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Infinite Product Group

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INFINITE PRODUCT GROUPS

by

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ABSTRACT

INFINITE PRODUCT GROUPS

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The theory of infinite multiplication has been studied in the case of the Hawaiian earring group, and has been seen to simplify the description of that group. In this paper we try to extend the theory of infinite multiplication to other groups and give a few examples of how this can be done. In particular, we discuss the theory as applied to symmetric groups and braid groups. We also give an equivalent definition to K. Eda's infinitary product as the fundamental group of a modified wedge product.

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Table of Contents

List of Figures

1 Introduction

In classical group theory, only finite multiplication is allowed. There are obvious reasons why this is the case, but this paper explores some of the possibilities of using infinite multiplication, how to make sense of it, and how it can in some cases simplify the description or "presentation" of a group.

In Section 2 we discuss the properties of the Hawaiian earring group that will be useful for the remainder of the paper, particularly in Section 3 where the theory of infinite multiplication will be generalized. In Section 4 we introduce notation for permutations and permutation groups and we give results that are part of classical group theory and that will be extended to the infinite multiplication case in Section 5.

We will give further examples of how infinite multiplication can be used by examining braid groups. Finite braid groups and classical braid group theory will be introduced in Section 6 and the results for infinite braid groups and the theory of infinite multiplication will be given in Section 7.

Finally, in Section 9, we discuss another way of thinking about infinite multiplication, which is based on the concept of Eda's infinitary product.

2 Hawaiian Earring

If we define $C(r, p)$ to be a circle in the plane centered at p with radius of r, then we call the space

$$E = \bigcup_{n \in \mathbb{N}} C(1/n, (1/n, 0))$$

the *Hawaiian earring* and its fundamental group $\pi_1(E, (0, 0))$ will be called the *Hawaiian earring group* and will be denoted by H. The space E and the group H have been studied extensively by K. Eda, J. Cannon, G. Conner, and others. The results

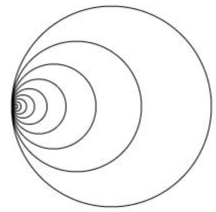


Figure 1: The Hawaiian earring

most pertinent to this paper will be found in [1]. We will cite those results here in this section and forgo the proofs. In the discussion of the Hawaiian earring we will talk about the inverse limit of free groups, so we define what we mean by that here.

Definition 2.1. Let F_n denote the free group on the generators $\{a_1, \ldots, a_n\}$. For $m \geq n$, define the map $\varphi_{m,n} : F_m \to F_n$ by

$$\varphi_{m,n}(a_k) = \begin{cases} a_k & \text{if } k \leq n \\ 1 & \text{if } k > n \end{cases}.$$

Unless stated otherwise, $\varprojlim F_n$ will mean the inverse limit of this system.

Definition 2.2. Given a reduced word $w \in F_n$ and a letter a_i, we define the i-weight of w to be the number of times the letter a_i appears in w—that is, the sum of the absolute values of the exponents—and we denote this $w_i(w)$.

Proposition 2.3. *The Hawaiian earring group H embeds in the inverse limit $\varprojlim F_n$. In particular, it can be identified with the elements of $\varprojlim F_n$ with bounded weights for all letters. That is, $(x_1, x_2, \ldots) \in \varprojlim F_n$ is in H if for each j, the sequence $\{w_j(x_i)\}_i$ is bounded.*

By Proposition 2.3, we may also define the i-weight of a word in the Hawaiian earring group. That is, for $h \in H$ and $i \in \mathbb{N}$, $w_i(h)$ is defined to be $\max\{w_i(h_j)\}$

(where h_j is the j^{th} coordinate of h). If we were to express h as a word in the letters $\{a_i\}$ rather than as a sequence of finite words, then we would see that a_i would appear exactly $w_i(h)$ times. Just as in the case with free groups, we insist that we only deal with reduced words, which by [1] exist and are unique.

We take a moment to note that elements of the inverse limit are really coherent sequences of words in free groups, but when we think of them as elements of the Hawaiian earring group it will be convenient to think of them as words themselves, rather than sequences of words. Thus the sequence $(a_1, a_1, a_1, \ldots)$ will be denoted simply a_1 and will represent the homotopy class of a map that wraps one time around the outer-most loop of the Hawaiian earring. Similarly, the sequence $(a_1, a_1a_2, a_1a_2a_3, \ldots)$ will be denoted $a_1a_2a_3\cdots$ and will represent the homotopy class of a map that wraps once around each loop, in order from outer-most inward. Since we identify H with a subgroup of $\varprojlim F_n$, we will use these two different notations interchangeably.

Also by Proposition 2.3 we see that there are natural maps $\psi_n : H \to F_n$ where

$$\psi_n(a_i) = \begin{cases} a_i & i \leq n \\ \\ 1 & 1 > n \end{cases}$$

and indeed ψ_n is an epimorphism. It is noted that $\psi_n = \pi_n|_H$, where $\pi_n : \prod F_i \to F_n$ is the canonical projection. Therefore, we will call this map the *projection* onto F_n or the n^{th} projection.

Now we topologize H as follows: give each free group F_n the discrete topology, so that the Hawaiian earring group H inherits its topology from the product space $\prod F_n$. It can be seen that a basis for this topology is given by the collection $\{\psi_n^{-1}(w) \mid n \in \mathbb{N}, w \in F_n\}$. Furthermore, this is a metrizable space and the topology is given by the metric

$$d(w,w') = \begin{cases} 0 & \text{if } w = w' \\ \\ 1/n & \text{if } \psi_n(w) \neq \psi_n(w') \text{ and } \psi_i(w) = \psi_i(w') \text{ for } i < n. \end{cases}$$

To ensure that this metric is well-defined, we insist that both of w and w' are reduced words. It is also important to note that Cannon and Conner have shown that for $w \neq w'$, there is an n such that $\psi_n(w) \neq \psi_n(w')$.

The significance of this topology lies in the fact that we can define infinite products very similarly to the way analysts define series. Series are said to converge if the sequence of partial sums converges. Likewise, we will say that an infinite product of words in H converges if every sequence of partial products converges. In this case, we call the product *legal* and in the case that no sequence of partial products converge we call the product *illegal*. That is, the sequence $\{a_1, a_1a_2, a_1a_2a_3, \ldots\}$ converges to the point $a_1a_2a_3\cdots$, so we say that the product $\prod a_i$ is legal. However, we would first do well to define what an infinite product is.

Definition 2.4. An *infinite product* on the Hawaiian earring group is a function $f : \mathbb{Q} \to H$. A subproduct $f|_A$ is *pseudo-finite* if $f(a) = 1$ for all but finitely many $a \in A$, and the *value* of the subproduct is $\nu(f|_A) = \prod\limits_{f(a)\neq 1} f(a)$, multiplied in the order dictated by A. A *subproduct chain* will be a sequence of sets $A_1 \subset A_2 \subset A_3 \subset \ldots$ whose union is $\mathbb{Q}$ and such that for each i, $f|_{A_i}$ is pseudo-finite. f is said to be *legal* if there is $h \in H$ such that for every subproduct chain, the sequence $\{\nu(f|_{A_i})\}$ converges to h. The *value* of f will be $\nu(f) = h$. Otherwise, f is said to be *illegal*. An infinite product $f : \mathbb{Q} \to H$ is said to be *fundamental* if $\operatorname{Im} f \subset \{1, a_1, a_1^{-1}, a_2, a_2^{-1}, \ldots\}$.

We concede that Cannon and Conner defined infinite products as functions from any countable totally-ordered set into the alphabet $A = \{a_1, a_2, \ldots\}$ and its inverse. However, it is noted that every countable order type embeds in the order type of

$\mathbb{Q}$, so it is sufficient to restrict the domain to be $\mathbb{Q}$ and to let 1 be in the image, as a sort of placeholder. Furthermore, since each element of H can be expressed as a fundamental product, given a product f that is legal but not fundamental we can construct a product g that is fundamental and has the same value as f. First we construct for each $q \in \mathbb{Q}$ a product $f_q : \mathbb{Q} \to H$ that has value $f(q)$, then we define $h : \mathbb{Q} \times \mathbb{Q} \to H$ by $h(q, r) = f_q(r)$. Since $\mathbb{Q} \times \mathbb{Q}$ with the lexicographic order has the same order type as $\mathbb{Q}$, let $j : \mathbb{Q} \to \mathbb{Q} \times \mathbb{Q}$ be a homeomorphism and define $g = h \circ j$. It is also noted that Cannon and Conner defined a product to be legal if the preimage of each letter was finite, and illegal otherwise. So we show that these definitions are equivalent.

Theorem 2.5. *A fundamental product f is legal if and only if $f^{-1}(a_i)$ and $f^{-1}(a_i^{-1})$ are finite for all i.*

Proof. Suppose that $f^{-1}(a_i)$ and $f^{-1}(a_i^{-1})$ are finite for all i. Then let $A_1, A_2, \ldots$ be a subproduct chain. Given $\varepsilon > 0$, let $N \in \mathbb{N}$ be such that $1/N < \varepsilon$. Since the set $X_N = f^{-1}(\{a_1, a_1^{-1}, \ldots, a_N, a_N^{-1}\})$ is finite, and the sequence A_i eventually exhausts all of $\mathbb{Q}$, we see that there is an M such that for $m > M$, $A_m \setminus A_M$ is disjoint from X_N. It then follows that $\psi_N(f|_{A_m}) = \psi_n(f|_{A_m})$ for all $m > M$ and for all $n > N$. This means that each coordinate is eventually constant and therefore we see that the sequence converges in the direct product $\prod F_i$. It is clear that this limit is in H, since each a_i is used only finitely often. Let h be this limit, and let $B_1, B_2, \ldots$ be another subproduct chain, which by the same argument converges to a point $h' \in H$. For a given $n \in \mathbb{N}$, we see that $\psi_n(h) = \psi_n(h')$ by choosing N such that $f^{-1}(\{a_1, a_1^{-1}, \ldots, a_n, a_n^{-1}\})$ is contained in both of B_N and A_N. Then it follows that $h = h'$. Therefore, f is legal.

Now we suppose that $f^{-1}(a_i)$ is infinite for some i. Let $A_1, A_2, \ldots$ be a subproduct chain and $h \in H$. Let $k = w_i(h)$. Since $f^{-1}(a_i)$ is infinite, we see that there is $N > i$ such that $w_i(\nu(f|_{A_N})) > k$. Therefore, we see that $\psi_N(\nu(f|_{A_n})) \neq \psi_N(h)$, therefore $d(h, \nu(f|_{A_N})) \geq 1/N$. However, it is also the case that $w_i(\nu(f|_{A_n})) \geq w_i(\nu(f|_{A_N}))$ for $n > N$, therefore, we see that $d(h, \nu(f|_{A_n}))$ for all $n > N$, and therefore $\nu(f|_{A_n})$ does not converge to h. Therefore f is illegal. $\square$

Another interesting thing about this topology on H is that the infinite-rank free group $F_\infty = \langle a_1, a_2, \ldots \rangle$ is dense in H. This is because $\psi_n(F_\infty) = F_n$, so F_∞ is in fact dense in the product space $\prod F_n$. Erin Summers [9] has shown that every endomorphism of H is determined by its action on the generators $\{a_i\}$. Since words in H are allowed to be infinitely long, we say that such a map is *infinitely multiplicative*. To formalize this definition, we note that if any set of words $\{x_i\}$ is mapped trivially, then any legal product of those words is also mapped trivially, and thus we see that the kernel of H is closed.

Definition 2.6. Let G be any group. Then a map $\varphi : H \to G$ is said to be *infinitely multiplicative* if $\ker \varphi$ is closed in H.

One last result that we will be using later is that the closure of a normal subgroup is still normal. This is seen with the following argument. Suppose that $\{w_n\}$ is a sequence in H converging to the point w. Then for any word $h \in H$ we see that $\{hw_nh^{-1}\} \to hwh^{-1}$. Therefore, if N is a normal subgroup and $\{w_n\} \subset N$ then $\{hw_nh^{-1}\} \subset N$, so we see that w is a limit point of N if and only if hwh^{-1} is, so $\bar{N}$ is normal. For this reason, given a set $A \subset H$, we say that the *big normal subgroup* generated by A is the closure of the normal subgroup $\langle\langle A \rangle\rangle$.

3 Infinite Product Structure

Here we give a formal definition for what an infinite product group is. Since we have already developed the theory of infinite multiplication on the Hawaiian earring, it makes sense to use that theory in our definition.

Definition 3.1. Let G be a group. An *infinite product structure* on G is an epimorphism $\varphi : H' \to G$, where H' is a subgroup of H containing F_∞. A group with an infinite product structure will be called an *infinite product group*. We say that G is *infinitely multiplicative* if φ is. We further say that G is *infinitely abelian* if $\ker \varphi$ also contains the commutator subgroup $[H, H]$ An element of H' will be called a *legal word* and all other words will be *illegal*. An *infinite product* on G will be a map $f : \mathbb{Q} \to G$. Given a word $h \in H'$, we call the *image product of h* the product $f = \varphi \circ g$ where g is a fundamental product representing h. Products in G that are image products of elements in H' will be called *legal products* and those that are not will be called *illegal*.

It is noted that in the above definition, an arbitrary choice must be made for defining an image product. However, we will define an equivalence relation on infinite products that will solve this problem by showing that any two choices for g will give equivalent results for f.

Definition 3.2. If $f : \mathbb{Q} \to G$ is an infinite product and $A \subset \mathbb{Q}$ then $f|_A$ is called a *subproduct*. If $f(a)$ is trivial for all but finitely many $a \in A$, then $f|_A$ is said to be *pseudo-finite*. In this case, we say that the *value* of the subproduct $f|_A$ is the value of the finite product $\prod_{f(a) \neq 1} f(a)$, taken in the order dictated by A, and we will denote it by $\nu(f|_A)$.

Definition 3.3. We say that an infinite product $f : \mathbb{Q} \to G$ admits a *cancellation* $*$

if there is a subset $A \subset \mathbb{Q}$ and a pairing $* : A \to A$ satisfying the following conditions:

1. $*$ is an involution of the set A.

2. $*$ is *complete* in the sense that $[a, a^*]_A = [a, a^*]_{\mathbb{Q}}$ for every $a \in A$ and *noncrossing* in the sense that $[a, a^*]_A = ([a, a^*]_A)^*$ for every $a \in A$.

3. $*$ is an *inverse pairing* in the sense that $f(a^*) = f(a)^{-1}$.

Definition 3.4. Suppose that f and g are both infinite products on the group G. Then we say that $f \sim g$ if there are partitions $\{A_i \mid i \in J \subset \mathbb{Q}\}$ and $\{B_i \mid i \in J' \subset \mathbb{Q}\}$ of $\mathbb{Q}$ and an order-preserving map $h : \mathbb{Q} \to \mathbb{Q}$ such that each of A_i, B_i is an interval in $\mathbb{Q}$, $A_i < A_j$ if and only if $B_i < B_j$, and for each i one of the following is true:

1. $f|_{A_i} = g|_{h(B_i)}$.

2. each of $f|_{A_i}$ and $g|_{h(B_i)}$ is pseudo-finite with the same value.

3. each of $f|_{A_i}$ and $g|_{h(B_i)}$ admits a cancellation or is pseudo-finite with value 1.

Then extend $\sim$ to be an equivalence relation.

It has been shown in [1] that two fundamental products f and g in the Hawaiian earring group represent the same word exactly when $f \sim g$. Thus to complete the proof that the choice of g in Definition 3.1 is not significant, we prove the following result.

Proposition 3.5. *Let H' be a subgroup of H and let G be a group with infinite product structure $\varphi : H' \to G$. If $f, g : \mathbb{Q} \to H$ are products representing the same word $h \in H'$, then $\varphi \circ f \sim \varphi \circ g$.*

Proof. The map φ preserves each of the three properties 1 – 3 in Definition 3.4. $\square$

We note that in the case of the Hawaiian earring group, all elements can be represented as fundamental products—that is, infinite words on the letters $\{a_i\}$. It is clear that the set $\{\varphi(a_i)\}$ also has this property—that all elements of G can be thought of as infinite products in the elements $\{\varphi(a_i)\}$ where each one is used only finitely often. Because this set in a way generates the group G, we have the following definition.

Definition 3.6. If φ is an infinite product structure, then the image $\varphi(\{a_i\})$ is called a *weak generating set* for G.

4 Classical Symmetric Group Theory

In this section we will establish the notation for symmetric groups that will be used in this paper. The results in this section are elementary and therefore the proofs will be withheld. They are merely stated to emphasize how they relate to their counterparts in the study of infinite multiplication.

Definition 4.1. A *permutation* on a set A is a bijective map $f : A \to A$. The set of all such permutations will be denoted by S_A and will be called the *symmetric group* or *permutation group* on A. The group law will be function composition. Since S_A only depends on the cardinality of A, given a cardinality α, we let S_α denote S_A where $|A| = \alpha$.

Definition 4.2. A permutation σ for which $\sigma(a) = b$, $\sigma(b) = a$ and $\sigma(c) = c$ for all $c \neq a, b$ is called a *transposition* and will be denoted by $\sigma = (a\ b)$. More generally, the notation $\sigma = (a_1\ a_2\ a_3\ \ldots\ a_n)$ will denote that $\sigma(a_i) = a_{i+1}$ for $1 \leq i < n$ and $\sigma(a_n) = a_1$, and $\sigma(b) = b$ for $b \neq a_i$. In this case σ will be called a *cycle*—more specifically, an n-cycle—and the notation will be called *cyclic notation.* Two cycles

$(a_1\ a_2\ \ldots a_n)$ and $(b_1\ b_2\ \ldots b_m)$ are said to be *disjoint* if the sets $\{a_1, a_2, \ldots, a_n\}$ and $\{b_1, b_2, \ldots, b_m\}$ are disjoint.

Definition 4.3. If $A = \{1, 2, 3, \ldots, n\}$ then we use the notation σ_k to denote $(k\ k{+}1)$. More generally, if A is well-ordered, then for any element $\alpha \in A$, we let σ_α denote the transposition that changes α with its immediate successor. Such a permutation will be called an *adjacent transposition*.

Lemma 4.4. *Every permutation in S_n has a representation as a product of disjoint cycles.*

Lemma 4.5. *Every permutation in S_n can be written as a product of adjacent transpositions.*

Lemma 4.6. *The group S_n has presentation*

$$\langle \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i^2, (\sigma_i\sigma_{i+1})^3, [\sigma_i, \sigma_j] \text{ for } |i-j| > 1 \rangle.$$

5 The Infinite Symmetric group $S_{\mathbb{N}}$

We would like to extend the results from Section 4 to infinite permutation groups, which we will do. First, however, we need to define one more type of cycle.

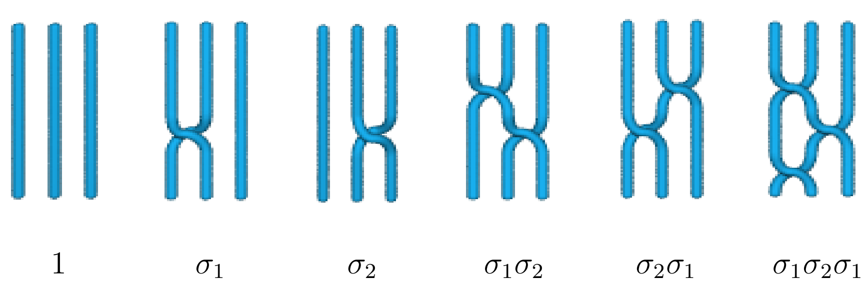


Figure 2: All elements of S_3 represented as braids.

Definition 5.1. Let $\sigma = (\ldots\ a_{-1}\ a_0\ a_1\ \ldots)$ denote that $\sigma(a_n) = a_{n+1}$ for all $n \in \mathbb{Z}$ and $\sigma(b) = b$ for $b \notin \{a_i\}$. Then we call σ an *infinite cycle*.

Lemma 5.2. *Let A be a nonempty set and let $\sigma \in S_A$. Then σ has a representation as a (possibly infinite) product of disjoint cycles—allowing infinite cycles.*

Proof. Well-order A. Let α be the least element. Construct the cycle containing α by considering the sequences $\{\sigma^i(\alpha)\}$ and $\{\sigma^{-i}(\alpha)\}$. If for some $i \in \mathbb{N}$, $\sigma^i(\alpha) = \alpha$, then $\sigma^{-i}(\alpha) = \alpha$, so there is a finite cycle $(\alpha\ \sigma(\alpha)\ \ldots\ \sigma^{i-1}(\alpha))$ containing α. Otherwise, we see that $(\ldots\ \sigma^{-1}(\alpha)\ \alpha\ \sigma(\alpha)\ \ldots)$ is an infinite cycle containing α. Now we proceed by induction. If there is an element of A that has not already been listed in a previous cycle, then let β be the least such element and write out the cycle containing β. Then we see by induction that every element of A will eventually be written and the result will be a (possibly infinite) list of cycles which define the permutation σ. $\square$

In the special case of the set of natural numbers, we also get the following result which generalizes Lemma 4.5.

Theorem 5.3. *For $\sigma \in S_{\mathbb{N}}$, σ has a representation as a (possibly infinite) product of adjacent transpositions such that each σ_i is used only finitely often.*

Proof. We use Lemma 5.2 to first express σ as a product of disjoint cycles. Then we show how each cycle can be written as a product of adjacent transpositions. For a finite cycle, we see that $(a_1\ a_2\ \ldots\ a_n) = (a_1\ a_2)(a_2\ a_3)\cdots(a_{n-1}\ a_n)$, and for infinite cycles $(\ldots\ a_{-1}\ a_0\ a_1\ \ldots) = \cdots(a_{-1}\ a_0)(a_0\ a_1)(a_1\ a_2)\cdots$. Now we only need to show that each transposition can be written as a product of adjacent transpositions. So we consider the transposition $(i\ j)$ and assume without loss of generality that $i < j$. Then

$$(i\ j) = \sigma_i\sigma_{i+1}\cdots\sigma_{j-2}\sigma_{j-1}\sigma_{j-2}\cdots\sigma_{i+1}\sigma_i.$$

To see that each σ_i is used finitely often, we place an upper bound on the number of times each one appears using this algorithm. For a given i, we see that in the above algorithm, σ_i appears only in the the factorization of $(n\ m)$ if $n \leq i \leq m$ (with one of the inequalities strict), and it appears at most twice in that factorization. Therefore, it appears at most twice each time a number smaller than i swaps with a number larger than i. Since there are only i numbers less than or equal to i, and in the first factorization from disjoint cycles, we see that each number appears at most twice, we conclude that σ_i is used no more than $4i$ times. $\square$

What is so significant about the above result? We have just shown that $\{\sigma_i\}$ is a weak generating set for $S_{\mathbb{N}}$. How do we define an infinite product structure on $S_{\mathbb{N}}$ and what products are legal? For this, we recall the presentation of finite symmetric groups, found in Lemma 4.6.

We use this concept to try to build a pseudo-presentation for $S_{\mathbb{N}}$ in the following way. Let N be the big normal subgroup of H generated by elements of the form

$$\left\{a_i^2, (a_i a_{i+1})^3, [a_i, a_j] \text{ for } |i-j| > 1\right\}.$$

Then let MG be the quotient H/N. We would hope that $MG \cong S_{\mathbb{N}}$ because it has the right relators and by Theorem 5.3, we see that every element of $S_{\mathbb{N}}$ can be expressed as a Hawaiian word. However, we look at the seemingly innocent word $\sigma_1\sigma_2\sigma_3 \cdots = (1\ 2)(2\ 3)(3\ 4)\cdots$. We see that this corresponds to the map $f : \mathbb{N} \to \mathbb{N}$ given by $f(n) = n + 1$, which means that no number maps to 1, and therefore this is not a permutation. Upon further investigation, we see that there are other words which are not even functions. Therefore, this assumption is false. However, we will realize $S_{\mathbb{N}}$ as a certain subroup of MG. The group MG will be called the *mutation group* on $\mathbb{N}$ since it contains elements that are not permutations, but the permutation group $S_{\mathbb{N}}$ is dense in MG (where $MG = H/N$ is given the quotient topology). Since the group MG can be thought of as all infinite words on the alphabet $\{\sigma_i\}$, we call MG the *Hawaiian closure* of $S_{\mathbb{N}}$.

The next natural question to ask is, given an element of MG, how do we decide whether it is a permutation? There is an important tool that we need to develop before we will be able to thoroughly answer this question.

Definition 5.4. Let $\Psi : H \to MG$ be the quotient map. For $g \in MG$ and $h \in \Psi^{-1}(g)$ we call the function $\Psi(\psi_n(h))$ the *n^{th} approximation of g by h* and denote it h_n. If there is $h \in \Psi^{-1}(g)$ such that the sequences $\{h_n(m)\}_n$ and $\{h_n^{-1}(m)\}_n$ are eventually constant for each $m \in \mathbb{N}$, then say that g is *bounded via h*. In this case we define the map $\sigma_{(g,h)} : \mathbb{N} \to \mathbb{N}$ by $\sigma_{(g,h)}(m) = \lim\limits_{n\to\infty} h_n(m)$.

Lemma 5.5. *If $g \in MG$ is bounded via h, then $\sigma_{(g,h)} \in S_{\mathbb{N}}$.*

Proof. By definition, since each sequence $\{h_n(m)\}_n$ is eventually constant, the map is well-defined. To see that it is a permutation, we note that $\sigma_{(g,h)}^{-1} = \sigma_{(g^{-1},h^{-1})}$. It is clear that g is bounded via h if and only if g^{-1} is bounded via h^{-1}, and that the maps are inverses follows from the fact that each h_n is a permutation. $\square$

We would now like to embed $S_{\mathbb{N}}$ in MG and to do so we will define a map $\phi : S_{\mathbb{N}} \to MG$ by $\phi(\sigma_i) = \Psi(a_i)$. To ensure that ϕ is well-defined, we first insist on a normal form for elements in $S_{\mathbb{N}}$. Namely, given $\tau \in S_{\mathbb{N}}$, run the algorithm in Theorem 5.3 to get a word in $\{\sigma_i\}$ representing τ. It is clear that there is only one possible output from this algorithm for each τ, so we call this a normal form and require that every element be written this way when applying the map ϕ. It is clear that ϕ is a homomorphism.

Proposition 5.6. *The map ϕ is injective. In particular, the image $\phi(S_{\mathbb{N}})$ is precisely the bounded elements of the mutation group MG.*

Proof. We first note that for $\tau \in S_{\mathbb{N}}$, using this method, $\phi(\tau)$ is bounded. We do this by first redefining the map ϕ to $\bar{\phi} : S_{\mathbb{N}} \to H$ by $\bar{\phi}(\sigma_i) = a_i$ and define $h = \bar{\phi}(\tau)$ so that $h \in \Psi^{-1}(\phi(\tau))$. Then we write τ as a product of disjoint cycles. Then given $m \in \mathbb{N}$, there is exactly one cycle containing m and all other cycles do not effect m. It follows that if we let $M = \max\{\tau(k), \tau^{-1}(k) \mid k \leq m\}$ then for all $n > M$, $h_n(m) = \tau(m)$ and $h_n^{-1}(m) = \tau^{-1}(m)$. Then it follows that $\phi(\tau)$ is bounded and that $\sigma_{\phi(\tau)} = \tau$, which shows that ϕ is injective. To see that $\phi(S_{\mathbb{N}})$ is the set of all bounded elements of MG, we note that $\phi(\sigma_g) = g$ for all bounded g. $\square$

6 Finite Braid Groups

We will be discussing finite and infinite braid groups in this paper, so we establish the notation and terminology that we will be using to do so. For ease of discussion, the set $\{(1,0), (2,0), \ldots, (n,0)\}$ will be denoted by X_n, and I will denote the closed unit interval $[0,1]$.

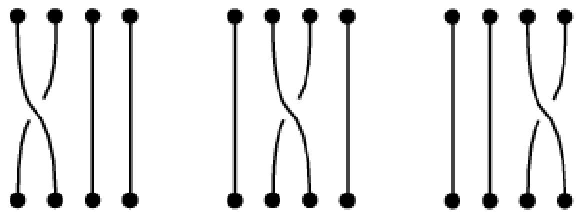
Definition 6.1. Given a fixed number $n \in \mathbb{N}$, a *braid on n strands* (or an n-braid) will be a continuous injective map $f : X_n \times I \to \mathbb{R}^2 \times I$ with the following properties:

1. $\pi_1(f(x,0)) = x$,

2. $\pi_1(f(x,1)) \in X_n$,

3. $\pi_2(f(x,t)) = t$ for all (x,t).

We will also call the image of f an n-braid. For each $1 \leq k \leq n$, we will call both the function $f\mid_{(k,0)\times I}$ and its image the k^{th} *strand* of the braid f. If f and g are braids,



σ_1 σ_2 σ_3

Figure 3: The first three elementary braids.

then define $f * g$ by

$$f * g(x,t) = \begin{cases} f(x, 2t) & t \in [0, 1/2] \\ g(x, 2t-1) & t \in [1/2, 1] \end{cases}.$$

In order to turn $*$ into a group law, we define the following equivalence relation.

Definition 6.2. If f and g are both n-braids, then we say that $f \sim g$ if there is a homotopy $H : X_n \times I \times I \to \mathbb{R}^2 \times I$ such that $H_0 = f$, $H_1 = g$, and H_t is a braid for all $t \in I$. In this case we say that the braids f and g are isotopic.

Now we are ready to define the braid group.

Definition 6.3. For a fixed $n \in \mathbb{N}$, define the braid group B_n to be the set of all equivalence classes of n-braids with the group law $[f][g] = [f * g]$.

Since it is well-known in classical braid group theory that this actually does turn B_n into a group, we will forgo the proof. Also for the ease of discussion, we will not distinguish between a braid, its image, and its equivalence class. In symmetric group theory we used the notation σ_i to denote the transposition $(i\ i+1)$. Here we will use it for the same idea, but as a braid rather than just a permutation. That is, σ_i will denote the braid that takes the i^{th} strand exactly once over the $(i+1)^{th}$ strand, as seen in Figure 3. Then we have the following classical result.

Proposition 6.4. *The braid group B_n is generated by $\{\sigma_1, \ldots, \sigma_{n-1}\}$ and has relators*

$$\left\{\sigma_i\sigma_{i+1}\sigma_i(\sigma_{i+1}\sigma_i\sigma_{i+1})^{-1}, [\sigma_i, \sigma_j] \mid |i-j| > 1\right\}.$$

Definition 6.5. Given a braid f we define its *endpoint evaluation* to be the permutation $\sigma_f \in S_n$ so that $\sigma_f(k) = \pi_1(f((k,0),1))$, where $\pi_1 : \mathbb{R}^2 \times I \to \mathbb{R}^2$ is the standard projection.

It is well-known that the above definition of endpoint evaluation gives an epimorphism $\varphi : B_n \to S_n$. The kernel of this map will be called the *pure braid group* and will be denoted by P_n. One more result that we will recall from classical braid group theory is that the braid group is isomorphic to the mapping class group of a punctured disk.

Definition 6.6. Let $X \subset \mathbb{R}^2$ be given. Define $\operatorname{Homeo}(X)$ to be the group of self-homeomorphisms of X. Let $\operatorname{Homeo}_0(X)$ be the self-homeomorphisms that are isotopic to the identity map. Then the *mapping class group* of X is the quotient $\operatorname{Homeo}(X)/\operatorname{Homeo}_0(X)$ and will be denoted by $\operatorname{MCG}(X)$.

Theorem 6.7. *The braid group B_n is isomorphic to $\operatorname{MCG}(\mathbb{R}^2 \setminus X_n)$.*

The last item to discuss is the idea of configuration spaces—that is, a space whose fundamental group is a given braid group.

Proposition 6.8. *Let $Z_n = \{(z_i) \in \mathbb{C}^n \mid z_i \neq z_j \text{ if } i \neq j\}$. Let S_n act on Z_n by permuting the coordinates naturally. Then $B_n \cong \pi_1(Z_n/S_n)$ and $P_n \cong \pi_1(Z_n)$.*

The way a braid is realized given an element of $\pi_1(Z_n)$ is that each coordinate traces out a path in $\mathbb{C}$. The restriction that each coordinate has a unique value prevents any of the strands from crossing each other. For the quotient space Z_n/S_n, we see that a loop can be thought of as a path in Z_n that ends at a permutation of its starting point.

7 Infinite Braid Groups

Definition 7.1. Let $X \subset \mathbb{R}^2$. Define the a *braid with strands at the points of* X to be a collection of continuous maps $\mathcal{F} = \{f_x : I \to \mathbb{R}^3 \mid x \in X\}$ with the following properties:

1. $f_x(0) = (x, 0)$,
2. $\pi_2(f_x(t)) = t$ for all t ($\pi_2 : \mathbb{R}^2 \times I \to I$ is the standard projection),
3. for all $t \in I$, $f_x(t) = f_y(t) \Rightarrow x = y$, and
4. the map $F : X \to X$ given by $F(x) = \pi_1(f_x(1))$ is a permutation.

The braid $\mathcal{F}$ is called *proper* if there is a continuous function $f : \mathbb{R}^3 \to \mathbb{R}^3$ such that $f(x, t) = f_x(t)$ for all $x \in X$ and all $t \in I$. Given braids $\mathcal{F}$ and $\mathcal{G}$ with strands in X, we say that $\mathcal{F} * \mathcal{G} = \{f_x * g_x\}$.

Definition 7.2. Two braids $\mathcal{F}$ and $\mathcal{G}$ with strands at $X \subset \mathbb{R}^2$ are said to be *isotopic* if for each $x \in X$ there is a homotopy $h_x : I \times I \to \mathbb{R}^2 \times I$ such that for all $t \in I$,

1. $h_x(t, 0) = f_x(t)$, $h_x(t, 1) = g_x(t)$ and
2. the collection $\{h_x(t, \cdot)\}$ is a braid.

In this case we write $\mathcal{F} \sim \mathcal{G}$.

The following are relatively elementary results and therefore the proof will be left as an exercise.

Proposition 7.3. *Braid isotopy $\sim$ is an equivalence relation.*

Proposition 7.4. *Given $X \subset \mathbb{R}^2$, the set $\{[\mathcal{F}]\}$ of equivalence classes of braids on X forms a group with multiplication $[\mathcal{F}][\mathcal{G}] = [\mathcal{F} * \mathcal{G}]$. Furthermore, the set $\{[\mathcal{F}] \mid \mathcal{F}$ is proper$\}$ is a subgroup.*

Given $X \subset \mathbb{R}^2$, we write B_X for the braid group and Pr_X for the proper braid group.

Proposition 7.5. *Given $X \subset \mathbb{R}^2$, the endpoint evaluation map $\varphi : B_X \to S_X$ is an epimorphism.*

The kernel of this map will be called the pure braid group and will be denoted P_X. The proper subgroup will be called the proper pure braid group and will be denoted PPr_X. We now extend the concept of configuration spaces to infinite braid groups. Unfortunately, the space $Z_\omega/S_\mathbb{N}$ is not Hausdorff and its fundamental group is not $B_\mathbb{N}$, but we use a slightly modified version of the fundamental group, similar to that of Fabel's [6].

Definition 7.6. Let $Z_\alpha = \{(z_i) \in \mathbb{C}^\alpha \mid z_i \neq z_j \text{ for } i \neq j\}$ and let S_α act on Z_α by permuting the coordinates. Given a fixed base point $z_0 \in Z_\alpha$, call a path from z_0 to a point σz_0 (for some $\sigma \in S_\alpha$) a *pseudo-loop* and let $PG(Z_\alpha, z_0)$ be the set of homotopy classes of all pseudo-loops. Given a pseudo-loop f, let σ_f denote the permutation such that $\sigma_f z_0$ is the endpoint of f. Given two pseudo-loops f and g, define the path $f * g$ by

$$f * g(t) = \begin{cases} f(2t) & t \in [0, 1/2] \\ \sigma_f g(2t-1) & t \in [1/2, 1] \end{cases}.$$

To justify that $f * g$ really is a path, we note that $\sigma_f g$ is continuous since it is merely a permutation of the coordinate functions of g, which must all be continuous. It is also easily seen that given pseudo-loops f and g, $\sigma_{f*g} = \sigma_f \sigma_g$.

Proposition 7.7. *The set $PG(Z_\alpha, z_0)$ is a group with multiplication $[f][g] = [f * g]$.*

Proof. Letting f_0 be the constant map $f_0(t) = z_0$, we see that $[f_0]$ acts as the identity. It is clear that $PG(Z_\alpha, z_0)$ is closed under multiplication as defined and it is also clear that multiplication is associative. Given $[f] \in PG(Z_\alpha, z_0)$, to see that it has an inverse we first construct the path $\bar{f}$ as in standard homotopy theory to be the reverse of f. That is $\bar{f}(t) = f(1-t)$. However, we see that this map does not begin at z_0, but rather at $\sigma_f z_0$ and therefore $[\bar{f}] \notin PG(Z_\alpha, z_0)$. To fix this problem, we define the path $\hat{f} = \sigma_f^{-1}\bar{f}$. Then it is clear that $[f * \hat{f}] = [\hat{f} * f] = 1$. $\square$

Theorem 7.8. *Given a set $X \subset \mathbb{R}^2$, let $z_0 \in Z_X$ be such that $\pi_x(z_0) = x$. Then $P_X \cong \pi_1(Z_X, z_0)$ and $B_X \cong PG(Z_X, z_0)$.*

Proof. The first result, for the pure braid groups, is an elementary extension of the finite case for configuration spaces. Therefore, we simply prove the second result. We will define a map $\varphi : B_X \to PG(Z_X, z_0)$ by first defining a map from the set of all braids on X to the set of all pseudo-loops in Z_X. Given a braid $\mathcal{F}$, define the pseudo-loop $g_\mathcal{F}$ by $g_\mathcal{F}(t) = (f_x(t))$, that is $\pi_x(g_\mathcal{F}) = f_x$. It is clear that $g_\mathcal{F}$ is continuous since each coordinate function is continuous. Furthermore, it is clear that $\sigma_{g_\mathcal{F}}$ is given by $\sigma_\mathcal{F}$, the endpoint evaluation of the braid $\mathcal{F}$. It is a straightforward argument to show that if $\mathcal{F} \sim \mathcal{G}$ then $g_\mathcal{F}$ and $g_\mathcal{G}$ are homotopic. Therefore, this defines an injective map $\varphi : B_X \to PG(Z_X, z_0)$. We see that this map is a homomorphism by noting that $g_{\mathcal{F}*\mathcal{G}} = g_\mathcal{F} * g_\mathcal{G}$. To show that it is an isomorphism, we give an inverse ψ which we construct in the following way. Given a pseudo-loop f, define the braid $\mathcal{F}_f = \{f_x\}$ by $f_x(t) = \pi_x(f(t))$. It is clear that given a braid $\mathcal{G}$, $\mathcal{F}_{g_\mathcal{G}} = \mathcal{G}$ and that given a pseudo-loop f, $g_{\mathcal{F}_f} = f$. Together with the fact that $f \sim g$ implies $\mathcal{F}_f \sim \mathcal{F}_g$, we get that ψ is an inverse. It is also clear that ψ is a homomorphism, so we get the isomorphism desired. $\square$

Corollary 7.9. *For sets $X, Y \subset \mathbb{R}^2$, if $|X| = |Y|$ then $B_X \cong B_Y$.*

Proof. We see that $Z_X = Z_Y$, and that $z_0 = (x)_x$ and $z_0' = (y)_y$ are merely different base points in Z_X, therefore there is a map $\varphi : PG(Z_X, z_0) \to PG(Z_Y, z_0')$ given by

$\varphi([f]) = [\alpha * f * \hat{\alpha}]$. A slight variation of the proof of the traditional change-of-base theorem will show that this map is indeed an isomorphism. Then by Theorem 7.8, we see that $B_X \cong B_Y$. $\square$

Now we explore some specific braid groups and their properties. Paul Fabel researched two different topological completions of the group B_∞, Artin's braid group on countably many strands. He found that one completion is the mapping class group of an infinitely punctured disk and the other is a much wilder braid group [6], [7]. We will discuss both of those in this paper.

Let $B_{\mathbb{N}}$ denote the braid group with strands at the points $X = \{(0, 1/n) \mid n \in \mathbb{N}\}$ and let $Pr_{\bar{\mathbb{N}}}$ be the proper braid group with strands at the points $\bar{X} = X \cup \{(0,0)\}$. Let $M_{\mathbb{N}}$ be the mapping class group $\mathrm{MCG}(\mathbb{R}^2 \setminus X)$. Then we see that $M_{\mathbb{N}}$ is the first of Fabel's braid groups, and we will see that $B_{\mathbb{N}}$ is the other.

Theorem 7.10. *There is a natural isomorphism $Pr_{\bar{\mathbb{N}}} \cong M_{\mathbb{N}}$.*

Proof. We first define a map F from the set of proper braids on $\bar{\mathbb{N}}$ to the set of self-homeomorphisms of $\mathbb{R}^2 \setminus \mathbb{N}$. Given a braid $g : \bar{\mathbb{N}} \times I \to \mathbb{R}^2 \times I$, extend g to $(\bar{\mathbb{N}} \times I) \cup (\mathbb{R}^2 \times 0)$ by letting $g|_{\mathbb{R}^2 \times 0}$ be the identity map. Then we see that we can complete g to all of $\mathbb{R}^2 \times I$. Define $F(g) = \pi_1 \circ g|_{\mathbb{R}^2 \setminus \mathbb{N}}$, so that $F(g)(x) = g(x, 1)$. Clearly $F(g * h) = F(g) \circ F(h)$ and $g \sim \mathrm{id}$ implies $F(g) \sim \mathrm{id}$, so we get a homomorphism $\varphi : Pr_{\bar{\mathbb{N}}} \to M_{\mathbb{N}}$. To show that φ has an inverse, define the map G so that given a self-homeomorphism f, we get a proper braid $G(f)$ as follows. The map f can be completed to a map on all of $\mathbb{R}^2$, which is nulisotopic in $\mathrm{Homeo}(\mathbb{R}^2)$, so let H be an isotopy from id to f. Then we define $G(f) = H|_{X \times I}$. It is clear that $G(f \circ g) = G(f) * G(g)$ and that $G(f) \sim \mathrm{id}$ if $f \sim \mathrm{id}$. Therefore, we get a homomorphism $\psi : M_{\mathbb{N}} \to Pr_{\bar{\mathbb{N}}}$. The proof is concluded by noting that $F(G(f)) \sim f$ and $G(F(f)) \sim f$. $\square$

We will use the notation σ_i consistent with finite braid group theory. Now we consider the subgroup $Pr'_{\bar{\mathbb{N}}} < Pr_{\bar{\mathbb{N}}}$ such that the limit strand is straight and will show that, just like with $S_{\mathbb{N}}$, every element of $Pr'_{\bar{\mathbb{N}}}$ can be expressed as an infinite word in the countable set $\{\sigma_i\}$. Now we note that by continuity, for any given i, there are only finitely many strands to the right of the i^{th} strand—that is, only finitely many that ever cross the i^{th} strand. Therefore, we can isotop f further so that anytime two strands cross, if the n^{th} strand is crossing the $(n+1)^{th}$ strand, they do so in the slice $(1/(n+1), 1/n) \times \mathbb{R} \times I$. The one exception is when a strand crosses all other strands, and therefore must cross the strand at the origin. In this case, we allow the crossing strand to go into the slice $(-\infty, 0) \times I$.

Now we note that there are only finitely many crossings in each slice and there are only countably many slices, so there must be a countable number of crossings. So we read from time 0 to time 1 and list all of the crossings in the order they occur. If there are multiple crossings at precisely the same time, we note that they must be more than one strand away from each other, so the order that we write does not matter. Then we see that we get a Hawaiian word in the letters $\{\sigma_i\}$. Since this word represents a legal element of the Hawaiian earring group, we can partition the interval into countably many subintervals I_α and define a path g so that $g|_{I_\alpha}$ is the standard loop around the circle $a_{i(\alpha)}$ in the Hawaiian earring.

We then define a braid $\hat{f}$ so that $\hat{f}|_{I_\alpha}$ is isotopic to $\sigma_{i(\alpha)}$. It is clear that any two braids that have the same crossings in the same order are isotopic, so we conclude that $f \sim \hat{f}$. Therefore, given a proper braid $f \in Pr_{\bar{\mathbb{N}}}$, we denote the word in $\{\sigma_i\}$ constructed here by $w(f)$. There are possibly infinitely many different such words, but we see that they are equivalent since the only choice to make was in which order to list the simultaneous crossings, which all commute infinitely with each other. Similarly to the way we defined the mutation group, we define the braid mutation group as follows.

Definition 7.11. Let N be the big normal subgroup of H generated by elements of the form

$$\left\{a_i a_{i+1} a_i (a_{i+1} a_i a_{i+1})^{-1}, [a_i, a_j] \mid |i-j| > 1\right\}.$$

Then let $\Phi : H \to H/N$ be the quotient map and let BMG be the quotient, which we will call the *braid mutation group.*

We see the reason for calling it the braid mutation group is that not every element represents a braid. Namely, the word $\sigma_1\sigma_2\sigma_3\cdots$ has the same problem as in the case of permutations, that it has no place to send the first strand. However, we get the following result linking the braid mutation group with $S_{\mathbb{N}}$ and $Pr'_{\bar{\mathbb{N}}}$.

Proposition 7.12. *The map $\phi : Pr'_{\bar{\mathbb{N}}} \to BMG$ given by $\phi(\sigma_i) = \Phi(a_i)$ is injective. Furthermore, if $\varphi : BMG \to MG$ is the standard quotient map then* $\operatorname{Im}\phi = \varphi^{-1}(MG)$*.*

Proof. From the above discussion, we see that the map is well-defined. It is therefore also a homomorphism and is injective since it respects the relators in $Pr_{\bar{\mathbb{N}}}$. Given a braid f with structure as defined above—that each crossing happens in its proper slice—we see that at any point between crossings all of the strands are at the points $\{(1/n, 0)\} \cup \{(0,0)\}$. Therefore if J is a subinterval of I whose endpoints are both between-crossing points, then we see that $f|_{\bar{\mathbb{N}}\times J}$ is also a pure braid. Therefore every contiguous subword of a proper braid is a proper braid □

8 Doubled Cone

The examples that we have thus far seen—with braid groups and symmetric groups—have all been cases of infinitely multiplicative groups. Meaning, that if there is a product f and any collection of factors of the product were trivial, then we could pretend

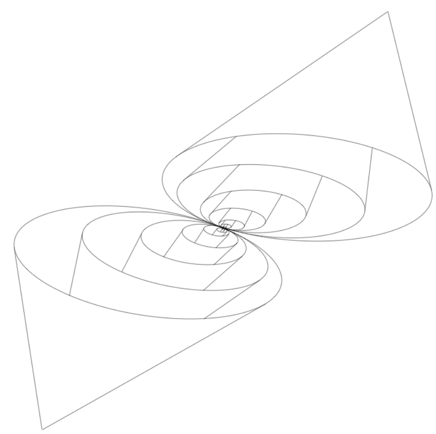


Figure 4: The doubled cone over the Hawaiian earring.

that they were not part of the product. To observe how wild infinite multiplication can get, we now look at examples where this is not the case.

Definition 8.1. Let C be the cone over the Hawaiian earring and let c_0 be the origin. Then let the space D be the wedge $C \vee_{c_0} C$. The space D will be called the *doubled cone over the Hawaiian earring* and its fundamental group $DH = \pi_1(D)$ will be called the *doubled cone group*.

Now, we see that the Hawaiian earring E embeds in the bases of these cones by mapping half of the loops to one cone and the other half to the other cone. That is, if we call the loops of the Hawaiian earring $\{a_i\}$ and the loops at the bases of each cone $\{b_i\}$ and $\{c_i\}$ respectively, then we map $f : a_{2i} \mapsto b_i$ and $f : a_{2i-1} \mapsto c_i$. This map induces a map f_* on fundamental group, which is clearly an epimorphism, so we get an infinite product structure on DH. It is clear that each of the loops $\{b_i, c_i\}$ is nulhomotopic, since it bounds a disk. It follows that $f_*(F_\infty) = \{1\}$. However, the group DH is not trivial. Indeed, it is uncountable. Since the map f_* is surjective, and F_∞ is dense in H we see that $\ker f_*$ is not closed and therefore f_* is not infinitely multiplicative. It is also clear that the infinite abelianization of DH is trivial, since the quotient map $\varphi : H \to AB(DH)$ would have the closure of F_∞ in its kernel, which is all of H.

9 Hawaiian Product

We have seen how the fundamental group of the Hawaiian earring can be thought of as an infinite extension of a free group. It is—intuitively—the free product on countably many generators where infinite multiplication is allowed, as long as only finitely many of each letter are used. We use this concept to define a type of generalization of the free product of groups.

We start this discussion with an example of a product that at first seems like it should work, but is indeed the wrong thing. Since a Hawaiian earring is a countable wedge of circles with a different topology—namely any neighborhood of the wedge point must contain all but finitely many of the circles—one might assume the way to tackle this problem is with the following construction. Given a countable collection of groups $\{G_n\}$ and a corresponding collection of spaces $\{X_n\}$ so that $\pi_1(X_n) = G_n$, define the space $X = \bigvee_n X_n$ with the topology that any neighborhood of the wedge point contains all but finitely many of the X_n. However, we use a simple example to show that this does not yield the desired construction. Let G_n be the trivial group and let X_n be the cone over the Hawaiian earring. Then we see that $\pi_1(X_n) = G_n = \{1\}$, but the space X contains a double cone over the Hawaiian earring, which has been shown to have uncountable fundamental group, so $\pi_1(X)$ is uncountable. Since one would expect any product of the trivial group to be trivial, we see that this definition would not work.

We now investigate this example in order to develop a construction that might yield an interesting product. We note that two coned Hawaiian earrings joined by an arc gives a contractible space, and is therefore simply connected. In general, if one takes a finite number of contractible spaces and attaches them by arcs, then the resulting space is contractible. With this in mind, we use the following construction.

Definition 9.1. Let $\{(X_i, x_i)\}$ be a collection of pointed spaces. Then for each X_i we construct $\tilde{X}_i$ by attaching a half-open interval to X_i, with the closed endpoint identified with x_i. We construct the space $X = \widetilde{\bigvee} X_i$ by adding one point x_0 to the disjoint union $\bigcup \tilde{X}_i$ so that a set $U \ni x_0$ is open if $U \cap X_i$ is open for all i and for all but finitely many i, $U \cap X_i = X_i$. That is, if each X_i is compact, then X is the one-point compactification of the disjoint union $\bigcup \tilde{X}_i$.

Definition 9.2. Given groups G_i and spaces (X_i, x_i) such that $G_i = \pi_1(X_i, x_i)$, we define the *Hawaiian product* $G = \widetilde{\prod} G_i$ by $G = \pi_1\left(\widetilde{\bigvee} X_i, x_0\right)$.

The reason we call this the Hawaiian product is that if we let $G_i = \mathbb{Z}$ and $X_i = S^1$, then the construction above of X yields a space homotopy equivalent to the Hawaiian earring. The first thing to prove is that this definition depends only on the groups themselves and not the spaces chosen. To do this, we prove that the Hawaiian product is the same as K. Eda's inifnitary product, which is discussed in Appendix A. The following theorem will also be proven in that appendix.

Theorem 9.3. *If $\{G_i\}$ is a collection of groups and $\{(X_i, x_i)\}$ is a collection of spaces such that $\pi_1(X_i, x_i) \cong G_i$ for all i, then $\pi_1\left(\widetilde{\bigvee} X_i, x_0\right) \cong \times\, G_i$.*

This theorem suffices to show that the Hawaiian product is indeed well-defined, which is expressed in the following corollary.

Corollary 9.4. *If $\{(Y_i, y_i)\}$ and $\{(X_i, x_i)\}$ are such that $\pi_1(X_i, x_i) \cong \pi_1(Y_i, y_i)$ for each i, then $\pi_1\left(\widetilde{\bigvee} X_i\right) = \pi_1\left(\widetilde{\bigvee} Y_i\right)$.*

This next result is merely a restatement of the fact that the infinitary product is an extension of the free product.

Corollary 9.5. *There is a natural monomorphism $\varphi : *G_i \to \widetilde{\prod} G_i$. Furthermore, if $\{G_i\}$ is a finite collection, then φ is an isomorphism.*

Now we use these tools to define an infinite product structure on a group.

Definition 9.6. Let G be a group and let $H = \widetilde{\prod}_{i \in G} G_i$ (where $G_i \cong G$). Then an infinite product structure on G is a homomorphism $F : H' \to G$ (where H' is a subgroup of H) that extends the natural map $f : *G_i \to G$. The words in H' will be called *legal words* and all the words in $H \setminus H'$ will be called *illegal words*.

We note that this definition is much more intuitive, since the elements of the Hawaiian product can be thought of as infinite words in G, and therefore infinite products actually look like infinite products, rather than in the case of the original definition of infinite product structure, where infinite products are elements of the Hawaiian earring group. However, the original definition is more closely in line with an extended definition of a group presentation, since one needs only list a set of relators and then perhaps describe some desired subset of the corresponding quotient.

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2

A Infinitary Product

Katsuya Eda defined an *infinitary product* of groups. In this paper we will define this infinitary product in a slightly different way, but so that the definitions agree.

Definition A.1. Let $\{G_i\}$ be a countable collection of groups. Then define an *infinite word* on the disjoint union $G = \bigcup G_i$ to be a function $f : \mathbb{Q} \to G$ such that the preimage $f^{-1}(G_i)$ is finite for each i. Let $\mathcal{W}(G)$ denote the set of all such functions.

The above defines infinite words on a collection of groups so that each group only has finitely many of its elements appear in any given word. This is reminiscent of the nature of the Hawaiian Earring group—that each letter in its weak generating set can be used only finitely often in any word. This is necessary because in each group G_i, only finite multiplication is defined. It also avoids the absurdity of infinite exponents. However, there are certain cases where we would want two infinite products to be the same. The simplest example is that we would like there to be only one identity element, so we would like to treat the identity in each group the same. Another thing is that we would like infinite words all of whose factors come from the same group to agree with finite multiplication. (In this case, obviously all but finitely many factors would be trivial, so the infinite word is really a finite word.) Finally, we would like to forget which indices we came from and only keep the order of the letters intact. For example, if $f \in \mathcal{W}(G)$, and $g(r) = f(r + s)$ for some constant $s \in \mathbb{Q}$, then we want to say that f and g are really the same word. Another thing is that we would like the trivial letter to always be trivial, so we can delete it wherever it appears no matter how many times it appears.

Definition A.2. Given a word $f \in \mathcal{W}(G)$ and a subset $A \subset \mathbb{Q}$, we say that $f \mid_A$ is a *pseudo-finite product* if $f(a) = 1$ for all but finitely many $a \in A$. If, in addition, there is an i for which $f(a) \in G_i$ for all $a \in A$ then we say that $f \mid_A$ is a product in G_i and we define as its value the finite product $g_1 g_2 \dots g_n$, where $g_1, \dots, g_n$ are the nonidentity elements of $f(A)$ and $i < j \Rightarrow g_i = f(q), g_j = f(r)$ with $q < r$ $(q, r \in A)$.

Definition A.3. We say that a word $f \in \mathcal{W}(G)$ admits a *cancellation* $*$ if there is a subset $A \subset \mathbb{Q}$ and a pairing $* : A \to A$ satisfying the following conditions:

1. $*$ is an involution of the set A.
2. $*$ is *complete* in the sense that $[a, a^*]_A = [a, a^*]_{\mathbb{Q}}$ for every $a \in A$ and *noncrossing* in the sense that $[a, a^*]_A = ([a, a^*]_A)^*$ for every $a \in A$.
3. $*$ is an *inverse pairing* in the sense that $f(a^*) = f(a)^{-1}$.

Definition A.4. Suppose that $f, g \in \mathcal{W}(G)$. Then we say that $f \sim g$ if there are partitions $\{A_i \mid i \in J \subset \mathbb{Q}\}$ and $\{B_i \mid i \in J' \subset \mathbb{Q}\}$ of $\mathbb{Q}$ and an order-preserving map $h : \mathbb{Q} \to \mathbb{Q}$ such that each of A_i, B_i is an interval in $\mathbb{Q}$, $A_i < A_j$ if and only if $B_i < B_j$, and for each i one of the following is true:

1. $f|_{A_i} = g|_{h(B_i)}$.
2. each of $f|_{A_i}$ and $g|_{h(B_i)}$ is pseudo-finite with the same value.
3. each of $f|_{A_i}$ and $g|_{h(B_i)}$ admits a cancellation or is pseudo-finite with value 1.

Then extend $\sim$ to be an equivalence relation.

Since each word in $\mathcal{W}(G)$ contains only finitely many of each G_i (excepting the identity), we see that given a word $f \in \mathcal{W}(G)$, there is a word $g \sim f$ such that whenever $g|_A$ is a pseudo-finite word in G_i and A is an interval in $\mathbb{Q}$, $g(a) \neq 1$ for only one $a \in A$. That is, we can evaluate any multiplication that takes place inside each individual group in the infinitary product.

Given a totally-ordered countable collection $\{f_\alpha \in \mathcal{W}(G) \mid \alpha \in J\}$, if for each i $f_\alpha^{-1}(G_i)$ is trivial for all but finitely many α, then we define the word $*f_\alpha$ as follows. First define the function $\bigsqcup f_\alpha : \mathbb{Q} \times J \to G$ by $\bigsqcup f_\alpha(q, \beta) = f_beta(q)$. Then we note that $\mathbb{Q} \times J$ with the lexicographic order has the same order type as $\mathbb{Q}$, so let $h : \mathbb{Q} \to \mathbb{Q} \times J$ be a homeomorphism and define $*f_\alpha = (\bigsqcup f_\alpha) \circ h$.

Definition A.5. Given a collection of group $\{G_i\}$, we define the infinitary product by $\times\, G_i = \{[f] \mid f \in \mathcal{W}(G)\}$ endow this set with multiplication given by $[f][g] = [f * g]$. In fact, allow infinite multiplication given by $\prod[f_\alpha] = [*f_\alpha]$.

Now we prove Theorem 9.3.

Theorem A.6. *If $\{G_i\}$ is a collection of groups and $\{(X_i, x_i)\}$ is a collection of spaces such that $\pi_1(X_i, x_i) \cong G_i$ for all i, then $\pi_1\left(\widetilde{\bigvee} X_i, x_0\right) \cong \times\, G_i$.*

Proof. In the construction of $\widetilde{\bigvee} X_i$ let p_i be the midpoint of the arc attached to X_i at x_i. Let f be a loop based at x_0. Then we see by continuity that for each i f travels from x_i to p_i only finitely often. Therefore, we may assume that each point between x_i and p_i has a finite preimage. Then we note that the wedge of arcs cut at the points $\{p_i\}$ is a tree and therefore contractible, so we can nulhomotop everything that f does in that region, and therefore assume that $f^{-1}(p_i)$ is finite for each i.

Next we see that the interval I can be broken into subintervals $\{I_\alpha = [a_\alpha, b_\alpha] \mid \alpha \in J\}$ such that $f(a_\alpha), f(b_\alpha) \in \{p_i\}$. If $f(a_\alpha) \neq f(b_\alpha)$, then we see that $f|_{I_\alpha}$ is nulhomotopic. Otherwise, $f|_{I_\alpha} \subset \tilde{X}_i$ (where $f(a_\alpha) = f(b_\alpha) = p_i$) and we see that

$[f|_{I_\alpha}] \in \pi_1(\tilde{X}_i, p_i) \cong \pi_1(X_i, x_i) \cong G_i$. Therefore, we define a function $F : J \to G$ by

$$F(\alpha) = \begin{cases} 1 & \text{if } f(a_\alpha) \neq f(b_\alpha) \\ [f|_{I_\alpha}]_{\tilde{X}_i} & \text{if } f(a_\alpha) = f(b_\alpha) = p_i \end{cases}.$$

Then we let $h : J \to \mathbb{Q}$ be an order-preserving map and define the function $\hat{F} : \mathbb{Q} \to G$ by $\hat{F}(q) = F(h^{-1}(q))$ if $q \in h(J)$ and $\hat{F}(q) = 1$ otherwise. We see that $\hat{F}^{-1}(G_i)$ is finite for all i by noting that $f^{-1}(p_i)$ is finite for all i. Therefore, $\hat{F} \in \mathcal{W}(G)$. Denote the map $f \mapsto \hat{F}$ by Ψ. By the definition described above, it is easy to see that if $f \sim g$ then $\Psi(f) \sim \Psi(g)$, so we can define the map $\psi : \pi_1\left(\widetilde{\bigvee} X_i, x_0\right) \to \times\, G_i$ by $\psi([f]) = [\Phi(f)]$.

Given an element $f \in \mathcal{W}(G)$ we define a loop in $\widetilde{X}_i$ as follows. Let $\{I_\alpha\}$ comprise the collection of intervals in the complement of the Cantor ternary set. This collection has the order type of the dyadic rationals, which is the same as $\mathbb{Q}$, so we will assume that $\{I_\alpha\}$ is indexed by $\mathbb{Q}$. For each nontrivial element $f(q)$, let f_q be a path in $\tilde{X}_{i(q)}$ representing the class $f(q)$. If $f(q)$ is trivial, then let f_q be the constant map x_0. Then we define the path F_f so that $F_f|_{I_q}$ is f_q appropriately scaled, and that $F_f(t) = x_0$ for t in the Cantor set. This is continuous since any neighborhood of a point in the image of F_f other than x_0 is contained inside a neighborhood whose preimage is $\operatorname{Int} I_q$ for some q, and any neighborhood of x_0 contains all but finitely many of the concatenated paths and it is easily seen that its preimage is open. Therefore, F_f is a loop. Denote the map $f \mapsto F_f$ by Φ.

We show that if $f \sim g$ then $\Phi(f) \sim \Phi(g)$. We do this by proving it in the case of each of 1 – 3 given in Definition A.4. So let $\{A_i\}$ and $\{B_i\}$ be partitions as dictated in that definition. We first define the set $I(X)$ to be the closure of the union $\bigcup_{q \in X} I_q$. In the case that $f|_{A_i} = g|_{h(B_i)}$ it is clear that $\Phi(f)|_{I(A_i)} \sim \Phi(g)|_{I(h(B_i))}$. In the case that each of f_{A_i} and $g|_{h(B_i)}$ is pseudo-finite with the same value, we see again that

$\Phi(f)|_{I(A_i)} \sim \Phi(g)|_{I(h(B_i))}$. In the last case of all, we see that each of $\Phi(f)|_{I(A_i)}$ and $\Phi(g)|_{I(h(B_i))}$ is nulhomotopic. Now to show that $\Phi(f) \sim \Phi(g)$, we insist that each individual homotopy take place in its respective space $\tilde{X}_i$. Then it follows that since any neighborhood of the base point contains all but finitely many of such homotopies, that the concatenated homotopy is also continuous.

So we define the map $\varphi : \divideontimes G_i \to \pi_1\left(\widetilde{\bigvee} X_i, x_0\right)$ by $\varphi([f]) = [\Phi(f)]$ and see that $\varphi \circ \psi$ is the identity on the Hawaiian product and $\psi \circ \varphi$ is the identity on the infinitary product, since $\Phi(\Psi(f)) \sim f$ and $\Psi(\Phi(g)) \sim g$. $\square$