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The Orbifold Landau-Ginzburg Conjecture for Unimodal and Bimodal Singularities

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by

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ABSTRACT

The Orbifold Landau-Ginzburg Conjecture for Unimodal and Bimodal Singularities

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The Orbifold Landau-Ginzburg Mirror Symmetry Conjecture states that for a quasi-homogeneous singularity W and a group G of symmetries of W, there is a dual singularity W^T and dual group G^T such that the orbifold A-model of W/G is isomorphic to the orbifold B-model of W^T/G^T. The Landau-Ginzburg A-model is the Frobenius algebra $\mathscr{H}_{W,G}$ constructed by Fan, Jarvis, and Ruan, and the B-model is the Orbifold Milnor ring of W^T. The unorbifolded conjecture has been verified for Arnol'd's list of simple, unimodal and bimodal quasi-homogeneous singularities with G the maximal diagonal symmetry group by Priddis, Krawitz, Bergin, Acosta, et al. [9], and by Fan-Shen [4] and Acosta [1] for all two dimensional invertible singularities and by Krawitz for all invertible singularities of 3 dimensions and greater in [8]. Based on this Krawitz posed the Orbifold Landau-Ginzburg Mirror Symmetry Conjecture, where the A-model is still the Frobenius algebra $\mathscr{H}_{W,G}$ constructed by Fan, Jarvis, and Ruan but constructed with respect to a proper subgroup G of the maximal group of symmetries G_W and the B-model is the orbifold Milnor ring of W^T orbifolded with respect to a non-trivial group K in SL_n of order $[G_W : \langle J \rangle]$. I verify this Orbifold Landau-Ginzburg Mirror Symmetry Conjecture for all unimodal and

bimodal quasi-homogeneous singularities in Arnol'd's list with $G = \langle J \rangle < G_W$, being the minimal admissible diagonal symmetry group. I also discuss some axioms and properties of these singularities.

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1 Introduction

1.1 Background and Motivation

In developing models in string theory, one often comes to a point where one must make an arbitrary choice between two alternatives. The two choices may lead to very different mathematical constructions–usually called an A-model and a B-model, and yet, since the choices are arbitrary, we expect the physics they describe to be the same. That means many of the final mathematical objects that are constructed should be equal, or isomorphic, or otherwise equivalent.

This phenomenon has led to many exciting new discoveries in algebraic and differential geometry. One such discovery is mirror symmetry. There are several types of mirror symmetry, but we are interested in the so-called Berglund-Huebsch mirror symmetry involving Landau-Ginzburg models.

The Landau-Ginzburg B-model is very well understood. Among other things, it takes a quasi-homogeneous polynomial, a polynomial with "weights" for each variable resulting in each term having "weighted" degree 1, with isolated singularities and associates to it the "Chiral ring", which is simply the Milnor ring $\mathbb{C}[x_1, \dots, x_n]/\left(\frac{\partial W}{\partial x_i}\right)$ of the singularity W.

Until very recently, no one knew how to construct the LG A-model mathematically. But in [5] the LG A-model was finally put on a solid mathematical foundation and many aspects of it were finally understood. Among other things, this A-model associates a ring to each quasi-homogeneous singularity called the FJRW ring. Both this FJRW ring and the (B-model) chiral ring are actually Frobenius algebras.

The general philosophy of mirror symmetry suggests that for a large class of polynomials W, there should be a corresponding mirror polynomial W^T so that the A-model of one is isomorphic to the B-model of the other. In the case of the B-model chiral ring and the A-model FJRW ring, the obvious conjecture is that for a large class of polynomials W, there is a choice of a mirror dual W^T so that $(W^T)^T = W$ and such that there is an isomorphism of Frobenius algebras between the FJRW ring of W and the chiral ring of W^T. Berglund and Huebsch described a construction of W^T for certain polynomials that was conjectured to provide the mirror dual.[3]

This conjecture was verified by Acosta in [1], Fan-Shen in [4] and Krawitz in [8]. That is, it was proved that the FJRW ring of W was isomorphic to the chiral (Milnor) ring of W^T and conversely. However, one key property of the A-model is that it depends not only on the singularity W but also on a group of admissible symmetries G. The FJRW ring actually depends heavily on the choice of the group G and in fact is graded by G.

The conjecture that was proved by Krawitz et al. was for the maximal symmetry group of W but did not involve any group on the B-side. However, recently an interesting physically motivated construction called "orbifolding" has been developed by Kaufmann and Krawitz for the B-side in [8]. For certain choices of a group of symmetries G of W, it constructs an orbifolded Chiral ring (orbifolded Milnor ring) which is graded by the group G. If the group is the trivial group, the construction reduces to the usual chiral (Milnor) ring.

Krawitz conjectured that for all the singularities that have an unorbifolded dual W^T described by Berglund-Huebsch and for all admissible groups G, there should be a dual group G^T so that the (A-model) FJRW ring for W and G is isomorphic as a Frobenius algebra, to the orbifolded (B-model) Chiral ring for W^T orbifolded by G^T.

In this paper I verify this conjecture for all unimodal and bimodal singularities with the minimal admissible group $\langle J \rangle$.

1.2 Overview of Results

In this paper I verify the Orbifold Landau-Ginzburg Mirror Symmetry Conjecture for Arnol'd's list of unimodal and bimodal singularities where $G = \langle J \rangle < G_W$ [2] based on certain restrictions to the correlators. These singularities are

Unimodal Singularities:

$$\begin{aligned} P_8 &: x^3 + y^3 + z^3 + axyz \\ X_9 &: x^4 + y^4 + bx^2y^2 \\ Q_{12} &: x^3 + y^5 + yz^2 \\ U_{12} &: x^3 + y^3 + z^4 \end{aligned}$$

Bimodal Singularities:

$$\begin{aligned} Z_{1,0} &: x^3y + y^7 \\ Q_{2,0} &: x^3 + xy^4 + yz^2 \\ S_{1,0} &: x^2z + yz^2 + y^5 \\ U_{16} &: x^3 + xz^2 + y^5 \\ Q_{16} &: x^3 + yz^2 + y^7 \end{aligned}$$

The original "unorbifolded" conjecture was proven for the simple and parabolic singularities in [5] and in [9] for Arnol'd's list of unimodal and bimodal singularities. As mentioned above, the complete conjecture was later proved by Fan-Shen, Acosta, and Krawitz in [1, 4, 8].

2 Construction

2.1 Review of Construction

For this paper W will always be a non-degenerate, quasi-homogeneous, invertible polynomial in variables $x_1, x_2, \ldots, x_N$.

Definition 2.1.1. A *quasi-homogeneous* polynomial W is a polynomial with weights $q_{x_1}, q_{x_2}, \ldots, q_{x_n}$ in $\mathbb{Q} \cap (0,1)$ such that any scalar $\lambda \in C^*$ satisfies

$$W(\lambda^{q_{x_1}} x_1, \lambda^{q_{x_2}} x_2, \ldots, \lambda^{q_{x_n}} x_n) = \lambda W(x_1, x_2, \ldots, x_n).$$

For example, the singularity known as $Q_{2,0}$, defined by the polynomial $x^3 + xy^4 + yz^2$, has weights $q_x = \frac{1}{3}, q_y = \frac{1}{6}, q_z = \frac{5}{12}$.

Definition 2.1.2. *Non-degeneracy* of a quasi-homogeneous polynomial requires that

- the weights be uniquely determined
- there is an isolated singularity at the origin.

Each quasi-homogeneous polynomial W determines a matrix of exponents B_W.

Definition 2.1.3. The ij entry of the B_W *matrix* is the exponent of x_j from the i-th monomial of the polynomial.

When the number of monomials equals the number of variables, the matrix B_W is square, and because of the non-degeneracy condition, B_W is invertible. In this case, rescaling the variables allows us to assume that all non-zero coefficients are 1, so the matrix completely determines the polynomial up to rescaling. As an example of this matrix representation $W \leftrightarrow B_W$, the singularity $Q_{2,0} : x^3 + xy^4 + yz^2$ has as its corresponding matrix

$$\begin{pmatrix} 3 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 1 & 2 \end{pmatrix}.$$

Definition 2.1.4. When W has the same number of variables as monomials, i.e., when B_W is square, we say that W is *invertible.*

Remark 2.1.5. It is known that when the weights are uniquely determined (as in our case), B_W has maximal rank, so B_W is an invertible matrix when it is square.

When W is invertible, the transpose matrix B_W^T corresponds to a different quasi-homogeneous polynomial. This new polynomial will be denoted W^T. Often W^T also has an isolated singularity at the origin.

Remark 2.1.6. For any invertible singularity, we can rescale the variables so that all non-zero coefficients are 1. Throughout this paper we will always make this rescaling.

Definition 2.1.7. For any invertible singularity W, the Berglund-Huebsch dual W^T is defined to be the polynomial with monomials determined by B^T (and with all non-zero coefficients equal to 1).

For example, $Q_{1,0}$ gives

$$B_{Q_{2,0}}^T = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 1 & 2 \end{pmatrix}^T = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

thus $Q_{2,0}^T : x^3y + y^4z + z^2$.

We need the Jacobean ideal to define both the A-model and the B-model rings.

Definition 2.1.8. The *Jacobian ideal* $\mathcal{J}$ is defined by

$$\mathcal{J} = \left(\frac{\partial W}{\partial x_1}, \frac{\partial W}{\partial x_2}, \dots, \frac{\partial W}{\partial x_N}\right).$$

Definition 2.1.9. The *Hessian of* W is defined by

$$\text{hess}(W) = \det\left(\tfrac{\partial^2 W}{\partial x_i \partial x_j}\right)$$

Definition 2.1.10. The *Milnor ring* $\mathcal{Q}_W$ of W, is defined to be

$$\mathcal{Q}_W := \mathbb{C}[x_1, x_2, \dots, x_N]/\mathcal{J}.$$

$\mathcal{Q}_W$ is finite dimensional as a vector space over $\mathbb{C}$ and the dimension as seen in [2] is

$$\mu = \prod_{j=1}^{N}\left(\frac{1}{q_j} - 1\right).$$

This ring $\mathcal{Q}_W$ is graded by weighted degree. The elements of the top degree form a one-dimensional subspace generated by hess(W). [9]

$\mathcal{Q}_W$ has a residue pairing $\langle f, g\rangle$ defined by

$$fg = \frac{\langle f, g\rangle}{\mu}\text{hess}(W) + \text{ lower order terms.}$$

for $f, g \in \mathcal{Q}_W$.

For $Q_{2,0}$ we see $\mathcal{J} = (3x^2 + y^4, 4xy^3 + z^2, 2yz)$ and so

$$\mathcal{Q}_{Q_{2,0}} = \mathbb{C}[x, y]/\mathcal{J} = \left\langle 1, x, x^2, y, y^2, y^3, z, z^2, xy, xy^2, x^2y, x^2y^2, xz, xz^2\right\rangle.$$

Definition 2.1.11. A *Frobenius algebra* is an algebra with a non-degnerate pairing $\langle , \rangle$ with the property that for all α, β, γ elements of the algebra we have $\langle \alpha\beta, \gamma \rangle = \langle \alpha, \beta\gamma \rangle$.

The Milnor ring with its residue pairing forms a graded Frobenius algebra.[9].

We will now define the construction of the (A-model) FJRW ring. To do this, we first need to choose an admissible group of diagonal symmetries.The choice of group determines the structure of the FJRW ring.

Definition 2.1.12. The *maximal group of diagonal symmetries* is given by

$$G_W = \left\{ (\alpha_1, \alpha_2, \ldots, \alpha_N) \subseteq (\mathbb{C}^*)^N \,|\, W(\alpha_1 x_1, \alpha_2 x_2, \ldots, \alpha_N x_N) = W(x_1, x_2, \ldots, x_N) \right\}.$$

Definition 2.1.13. For a quasi-homogeneous polynomial with weights $\{q_{x_i}\}$,the *exponential grading element* is $J = (e^{2\pi i q_{x_1}}, e^{2\pi i q_{x_2}}, \ldots, e^{2\pi i q_{x_N}})$.

G_W always contains the exponential grading element. In [9] the maximal symmetry group G_W was always used and corresponds on the B-side to the trivial group (the "unorbifolded" case). It is known that the group $\langle J \rangle$ is always admissible [5]. The computations in this paper always will use the cyclic group $\langle J \rangle$ generated by the exponential grading element.

Recall Krawitz conjectured that for all the singularities that have an unorbifolded dual W^T described by Berglund-Huebsch and for all admissible groups G, there should be a dual group G^T so that the (A-model) FJRW ring for W and G is isomorphic as a Frobenius algebra, to the orbifolded (B-model) Milnor ring.

Definition 2.1.14. For $h \in G$, $\operatorname{Fix} h \subset \mathbb{C}^N$ is the *fixed locus of* h. The dimension of this fixed locus will be denoted as N_h.

Definition 2.1.15. For any admissible group G and for each $h \in G$ we define

$$\mathscr{H}_h := \mathscr{Q}_{W|_{\mathrm{Fix}\, h}} \cdot \omega$$

where $\omega = dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_{N_h}}$ is the natural choice of volume form.

Note: The FJRW construction uses middle-dimensional relative homology of a Milnor fibration, but that construction is isomorphic to this one.

Definition 2.1.16. Choose a cyclic admissible group $G \leq G_W$ with generator a. If $Fix(a^k) = \{0\}$ then we define

$$e_k = 1 \in \mathscr{H}_{a^k} \cong \mathbb{C},$$

and if $\mathrm{Fix}\, a^k = \mathbb{C}x_{i_1} \oplus \cdots \oplus \mathbb{C}x_{i_{N_a}}$ define

$$e_k = dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_{N_a}} \in \mathscr{H}_{g^k}.$$

Note that for $a = J$ we have $\mathrm{Fix}(J^1) = \{0\}$.

Consider $Q_{2,0}$ again where $G = \langle J \rangle = \left\langle (e^{2\pi i \frac{1}{3}}, e^{2\pi i \frac{1}{6}}, e^{2\pi i \frac{5}{12}}) \right\rangle$.

$$\mathscr{H}_{J^k} = \begin{cases} \langle e_0, xe_0, x^2e_0, ye_0, y^2e_0, y^3e_0, ze_0, z^2e_0, xye_0, xy^2e_0, x^2ye_0, x^2y^2e_0, xze_0, xz^2e_0 \rangle & k = 0 \\ \langle e_6, xe_6, x^2e_6, ye_6, y^2e_6, y^3e_6, xye_6, xy^2e_6, x^2ye_6, x^2y^2e_6 \rangle & k = 6 \\ \langle e_k, xe_k \rangle & k = 3, 9 \\ \langle e_k \rangle & \text{otherwise.} \end{cases}$$

The group G acts on $\mathscr{H}_h$ by acting on the coordinates. We define the *h-sector* $\mathscr{H}_h^G$ to be the vector space of G-invariants of $\mathscr{H}_h$. The underlying vector space, often called the *state space*, of the FJRW-ring is defined to be

$$\mathscr{H}_{W,G} := \left(\bigoplus_{h \in G} \mathscr{H}_h \right)^G .$$

For $Q_{2,0}$ this vector space is $\mathscr{H}_{Q_{2,0},\langle J \rangle} = \langle e_1, e_2, e_4, e_5, y^3 e_6, xye_6, e_7, e_8, e_{10}, e_{11} \rangle$.

Definition 2.1.17. For each $h \in G$ we define $\Theta_i^h \in \mathbb{Q} \cap [0,1)$ by the fact that h can be uniquely expressed as

$$h = (e^{2\pi i \Theta_1^h}, e^{2\pi i \Theta_2^h}, \dots, e^{2\pi i \Theta_N^h})$$

Having considered Θ we can now talk about the W-degree of an element.

Definition 2.1.18. For any $h \in G$ and and α_h in the h-sector H_h^G, the *W-degree* of α_h is defined by

$$\deg_W(\alpha_h) := N_h + 2 \sum_{j=1}^{N} (\Theta_j^h - q_j) \tag{1}$$

when $\alpha_h \in (\mathscr{H}_h)^G$.

The space $\mathscr{H}_{W,G}$ is a complex vector space that is $\mathbb{Q}$-graded by this W-degree. Clearly the W-degree only depends on the G-grading.

Now we wish to define a pairing on the state space $\mathscr{H}_{W,G}$. To do this, note first that we have an isomorphism $I : \mathscr{H}_h \longrightarrow \mathscr{H}_{h^{-1}}$.

Definition 2.1.19. Define a pairing on $\mathscr{H}_h^G \otimes H_{h^{-1}}^G$ by $\langle \alpha, I^{-1}(\beta) \rangle$ for $\alpha \in \mathscr{H}_h^G$ and $\beta \in \mathscr{H}_{h^{-1}}^G$, and extend the pairing linearly to all of $H_{W,G}$. It can be shown that this pairing is non-degenerate on $\mathscr{H}_{W,G}$.

For a given choice of basis we denote by $\eta_{\alpha,\beta}$ the matrix representation of the pairing and by $\eta^{\alpha,\beta}$ the inverse of that matrix.

The multiplication for the Frobenius algebra is determined by the FJRW cohomological field theory.[5] This field theory produces classes $\Lambda^W_{g,n}(\alpha_1, \alpha_2, \dots, \alpha_n) \in H^*(\overline{\mathscr{M}}_{g,n})$ where $\overline{\mathscr{M}}_{g,n}$ is the stack of stable curves of genus g with n marked points. The classes $\Lambda^W_{g,n}$ have complex codimension D for each n-tuple $(\alpha_1, \alpha_2, \dots, \alpha_n) \in (\mathscr{H}_{W,G})^n$ where

$$D := \hat{c}_W(g-1) + \frac{1}{2}\sum_{i=1}^{n} \deg_W(\alpha_i)$$

and where

$$\hat{c}_W := \sum_i (1 - 2q_{x_i})$$

We do not need the entire cohomological field theory to define the FJRW ring, but we can use the genus-zero, three-point classes to define *correlators* which will determine the structure constants of the algebra.

Definition 2.1.20. We define the three-point correlators as follows:

$$\langle \alpha_1, \alpha_2, \alpha_3 \rangle_0^W := \int_{\overline{\mathscr{M}}_{0,3}} \Lambda^W_{0,3}(\alpha_1, \alpha_2, \alpha_3).$$

It is easy to see that $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ is nonzero only when its codimension D is zero because $\overline{\mathscr{M}}_{0,3}$ is a point.

When $g = 0$ and $n = 3$, then $D = 0$ if and only if $\sum_{i=1}^3 \deg_W \alpha_i = 2\hat{c}_W$.

The ring structure is given by these three-point correlators. Given $r, s \in \mathscr{H}_{W,G}$, their product is defined to be

$$r * s := \sum_{\alpha,\beta} \langle r, s, \alpha \rangle \, \eta^{\alpha,\beta} \beta \tag{2}$$

where the sum is taken over all choices of α and β in a fixed basis of $\mathscr{H}_{W,G}$.[5]

In [5] it is proved that the classes $\Lambda^W_{g,n}$ satisfy certain axioms that facilitate their computation. Below we provide a simplified form of these axioms that applies in the cases that we need to compute.

Axiom 1. *Dimension: If* $D \notin \frac{1}{2}\mathbb{Z}$*, then* $\Lambda^W_{g,n}(\alpha_1, \alpha_2, \ldots, \alpha_n) = 0$*. Otherwise,* D *is the complex codimension of the class* $\Lambda^W_{g,n}(\alpha_1, \alpha_2, \ldots, \alpha_n)$*. In particular, if* $g = 0$ *and* $n = 3$*, then* $\langle \alpha_1, \alpha_2, \alpha_3 \rangle = 0$ *unless* $D = 0$*.*

Axiom 2. *Symmetry: Let* $\sigma \in S_3$*. Then*

$$\langle \alpha_1, \alpha_2, \alpha_3 \rangle = \left\langle \alpha_{\sigma(1)}, \alpha_{\sigma(2)}, \alpha_{\sigma(3)} \right\rangle$$

The next few axioms rely on the degrees of line bundles $\mathscr{L}_1, \ldots, \mathscr{L}_N$ endowing an orbicurve with a so-called W*-structure*; however, this can be reduced to a simple numerical criterion. Consider the class $\Lambda^W_{g,n}(\alpha_1, \alpha_2, \ldots, \alpha_k)$, with $\alpha_j \in (\mathscr{H}_{h_j})^G$ for each j. For each variable x_j, define l_j by

$$l_j = q_j(2g - 2 + k) - \sum_{i=1}^{k} \Theta_j^{h_i}$$

Axiom 3. *Integer degrees: If* $l_j \notin \mathbb{Z}$ *for some* $j \in \{1 \ldots, N\}$*, then* $\Lambda^W_{g,n}(\alpha_1, \alpha_2, \ldots, \alpha_n) = 0$*.*

Axiom 4. *Concavity: If* $l_j < 0$ *for all* $j \in \{1, 2, 3\}$*, then* $\langle \alpha_1, \alpha_2, \alpha_3 \rangle = 1$*.*

The next axiom is related to the Witten map:

$$\mathcal{W} : \bigoplus_{j=1}^{N} \mathbb{C}^{h_j^0} \to \bigoplus_{j=1}^{N} \mathbb{C}^{h_j^1}$$
$$\mathcal{W} = \left(\overline{\frac{\partial W}{\partial x_1}}, \overline{\frac{\partial W}{\partial x_2}}, \ldots, \overline{\frac{\partial W}{\partial x_N}} \right)$$

where h_j^0 and h_j^1 are defined by

$$h_j^0 := \begin{cases} 0 & \text{if } l_j < 0 \\ l_j + 1 & \text{if } l_j \geq 0 \end{cases}$$

$$h_j^1 := \begin{cases} -l_j - 1 & \text{if } l_j < 0 \\ 0 & \text{if } l_j \geq 0 \end{cases}$$

so that both are non-negative integers satisfying $h_j^0 - h_j^1 = l_j + 1$. The fact that the Witten map is well-defined is a consequence of the geometric conditions on the $\mathscr{L}_j$ considered in [5]. For further details, we refer readers to the original paper.

In $\overline{\mathscr{M}}_{g,n}$, if $\Lambda^W_{g,n}(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a class of codimension zero, then these classes are constant and so, abusing notation, we will simply consider $\Lambda^W_{g,n}(\alpha_1, \alpha_2, \ldots, \alpha_n)$ to be a complex number. We will use this convention through the rest of the thesis.

Axiom 5. *Index Zero: Consider the class* $\Lambda^W_{g,n}(\alpha_1, \alpha_2, \ldots, \alpha_n)$*, with* $\alpha_i \in \mathscr{H}_{\gamma_i,G}$*. If* $\operatorname{Fix}\gamma_i = \{0\}$ *for each* $i \in \{1, 2, \ldots, n\}$ *and*

$$\sum_{j=1}^{N}(h_j^0 - h_j^1) = 0,$$

then $\Lambda_{g,n}(\alpha_1, \alpha_2, \ldots, \alpha_n)$ *is of codimension zero, and* $\Lambda^W_{g,n}(\alpha_1, \alpha_2, \ldots, \alpha_n)$ *is equal to the degree of the Witten map.*

Axiom 6. *Composition: If the four-point class* $\Lambda^W_{g,n}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ *is of codimension zero, then it decomposes as sums of three-point correlators in the following way:*

$$\Lambda^W_{0,4}(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \sum_{\beta,\delta} \langle \alpha_1, \alpha_2, \beta \rangle \, \eta^{\beta,\delta} \, \langle \delta, \alpha_3, \alpha_4 \rangle = \sum_{\beta,\delta} \langle \alpha_1, \alpha_3, \beta \rangle \, \eta^{\beta,\delta} \, \langle \delta, \alpha_2, \alpha_4 \rangle .$$

Note that $\operatorname{Fix} J = \{0\}$ so $\mathscr{H}_J \cong \mathbb{C}$. Let $\mathbb{1}$ be the element in $\mathscr{H}_J$ corresponting to $1 \in \mathbb{C}$. This element has $\deg_W(\mathbb{1}) = 0$ and it turns out to be the identity element in the FJRW-ring. The next axiom deals with this element.

Axiom 7. *Pairing: For* $\alpha_1, \alpha_2 \in \mathscr{H}_{W,G}$*,* $\langle \alpha_1, \alpha_2, \mathbb{1} \rangle = \eta_{\alpha_1,\alpha_2}$*.*

Axiom 8. *Sums of singularities: If $W_1 \in \mathbb{C}[x_1, \dots, x_r]$ and $W_2 \in \mathbb{C}[y_1, \dots, y_s]$ are two non-degenerate, quasi-homogeneous polynomials with maximal symmetry groups G_1 and G_2, then the maximal symmetry group of $W = W_1 + W_2$ is $G = G_1 \times G_2$, and there is an isomorphism of Frobenius algebras*

$$\mathscr{H}_{W,G} \cong \mathscr{H}_{W_1,G_{W_1}} \otimes \mathscr{H}_{W_2,G_{W_2}}$$

2.2 Orbifolded B-model construction

In the original "unorbifolded" Landau-Ginzburg conjecture the B-model of a singularity W is simply the Milnor ring of W, that is $\mathbb{C}[x_1, \dots, x_n]/\left(\frac{\partial W}{\partial x_i}\right)$. This Milnor ring is the same as orbifolding by the trivial group. Orbifolding for the B-models is a very similar construction for the FJRW-ring of the A-model. First we must choose a group K such that $K < G_W \cap \mathrm{SL}_n$ and find the fixed locus of every element in K. Restricting W to each fixed locus we can find the Milnor ring of that restriction. Using the same K-action as in the A-model, one may compute invariants of each of these restricted Milnor rings and sum these sectors over all the elements in K. This will give us the underlying vector space of the B-model orbifolded chiral ring, but we still need to define the multiplication in this new algebra.

2.2.1 Orbifold B-side multiplication

As discussed earlier, although the B-side as a vector space has been around for some time, its structure as a ring has only recently been developed. This section describes the B-side multiplication which was investigated in general by Kaufmann [6] and explicitly written out by Marc Krawitz in [8].

The underlying vector space of the Landau-Ginzburg orbifold B-model of W/G is defined to be

$$\mathcal{Q} = \bigoplus_{g \in K} \mathcal{Q}_g$$

where $\mathcal{Q}$ is a G-graded $\mathbb{C}$-vector space.

Now we are ready to define the multiplication on the B-model orbifolded chiral ring.

Definition 2.2.1. For $g \in K$, let $I_g = \{ i \mid g_i = 1_g \}$ and when let $N_g := \dim(\mathrm{Fix}(g))$.

$$\gamma_{g,h} \frac{\mathrm{Hess} W|_{\mathrm{Fix}(g) \cap \mathrm{Fix}(h)}}{\dim(\mathrm{Fix}(g) \cap \mathrm{Fix}(h))} = \begin{cases} \frac{\mathrm{Hess} W|_{\mathrm{Fix}(gh)}}{\dim(\mathrm{Fix}(gh))} & \text{If } I_g \cup I_h \cup I_{gh} = \{1, 2, \ldots, n\} \\ 0 & \text{otherwise.} \end{cases}$$

where $\frac{\mathrm{Hess}}{\dim} := 1$ if $\mathrm{Fix} = 0$.

So $\gamma_{g,h}$ is given by the determinant of the hessian of W on the newly fixed locus, provided each variable is fixed by at least one of g, h and gh. If $g = id$ or $h = id$ the newly fixed locus is empty then by convention the determinant of the empty (0×0) matrix is 0.

Let b_g denote the element 1 in the milnor ring Q_g.

Definition 2.2.2. We define the multiplication of the elements b_g and $b_h \in \mathscr{Q}$ by $b_g \star b_h = \gamma_{g,h} b_{gh}$ and extend to the rest of $\mathscr{Q}$ in the obvious way.

Note that if $e \in K$ is the identity in K then b_e is the multiplicative identity for this multiplication. This follows from the fact that

$\gamma_{e,g} = 1 = \gamma_{g,e}$.

This multiplication is associative, which was proved by Kaufmann in [6]-[7]. It suffices to check $\gamma_{g,h}\gamma_{gh,k} = \gamma_{g,hk}\gamma_{h,k}$.

The orbifolded Milnor ring of $Q^T_{2,0} : x^3y + y^4z + z^2$ is

$$\langle b_0, zb_0, x^2b_0, y^2b_0, xyb_0, y^2zb_0, xy^3b_0, xyzb_0, xy^3zb_0, b_1 \rangle$$

As an example of this multiplication where $\langle J \rangle^T = \langle (\alpha, \alpha, 1) \rangle$ when $\alpha^2 = 1$ consider

$b_1 * b_1 = \gamma_{1,1} b_0$ where

$$\begin{aligned}
\gamma_{1,1} \frac{\text{Hess} Q^T_{2,0}|_{\text{Fix}((\alpha,\alpha,1)) \cap \text{Fix}((\alpha,\alpha,1))}}{\dim(\text{Fix}((\alpha,\alpha,1)) \cap \text{Fix}((\alpha,\alpha,1)))} &= \frac{\text{Hess} Q^T_{2,0}|_{\text{Fix}((\alpha,\alpha,1)(\alpha,\alpha,1))}}{\dim(\text{Fix}((\alpha,\alpha,1)(\alpha,\alpha,1)))} \\
\gamma_{1,1} \frac{\text{Hess} Q^T_{2,0}|_{\mathbb{C}_z}}{\dim(\mathbb{C}_z \cap \mathbb{C}_z)} &= \frac{\text{Hess} Q^T_{2,0}|_{\mathbb{C}^3}}{\dim(\mathbb{C}^3)} \\
\gamma_{1,1} \frac{\text{Hess} Q^T_{2,0}|_{\mathbb{C}_z}}{1} &= \frac{\text{Hess} Q^T_{2,0}|_{\mathbb{C}^2}}{3} \\
\gamma_{1,1} 2 &= \frac{408 x y^3 z}{3} \\
\gamma_{1,1} &= \frac{204 x y^3 z}{3}.
\end{aligned}$$

So $b_1 * b_1 = \gamma_{1,1} b_0 = \frac{204}{3} x y^3 z b_0$.

2.3 Additional Notation

This paper discusses the Orbifold Landau-Ginzburg mirror symmetry conjecture for the invertible unimodal and bimodal singularities where the A side is orbifolded by $G = \langle J \rangle$ and the B side is orbifolded by $G^T \cap \text{SL}_n$, where $|G^T| = [G_W : \langle J \rangle]$.

Definition 2.3.1. A loop and a chain are polynomials defined as

$$W_{\text{loop}} := x_1^{a_1} x_2 + x_2^{a_2} x_3 + \cdots + x_n^{a_n} x_1$$

$$W_{\text{chain}} := x_1^{a_1} x_2 + x_2^{a_2} x_3 + \cdots + x_n^{a_n}$$

Definition 2.3.2. A singularity is *irreducible* if the polynomial associated to it is a loop or a chain.

Proposition 2.3.3. *The maximal symmetry group G_W is cyclic when W is an irreducible quasi-homogeneous invertible non-degenerate singularity.*

Proof. Since the singularity is irreducible it must either be a loop or a chain. Notice that the only difference between the loop and the chain is the last term.

Choose a $(\beta_1, \beta_2, \ldots, \beta_n) \in G_W$. By definition of G_W we have

$W(\beta_1 x_1, \beta_2 x_2, \ldots, \beta_n x_n) = W(x_1, x_2, \ldots, x_n)$. Therefore $\beta_i^{a_i}\beta_{i+1} = 1$ for every $1 \leq i \leq n-1$. Using this we can get each β_i entirely in terms of β_1 by $\beta_i = \beta_1^{\prod_{k=1}^{i-1}(-1)^{i-1}a_k}$.

So $(\beta_1, \beta_2, \ldots, \beta_n) = \left(\beta_1, \beta_1^{-a_1}, \beta_1^{a_1 a_2}, \ldots, \beta_1^{\prod_{k=1}^{n-1}(-1)^{n-1}a_k}\right)$.

For a loop $\beta_1 \in \mathbb{C}^*$ has order that divides $|1 + (-1)^{n-1}\prod_{k=1}^{n} a_k|$.

For a chain $\beta_1 \in \mathbb{C}^*$ has order that divides $\prod_{k=1}^{n} a_k$.

Let $\alpha_1 \in \mathbb{C}^*$ have order $|1 + (-1)^{n-1}\prod_{k=1}^{n} a_k|$ for a loop and have order $\prod_{k=1}^{n} a_k$ for a chain. Clearly $\left(\alpha_1, \alpha_1^{-a_1}, \alpha_1^{a_1 a_2}, \ldots, \alpha_1^{\prod_{k=1}^{n-1}(-1)^{n-1}a_k}\right)$ is an element of G_W.

We can see that β_1 would have to be some power q of α_1, giving

$$\begin{aligned}
(\beta_1, \beta_2, \ldots, \beta_n) &= \left(\beta_1, \beta_1^{-a_1}, \beta_1^{a_1 a_2}, \ldots, \beta_1^{\prod_{k=1}^{n-1}(-1)^{n-1}a_k}\right) \\
&= \left(\alpha_1^q, (\alpha_1^q)^{-a_1}, (\alpha_1^q)^{a_1 a_2}, \ldots, (\alpha_1^q)^{\prod_{k=1}^{n-1}(-1)^{n-1}a_k}\right) \\
&= \left(\alpha_1^q, (\alpha_1^{-a_1})^q, (\alpha_1^{a_1 a_2})^q, \ldots, (\alpha_1^{\prod_{k=1}^{n-1}(-1)^{n-1}a_k})^q\right) \\
&= \left(\alpha_1, \alpha_1^{-a_1}, \alpha_1^{a_1 a_2}, \ldots, \alpha_1^{\prod_{k=1}^{n-1}(-1)^{n-1}a_k}\right)^q.
\end{aligned}$$

Therefore G_W is cyclic with $\left(\alpha_1, \alpha_1^{-a_1}, \alpha_1^{a_1 a_2}, \ldots, \alpha_1^{\prod_{k=1}^{n-1}(-1)^{n-1}a_k}\right)$ as a generator. □

There are many non-degenerate invertible singularities that are reducible. These singularities are sums of loops and chains. G_W is a product of the cyclic groups for these loops and chains by axiom 8.

2.3.1 Orbifold Example $Q_{2,0}$

The charges for $Q_{2,0} = x^3 + xy^4 + yz^2$ are $q_x = \frac{1}{3}$ $q_y = \frac{1}{6}$, $q_z = \frac{5}{12}$. Therefore

$$\langle J \rangle = \left\langle \left(e^{\frac{2\pi i}{3}}, e^{\frac{2\pi i}{6}}, e^{\frac{10\pi i}{12}} \right) \right\rangle \cong \mathbb{Z}_{12}$$

For the orbifold B-side we will need to know the index of $\langle J \rangle$ in the maximal symmetry group of $Q_{2,0}$. In order to find the maximal symmetry group we consider $G_{Q_{2,0}} = \langle (\alpha, \beta, \gamma) \rangle$ such that $(\alpha x)^3 + (\alpha x)(\beta y)^4 + (\beta y)(\gamma z)^2 = x^3 + xy^4 + yz^2$. Thus we have $\alpha^3 = \alpha\beta^4 = \beta\gamma^2 = 1$ and $G_{Q_{2,0}} = \langle (\gamma^8, \gamma^{-2}, \gamma) \rangle \cong \mathbb{Z}_{24}$ when $\gamma^{24} = 1$

In order to find the fixed locus of J^k we consider what variables are fixed for $k \in \mathbb{Z}_{24} = \{0, 1, 2 \ldots, 23\}$. Everything will be fixed for $k = 0$ since $J^0 = (1, 1, 1)$. When $k = 6$ we have $J^6 = (1, 1, e^{5\pi i})$, so only the x and y values are fixed. Following this pattern for all values of k we get the following fixed locus

$$\text{Fix}J^k = \begin{cases} \mathbb{C}^3 & k = 0 \\ \mathbb{C}^2_{xy} & k = 6 \\ \mathbb{C}_x & k = 3, 9 \\ 0 & \text{otherwise.} \end{cases}$$

Restricting $Q_{2,0}$ to the fixed locus gives us

$$Q_{2,0}|_{\text{Fix}J^k} = \begin{cases} x^3 + xy^4 + yz^2 & k = 0 \\ x^3 + xy^4 & k = 6 \\ x^3 & k = 3, 9 \\ 0 & \text{otherwise.} \end{cases}$$

and thus the Milnor rings for these values of k gives

$$\mathcal{Q}|_{\mathrm{Fix}J^k} = \begin{cases} \langle 1, x, x^2, y, y^2, y^3, z, z^2, xy, xy^2, x^2y, x^2y^2, xz, xz^2 \rangle & k = 0 \\ \langle 1, x, x^2, y, y^2, y^3, xy, xy^2, x^2y, x^2y^2 \rangle & k = 6 \\ \langle 1, x \rangle & k = 3, 9 \\ \langle 1 \rangle & \text{otherwise.} \end{cases}$$

In order to find $\langle J \rangle$ invariants we consider $x^i y^j z^k dx \wedge dy \wedge dz$. This element is invariant if and only if $\frac{i}{3} + \frac{j}{6} + \frac{5k}{12} + \frac{1}{3} + \frac{1}{6} + \frac{5}{12} = 0 \bmod 1$. Similarly the element $x^i y^j dx \wedge dy$ is invariant if and only if $\frac{i}{3} + \frac{j}{6} + \frac{1}{3} + \frac{1}{6} = 0 \bmod 1$. So from the sector $k = 6$ we get the invariants $y^3 e_6$ and xye_6. The element 1 in each k sector will be expressed as e_k. For the sectors $k = 0, 3, 9$ there are no invariant elements. For all other k-sectors the only invariant element is e_k. Thus this gives us our table of elements

k	1	2	4	5	6	7	8	10	11
$\deg_W$	0	$\frac{11}{6}$	$\frac{3}{2}$	$\frac{4}{3}$	$\frac{7}{6}, \frac{7}{6}$	1	$\frac{5}{6}$	$\frac{1}{2}$	$\frac{7}{3}$
invariants	e_1	e_2	e_4	e_5	y^3e_6, xye_6	e_7	e_8	e_{10}	e_{11}

Table 1: $Q_{2,0}$ example for table of A side elements.

The only nonzero correlators by Axiom 1 are these the following.

Concavity axiom:

$\langle e_1, e_1, e_{11} \rangle$, $\langle e_1, e_7, e_5 \rangle$, $\langle e_1, e_8, e_4 \rangle$, $\langle e_1, e_{10}, e_2 \rangle$, $\langle e_{10}, e_8, e_7 \rangle$, $\langle e, e, e \rangle$ all equal 1.

Pairing axiom:

$\langle e_1, y^3e_6, y^3e_6 \rangle = -\frac{1}{4}$

$\langle e_1, xye_6, xye_6 \rangle = \frac{1}{12}$

Index zero axiom:

$\langle e_{10}, e_{10}, e_5 \rangle = -2$

Thus using these correlators we can compute all of the multiplication for this ring. Since $e_1 = \mathbf{1}$ is the identity in the ring multiplication with e_1 is trivial. The upper-half multiplication table is given as

	e_1	e_{10}	e_8	e_7	xye_6	y^3e_6	e_5	e_4	e_2	e_{11}
e_1	e_1	e_{10}	e_8	e_7	xye_6	y^3e_6	e_5	e_4	e_2	e_{11}
e_{10}		$-2e_7$	e_5	e_4	0	0	$-2e_2$	0	e_{11}	0
e_8			0	e_2	0	0	0	e_{11}	0	0
e_7				0	0	0	e_{11}	0	0	0
xye_6					$\frac{1}{12}e_{11}$	0	0	0	0	0
y^3e_6						$-\frac{1}{4}e_{11}$	0	0	0	0
e_5							0	0	0	0
e_4								0	0	0
e_2									0	0
e_{11}										0

Table 2: $Q_{2,0}$ example for multiplication table of A side elements.

Now we will construct the Orbifold B side. We can easily see that $Q_{2,0}^T = x^3y+y^4z+z^2$ and its maximal symmetry group is also $G_{Q_{2,0}}^T = \langle(\alpha, \alpha^{-3}, \alpha^{12})\rangle \cong \mathbb{Z}_{24}$ where $\alpha^{24} = 1$ The group K by which the B side is orbifolded must be of order $[G_W : \langle J\rangle]$ in G_W^T and also in $\mathrm{SL}_3(\mathbb{C})$. Since $G_{Q_{2,0}} \cong \mathbb{Z}_{24}$ and $\langle J\rangle \cong \mathbb{Z}_{12}$, $[G_{Q_{2,0}} : \langle J\rangle] = 2$. Therefore one such K is $K = \langle(\alpha^{12}, \alpha^{12}, 1)\rangle = \langle(\beta, \beta, 1)\rangle$ where $\beta^2 = 1$. For notational ease we denote K as $K = \langle m\rangle$.

Computing the fixed and Milnor ring locus in a similar way as before we get

$$\mathrm{Fix} m^k = \begin{cases} \mathbb{C}^2 & k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathcal{Q}|_{\mathrm{Fix} m^k} = \begin{cases} \langle 1, x, x^2, y, y^2, y^3, z, xy, xy^2, xy^3, xz, yz, y^2z, y^3z, xyz, xy^2z, xy^3z\rangle & k = 0 \\ \langle 1\rangle & \text{otherwise.} \end{cases}$$

Also computing K-invariance we get the following elements in Table 3 to be invariant.

k	0	1
$\deg_W$	$0, \frac{1}{2}, \frac{7}{12}, \frac{1}{4}, \frac{5}{12}, \frac{3}{4}, \frac{2}{3}, \frac{11}{12}, \frac{7}{6}$	$\frac{7}{12}$
invariants	$b_0, zb_0, x^2b_0, y^2b_0, xyb_0, y^2zb_0, xy^3b_0, xyzb_0, xy^3zb_0$	b_1

Table 3: $Q_{2,0}$ example for table of B side elements.

Let b_k represent the element 1 in the respective k sector.

Now for the B side multiplication, since b_0 is the identity in the ring its multiplication is trivial. In the Orbifold B-model multiplication section we already walked through a multiplication example for $Q_{2,0}^T$. Following this same process for all pairs we get

	b_0	y^2b_0	xyb_0	z_0	b_1	x^2b_0	xy^3b_0	y^2zb_0	$xyzb_0$	xy^3zb_0
b_0	b_0	y^2b_0	xyb_0	z_0	b_1	x^2b_0	xy^3b_0	y^2zb_0	$xyzb_0$	xy^3zb_0
y^2b_0		$-2zb_0$	xy^3b_0	y^2zb_0	0	0	$-2xyzb_0$	0	xy^3zb_0	0
xyb_0			0	$xyzb_0$	0	0	0	xy^3zb_0	0	0
z_0				0	0	0	xy^3zb_0	0	0	0
b_1					$\frac{204}{3}xy^3zb_0$	0	0	0	0	0
x^2b_0						$-4xy^3zb_0$	0	0	0	0
xy^3b_0							0	0	0	0
y^2zb_0								0	0	0
$xyzb_0$									0	0
xy^3zb_0										0

Now the FJRW ring for $Q_{2,0}$ and the orbifold ring for $Q_{2,0}^T$ are isomorphic as vector spaces just by sharing the same dimension and corresponding degrees. Therefore if they were to have the same multiplication table that would be enough to prove they are isomorphic as rings. So instead of having b_1 and x^2b_0 as elements we can scale them to be $\frac{1}{4\sqrt{51}}b_1$ and $\frac{1}{4}x^2b_0$ giving the upper half of the multiplication table as seen in Table 4.

	b_0	y^2b_0	xyb_0	z_0	$\frac{1}{4\sqrt{51}}b_1$	$\frac{1}{4}x^2b_0$	xy^3b_0	y^2zb_0	$xyzb_0$	xy^3zb_0
b_0	b_0	y^2b_0	xyb_0	z_0	$\frac{1}{4\sqrt{51}}b_1$	$\frac{1}{4}x^2b_0$	xy^3b_0	y^2zb_0	$xyzb_0$	xy^3zb_0
y^2b_0		$-2zb_0$	xy^3b_0	y^2zb_0	0	0	$-2xyzb_0$	0	xy^3zb_0	0
xyb_0			0	$xyzb_0$	0	0	0	xy^3b_0	0	0
z_0				0	0	0	xy^3zb_0	0	0	0
$\frac{1}{4\sqrt{51}}b_1$					$\frac{1}{12}xy^3zb_0$	0	0	0	0	0
$\frac{1}{4}x^2b_0$						$-\frac{1}{4}xy^3zb_0$	0	0	0	0
xy^3b_0							0	0	0	0
y^2zb_0								0	0	0
$xyzb_0$									0	0
xy^3zb_0										0

Table 4: $Q_{2,0}$ example for multiplication table of B side elements.

Since the FJRW ring A model multiplication table and the Chiral ring B-model multiplication table match exactly the rings are isomorphic.

2.4 Format of results

For each singularity, the information will be displayed in the following pattern:

- The name of the singularity will be given and also the polynomial that defines it, the Jacobian ideal, the weights associated to each variable, and the central charge. Also given will be the symmetry group used in the construction, $\langle J \rangle$.
- The fixed locus will be described for each group element.
- A basis for the Milnor ring of W restricted to each fixed locus will be given.
- Sectors with non-trivial J-invariants will be displayed in a table including the invariant elements and their W-degrees.
- Values of the three-point correlators that are not required to vanish by Axioms 1 and 2 will be given. There are some correlators that cannot be computed

from the axioms alone. These will be given variable labels.

- Multiplication table for both the A-side and B-side singularities will be given. For many singularities a system of equations will be shown in order to match these multiplication tables for an isomorphism. The solution to the systems will be given.

3 Computations

The examples are taken from the unimodal and bimodal singularities listed by Arnol'd. Many of these singularities are quasi-homogeneous only after fixing specific parameter values. This will be done without further comment.

3.1 Unimodal singularities

3.1.1 P_8

P_8 is normally the singularity $x^3 + y^3 + z^3 + axyz$ however this is not invertible. We will continue for the case where $a = 0$ making P_8 invertible.

A model: $P_8 : x^3 + y^3 + z^3$

$$\mathcal{J} = \langle 3x^2, 3y^2, 3z^2 \rangle$$

$$q_x = \frac{1}{3}, q_y = \frac{1}{3}, q_z = \frac{1}{3}$$

$$G_{P_8} = \langle (\alpha, \beta, \gamma) \rangle \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \text{ when } \alpha^3 = \beta^3 = \gamma^3$$

$$\text{Fix} J^k = \begin{cases} \mathbb{C}^3 & k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathcal{Q}|_{\mathrm{Fix}J^k} = \begin{cases} \langle 1, x, y, z, xy, xz, yz, xyz \rangle & k = 0 \\ \langle 1 \rangle & \text{otherwise.} \end{cases}$$

k	0	1	2
$\deg_W$	$1, 1$	0	2
invariants	$e_0, xyze_0$	e_1	e_2

Table 5: P_8 A side elements.

non-zero correlators:

Concavity axiom:

$\langle e_1, e_1, e_2 \rangle = 1$

Pairing axiom:

$\langle e_1, e_0, xyze_0 \rangle = \frac{1}{27}$

	e_1	e_0	$xyze_0$	e_2
e_1	e_1	e_0	$xyze_0$	e_2
e_0		0	$\frac{1}{27}e_2$	0
$xyze_0$			0	0
e_2				0

Table 6: P_8 A side multiplication.

<u>B model</u>:$P_8^T : x^3 + y^3 + z^3$

$$\mathcal{J} = \langle 3x^2, 3y^2, 3z^2 \rangle$$

$$q_x = \frac{1}{3}, q_y = \frac{1}{3}, q_z = \frac{1}{3}$$

$$K = \langle (\alpha^2, 1, \alpha), (\alpha^2, \alpha, 1) \rangle = \langle m, n \rangle \text{ when } \alpha^3 = 1$$

$$\mathrm{Fix}(m^k n^j) = \begin{cases} \mathbb{C}^3 & k=0, j=0 \\ \mathbb{C}_z & k=0, j=1 \\ \mathbb{C}_z & k=0, j=2 \\ \mathbb{C}_y & k=1, j=0 \\ 0 & k=1, j=1 \\ \mathbb{C}_x & k=1, j=2 \\ \mathbb{C}_y & k=2, j=0 \\ \mathbb{C}_x & k=2, j=1 \\ 0 & k=2, j=2 \end{cases}$$

$$\mathcal{Q}|_{\mathrm{Fix}(m^k n^j)} = \begin{cases} \langle 1, x, y, z, xy, xz, yz, xyz \rangle & k=0, j=0 \\ \langle 1, z \rangle & k=0, j=1 \\ \langle 1, z \rangle & k=0, j=2 \\ \langle 1, y \rangle & k=1, j=0 \\ \langle 1 \rangle & k=1, j=1 \\ \langle 1, x \rangle & k=1, j=2 \\ \langle 1, y \rangle & k=2, j=0 \\ \langle 1, x \rangle & k=2, j=1 \\ \langle 1 \rangle & k=2, j=2 \end{cases}$$

k, j	$0, 0$	$1, 1$	$2, 2$
$\deg_W$	$0, 1$	$\frac{1}{2}$	$\frac{1}{2}$
invariants	$b_{0,0}, xyzb_{0,0}$	$b_{1,1}$	$b_{2,2}$

Table 7: P_8 B side elements.

	$b_{0,0}$	$b_{1,1}$	$\frac{1}{1944}b_{2,2}$	$xyzb_{0,0}$
$b_{0,0}$	$b_{0,0}$	$b_{1,1}$	$\frac{1}{1944}b_{2,2}$	$xyzb_{0,0}$
$b_{1,1}$		0	$\frac{1}{27}xyzb_{0,0}$	0
$\frac{1}{1944}b_{2,2}$			0	0
$xyzb_{0,0}$				0

Table 8: P_8 B side multiplication.

3.1.2 X_9

For $X_9 : x^4 + y^4 + bx^2y^2$, X_9 is not invertible as written so we must have $b = 0$.

Amodel: $X_9 : x^4 + y^4$

$$\mathcal{J} = \langle 4x^3, 4y^3 \rangle$$

$$q_x = \frac{1}{4}, q_y = \frac{1}{4}$$

$$G_{X_9} = \langle \alpha, \beta \rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_4 \text{ when } \alpha^4 = \beta^4 = 1$$

$$\text{Fix}J^k = \begin{cases} \mathbb{C}^2 & k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathcal{Q}|_{\text{Fix}J^k} = \begin{cases} \langle 1, x, x^2, y, y^2, xy, xy^2, x^2y, x^2y^2 \rangle & k = 0 \\ \langle 1 \rangle & \text{otherwise.} \end{cases}$$

k	0	1	2	3
$\deg_W$	$1, 1, 1$	0	1	2
invariants	x^2e_0, y^2e_0, xye_0	e_1	e_2	e_3

Table 9: X_9 A side elements.

non-zero correlators:

Concavity axiom:

$\langle e_1, e_1, e_3 \rangle$ and $\langle e_1, e_2, e_2 \rangle$ both equal 1.

Pairing axiom:

$\langle e_1, x^2e_0, y^2e_0\rangle$ and $\langle e_1, xye_0, xye_0\rangle$ both equal $\frac{1}{16}$.

	e_1	x^2e_0	y^2e_0	xye_0	e_2	e_3
e_1	e_1	x^2e_0	y^2e_0	xye_0	e_2	e_3
x^2e_0		0	$\frac{1}{16}e_3$	0	0	0
y^2e_0			0	0	0	0
xye_0				$\frac{1}{16}e_3$	0	0
e_2					e_3	0
e_3						0

Table 10: X_9 A side multiplication.

Bmodel: $X_9^T : x^4 + y^4$

$$\mathcal{J} = \langle 4x^3, 4y^3\rangle$$

$$q_x = \frac{1}{4}, q_y = \frac{1}{4}$$

$$K = \langle m\rangle = \langle \alpha^3, \beta\rangle \text{ when } \alpha^4 = \beta^4 = 1$$

$$\text{Fix}m^k = \begin{cases} \mathbb{C}^4 & k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathcal{Q}|_{\text{Fix}m^k} = \begin{cases} \langle 1, x, x^2, y, y^2, xy, xy^2, x^2y, x^2y^2\rangle & k = 0 \\ \langle 1\rangle & \text{otherwise.} \end{cases}$$

k	0	1	2	3
$\deg_W$	$0, \frac{1}{2}, 1$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
invariants	$b_0, xyb_0, x^2y^2b_0$	b_1	b_2	b_3

Table 11: X_9 B side elements.

	b_0	$\frac{1}{24\sqrt{2}}b_1$	b_3	$\frac{1}{24\sqrt{2}}b_2$	xyb_0	$x^2y^2b_0$
b_0	b_0	$\frac{1}{24\sqrt{2}}b_1$	b_3	$\frac{1}{24\sqrt{2}}b_2$	xyb_0	$x^2y^2b_0$
$\frac{1}{24\sqrt{2}}b_1$		0	$\frac{1}{16}x^2y^2b_0$	0	0	0
b_3			0	0	0	0
$\frac{1}{24\sqrt{2}}b_2$				$\frac{1}{16}x^2y^2b_0$	0	0
xyb_0					$x^2y^2b_0$	0
$x^2y^2b_0$						0

Table 12: X_9 B side multiplication.

3.1.3 Q_{12}

A model: $Q_{12} : x^3 + y^5 + yz^2$

$$\mathcal{J} = \langle 3x^2, 5y^4 + z^2, 2yz\rangle$$

$$q_x = \frac{1}{3}, q_y = \frac{1}{5}, q_z = \frac{2}{5}$$

$$G_{Q_{12}} = \langle(\alpha, \gamma^{-2}, \gamma)\rangle \cong \mathbb{Z}_{30} \text{ when } \alpha^3 = \gamma^{10} = 1$$

$$\mathrm{Fix}J^k = \begin{cases} \mathbb{C}^3 & k = 0 \\ \mathbb{C}_x & 3|k \\ \mathbb{C}^2_{yz} & 5|k \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathcal{Q}|_{\mathrm{Fix}J^k} = \begin{cases} \langle 1, x, y, y^2, y^3, y^4, z, xy, xy^2, xy^3, xy^4, xz\rangle & k = 0 \\ \langle 1, x\rangle & 3|k \\ \langle 1, y, y^2, y^3, y^4, z\rangle & 5|k \\ \langle 1\rangle & \text{otherwise.} \end{cases}$$

k	1	2	4	5	7	8	10	11	13	14
$\deg_W$	0	$\frac{28}{15}$	$\frac{8}{5}$	$\frac{22}{15}, \frac{22}{15}$	$\frac{6}{5}$	$\frac{16}{15}$	$\frac{4}{5}, \frac{4}{5}$	$\frac{2}{3}$	$\frac{2}{5}$	$\frac{34}{15}$
invariants	e_1	e_2	e_4	y^2e_5, ze_5	e_7	e_8	y^2e_{10}, ze_{10}	e_{11}	e_{13}	e_{14}

Table 13: Q_{12} A side elements.

Potential non-zero correlators:

Concavity axiom:

$\langle e_1, e_1, e_{14}\rangle$, $\langle e_1, e_8, e_7\rangle$, $\langle e_1, e_{11}, e_4\rangle$, $\langle e_1, e_{13}, e_2\rangle$ $\langle e_{13}, e_{11}, e_7\rangle$ all equal 1

Pairing axiom:

$\langle e_1, y^2e_{10}, y^2e_5\rangle = \frac{1}{10}$

$\langle e_1, ze_{10}, ze_5\rangle = -\frac{1}{2}$

Correlator equation:

$-2 = -2\langle e_{13}, e_{13}, ze_5\rangle\langle e_{13}, ze_{10}, e_8\rangle + 10\langle e_{13}, e_{13}, y^2e_5\rangle\langle e_{13}, y^2e_{10}, e_8\rangle$

Correlators we cannot determine with axioms alone:

$\langle e_{13}, e_{13}, ze_5\rangle = a_1$

$\langle e_{13}, ze_{10}, e_8\rangle = a_2$

$\langle e_{13}, e_{13}, y^2e_5\rangle = a_3$

$\langle e_{13}, y^2e_{10}, e_8\rangle = a_4$

$\langle e_{11}, y^2e_{10}, y^2e_{10}\rangle = a_5$

$\langle e_{11}, ze_{10}, y^2e_{10}\rangle = a_6$

$\langle e_{11}, ze_{10}, ze_{10}\rangle = a_7$

	e_1	e_{13}	e_{11}	ze_{10}	y^2e_{10}	e_8	e_7	ze_5	y^2e_5	e_4	e_2	e_{14}
e_1	e_1	e_{13}	e_{11}	ze_{10}	y^2e_{10}	e_8	e_7	ze_5	y^2e_5	e_4	e_2	e_{14}
e_{13}		$-2a_1ze_{10} + 10a_3y^2e_{10}$	e_8	a_2e_7	a_4e_7	$-2a_2ze_5 + 10a_4y^2e_5$	e_4	a_1e_2	a_3e_2	0	e_{14}	0
e_{11}			0	$-2a_7ze_5 + 10a_6y^2e_5$	$-2a_6ze_5 + 10a_5y^2e_5$	0	e_2	0	0	e_14	0	0
ze_{10}				a_7e_4	a_6e_4	a_2e_2	0	$-\frac{1}{2}e_{14}$	0	0	0	0
y^2e_{10}					a_5e_4	a_4e_2	0	0	$\frac{1}{10}e_{14}$	0	0	0
e_8						0	e_{14}	0	0	0	0	0
e_7							0	0	0	0	0	0
ze_5								0	0	0	0	0
y^2e_5									0	0	0	0
e_4										0	0	0
e_2											0	0
e_{14}												0

Table 14: Q_{12} A side multiplication.

B model: $Q_{12}^T : x^3 + y^5 z + z^2$

$$\mathcal{J} = \langle 3x^2, 5y^4 z, y^5 + 2z \rangle$$

$$q_x = \frac{1}{3}, q_y = \frac{1}{10}, q_z = \frac{1}{2}$$

$$K = \langle m \rangle = \langle (1, \beta^{15}, \beta^{15}) \rangle \text{ when } \beta^{10} = 1$$

$$\text{Fix} m^k = \begin{cases} \mathbb{C}^3 & k = 0 \\ \mathbb{C}_x & k = 1 \end{cases}$$

$$\mathcal{Q}|_{\text{Fix} m^k} = \begin{cases} \langle 1, x, y, y^2, y^3, y^4, z, xy, xy^2, xy^3, xy^4, xz, yz, y^2z, y^3z, xyz, xy^2z, xy^3z \rangle & k = 0 \\ \langle 1, x \rangle & k = 1 \end{cases}$$

k	0	1
$\deg_W$	$0, \frac{1}{3}, \frac{1}{5}, \frac{2}{5}, \frac{8}{15}, \frac{3}{5}, \frac{4}{5}, \frac{11}{15}, \frac{14}{15}, \frac{17}{15}$	$\frac{2}{5}, \frac{11}{15}$
invariants	$b_0, xb_0, y^2b_0, y^4b_0, xy^2b_0, xy^4b_0, yzb_0, y^3zb_0, xyzb_0, xy^3zb_0$	b_1, xb_1

Table 15: Q_{12} B side elements.

For Table 16 we will let

$$\begin{aligned} \alpha &= eb_1 + fy^4b_0 \\ \beta &= gb_1 + hy^4b_0 \\ \gamma &= axb_1 + bxy^4b_0 \\ \theta &= cxb_1 + dxy^4b_0. \end{aligned}$$

	b_0	y^2b_0	xb_0	α	β	xy^2b_0	yzb_0	γ	θ	y^3zb_0	$xyzb_0$	xy^3zb_0
b_0	b_0	y^2b_0	xb_0	β	xy^2b_0	yzb_0	γ	θ	y^3zb_0	$xyzb_0$	xy^3zb_0	
y^2b_0		$\frac{-g}{eh-gf}\alpha+$ $\frac{e}{eh-gf}\beta$	xy^2b_0	$-2fyzb_0$	$-2hyzb_0$	$\frac{-c}{ad-cb}\gamma+$ $\frac{a}{ad-cb}\theta$	y^3zb_0	$-2bxyzb_0$	$-2dxyzb_0$	0	xy^3zb_0	0
xb_0			0	$\frac{de-fc}{ad-cb}\gamma+$ $\frac{af-eb}{ad-cb}\theta$	$\frac{dg-hc}{ad-cb}\gamma+$ $\frac{ah-gb}{ad-cb}\theta$	0	$xyzb_0$	0	0	xy^3zb_0	0	0
α				$(30e^2-2f^2)y^3zb_0$	$(30eg-2fh)y^3zb_0$	$-2fxyzb_0$	0	$-\frac{1}{2}xy^3zb_0$	0	0	0	0
β					$(30g^2-2h^2)y^3zb_0$	$-2hxyzb_0$	0	0	$\frac{1}{10}xy^3zb_0$	0	0	0
xy^2b_0						0	xy^3zb_0	0	0	0	0	0
yzb_0							0	0	0	0	0	0
γ								0	0	0	0	0
θ									0	0	0	0
y^3zb_0										0	0	0
$xyzb_0$											0	0
xy^3zb_0												0

Table 16: Q_{12} B side multiplication

System of equations found by matching the multiplication tables:

$$
\begin{aligned}
a_1 &= -2b = \frac{g}{2(eh - gf)} \\
a_2 &= -2f = \frac{c}{2(ad - cb)} \\
a_3 &= -2d = \frac{e}{10(eh - gf)} \\
a_4 &= -2h = \frac{a}{10(ad - cb)} \\
a_5 &= 30g^2 - 2h^2 = \frac{ah - gb}{10(ad - cb)} \\
a_6 &= 30eg - 2fh = \frac{af - eb}{10(ad - cb)} = \frac{dg - hc}{-2(ad - cb)} \\
a_7 &= 30e^2 - 2f^2 = \frac{de - fc}{-2(ad - cb)} \\
30ae - 2fb &= -\frac{1}{2} \\
30ce - 2df &= 0 \\
30ag - 2bf &= 0 \\
30cg - 2dh &= \frac{1}{10}
\end{aligned}
$$

Solution to these equations in terms of the a_is:

$$
\begin{aligned}
a &= \frac{a_4 a_1}{2\sqrt{30a_5 + 15a_4^2}} \\
b &= -\frac{a_1}{2} \\
c &= \frac{a_2 a_1}{10\sqrt{30a_5 + 15a_4^2}} \\
d &= -\frac{a_3}{2} \\
e &= \frac{a_3\sqrt{30 * a_5 + 15 * a_4^2}}{6a_1} \\
f &= -\frac{a_2}{2}
\end{aligned}
$$

$$
\begin{aligned}
g &= \frac{1}{30}\sqrt{30a_5 + 15a_4^2} \\
h &= -\frac{a_4}{2}
\end{aligned}
$$

With the relations:

$$
\begin{aligned}
a_3 &= \frac{a_1a_2 - 1}{5a_4} \\
a_6 &= \frac{2a_1a_2a_5 - 2a_5 - a_4^2}{2a_1a_4} \\
a_7 &= \frac{2a_2^2a_1^2a_5 - 4a_1a_2a_5 - 2a_2a_4^2a_1 + 2a_5 + a_4^2}{2a_4^2a_1^2}
\end{aligned}
$$

3.1.4 U_{12}

A model: $U_{12} : x^3 + y^3 + z^4$

$$\mathcal{J} = \langle 3x^2, 3y^2, 4z^3 \rangle$$

$$q_x = \frac{1}{3}, q_y = \frac{1}{3}, q_z = \frac{1}{4}$$

$$G_{U_{12}} = \langle(\alpha, \beta, \gamma)\rangle \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_4 \text{ when } \alpha^3 = \beta^3 = \gamma^4 = 1$$

$$\text{Fix}J^k = \begin{cases} \mathbb{C}^3 & k = 0 \\ \mathbb{C}^2_{xy} & 3|k \\ \mathbb{C}_z & 4|k \\ 0 & \text{else.} \end{cases}$$

$$\mathcal{Q}|_{\text{Fix}J^k} = \begin{cases} \langle 1, x, y, z, z^2, xy, xz, xz^2, yz, yz^2, xyz, xyz^2 \rangle & k = 0 \\ \langle 1, x, y, xy \rangle & 3|k \\ \langle 1, z, z^2 \rangle & 4|k \\ \langle 1 \rangle & \text{else.} \end{cases}$$

k	1	2	3	5	6	7	9	10	11
$\deg_W$	0	$\frac{11}{6}$	$\frac{5}{3}, \frac{5}{3}$	$\frac{4}{3}$	$\frac{7}{6}, \frac{7}{6}$	1	$\frac{2}{3}, \frac{2}{3}$	$\frac{1}{2}$	$\frac{7}{3}$
invariants	e_1	e_2	xe_3, ye_3	e_5	xe_6, ye_6	e_7	xe_9, ye_9	e_{10}	e_{11}

Table 17: U_{12} A side elements.

Potential non-zero correlators:

Concavity axiom:

$\langle e_1, e_1, e_{11}\rangle$, $\langle e_1, e_7, e_5\rangle$, $\langle e_1, e_{10}, e_2\rangle$, $\langle e_{10}, e_{10}, e_5\rangle$ are all equal to 1

Pairing axiom:

$\langle e_1, xe_9, ye_3\rangle$, $\langle e_1, ye_6, xe_6\rangle$, $\langle e_1, ye_9, xe_3\rangle$ are all equal to $\frac{1}{9}$

Correlators that cannot be computed with the axioms alone:

$\langle e_{10}, xe_9, xe_6\rangle = a_1$

$\langle e_{10}, xe_9, ye_6\rangle = a_2$

$\langle e_{10}, ye_9, xe_6\rangle = a_3$

$\langle e_{10}, ye_9, ye_6\rangle = a_4$

$\langle xe_9, xe_9, e_7\rangle = a_5$

$\langle xe_9, ye_9, e_7\rangle = a_6$

$\langle ye_9, ye_9, e_7\rangle = a_7$

	e_1	e_{10}	ye_9	xe_9	e_7	ye_6	xe_6	e_5	ye_3	xe_3	e_2	e_{11}
e_1	e_1	e_{10}	ye_9	xe_9	e_7	ye_6	xe_6	e_5	ye_3	xe_3	e_2	e_{11}
e_{10}		e_7	$9a_3ye_6 + 9a_4xe_6$	$9a_1ye_6 + 9a_2xe_6$	0	$9a_2ye_3 + 9a_4xe_3$	$9a_1ye_3 + 9a_3xe_3$	e_2	0	0	e_{11}	0
ye_9			a_7e_5	a_6e_5	$9a_6ye_3 + 9a_7xe_3$	a_4e_2	a_3e_2	0	0	$\frac{1}{9}e_{11}$	0	0
xe_9				a_5e_5	$9a_5ye_3 + 9a_6xe_3$	a_2e_2	a_1e_2	0	$\frac{1}{9}e_{11}$	0	0	0
e_7					0	0	0	e_{11}	0	0	0	0
ye_6						0	$\frac{1}{9}e_{11}$	0	0	0	0	0
xe_6							0	0	0	0	0	0
e_5								0	0	0	0	0
ye_3									0	0	0	0
xe_3										0	0	0
e_2											0	0
e_{11}												0

Table 18: U_{12} A side multiplication.

<u>B model</u>: $U_{12}^T : x^3 + y^3 + z^4$

$$\mathcal{J} = \langle 3x^2, 3y^2, 4z^3 \rangle$$

$$q_x = \frac{1}{3}, q_y = \frac{1}{3}, q_z = \frac{1}{4}$$

$$K = \langle m \rangle = \langle (\alpha^2, \beta, 1) \rangle \text{ when } \alpha^3 = \beta^3 = \gamma^4 = 1$$

$$\text{Fix} m^k = \begin{cases} \mathbb{C}^3 & k = 0 \\ \mathbb{C}_z & k = 1, 2 \end{cases}$$

$$\mathcal{Q}|_{\text{Fix}m^k} = \begin{cases} \langle 1, x, y, z, z^2, xy, xz, xz^2, yz, yz^2, xyz, xyz^2 \rangle & k = 0 \\ \langle 1, z, z^2 \rangle & k = 1, 2 \end{cases}$$

k	0	1	2
$\deg_W$	$0, \frac{1}{4}, \frac{1}{2}, \frac{2}{3}, \frac{11}{12}, \frac{7}{6}$	$\frac{1}{3}, \frac{7}{12}, \frac{5}{6}$	$\frac{1}{3}, \frac{7}{12}, \frac{5}{6}$
invariants	$b_0, zb_0, z^2b_0, xyb_0, xyzb_0, xyz^2b_0$	b_1, zb_1, z^2b_1	b_2, zb_2, z^2b_2

Table 19: U_{12} B side elements.

For Table 20 we will let

$$\begin{aligned} \alpha &= ab_2 + bb_1 \\ \beta &= cb_2 + db_1 \\ \gamma &= ezb_2 + fzb_1 \\ \theta &= gzb_2 + hzb_1 \\ \eta &= iz^2b_2 + jz^2b_1 \\ \mu &= kz^2b_2 + lz^2b_1. \end{aligned}$$

	b_0	zb_0	α	β	z^2b_0	γ	θ	xyb_0	η	μ	$xyzb_0$	xyz^2b_0
b_0	b_0	zb_0	α	β	z^2b_0	γ	θ	xyb_0	η	μ	$xyzb_0$	xyz^2b_0
zb_0		z^2b_0	$\frac{ha-gb}{he-gf}\gamma+\frac{eb-af}{he-gf}\theta$	$\frac{hc-gd}{he-gf}\gamma+\frac{ed-cf}{he-gf}\theta$	0	$\frac{el-kf}{il-jk}\eta+\frac{if-je}{il-jk}\mu$	$\frac{gl-kh}{il-jk}\eta+\frac{ih-jg}{il-jk}\mu$	$xyzb_0$	0	0	xyz^2b_0	0
α			$24abxyb_0$	$12(ad+bc)xyb_0$	$\frac{al-kb}{il-jk}\eta+\frac{ib-ja}{il-jk}\mu$	$12(af+be)xyzb_0$	$12(ah+bg)xyzb_0$	0	0	$\frac{1}{9}xyz^2b_0$	0	0
β				$24cdxyb_0$	$\frac{cl-kd}{il-jk}\eta+\frac{id-jc}{il-jk}\mu$	$12(cf-de)xyzb_0$	$12(ch+dg)xyzb_0$	0	$\frac{1}{9}xyz^2b_0$	0	0	0
z^2b_0					0	0	0	xyz^2b_0	0	0	0	0
γ						0	$\frac{1}{9}xyz^2b_0$	0	0	0	0	0
θ							0	0	0	0	0	0
xyb_0								0	0	0	0	0
η									0	0	0	0
μ										0	0	0
$xyzb_0$											0	0
xyz^2b_0												0

Table 20: U_{12} B side multiplication.

System of equations found by matching the multiplication tables:

$$
\begin{aligned}
a_1 &= \frac{hc-gd}{9(he-gf)} = \frac{gl-kh}{9(il-jk)} = 12(ch+dg) \\
a_2 &= \frac{ed-cf}{9(he-gf)} = \frac{el-kf}{9(il-jk)} = 12(cf+de) \\
a_3 &= \frac{ha-gb}{9(he-gf)} = \frac{ih-jg}{9(il-jk)} = 12(ah+bg) \\
a_4 &= \frac{eb-af}{9(he-gf)} = \frac{if-je}{9(il-jk)} = 12(af+be) \\
a_5 &= 24cd = \frac{cl-kd}{9(il-jk)} \\
a_6 &= 12(ad+bc) = \frac{al-kb}{9(il-jk)} = \frac{id-jc}{9(il-jk)} \\
a_7 &= 24ab = \frac{ib-ja}{9(il-jk)} \\
12(aj+bi) &= 0 \\
12(al+bk) &= \frac{1}{9} \\
12(cj+di) &= \frac{1}{9} \\
12(cl+dk) &= 0 \\
24ef &= 0 \\
24gh &= 0 \\
12(eh+fg) &= \frac{1}{9}
\end{aligned}
$$

Solution to this system in terms of the a_is:

$$
\begin{aligned}
a &= \frac{a_3}{12h} \\
b &= 9ha_4 \\
c &= \frac{a_1}{12h} \\
d &= 9ha_2
\end{aligned}
$$

$$
\begin{aligned}
e &= \frac{1}{108h} \\
f &= 0 \\
g &= 0 \\
h &= h \\
i &= \frac{a_1}{972h(a_1a_4 - a_3a_2)} \\
j &= \frac{ha_4}{9(a_1a_4 - a_3a_2)} \\
k &= \frac{a_1}{972h(a_1a_4 - a_3a_2)} \\
l &= -\frac{ha_2}{9(a_1a_4 - a_3a_2)}
\end{aligned}
$$

With the relations:

$$
\begin{aligned}
a_5 &= 18a_2a_1 \\
a_6 &= 9a_4a_1 + 9a_3a_2 \\
a_7 &= 18a_3a_4
\end{aligned}
$$

3.2 Bimodal Singularities

3.2.1 $Z_{1,0}$

A model: $Z_{1,0} : yx^3 + y^7$

$$\mathcal{J} = \langle 3x^2y, x^3 + 7y^6 \rangle$$

$$q_x = \frac{2}{7}, q_y = \frac{1}{7}$$

$$G_{Z_{1,0}} = \langle (\alpha, \alpha^{-3}) \rangle \cong \mathbb{Z}_{21} \text{ when } \alpha^{21} = 1$$

$$\text{Fix} J^k = \begin{cases} \mathbb{C}^2 & k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathcal{Q}|_{\text{Fix}J^k} = \begin{cases} \langle 1, x, x^2, y.y^2, y^3, y^4, y^5, y^6, xy, xy^2, xy^3, xy^4, & \\ xy^5, xy^6 \rangle & k = 0 \\ \langle 1 \rangle & \text{otherwise.} \end{cases}$$

k	0	1	2	3	4	5	6
$\deg_W$	$\frac{8}{7}, \frac{8}{7}, \frac{8}{7}$	0	$\frac{6}{7}$	$\frac{12}{7}$	$\frac{4}{7}$	$\frac{10}{7}$	$\frac{16}{7}$
invariants	x^2e_0, y^4e_0, xy^2e_0	e_1	e_2	e_3	e_4	e_5	e_6

Table 21: $Z_{1,0}$ A side elements.

Potential non-zero correlators:

Concavity axiom:

$\langle e_1, e_1, e_6 \rangle$, $\langle e_1, e_2, e_5 \rangle$, $\langle e_1, e_4, e_3 \rangle$, $\langle e_4, e_2, e_2 \rangle$ all equal 1.

Pairing axiom:

$\langle e_1, x^2e_0, x^2e_0 \rangle = -\frac{1}{3}$

$\langle e_1, xy^2e_0, y^4e_0 \rangle = \frac{1}{21}$

Correlators that cannot be computed from the axioms alone:

$\langle e_4, e_4, x^2e_0 \rangle = a_1$

$\langle e_4, e_4, y^4e_0 \rangle = a_2$

$\langle e_4, e_4, xy^2e_0 \rangle = a_3$

Correlator equation:

$$-3 = -3\langle e_4, e_4, x^2e_0\rangle^2 + 42\langle e_4, e_4, y^4e_0\rangle\langle e_4, e_4, xy^2e_0\rangle$$

	e_1	e_4	e_2	x^2e_0	xy^2e_0	y^4e_0	e_5	e_3	e_6
e_1	e_1	e_4	e_2	x^2e_0	xy^2e_0	y^4e_0	e_5	e_3	e_6
e_4		$-3a_1x^2e_0 + 21a_2xy^2e_0 + 21a_3y^4e_0$	e_5	a_1e_3	a_3e_3	a_2e_3	0	e_6	0
e_2			e_3	0	0	0	e_6	0	0
x^2e_0				$-\frac{1}{3}e_6$	0	0	0	0	0
xy^2e_0					0	$\frac{1}{21}e_6$	0	0	0
y^4e_0						0	0	0	0
e_5							0	0	0
e_3								0	0
e_6									0

Table 22: $Z_{1,0}$ A side multiplication.

B model: $Z^T_{1,0} : x^3 + xy^7$

$$\mathcal{J} = \langle 3x^2 + y^7, 7xy^6\rangle$$

$$q_x = \frac{1}{3}, q_y = \frac{2}{21}$$

$$K = \langle m\rangle = \langle(\beta^{14}, \beta^7)\rangle \text{ when } \beta^{21} = 1$$

$$\text{Fix}m^k = \begin{cases} \mathbb{C}^2 & k = 0 \\ 0 & k = 1, 2 \end{cases}$$

$$\mathcal{Q}|_{\text{Fix}m^k} = \begin{cases} \langle 1, x, x^2, y, y^2, y^3, y^4, y^5, y^6, xy, xy^2, xy^3, xy^4, xy^5, & \\ x^2y, x^2y^2, x^2y^3, x^2y^4, x^2y^5\rangle & k = 0 \\ \langle 1\rangle & k = 1, 2 \end{cases}$$

k	0	1	2
$\deg_W$	$0, \frac{2}{7}, \frac{4}{7}, \frac{3}{7}, \frac{5}{7}, \frac{6}{7}, \frac{8}{7}$	$\frac{4}{7}$	$\frac{4}{7}$
invariants	$b_0, y^3b_0, y^6b_0, xyb_0, xy^4b_0, x^2y^2b_0, x^2y^5b_0$	b_1	b_2

Table 23: $Z_{1,0}$ B side elements.

For Table 24 let

$$\begin{aligned}\alpha &= ab_1 + bb_2 + cy^6 b_0 \\ \beta &= db_1 + eb_2 + fy^6 b_0 \\ \gamma &= gb_1 + hb_2 + iy^6 b_0\end{aligned}$$

	b_0	y^3b_0	xyb_0	α	β	γ	xy^4b_0	$x^2y^2b_0$	$x^2y^5b_0$
b_0	b_0	y^3b_0	xyb_0	α	β	γ	xy^4b_0	$x^2y^2b_0$	$x^2y^5b_0$
y^3b_0		$\frac{-eg+hd}{-ceg+chd-fha+fbg+iae-ibd}\alpha - \frac{ha-gb}{-ceg+chd-fha+fbg+iae-ibd}\beta + \frac{ae-bd}{-ceg+chd-fha+fbg+iae-ibd}\gamma$	xy^4b_0	$-3cx^2y^2b_0$	$-3fx^2y^2b_0$	$-3ix^2y^2b_0$	0	$x^2y^5b_0$	0
xyb_0			$x^2y^2b_0$	0	0	0	$x^2y^5b_0$	0	0
α				$-\frac{1}{3}x^2y^5b_0$	0	0	0	0	0
β					0	$\frac{1}{21}x^2y^5b_0$	0	0	0
γ						0	0	0	0
xy^4b_0							0	0	0
$x^2y^2b_0$								0	0
$x^2y^5b_0$									0

Table 24: $Z_{1,0}$ B side multiplication.

System of equations found by matching the multiplication tables and correlator equations:

$$\begin{aligned}
399ab - 3c^2 &= -\frac{1}{3} \\
\frac{399}{2}(ae + bd) - 3cf &= 0 \\
\frac{399}{2}(ah + bg) - 3ci &= 0 \\
399de - 3f^2 &= 0 \\
\frac{399}{2}(dh + eg) - 3fi &= \frac{1}{21} \\
399gh - 3i^2 &= 0 \\
-3 &= -27c^2 + 378fi \\
a_1 = -3c &= \frac{-eg + hd}{-3(-ceg + chd - fah + fbg + iae - ibd)} \\
a_2 = -3i &= -\frac{ah - bg}{21(-ceg + chd - fah + fbg + iae - ibd)} \\
a_3 = -3f &= \frac{ae - bd}{21(-ceg + chd - fah + fbg + iae - ibd)}
\end{aligned}$$

Solution in terms of the a_is:

$$\begin{aligned}
a &= \frac{a_1 a_2 - a_2}{1197h} \\
b &= \frac{a_1 h + h}{a_2} \\
c &= -\frac{a_1}{3} \\
d &= \frac{(a_1 - 1)^2}{16758h} \\
e &= \frac{h(a_1 + 1)^2}{14a_2^2} \\
f &= -\frac{a_3}{3} \\
g &= \frac{a_2^2}{1197h}
\end{aligned}$$

$$h = h$$
$$i = -\frac{a_2}{3}$$

With the relation:

$$a_3 = \frac{a_1^2 - 1}{14a_2}$$

3.2.2 $Q_{2,0}$

A model: $Q_{2,0} : x^3 + xy^4 + yz^2$

$$\mathcal{J} = \langle 3x^2 + y^4, 4xy^3 + z^2, 2yz \rangle$$

$$q_x = \frac{1}{3}, q_y = \frac{1}{6}, q_z = \frac{5}{12}$$

$$G_{Q_{2,0}} = \langle (\gamma^8, \gamma^{-2}, \gamma) \rangle \cong \mathbb{Z}_{24} \text{ when } \gamma^{24} = 1$$

$$\text{Fix} J^k = \begin{cases} \mathbb{C}^3 & k = 0 \\ \mathbb{C}^2_{xy} & k = 6 \\ \mathbb{C}_x & k = 3, 9 \\ 0 \text{otherwise.} \end{cases}$$

$$\mathcal{Q}|_{\text{Fix}J^k} = \begin{cases} \langle 1, x, x^2, y, y^2, y^3, z, z^2, xy, xy^2, x^2y, x^2y^2, xz, xz^2 \rangle & k = 0 \\ \langle 1, x, x^2, y, y^2, y^3, xy, xy^2, x^2y, x^2y^2 \rangle & k = 6 \\ \langle 1, x \rangle & k = 3, 9 \\ \langle 1 \rangle & \text{otherwise.} \end{cases}$$

k	1	2	4	5	6	7	8	10	11
$\deg_W$	0	$\frac{11}{6}$	$\frac{3}{2}$	$\frac{4}{3}$	$\frac{7}{6}, \frac{7}{6}$	1	$\frac{5}{6}$	$\frac{1}{2}$	$\frac{7}{3}$
invariants	e_1	e_2	e_4	e_5	y^3e_6, xye_6	e_7	e_8	e_{10}	e_{11}

Table 25: $Q_{2,0}$ A side elements.

Non-zero correlators:

Concavity axiom:

$\langle e_1, e_1, e_{11}\rangle$, $\langle e_1, e_7, e_5\rangle$, $\langle e_1, e_8, e_4\rangle$, $\langle e_1, e_{10}, e_2\rangle$, $\langle e_{10}, e_8, e_7\rangle$, $\langle e, e, e\rangle$ all equal 1.

Pairing axiom:

$\langle e_1, y^3e_6, y^3e_6\rangle = -\frac{1}{4}$

$\langle e_1, xye_6, xye_6\rangle = \frac{1}{12}$

Index zero axiom:

$\langle e_{10}, e_{10}, e_5\rangle = -2$

	e_1	e_{10}	e_8	e_7	xye_6	y^3e_6	e_5	e_4	e_2	e_{11}
e_1	e_1	e_{10}	e_8	e_7	xye_6	y^3e_6	e_5	e_4	e_2	e_{11}
e_{10}		$-2e_7$	e_5	e_4	0	0	$-2e_2$	0	e_{11}	0
e_8			0	e_2	0	0	0	e_{11}	0	0
e_7				0	0	0	e_{11}	0	0	0
xye_6					$\frac{1}{12}e_{11}$	0	0	0	0	0
y^3e_6						$-\frac{1}{4}e_{11}$	0	0	0	0
e_5							0	0	0	0
e_4								0	0	0
e_2									0	0
e_{11}										0

Table 26: $Q_{2,0}$ A side multiplication.

B model:$Q_{2,0}^T : x^3y + y^4z + z^2$

$$\mathcal{J} = \langle 3x^2y, x^3 + 4y^3z, y^4 + 2z \rangle$$

$$q_x = \frac{7}{24}, q_y = \frac{1}{8}, q_z = \frac{1}{2}$$

$$K = \langle m \rangle = \langle \alpha, \alpha, 0 \rangle \text{ when } \alpha^2 = 1$$

$$\text{Fix} m^k = \begin{cases} \mathbb{C}^2 & k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathcal{Q}|_{\text{Fix} m^k} = \begin{cases} \langle 1, x, x^2, y, y^2, y^3, z, xy, xy^2, xy^3, xz, yz, y^2z, y^3z, xyz, xy^2z, xy^3z \rangle & k = 0 \\ \langle 1 \rangle & \text{otherwise.} \end{cases}$$

k	0	1
$\deg_W$	$0, \frac{1}{2}, \frac{7}{12}, \frac{1}{4}, \frac{5}{12}, \frac{3}{4}, \frac{2}{3}, \frac{11}{12}, \frac{7}{6}$	$\frac{7}{12}$
invariants	$b_0, zb_0, x^2b_0, y^2b_0, xyb_0, y^2zb_0, xy^3b_0, xyzb_0, xy^3zb_0$	b_1

Table 27: $Q_{2,0}$ B side elements.

	b_0	y^2b_0	xyb_0	z_0	$\frac{1}{4\sqrt{51}}b_1$	$\frac{1}{4}x^2b_0$	xy^3b_0	y^2zb_0	$xyzb_0$	xy^3zb_0
b_0	b_0	y^2b_0	xyb_0	z_0	$\frac{1}{4\sqrt{51}}b_1$	$\frac{1}{4}x^2b_0$	xy^3b_0	y^2zb_0	$xyzb_0$	xy^3zb_0
y^2b_0		$-2zb_0$	xy^3b_0	y^2zb_0	0	0	$-2xyzb_0$	0	xy^3zb_0	0
xyb_0			0	$xyzb_0$	0	0	0	xy^3zb_0	0	0
z_0				0	0	0	xy^3zb_0	0	0	0
$\frac{1}{4\sqrt{51}}b_1$					$\frac{1}{12}xy^3zb_0$	0	0	0	0	0
$\frac{1}{4}x^2b_0$						$-\frac{1}{4}xy^3zb_0$	0	0	0	0
xy^3b_0							0	0	0	0
y^2zb_0								0	0	0
$xyzb_0$									0	0
xy^3zb_0										0

Table 28: $Q_{2,0}$ B side multiplication.

3.2.3 $S_{1,0}$

A model: $S_{1,0} : zx^2 + yz^2 + y^5$

$$\mathcal{J} = \langle 2zx, z^2 + 5y^4, x^2 + 2yz\rangle$$

$$q_x = \frac{3}{10}, q_y = \frac{1}{5}, q_z = \frac{2}{5}$$

$$G_{S_{1,0}} = \langle(\alpha, \alpha^4, \alpha^{-2})\rangle \cong \mathbb{Z}_{20} \text{ when } \alpha^{20} = 1$$

$$\text{Fix}J^k = \begin{cases} \mathbb{C}^3 & k = 0 \\ \mathbb{C}^2_{yz} & 5|k \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathcal{Q}|_{\text{Fix}J^k} = \begin{cases} \langle 1, x, x^2, y, y^2, y^3, y^4, z, xy, xy^2, xy^3, x^2y, x^2y^2, x^2y^3\rangle & k = 0 \\ \langle 1, y, y^2, y^3, y^4, z\rangle & 5|k \\ \langle 1\rangle & \text{otherwise.} \end{cases}$$

k	1	2	3	4	5	6	7	8	9
$\deg_W$	0	$\frac{9}{5}$	$\frac{8}{5}$	$\frac{7}{5}$	$\frac{6}{5}, \frac{6}{5}$	$\frac{4}{5}$	$\frac{4}{5}$	$\frac{3}{5}$	$\frac{12}{5}$
invariants	e_1	e_2	e_3	e_4	y^2e_5, ze_5	e_6	e_7	e_8	e_9

Table 29: $S_{1,0}$ A side elements.

Potential non-zero correlators:

Concavity axiom:

$\langle e_1, e_1, e_9\rangle$, $\langle e_1, e_6, e_4\rangle$, $\langle e_1, e_7, e_3\rangle$, $\langle e_1, e_8, e_2\rangle$, $\langle e_8, e_7, e_6\rangle$, are all equal to 1.

Pairing axiom:

$\langle e_1, y^2e_5, y^2e_5\rangle = \frac{1}{10}$

$\langle e_1, ze_5, ze_5\rangle = -\frac{1}{2}$

Index zero axiom:

$\langle e_7, e_7, e_7\rangle = -2$

Correlators we cannot compute with the axioms alone:

$\langle e_8, e_8, ze_5\rangle = a_1$

$\langle e_8, e_8, y^2e_5\rangle = a_2$

Correlator equation:

$-2 = -2\langle e_8, e_8, ze_5\rangle^2 + 10\langle e_8, e_8, y^2e_5\rangle^2$

	e_1	e_8	e_7	e_6	ze_5	y^2e_5	e_4	e_3	e_2	e_9
e_1	e_1	e_8	e_7	e_6	ze_5	y^2e_5	e_4	e_3	e_2	e_9
e_8		$-2a_1ze_5 + 10a_2y^2e_5$	e_4	e_3	a_1e_2	a_2e_2	0	0	e_9	0
e_7			$-2e_3$	e_2	0	0	0	e_9	0	0
e_6				0	0	0	e_9	0	0	0
ze_5					$-\frac{1}{2}e_9$	0	0	0	0	0
y^2e_5						$\frac{1}{10}e_9$	0	0	0	0
e_4							0	0	0	0
e_3								0	0	0
e_2									0	0
e_9										0

Table 30: $S_{1,0}$ A side multiplication.

B model:$S_{1,0} : x^2 + yz^5 + xy^2$

$$\mathcal{J} = \langle 2x + y^2, z^5 + 2xy, 5yz^4\rangle$$

$$q_x = \frac{1}{2}, q_y = \frac{1}{4}, q_z = \frac{3}{20}$$

$$K = \langle m\rangle = \langle 1, \beta, \beta\rangle \text{ where } \beta^2 = 1$$

$$\mathrm{Fix} m^k = \begin{cases} \mathbb{C}^3 & k = 0 \\ \mathbb{C}_x & k = 2, 6, 10, 14, 18 \\ \mathbb{C}^2_{xy} & k = 4, 8, 12, 16 \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathcal{Q}|_{\mathrm{Fix} m^k} = \begin{cases} \langle 1, x, y, z, z^2, z^3, z^4, xy, xz, xz^2, xz^3, yz, yz^2, yz^3, & \\ xyz, xyz^2, xyz^3 \rangle & k = 0 \\ \langle 1 \rangle & k = 2, 6, 10, 14, 18 \\ \langle 1, x, y \rangle & k = 4, 8, 12, 16 \\ \langle 1 \rangle & \text{otherwise.} \end{cases}$$

k	0	1
$\deg_W$	$0, \frac{1}{2}, \frac{3}{10}, \frac{3}{5}, \frac{9}{10}, \frac{6}{5}, \frac{2}{5}, \frac{7}{10}, \frac{4}{5}$	$\frac{3}{5}$
invariants	$1, x, z^2, z^4, xyz, xyz^3, yz, yz^3, xz^2$	1

Table 31: $S_{1,0}$ B side elements.

	b_0	z^2b_0	yzb_0	xb_0	$ab_1+bz^4b_0$	$cb_1+dz^4b_0$	yz^3b_0	xz^2b_0	$xyzb_0$	xyz^3b_0
b_0	b_0	z^2b_0	yzb_0	xb_0	$ab_1+bz^4b_0$	$cb_1+dz^4b_0$	yz^3b_0	xz^2b_0	$xyzb_0$	xyz^3b_0
z^2b_0		$\frac{-c}{ad-bc}(ab_1+bz^4b_0)+\frac{a}{ad-bc}(cb_1+dz^4b_0)$	yz^3b_0	$y^2z^2b_0$	$-2bxyzb_0$	$-2dxyzb_0$	0	0	xyz^3b_0	0
yzb_0			$-2xz^2b_0$	$xyzb_0$	0	0	0	xyz^3b_0	0	0
xb_0				0	0	0	xyz^3b_0	0	0	0
$ab_1+bz^4b_0$					$-\frac{1}{2}xyz^3b_0$	0	0	0	0	0
$cb_1+dz^4b_0$						$\frac{1}{10}xyz^3b_0$	0	0	0	0
yz^3b_0							0	0	0	0
xz^2b_0								0	0	0
$xyzb_0$									0	0
xyz^3b_0										0

Table 32: $S_{1,0}$ B side multiplication.

System of equations found by matching the multiplication tables:

$$\begin{aligned}
a_1 &= -2b = \frac{c}{2(ad-bc)}\\
a_2 &= -2d = \frac{a}{10(ad-bc)}\\
\frac{170}{3}a^2 - 2b^2 &= -\frac{1}{2}\\
\frac{170}{3}ac - 2bd &= 0\\
\frac{170}{3}c^2 - 2d^2 &= \frac{1}{10}\\
-2 &= -8b^2 + 40d^2
\end{aligned}$$

Solution in terms of the a_is:

$$\begin{aligned}
a &= -\frac{1}{34}a_2\sqrt{51}\\
b &= -\frac{a_1}{2}\\
c &= -\frac{\sqrt{51}a_1}{170}\\
d &= -\frac{a_2}{2}
\end{aligned}$$

With the relation:

$$a_1 = -\sqrt{1+5a_2^2}$$

3.2.4 U_{16}

A model: $U_{16} : x^3 + z^2x + y^5$

$$\mathcal{J} = \langle 3x^2 + z^2, 5y^4, 2zx\rangle$$

$$q_x = \frac{1}{3}, q_y = \frac{1}{5}, q_z = \frac{1}{3}$$

$$G_{U_{16}} = \langle(\gamma^{-2}, \beta, \gamma)\rangle \cong \mathbb{Z}_{30} \text{ when } \beta^5 = \gamma^6 = 1$$

$$\mathrm{Fix}J^k = \begin{cases} \mathbb{C}^3 & k=0 \\ \mathbb{C}^2_{xz} & 3|k \\ \mathbb{C}_y & 5|k \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathcal{Q}|_{\mathrm{Fix}J^k} = \begin{cases} \langle 1,x,x^2,y,y^2,y^3,z,xy,xy^2,xy^3,x^2y,x^2y^2,x^2y^3, & \\ yz,y^2z,y^3z\rangle & k=0 \\ \langle 1,x,x^2,z\rangle & 3|k \\ \langle 1,y,y^2,y^3\rangle & 5|k \\ \langle 1\rangle & \text{otherwise.} \end{cases}$$

k	1	2	3	4	6	7	8	9	11	12	13	14
$\deg_W$	0	$\frac{26}{15}$	$\frac{22}{15},\frac{22}{15}$	$\frac{6}{5}$	$\frac{2}{3},\frac{2}{3}$	$\frac{2}{5}$	$\frac{32}{15}$	$\frac{28}{15},\frac{28}{15}$	$\frac{4}{3}$	$\frac{16}{15},\frac{16}{15}$	$\frac{4}{5}$	$\frac{38}{15}$
invariants	e_1	e_2	xe_3, ze_3	e_4	xe_6, ze_6	e_7	e_8	xe_9, ze_9	e_{11}	xe_{12}, ze_{12}	e_{13}	e_{14}

Table 33: U_{16} A side elements.

Potential non-zero correlators:

Concavity axiom:

$\langle e_1,e_1,e_{14}\rangle$, $\langle e_1,e_4,e_{11}\rangle$, $\langle e_1,e_7,e_8\rangle$, $\langle e_1,e_{13},e_2\rangle$, $\langle e_7,e_7,e_2\rangle$, $\langle e_7,e_{13},e_{11}\rangle$ all equal 1.

Pairing axiom:

$\langle e_1,xe_6,xe_9\rangle$ and $\langle e_1,xe_{12},xe_3\rangle$ both equal $\frac{1}{6}$.

$\langle e_1,ze_6,ze_9\rangle$ and $\langle e_1,ze_{12},ze_3\rangle$ both equal $-\frac{1}{2}$.

Correlators we cannot compute with the axioms alone:

$\langle e_7,xe_6,xe_3\rangle = a_1$

$\langle e_7,xe_6,ze_3\rangle = a_2$

$$\langle e_7, xe_{12}, xe_{12}\rangle = a_3$$

$$\langle e_7, xe_{12}, ze_{12}\rangle = a_4$$

$$\langle e_7, ze_6, xe_3\rangle = a_5$$

$$\langle e_7, ze_6, ze_3\rangle = a_6$$

$$\langle e_7, ze_{12}, ze_{12}\rangle = a_7$$

$$\langle xe_6, e_{13}, xe_{12}\rangle = a_8$$

$$\langle xe_6, e_{13}, ze_{12}\rangle = a_9$$

$$\langle xe_6, xe_6, e_4\rangle = a_{10}$$

$$\langle xe_6, ze_6, e_4\rangle = a_{11}$$

$$\langle ze_6, e_{13}, xe_{12}\rangle = a_{12}$$

$$\langle ze_6, e_{13}, ze_{12}\rangle = a_{13}$$

$$\langle ze_6, ze_6, e_4\rangle = a_{14}$$

	e_1	e_7	ze_6	xe_6	e_{13}	ze_{12}	xe_{12}	e_4
e_1	e_1	e_7	ze_6	xe_6	e_{13}	ze_{12}	xe_{12}	e_4
e_7		e_{13}	$-2a_6ze_{12}+$ $6a_5xe_{12}$	$-2a_2ze_{12}+$ $6a_1xe_{12}$	e_4	$-2a_7ze_3+$ $6a_4xe_3$	$-2a_4ze_3+$ $6a_3xe_3$	0
ze_6			$a_{14}e_{11}$	$a_{11}e_{11}$	$-2a_{13}ze_3+$ $6a_{12}xe_3$	$a_{13}e_2$	$a_{12}e_2$	$-2a_{13}ze_6+$ $6a_{11}xe_6$
xe_6				$a_{10}e_{11}$	$-2a_9ze_3+$ $6a_8xe_3$	a_9e_2	a_8e_2	$-2a_{11}ze_6+$ $6a_{10}xe_6$
e_{13}					0	$-2a_{13}ze_9+$ $6a_9xe_9$	$-2a_{12}ze_9+$ $6a_8xe_9$	0
ze_{12}						a_7e_8	a_4e_8	0
xe_{12}							a_3e_8	0
e_4								0

	e_{11}	ze_3	xe_3	e_2	ze_9	xe_9	e_8	e_{14}
e_1	e_{11}	ze_3	xe_3	e_2	ze_9	xe_9	e_8	e_{14}
e_7	e_2	$-2a_6ze_6+$ $6a_2xe_6$	$-2a_5ze_6+$ $6a_1xe_6$	e_8	0	0	e_{14}	0
ze_6	0	a_6e_8	a_5e_8	0	$-\frac{1}{2}e_{14}$	0	0	0
xe_6	0	a_2e_8	a_1e_8	0	0	$\frac{1}{6}e_{14}$	0	0
e_{13}	e_8	0	0	e_{14}	0	0	0	0
ze_{12}	0	$-\frac{1}{2}e_{14}$	0	0	0	0	0	0
xe_{12}	0	0	$\frac{1}{6}e_{14}$	0	0	0	0	0
e_4	e_{14}	0	0	0	0	0	0	0
e_{11}	0	0	0	0	0	0	0	0
ze_3		0	0	0	0	0	0	0
xe_3			0	0	0	0	0	0
e_2				0	0	0	0	0
ze_9					0	0	0	0
xe_9						0	0	0
e_8							0	0
e_{14}								0

Table 34: U_{16} A side multiplication.

B model: $U_{16}^T : x^3y + y^2 + z^5$

$$\mathcal{J} = \langle 3x^2y, x^3 + 2y, 5z^4 \rangle$$

$$q_x = \frac{1}{6}, q_y = \frac{1}{2}, q_z = \frac{1}{5}$$

$$K = \langle m \rangle = \langle (\alpha^{15}, \alpha^{15}, 1) \rangle \text{ when } \alpha^6 = \gamma^5 = 1$$

$$\text{Fix} m^k = \begin{cases} \mathbb{C}^3 & k = 0 \\ \mathbb{C}_z & k = 1 \end{cases}$$

$$\mathcal{Q}|_{\text{Fix}m^k} = \begin{cases} \langle 1, x, x^2, y, z, z^2, z^3, xy, xz, xz^2, xz^3, x^2z, x^2z^2, x^2z^3, yz, yz^2, & \\ yz^3, xyz, xyz^2, xyz^3 \rangle & k = 0 \\ \langle 1, z, z^2, z^3 \rangle & k = 1 \end{cases}$$

k	0	1
$\deg_W$	$0, \frac{1}{3}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{2}{3}, \frac{8}{15}, \frac{11}{15}, \frac{14}{15}, \frac{13}{15}, \frac{16}{15}, \frac{19}{15}$	$\frac{1}{3}, \frac{8}{15}, \frac{11}{15}, \frac{14}{15}$
invariants	$b_0, x^2b_0, zb_0, z^2b_0, z^3b_0, xyb_0, x^2zb_0, x^2z^2b_0, x^2z^3b_0, xyzb_0, xyz^2b_0, xyz^3b_0$	$b_1, zb_1, z^2b_1, z^3b_1$

Table 35: U_{16} B side elements.

For Table 36 let

$$\begin{aligned} \alpha &= ab_1 + bx^2b_0 \\ \beta &= cb_1 + dx^2b_0 \\ \gamma &= ezb_1 + fx^2zb_0 \\ \mu &= gzb_1 + hx^2zb_0 \\ \eta &= iz^2b_1 + jx^2z^2b_0 \end{aligned}$$

$$\delta = kz^2b_1 + lx^2z^2b_0$$

$$\epsilon = mz^3b_1 + nx^2z^3b_0$$

$$\psi = oz^3b_1 + px^2z^3b_0$$

	b_0	zb_0	α	β	z^2b_0
b_0	b_0	zb_0	α	β	z^2b_0
zb_0		z^2b_0	$\frac{ha-bg}{eh-fg}(ezb_1 + fx^2b_0)+$ $\frac{eb-af}{eh-fg}\mu$	$\frac{hc-dg}{eh-fg}\gamma+$ $\frac{ed-cf}{eh-fg}\mu$	z^3b_0
α			$(10a^2 - 2b^2)xyb_0$	$(3ac - 2bd)xyb_0$	$\frac{al-kb}{il-jk}\eta+$ $\frac{aj-bi}{il-jk}\delta$
β				$(3 - c^2 - 2d^2)xyb_0$	$\frac{cl-kd}{il-jk}\eta+$ $\frac{ik-jc}{il-jk}\delta$
z^2b_0					0

	γ	μ	z^3b_0	xyb_0	η
b_0	γ	μ	z^3b_0	xyb_0	η
zb_0	$\frac{el-kf}{il-jk}\eta+\frac{if-je}{il-jk}\delta$	$\frac{gl-kh}{il-jk}\eta+\frac{ih-jg}{il-jk}\delta$	0	$xyzb_0$	$\frac{pi-oj}{pm-on}\epsilon+\frac{mj-in}{pm-on}\psi$
α	$(10ae-2bf)xyzb_0$	$(10ag-2bh)xyzb_0$	$\frac{pa-ob}{pm-on}\epsilon+\frac{mb-an}{pm-on}\psi$	0	$(10ai-2bj)xyz^2b_0$
β	$(10ce-2df)xyzb_0$	$(10cg-2dh)xyzb_0$	$\frac{pc-od}{pm-on}\epsilon+\frac{md-cn}{pm-on}\psi$	0	$(10ci-2dj)xyz^2b_0$
z^2b_0	$\frac{pe-of}{pm-on}\epsilon+\frac{mf-en}{pm-on}\psi$	$\frac{pg-oh}{pm-on}\epsilon+\frac{mh-gn}{pm-on}\psi$	0	xyz^2b_0	0
γ	$(10e^2-2f^2)xyz^2b_0$	$(10eg-2fh)xyz^2b_0$	0	0	$-\frac{1}{2}xyz^3b_0$
μ		$(10g^2-2h^2)xyz^2b_0$	0	0	0
z^3b_0			0	xyz^3b_0	0
xyb_0				0	0
η					0

	δ	$xyzb_0$	ϵ	ψ	xyz^2b_0	xyz^3b_0
b_0	δ	$xyzb_0$	ϵ	ψ	xyz^2b_0	xyz^3b_0
zb_0	$\frac{pk-ol}{pm-on}\epsilon+\frac{ml-kn}{pm-on}\psi$	xyz^2b_0	0	0	xyz^3b_0	0
α	$(10ak-2bl)xyz^2b_0$	0	$-\frac{1}{2}xyz^3b_0$	0	0	0
β	$(10ck-2dl)xyz^2b_0$	0	0	$\frac{1}{6}xyz^3b_0$	0	0
z^2b_0	0	xyz^3b_0	0	0	0	0
γ	0	0	0	0	0	0
μ	$\frac{1}{6}xyz^3b_0$	0	0	0	0	0
z^3b_0	0	0	0	0	0	0
xyb_0	0	0	0	0	0	0
η	0	0	0	0	0	0
δ	0	0	0	0	0	0
$xyzb_0$		0	0	0	0	0
ϵ			0	0	0	0
ψ				0	0	0
xyz^2b_0					0	0
xyz^3b_0						0

Table 36: U_{16} B side multiplication.

System of equations found by matching the multiplication tables:

$$
\begin{aligned}
a_1 &= -\frac{-ed+cf}{6(eh-fg)} = -\frac{-ml+kn}{6(-on+pm)} = 10ck-2dl \\
a_2 &= \frac{-dg+hc}{-2(eh-fg)} = -\frac{-mj+in}{6(-on+pm)} = ci-2dj \\
a_3 &= -\frac{-ih+jg}{6(il-jk)} = 30g^2-2h^2 \\
a_4 &= -\frac{-if+je}{6(il-jk)} = \frac{-kh+gl}{-2(il-jk)} = 10eg-2fh \\
a_5 &= -\frac{-eb+af}{6(eh-fg)} = \frac{-ol+pk}{-2(-on+pm)} = 10ak-2bl \\
a_6 &= \frac{-bg+ha}{-2(eh-fg)} = \frac{-oj+pi}{-2(-on+pm)} = 10ai-2bj \\
a_7 &= \frac{-kf+el}{-2(il-jk)} = 10e^2-2f^2 \\
a_8 &= -\frac{-id+jc}{6(il-jk)} = 10cg-2dh = -\frac{-mh+gn}{6(-on+pm)} \\
a_9 &= \frac{-kd+cl}{-2(il-jk)} = 10ce-2df = -\frac{-mf+en}{6(-on+pm)} \\
a_{10} &= 10c^2-2d^2 = -\frac{-md+cn}{6(-on+pm)} \\
a_{11} &= 10ac-2bd = -\frac{-mb+an}{6(-on+pm)} = \frac{-od+pc}{-2(-on+pm)} \\
a_{12} &= -\frac{-ib+ja}{6(il-jk)} = 10ag-2bh = \frac{-oh+pg}{-2(-on+pm)} \\
a_{13} &= \frac{-kb+al}{-2(il-jk)} = 10ae-2bf = \frac{-of+pe}{-2(-on+pm)} \\
a_{14} &= 10a^2-2b^2 = \frac{-ob+pa}{-2(-on+pm)} \\
10am-2bn &= -\frac{1}{2} \\
10ao-2bp &= 0 \\
10cm-2dn &= 0 \\
10co-2dp &= \frac{1}{6}
\end{aligned}
$$

$$
\begin{aligned}
10ei - 2fj &= -\frac{1}{2} \\
10gi - 2hj &= 0 \\
10ek - 2fl &= 0 \\
10gk - 2hl &= \frac{1}{6}
\end{aligned}
$$

Solution to this system of equations in terms of the a_is:

$$
\begin{aligned}
a &= \frac{a_5\sqrt{2(-5a_10a_7 + 5a_9^2)}}{10a_1\sqrt{-a_7}} \\
b &= \frac{a_13\sqrt{2}}{2\sqrt{-a_7}} \\
c &= \frac{\sqrt{2(-5a_10a_7 + 5a_9^2)}}{10\sqrt{-a_7}} \\
d &= \frac{a_9\sqrt{2}}{1\sqrt{-a_7}} \\
e &= 0 \\
f &= -\frac{1}{2} sqrt{-2a_7} \\
g &= \frac{\sqrt{2(-5a_10a_7 + 5a_9^2)}}{60a_1\sqrt{-a_7}} \\
h &= \frac{a_4\sqrt{2}}{2\sqrt{-a_7}} \\
i &= -\frac{3a_1a_4\sqrt{2}}{2\sqrt{-a_7(-5a_10a_7 + 5a_9^2)}} \\
j &= -\frac{\sqrt{2}}{4\sqrt{-a_7}} \\
k &= \frac{a_1\sqrt{-2a_7}}{2\sqrt{-5a_10a_7 + 5a_9^2}}
\end{aligned}
$$

$$
\begin{aligned}
l &= 0 \\
m &= \frac{a_1 a_9 \sqrt{-2a_7}}{4\sqrt{-5a_10a_7 + 5a_9^2}(-a_5a_9 + a_1a_13)} \\
n &= \frac{a_1\sqrt{-2a_7}}{4(-a_5a_9 + a_1a_13)} \\
o &= \frac{a_1a_{13}\sqrt{-2a_7}}{12\sqrt{-5a_10a_7 + 5a_9^2}(-a_5a_9 + a_1a_13)} \\
p &= \frac{a_5\sqrt{-2a_7}}{12(-a_5a_9 + a_1a_13)}
\end{aligned}
$$

With the relations:

$$
\begin{aligned}
a_2 &= \frac{-a_9 + 6a_4a_1}{2a_7} \\
a_3 &= \frac{36a_4^2a_1^2 + a_10a_7 - a_9^2}{36(a_7a_1^2)} \\
a_6 &= \frac{-a_13 + 6a_4a_5}{2a_7} \\
a_8 &= \frac{6a_4a_1a_9 + a_10a_7 - a_9^2}{6(a_7a_1)}
\end{aligned}
$$

3.2.5 Q_{16}

A model: $Q_{16} : x^3 + yz^2 + y^7$

$$\mathcal{J} = \langle 3x^2, z^2 + 7y^6, 2yz \rangle$$

$$q_x = \frac{1}{3}, q_y = \frac{1}{7}, q_z = \frac{3}{7}$$

$$G_{Q_{16}} = \langle (\alpha, \gamma^{-2}, \gamma) \rangle \cong \mathbb{Z}_{42} \text{ when } \alpha^3 = \gamma^{14}$$

$$\mathrm{Fix} J^k = \begin{cases} \mathbb{C}^3 & k = 0 \\ \mathbb{C}^2_{yz} & 7|k \\ \mathbb{C}_x & 3|k \\ 0 & \text{else.} \end{cases}$$

$$\mathcal{Q}|_{\mathrm{Fix} J^k} = \begin{cases} \langle 1, x, y, y^2, y^3, y^4, y^5, y^6, z, xy, xy^2, xy^3, xy^4, xy^5, xy^6, xz \rangle & k = 0 \\ \langle 1, y, y^2, y^3, y^4, y^5, y^6, z \rangle & 7|k \\ \langle 1, x \rangle & 3|k \\ \langle 1 \rangle & \text{else.} \end{cases}$$

k	1	2	4	5	7	8	10	11	13	14	16	17	19	20
$\deg_W$	0	$\frac{38}{21}$	$\frac{10}{7}$	$\frac{26}{21}$	$\frac{6}{7}, \frac{6}{7}$	$\frac{2}{3}$	$\frac{2}{7}$	$\frac{44}{21}$	$\frac{12}{7}$	$\frac{32}{21}\frac{32}{21}$	$\frac{8}{7}$	$\frac{20}{21}$	$\frac{4}{7}$	$\frac{50}{21}$
invariants	e_1	e_2	e_4	e_5	y^3e_7, ze_7	e_8	e_{10}	e_{11}	e_{13}	y^3e_{14}, ze_{14}	e_{16}	e_{17}	e_{19}	e_{20}

Table 37: Q_{16} A side elements.

Potential non-zero correlators:

Concavity axiom:

$\langle e_1, e_1, e_{20}\rangle$, $\langle e_1, e_8, e_{13}\rangle$, $\langle e_1, e_{10}, e_{11}\rangle$, $\langle e_1, e_{16}, e_5\rangle$, $\langle e_1, e_{17}, e_4\rangle$, $\langle e_1, e_{19}, e_2\rangle$, $\langle e_{10}, e_8, e_4\rangle$, $\langle e_{10}, e_{10}, e_2\rangle$, $\langle e_{10}, e_{17}, e_{16}\rangle$, $\langle e_{19}, e_8, e_{16}\rangle$, $\langle e, e, e\rangle$ are all equal to 1.

Pairing axiom:

$\langle e_1, y^3e_7, y^3e_{14}\rangle = \frac{1}{14}$

$\langle e_1, ze_7, ze_{14}\rangle = -\frac{1}{2}$

Index zero axiom:

$\langle e_{19}, e_{19}, e_5\rangle = -2$

$$\langle e_{10}, e_{19}, ze_{14} \rangle = a_1$$

$$\langle e_{10}, ze_7, e_5 \rangle = a_2$$

$$\langle e_{10}, e_{19}, y^3 e_{14} \rangle = a_3$$

$$\langle e_{10}, y^3 e_7, e_5 \rangle = a_4$$

$$\langle e_{19}, ze_7, e_{17} \rangle = a_5$$

$$\langle e_{19}, y^3 e_7, e_{17} \rangle = a_6$$

$$\langle e_8, y^3 e_7, y^3 e_7 \rangle = a_7$$

$$\langle e_8, ze_7, y^3 e_7 \rangle = a_8$$

$$\langle e_8, ze_7, ze_7 \rangle = a_9$$

	e_1	e_{10}	e_{19}	e_8	ze_7	y^3e_7	e_{17}	e_{16}	e_5	e_4	ze_{14}	y^3e_{14}	e_{13}	e_2	e_{11}	e_{20}
e_1	e_1	e_{10}	e_{19}	e_8	ze_7	y^3e_7	e_{17}	e_{16}	e_5	e_4	ze_{14}	y^3e_{14}	e_{13}	e_2	e_{11}	e_{20}
e_{10}		e_{19}	$-2a_1ze_7+$ $14a_3y^3e_7$	e_7	a_2e_{16}	a_4e_{16}	e_5	e_4	$-2a_2ze_{14}+$ $14a_4y^3e_{14}$	$e_1 3$	a_1e_2	a_3e_2	0	e_{11}	$e_2 0$	0
e_{19}			$-2e_{16}$	e_5	a_5e_4	a_6e_4	$-2a_5ze_{14}+$ $14a_6y^3e_{14}$	e_{13}	$-2e_2$	0	a_1e_{11}	a_3e_{11}	0	e_{20}	0	0
e_8				0	$-2a_9ze_{14}+$ $14a_8y^3e_{14}$	$-2a_8ze_{14}+$ $14a_7y^3e_{14}$	0	e_2	0	e_{11}	0	0	e_{20}	0	0	0
ze_7					a_9e_{13}	a_8e_{13}	a_5e_2	0	a_2e_{11}	0	$-\frac{1}{2}e_{20}$	0	0	0	0	0
y^3e_7						a_7e_{13}	a_6e_2	0	a_4e_{11}	0	0	$\frac{1}{14}e_{20}$	0	0	0	0
e_{17}							0	e_{11}	0	e_{20}	0	0	0	0	0	0
e_{16}								0	e_{20}	0	0	0	0	0	0	0
e_5									0	0	0	0	0	0	0	0
e_4										0	0	0	0	0	0	0
ze_{14}											0	0	0	0	0	0
y^3e_{14}												0	0	0	0	0
e_{13}													0	0	0	0
e_2														0	0	0
e_{11}															0	0
e_{20}																0

Table 38: Q_{16} A side multiplication.

<u>B model</u>: $Q_{16}^T : x^3 + yz^7 + y^2$

$$\mathcal{J} = \langle 3x^2, z^7 + 2y, 7yz^6 \rangle$$

$$q_x = \frac{1}{3}, q_y = \frac{1}{2}, q_z = \frac{1}{14}$$

$$K = \langle m \rangle = \langle (1, \gamma^{21}, \gamma^{21}) \rangle \text{ when } \alpha^3 = \gamma^{14} = 1$$

$$\text{Fix} m^k = \begin{cases} \mathbb{C}^3 & k = 0 \\ \mathbb{C}_x & k = 1 \end{cases}$$

$$\mathcal{Q}|_{\text{Fix}m^k} = \begin{cases} \langle 1, x, y, z, z^2, z^3, z^4, z^5, z^6, xz, xz^2, xz^3, xz^4, xz^5, xz^6, & \\ yz, yz^2, yz^3, yz^4, yz^5, xyz, xyz^2, xyz^3, xyz^4, xyz^5 \rangle & k = 0 \\ \langle 1, x \rangle & k = 1 \end{cases}$$

k	0	1
$\deg_W$	$0, \frac{1}{3}, \frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{10}{21}, \frac{13}{21}, \frac{16}{21}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}, \frac{19}{21}, \frac{22}{21}, \frac{25}{21}$	$\frac{3}{7}, \frac{16}{21}$
invariants	$b_0, xb_0, z^2b_0, z^4b_0, z^6b_0, xz^2b_0, xz^4b_0, xz^6b_0, yzb_0, yz^3b_0, yz^5b_0, xyzb_0, xyz^3b_0, xyz^5b_0$	b_1, xb_1

Table 39: Q_{16} B side elements.

For Table 40 we will let

$$\begin{aligned} \alpha &= ab_1 + bz^6b_0 \\ \beta &= cb_1 + dz^6b_0 \\ \gamma &= exb_1 + fxz^6b_0 \\ \eta &= gxb_1 + hxz^6b_0 \end{aligned}$$

	b_0	z^2b_0	z^4b_0	xb_0	α	β	xz^2b_0	yzb_0
b_0	b_0	z^2b_0	z^4b_0	xb_0	α	β	xz^2b_0	yzb_0
z^2b_0		z^4b_0	$\frac{-c}{ad-bc}\alpha+ \frac{a}{ad-bc}\beta$	xz^2b_0	$-2byzb_0$	$-2dyzb_0$	xz^4b_0	yz^3b_0
z^4b_0			$-2yzb_0$	xz^4b_0	$-2byz^3b_0$	$-2dyz^3b_0$	$\frac{-g}{he-gf}\gamma+ \frac{e}{he-gf}\eta$	yz^5b_0
xb_0				0	$\frac{ha-gb}{he-gf}\gamma+ \frac{eb-af}{he-gf}\eta$	$\frac{hc-gd}{he-gf}\gamma+ \frac{ed-cf}{he-gf}\eta$	0	$xyzb_0$
α					$(\frac{182}{3}a^2-2b^2)yz^5b_0$	$(\frac{182}{3}ac-2bd)yz^5b_0$	$-2bxyzb_0$	0
β						$(\frac{182}{3}c^2-2d^2)yz^5b_0$	$-2dxyzb_0$	0
xz^2b_0							0	xyz^3b_0
yzb_0								0

	xz^4b_0	yz^3b_0	γ	η	yz^5b_0	$xyzb_0$	xyz^3b_0	xyz^5b_0
b_0	xz^4b_0	yz^3b_0	γ	η	yz^5b_0	$xyzb_0$	xyz^3b_0	xyz^5b_0
z^2b_0	$\frac{-g}{he-gf}\gamma+ \frac{e}{he-gf}\eta$	yz^5b_0	$-2fxyzb_0$	$-2hxyzb_0$	0	xyz^3b_0	xyz^5b_0	0
z^4b_0	$-2xyzb_0$	0	$-2fxyz^3b_0$	$-2hxyz^3b_0$	0	xyz^5b_0	0	0
xb_0	0	xyz^3b_0	0	0	xyz^5b_0	0	0	0
α	$-2bxyz^3b_0$	0	$-\frac{1}{2}xyz^5b_0$	0	0	0	0	0
β	$-2dxyz^3b_0$	0	0	$\frac{1}{14}xyz^5b_0$	0	0	0	0
xz^2b_0	0	xyz^5b_0	0	0	0	0	0	0
yzb_0	xyz^5b_0	0	0	0	0	0	0	0
xz^4b_0	0	0	0	0	0	0	0	0
yz^3b_0		0	0	0	0	0	0	0
γ			0	0	0	0	0	0
η				0	0	0	0	0
yz^5b_0					0	0	0	0
$xyzb_0$						0	0	0
xyz^3b_0							0	0
xyz^5b_0								0

Table 40: Q_{16} B side multiplication.

System of equations found by matching the multiplication tables:

$$\begin{aligned}
a_1 &= -2f = \frac{c}{2(ad-bc)} \\
a_2 &= -2b = \frac{g}{2(he-gf)} \\
a_3 &= -2h = \frac{a}{14(ad-bc)} \\
a_4 &= -2d = \frac{e}{14(he-gf)} \\
a_5 &= -2b = \frac{g}{2(he-gf)} \\
a_6 &= -2d = \frac{e}{14(he-gf)} \\
a_7 &= \frac{ed-cf}{14(he-gf)} = \frac{182}{3}c^2 - 2d^2 \\
a_8 &= \frac{eb-af}{14(he-gf)} = \frac{hc-gd}{-2(he-gf)} = \frac{182}{3}ac - 2bd \\
a_9 &= \frac{ha-gb}{-2(he-gf)} = \frac{182}{3}a^2 - 2b^2 \\
\frac{182}{3}ae - 2bf &= -\frac{1}{2} \\
\frac{182}{3}ag - 2bh &= 0 \\
\frac{182}{3}ce - 2df &= 0 \\
\frac{182}{3}cg - 2dh &= \frac{1}{14} \\
-2 &= -8fb + 56hd
\end{aligned}$$

Solution of the system of equations in terms of the a_is:

$$\begin{aligned}
a &= \frac{1}{182}\sqrt{546a_9 + 273a_2^2} \\
b &= -\frac{a_2}{2} \\
c &= \frac{3a_1\sqrt{2a_9 + a_2^2}}{14a_3\sqrt{273}} \\
d &= -\frac{a_4}{2}
\end{aligned}$$

$$
\begin{aligned}
e &= \frac{21a_3a_4}{2\sqrt{546a_9+273a_2^2}} \\
f &= -\frac{a_1}{2} \\
g &= \frac{3a_3a_2}{2\sqrt{546a_9+273a_2^2}} \\
h &= -\frac{a_3}{2}
\end{aligned}
$$

With the relations:

$$
\begin{aligned}
a_1 &= \frac{7a_3a_4+1}{a_2} \\
a_7 &= \frac{a_1a_4a_9+a_2a_4+a_8}{7a_2a_3} \\
a_8 &= \frac{2a_1a_9+a_2}{14a_3}
\end{aligned}
$$

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