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Applications of Descriptive Set Theory in Homotopy Theory

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Abstract

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This thesis presents new theorems in homotopy theory, in particular it generalizes a theorem of Saharon Shelah [1]. We employ a technique used by Janusz Pawlikowski [2] to show that certain Peano continua have a least nontrivial homotopy group that is finitely presented or of cardinality continuum. We also use this technique to give some relative consistency results.

Keywords: compact metric space, descriptive set theory, homotopy theory

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Contents

Chapter 1. Introduction

In this thesis we shall be proving results within the framework of Zermelo-Fraenkel set theory with the Axiom of Choice, which framework we shall denote ZFC. The result in chapter 2 will be proven entirely within this framework, while those in chapter 3 and chapter 4 will use stronger set theoretic axioms. We will introduce concepts in set theory as needed in the body of the text. Where we prove a theorem assuming extra set theoretic axioms, we shall indicate the axioms we assume in the statement of the theorem (as in Theorem 3.3, for example).

Given a set X, we let $\|X\|$ denote the cardinality of X and shall use the standard notation $\|\mathbb{R}\| = c$, calling this particular cardinality continuum. By CH we mean the Continuum Hypothesis, that is, the statement that there does not exist a cardinal κ such that $\aleph_0 < \kappa < c$, and by $\neg$ CH we mean the negation of the Continuum Hypothesis. Where appropriate, we shall think of a cardinal number as the least ordinal number of that particular cardinality (which is often the way in which the cardinals are defined). Our set $\mathbb{N}$ of natural numbers shall be such that $0 \in \mathbb{N}$. If Y is a subset of some set X that is understood by context, we shall let $Y^c = X - Y$ be our notation for the complement of Y in X (we shall not have to use continuum products of sets). If X is a set, let $\mathcal{P}(X)$ be the powerset of X.

We shall be discussing homotopy groups of metric spaces and assume some familiarity with algebra and topology, including nerves of covers.

Definition 1.1. An n-loop in a space X at $x_0 \in X$ is a continuous mapping $f : [0,1]^n \to X$ such that $f|_{\partial[0,1]^n} = x_0$. We say two n-loops f and g are homotopic if there exists a continuous function $F : [0,1]^{n+1} \to X$ such that $F(t_1, \ldots, t_n, 0) = f(t_1, \ldots, t_n)$, $F(t_1, \ldots, t_n, 1) = g(t_1, \ldots, t_n)$ and $F(t_1, \ldots, t_n, t) = x_0$ for $(t_1, \ldots, t_n) \in \partial[0,1]^n$. Such an F is called a homotopy from f to g. An n-loop is trivial if it is homotopic to a constant map, it is called essential otherwise. If X is a metric space with metric d, the diameter of a set $A \subseteq X$, $\operatorname{diam}(A)$, is given by $\sup\{d(x,y) : x, y \in A\}$. If f is an n-loop at x_0 in the metric space X, we shall say that the diameter of f is given by $\operatorname{diam}(f([0,1]^n))$.

For a specified space X and $x_0 \in X$, we define the operation of concatenation of n-loops $f * g$ at x_0 in the following way:

$$(f * g)(t_1, \ldots t_n) = \begin{cases} f(t_1, \ldots, t_{n-1}, 2t_n) \text{ if } t_n \leq \frac{1}{2} \\ g(t_1, \ldots, t_{n-1}, 2(t_n - \frac{1}{2})) \text{ if } t_n \geq \frac{1}{2} \end{cases}$$

It is straightforward to check that the set of all n-loops at x_0 mod homotopy forms a group in the following way: $[f] * [g] = [f * g]$, where the second $*$ is concatenation. This group is denoted $\pi_n(X, x_0)$ and is abelian for $n \geq 2$. Any continuous map $f : X \to Y$ that takes $x_0 \in X$ to $y_0 \in Y$ induces a homomorphism $f_* : \pi_n(X, x_0) \to \pi_n(Y, y_0)$ in the obvious way: $f_*([g]) = [f \circ g]$. If X is path connected, we have that for any $x_0, x_1 \in X$, the groups $\pi_n(X, x_0)$ and $\pi_n(X, x_1)$ are isomorphic so that we can drop the x_0 and write for convenience $\pi_n(X)$. We call $\pi_1(X)$ the fundamental group of X.

Chapter 2. The n+1 Homotopy Group of an n-connected Peano Continuum

Definition 2.1. A path connected space X is n-connected if $\pi_i(X)$ is trivial for $i \leq n$. A path connected space X is locally k-connected if for every $x \in X$ and open neighborhood U of x there exists a path connected open neighborhood $V \subseteq U$ of x such that the map induced by inclusion $\pi_i(V) \to \pi_i(U)$ is the zero map for all $i \leq k$.

Note that any n-connected, locally n-connected compact metric space is a Peano continuum. In this chapter, we shall prove the following theorem:

Theorem 2.2. *Suppose that X is a compact metric space that is n-connected and locally n-connected, $n \geq 1$. Then $\pi_{n+1}(X)$ is finitely presented or uncountable.*

We point out that this result is fairly sharp. We can't drop compactness, for $\pi_2(\mathbb{R}^3 - \mathbb{N} \times \{0\} \times \{0\})$ is countable and not finitely generated. We can't drop n-connected, for we can consider π_2 of the wedge of S^2 and S^1. We also can't drop locally connected. Consider the suspension over the space X that is the closure in $\mathbb{R}^2$ of the union over $n \in \mathbb{N}$ of circles in $\mathbb{R}^2$ centered at $(1 - \frac{1}{n+2}, 0)$ of radius $1 - \frac{1}{n+2}$. The π_2 of this space is countable, not finitely presented.

Through Theorem 2.10, we fix a compact metric space, (X, ρ), that is n-connected and locally n-connected with $n \geq 1$, with π_{n+1} not finitely presented. As we will be dealing with simplices, we will say that given a k-simplex $\Delta = [v_0, \dots, v_k]$, the set $[v_0, \dots, \hat{v}_i, \dots, v_k]$ is the $k-1$ dimensional face of Δ that does not have v_i as a vertex.

2.1 A Necessary Theorem

Before proving Theorem 2.2, we prove the following:

Theorem 2.3. *If (X, ρ) is an $n-$connected, locally $n-$connected compact metric space with $\pi_{n+1}(X)$ not finitely presented, there exists a point $x \in X$ such that for all $k \in \mathbb{N}$ there exists an essential $n+1$-loop f_k at x of diameter less than $\frac{1}{2^k}$.*

Proof. Suppose the conclusion fails. We know that for each $x \in X$ there exists δ_x such that $B(x, \delta_x)$ contains no nontrivial $n+1$-loops. As $\{B(x, \delta_x)\}_{x \in X}$ is an open cover of X, select a Lebesgue number γ. Let $\epsilon_1 = \gamma 100^{-n}$ and $\mathcal{B}_1$ be a finite cover of X by balls of radius ϵ_1. Pick a finite refinement, $\mathcal{U}_2$, of $\mathcal{B}_1$ of open sets such that for each $U \in \mathcal{U}_2$ there exists a set $O \in \mathcal{B}_1$ such that the map induced by inclusion $\pi_i(U) \to \pi_i(O)$ is trivial for all $i \leq n$. The collection $\mathcal{U}_2$ has a Lebesgue number, δ_2. Let $\epsilon_2 = \delta_2 100^{-n}$. Let $\mathcal{B}_2$ be a finite cover of X by open balls of radius ϵ_2. Iterate this process to get finite open covers $\mathcal{U}_2, \dots, \mathcal{U}_{2n+4}$ and

$\mathcal{B}_1, \ldots, \mathcal{B}_{2n+4}$ such that $\mathcal{U}_{k+1}$ refines $\mathcal{B}_k$, $\mathcal{B}_k$ refines $\mathcal{U}_k$, each cover $\mathcal{U}_k$ has a selected Lebesgue number δ_k, $\epsilon_k = \delta_k 100^{-n}$ for $k > 1$, for each $U \in \mathcal{U}_{k+1}$ there exists $O \in \mathcal{B}_k$ such that for all $i \leq n$ the map $\pi_i(U) \to \pi_i(O)$ is trivial, and each ball in $\mathcal{B}_k$ has radius ϵ_k.

Let N be the nerve of $\mathcal{B}_{2n+4} = \{B_i\}_{i=0}^{l-1}$ and $\mu : X \to [0,1]^l$ be a partition of unity for $\mathcal{B}_{2n+4}$. Let $f : X \to N$ be given by μ composed with the baricentric map to N. We define a map g from $N^{(n+1)}$ (the n+1-skeleton of N) to X inductively. For each $x \in N^{(0)}$ we have a set in $\mathcal{B}_{2n+4}$ which x essentially represents. Select a point in this set and let g take x to this point. Suppose $[x, y]$ is a 1-simplex in the 1-skeleton, $g(x) \in B_x \in \mathcal{B}_{2n+4}$ and $g(y) \in B_y \in \mathcal{B}_{2n+4}$. We have that $B_x \cap B_y \neq \emptyset$, so $\mathrm{diam}(B_x \cup B_y) \leq 2\epsilon_{2n+4} + 2\epsilon_{2n+4} < \delta_{2n+4}$, thus $\rho(g(x), g(y)) < \delta_{2n+4}$. Then $g(x)$ and $g(y)$ lie in a common element of $\mathcal{U}_{2n+4}$, so we extend g to map $[x, y]$ as a path in this element. If $[x, y, z]$ is a 2-simplex in $N^{(2)}$, then $g([x, y]) \cup g([y, z]) \cup g([x, z])$ is in the union of three elements of $\mathcal{U}_{2n+4}$ which have pairwise nonempty intersection. Then $\mathrm{diam}(g([x, y]) \cup g([y, z]) \cup g([x, z])) \leq 2\epsilon_{2n+3} + 2\epsilon_{2n+3} + 2\epsilon_{2n+3} < \delta_{2n+3}$, so $g([x, y]) \cup g([y, z]) \cup g([x, z])$ lies entirely within an element of $\mathcal{U}_{2n+3}$. By our choice of covers, we extend g to $[x, y, z]$ so that $g([x, y, z])$ lies entirely within an element of $\mathcal{B}_{2n+2}$. In general, suppose that we have extended g to $N^{(k)}$, $1 < k < n + 1$, in this way and suppose $[x_0, x_1, \ldots, x_{k+1}]$ is a $k + 1$-simplex in $N^{(k+1)}$. Notice that $\mathrm{diam}(g([\hat{x_0}, x_1, \ldots, x_{k+1}]) \cup g([x_0, \hat{x_1}, x_2, \ldots, x_{k+1}]) \cup \cdots \cup g([x_0, x_1, \ldots, x_k, \hat{x}_{k+1}])) \leq 2\epsilon_{2n+4-k} \cdot (k + 1) \leq 100^k \epsilon_{2n+4-k} = \delta_{2n+4-k}$. Then this union is in an element of $\mathcal{U}_{2n+4-k}$, so by our choice of covers, we know that that g can be extended to $[x_0, \ldots, x_{k+1}]$ entirely within an element of $\mathcal{B}_{2n+4-(k+1)}$. In the end, we map the $n + 1$-simplices so that each image is contained entirely within an element of $\mathcal{B}_{n+3}$.

Notice that if Δ^{n+2} is an n+2-simplex in our simplicial complex N, then $\mathrm{diam}(g(\partial(\Delta^{n+2}))) \leq 2(n + 3)\epsilon_{n+3} < \gamma$, so the boundary of any $n + 2$-simplex in N maps nulhomotopically into X. Thus g induces a map on $H_{n+1}(N)$, and we shall call the induced map $g_* : H_{n+1}(N) \to H_{n+1}(X)$. The map f given by baricentric coordinates induces a map $f_* : H_{n+1}(X) \to H_{n+1}(N)$. We show that the composition $g_* \circ f_* : H_{n+1}(X) \to H_{n+1}(X)$ is identity, so that f_* is injective, so $H_{n+1}(X)$ can be thought of as a subgroup of $H_{n+1}(N)$, which is finitely generated abelian. Thus $H_{n+1}(X)$ is finitely generated abelian, so $H_{n+1}(X)$ is finitely presented, and we will have a contradiction.

To see that $g_* \circ f_*$ is identity, we notice that since X is $n-$connected, we have that the Hurewitz map $\mathcal{H} : \pi_{n+1}(X) \to H_{n+1}(X)$ is an isomorphism, in particular it is onto. Thus any element of $H_{n+1}(X)$ is represented by a map $\sigma : S^{n+1} \to X$. We therefore let $\sigma : S^{n+1} \to X$ be continuous. Now $f \circ \sigma$ is homotopic to a map $\sigma' : S^{n+1} \to N^{(n+1)}$ satisfying the following:

(i) If $(f \circ \sigma)(x) \in N^{(n+1)}$, then $(f \circ \sigma)(x) = \sigma'(x)$

(ii) If $(f \circ \sigma)(x) \notin N^{(n+1)}$, then $\sigma'(x)$ lies in the boundary of the cell in which $(f \circ \sigma)(x)$ lies.

If we can show that $g \circ \sigma'$ is homotopic to σ, then we will have shown that $g_* \circ f_*$ is identity, and we will be done. Towards this, notice that if $y \in N^{(0)}$ and $x \in \mathrm{Star}(y) \cap N^{(n+1)}$, then by construction we have that $\rho(g(y), g(x)) < 2\epsilon_{n+3}$. Then for $x \in S^{n+1}$, we have that $\rho(\sigma(x), (g \circ \sigma')(x)) < 2\epsilon_{n+3} + 2\epsilon_{2n+3} < \delta_{n+3}$. By uniform continuity, tesselate S^{n+1} by $n+1$-simplices so that for each $n+1$-simplex Δ, $\mathrm{diam}(g \circ \sigma'(\Delta) \cup \sigma(\Delta)) < \delta_{n+3}$. This induces a simplicial complex structure on S^{n+1}. Call the k-skeleton of this structure $(S^{n+1})^{(k)}$.

We define by induction a continuous function $F : S^{n+1} \times [0,1] \to X$ such that $F|_{S^{n+1} \times \{0\}} = \sigma$ and $F|_{S^{n+1} \times \{1\}} = g \circ \sigma'$. The function F will give the homotopy that we seek. We start defining F on $S^{n+1} \times \{0\} \cup S^{n+1} \times \{1\}$ so as to satisfy the conditions in the previous sentence. Given $x \in (S^{n+1})^{(0)}$, we know that because $g \circ \sigma'(x)$ and $\sigma(x)$ are within δ_{n+3} of each other, they lie in a common element of $\mathcal{U}_{n+3}$. As this element is path connected, we can extend the map F so that $F|_{\{x\} \times [0,1]}$ lies entirely within this element of $\mathcal{U}_{n+3}$. Do this for all elements of the 0-skeleton of S^{n+1}.

In general, suppose that we have defined F for $(S^{n+1})^{(k)} \times [0,1]$, with $k < n$, so that given any k-simplex Δ in S^{n+1} we have that $F(\Delta \times [0,1])$ is completely contained in an element of $\mathcal{B}_{n+2-(k+1)}$. Let a $k+1$-simplex $\Delta = [v_0, \ldots, v_k]$ in S^{n+1} be given. We have that

$$\operatorname{diam}(F(\Delta\times\{0\})\cup F(\Delta\times\{1\})\cup F([\hat{v_0},v_1,\ldots,v_{k+1}]\times[0,1])\cup\cdots\cup F([v_0,v_1,\ldots,\hat{v_{k+1}}]\times[0,1])) \leq$$

$$2\delta_{n+3}+2(k+1)\epsilon_{n+2-(k+1)}$$

$$\leq 2(k+3)\epsilon_{n+2-(k+1)} < \delta_{n+2-(k+1)}$$

Thus $F(\Delta\times\{0\})\cup F(\Delta\times\{1\})\cup F([\hat{v_0},v_1,\ldots,v_{k+1}]\times[0,1])\cup\cdots\cup F([v_0,v_1,\ldots,\hat{v_{k+1}}]\times[0,1])$ is completely contained in an element of $\mathcal{U}_{n+2-(k+1)}$, so we can continuously extend F to $\Delta\times[0,1]$ so that $F(\Delta\times[0,1])$ lies entirely in an element of $\mathcal{B}_{n+2-(k+2)}$. Do this for all $k+1$-simplices. In this way we extend F to $(S^{n+1})^{(n)}$ so that for each n-simplex Δ in S^{n+1}, $F(\Delta\times[0,1])$ is contained in an element of $\mathcal{B}_1$. Thus, given an $n+1$-simplex $\Delta=[v_0,\ldots,v_{n+1}]$ in S^{n+1}, we have that

$$\operatorname{diam}(F(\Delta\times\{0\})\cup F(\Delta\times\{1\})\cup F([\hat{v_0},v_1,\ldots,v_{n+1}]\times[0,1])\cup\cdots\cup F([v_0,v_1,\ldots,\hat{v_{n+1}}]\times[0,1]))$$

$$\leq 2\delta_{n+3}+(n+1)\epsilon_1$$

$$\leq (n+3)\epsilon_1 < \gamma$$

Since any mapping of an $n+1$-sphere into X of diameter less than γ is nulhomotopic in X, we can complete F to $\Delta\times[0,1]$. Do this for each $n+1$ simplex in S^{n+1}. Thus we have completed the map $F: S^{n+1}\times[0,1]\to X$ and we are done with our proof.

□

2.2 The Proof of Theorem 2.2 and a Family of Examples

We now fix $x\in X$ as specified by the previous theorem. Select a sequence of essential $n+1$-loops f_0, f_1, … at x such that $\operatorname{diam}(f_i)\leq\frac{1}{2^i}$. Let f_k^1 be f_k and f_k^0 be the constant $n+1$-loop at x. For $\alpha\in\{0,1\}^{\mathbb{N}}$, let $f^\alpha=f_0^{\alpha(0)}*f_1^{\alpha(1)}*f_2^{\alpha(2)}*\cdots$. This can be thought of as the (pointwise) limit of a Cauchy sequence in the complete metric space $\mathcal{C}([0,1],X)$, the metric being the sup metric. We define an equivalence relation E on $\{0,1\}^{\mathbb{N}}$ as follows: $\alpha\sim\beta$ iff f^α is homotopic to f^β. We have the following lemma, given in [2]:

Lemma 2.4. *If α and β differ at exactly one point, i.e. $\alpha(n) \neq \beta(n)$ and $\alpha(m) = \beta(m)$ for $m \neq n$, then $\alpha \nsim \beta$.*

Proof. If α and β differ exactly at $n \in \mathbb{N}$, then we know that $f_0^\alpha * \cdots f_{n-1}^\alpha$ is homotopic to $f_0^\beta * \cdots f_{n-1}^\beta$ and $f_{n+1}^\alpha * f_{n+2}^\alpha \cdots$ is homotopic to $f_{n+1}^\beta * f_{n+2}^\beta \cdots$. Supposing that $\alpha \sim \beta$, we apply cancelations on the right and left so that f_n^α is homotopic to f_n^β, so that f_n is trivial, a contradiction. $\square$

Definition 2.5. [3] We say a space Y is Polish if it is completely metrizable and separable (for example, the space $\{0,1\}^{\mathbb{N}}$ under the product topology, which is homeomorphic to the Cantor set.) Recall that a set A in a topological space Y is meager if its closure has empty interior and comeager if $Y - A$ is meager. A set $A \subseteq Y$ satisfies the property of Baire if there exists an open set $O \subseteq Y$ such that $A \Delta O = (A \cup O) - (A \cap O)$ is meager (and say that A is comeager in O.) If Y is a Polish space, we say that $A \subseteq Y$ is analytic if there exists a Polish space Z and a closed set $D \subseteq Y \times Z$ such that A is the projection of D in Y. Analytic sets are preserved under continuous preimages and satisfy the property of Baire. A subset of a Polish space is perfect if it is closed, nonempty and has no isolated points. Such sets always have the cardinality of the continuum.

It will be useful to consider the equivalence relation E as a subspace of $\{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$. We use the following consequence of the Kuratowski-Ulam Theorem [4]:

Lemma 2.6. *If Y is a Polish space and $A \subseteq Y \times Y$ is comeager in $U \times V$, with U and V open sets in Y, then $\{y \in U : \{z \in V : (y,z) \in A\}$ is comeager in $V\}$ is comeager in U. Also, if $A \subseteq Y \times Y$ is meager, then $\{y \in Y : \{z \in Y : (y,z) \in A\}$ is meager in $Y\}$ is comeager in Y.*

Lemma 2.7. *The relation E when considered as a subset of the space $\{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$ has the property of Baire.*

Proof. Let H be the space of $n+1$-loops at x and $\mathbb{H}$ be the space of homotopies of $n+1$-loops at x, each having the topology induced by the sup metric. Letting $D = \{(f, g, F) \in H \times H \times \mathbb{H} : F \text{ homotopes } f \text{ to } g\}$, we see that D is closed in the topology on $H \times H \times \mathbb{H}$. Projecting to $H \times H$ gives the homotopy equivalence relation, which is therefore analytic. The function from $\{0,1\}^{\mathbb{N}}$ to H that takes α to f^{α} is continuous. The relation E is then the preimage under a continuous function of the homotopy equivalence relation on $H \times H$. Then E is analytic in $\{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$. Thus E has the property of Baire. $\square$

The following proof is due to Pawlikowski [2]:

Lemma 2.8. *If $E \subseteq \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$ is an equivalence relation which has the Baire property and if $\neg xEy$ whenever x and y differ by exactly one coordinate, then E is meager.*

Proof. If E is non-meager, then it is comeager in a neighborhood in $\{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$ of the form $U \times V$, with U and V open in $\{0,1\}^{\mathbb{N}}$ generated by some finite $0,1$ sequence. By Lemma 2.6 theorem,

$$A = \{\alpha \in U : \{\beta \in V : \alpha E \beta\} \text{ is comeager in } V\}$$

is comeager in U. Let $n \in \mathbb{N}$ be greater than the length of a generator of U. We let $\Phi : U \to U$ be given by $\Phi(\alpha)(n) = 1 - \alpha(n)$ and $\Phi(\alpha)(i) = \alpha(i)$ otherwise. Notice that Φ is a homeomorphism of U to itself. Then we can choose $\alpha \in A \cap \Phi(A)$. Letting $\gamma = \Phi(\alpha)$, we see that γ and α differ only at n, so that $\neg \alpha E \gamma$. From the definition of A, we have comeagerly many $\beta \in V$ such that $\alpha E \beta$. Since $\gamma \in A$, the same can be said of γ. Then for some β, we have $\beta E \alpha$ and $\beta E \gamma$, implying $\gamma E \alpha$, a contradiction. $\square$

The next result concludes the proof of Theorem 2.2.

Lemma 2.9. *There exists a set $Y \subseteq \{0,1\}^{\mathbb{N}}$ such that $\|Y\| = \aleph_1$ and for distinct α and β in Y we have $\neg \alpha E \beta$.*

Proof. For $\alpha \in \{0,1\}^{\mathbb{N}}$ we write E^α as the equivalence class of α. As E is meager, we have by Lemma 2.6 that $J = \{\alpha \in \{0,1\}^{\mathbb{N}} : E^\alpha \text{ is meager }\}$ is comeager in $\{0,1\}^{\mathbb{N}}$. Then by transfinite induction we define a sequence $\{\alpha_i\}_{i<\omega_1}$ by picking α_j such $\alpha_j \in J - \bigcup_{i<j} E^{\alpha_i}$. Let $Y = \{\alpha_i\}_{i<\omega_1}$.

□

To see that there is actually a continuum of pairwise non-homotopic loops, we invoke the following theorem of Mycielski [5]:

Theorem 2.10. *Suppose that Y is a Polish space with no isolated points and that $R \subseteq Y \times Y$ is meager. Then there exists a perfect set $P \subseteq Y$ such that if α and β are distinct points of P then $(\alpha, \beta) \notin R$.*

For an application of Theorem 2.2, we consider the classic Barratt-Milnor space:

Example 2.11. Let $X \subseteq \mathbb{R}^3$ be the union of spheres of radius $\frac{1}{n}$ centered at $(\frac{1}{n}, 0, 0)$, with $n \in \mathbb{N}$. Then $\pi_2(X)$ is uncountable.

Proof. It is clear that X is path connected, locally path connected and compact. It is a straightforward exercise to verify that the fundamental group is trivial and locally trivial. Also, the hypotheses of Theorem 2.3 apply to the space X, and so by our proof of Theorem 2.2, we have that $\pi_2(X)$ is uncountable. □

A similar proof demonstrates that letting X be the union over $n \in \mathbb{N}$ of $k-1$ spheres in $\mathbb{R}^k$ centered at $(\frac{1}{n}, 0, \ldots, 0)$ of radius $\frac{1}{n}$, we have $\pi_{k-1}(X)$ is uncountable.

2.3 An Instructive Example

We show that in Lemma 2.8, the hypothesis that E has the Baire property cannot be dropped outright.

Example 2.12. There exists an equivalence relation E on $\{0,1\}^{\mathbb{N}}$ such that $\neg \alpha E \beta$ whenever α and β differ by exactly one coordinate, E does not have the property of Baire, and E has exactly two equivalence classes.

Proof. Given $\alpha \in \{0,1\}^{\mathbb{N}}$ and $n \in \mathbb{N}$, we define $\alpha \upharpoonright n \in \{0,1\}^{\mathbb{N}}$ as follows:

$$\alpha \upharpoonright n(m) = \begin{cases} \alpha(m) \text{ if } m \neq n \\ 1 - \alpha(n) \text{ if } m = n \end{cases}$$

We define sets $A, B \subseteq \{0,1\}^{\mathbb{N}}$ satisfying the following:

(i) $A^c = B$

(ii) Given $\alpha \in A$, $\beta \in B$ and $n \in \mathbb{N}$, we have that $\alpha \upharpoonright n \in B$ and $\beta \upharpoonright n \in A$

Once we have completed this construction, we will have that $E = (A \times A) \cup (B \times B)$ is an equivalence relation with two classes such that if α and β in $\{0,1\}^{\mathbb{N}}$ differ by one coordinate then $\neg \alpha E \beta$. If E were to have the Baire property, it would be meager by Lemma 2.8, and so have uncountably many equivalence classes by Lemma 2.9 or Theorem 2.10, a contradiction.

We well-order $\{0,1\}^{\mathbb{N}}$ by the continuum, $\{0,1\}^{\mathbb{N}} = \{\alpha_\beta\}_{\beta < c}$. Define $\Phi : \{0,1\}^{\mathbb{N}} \to \mathcal{P}(\{0,1\}^{\mathbb{N}})$ by $\Phi(\alpha) = \{\beta \in \{0,1\}^{\mathbb{N}} : \beta = \alpha \upharpoonright n \text{ for some } n \in \mathbb{N}\}$. We let $J_0 = \{\alpha_0\}$ and for $n \in \mathbb{N}$, $J_{n+1} = \Phi(J_n)$. Let $A_0 = \bigcup_{n \in \mathbb{N}} J_{2n}$ and $B_0 = \bigcup_{n \in \mathbb{N}} J_{2n+1}$. Suppose that we have defined A_γ and B_γ for all $\gamma < \delta < c$. Select $\alpha_\epsilon \in \{0,1\}^{\mathbb{N}} - (\bigcup_{\gamma < \delta}(A_\gamma \cup B_\gamma))$ of smallest index. Let $J_0 = \bigcup_{\gamma < \delta} A_\gamma \cup \{\alpha_\epsilon\}$, and for $n \in \mathbb{N}$, $J_{n+1} = \Phi(J_n)$. Let $A_\delta = \bigcup_{n \in \mathbb{N}} J_{2n}$ and $B_\delta = \bigcup_{n \in \mathbb{N}} J_{2n+1}$. Let $A = \bigcup_{\delta < c} A_\delta$ and $B = \bigcup_{\delta < c} B_\delta$ and it is clear that A and B satisfy our specifications.

$\square$

Chapter 3. A Consistency Result

Definition 3.1. We say that a space X is κ-separable if there exists a dense subset $Q \subseteq X$ such that $\|Q\| \leq \kappa$.

In this and the following chapter, we shall make use of the following axiom, which we shall abbreviate MA:

Definition 3.2. [6] Martin's Axiom is the assertion that if B is a Boolean algebra satisfying the countable chain condition and S is any family of subsets of B (each of member of which has a join), such that $\|S\| < c$, then there is an S-complete ultrafilter in B.

We will not define the terms used in the previous definition, as they will not come up in our proofs. We only write down the axiom for the sake of completeness. We are interested in the relative consistency of Martin's Axiom with ZFC, that is, if the Zermelo-Fraenkel Axioms are consistent, then so is the axiom system ZFC + MA. This is proven in [6].

In this chapter, we prove the following:

Theorem 3.3. *(ZFC +MA) If X is a κ-separable complete metric space, with $\kappa < c$ and there exists a point $x \in X$ and a sequence of essential n-loops, $f_0, f_1, \ldots$ at x such that* $\operatorname{diam}(f_i) < 2^{-i}$*, then $\|\pi_n(X, x_0)\| = c$.*

Thus, in any set theoretic universe which satisfies Martin's Axiom, the above theorem is true. Any universe in which the Continuum Hypothesis holds has Martin's Axiom. However, there are universes in which the Continuum Hypothesis fails and Martin's Axiom holds [6].

3.1 More Background in Descriptive Set Theory

To be concise, we shall use $\mathbb{BP}$ to denote the σ-algebra of sets that satisfy the Baire property (the space should be clear from context). We shall say that an algebra of sets $\mathcal{A}$ is a κ-algebra if it is closed under unions of κ-many sets (so that a σ-algebra is an $\aleph_0$-algebra). Given a topological space X, a set $Y \subseteq X$ is κ-Borel if it is in the least κ-algebra generated by the open sets.

Considering a cardinal κ as a set, we give it the discrete topology and take the countably infinite product $\kappa^{\mathbb{N}}$. This space is completely metrizable, and κ-separable. Given a Polish space X, we say that a set $Y \subseteq X$ is κ-Souslin if there exists a closed set $D \subseteq X \times \kappa^{\mathbb{N}}$ such that Y is the projection of D to X (so that analytic sets are $\aleph_0$-Souslin). Given a set X, a collection $\mathcal{I} \subseteq \mathcal{P}(X)$ is a κ-ideal if it is closed under subsets and κ-unions. If $\mathcal{A}$ is a λ -algebra of sets on X, then we say that $P \subseteq X$ is in $\mathcal{A}$ modulo $\mathcal{I}$ if there is some $P^* \in \mathcal{A}$ such that $P \Delta P^* \in \mathcal{I}$.

Theorem 3.4. *(ZFC + MA) [7] If $\kappa < c$, then $\mathbb{BP}$ is a κ-algebra. Also, the ideal given by meager sets is a κ-ideal.*

Theorem 3.5. *[8] If X is a perfect Polish space (has no isolated points) and $\mathcal{I}$ is the κ-ideal of sets generated by the σ-ideal of meager sets, then every κ-Souslin subset of X is κ-Borel modulo $\mathcal{I}$.*

It is easy to see how we will want to combine these two theorems:

Corollary 3.6. *(ZFC + MA) If $\kappa < c$ and X is a perfect Polish space, then all κ-Souslin sets are in $\mathbb{BP}$.*

3.2 A Lemma

Lemma 3.7. *If a complete metric space $\mathcal{M}$ is κ-Lindelof, then there exists a continuous surjection $f : \kappa^{\mathbb{N}} \to \mathcal{M}$.*

Proof. Fix a dense set $\{r_\gamma\}_{\gamma<\kappa}$ in $\mathcal{M}$. For each $\alpha \in \kappa^{\mathbb{N}}$, we give the sequence $\{x_n^\alpha\}$ as follows: $x_0 = r_{\alpha(0)}$ and

$$x_{n+1} = \begin{cases} r_{\alpha(n+1)} \text{ if } d(x_n, r_{\alpha(n+1)}) < 2^{-n} \\ x_n \text{ otherwise} \end{cases}$$

By construction, we have that for each $\alpha \in \kappa^{\mathbb{N}}$ the sequence $\{x_n^\alpha\}$ is Cauchy, and so converges to what we call $f(\alpha)$. It is easy to check that f is a continuous surjection. $\square$

3.3 The Proof of Theorem 3.3

To prove Theorem 3.3, we assume Martin's Axiom, fix a space X, point $x \in X$ and essential n-loops $f_0, f_1, \ldots$ as given by the hypotheses. To see that $\|\pi_n(X, x_0)\| \leq c$, we notice that there are at most c-many continuous functions from $[0,1]$ to X. We need to show that there are at least c-many elements in $\pi_n(X, x_0)$. We proceed as in chapter 1.

Given $\alpha \in \{0,1\}^{\mathbb{N}}$, let f_k^α be f_k if $\alpha(k) = 1$, and be the constant loop at x_0 otherwise. For $\alpha \in \{0,1\}^{\mathbb{N}}$, let $f^\alpha = f_0^{\alpha(0)} * f_1^{\alpha(1)} * f_2^{\alpha(2)} * \cdots$. This can be thought of the (pointwise) limit of a Cauchy sequence in the complete metric space $C([0,1], X)$, the metric being the sup metric. We define an equivalence relation E on $\{0,1\}^{\mathbb{N}}$ as follows: $\alpha \sim \beta$ iff f^α is homotopic to f^β. Notice that the conclusion of Lemma 2.4 holds in this situation. We again consider the equivalence relation as a subset of $\{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$, calling it E again. By our proof of Theorem 2.2, we need only prove that $E \in \mathbb{BP}$. We shall show that E is κ-Souslin, so that by Corollary 3.6 we will be done.

Define H to be the set of all n-loops at x_0 and $\mathbb{H}$ to be the set of all continuous functions $g : [0,1]^{n+1} \to X$ such that $g(\partial[0,1]^{n+1}) = \{x_0\}$. Define $F : \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}} \to H \times H$ by $(\alpha, \beta) \mapsto (f^\alpha, f^\beta)$. This is clearly continuous. Then $Graph(F)$ is closed in $\{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}} \times H \times H$. Then $Graph(F) \times \mathbb{H}$ is closed in $\{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}} \times H \times H \times \mathbb{H}$. Let $D \subseteq H \times H \times \mathbb{H}$ be given by

$$(f, g, T) \in D \Leftrightarrow T \text{ homotopes } f \text{ to } g$$

It is clear that D is closed. Thus $(\{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}} \times D) \cap (Graph(F) \times \mathbb{H})$ is closed.

Certainly $H \times H \times \mathbb{H}$ is κ-separable, so let $f : \kappa^{\mathbb{N}} \to H \times H \times \mathbb{H}$ be the surjection given by Lemma 3.7. Extend f to $f : \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}} \times \kappa^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}} \times H \times H \times \mathbb{H}$ in the obvious way. Then $f^{-1}((\{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}} \times D) \cap (Graph(F) \times \mathbb{H}))$ is closed and E is the the projection of this preimage onto $(\{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$. Thus E is κ-Souslin.

Chapter 4. Another Consistency Result

In this chapter we shall prove a theorem related to a question posed by my advisor and Dr. Cannon. We first define the following:

Definition 4.1. If X is a metric space and $x \in X$, we say that X is nonhomotopically Hausdorff at x if there exists an essential 1-loop at x that can be homotoped to have arbitrarily small diameter.

One can ask the question: If X is a separable, locally connected metric space that is nonhomotopically Hausdorff at x, is it the case that $\pi_1(X, x)$ is uncountable?

Pawlikoski answered the question in the affirmative if X is complete, without the assumption that X is locally connected [2].

We shall prove a partial result, giving the consistency of this result for a certain class of spaces.

4.1 Even More Descriptive Set Theory

Definition 4.2. (The Lusin Pointclasses) [8] For a fixed Polish space X, we define Σ^1_1 to be the class of all analytic sets of X, $\Pi^1_1 = \{Y \subseteq X : Y^c \text{ is a } \Sigma^1_1 \text{ set}\}$, and $\Delta^1_1 = \Sigma^1_1 \cap \Pi^1_1$. Having defined Σ^1_n and Π^1_n for $n \in \mathbb{N}$, we let

$$\Sigma_{n+1} = \{Y \subseteq X : \text{there exists a Polish space } Z, \text{ a } \Pi^1_n \text{ subset } W \text{ of } Z \text{ and continuous map } f : Z \to X \text{ such that } f(W) = Y\}$$

$$\Pi^1_{n+1} = \{Y \subseteq X : Y^c \text{ is a } \Sigma^1_{n+1} \text{ set}\}$$

$$\Delta^1_{n+1} = \Sigma^1_{n+1} \cap \Pi^1_{n+1}$$

The Lusin pointclasses are often given in the following diagram:

$$\begin{array}{ccccccccc}
 & \Sigma^1_1 & & \Sigma^1_2 & & \Sigma^1_3 & & \Sigma^1_4 & \cdots \\
\Delta^1_1 & & \Delta^1_2 & & \Delta^1_3 & & \Delta^1_4 & & \\
 & \Pi^1_1 & & \Pi^1_2 & & \Pi^1_3 & & \Pi^1_4 & \cdots
\end{array}$$

A class that lies to the left of another class is contained in that class (e.g. $\Pi^1_1 \subseteq \Delta^1_2$). The class Δ^1_1 is the class of Borel sets. The Lusin pointclasses are closed under continuous preimages, countable unions, countable intersections, and under cross products with Polish spaces, i.e. if X and Y are Polish spaces and $A \subseteq Y$ is of pointclass $\mathcal{J}$ in Y, then $X \times A$ is of pointclass $\mathcal{J}$ in $X \times Y$. Also, the pointclasses Σ^1_n are each closed under continuous images.

Definition 4.3. If $\mathcal{J}$ is a Lusin pointclass, we shall say that a space X is standard $\mathcal{J}$ if it is homeomorphic to a $\mathcal{J}$ subspace of a Polish space.

We shall make use of the following:

Theorem 4.4. *[9] If X is a Polish space and $Y \subseteq X$ is a Σ^1_2 subset, then we can write $Y = \bigcup_{\alpha<\aleph_1} B_\alpha$, where each B_α is a Borel subset of X.*

4.2 Statement of our Consistency Result and its Proof

Our result is the following:

Theorem 4.5. *(ZFC+MA + ¬ CH) Suppose Y is a standard Π^1_1 space and has a point x and a sequence of essential n-loops $f_0, f_1, f_2, \ldots$ such that given any open neighborhood O of x, the images of all but finitely many of the f_i are contained in O. Then $\|\pi_n(Y, x)\| = c$.*

We know that if the Zermelo-Fraenkel Axioms are consistent, then so is ZFC + MA + $\neg$ CH [6], so we will have shown that the following statement is also consistent with ZFC:

If X is a standard Borel space that is nonhomotopically Hausdorff at x, then

$$\|\pi_1(X,x)\| = c.$$

Proof. (of Theorem 4.5) We assume ZFC + MA + $\neg$ CH and fix Y, x and a set of n-loops at x $\{f_i\}_{i\in\mathbb{N}}$ as given by the hypotheses. Let Y be a Π^1_1 subspace of a Polish space X. We proceed as in Theorem 3.3. It is a straightforward argument in transfinite combinatorics to see that $\|\pi_n(Y,x)\| \leq c$. Given $\alpha \in \{0,1\}^{\mathbb{N}}$, let f^α_k be f_k if $\alpha(k) = 1$, and be the constant loop at x_0 otherwise. For $\alpha \in \{0,1\}^{\mathbb{N}}$, let $f^\alpha = f_0^{\alpha(0)} * f_1^{\alpha(1)} * f_2^{\alpha(2)} * \cdots$. We define an equivalence relation E on $\{0,1\}^{\mathbb{N}}$ as follows: $\alpha \sim \beta$ iff f^α is homotopic to f^β in Y. Notice that the conclusion of Lemma 2.4 holds in this situation. We again consider the equivalence relation as a subset of $\{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$, calling it E again. By our proof of Theorem 2.2, we need only prove that $E \in \mathbb{BP}$. For this, we show that E is Σ^1_2. Taking this for granted, we will know that E is a union of $\aleph_1$-many Borel sets, and since $\aleph_1 < c$ and we are assuming Martin's axiom, we have by Theorem 3.4 that $E \in \mathbb{BP}$.

To see that E is Σ^1_2, we again let H be the space of all n-loops at x in the space X, and $\mathbb{H}$ be the space of all homotopies of n-loops at x in the space X. Let

$$J \subseteq H \times H \times \mathbb{H} \times [0,1]^n \times [0,1]^n \times [0,1]^{n+1} \times X \times X \times X$$

be the graph of the continuous function $(f, g, F, \lambda_0, \lambda_1, \lambda_2) \mapsto (f(\lambda_1), g(\lambda_1), f(\lambda_3))$, so that J is closed. Let

$$\begin{gathered} K = H \times H \times \mathbb{H} \times [0,1]^n \times [0,1]^n \times [0,1]^{n+1} \times Y^c \times X \times X \\ \bigcup \\ H \times H \times \mathbb{H} \times [0,1]^n \times [0,1]^n \times [0,1]^{n+1} \times X \times Y^c \times X \\ \bigcup \\ H \times H \times \mathbb{H} \times [0,1]^n \times [0,1]^n \times [0,1]^{n+1} \times X \times X \times Y^c \end{gathered}$$

As K is a finite intersection of analytic (Σ^1_1) sets, we have that K is analytic. Then $J \cap K$ is analytic. Then the projection of $J \cap K$ into $H \times H \times \mathbb{H}$ is analytic, and the complement of this projection, which complement we call G, is a Π^1_1 set in $H \times H \times \mathbb{H}$. Notice that $G = \{(f, g, F) \in H \times H \times \mathbb{H} : f([0,1]^n) \cup g([0,1]^n) \cup F([0,1]^{n+1}) \subseteq Y\}$. Letting $D = \{(f, g, F) \in H \times H \times \mathbb{H} : F \text{ homotopes } f \text{ to } g\}$ as before, we have that $G \cap D$ is Π^1_1 as a finite intersection of two Π^1_1 sets. Projecting $G \cap J$ into $H \times H$ we get a Σ^1_2 set, and taking the preimage of this set into $\{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$ under the continuous map $(\alpha, \beta) \mapsto (f^\alpha, f^\beta)$ we get our relation E, and so we are done, since the Lusin pointclasses are closed under continuous preimages. $\square$

To see that we cannot drop strong set theoretic assumptions for our method of proof, we mention that Σ^1_2 sets need not have the property of Baire (such is the case in Goedel's constructible universe $\mathcal{L}$).

4.3 A Large Cardinal Result

We finish this chapter by noting that if we assume the hypotheses of Theorem 4.5, substituting Π^1_n in the hypotheses for Π^1_1, then we can show that the equivalence relation that we call E is Σ^1_{n+1} in almost exactly the same way.

Definition 4.6. If X is a Polish space, then we define the class of projective sets, $\mathbb{P}$, as follows:

$$\mathbb{P} = \bigcup_{i=1}^{\infty} \Sigma^1_i = \bigcup_{i=1}^{\infty} \Pi^1_i = \bigcup_{i=1}^{\infty} \Delta^1_i$$

We say that a space is standard projective if it is a homeomorph of a projective subspace of a Polish space.

Definition 4.7. A cardinal κ is called inaccessible if it is uncountable, regular (for any $\lambda < \kappa$, there is no unbounded mapping $f : \lambda \to \kappa$) and for each $\lambda < \kappa$ we have that $\|\mathcal{P}(\lambda)\| < \kappa$.

We call the assertion that there exist inaccessible cardinals IC. We have the following consistency theorem:

Theorem 4.8. *[8] If the theory ZFC + IC is consistent, then the following statement is consistent with ZFC:*

If X is a Polish space, then every projective subset of X has the property of Baire.

We finally claim:

Corollary 4.9. *(ZFC + IC is consistent) The following statement is consistent with ZFC:*

Assume X is standard projective and has a point x and a sequence of essential n-loops $f_0, f_1, f_2, \ldots$ at x such that given any open neighborhood O of x, the images of all but finitely many of the f_i are contained in O. Then $\|\pi_n(Y, x)\| = c$.

Proof. Assume the hypotheses, including ZFC + IC is consistent. Then by Theorem 4.8, we can assume that any projective set in a Polish space is in $\mathbb{BP}$. Certainly $\|\pi_n(X, x)\| \leq c$. As X is projective, it is in Π^1_n for some n, so the equivalence relation E as defined in Theorem 4.5 is Σ^1_{n+1}, so $E \in \mathbb{BP}$, so we are done by the proof of Theorem 2.2.

□

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