know $\mathcal{I}(S_4) = T[3,4] \cup \{(1,1)\}$ and $\mathcal{I}(\text{H-graph}) = T[5,6] \cup \{(3,1),(2,2),(1,3)\}$ so by Corollary 4.14, repeated edge subdivision yields $\{(n-3,1), \cdots, (1,n-3)\} \subseteq \mathcal{I}(T)$. Now since T is a tree, $\text{mr}_+(T) = n-1$ by Theorem 1.20, so $(n-2,0),(0,n-2) \notin \mathcal{I}(G)$ and the result follows. $\square$

Theorems 4.28, 4.29, and 4.30 thus give, by Theorem 4.25, the inertia sets of all graphs whose minimum rank is $n-2$. In particular, the inertia sets of each of these graphs contain all points that are not already ruled out by the constraints imposed by the minimum rank and the minimum positive semidefinite rank. We can also see that graphs satisfying the hypotheses of Theorems 4.28 and 4.29 are exactly those graphs whose inertia set is the trapezoid $T[n-2,n]$. This motivates the following:

Question. Given a subset of $I \subseteq \mathbb{N}^2$, can we characterize all graphs G such that $\mathcal{I}(G) = I$?

We suspect that this question could be very difficult to answer without results similar to Theorem 4.25 that characterize all graphs with a given minimum rank or maximum nullity.

Chapter 5. Some Results on the Inverse Eigenvalue Problem

We have seen how knowledge of the minimum rank of a graph is helpful in determining what partial inertias can be achieved by the graph. In this chapter, we will investigate what the minimum rank and inertia set can tell us about possible eigenvalues of a graph. Very little has been done on this problem, and our results are indicative of many of the difficulties that arise in this problem. We will be concerned primarily with the possible eigenvalues that can be attained by minimum rank matrices for a graph.

We note in passing one result on the inverse eigenvalue problem for paths proven by Hald in [1] (although we will word it slightly differently).

Theorem 5.1. *[1, Theorem 2] Given any n distinct real numbers, there is a matrix in $\mathcal{S}(P_n)$ with those numbers as its eigenvalues.*

Since $M(P_n) = 1$, 0 can never be an eigenvalue of multiplicity 2 for a path, and hence no number can be an eigenvalue of multiplicity 2 for a path. Thus this theorem completely characterizes the possible eigenvalues for a path.

Hald's proof depends on the solution to recurrence relations involving the characteristic polynomials for matrices corresponding to a path. This proof is very specific to the structure of these matrices, and is thus very hard to apply to other graphs. We will need to turn to other techniques to handle such graphs. We will look particularly at graphs whose minimum rank is two.

Before beginning, we include some standard results from matrix theory.

Lemma 5.2. *A symmetric $n \times n$ matrix M has partial inertia (r, s) with $r + s = k$ if and* *only if M has a factorization*

$$M = A \begin{bmatrix} 1 & & & & & \\ & \ddots & & & 0 & \\ & & 1 & & & \\ & & & -1 & & \\ & 0 & & & \ddots & \\ & & & & & -1 \end{bmatrix} A^T$$

where A is an $n \times k$ matrix, and we have 1 occurring r times on the diagonal, and -1 occurring s times.

Lemma 5.3. *[21, Theorem 1.3.20] Suppose A is $m \times n$ and B $n \times m$ with $m \leq n$. Then BA has the same eigenvalues as AB, counting multiplicity, together with an additional $n - m$ eigenvalues equal to 0.*

Lemma 5.4. *[21, Theorem 2.5.4] Given a matrix A, the* Frobenius norm $||A||_F$ *is defined to be $\sum_{i,j} |a_{ij}|^2$. If A is symmetric with eigenvalues $\lambda_1, \cdots, \lambda_n$, then*

$$||A||_F^2 = \sum_{i=1}^{n} |\lambda_i|^2.$$

5.1 Stars

Let S_n denote the star on n vertices, $n \geq 3$. We know that $\mathrm{mr}(S_n) = 2$ and that for $n \geq 4$, $(1, 1)$ is the only partial inertia on the minimum rank line for S_n (see Example 4.16). Assume $n \geq 4$. Let

$$A = \begin{bmatrix} d & b \\ b^T & 0 \end{bmatrix} \in \mathcal{S}(S_n)$$

where d is a real number and $b \in \mathbb{R}^{n-1}$. Notice that $\operatorname{rank} A = 2 = \operatorname{mr}(S_n)$. Let $p(x)$ denote the characteristic polynomial of A. Notice that $\operatorname{tr}(A) = d$ so the coefficient of x^{n-1} in p is $-d$. Also note that the only nonzero 2×2 principal minors of A are of the form

$$\begin{vmatrix} d & b_i \\ b_i & 0 \end{vmatrix} = -b_i^2$$

so the coefficient of x^{n-2} in p is $-(b_1^2 + \cdots + b_{n-1}^2) = -||b||^2$. Note that any larger principal minors are 0 since A has rank 2. Thus we have the characteristic polynomial of A is

$$x^n - dx^{n-1} - ||b||^2 x^{n-2} = x^{n-2}(x^2 - dx - ||b||^2).$$

So A has eigenvalue 0 with multiplicity $n-2$. If r, s are the nonzero eigenvalues of A, we see that $r + s = d$, and $rs = -||b||^2$. Therefore, r and s must have opposite signs, which we knew from the inertia set for a star.

Now, if we are given any two numbers r, s with opposite signs, then we see that if we choose any $b \in \mathbb{R}^{n-1}$, none of whose entries are 0, such that $||b||^2 = |rs|$, and let $d = r + s$, then the eigenvalues of the matrix A above are r and s. So we can find a rank minimizing matrix in $\mathcal{S}(S_n)$ that has any two eigenvalues of opposite sign that we wish. In other words, the only restriction on the possible eigenvalues for a minimum rank matrix in $\mathcal{S}(S_n)$ is the restriction given by the inertia set.

For S_3, we also get inertia $(2, 0)$ and $(0, 2)$, but notice that S_3 is just a path P_3, so the inverse eigenvalue problem for this graph follows from the result on paths in Theorem 5.1, although it will also follow from what we will do in the next section.

The star S_2 is just the clique K_2. But of course, the inverse eigenvalue problem for rank minimizing matrices for cliques is trivial. The inertia must be either $(1, 0)$ of $(0, 1)$, so simply take a multiple of the all ones matrix J or of $-J$ to attain any positive or any negative eigenvalue that we wish.

In summary, any star can achieve any spectrum allowed by the inertia set for the graph.

5.2 Graphs of Clique Cover Number Two

Let G be a graph on n vertices such that $\mathrm{cc}(G) = 2$. Then G can be thought of as the union of 2 cliques. Since $\mathrm{cc}(G) = 2$ and G is not a complete graph, we can see that $\mathrm{mr}(G) = 2$, and by Theorem 4.18, the inertia set $\mathcal{I}(G)$ is the trapezoid $T[2, n]$. Suppose the two cliques have l vertices in common, the first has k other vertices, and the other has m other vertices. So G has a total of $n = k + l + m$ vertices. Number the vertices of G so that the first k are the k vertices only in the first clique, the next l are the l vertices of the overlap, and the last m are the vertices only in the other clique. We will first look at the positive semidefinite (partial inertia $(2, 0)$) case.

Consider the matrix

$$A = \begin{bmatrix} \frac{1}{\sqrt{k}} & 0 \\ \vdots & \vdots \\ \frac{1}{\sqrt{k}} & 0 \\ x & x \\ \vdots & \vdots \\ x & x \\ 0 & \frac{1}{\sqrt{m}} \\ \vdots & \vdots \\ 0 & \frac{1}{\sqrt{m}} \end{bmatrix}$$

where the first row is repeated k times, the next l times, and the last m times, $x \neq 0$. Then $AA^T \in \mathcal{S}(G)$, has rank 2, and is positive semidefinite.

Now, notice that

$$A^T A = \begin{bmatrix} 1 + lx^2 & lx^2 \\ lx^2 & 1 + lx^2 \end{bmatrix} = \begin{bmatrix} lx^2 & lx^2 \\ lx^2 & lx^2 \end{bmatrix} + I.$$

The matrix $A^T A$ has the same nonzero eigenvalues as AA^T by Lemma 5.3. Since

$$\begin{bmatrix} lx^2 & lx^2 \\ lx^2 & lx^2 \end{bmatrix}$$

has rank 1 and trace $2lx^2$, its eigenvalues are 0 and $2lx^2$. Therefore the eigenvalues of $A^T A$ are $1, 1 + 2lx^2$, and so these are the nonzero eigenvalues of AA^T as well. Then by scaling, we can get any two distinct positive eigenvalues we wish for rank 2 matrices in $\mathcal{S}(G)$.

We now examine whether it is possible to have two equal positive eigenvalues for a rank 2 matrix in $\mathcal{S}(G)$. Suppose M is any positive semidefinite matrix of rank two in $\mathcal{S}(G)$. Then M can be decomposed as the product BB^T where B is a $n \times 2$ matrix. Note that since $BB^T \in \mathcal{S}(G)$, the first k rows of B must be orthogonal to the last m rows. So the first k rows are all multiples of the first row, and the last m rows are all multiples of the last row. Let Q be the rotation matrix on $\mathbb{R}^2$ that rotates, by multiplication on the right, these two orthogonal row vectors into the position so that the first is of the form $(a, 0)$ and the other of the form $(0, b)$, $a, b \neq 0$. Then $M = BQQ^T B^T$ and the columns of BQ are supported on the cliques. Thus, without loss of generality, we may assume that M can be written as a product BB^T where the columns of B are supported on the cliques.

Case 1: Suppose $l = 1$, that is, G is the vertex sum of two cliques. Then

$$B^T = \begin{bmatrix} u & x & 0 \\ 0 & y & v \end{bmatrix}$$

where $u \in \mathbb{R}^k$, $v \in \mathbb{R}^m$. Then

$$B^T B = \begin{bmatrix} u \cdot u + x^2 & xy \\ xy & v \cdot v + y^2 \end{bmatrix}.$$

Suppose that BB^T has a positive eigenvalue of multiplicity two. Then $B^T B$ does as well. But this is a 2×2 matrix, so $B^T B$ must be similar to a multiple of the identity, and therefore must be a multiple of the identity. Therefore $xy = 0$ which implies $x = 0$ or $y = 0$. But then when we multiply BB^T we will get 0 in entries where there is an edge adjacent to the cut vertex. This matrix is no longer in $\mathcal{S}(G)$, a contradiction. So in this case, we cannot have a positive eigenvalue of multiplicity two.

Case 2: Suppose $l > 1$. Choose B so that

$$B^T = \begin{bmatrix} u_1 & \cdots & u_k & 1 & \cdots & 1 & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & a & \cdots & a & -(l-1)a & v_1 & \cdots & v_m \end{bmatrix}$$

where a is chosen so that there will be no cancellation when we multiply BB^T (we need $1-(l-1)a^2 \neq 0$), and $u_1, \cdots, u_k, v_1, \cdots, v_m$ are chosen so that the vectors $(u_1, \cdots, u_k, 1, \cdots, 1, 1, 0, \cdots, 0)$ and $(0, \cdots, 0, a, \cdots, a, -(l-1)a, v_1, \cdots, v_m)$ have the same magnitude. Let b denote that magnitude. Then

$$B^T B = \begin{bmatrix} b^2 & 0 \\ 0 & b^2 \end{bmatrix}.$$

This has a positive eigenvalue of multiplicity two, thus BB^T does as well. Our choice of a, and the fact that $l > 1$, guarantees that $BB^T \in \mathcal{S}(G)$. By scaling, we can achieve any positive eigenvalue of multiplicity two that we wish.

In summary, if G is a graph with $\mathrm{cc}(G) = 2$, then we can find a positive semidefinite matrix in $\mathcal{S}(G)$ with rank two that has any two distinct positive eigenvalues we wish. If G does not have a cut vertex, then we do not need to require them to be distinct, but if G does have a cut vertex, we need that restriction.

Of course, the analogous result for the negative semidefinite case follows from this as well.

Now we will look at rank 2 matrices in $\mathcal{S}(G)$ that have partial inertia $(1,1)$, that is, exactly one positive and one negative eigenvalue. Consider

$$\begin{bmatrix} a_1 & 0 \\ \vdots & \vdots \\ a_k & 0 \\ b_1 & c_1 \\ \vdots & \vdots \\ b_l & c_l \\ 0 & d_1 \\ \vdots & \vdots \\ 0 & d_m \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a_1 & \cdots & a_k & b_1 & \cdots & b_l & 0 & \cdots & 0 \\ 0 & \cdots & 0 & c_1 & \cdots & c_l & d_1 & \cdots & d_m \end{bmatrix}.$$

This has rank two and partial inertia $(1,1)$, and will be in $\mathcal{S}(G)$ provided $b_ib_j - c_ic_j \neq 0$ for $1 \leq i < j \leq l$. Let A denote the first matrix, and M the matrix in the middle. Multiplying in the other order gives

$$MA^TA = \begin{bmatrix} a_1^2 + \cdots + a_k^2 + b_1^2 + \cdots + b_l^2 & b_1c_1 + \cdots + b_lc_l \\ -(b_1c_1 + \cdots + b_lc_l) & -(c_1^2 + \cdots + c_l^2 + d_1^2 + \cdots + d_m^2) \end{bmatrix}$$

which has the same non-zero eigenvalues as AMA^T. Let $x > 0, t \geq 0$, and choose the entries of A as follows:

$$\begin{aligned}
a_i &= \sqrt{\frac{1}{k}x + \frac{1}{k}t}, \quad i = 1, \cdots, k \\
b_i &= \sqrt{\frac{2}{l}x}, \quad i = 1, \cdots, l \\
c_i &= \sqrt{\frac{1}{2l}x}, \quad i = 1, \cdots, l \\
d_i &= \sqrt{\frac{5}{2m}x}, \quad i = 1, \cdots, m.
\end{aligned}$$

We make this choice so that the b_i and c_i are distinct, ensuring that AMA^T is in $\mathcal{S}(G)$, and so that MA^TA will be relatively nice. This gives

$$MA^TA = \begin{bmatrix} 3x+t & x \\ -x & -3x \end{bmatrix}.$$

This has characteristic polynomial

$$\lambda^2 - t\lambda - (8x^2 + 3xt)$$

so the eigenvalues are

$$\frac{1}{2}(t \pm \sqrt{t^2 + 12xt + 32x^2}).$$

Note that t is the trace of the matrix. For $t \geq 0$, the $+$ always gives a positive number and the $-$ a negative number (we knew this at the outset since a matrix of the form we chose must have partial inertia $(1,1)$). The gap between the two eigenvalues is then $\sqrt{t^2 + 12xt + 32x^2}$. Given any $\lambda_1 > 0$ and $\lambda_2 < 0$, where without loss of generality, $|\lambda_2| \leq \lambda_1$, set $t = \lambda_1 + \lambda_2$ and choose x so that $\sqrt{t^2 + 12xt + 32x^2} = \lambda_1 - \lambda_2$. Notice that since $\lambda_2 < 0$, this is larger than $t = \lambda_1 + \lambda_2$. Thus this choice of x is possible since the range of $\sqrt{t^2 + 12xt + 32x^2}$ for $x \in (0, \infty)$ is (t, ∞). This choice of x and t gives λ_1 and λ_2 as eigenvalues.

5.3 Complete Bipartite Graphs

We will now look at complete bipartite graphs $K_{m,n}$, which also have minimum rank 2. It was shown in [4, Theorem 4.2] that if either of $m, n \geq 3$, then $\mathrm{mr}_+(K_{m,n}) \geq 3$ as well, so $(1,1)$ is the only point in the inertia set on the minimum rank line. The complete bipartite graphs $K_{1,m}$ are stars, which we have already done, so we need to consider the class $K_{m,n}$, $m, n \geq 2$. We will look specifically at $K_{2,2}$ later.

First, we will look at possible eigenvalues for minimum rank matrices for a complete bipartite graph $K_{2,m}$. For $m \geq 3$ such a matrix must have rank 2 and partial inertia $(1,1)$. Hence, it has a factorization

$$A \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} A^T$$

where A is an $(m+2) \times 2$ matrix. Consider

$$A = \begin{bmatrix} \sqrt{t+s} & 0 \\ 0 & \sqrt{s} \\ \sqrt{s} & -\sqrt{s} \\ \vdots & \vdots \\ \sqrt{s} & -\sqrt{s} \end{bmatrix}$$

for $s > 0$ and $t \geq 0$. Then

$$A \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} A^T = \begin{bmatrix} t+s & 0 & \sqrt{s(s+t)} & \cdots & \sqrt{s(s+t)} \\ 0 & -s & s & \cdots & s \\ \sqrt{s(s+t)} & s & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \\ \sqrt{s(s+t)} & s & 0 & \cdots & 0 \end{bmatrix} \in \mathcal{S}(K_{2,m}).$$

Multiplying in the other order, we have

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} A^T A = \begin{bmatrix} t+(m+1)s & -ms \\ ms & -(m+1)s \end{bmatrix}$$

which has the same nonzero eigenvalues. This has characteristic polynomial

$$\lambda^2 - t\lambda - ((m+1)st + (2m+1)s^2)$$

so the eigenvalues are

$$\frac{1}{2}\left(t \pm \sqrt{t^2 + 4((m+1)st + (2m+1)s^2)}\right).$$

Note that t here is the trace of the matrix. So, for $\lambda_1 > 0$ and $\lambda_2 < 0$, where, without loss of generality, assume $|\lambda_2| \leq \lambda_1$, set $t = \lambda_1 + \lambda_2$ and then choose s so that the gap between the eigenvalues, $\sqrt{t^2 + 4((m+1)st + (2m+1)s^2)} = \lambda_1 - \lambda_2$. This can be done because the range of $\sqrt{t^2 + 4((m+1)st + (2m+1)s^2)}$ for $s \in (0, \infty)$ is (t, ∞), and since λ_2 is negative, $\lambda_1 - \lambda_2 > \lambda_1 + \lambda_2 = t$. This choice of s and t will then guarantee λ_1 and λ_2 as eigenvalues.

Thus, we can find a minimum rank matrix in $\mathcal{S}(K_{2,m})$ attaining any positive and any negative eigenvalue that we wish. In other words, the inertia set gives the only restriction on possible eigenvalues for the minimum rank case.

Now we will consider the general complete bipartite graph $K_{m,n}$ where both $m, n \geq 3$. This case is actually much more restricted in terms of what possible eigenvalues can be attained. We will first argue that each diagonal entry of a rank minimizing matrix for such a graph must be zero. Label the vertices of $K_{m,n}$ so that the first m are the independent set of size m, and the remaining n are the independent set of size n. Then any matrix M in

$\mathcal{S}(K_{m,n})$ is of the form

$$M = \begin{bmatrix} D_1 & A \\ A^T & D_2 \end{bmatrix}$$

where D_1, D_2 are square diagonal matrices of size m and n respectively, and A is an $m \times n$ block, all of whose entries are nonzero. Suppose M has rank 2. Notice that if three (or more) diagonal entries within a single set of the bipartition are nonzero, then taking the three corresponding rows and columns, we get the following submatrix in M:

$$\begin{bmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix}$$

where the $*$ indicates a nonzero entry (but not necessarily the same nonzero entry each time). This has rank three, contradicting that M has rank 2. So M can have at most two nonzero diagonals within a given set. Suppose M has exactly two nonzero diagonal entries within a given set. Then take the submatrix of M corresponding to those two rows and columns, some other row from within the same set (here is where we need $m, n \geq 3$), and a column of A whose entries are taken from these rows. Then we get a submatrix of the form

$$\begin{bmatrix} 0 & 0 & * \\ * & 0 & * \\ 0 & * & * \end{bmatrix}$$

or some permutation of this matrix. This is a combinatorially invertible form, again contradicting that M has rank 2. Now suppose M has exactly one nonzero diagonal entry within a given set. Then there is a submatrix of the form

$$\begin{bmatrix} 0 & * & * \\ * & * & 0 \\ * & 0 & 0 \end{bmatrix}$$

or some permutation of this. This is again combinatorially invertible, contradicting that M has rank 2.

Thus M must have all diagonal entries equal to 0. This implies that the trace of M is 0, so the two nonzero eigenvalues of M must be opposites. Taking the appropriate multiple of M, we can attain any two opposite eigenvalues we wish for a rank 2 matrix in $\mathcal{S}(K_{m,n})$.

More specifically,

$$M = \begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix}$$

has eigenvalues $\lambda, -\lambda$, and 0 with multiplicity $m + n - 2$ if and only if A has rank 1 and $||A||_F^2 = \lambda^2$ (see Lemma 5.4). So we not only know that all such spectra are attainable, but we also know every matrix in $\mathcal{S}(K_{m,n})$ that attains it.

We have thus done the inertia $(1, 1)$ case for every complete bipartite graph ($K_{1,n}$ is a star which we did previously). However, there is one complete bipartite graph, $K_{2,2}(= C_4)$ that has $(2, 0)$ and $(0, 2)$ in its inertia set as well. To handle this case, consider the matrix

$$A = \begin{bmatrix} t & 0 \\ 0 & 1 \\ 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

Then

$$AA^T = \begin{bmatrix} t^2 & 0 & t & -t \\ 0 & 1 & 1 & 1 \\ t & 1 & 2 & 0 \\ -t & 1 & 0 & 2 \end{bmatrix}.$$

This is in $\mathcal{S}(K_{2,2})$, is positive semidefinite, and has rank 2.

$$A^T A = \begin{bmatrix} t^2+2 & 0 \\ 0 & 3 \end{bmatrix}$$

has the same nonzero eigenvalues. The eigenvalues are clearly 3 and t^2+2. Then for $t \in [1, \infty)$, we get eigenvalues that are arbitrarily far apart. Multiplying by the appropriate scalar, we can attain any set of two positive eigenvalues that we wish. More specifically, given positive real numbers $\lambda_1, \lambda_2, \lambda_1 \geq \lambda_2$, set

$$A = \begin{bmatrix} \sqrt{\lambda_1 - \frac{2\lambda_2}{3}} & 0 \\ 0 & \sqrt{\frac{\lambda_2}{3}} \\ \sqrt{\frac{\lambda_2}{3}} & \sqrt{\frac{\lambda_2}{3}} \\ -\sqrt{\frac{\lambda_2}{3}} & \sqrt{\frac{\lambda_2}{3}} \end{bmatrix}.$$

Then $AA^T \in \mathcal{S}(K_{2,2})$ and

$$A^T A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

so the positive eigenvalues of AA^T are λ_1 and λ_2. Of course, the negative semidefinite case follows from this as well.

5.4 Clique-Stars

Another important class of graphs whose minimum rank is 2 is the class of *clique-stars*:

Definition 5.5. The *clique-star* $KS_{m,n}$ is the graph $K_m \vee nK_1$.

The clique-star $KS_{m,n}$ can be thought of as the complete bipartite graph $K_{m,n}$ with the independent set of size m filled in to form a clique. Notice that for $n \leq 2$, $KS_{m,n}$ is either a clique or a graph of clique cover number 2, so we have already done those cases. We will thus consider only the case $n \geq 3$. It was shown in [4, Theorem 4.4] that for $n \geq 3$, $(1,1)$ is the only point on the minimum rank line for $KS_{m,n}$.

A rank minimizing matrix can thus be factored as

$$A \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} A^T$$

where A is an $(m+n) \times 2$ matrix. Consider,

$$A = \begin{bmatrix} \sqrt{\frac{t}{m}} & 0 \\ \vdots & \vdots \\ \sqrt{\frac{t}{m}} & 0 \\ r & r \\ \vdots & \vdots \\ r & r \end{bmatrix}$$

where $t > 0$, $r \neq 0$, the first row is repeated m times, and the other row n times. Then

$$A \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} A^T = \begin{bmatrix} \frac{t}{m} & \cdots & \frac{t}{m} & r\sqrt{\frac{t}{m}} & \cdots & r\sqrt{\frac{t}{m}} \\ \vdots & & \vdots & \vdots & \cdots & \vdots \\ \frac{t}{m} & \cdots & \frac{t}{m} & r\sqrt{\frac{t}{m}} & \cdots & r\sqrt{\frac{t}{m}} \\ r\sqrt{\frac{t}{m}} & \cdots & r\sqrt{\frac{t}{m}} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \ddots & \\ r\sqrt{\frac{t}{m}} & \cdots & r\sqrt{\frac{t}{m}} & 0 & \cdots & 0 \end{bmatrix} \in \mathcal{S}(KS_{m,n}).$$

Multiplying in the other order, we have

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} A^T A = \begin{bmatrix} t + nr^2 & nr^2 \\ -nr^2 & -nr^2 \end{bmatrix}$$

which has the same nonzero eigenvalues. This has characteristic polynomial

$$\lambda^2 - t\lambda - nr^2 t$$

so the non-zero eigenvalues are

$$\frac{1}{2}(t \pm \sqrt{t^2 + 4nr^2 t}).$$

Note that t here is the trace of the matrix. Suppose $\lambda_1 > 0$ and $\lambda_2 < 0$, where, without loss of generality, we assume $|\lambda_2| \leq \lambda_1$. First suppose $|\lambda_2| < \lambda_1$. Set $t = \lambda_1 + \lambda_2$ and then choose r so that the gap between the eigenvalues, $\sqrt{t^2 + 4nr^2 t} = \lambda_1 - \lambda_2$. This can be done because the range of $\sqrt{t^2 + 4nr^2 t}$ for $r \in (0, \infty)$ is (t, ∞), and since λ_2 is negative, $\lambda_1 - \lambda_2 > \lambda_1 + \lambda_2 = t$. This choice of r and t will then guarantee λ_1 and λ_2 as eigenvalues.

Now suppose $\lambda_1 = -\lambda_2$. Consider the matrix

$$M = \begin{bmatrix} m-1 & 1 & \cdots & 1 & \sqrt{m}-1 & \cdots & \sqrt{m}-1 \\ 1 & -1 & \cdots & -1 & 1 & & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 1 & -1 & \cdots & -1 & 1 & & 1 \\ \sqrt{m}-1 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ \vdots & & \vdots & & & \ddots & \\ \sqrt{m}-1 & 1 & \cdots & 1 & 0 & \cdots & 0 \end{bmatrix} \in \mathcal{S}(KS_{m,n}).$$

This matrix has only 3 distinct rows, and multiplying the second row by $-(\sqrt{m}-1)$ and adding it to the first row gives $\sqrt{m}$ times the last row. Thus, M has rank 2, and has trace 0 by construction. Thus, it has one positive and one negative eigenvalue which are opposite. So by multiplying M by the appropriate scalar, we can attain any two opposite eigenvalues we wish.

In summary, for minimum rank matrices for $KS_{m,n}$, we can have any positive and any negative eigenvalues we wish. In other words, the inertia set gives the only restrictions on possible eigenvalues.

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