2011-06-20

A Multi-Frequency Inverse Source Problem for the Helmholtz Equation

Sebastian Ignacio Acosta
Brigham Young University - Provo

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Sebastian Acosta

A thesis submitted to the faculty of
Brigham Young University
in partial fulfillment of the requirements for the degree of

Master of Science

Vianey Villamizar, Chair
Shue-Sum Chow
Tiancheng Ouyang

Department of Mathematics

Brigham Young University

August 2011

Abstract

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The inverse source problem for the Helmholtz equation is studied. An unknown source is to be identified from the knowledge of its radiated wave. The focus is placed on the effect that multi-frequency data has on establishing uniqueness. In particular, we prove that data obtained from finitely many frequencies is not sufficient. On the other hand, if the frequency varies within an open interval of the positive real line, then the source is determined uniquely. An algorithm for the reconstruction of the source using multi-frequency data is proposed. The algorithm is based on an incomplete Fourier transform of the measured data and we establish an error estimate under certain regularity assumptions on the source function. We conclude that multi-frequency data not only leads to uniqueness for the inverse source problem, but in fact it contributes with a stability result for the reconstruction of an unknown source.

Keywords: Inverse source problem, Helmholtz equation, multi-frequency.

Acknowledgments

First and foremost, I would like to thank my advisor, Professor Vianey Villamizar, for his guidance, enthusiasm and friendship. He has introduced me to the fascinating world of applied mathematics, and has had a sincere interest in my academic and personal development. Working with him has been an immense privilege and I look forward to our future collaborations. I also thank Dr. Villamizar for sparking my interest in the topic of this thesis. *Indeed, wave propagation problems have provided a beautiful training ground for budding applied mathematicians to sharpen their skills in analysis* [1, p. 45].

My education at BYU has been incredibly enhanced by the mentoring environment developed by the Department of Mathematics. I thank all the department faculty and staff responsible for the undergraduate and graduate research programs. I thankfully acknowledge the financial support provided by the BYU Graduate Research Fellowship and the Department of Mathematics graduate tuition award.

I wish to express my gratitude to Professors Sum Chow and Tiancheng Ouyang for serving as members of my Master thesis committee. I thank Professor Chow for his invaluable course on numerical linear algebra. For actively participating in our seminar on integral equations and for providing insightful comments about our work on inverse problems. I also thank James Taylor for his indispensable participation in our research project and fruitful discussions which helped me understand the subject more clearly.

Finally, I want to express my love and gratitude to my wonderful wife, Gabriela, and daughter, Naomi. They are the source of my motivation and joy. This work would not have been possible without their constant support, patience, encouragement and love. I thank my parents and brothers as well, for supporting me in all my endeavors.

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Chapter 1. Introduction

I am not able to learn any mathematics unless I can see some problem I am going to solve with mathematics ...

Steven Weinberg

The greatest mathematicians, such as Archimedes, Newton, and Gauss, always united theory and application in equal measure.

Felix Klein

Over the last 25 years, the study of inverse problems has experienced tremendous progress from the mathematical point of view. This growth is reflected in the appearance of the following four academic journals: Inverse Problems (1985), Journal of Inverse and Ill-Posed Problems (1993), Inverse Problems in Science and Engineering (2004), and Inverse Problems and Imaging (2007), as well as in the publication of several applied mathematics monographs such as [2, 3, 4, 5, 6, 7, 8, 9], and some topical reviews on inverse problems for differential equations [10, 11, 12, 13, 14, 15, 16, 17, 18]. This mathematical activity has been motivated by the fundamental role that inverse problems for partial differential equations play in many technological areas such as radar, sonar and satellite imaging [19, 20], geological prospection [13, 21, 22, 23, 24, 25], astronomical exploration [26], medical imaging [21, 23, 25, 27, 28, 29], systems biology [29, 30, 31], and nondestructive testing [13, 21, 25]. In the present work, we study the inverse source problem for the Helmholtz equation which has particular applications in medical imaging techniques such as electroencephalography [32, 33], magnetoencephalography [34, 35, 36, 37], photoacoustic tomography [38], and optical tomography [28]. In addition, this inverse source problem is of great interest in theoretical physics. It has been a subject of study concerning the physics of invisibility, radiation reaction, and the quantum mechanical stability of elementary particles. For a good review see [39].

The relationship between a direct problem and its inverse is described by Keller [40, p. 107] as follows,

> *We call two problems inverses of one another if the formulation of each involves all or part of the solution of the other. Often, for historical reasons, one of the two problems has been studied extensively for some time, while the other is newer and not so well understood. In such cases, the former is called the direct problem, while the latter is called the inverse problem.*

The direct source problem for the Helmholtz equation is, given the forcing function F, to find the radiated wave field. On the other hand, the inverse source problem starts with the knowledge of the radiated wave U on some surface Γ enclosing the support of the source F and asks for the nature of this source.

The direct source problem has been thoroughly investigated over the past centuries and a huge amount of information is available in the literature. We review the main mathematical results in Chapter 2 and provide plenty of references therein. In contrast, the inverse source problem has received much less attention especially in connection with a solid mathematical foundation. The reason for this is that this inverse problem is inherently ill-posed in terms of uniqueness. More specifically, due to the existence of non-radiating sources and the linearity of the inverse source problem, an infinity of solutions can be obtained by adding any of the non-radiating sources to a given solution. Hence, it is impossible for the true source to be uniquely reconstructed from a single set of measurements on the surface Γ. This phenomenon, to be reviewed in Chapter 3, holds true for problems governed by the Helmholtz (acoustics), Maxwell (electrodynamics) and Laplace (gravimetry) equations [2, 3, 41, 37, 42, 43, 44]. For comprehensive studies of such ill-posed problems and regularization methods, we refer the reader to [5, 45, 46, 47, 48].

We assume the presence of a source function F with its support in a bounded open set

$\Omega \subset \mathbb{R}^3$ with smooth boundary Γ. The radiated field U satisfies the following problem

$$\Delta U(x,k) + k^2 U(x,k) = -F(x,k) \quad \text{in} \quad \mathbb{R}^3, \tag{1.1}$$

$$\lim_{r\to\infty} r\left(\frac{\partial U(x,k)}{\partial r} - ikU(x,k)\right) = 0, \tag{1.2}$$

where $r = |x|$. The limit in (1.2) is known as the Sommerfeld radiation condition. Here $k > 0$ is the *wavenumber*, although, we will refer to it as the *frequency*. Notice that $U = U(x,k)$ depends on the frequency since k appears in the left-hand side of the Helmholtz equation and $F = F(x,k)$ depends on k as well. In the present work, we only consider separable sources of the form

$$F(x,k) = g(k)f(x), \tag{1.3}$$

where $g(k) \neq 0$ is a known function. Hence we can define $u(x,k) = U(x,k)/g(k)$ which will satisfy a new problem with a frequency-independent source f given by

$$\Delta u(x,k) + k^2 u(x,k) = -f(x) \quad \text{in} \quad \mathbb{R}^3, \tag{1.4}$$

$$\lim_{r\to\infty} r\left(\frac{\partial u(x,k)}{\partial r} - iku(x,k)\right) = 0. \tag{1.5}$$

The multi-frequency inverse problem consists of identifying f from measurements of the radiated field u on Γ for all frequencies $k \in \mathcal{K}$ where $\mathcal{K} \subset \mathbb{R}_+$ is the set of admissible frequencies. This research topic is motivated by applications in acoustic, elastic and electromagnetic remote sensing where the wave frequency is a user controlled parameter and the spatial profile of the source function is frequency-independent.

The focus of this work is placed on the effect that multi-frequency data has on establishing uniqueness, i.e., whether the unknown source can be identified from the knowledge of multi-frequency measurements of the radiated waves. One of the major contributions of our work, found in Chapter 3, is a proof that multi-frequency data does indeed determine the unknown source uniquely. It is proper to mention here that, in the course of the present work, we found that Eller and Valdivia [49], and Bao et al. [50] have recently investigated the same problem. In terms of uniqueness, Eller and Valdivia proved that the knowledge of $u(x,k)$ for all $x \in \Gamma$ and all $k \in \mathcal{K}$ is sufficient to recover f uniquely if $\mathcal{K} = \mathbb{R}_+$. In fact, their proof requires data obtained at frequencies coinciding with the Dirichlet eigenvalues of the negative Laplacian in the region Ω. Bao et al. improved the uniqueness result by showing that measured data from a set of frequencies with an accumulation point is enough the identify the source uniquely. We complement their results in two directions. First, we prove that if $\mathcal{K}$ is any open interval of the positive real line, then the source is determined uniquely. On the other hand, we prove that data obtained from finitely many frequencies is not sufficient to recover f uniquely, that is, we show the existence of sources that do not radiate at a finite number of distinct frequencies. We also mention that the unique determination of the location and intensity of point sources has been shown in [33] for data obtained at a single frequency.

We also characterize the set of non-radiating and purely-radiating sources, and discuss the idea of minimum-norm solutions for the inverse problem at a fixed single frequency. As a consequence, rigorous results are obtained concerning the unique decomposition of any source $f \in L^2(\Omega)$ as the superposition of a non-radiating component f_N and a purely-radiating part f_P which happens to be the minimum-L^2-norm solution to the inverse problem. Results in this direction were initiated by Marengo, Devaney and Ziolkowski [44, 51, 52] among others, and rigorously established by Albanese and Monk for Maxwell equations [37].

An algorithm for the reconstruction of the source using multi-frequency data is proposed. This is done in Chapter 4. We assume that the input data is available for a range of frequencies belonging to an interval $\mathcal{K} = (0, K)$. The algorithm is based on an incomplete Fourier transform of the measured data and we establish an error estimate under certain regularity assumptions on the source function f. We conclude that multi-frequency data not only leads to uniqueness for the inverse source problem, but in fact it contributes with a stability result for the reconstruction of an unknown source.

Chapter 2. Direct Source Problem

I am convinced that it will be possible to get these existence proofs by a general basic idea ... Perhaps it will then also be possible to answer the question of whether or not every regular variational problem possesses a solution if, with regard to boundary conditions, certain assumptions are fulfilled and if, when necessary, one sensibly generalizes the concept of solution.

David Hilbert

In this chapter we start by introducing the classical formulation of the direct source problem for the Helmholtz equation in $\mathbb{R}^3$. In view of the inverse source problem to be discussed in the next chapter, we consider a source with compact support. Hence, the boundary value problem can be equivalently posed in a bounded domain by employing the Dirichlet-to-Neumann (DtN) map as an artificial boundary condition imposed on the boundary of the truncated domain. The well-posedness of the classical formulation is briefly discussed in this chapter.

Then, we define and prove the well-posedness of the variational formulation of the direct source problem for the Helmholtz equation. We will base our study of the variational problem on the excellent books by Gilbarg and Trudinger [53] and McLean [54]. The basic tools necessary to prove existence and uniqueness of the sought solutions are based on some functional analytical tools stated in Appendix A. These essential tools are the Sobolev embedding theorems, the Lax-Milgram theorem, and the Riesz-Fredholm theory for compact perturbations of the identity operator. In Appendix B, we also review the application of Green's identities to functions in the appropriate Sobolev spaces which are employed in the proof of uniqueness.

2.1 Classical Formulation

Here we consider the boundary value problem (BVP) for the Helmholtz equation in all of $\mathbb{R}^3$. We assume the presence of a source function f with its support in a bounded open set $\Omega \subset \mathbb{R}^3$. The domain Ω is assumed to have a smooth boundary Γ and a connected complement. The classical BVP is formulated as follows.

Problem 2.1. *Given a source function f with support in Ω, find a function $u \in C^2(\mathbb{R}^3)$ satisfying*

$$\Delta u + k^2 u = -f \quad \text{in} \quad \mathbb{R}^3, \tag{2.1}$$

$$\lim_{r\to\infty} r\left(\frac{\partial u}{\partial r} - iku\right) = 0, \tag{2.2}$$

where $r = |x|$. The limit in (2.2), known as the Sommerfeld radiation condition, is assumed to hold uniformly in all directions $x/|x|$.

This problem has been shown to have a unique classical solution if the source function f is sufficiently smooth. In fact, if $f \in C^{0,\alpha}(\mathbb{R}^3)$ then the unique solution $u \in C^{2,\alpha}(\mathbb{R}^3)$ is explicitly given by,

$$u(x) = \int_\Omega G(x,y)f(y)dy, \qquad x \in \mathbb{R}^3, \tag{2.3}$$

where $G(x,y)$ is the well-known fundamental solution (or free-space Green's function) for the Helmholtz equation,

$$G(x,y) = \frac{1}{4\pi}\frac{e^{ik|x-y|}}{|x-y|}, \qquad x \neq y. \tag{2.4}$$

For the proof on the regularity of the solution (2.3) see [4, Theorem 8.1] or [53, Lemmas 4.1-4.2]. Uniqueness is proven by combining [55, Theorem 3.1] and [55, Theorem 3.3].

It is possible to formulate an equivalent BVP in the bounded domain Ω by replacing the

Sommerfeld radiation condition at infinity by an artificial boundary condition on Γ. This boundary condition is known as the Dirichlet-to-Neumann (DtN) condition. Surprisingly, it renders a BVP in Ω whose solution coincides exactly with the restriction to Ω of the solution of the Problem 2.1 originally defined in all of $\mathbb{R}^3$. We will rigorously establish this result in Corollary 2.13 in the framework of variational problems.

The DtN map has become a powerful tool to numerically handle BVPs defined in unbounded domains. For excellent references concerning the DtN map see [56, 57, 58, 59, 60, 61]. However, the DtN map not only has a great value in numerical settings. It is also employed as a theoretical tool for various direct and inverse problems. For instance, it is used in the study of surface potentials for acoustic scattering [4, Section 3.2], and the mathematical analysis of electrical impedance tomography [62, Chapter 8], [63, 64]. We will also take advantage of the DtN map to set up the weak counterpart of the Problem 2.1. It will yield the possibility to define the variational problem in a bounded domain where the compactness of the Sobolev embeddings is valid. For completeness, we state the classical formulation of the source problem based on the DtN boundary condition imposed on the surface Γ.

Problem 2.2. *Given a source function f, find a function $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ satisfying*

$$\Delta u + k^2 u = -f \quad \textit{in} \quad \Omega, \tag{2.5}$$

$$\frac{\partial u}{\partial \nu} - Mu = 0 \quad \textit{on} \quad \Gamma, \tag{2.6}$$

where ν denotes the outward unit normal vector on Γ, and $M : C(\Gamma) \to C(\Gamma)$ denotes the Dirichlet-to-Neumann map which transfers the boundary values of a radiating solution of the Helmholtz equation in $\mathbb{R}^3 \setminus \Omega$ into its normal derivative at the boundary Γ.

2.2 Variational Formulation

For the direct source problem governed by Helmholtz equation in a bounded domain Ω with smooth boundary Γ, the natural choice of trial and test spaces is the Sobolev space $H^1(\Omega)$.

The variational counterpart of the Classical Problem 2.2 is defined as follows.

Problem 2.3 (Direct Problem)**.** *Given a source function $f \in L^2(\Omega)$, find a function $u \in H^1(\Omega)$ satisfying*

$$B(u,v) = \langle f, v\rangle_{L^2(\Omega)}, \qquad \text{for all} \quad v \in H^1(\Omega) \tag{2.7}$$

where the sesquilinear form $B : H^1(\Omega) \times H^1(\Omega) \to \mathbb{C}$ is defined as follows,

$$B(u,v) = \langle \nabla u, \nabla v\rangle_{L^2(\Omega)} - k^2 \langle u, v\rangle_{L^2(\Omega)} - \langle MTu, Tv\rangle_{L^2(\Gamma)}. \tag{2.8}$$

Here T is the trace operator from Theorem B.1 in the Appendix, and $M : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ denotes the Dirichlet-to-Neumann map which transfers the boundary values of a radiating solution of the Helmholtz equation in $\mathbb{R}^3 \setminus \Omega$ into its normal derivative at the boundary Γ.

In order to study the well-posedness of this variational problem, we need to analyze the properties of the sesquilinear form B. We start by stating the properties of the Dirichlet-to-Neumann operator [59, 61]. The Dirichlet-to-Neumann operator is also known in the literature as the Steklov-Poincaré operator [54, Chapter 4].

Lemma 2.4. *The Dirichlet-to-Neumann operator $M : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is a well-defined bounded linear operator. Furthermore, it satisfies the following property: Given $w \in H^{1/2}(\Gamma)$, if $\mathfrak{Im}\langle Mw, w\rangle_{L^2(\Gamma)} = 0$ then $w = 0$. The adjoint operator $M^* : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ enjoys the same property.*

Proof. This follows from the well-posedness of the exterior scattering problem in $H^1_{\mathrm{loc}}(\mathbb{R}^3 \setminus \Omega)$ for Dirichlet boundary data in $H^{1/2}(\Gamma)$. See [4, Chapter 3], [54, Chapters 4,9] or [65, Chapter 2]. The boundedness of M follows from the stability of the exterior Dirichlet problem for the Helmholtz equation. See [4, Chapter 3] or [65, Chapter 2], or a good study of the Steklov-Poincaré operator found in [54, Chapter 4].

Now let u be the solution to the exterior scattering problem with Dirichlet boundary data $w \in H^{1/2}(\Gamma)$ and assume that $\mathfrak{Im}\langle Mw, w\rangle_{L^2(\Gamma)} = 0$. By definition of the DtN operator we have that $D_\nu u = Mw$. Then $\mathfrak{Im}\langle D_\nu u, Tu\rangle_{L^2(\Gamma)} = 0$ and by [4, Theorem 2.12] or [54, Lemma 9.9] we conclude that $u = 0$. This implies that $w = Tu = 0$.

Regarding the adjoint operator M^*, notice that if $\mathfrak{Im}\langle M^*w, w\rangle_{L^2(\Gamma)} = 0$ then we have $\mathfrak{Im}\langle Mw, w\rangle_{L^2(\Gamma)} = 0$ and from the previous part of this lemma, it follows that $w = 0$. $\square$

Now we prove that the sesquilinear form B defined in (2.8) is bounded.

Lemma 2.5. *The sesquilinear form $B : H^1(\Omega) \times H^1(\Omega) \to \mathbb{C}$ defined in (2.8) is bounded, i.e., there exists a constant C such that $|B(u, v)| \leq C\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}$ for all $u, v \in H^1(\Omega)$.*

Proof. Let $u, v \in H^1(\Omega)$. Then

$$\begin{aligned} |B(u,v)| &\leq \|\nabla u\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)} + k^2\|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} + \|M\|\|T\|\|u\|_{H^1(\Omega)}\|T\|\|v\|_{H^1(\Omega)} \\ &\leq (1 + k^2 + \|M\|\|T\|^2)\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}, \end{aligned}$$

where we have used the boundedness of the operators M and T as stated by Lemma 2.4 and the Trace Theorem B.1 in the Appendix. This concludes the proof. $\square$

This estimate is significant from the point of view of the existence theory for the Problem 2.3. By virtue of the Lax-Milgram Theorem A.1 in the Appendix, we can re-express the Direct Problem 2.3 as an operator equation. Notice that

$$B(u, v) = \langle u, v\rangle_{H^1(\Omega)} - (1 + k^2)\langle u, v\rangle_{L^2(\Omega)} - \langle MTu, Tv\rangle_{L^2(\Gamma)},$$

and that both $\langle u, v\rangle_{L^2(\Omega)}$ and $\langle MTu, Tv\rangle_{L^2(\Gamma)}$ define bounded sesquilinear forms on $H^1(\Omega) \times H^1(\Omega)$ as shown in Lemma 2.5. Hence by the first part of the Lax-Milgram Theorem A.1 (which is a direct consequence of the Riesz representation theorem), there exist bounded linear operators $A, C : H^1(\Omega) \to H^1(\Omega)$ and a unique element $F \in H^1(\Omega)$ such that

$$\langle Au, v\rangle_{H^1(\Omega)} = \langle u, v\rangle_{L^2(\Omega)} \quad \text{for all} \quad u, v \in H^1(\Omega), \tag{2.9}$$

$$\langle Cu, v\rangle_{H^1(\Omega)} = \langle MTu, Tv\rangle_{L^2(\Gamma)} \quad \text{for all} \quad u, v \in H^1(\Omega), \tag{2.10}$$

$$\langle F, v\rangle_{H^1(\Omega)} = \langle f, v\rangle_{L^2(\Omega)} \quad \text{for all} \quad v \in H^1(\Omega). \tag{2.11}$$

Hence, we can re-write (2.7) as follows,

$$\langle (I - (1 + k^2)A - C)u, v\rangle_{H^1(\Omega)} = \langle F, v\rangle_{H^1(\Omega)} \quad \text{for all} \quad v \in H^1(\Omega).$$

As is commonly done in the study of the well-posedness of variational problems, we have proven that the Direct Problem 2.3 is equivalent to the following one.

Problem 2.6 (Operator Problem). *Find $u \in H^1(\Omega)$ satisfying*

$$(I - (1 + k^2)A - C)u = F, \tag{2.12}$$

where A, C and F are define by (2.9)-(2.11).

2.3 Well-Posedness

In this section we proceed to prove the solvability of the Direct Problem 2.3. The proof of the existence of a solution will be based on the Riesz-Fredholm Theorem A.5 for operator equations of the second kind. See Appendix A. Hence, we shall prove uniqueness first and then show that the hypothesis of the Riesz-Fredholm theorem is indeed satisfied. Hence, the Direct Problem 2.3 will be shown to enjoy the existence and uniqueness of a solution, as well as, its stability on the forcing data. First, however, we need the following fundamental result concerning the regularity of the variational solution to the Problem 2.3. A proof can be found in [54, Chapters 4,7]. See also [53, Chapter 8], [66, Chapter 9], [67, Section 6.2] and [68, Chapter 8].

Theorem 2.7 (Regularity)**.** *Let $u \in H^1(\Omega)$ be a variational solution of the problem*

$$\Delta u + k^2 u = -f \quad \text{in} \quad \Omega, \quad \text{and} \quad \frac{\partial u}{\partial \nu} = g \quad \text{on} \quad \Gamma,$$

for $f \in L^2(\Omega)$ and $g \in H^{1/2}(\Gamma)$. Assume also that Γ is of class C^2. Then $u \in H^2(\Omega)$ and in fact $\Delta u + k^2 u = -f$ a.e. in Ω (u is a strong solution). In general, if $f \in H^k(\Omega)$ and $g \in H^{k+1/2}(\Gamma)$ for $k \geq 0$ and Γ is C^{k+2} then $u \in H^{k+2}(\Omega)$. Furthermore, if $f = 0$ and $g \in C(\Gamma)$ then u is analytic and it is a classical solution to the Helmholtz equation in Ω.

Theorem 2.8 (Uniqueness)**.** *The Direct Problem 2.3 has at most one solution.*

Proof. We have to show that all solutions to the homogeneous problem ($f = 0$) vanish. Suppose $u \in H^1(\Omega)$ is a solution to the homogeneous Direct Problem 2.3. Then

$$-\,\mathfrak{Im}\; B(u,u) = \mathfrak{Im}\; \langle MTu, Tu \rangle_{L^2(\Gamma)} = 0.$$

By Lemma 2.4 then $Tu = 0$ which implies that $MTu = 0$.

Now from the regularity result summarized in Theorem 2.7, we have that $u \in H^2(\Omega)$ and we are in position to employ Green's first identity as follows,

$$\langle \Delta u, v \rangle_{L^2(\Omega)} + \langle \nabla u, \nabla v \rangle_{L^2(\Omega)} = \langle D_\nu u, Tv \rangle_{L^2(\Gamma)}, \qquad \text{for all} \quad v \in H^1(\Omega).$$

Combining the above equation with (2.7) we obtain

$$\langle \Delta u + k^2 u, v \rangle_{L^2(\Omega)} = \langle D_\nu u - MTu, Tv \rangle_{L^2(\Gamma)}, \qquad \text{for all} \quad v \in H^1(\Omega).$$

This last equation being particularly true for all $v \in H_0^1(\Omega)$ which implies that $\Delta u + k^2 u = 0$ in the L^2-sense since $H_0^1(\Omega)$ is dense in $L^2(\Omega)$. Then from the same equation we conclude that

$$\langle D_\nu u - MTu, Tv\rangle_{L^2(\Gamma)} = 0, \qquad \text{for all} \quad v \in H^1(\Omega).$$

Now, from Theorem B.1 in the Appendix, we learned that the trace operator $T : H^1(\Omega) \to H^{1/2}(\Gamma)$ is surjective. Since $H^{1/2}(\Gamma)$ is dense in $L^2(\Gamma)$, we conclude that $D_\nu u = MTu$ in the L^2-sense. We combine these last results with Green's third identity (Theorem B.5 in the Appendix) to obtain,

$$\begin{aligned} u &= \mathcal{S}D_\nu u - \mathcal{K}Tu - \mathcal{G}(\Delta u + k^2 u) \\ &= \mathcal{S}MTu - \mathcal{K}Tu - \mathcal{G}(\Delta u + k^2 u). \end{aligned}$$

The right-hand side above vanishes identically since we have already shown that $Tu = 0$ and $\Delta u + k^2 u = 0$. This concludes the proof. □

Now we take advantage of the equivalence between the original Direct Problem 2.3 and Operator Problem 2.6. The goal is to use the Riesz-Fredholm theory for operator equations of the second kind to prove the existence of a unique solution for (2.12). Since we have already proven uniqueness, it only remains to show that the hypothesis of the Riesz-Fredholm Theorem A.5 is satisfied (see Appendix A). In other words, we only need to show that the operators A and C defined in (2.9)-(2.10) are compact. This we do without further ado.

Lemma 2.9. *The operator $A : H^1(\Omega) \to H^1(\Omega)$ is compact.*

Proof. Let $\{u_n\} \subset H^1(\Omega)$ be a sequence bounded in the H^1-norm. By the weak-compactness (Theorem A.3) of the Hilbert space $H^1(\Omega)$ then it has a weakly convergent subsequence. Allowing some abuse of notation, we denote it by $\{u_n\}$. Hence, there exists $u \in H^1(\Omega)$ such that

$$\langle u_n, v\rangle_{H^1(\Omega)} \to \langle u, v\rangle_{H^1(\Omega)} \quad \text{as} \quad n \to \infty, \quad \text{for all} \quad v \in H^1(\Omega).$$

Now we claim that $\{Au_n\}$ converges to Au in the H^1-norm which would establish the compactness of the operator A.

$$\begin{aligned}
\|Au_n - Au\|^2_{H^1(\Omega)} &= \langle Au_n, Au_n\rangle_{H^1(\Omega)} - 2\mathfrak{Re}\langle Au_n, Au\rangle_{H^1(\Omega)} + \langle Au, Au\rangle_{H^1(\Omega)} \\
&\leq |\langle Au_n, Au_n - Au\rangle_{H^1(\Omega)}| + |\langle Au - Au_n, Au\rangle_{H^1(\Omega)}| \\
&= |\langle u_n, Au_n - Au\rangle_{L^2(\Omega)}| + |\langle Au - Au_n, Au\rangle_{H^1(\Omega)}| \\
&\leq \|u_n\|_{L^2(\Omega)}\|Au_n - Au\|_{L^2(\Omega)} + |\langle Au - Au_n, Au\rangle_{H^1(\Omega)}|.
\end{aligned}$$

Here we have used (2.9) and the Cauchy-Schwarz inequality. Notice now that the second term on the right-hand side approaches zero since $\{Au_n\}$ is weakly convergent to Au (Theorem A.4). Moreover, $\{u_n\}$ is a bounded sequence in both the H^1- and L^2-norms. The operator $A : H^1(\Omega) \to H^1(\Omega)$ is bounded and since $H^1(\Omega)$ is compactly embedded into $L^2(\Omega)$ (Theorem A.6) then $A : H^1(\Omega) \to L^2(\Omega)$ is compact. From Theorem A.4 it follows that the first term above approaches zero as well. This conclude s the proof. $\square$

Lemma 2.10. *The operator $C : H^1(\Omega) \to H^1(\Omega)$ is compact.*

Proof. This proof is very similar to the preceding one. Let $\{u_n\} \subset H^1(\Omega)$ be a sequence bounded in the H^1-norm. By the weak-compactness (Theorem A.3) of the Hilbert space $H^1(\Omega)$ then it has a weakly convergent subsequence. Allowing some abuse of notation, we denote it by $\{u_n\}$. Hence, there exists $u \in H^1(\Omega)$ such that $u_n \rightharpoonup u$ in the H^1-sense. Now we claim that $\{Cu_n\}$ converges to Cu in the H^1-norm.

$$\begin{aligned}
\|Cu_n - Cu\|^2_{H^1(\Omega)} &= \langle Cu_n, Cu_n\rangle_{H^1(\Omega)} - 2\mathfrak{Re}\langle Cu_n, Cu\rangle_{H^1(\Omega)} + \langle Cu, Cu\rangle_{H^1(\Omega)} \\
&\leq |\langle Cu_n, Cu_n - Cu\rangle_{H^1(\Omega)}| + |\langle Cu - Cu_n, Cu\rangle_{H^1(\Omega)}| \\
&= |\langle MTu_n, Tu_n - Tu\rangle_{L^2(\Omega)}| + |\langle Cu - Cu_n, Cu\rangle_{H^1(\Omega)}| \\
&\leq \|M\|\|T\|\|u_n\|_{H^1(\Omega)}\|Tu_n - Tu\|_{L^2(\Omega)} + |\langle Cu - Cu_n, Cu\rangle_{H^1(\Omega)}|.
\end{aligned}$$

Here we have used (2.10). Notice now that the second term approaches zero since $\{Cu_n\}$ is weakly convergent to Cu (Theorem A.4). Moreover, $\{u_n\}$ is a bounded sequence in the H^1-norm. The trace operator $T : H^1(\Omega) \to H^{1/2}(\Gamma)$ is bounded and since $H^{1/2}(\Gamma)$ is compactly embedded into $L^2(\Gamma)$ (Theorem A.6), then $T : H^1(\Omega) \to L^2(\Gamma)$ is compact. From Theorem A.4 it follows that the first term approaches zero as well. This establishes the compactness of the operator C. □

Finally we are ready to conclude the well-posedness of the Direct Problem 2.3 or equivalently of the Operator Problem 2.6.

Theorem 2.11 (Well-Posedness). *For each $f \in L^2(\Omega)$, the Direct Problem 2.3 has a unique solution $u \in H^1(\Omega)$ and it depends continuously on the forcing data f, that is, there exists a constant K such that*

$$\|u\|_{H^1(\Omega)} \leq K\|f\|_{L^2(\Omega)}.$$

Proof. The proof relies on checking that the hypothesis of the Riesz-Fredholm Theorem A.5 is satisfied so that it can be applied to the Operator Problem 2.6. Indeed, this is the case since it follows from Lemmas 2.9-2.10. The injectivity of the operator $(I - (1 + k^2)A - C)$ is established by the uniqueness Theorem 2.8. Hence, we conclude that the Direct Problem 2.3 is well-posed. □

It is proper to mention here that the unique solution to the source problem for the Helmholtz equation in $\mathbb{R}^3$ can be written explicitly as a volume potential just like in the classical sense. See Section 2.1. Recall that the volume potential of f is the function $u = \mathcal{G}f$ defined by the following expression,

$$u(x) = (\mathcal{G}f)(x) = \int_{\mathbb{R}^3} G(x, y)f(y)dy, \qquad x \in \mathbb{R}^3, \tag{2.13}$$

where G is the fundamental solution to the Helmholtz equation defined by (2.4) and $f \in L^2(\mathbb{R}^3)$ is assumed to be compactly supported. The following theorem proves that (2.13) is indeed a *strong* solution for both Problems 2.1 and 2.2.

Theorem 2.12. *Let $f \in L^2(\mathbb{R}^3)$ be compactly supported, and u be the volume potential of f as defined by (2.13). Then $u \in H^2_{loc}(\mathbb{R}^3)$ and $\Delta u + k^2 u = -f$ a.e. in $\mathbb{R}^3$. In addition, u satisfies the Sommerfeld radiation condition (2.2) and the Dirichlet-to-Neumann equation (2.6) on any surface that encloses the support of f. Hence u is a* strong *solution for both Problems 2.1 and 2.2.*

Proof. The proof of the regularity of the volume potential can be found in [53, Theorem 9.9], [4, Theorem 8.2] and [69]. Now, since G satisfies the Sommerfeld radiation condition then so does u. Hence u is a *strong* solution for Problem 2.1.

Now u is a radiating solution to the homogeneous Helmholtz equation in the exterior of any surface that encloses the support of f. By definition of the DtN map, then u satisfies equation (2.6) on that surface. Therefore, u is a *strong* solution for Problem 2.2. □

The first consequence of the above theorem is that the volume potential (2.13) is indeed the unique solution for the variational Direct Problem 2.3. This follows from the fact that a strong solution of a BVP can easily be shown to satisfy the variational counterpart of the same problem. The next implication is concerning the DtN map. Since the volume potential (2.13) is simultaneously the unique solution for both the original Problem 2.1 defined in all of $\mathbb{R}^3$ and the variational Direct Problem 2.3 defined in Ω, then we have shown the following assertion.

Corollary 2.13. *The unique solution for the variational Direct Problem 2.3 coincides with the restriction of the unique strong solution for the original Problem 2.1 to the domain Ω.*

Chapter 3. Inverse Source Problem

A lack of information cannot be remedied by any mathematical trickery !

Cornelius Lanczos

The purpose of this chapter is to investigate the inverse source problem for the Helmholtz equation. Recall that given a source f and a frequency $k > 0$ there exists a unique solution $u \in H^1(\Omega)$ that satisfies the Direct Problem 2.3. When needed, we will make explicit reference to the dependence of u on the frequency k by writing $u = u_k$ or $u = u(x, k)$. We will consider an admissible set of frequencies denoted by $\mathcal{K} \subset \mathbb{R}_+$. With this notation we are ready to define the inverse problem.

Problem 3.1 (Inverse Source Problem)**.** *Let $u_k \in H^1(\Omega)$ be the solution to the Direct Problem 2.3 for given frequency $k \in \mathcal{K}$ and some unknown source $f \in L^2(\Omega)$. The inverse source problem is, given the traces Tu_k on Γ, find the source f.*

The focus is placed on the effect that multi-frequency data has on establishing uniqueness, i.e., whether the unknown source can be identified from the knowledge of multi-frequency measurements of the radiated waves. One of the major contributions of this work is a proof that multi-frequency data does indeed determine the unknown source uniquely. We answer the following question: How much data is needed to determine the source ? In particular, we prove that data obtained from finitely many frequencies is not sufficient, that is, we show the existence of sources that do not radiate at any finite number of distinct frequencies. On the other hand, if the frequency varies within an open interval of the positive real line, then the source is determined uniquely.

We start this chapter by characterizing the set of non-radiating and introducing the idea of minimum-norm solutions for the inverse problem. Then, we proceed to characterize the set of purely-radiating sources. These concepts are made mathematically precise using the variational setting laid out in Chapter 2. As a consequence, we reproduce the analogue results obtained by Albanese and Monk for Maxwell's equations [37] and previously investigated by Marengo, Devaney and Ziolkowski [44, 51, 52] among others.

3.1 Non-Uniqueness and Minimum-Norm Solutions

We begin this section by giving a precise definition of non-radiating sources for the Helmholtz equation.

Definition 3.2 (Non-Radiating Source)**.** A source $f \in L^2(\Omega)$ is said to be *non-radiating* at a frequency k if the solution $u_k \in H^1(\Omega)$ of the Direct Problem 2.3 corresponding to this source f is such that $Tu_k = 0$ on Γ. Let $N(\Omega, k)$ denote the set of all non-radiating sources for the Helmholtz equation at the frequency k.

In this section we seek the following orthogonal decomposition of $L^2(\Omega)$,

$$L^2(\Omega) = N(\Omega, k) \oplus N(\Omega, k)^{\perp} \tag{3.1}$$

This decomposition is valid if $N(\Omega, k)$ is a closed subspace of $L^2(\Omega)$. This is the case because $N(\Omega, k)$ is the nullspace of the operator $T \circ S_k : L^2(\Omega) \to H^{1/2}(\Gamma)$ where T is the trace operator given by Theorem B.1 and $S_k : L^2(\Omega) \to H^1(\Omega)$ represents the *solution operator* for the Direct Problem 2.3, i.e., it maps the source f into the solution u_k of the direct problem. From the well-posedness Theorem 2.11 it follows that S_k is a bounded operator. Since both operators T and S_k are bounded, then $N(\Omega, k) = \text{null}(T \circ S_k)$ is a closed subspace of $L^2(\Omega)$ and the orthogonal decomposition (3.1) is well-defined. As a consequence, we have the following definition for purely-radiating sources.

Definition 3.3 (Purely-Radiating Source)**.** A source $f \in L^2(\Omega)$ is said to be *purely-radiating* at a frequency k if $f \in N(\Omega, k)^{\perp}$.

The goal of this section is to characterize $N(\Omega, k)$ in an interesting way and show that it is not a trivial vector space. We begin by defining the following set

$$\mathcal{N}(\Omega, k) = \big\{ g \in L^2(\Omega) \ \text{ such that } g = \Delta w + k^2 w \ \text{ for some } \ w \in C_c^{\infty}(\Omega) \big\}. \tag{3.2}$$

It is clear that $\mathcal{N}(\Omega, k) \subset N(\Omega, k)$ because if $g \in \mathcal{N}(\Omega, k)$ so that $g = \Delta w + k^2 w$ for some $w \in C_c^{\infty}(\Omega)$ then $Tw = 0$ on Γ, and by definition $g \in N(\Omega, k)$. Moreover, $\mathcal{N}(\Omega, k)$ is a non-trivial vector space since every nonzero function in $C_c^{\infty}(\Omega)$ does not satisfy the Helmholtz equation in Ω. This follows from the fact that a solution to the Helmholtz equation is an analytic function. This simple argument shows the existence of non-radiating sources for the Helmholtz equation. Similarly, it is possible to show the existence of non-radiating sources for many other linear partial differential equations such as the Maxwell, Schrödinger, Laplace, Stokes, and Navier equations.

Remark 3.4 (Non-Uniqueness)**.** The fact that $\mathcal{N}(\Omega, k) \subset N(\Omega, k)$ is a non-trivial space implies that the inverse source problem cannot be solved uniquely given boundary data at a single frequency. This is one of the major difficulties encountered in the mathematical study of the inverse source problem.

The first goal of this section is to show that $\mathcal{N}(\Omega, k)$ characterizes $N(\Omega, k)$ in the sense that $\overline{\mathcal{N}(\Omega, k)} = N(\Omega, k)$ which we proceed to prove.

Theorem 3.5. *The set $\mathcal{N}(\Omega, k)$ is dense in $N(\Omega, k)$ in the $L^2(\Omega)$-norm.*

Proof. Let $f \in N(\Omega, k)$. From the definition of $N(\Omega, k)$ and the regularity theorems [54, Theorem 4.18] or [53, Theorem 8.12], we obtain a strong solution $u_k \in H_0^2(\Omega)$ so that $\Delta u_k + k^2 u_k = f$ a.e. and $Tu_k = 0$ on Γ. Now since $C_c^{\infty}(\Omega)$ is dense in $H_0^2(\Omega)$ then there exists a sequence $\{w^{(n)}\} \subset C_c^{\infty}(\Omega)$ converging to u_k in the $H_0^2(\Omega)$-norm. Let $g^{(n)} =$

$\Delta w^{(n)} + k^2 w^{(n)} \in \mathcal{N}(\Omega, k)$. It is clear that the Helmholtz operator $(\Delta + k^2) : H_0^2(\Omega) \to L^2(\Omega)$ is bounded, which means that

$$\lim_{n\to\infty} g^{(n)} = \lim_{n\to\infty} (\Delta w^{(n)} + k^2 w^{(n)}) = \Delta u_k + k^2 u_k = f \quad \text{(in the } L^2(\Omega)\text{-norm).}$$

This concludes the proof. □

Another goal of this section is to prove that data obtained from finitely many frequencies is not sufficient to recover f uniquely. We want to show the existence of nontrivial sources that do not radiate at a finite number of distinct frequencies. This is done in a manner similar to that of the definition of the non-radiating sources found in the set $\mathcal{N}(\Omega, k)$ as given by (3.2).

Let $L_{k_1} = \Delta + k_1^2$ be the Helmholtz operator at a frequency k_1. Similarly define L_{k_2} for a frequency $k_2 \neq k_1$. Now let $w \in C_c^\infty(\Omega)$ be such that $L_{k_1} L_{k_2} w \neq 0$, and define $g = L_{k_1} L_{k_2} w$, $w_1 = L_{k_2} w$ and $w_2 = L_{k_1} w$. Notice that all three $g, w_1, w_2 \in C_c^\infty(\Omega)$. Also notice that the Helmholtz operators commute, that is, $L_{k_1} L_{k_2} w = L_{k_2} L_{k_1} w = g$. Hence, the source g is non-radiating at the two distinct frequencies k_1 and k_2 because it gives rise to the wave fields w_1 at frequency k_1 and w_2 at frequency k_2 such that $Tw_1 = Tw_2 = 0$. This argument can easily be extended to an arbitrary finite number of frequencies. Hence, we have proven the following assertion.

Theorem 3.6. *There exist nontrivial sources in $L^2(\Omega)$ that do not radiate at a finite number of frequencies. Equivalently, for any finite set of frequencies $\mathcal{K} = \{k_1, k_2, ..., k_n\}$, there exists $g \in L^2(\Omega)$ such that*

$$g \in \bigcap_{k \in \mathcal{K}} N(\Omega, k) \quad \textit{and} \quad g \neq 0.$$

Hence, concerning the multi-frequency Inverse Source Problem 3.1, data obtained from finitely many frequencies is not sufficient to recover the unknown source f uniquely.

We can now address the concept of minimum-norm solutions for the inverse problem. Notice that if f solves the inverse source problem at a frequency k, then $f + g$ for any $g \in N(\Omega, k)$ will solve it as well. Out of the infinitely many solutions to the inverse problem we can select the one that has minimum L^2-norm. This is a consequence of the well-known best approximation theorems for Hilbert spaces [45, Section 1.5]. We state this in the form of a corollary.

Corollary 3.7 (Minimum-Norm Solution)**.** *Given an arbitrary source $f \in L^2(\Omega)$ there exist unique decomposition $f = f_N + f_P$ with $f_N \in N(\Omega, k)$ and $f_P \in N(\Omega, k)^\perp$, such that f_P is the minimum-norm solution to the inverse source problem associated with f.*

Proof. From the orthogonal decomposition (3.1) we obtain $f = f_N + f_P$ with $f_N \in N(\Omega, k)$ and $f_P \in N(\Omega, k)^\perp$. Now notice that all solutions to the inverse problem associated with f have the form $h = f_P + g$ for $g \in N(\Omega, k)$ and from the orthogonal decomposition we have that $\|h\|^2 = \|f_P\|^2 + \|g\|^2$. So the norm of h is minimized when $g = 0$, that is f_P is the minimum-norm solution of the inverse problem. □

3.2 Multi-Frequency Uniqueness

In this section we study the establishment of uniqueness for the Inverse Source Problem 3.1 when measurements of the wave field u_k on the surface Γ are considered for many frequencies $k \in \mathcal{K}$. The goal is to prove that the unknown source f can be identified uniquely if $\mathcal{K}$ is any open interval of the positive real line.

Recall the variational Direct Problem 2.3 satisfied by the wave field $u_k \in H^1(\Omega)$ associated with the unknown source $f \in L^2(\Omega)$,

$$\langle \nabla u_k, \nabla v \rangle_{L^2(\Omega)} - k^2 \langle u_k, v \rangle_{L^2(\Omega)} - \langle M T u_k, T v \rangle_{L^2(\Gamma)} = \langle f, v \rangle_{L^2(\Omega)} \quad \text{for all} \quad v \in H^1(\Omega).$$

We may choose $v = v_k$ to be *plane waves* of the form $v_k = e^{ik\hat{x}\cdot z}$, traveling in the direction $\hat{x} \in \mathbb{S}^2$ with frequency $k > 0$. The expression above then renders the Fourier transform $\mathcal{F}f$

of the unknown function f. After integration by parts to pass the all of the derivatives onto the function v_k, we obtain,

$$(\mathcal{F}f)(k\hat{x}) = (2\pi)^{-3/2}\big(\langle Tu_k, D_\nu v_k\rangle_{L^2(\Gamma)} - \langle MTu_k, Tv_k\rangle_{L^2(\Gamma)}\big), \quad k\hat{x} \in \mathbb{R}^3. \tag{3.3}$$

This last expression gives rise to a reconstruction algorithm based on the inverse Fourier transform as described in Chapter 4. Another useful choice for the functions $v_k \in H^1(\Omega)$ is the set of Dirichlet eigenfunctions of the Laplacian in the region Ω or any other domain containing Ω. Then, the expression (3.10) yields the Fourier coefficients of the unknown source f. This is the foundation of the reconstruction algorithm devised in [49].

We are now ready to prove the uniqueness result.

Theorem 3.8 (Uniqueness)**.** *Let $\mathcal{K} \subset \mathbb{R}_+$ be an open interval. Suppose that $f^{(1)}, f^{(2)} \in L^2(\Omega)$ are two sources such that their radiated waves (solutions of the Direct Problem 2.3) coincide on the surface Γ for all frequencies $k \in \mathcal{K}$. Then $f^{(1)} = f^{(2)}$.*

Proof. Let $u_k^{(1)}$ and $u_k^{(2)}$ be the wave fields radiated at frequencies $k \in \mathcal{K}$ by $f^{(1)}$ and $f^{(2)}$, respectively. Set $f = f^{(1)} - f^{(2)}$ and $u = u_k^{(1)} - u_k^{(2)}$, and notice that by the linearity of the Helmholtz equation, then u_k is the solution of the Direct Problem 2.3 for the source f. The goal is to show that $f = 0$. Since the wave fields $u_k^{(1)}$ and $u_k^{(2)}$ coincide on the surface Γ, then we have that $Tu_k = 0$ for all $k \in \mathcal{K}$. Now, from the identity (3.3) we obtain that the Fourier transform $(\mathcal{F}f)(k\hat{x}) = 0$ for all $k \in \mathcal{K}$ and all $\hat{x} \in \mathbb{S}^2$. It is known by the Paley-Wiener theorem [70, Section 2.8] that the Fourier transform of a function with compact support is analytic. Since $\mathcal{F}f$ vanishes in the open region $(\mathbb{S}^2 \times \mathcal{K}) \subset \mathbb{R}^3$, then it vanishes identically in all of $\mathbb{R}^3$. By simply applying the inverse Fourier transform to (3.3) we obtain that $f = 0$ as desired. $\square$

3.3 Variational Characterization of the Unknown Source

We now turn to the characterization of the set $N(\Omega, k)^{\perp}$. The approach follows Albanese and Monk [37]. For that reason, we define a variational problem whose solutions will be shown to be dense in $N(\Omega, k)^{\perp}$. This new problem will be called *adjoint* to the original Direct Problem 2.3.

Problem 3.9 (Adjoint Problem). *Given boundary data $\eta \in L^2(\Gamma)$, find a function $\psi \in H^1(\Omega)$ satisfying*

$$A(\psi, \phi) = \langle \eta, T\phi \rangle_{L^2(\Omega)}, \qquad \textit{for all} \quad \phi \in H^1(\Omega), \tag{3.4}$$

where the sesquilinear form $A : H^1(\Omega) \times H^1(\Omega) \to \mathbb{C}$ is defined as follows,

$$A(\psi, \phi) = \langle \nabla\psi, \nabla\phi \rangle_{L^2(\Omega)} - k^2 \langle \psi, \phi \rangle_{L^2(\Omega)} - \langle M^* T\psi, T\phi \rangle_{L^2(\Gamma)}. \tag{3.5}$$

Here T is the trace operator from Theorem B.1, and $M^* : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ denotes the adjoint of the Dirichlet-to-Neumann operator. Notice also that ψ would depend on the frequency k, so we may write $\psi = \psi_k$.

Let us denote the set of solutions of the Adjoint Problem 3.9 by

$$\begin{aligned} \mathcal{P}(\Omega, k) = \big\{ &\psi \in H^1(\Omega) \text{ a solution of Problem 3.9 for some } \eta \in L^2(\Gamma) \\ &\text{and fixed frequency } k \big\}, \end{aligned} \tag{3.6}$$

$$P(\Omega, k) = \overline{\mathcal{P}(\Omega, k)} \quad \text{(closure in the } L^2(\Omega)\text{-norm)}. \tag{3.7}$$

In order for $\mathcal{P}(\Omega, k)$ to be well-defined it is necessary to show that the Adjoint Problem 3.9 is well-posed. This we do in the form of a lemma.

Lemma 3.10. *The Adjoint Problem 3.9 is well-posed.*

Proof. The well-posedness of the Adjoint Problem 3.9 can be proven in a completely analogous way to the proof of the well-posedness of Problem 2.3. In fact, Lemma 2.4 provides the key step to prove uniqueness since M and M^* satisfy the same crucial property (see Theorem 2.8 for proof of uniqueness). Then from the Riesz-Fredholm Theorem A.5 we obtain that for each element $\eta \in L^2(\Gamma)$ there exists a unique function $\psi \in H^1(\Omega)$ that satisfies the Adjoint Problem 3.9. Hence $\mathcal{P}(\Omega, k)$ is a well-defined set. □

As done in [37] for Maxwell's equations, the Adjoint Problem 3.9 can be used to obtain a variational characterization of the unknown source f. This is done by developing a link between f and the boundary measurements Tu_k on the surface Γ. Let $\psi_k \in H^1(\Omega)$ be a solution to the Adjoint Problem 3.9 for some $\eta \in L^2(\Gamma)$. Then

$$A(\psi_k, \phi) = \langle \eta, T\phi \rangle_{L^2(\Gamma)}, \qquad \text{for all} \quad \phi \in H^1(\Omega), \tag{3.8}$$

Denote by $u_k \in H^1(\Omega)$, the solution to the original Direct Problem 2.3 for the unknown source f. Then we also have,

$$B(u_k, v) = \langle f, v \rangle_{L^2(\Omega)}, \qquad \text{for all} \quad v \in H^1(\Omega). \tag{3.9}$$

Now, we let $\phi = u_k$ and $v = \psi_k$, and the combination of (3.8) and (3.9) renders,

$$\langle f, \psi_k \rangle_{L^2(\Omega)} = \langle Tu_k, \eta \rangle_{L^2(\Gamma)}, \qquad \text{for all} \quad \psi_k \in \mathcal{P}(\Omega, k). \tag{3.10}$$

The above variational characterization of the unknown source f can be employed to obtain the projection of f on the space $P(\Omega, k)$. This is important as we shall show that $P(\Omega, k)$ is in fact the set of all purely-radiating sources. In other words, expression (3.10) may be used to compute the minimum-norm solution of the inverse source problem as asserted by Corollary 3.7.

The definition of the set $P(\Omega, k)$ as a closed subspace of $L^2(\Omega)$ allows us to establish another orthogonal decomposition

$$L^2(\Omega) = P(\Omega, k) \oplus P(\Omega, k)^{\perp}. \tag{3.11}$$

Now we proceed to characterize the space $N(\Omega, k)^{\perp}$ by proving that $N(\Omega, k)^{\perp} = P(\Omega, k)$ or equivalently that $N(\Omega, k) = P(\Omega, k)^{\perp}$

Theorem 3.11. *A function $f \in L^2(\Omega)$ belongs to $P(\Omega, k)^{\perp}$ if and only if it belongs to $N(\Omega, k)$.*

Proof. Let $f \in L^2(\Omega)$ be arbitrary. The proof relies on the variational characterization (3.10) where $u \in H^1(\Omega)$ is the unique solution to the Direct Problem 2.3.

Now assume that $f \in P(\Omega, k)^{\perp}$. In (3.10) let $\eta = Tu_k \in H^{1/2}(\Gamma) \subset L^2(\Gamma)$ and $\psi_k \in \mathcal{P}(\Omega, k)$ be the associated solution of the Adjoint Problem 3.9. Then it follows that $Tu_k = 0$ which means that $f \in N(\Omega, k)$. Conversely, if $f \in N(\Omega, k)$ then $Tu_k = 0$ by definition. It follows from (3.10) that $\langle f, \psi_k \rangle_{L^2(\Omega)} = 0$ for all $\psi_k \in \mathcal{P}(\Omega, k)$. Hence $f \in P(\Omega, k)^{\perp}$ since $\mathcal{P}(\Omega, k)$ is dense in $P(\Omega, k)$. $\square$

In summary, we conclude from Theorem 3.11 that the orthogonal decompositions (3.1) and (3.11) are the same because $N(\Omega, k) = P(\Omega, k)^{\perp}$ and $P(\Omega, k) = N(\Omega, k)^{\perp}$. In addition, the variational characterization (3.10) can be employed to obtain the projection of the unknown source f on the space $P(\Omega, k)$ and this projection coincides with the minimum-norm solution of the inverse source problem.

Finally, we wish to mention that the Adjoint Problem 3.9 is just the variational formulation of the following boundary value problem,

$$\begin{aligned} &\Delta\psi + k^2\psi = 0 \quad \text{in} \quad \Omega, \\ &\frac{\partial \psi}{\partial \nu} - M^*\psi = \eta \quad \text{on} \quad \Gamma. \end{aligned}$$

This means that purely-radiating sources for the Helmholtz equation are themselves (approximate) weak solutions to the Helmholtz equation. This is the subject of several papers [37, 44, 51, 52].

Chapter 4. Reconstruction Algorithm

I owe a lot to my engineering training because it taught me to tolerate approximations. Previously to that I thought ... one should just concentrate on exact equations all the time.

P.A.M. Dirac

In this chapter we develop a simple algorithm to obtain an approximate reconstruction of the source f. The algorithm is based on the Fourier transform with incomplete data. As more data is available with higher frequency content, then the algorithm is shown to render a convergent reconstruction of the true source f. Furthermore, if f is assumed to belong to a Sobolev space $H^s(\mathbb{R}^3)$ for some $s > 0$, then the rate of convergence is estimated.

4.1 Reconstruction via Fourier Transform

The proof of uniqueness for Theorem 3.8 can be employed to develop an algorithm to solve the Inverse Source Problem 3.1. We consider the problem where the traces Tu_k are available for frequencies $k \in (0, K)$. This is the case in many applications where the frequency cannot exceed a given threshold. The use of the relationship (3.3) translates into the knowledge of the Fourier transform $(\mathcal{F}f)(k\hat{x})$ for all $\hat{x} \in \mathbb{S}^2$ and all $k \in (0, K)$, that is,

$$(\mathcal{F}f)(k\hat{x}) = (2\pi)^{-3/2} \int_\Gamma \left[u(y,k) \frac{\partial e^{-ik\hat{x}\cdot y}}{\partial \nu(y)} - Mu(y,k) e^{-ik\hat{x}\cdot y} \right] dS(y). \tag{4.1}$$

In order to obtain an approximation of the Fourier transform we would have to evaluate the DtN map. Fortunately there are many efficient algorithms to evaluate the DtN map or good approximations of it called *absorbing boundary conditions*. See for instance [56, 57, 58, 59, 60, 61].

Now, if we had the availability of $(\mathcal{F}f)(x)$ for all $x \in \mathbb{R}^3$ then we would recover the unknown source by simply applying the inverse Fourier transform, i.e.,

$$f = \mathcal{F}^{-1}\mathcal{F}f. \tag{4.2}$$

However, in practice we only know $(\mathcal{F}f)(x)$ for $|x| < K$. In other words, we have knowledge of the function $\chi_{B(K)}\mathcal{F}f$, where $\chi_{B(K)}$ is the characteristic function of the open ball $B(K) \subset \mathbb{R}^3$ with center at the origin and radius K. A natural and simple algorithm is to define an approximate reconstructed source f_K as follows,

$$f_K = \mathcal{F}^{-1}\big(\chi_{B(K)}\mathcal{F}f\big). \tag{4.3}$$

Hence, the reconstruction algorithm is simply an incomplete inverse Fourier transform.

4.2 Multi-Frequency Stability

The goal of this section is to obtain an error bound for the source reconstruction algorithm described in Section 4.1. The expression (4.3) renders an approximate reconstructed source given by $f_K = \mathcal{F}^{-1}\big(\chi_{B(K)}\mathcal{F}f\big)$. Notice that we cannot guarantee f_K to have its support within Ω. The idea is to estimate the error of the reconstruction in the $L^2(\mathbb{R}^3)$-norm. We have the following error bound under certain regularity assumptions on the true source f. At the same time, we would like to determine the influence of noise or error in the measurements. Hence, we define the recovered source as

$$f_{K,\epsilon} = \mathcal{F}^{-1}\big(\chi_{B(K)}\mathcal{F}f + \epsilon\big), \tag{4.4}$$

where $\epsilon \in L^2(\mathbb{R}^3)$ represents the error in the measurements. We would like to show that $f_{K,\epsilon}$ approaches f as $K \to \infty$ and $\epsilon \to 0$. This we do in the form of a theorem.

Theorem 4.1. *Let f and $f_{K,\epsilon}$ be given by (4.2) and (4.4) respectively. In addition, assume that $f \in H^s(\mathbb{R}^3)$ for some $s > 0$. Then there exists a constant C such that*

$$\|f - f_{K,\epsilon}\|_{L^2(\mathbb{R}^3)} \leq \frac{C}{1+K^s}\|f\|_{H^s(\mathbb{R}^3)} + \|\epsilon\|_{L^2(\mathbb{R}^3)}. \tag{4.5}$$

Proof. Consider the following estimates.

$$\begin{aligned}
\|f - f_{K,\epsilon}\|_{L^2(\mathbb{R}^3)} &= \|\mathcal{F}^{-1}\big(\mathcal{F}f - \chi_{B(K)}\mathcal{F}f - \epsilon\big)\|_{L^2(\mathbb{R}^3)} \\
&= \|\frac{(1-\chi_{B(K)})}{(1+|y|^s)}(1+|y|^s)\mathcal{F}f - \epsilon\|_{L^2(\mathbb{R}^3)} \\
&\leq \sup_{y\in\mathbb{R}^3}\left\{\frac{(1-\chi_{B(K)})}{(1+|y|^s)}\right\}\|(1+|y|^s)\mathcal{F}f\|_{L^2(\mathbb{R}^3)} + \|\epsilon\|_{L^2(\mathbb{R}^3)} \\
&\leq \frac{C}{1+K^s}\|f\|_{H^s(\mathbb{R}^3)} + \|\epsilon\|_{L^2(\mathbb{R}^3)}.
\end{aligned}$$

Here we have used (4.2) and (4.4), the fact the the Fourier transform is unitary, and the characterization of the Sobolev spaces $H^s(\mathbb{R}^3)$ in terms of the Fourier transform as described by Theorem A.8 in the Appendix. □

We have established an error estimate under certain regularity assumptions on the source function f. We conclude that multi-frequency data not only leads to uniqueness for the inverse source problem, but in fact it contributes with a stability result for the reconstruction of an unknown source.

4.3 Explicit Examples

In this short section we perform the reconstruction of specific sources using the method described in Section 4.1. For simplicity, we only consider spherically symmetric sources for which the Fourier transform in $\mathbb{R}^3$ reduces to

$$(\mathcal{F}f)(k) = \frac{2}{(2\pi)^{1/2}} \int_0^\infty f(r) \frac{r\sin(rk)}{k} dr, \qquad 0 \leq k.$$

4.3.1 Example 1. Here we set $f = \chi_B$, the characteristic function of the open ball $B \subset \mathbb{R}^3$ with center at the origin and unit radius. We also choose $\Omega = B$. A straightforward calculation shows that

$$(\mathcal{F}f)(k) = \frac{2}{(2\pi)^{1/2}} \int_0^\infty f(r) \frac{r\sin(rk)}{k} dr = \frac{2}{(2\pi)^{1/2}} \frac{\sin k - k\cos k}{k^3}.$$

Now we apply an incomplete inverse Fourier transform to obtain the reconstructed source f_K as defined by (4.3) as follows,

$$\begin{aligned} f_K(r) &= \frac{2}{(2\pi)^{1/2}} \int_0^K (\mathcal{F}f)(k) \frac{k\sin(kr)}{r} dk = \frac{2}{\pi} \int_0^K \frac{(\sin k - k\cos k)}{k^2} \frac{\sin(kr)}{r} dk \\ &= \frac{1}{\pi} \left(S(K(r+1)) - S(K(r-1)) - 2\sin K \frac{\sin(Kr)}{Kr} \right) \end{aligned}$$

where S represents the sine-integral $S(t) = \int_0^t \frac{\sin s}{s} ds$. The comparison between the true source and its reconstruction for various values of K is shown in Figure 4.1 (left).

4.3.2 Example 2. Here we choose the following source function

$$f(r) = \begin{cases} 1 - r, & r \leq 1; \\ 0, & r > 1. \end{cases}$$

As opposed to the first example, here the source is continuous. The integration to obtain the Fourier transform $\mathcal{F}f$ and the reconstructed source f_K is performed numerically using Maple 13. The comparison between the true source and its reconstruction for various values of K is shown in Figure 4.1 (right).

Figure 4.1: Reconstructed sources for Example 1 (left) and Example 2 (right).

Appendix A. Functional Analytical Tools

We start this section by stating the Lax-Milgram theorem. The proof can be found in [46, Theorem 13.26], [71, Proposition 1.2.41], [54, Lemma 2.32], [53, Theorem 5.8], and the original paper by Lax and Milgram [72].

Theorem A.1 (Lax-Milgram). *Let H be a Hilbert space and $B : H \times H \to \mathbb{C}$ a bilinear (or sesquilinear) form.*

(i) If the bilinear form B is bounded (there is a constant c such that $|B(u, v)| \leq c\|u\|\|v\|$ for all $u, v \in H$), then there is a bounded linear operator $A : H \to H$ such that $\|A\| \leq c$ and $B(u, v) = \langle Au, v\rangle$ for all $u, v \in H$.

(ii) If in addition the bilinear form B is coercive (there is a positive constant d such that $d\|u\|^2 \leq B(u, u)$ for all $u \in H$), then A has a bounded inverse and $\|A^{-1}\| \leq 1/d$.

As we shall see later, the nature of the Helmholtz equation will yield a sesquilinear form that is not coercive. Hence, we can only employ the first part of the Lax-Milgram theorem.

For a given bounded linear operator between Hilbert spaces, the existence of its adjoint operator is guaranteed by the following result. For a proof see [45, Theorem 4.9].

Theorem A.2 (Adjoint Operator). *Let X and Y be Hilbert spaces, and let $A : X \to Y$ be a bounded linear operator. Then there exists a uniquely determined linear operator $A^* : Y \to X$ with the property $\langle Au, v\rangle_Y = \langle u, A^*v\rangle_X$ for all $u \in X$ and $v \in Y$. The operator A^* is bounded and $\|A^*\| = \|A\|$.*

Now we state a series of important theorem from functional analysis. We start with a convergent sequence selection criterion for Hilbert spaces. This criterion allows us a select a weakly convergent sequence whose point of convergence becomes the candidate solution for many problems. A proof can be found in [71, Theorem 2.1.25], [53, Theorem 5.12], and [54, Theorem 2.31]. For its generalization to a reflexive space see [73] or [74].

Theorem A.3 (Weak Compactness)**.** *Let H be a Hilbert space. Every bounded sequence contains a weakly convergent subsequence.*

The following theorem concern the behavior of sequences under the image of bounded and compact linear operators. Proofs can be found any book on functional analysis. See [71, 45, 73, 74].

Theorem A.4. *Let X and Y be normed linear spaces and $A : X \to Y$ be a bounded linear operator. Then,*

(i) If $u_n \rightharpoonup u$ then $Au_n \rightharpoonup Au$.

(ii) If $u_n \to u$ then $Au_n \to Au$.

(iii) If A is compact and $u_n \rightharpoonup u$ then $Au_n \to Au$.

One of them most important tools in proving the well-posedness of variational linear problems is the Riesz-Fredholm theory for operator equations of the second kind. The proof to the theorem to be stated can be found in many textbooks under the subject of the Fredholm alternative or the Riesz-Schauder theory. We refer to [45, Chapters 3-4], [74, Chapter 5], and [71, Chapter 2].

Theorem A.5 (Riesz-Fredholm)**.** *Let $A : X \to X$ be a compact linear operator on a normed space X. Then $I - A$ is injective if and only if it is surjective. If $I - A$ is injective (and therefore bijective), then the inverse operator $(I - A)^{-1} : X \to X$ is bounded.*

Now we state the Sobolev embedding theorems for bounded open regions of $\mathbb{R}^n$ with Lipschitz boundaries. The proofs can be found in [54, Chapter 3], [75, Chapter 6], [68, Chapter 5], [69, Chapter 10], and [53, Chapter 7].

Theorem A.6 (Rellich-Kondrachov)**.** *Let Ω be a bounded open set in $\mathbb{R}^n$ with locally Lipschitz boundary, $k \in \mathbb{N}$ and $p \in [1, \infty)$.*

(i) If $k < n/p$ and $q \in [1, p^*)$ where $p^* = pn/(n-kp)$ known as the critical Sobolev exponent. Then the embedding of $W^{k,p}(\Omega)$ into $L^q(\Omega)$ is compact.

(ii) If $k = n/p$, then the embedding of $W^{k,p}(\Omega)$ into $L^q(\Omega)$ is compact for all $q \in [1, \infty)$.

(iii) If $k > n/p$, then the embedding of $W^{k,p}(\Omega)$ into $C^{0,\alpha}(\overline{\Omega})$ is compact for all $0 < \alpha < k - n/p$.

This Sobolev embedding theorem also holds for Sobolev spaces $W^{k,p}(M)$ on other suitable regions M. In particular it holds when M is a compact Riemannian manifold without boundary or with a Lipschitz boundary [67].

The last tool that we include in this Appendix is the Fourier transform and its connection with the Sobolev spaces $H^s(\mathbb{R}^n)$. The theorem following the below definition of the Fourier transform is found in [68, Section 5.8.5].

Definition A.7. If $f \in L^1(\mathbb{R}^n)$, we define the *Fourier transform* of f by

$$(\mathcal{F}f)(y) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot y} f(x)dx, \qquad y \in \mathbb{R}^n,$$

and its *inverse Fourier transform* by

$$(\mathcal{F}^{-1}f)(y) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{ix\cdot y} f(x)dx, \qquad y \in \mathbb{R}^n.$$

Using Plancherel's theorem we find that the Fourier transform is densely defined in the space $L^2(\mathbb{R}^n)$.

Theorem A.8 (Characterization of H^s by Fourier transform)**.** *Let $s > 0$, then a function $f \in L^2(\mathbb{R}^n)$ belongs to $H^s(\mathbb{R}^n)$ if and only if $(1 + |y|^s)\mathcal{F}f$ belongs to $L^2(\mathbb{R}^n)$. In addition, there exists a constant C such that*

$$\frac{1}{C}\|f\|_{H^s(\mathbb{R}^n)} \leq \|(1 + |y|^s)\mathcal{F}f\|_{L^2(\mathbb{R}^n)} \leq C\|f\|_{H^s(\mathbb{R}^n)}, \quad \textit{for each } f \in H^s(\mathbb{R}^n).$$

Appendix B. Green's Identities

In this section we simply state Green's first, second, and third identities in the context of Sobolev spaces for bounded domains with Lipschitz boundaries. These identities become crucial in the proof of the well-posedness of the variational formulation for the Helmholtz equation. The proofs for the first and second Green's identities are found in [54, Lemma 4.1-4.2]. In order to introduce these identities in the proper setting we need to define the trace and normal derivative operators [54, Chapter 3-4].

Theorem B.1 (Trace)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary Γ. Define the trace operator $T : C^\infty(\overline{\Omega}) \to C(\Gamma)$ by $Tu = u|_\Gamma$. Then T has a unique extension to a surjective bounded linear operator $T : H^s(\Omega) \to H^{s-1/2}(\Gamma)$ for $\frac{1}{2} < s \leq 1$.*

Theorem B.2 (Normal Derivative)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary Γ. Define the normal derivative operator $D_\nu : C^\infty(\overline{\Omega}) \to C(\Gamma)$ by $D_\nu u = \partial_\nu u|_\Gamma$, where ν denotes the outward normal to Γ. Then D_ν has a unique extension to a bounded linear operator $D_\nu : H^{s+1}(\Omega) \to H^{s-1/2}(\Gamma)$ for $\frac{1}{2} < s \leq 1$.*

Now we proceed to state Green's first, second, and third identities.

Theorem B.3 (Green's First Identity)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary Γ, $u \in H^2(\Omega)$ and $v \in H^1(\Omega)$. Then*

$$\langle \Delta u, v \rangle_{L^2(\Omega)} + \langle \nabla u, \nabla v \rangle_{L^2(\Omega)} = \langle D_\nu u, Tv \rangle_{L^2(\Gamma)}.$$

Theorem B.4 (Green's Second Identity)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary Γ, and $u, v \in H^2(\Omega)$. Then*

$$\langle \Delta u, v \rangle_{L^2(\Omega)} - \langle u, \Delta v \rangle_{L^2(\Omega)} = \langle D_\nu u, Tv \rangle_{L^2(\Gamma)} - \langle Tu, D_\nu v \rangle_{L^2(\Gamma)}.$$

We also state Green's third identity for the Helmholtz equation using its fundamental solution G defined in (2.4). The proof can be found in [54, Theorem 6.10]. This identity can be conveniently expressed in terms of the Green's volume potential $\mathcal{G}$ given by

$$(\mathcal{G}u)(x) = \int_{\Omega} G(x,y)u(y)dy, \qquad x \in \Omega, \qquad u \in L^2(\Omega),$$

and the single- and double-layer surface potentials $\mathcal{S}$ and $\mathcal{K}$ given by

$$(\mathcal{S}w)(x) = \int_{\Gamma} G(x,y)w(y)dS(y), \qquad x \in \Omega, \qquad w \in L^2(\Gamma),$$
$$(\mathcal{K}w)(x) = \int_{\Gamma} \frac{\partial G(x,y)}{\partial \nu(y)} w(y)dS(y), \qquad x \in \Omega, \qquad w \in L^2(\Gamma).$$

Theorem B.5 (Green's Third Identity)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary Γ, and $u \in H^2(\Omega)$. Then we have Green's formula,*

$$u = \mathcal{S}D_\nu u - \mathcal{K}Tu - \mathcal{G}(\Delta u + k^2 u).$$

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