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Mirror Symmetry for Non-Abelian Landau-Ginzburg Models

Matthew Michael Williams
Brigham Young University

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Matthew Michael Williams

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Nathan Priddis, Chair
Tyler Jarvis
Pace Nielsen

Department of Mathematics

Brigham Young University

ABSTRACT

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Matthew Michael Williams
Department of Mathematics, BYU
Master of Science

We consider Landau-Ginzburg models stemming from non-abelian groups comprised of non-diagonal symmetries, and we describe a rule for the mirror LG model. In particular, we present the *non-abelian dual group* $G^{\star}$, which serves as the appropriate choice of group for the mirror LG model. We also describe an explicit mirror map between the A-model and the B-model state spaces for two examples. Further, we prove that this mirror map is an isomorphism between the untwisted broad sectors and the narrow diagonal sectors in general.

Keywords: algebraic geometry, mirror symmetry

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CONTENTS

CHAPTER 1. INTRODUCTION

Mirror symmetry is most easily explained for Calabi-Yau manifolds. The physics of string theory produces an A-model and a B-model for each Calabi-Yau manifold, so these come in dual pairs. Mirror symmetry essentially says that the A-model for a Calabi-Yau manifold is "the same" as the B-model on its mirror dual, meaning they produce the same physics.

Instead of working with Calabi-Yau manifolds, physics predicts that one can work with what is called a Landau-Ginzburg model instead, which is computationally more efficient. Landau-Ginzburg models are built from an *invertible polynomial* W and a group $G \leq G_W^{max}$, both of which we describe later. One of the important structures of a Landau-Ginzburg model—both for the A-model and B-model—is that of a vector space called the *state space*. This can also be given the structure of a Frobenius algebra or a Frobenius manifold. The *Landau-Ginzburg (LG) Mirror Symmetry Conjecture* predicts that for an invertible polynomial W with a group G of *admissible symmetries* of W, there is a dual polynomial W^T and dual group G^T of symmetries defined by W^T such that the Landau-Ginzburg A-model for the pair (W, G) is isomorphic to the Landau-Ginzburg B-model for the pair (W^T, G^T) (see Berglund–Hübsch [3] or Krawitz [7]).

In the past, mathematicians have primarily studied LG models of pairs (W, G) where G is an abelian group comprised of so-called *diagonal symmetries* (see Francis/Jarvis [6]). There has been much interest in understanding the mirror symmetry for when G is non-abelian, but until now there has not been a clear way to determine the mirror model. Formerly, the dual group of G was only defined when G was a group of diagonal symmetries.

In this paper, we describe the *non-abelian dual group* $G^\star$, which extends the Landau-Ginzburg Mirror Symmetry Conjecture to LG models built from non-abelian groups. We detail the construction of the A- and B-model state spaces, and for two examples provide an explicit isomorphism between them. Furthermore, we prove that certain natural subspaces are isomorphic in general for particular polynomials W and non-abelian groups G. This has several similarities with the mirror map defined by Krawitz [7] for abelian LG models. There are still many hurdles for considering structures beyond vector spaces, the foremost being the lack of definition of even a Frobenius product on the B-side. This is a possible direction for future work.

This construction of $G^\star$ was also discovered independently by Ebeling and Gusein-Zade (see [5], [4]). Much of their work focuses on proving when the mirror map isomorphism exists, while our work pertains more to the actual construction of the mirror map between the A- and B-models.

Chapter 2. Preliminary Definitions

In this chapter, we begin by introducing some definitions that will be vital to the construction of the A- and B-model state spaces. As mentioned earlier, every Landau-Ginzburg model stems from a polynomial W and a group G, but both of these are subject to certain conditions which we will detail here.

2.1 Invertible Polynomials

Definition 2.1. A polynomial is *quasihomogeneous* if there exist positive rational numbers $(q_1, \ldots, q_n)$ so that for every $c \in \mathbb{C}^*$, we have

$$W(c^{q_1}x_1, \ldots, c^{q_n}x_n) = cW(x_1, \ldots, x_n).$$

The numbers $(q_1, \ldots, q_n)$ are called the *weights* of the polynomial W. These will be used later to construct an important symmetry of W called the *exponential grading operator*, which we will denote by j_W.

Definition 2.2. A quasihomogeneous polynomial $W : \mathbb{C}^n \to \mathbb{C}$ is *nondegenerate* if it has an isolated critical point at the origin, and it contains no monomials of the form $x_i x_j$ for $i \neq j$.

This definition implies that the weights $(q_1, \ldots, q_n)$ of W are uniquely determined and $q_i \in (0, \frac{1}{2}) \cap \mathbb{Q}$ for all i.

Definition 2.3. A quasihomogeneous, nondegenerate polynomial is *invertible* if the polynomial has the same number of monomials as variables.

Example 2.4. Consider the polynomial $W : \mathbb{C}^4 \to \mathbb{C}$ defined by $W = x_1^4 + x_2^4 + x_3^4 + x_4^4$. First, W is nondegenerate since it has a unique critical point at $(0, 0, 0, 0)$. Next, note that W is quasihomogeneous with weights $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ since for every $c \in \mathbb{C}^*$, we have

$$\begin{aligned} W(c^{\frac{1}{4}}x_1, c^{\frac{1}{4}}x_2, c^{\frac{1}{4}}x_3, c^{\frac{1}{4}}x_4) &= (c^{\frac{1}{4}}x_1)^4 + (c^{\frac{1}{4}}x_2)^4 + (c^{\frac{1}{4}}x_3)^4 + (c^{\frac{1}{4}}x_4)^4 \\ &= cx_1^4 + cx_2^4 + cx_3^4 + cx_4^4 \\ &= c(x_1^4 + x_2^4 + x_3^4 + x_4^4) \\ &= cW(x_1, x_2, x_3, x_4). \end{aligned}$$

Clearly this choice of weights is unique. Also, we can see that W has four monomials and four variables, hence W is invertible. We will continue to work with the above polynomial throughout this paper. This is known as a *Fermat* polynomial.

Theorem 2.5 (Kreuzer/Sharke [8])**.** *Any invertible quasihomogeneous polynomial is a decoupled sum of polynomials of one of the following three atomic types:*

Fermat type: $x_1^{a_1}$

Chain type: $x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_n^{a_n} (n \geq 1)$

Loop type: $x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_n^{a_n}x_1 (n \geq 2)$

Example 2.6. An example of a chain polynomial is $W = x_1^3 x_2 + x_2^2 x_3 + x_3^2$ with weights $(\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$. An example of a loop polynomial is $W = x_1^2 x_2 + x_2^2 x_3 + x_3^2 x_1$, which has weights $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

2.2 MAXIMAL SYMMETRY GROUP

Definition 2.7 (Mukai [9])**.** Let $W : \mathbb{C}^n \to \mathbb{C}$ be an invertible polynomial with weights $(q_1, \ldots, q_n)$. Then the *maximal symmetry group of* W, denoted G_W^{max}, is defined as follows:

$$G_W^{max} := \{g \in \mathrm{GL}_n(\mathbb{C}) | (g \cdot W)(x_1, \ldots, x_n) = W(x_1, \ldots, x_n)$$
$$\text{and } g_{ij} = 0 \text{ if } q_i \neq q_j\}$$

The condition $g_{ij} = 0$ if $q_i \neq q_j$ implies that g has a block diagonal form. This is equivalent to the condition that each g commutes with the action of $c \in \mathbb{C}^*$ where c acts on $(x_1, \ldots, x_n)$ by $c \cdot (x_1, \ldots, x_n) = (c^{q_1} x_1, \ldots, c^{q_n} x_n)$ (Mukai [9], pointed out by Y. Ruan).

The *diagonal symmetry group of* W is the group of diagonal linear transformations, defined

$$G_W^{diag} := \{(g_1, \ldots, g_n) \in (\mathbb{C}^*)^n | W(g_1 x_1, \ldots, g_n x_n) = W(x_1, \ldots, x_n)\}.$$

The second definition is the standard definition of diagonal symmetries, (Francis/Jarvis [6]). Note that G_W^{diag} can be viewed as a subgroup of G_W^{max} via diagonal matrices. It is a standard fact that for $g = (g_1, \ldots, g_n) \in G_W^{diag}$ the entries g_i as above are roots of unity (Artebani/Boissière/Sarti [1]). For simplicity, we will typically represent these symmetries additively as n-tuples of rational numbers as follows:

$$(e^{2\pi i a_1}, \ldots, e^{2\pi i a_n}) \leftrightarrow (a_1, \ldots, a_n) \in (\mathbb{Q}/\mathbb{Z})^n$$

It is a fact that G_W^{diag} is generated by the entries of the inverse of the *exponent matrix* A_W (Artebani/Boissière/Sarti [1] or Krawitz [7]), which we define below. Furthermore, one can see that the exponential grading operator $j_W = (q_1, \ldots, q_n)$ is an element of G_W^{diag}, where $q_1, \ldots, q_n$ are the weights of W.

Two other important subgroups of G_W^{max} are J_W and SL_W^{diag}. The group J_W is the group generated by j_W. The group SL_W^{diag} is the group of matrices in G_W^{diag} whose determinant is

1. Explicitly, we write these groups as follows:

$$J_W = \langle j_W \rangle = \langle (q_1, \dots, q_n) \rangle$$
$$\mathrm{SL}_W^{diag} = \mathrm{SL}(n, \mathbb{C}) \cap G_W^{diag}$$

Example 2.8. For $W = x_1^4 + x_2^4 + x_3^4 + x_4^4$, we have

$$J_W = \langle (\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}) \rangle \text{ and } \mathrm{SL}_W^{diag} = \langle (\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}), (\tfrac{2}{4}, \tfrac{1}{4}, \tfrac{1}{4}, 0), (\tfrac{1}{4}, \tfrac{2}{4}, \tfrac{1}{4}, 0) \rangle.$$

BHK mirror symmetry associates to an LG model (W, G) another LG model (W^T, G^T), which we work towards next. These two LG models are the *dual* of each other under BHK mirror symmetry, discovered by Berglund–Hübsch [3] and Krawitz [7], which we will show explicitly for this choice of W in Example 2.13.

2.3 Dual Polynomials and Dual Groups

Definition 2.9. Let W be an invertible polynomial. If we write $W = \sum_{i=1}^n \prod_{j=1}^n x_j^{a_{ij}}$, then the associated *exponent matrix* is defined to be $A_W = (a_{ij})$. The *dual polynomial* W^T is the invertible polynomial defined by the matrix A_W^T.

Example 2.10. For $W = x_1^4 + x_2^4 + x_3^4 + x_4^4$, we have

$$A_W = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} = A_W^T.$$

Hence in this case, $W^T = W$.

Example 2.11. While the dual polynomial is invariant when W is a Fermat polynomial as in the previous example, this isn't always the case. If W is the chain polynomial $W =$

$x_1^3x_2 + x_2^2x_3 + x_3^2$ from Example 2.6, then

$$A_W = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}, \text{ so } A_W^T = \begin{pmatrix} 3 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}.$$

In this example, we see that $W^T = x_1^3 + x_1x_2^2 + x_2x_3^2$. Notice that W^T is also invertible and that its weights are $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

Note that the exponent matrix A_W from Definition 2.9 is only defined up to a reordering of rows. For instance, if we write W from Example 2.11 as $W = x_2^2x_3 + x_3^2 + x_1^3x_2$, then

$$A_W = \begin{pmatrix} 0 & 2 & 1 \\ 0 & 0 & 2 \\ 3 & 1 & 0 \end{pmatrix}, \text{ so } A_W^T = \begin{pmatrix} 0 & 0 & 3 \\ 2 & 0 & 1 \\ 1 & 2 & 0 \end{pmatrix}.$$

Thus $W^T = x_3^3 + x_1^2x_3 + x_1x_2^2$, but this is just a reordering of the variables from W^T in Example 2.11, so they are effectively the same.

Definition 2.12. The *dual group* of a subgroup $G \leq G_W^{diag}$ is the set

$$G^T = \{g \in G_{W^T}^{diag} | g A_W h^T \in \mathbb{Z} \text{ for all } h \in G\},$$

where we consider g and h in their additive form as row vectors.

Example 2.13. After Example 2.8, we claimed that the dual group of J_W is SL_W^{diag} for $W = x_1^4 + x_2^4 + x_3^4 + x_4^4 = W^T$. Observe

$$(J_W)^T = \{g \in G_{W^T}^{diag} | g A_W h^T \in \mathbb{Z} \text{ for all } h \in J_W\}.$$

Let $g \in G_{W^T}^{diag}$ and $h \in J_W$, then $g = (\frac{a_1}{4}, \frac{a_2}{4}, \frac{a_3}{4}, \frac{a_4}{4})$ and $h = (\frac{b}{4}, \frac{b}{4}, \frac{b}{4}, \frac{b}{4})$ where $a_1, a_2, a_3, a_4, b \in$

$\{0,1,2,3\}$. Then

$$gA_Wh^T = (\tfrac{a_1}{4}, \tfrac{a_2}{4}, \tfrac{a_3}{4}, \tfrac{a_4}{4}) \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} (\tfrac{b}{4}, \tfrac{b}{4}, \tfrac{b}{4}, \tfrac{b}{4})^T$$

$$= \tfrac{a_1 b}{4} + \tfrac{a_2 b}{4} + \tfrac{a_3 b}{4} + \tfrac{a_4 b}{4}$$

$$= b(\tfrac{a_1}{4} + \tfrac{a_2}{4} + \tfrac{a_3}{4} + \tfrac{a_4}{4}).$$

This value is an integer for all $b \in \{0,1,2,3\}$ if and only if $(\frac{a_1}{4} + \frac{a_2}{4} + \frac{a_3}{4} + \frac{a_4}{4}) \in \mathbb{Z}$, implying $g \in \mathrm{SL}^{diag}_{W^T}$. Hence $(J_W)^T = \mathrm{SL}^{diag}_{W^T}$. In fact, it is true that $(J_W)^T = \mathrm{SL}^{diag}_{W^T}$ for any choice of invertible polynomial W (Artebani/Boissière/Sarti [1]).

As mentioned previously, most of the work done with Landau-Ginzburg models has been with subgroups of G^{diag}_W. Next, we consider a group with a permutation as one of its generators, which is a non-diagonal symmetry.

Example 2.14. With $W = x_1^4 + x_2^4 + x_3^4 + x_4^4$ as before, consider the subgroup

$$G = \langle j_W, (123) \rangle \leq G^{max}_W, \text{ where } (123) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Here (123) permutes the variables x_1, x_2, and x_3. Even though G contains non-diagonal matrices, it is actually still abelian since the generators commute. We can see this because j_W is a constant diagonal matrix, and so it lies in the center of $\mathrm{GL}(n, \mathbb{C})$.

Although G is abelian in the above example, we cannot use the previously mentioned definition for G^T since G is not a subset of G^{diag}_W, as required by Definition 2.12. In order to define the dual group we need to define the *non-abelian dual group*.

2.4 THE NON-ABELIAN DUAL GROUP

Definition 2.15. An element of G_W^{max} is called a *pure permutation* if it acts on $\mathbb{C}[x_1, \ldots, x_n]$ by simply permuting the variables.

Notice that a pure permutation can only permute variables that have the same weight with respect to W. We are now ready to define the non-abelian dual group $G^\star$.

Definition 2.16. Let $G \leq G_W^{max}$ be a group of the form

$$G = K \cdot H,$$

where $K \leq G$ is the subgroup of pure even permutations and $H \leq G \cap G_W^{diag}$. This product should be thought of as a subgroup of $\mathrm{GL}(n, \mathbb{C})$. We define the *non-abelian dual group* of G to be

$$G^\star = K \cdot H^T \leq \mathrm{GL}(n, \mathbb{C}).$$

Example 2.17. If we consider $G = \langle j_W, (123) \rangle \leq G_W^{max}$ from Example 2.14, then

$$G^\star = \big\langle (123) \big\rangle \cdot (J_W)^T = \big\langle (123) \big\rangle \cdot \mathrm{SL}_{W^T}^{diag}.$$

Explicitly, the elements of $G^\star$ are of the form $(123)^k(\frac{a_1}{4}, \frac{a_2}{4}, \frac{a_3}{4}, \frac{a_4}{4})$, where $a_1 + a_2 + a_3 + a_4 \in 4\mathbb{Z}$ and $0 \leq k \leq 2$. As stated earlier, in this example $G^\star$ is non-abelian. For instance, consider the products of $(123)(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, 0)$ and $(132)(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, 0) \in G^\star$ in both ways. Observe

$$(123)(\tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{4}, 0) \cdot (132)(\tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{4}, 0) = (\tfrac{3}{4}, \tfrac{1}{2}, \tfrac{3}{4}, 0), \text{ whereas}$$

$$(132)(\tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{4}, 0) \cdot (123)(\tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{4}, 0) = (\tfrac{3}{4}, \tfrac{3}{4}, \tfrac{1}{2}, 0).$$

Now we have defined a rule relating two LG models (W, G) and $(W^T, G^\star)$. In the next sections we will construct the A- and B-model state spaces.

The A-model vector space is refered to as the *state space*. The construction for the state space, as shown by Basalaev, Takahashi, and Werner [2] and Mukai [9], requires an invertible polynomial W and an *admissible subgroup* of G_W^{max}, which we will define shortly. One other important piece is the Milnor ring, which we now define.

Definition 3.1. The Milnor ring of a polynomial W is defined to be

$$\mathcal{Q}_W = \frac{\mathbb{C}[x_1, \ldots, x_n]}{\left(\frac{\partial W}{\partial x_1}, \ldots, \frac{\partial W}{\partial x_n}\right)}.$$

Definition 3.2. Let W be a nondegenerate quasihomogenenous polynomial with unique weights $(q_1, \ldots, q_n)$, and let G be a subgroup of G_W^{max}. Then G is *admissible* if G contains $j_W = (q_1, \ldots, q_n)$.

Definition 3.3. Given an admissible group G and an element $g \in G$, we let $\text{Fix}(g)$ denote the subspace of $\mathbb{C}^n$ which is fixed by g

$$\text{Fix}(g) = \{(a_1, \ldots, a_N) | g(a_1, \ldots, a_N) = (a_1, \ldots, a_N)\}.$$

To find $\text{Fix}(g)$ we look for eigenvectors of g with an eigenvalue of 1, and the span of these vectors will define $\text{Fix}(g)$. We also write

$$W_g = W|_{\text{Fix}(g)}$$

to denote the polynomial W restricted to $\text{Fix}(g)$.

Definition 3.4. Let W be an invertible polynomial and G be an admissible subgroup of

G_W^{max}. The *state space* for the A-model is defined as

$$\mathcal{A}_{W,G} = \left(\bigoplus_{g \in G} \mathcal{Q}_{W_g} \cdot \omega_g \right)^G ,$$

where ω_g is a volume form on the fixed locus of g.

We will use the notation $\lfloor P, g \rceil$ to denote an element of $\mathcal{Q}_{W_g} \cdot \omega_g$, often suppressing the volume form where convenient. The volume form can be easily determined by g. We can form a basis of $\mathcal{A}_{W,G}$ using sums of the form

$$\sum_{g_i \in [g]} \lfloor P, g_i \rceil ,$$

where g_i are the group elements in the same conjugacy class $[g]$ of G, and $P \in \mathcal{Q}_{W_{g_i}}$.

When G is abelian, we can rewrite the state space definition as

$$\mathcal{A}_{W,G} = \bigoplus_{g \in G} (\mathcal{Q}_{W_g} \cdot \omega_g)^G$$

as the action of G preserves each summand. But if G is non-abelian, then for $h \in G$,

$$h \cdot \left(\mathcal{Q}_{W_g} \cdot \omega_g \right) \subseteq \mathcal{Q}_{W_{h^{-1}gh}} \cdot \omega_{h^{-1}gh}. \qquad (*)$$

3.1 Constructing an A-model State Space

Example 3.5. Let $W = x_1^4 + x_2^4 + x_3^4 + x_4^4$ and $G = \langle j_W, (123) \rangle$. We will determine a basis for $\mathcal{A}_{W,G}$. Since in this case, G is an abelian group, the conjugacy class for each $g \in G$ contains only g. Hence we can choose a basis of $\mathcal{A}_{W,G}$ consisting of elements of the form $\lfloor P, g \rceil$ (i.e. single terms, instead of sums, although P may have more than one summand). The elements of G can be expressed as $(123)^a j_W^b$ with $0 \leq a \leq 2$ and $0 \leq b \leq 3$. For each $g \in G$, we will need to find the basis elements of $(\mathcal{Q}_{W_g} \cdot \omega_g)^G$. The choices of g can be broken down into three different cases.

Case 1: $g = (0,0,0,0)$

When $g = (0,0,0,0)$, then $W_g = W$, and the Milnor ring of W_g is

$$\mathcal{Q}_{W_g} = \mathcal{Q}_W = \frac{\mathbb{C}[x_1, x_2, x_3, x_4]}{\langle 4x_1^3, 4x_2^3, 4x_3^3, 4x_4^3 \rangle} = \frac{\mathbb{C}[x_1, x_2, x_3, x_4]}{\langle x_1^3, x_2^3, x_3^3, x_4^3 \rangle}.$$

The elements of $\mathcal{Q}_W$ are sums of elements in the set $\{x_1^a x_2^b x_3^c x_4^d | 0 \leq a, b, c, d \leq 2\}$. The volume form ω_g in this case is $dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4$. To find the elements of $(\mathcal{Q}_W \cdot (dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4))^G$ we look for $p(x) \in \mathcal{Q}_W$ such that $p(x) \cdot (dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4)$ is invariant under j_W and (123), the generators of G. The volume form is invariant under j_W since

$$(e^{\frac{2\pi i}{4}} dx_1) \wedge (e^{\frac{2\pi i}{4}} dx_2) \wedge (e^{\frac{2\pi i}{4}} dx_3) \wedge (e^{\frac{2\pi i}{4}} dx_4) = dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4.$$

It is also invariant under (123) since

$$dx_2 \wedge dx_3 \wedge dx_1 \wedge dx_4 = -(dx_2 \wedge dx_1 \wedge dx_3 \wedge dx_4) = dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4.$$

Thus, in this case we only need to be concerned that the actual polynomial $p(x)$ is invariant under j_W and (123).

In order to be invariant under (123), the polynomial must be symmetric with respect to x_1, x_2, and x_3 and polynomials invariant under j_W must have exponents in each term sum to a multiple of 4; for example, the polynomial $x_1 x_2 x_3 x_4 \in \mathcal{Q}_W$ is invariant under both j_W and (123). The basis elements of $(\mathcal{Q}_W \cdot \omega_g)^G$ can also be sums of monomials in

$\mathcal{Q}_W \cdot \omega_g$. Consider $x_1^2x_4^2 + x_2^2x_4^2 + x_3^2x_4^2 \in \mathcal{Q}_W$. Applying j_W to $x_1^2x_4^2 + x_2^2x_4^2 + x_3^2x_4^2$ gives

$$\begin{aligned}(e^{\frac{2\pi i}{4}}x_1)^2(e^{\frac{2\pi i}{4}}x_4)^2 + (e^{\frac{2\pi i}{4}}x_2)^2(e^{\frac{2\pi i}{4}}x_4)^2 + (e^{\frac{2\pi i}{4}}x_3)^2(e^{\frac{2\pi i}{4}}x_4)^2 \\ = (e^{\frac{4\pi i}{4}}x_1^2)(e^{\frac{4\pi i}{4}}x_4^2) + (e^{\frac{4\pi i}{4}}x_2^2)(e^{\frac{4\pi i}{4}}x_4^2) + (e^{\frac{4\pi i}{4}}x_3^2)(e^{\frac{4\pi i}{4}}x_4^2) \\ = x_1^2x_4^2 + x_2^2x_4^2 + x_3^2x_4^2.\end{aligned}$$

Applying (123) gives

$$x_2^2x_4^2 + x_3^2x_4^2 + x_1^2x_4^2 = x_1^2x_4^2 + x_2^2x_4^2 + x_3^2x_4^2,$$

and thus this element of the Milnor ring (including its volume form) is invariant under all the generators of G, so it is invariant under G.

In the same way, we find that the invariant elements of the Milnor ring in the identity sector are of the form $P = p(x) \cdot (dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4)$, where $p(x)$ is one of the following polynomials:

$$1$$
$$x_1x_2x_3x_4$$
$$x_1^2x_2^2x_3^2x_4^2$$
$$x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2$$
$$x_1x_2x_3^2 + x_1^2x_2x_3 + x_1x_2^2x_3$$
$$x_1x_2x_4^2 + x_1x_3x_4^2 + x_2x_3x_4^2$$
$$x_1x_2^2x_4 + x_2x_3^2x_4 + x_3x_1^2x_4$$
$$x_1^2x_2x_4 + x_2^2x_3x_4 + x_3^2x_1x_4$$
$$x_1^2x_4^2 + x_2^2x_4^2 + x_3^2x_4^2$$

The 9 dimensional vector space generated by these elements is called the *untwisted broad* *sector* of $\mathcal{A}_{W,G}$, where all the eigenvectors of g have an eigenvalue of 1.

Case 2: $g = (123)$ **or** $g = (132)$

Let

$$g = (123) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

To find Fix(123), we look for eigenvectors of (123) with an eigenvalue of 1. Diagonalizing (123) gives $(123) = QDQ^{-1}$, where

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{\frac{4\pi i}{3}} & 0 \\ 0 & 0 & 0 & e^{\frac{2\pi i}{3}} \end{bmatrix} \text{ and } Q = \begin{bmatrix} 0 & 1 & e^{\frac{4\pi i}{3}} & e^{\frac{2\pi i}{3}} \\ 0 & 1 & e^{\frac{2\pi i}{3}} & e^{\frac{4\pi i}{3}} \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Thus the eigenvectors with eigenvalue 1 are $(1,1,1,0)$ and $(0,0,0,1)$, and the span of these two vectors is Fix(123). If we call the coordinates of these two vectors y_1 and y_4, then we have $W_g = c_1 y_1^4 + y_4^4$ for some constant c_1. The value of c_1 does not matter for our purposes, since $\mathcal{Q}_{W_g}$ is simply $\mathbb{C}[y_1, y_4]/(y_1^3, y_4^3)$. The volume form here is $dy_1 \wedge dy_4 = (dx_1 + dx_2 + dx_3) \wedge dx_4$. This is invariant under (123), which acts trivially when considering only y_1 and y_4. However, this volume form is not invariant under j_W since

$$(e^{\frac{2\pi i}{4}} dy_1) \wedge (e^{\frac{2\pi i}{4}} dy_4) = -(dy_1 \wedge dy_4) \neq dy_1 \wedge dy_4.$$

To balance this, in order for an element of $\mathcal{Q}_{W_g} \cdot \omega_g$ to be invariant under j_W, the polynomial $p(x)$ must have each term be degree equal to 2 (mod 4). This means the degree must be 2 since the elements of $\mathbb{C}[y_1, y_4]/(y_1^3, y_2^3)$ have the exponents on y_1 and y_4 capped at 2.

This gives us three anti-invariant polynomials:

$$y_1^2 = (x_1 + x_2 + x_3)^2$$
$$y_1 y_4 = (x_1 + x_2 + x_3)x_4$$
$$y_4^2 = x_4^2$$

Each one of these, together with the volume form, is another element in the basis of $\mathcal{A}_{W,G}$.

The case of $g = (132)$ is almost identical. The matrix Q will be different, but the vectors $(0,0,0,1)$ and $(1,1,1,0)$ still correspond to the eigenvalues of 1, and they produce the same Milnor ring, volume form, and invariant entries. The two 3 dimensional vector spaces produced by (123) and (132) are known as *twisted broad sectors*, where some, but not all of the eigenvectors of g have an eigenvalue of 1.

Case 3: Other Values of g

The eigenvalues of j_W are all $e^{\frac{2\pi i}{4}}$, so $g = j_W$ has trivial fixed locus. Thus $W|_{\text{Fix}(j_W)} = 0$. This implies that for $g = j_W$, we get $\mathcal{Q}_{W_g} \cdot \omega_g \cong \mathbb{C}$. There is a natural basis for $\mathbb{C}$, namely 1, so this sector only produces a single basis element of $\mathcal{A}_{W,G}$, being $\lfloor 1, j_W \rceil$. Sectors with $\text{Fix}(g) = 0$ are called *narrow sectors*. The action of G on these narrow sectors is trivial, so each contributes to the basis. Similarly, $(j_W)^2$ and $(j_W)^3$ produce narrow sectors as well.

Next, we look at $g = (123)j_W$. As seen in case 2, the eigenvalues of (123) are $1, 1, e^{\frac{4\pi i}{3}}$, and $e^{\frac{2\pi i}{3}}$, so the eigenvalues of $(123)j_W$ are $e^{\frac{2\pi i}{4}}, e^{\frac{2\pi i}{4}}, e^{\frac{22\pi i}{12}}$, and $e^{\frac{14\pi i}{12}}$. None of these are 1, so $(123)j_W$ produces another narrow sector. Similarly, we can see that $(123)j_W$, $(123)(j_W)^2$, $(123)(j_W)^3$, $(132)(j_W)$, $(132)(j_W)^2$, and $(132)(j_W)^3$ have no eigenvalues equal to 1, so they are also narrow sectors. In total, there are 9 narrow sectors in $\mathcal{A}_{W,G}$.

To conclude this example, we have found that there are 9 narrow sectors, the untwisted broad sector has dimension 9, and the two twisted broad sectors from (123) and (132) each contribute dimension 3 to the state space. Hence $\mathcal{A}_{W,G}$ has dimension 24.

3.2 A-MODEL BIGRADING

The A-model can also be given a bigrading which we will see is also preserved under mirror symmetry. This bigrading is similar to the Hodge grading for Calabi-Yau manifolds. Since mirror symmetry for Calabi-Yau manifolds rotates the Hodge diamond, we expect some similar phenomenon for LG models.

Definition 3.6 (Mukai [9]). Let G be a finite subgroup of the symmetry group of some non-degenerate quasihomogeneous polynomial in $\mathbb{C}[x_1, \dots, x_n]$. We define the *age* of $g \in G$ as

$$\text{age } g = \frac{1}{2\pi i} \sum_{j=1}^{n} \log(\lambda_j),$$

where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of g and the branch of the logarithmic function for $z \in \mathbb{C}^*$ such that $|z| = 1$ is chosen to satisfy $0 \leq \log(z) < 2\pi i$.

Example 3.7. When g is diagonal, then λ_j will take on the value of the sole entry in the j^{th} column (when we view g multiplicitively). Since the entries are all of the form $e^{2\pi i(a_j)}$, the age of g is just $\sum_{j=1}^{n} a_j$. Hence for diagonal symmetries, we can simply sum the entries when they are written in additive form. For example, for $j_W = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ from the A-model vector space in the previous section, we have

$$\text{age}(j_W) = \tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{4} = 1.$$

Definition 3.8. The A-model has a *bigrading*, defined to be the ordered pair

$$(\deg P + \text{age } g - \text{age } j_W, N_g - \deg P + \text{age } g - \text{age } j_W),$$

where N_g is the dimension of $\text{Fix}(g)$, and $\deg P$ is the weighted degree. The weighted degree of P is the standard way you would think of the degree of a polynomial, but divided by the weights of W for each variable. In this notation, note that the volume form ω_g contributes to $\deg P$.

Example 3.9. Let's continue with the A-model vector space from the previous section to turn it into a bigraded vector space. Since $\text{age}(j_W) = 1$ as mentioned in Example 3.7, the bidegree for each element reduces to

$$(\deg P + \text{age } g - 1, N_g - \deg P + \text{age } g - 1),$$

and we can break up the process into the same three cases as before depending on g.

Case 1: g = (0, 0, 0, 0)

When g is the identity, we get age $g = 0$ and $N_g = 4$, so the bidegree simplifies to

$$(\deg P + 0 - 1, 4 - \deg P + 0 - 1) = (\deg P - 1, 3 - \deg P),$$

and the bidegree is dependent only on deg P. Recall that there were 9 polynomials in our basis for this choice of g. A few examples of deg P can be seen below. The weights of W are $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$, which is why we divide by 4 in the following computations. In particular the volume form will have weighted degree of $\frac{4}{4} = 1$, so $\deg P = \deg p(x) + 1$.

$$\deg(1 \cdot (dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4)) = \deg(1) + 1 = \frac{0}{4} + 1 = 1$$
$$\deg(x_1x_2x_3x_4 \cdot (dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4)) = \deg(x_1x_2x_3x_4) + 1 = \frac{4}{4} + 1 = 2$$
$$\deg(x_1^2x_2^2x_3^2x_4^2 \cdot (dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4)) = \deg(x_1^2x_2^2x_3^2x_4^2) + 1 = \frac{8}{4} + 1 = 3$$
$$\deg(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2 \cdot (dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4)) = \deg(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2) + 1 = \frac{4}{4} + 1 = 2$$

The rest of the polynomials in this sector have degree 2. We list in the following table the bidegree for all of the basis elements in this sector.

Basis element	**Bidegree**
$\lfloor 1, (0,0,0,0) \rceil$	$(0,2)$
$\lfloor x_1x_2x_3x_4, (0,0,0,0) \rceil$	$(1,1)$
$\lfloor x_1^2x_2^2x_3^2x_4^2, (0,0,0,0) \rceil$	$(2,0)$
$\lfloor x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2, (0,0,0,0) \rceil$	$(1,1)$
$\lfloor x_1x_2x_3^2 + x_1^2x_2x_3 + x_1x_2^2x_3, (0,0,0,0) \rceil$	$(1,1)$
$\lfloor x_1x_2x_4^2 + x_1x_3x_4^2 + x_2x_3x_4^2, (0,0,0,0) \rceil$	$(1,1)$
$\lfloor x_1x_2^2x_4 + x_2x_3^2x_4 + x_3x_1^2x_4, (0,0,0,0) \rceil$	$(1,1)$
$\lfloor x_1^2x_2x_4 + x_2^2x_3x_4 + x_3^2x_1x_4, (0,0,0,0) \rceil$	$(1,1)$
$\lfloor x_1^2x_4^2 + x_2^2x_4^2 + x_3^2x_4^2, (0,0,0,0) \rceil$	$(1,1)$

Case 2: $g = (123)$ **or** $g = (132)$

Recall that in this case, the fixed locus was spanned by the two vectors $y_1 = x_1 + x_2 + x_3$ and $y_4 = x_4$, so $N_g = 2$. To find age((123)) and age((132)), recall that the eigenvalues of (123) and (132) are $1, 1, e^{\frac{2\pi i}{3}}$, and $e^{\frac{4\pi i}{3}}$. Then

$$\begin{aligned}\frac{1}{2\pi i}\sum_{j=1}^{n}\log(\lambda_j) &= \frac{1}{2\pi i}(\log(1) + \log(1) + \log(e^{\frac{2\pi i}{3}}) + \log(e^{\frac{4\pi i}{3}})) \\ &= \frac{1}{2\pi i}(0 + 0 + \frac{2\pi i}{3} + \frac{4\pi i}{3}) \\ &= \frac{1}{2\pi i}(2\pi i) = 1,\end{aligned}$$

so age((123)) and age((132)) are both 1. Next we need to find deg P for the three polynomials we found earlier. It is easiest to think about the polyomials as elements of the Milnor ring $\mathbb{C}[y_1, y_4]/(y_1^3, y_4^3)$. There were 3 elements in the G-invariant subspace of the

Milnor ring, and their degrees are computed below.

$$\deg(y_1^2 \cdot (dy_1 \wedge dy_4)) = 1$$
$$\deg(y_1 y_4 \cdot (dy_1 \wedge dy_4)) = 1$$
$$\deg(y_4^2 \cdot (dy_1 \wedge dy_4)) = 1$$

Thus the bidegree of each of these elements is the same, which is

$$(\deg P + \text{age } g - 1, N_g - \deg P + \text{age } g - 1) = (1 + 1 - 1, 2 - 1 + 1 - 1) = (1, 1).$$

Case 3: Other Values of g

For all other choices of g, we know that g creates a narrow sector. So the fixed locus has dimension 0, that is, $N_g = 0$. The formula for bidegree thus reduces to

$$(\text{age } g - \text{age } j_W, \text{age } g - \text{age } j_W) = (\text{age } g - 1, \text{age } g - 1).$$

Hence in this case, the only thing we need to actually compute is age g. When g is a multiple of (j_W), we can simply add up the components to get

$$\text{age}(j_W) = 1, \text{age}((j_W)^2) = 2, \text{ and } \text{age}((j_W)^3) = 3,$$

and their bidegree is shown below.

Basis element	**Bidegree**
$\lfloor 1, j_W \rceil$	$(0, 0)$
$\lfloor 1, (j_W)^2 \rceil$	$(1, 1)$
$\lfloor 1, (j_W)^3 \rceil$	$(2, 2)$

The rest of the elements are non-diagonal, so we must find the eigenvalues as in the previous case. The resulting age is the same for all of them, which is 2. Thus the bidegree for the rest of the narrow sectors is $(1,1)$. We have now covered all cases, and in following table, we can see all basis elements and their bidegree.

A-model basis element	**Bidegree**
$\lfloor 1, (0,0,0,0) \rceil$	$(0,2)$
$\lfloor x_1x_2x_3x_4, (0,0,0,0) \rceil$	$(1,1)$
$\lfloor x_1^2x_2^2x_3^2x_4^2, (0,0,0,0) \rceil$	$(2,0)$
$\lfloor x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2, (0,0,0,0) \rceil$	$(1,1)$
$\lfloor x_1x_2x_3^2 + x_1^2x_2x_3 + x_1x_2^2x_3, (0,0,0,0) \rceil$	$(1,1)$
$\lfloor x_1x_2x_4^2 + x_1x_3x_4^2 + x_2x_3x_4^2, (0,0,0,0) \rceil$	$(1,1)$
$\lfloor x_1x_2^2x_4 + x_2x_3^2x_4 + x_1^2x_3x_4, (0,0,0,0) \rceil$	$(1,1)$
$\lfloor x_1^2x_2x_4 + x_2^2x_3x_4 + x_1x_3^2x_4, (0,0,0,0) \rceil$	$(1,1)$
$\lfloor x_1^2x_4^2 + x_2^2x_4^2 + x_3^2x_4^2, (0,0,0,0) \rceil$	$(1,1)$
$\lfloor (x_1 + x_2 + x_3)^2, (123) \rceil$	$(1,1)$
$\lfloor (x_1 + x_2 + x_3)x_4, (123) \rceil$	$(1,1)$
$\lfloor x_4^2, (123) \rceil$	$(1,1)$
$\lfloor (x_1 + x_2 + x_3)^2, (132) \rceil$	$(1,1)$
$\lfloor (x_1 + x_2 + x_3)x_4, (132) \rceil$	$(1,1)$
$\lfloor x_4^2, (132) \rceil$	$(1,1)$
$\lfloor 1, j_W \rceil$	$(0,0)$
$\lfloor 1, (j_W)^2 \rceil$	$(1,1)$
$\lfloor 1, (j_W)^3 \rceil$	$(2,2)$
$\lfloor 1, (123)j_W \rceil$	$(1,1)$
$\lfloor 1, (123)(j_W)^2 \rceil$	$(1,1)$
$\lfloor 1, (123)(j_W)^3 \rceil$	$(1,1)$
$\lfloor 1, (132)j_W \rceil$	$(1,1)$
$\lfloor 1, (132)(j_W)^2 \rceil$	$(1,1)$
$\lfloor 1, (132)(j_W)^3 \rceil$	$(1,1)$

If we arrange these as a Hodge diamond, we have

$$\begin{array}{ccc} & 1 & \\ 1 & 20 & 1 \\ & 1 & \end{array}$$

The reader may notice that this is the Hodge diamond of a K3 surface.

During this section, when we considered the bidegree of a polynomial P, where P was a sum of terms, we only needed to check the degree of a single term. This is because the bidegree of a polynomial is unchanged when acted upon by a symmetry in G, which is a fact we shall prove next. This will be a very useful fact to have in Theorem 5.2, where we will prove that the so called *mirror map* is a bigraded vector space isomorphism.

Lemma 3.10. *Given $h \in G_W^{max}$, and $\lfloor P, g \rceil \in \mathcal{A}_{W,G}$, the element $h \cdot \lfloor P, g \rceil$ has the same bidegree as $\lfloor P, g \rceil$.*

Proof. First, recall the A-model bigrading from Definition 3.8:

$$(\deg P + \operatorname{age} g - \operatorname{age} j_W, N_g - \deg P + \operatorname{age} g - \operatorname{age} j_W).$$

We are also going to rely on (∗) from right after Defintion 3.4, which says that

$$h \cdot \lfloor P, g \rceil = \lfloor h \cdot P, h^{-1} g h \rceil.$$

Note that $\operatorname{age} j_W$ will clearly be unaffected by the action of h on $\lfloor P, g \rceil$. We aim to show that $\operatorname{age}(h^{-1}gh) = \operatorname{age} g$, $N_{h^{-1}gh} = N_g$, and $\deg(h \cdot P) = \deg P$. Recall from Definition 3.6 that the age of g is dependent only on the eigenvalues of g. Since g and $h^{-1}gh$ are similar matrices, they must have the same eigenvalues, so $\operatorname{age}(h^{-1}gh) = \operatorname{age} g$. This also gives us $N_{(h^{-1}gh)} = N_g$, since N_g is the number of eigenvalues of g which are equal to one. To show that $\deg(h \cdot P) = \deg P$, we will have two cases for h: either h is a pure permutation or h is a diagonal symmetry. A third case would be when h is a product of a pure permutation and a diagonal symmetry, but this follows from the previous two cases.

Case 1: Suppose h is a pure permutation. Then $h \cdot P$ simply renames the indexes of the variables which will not change the degree at all. Recall from Definition 2.7 that the elements of G_W^{max} only permute variables with the same weight. The degree of the volume form is also unaffected for the same reason. Thus $\deg(h \cdot P) = \deg P$.

Case 2: Suppose h is a diagonal symmetry, meaning it is of the form $(a_1, a_2, \ldots, a_n)$, where $a_i \in \mathbb{Q}/\mathbb{Z}$ for all i. Then $h \cdot P = cP$ for some $c \in \mathbb{C}^*$, which would have the same degree as P, again implying that $\deg(h \cdot P) = \deg P$.

Thus $h \cdot \lfloor P, g \rceil$ has the same bidegree as $\lfloor P, g \rceil$ in $\mathcal{A}_{W,G}$. □

Chapter 4. The B-model State Space

Having constructed the A-model as a bigraded vector space, we can begin our construction of the B-model. We expect the B-model for $(W^T, G^\star)$ to be isomorphic to the A-model for (W, G); in our example, this means that the B-model should also have dimension 24 with the same bidegree as the A-model.

Definition 4.1. Let W be an invertible polynomial and $H \leq \mathrm{SL}_W^{diag}$. The *state space* for the B-model is defined as

$$\mathcal{B}_{W,H} = \left(\bigoplus_{h \in H} \mathcal{Q}_{W_h} \cdot \omega_h \right)^H,$$

where ω_h is a volume form on the fixed locus of h.

This is exactly analogous to Definition 3.4, except that the associated group H has different requirements than the group G used for the A-model. If we use (W, G) to construct the A-model with $G \leq G_W^{diag}$, then Krawitz [7] showed that the B-model state space associated to (W^T, G^T) will be isomorphic to the A-model state space for (W, G). While the state spaces have similar definitions, the grading and product structures are very different, although we won't explore the product structure here in this thesis.

For groups of non-diagonal matrices, in order for mirror symmetry to hold we replace G^T by the non-abelian dual group $G^\star$, defined in Definition 2.16.

4.1 CONSTRUCTING A B-MODEL STATE SPACE

Example 4.2. Let $W = x_1^4 + x_2^4 + x_3^4 + x_4^4, G = \langle j_W, (123)\rangle$, and $H = G^\star$. Recall from Example 2.17 that $G^\star = \langle(123)\rangle \cdot \mathrm{SL}_W^{diag}$. The elements of SL_W^{diag} are of the form $(123)^k(\frac{a_1}{4}, \frac{a_2}{4}, \frac{a_3}{4}, \frac{a_4}{4})$; again, the notation $(\frac{a_1}{4}, \frac{a_2}{4}, \frac{a_3}{4}, \frac{a_4}{4})$ refers to a 4x4 diagonal matrix with diagonal entries on the complex unit circle. The entries also satisfy $4|(a_1 + a_2 + a_3 + a_4)$—the requirement to be in $\mathrm{SL}(4, \mathbb{C})$. Alternately, the elements are generated by (123), j_W, K, and L, where $j_W = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$, $K = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, 0)$, and $L = (\frac{1}{4}, \frac{1}{2}, \frac{1}{4}, 0)$.

As we begin to construct $\mathcal{B}_{W,G^\star}$, we need to pay attention to centralizers and conjugacy classes. Recall the property $h \cdot \mathcal{A}_{W_g} \subseteq \mathcal{A}_{W_{h^{-1}gh}}$ ((∗) under Definition 3.4). On the A-side, j_W commuted with (123), so the centralizer of every element was G and the conjugacy class of every element was itself. That is not the case for $G^\star$.

Case 1: $g = (0, 0, 0, 0)$

Given that $W^T = W$, the Milnor ring here will be exactly the same as in case 1 of Example 4.2. However, the list of polynomials invariant under $G^\star$ will not be the same as that for G, since $G^\star$ has different generators. Since $(123), j_W \in G^\star$, this list of polynomials will be a subset of the 9 from earlier, but we also need to check if those 9 polynomials are invariant under $K = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, 0)$, and $L = (\frac{1}{4}, \frac{1}{2}, \frac{1}{4}, 0)$ as well. The only polynomials that will work are those where each monomial has the same exponent for x_1, x_2, and x_3. An example of a polynomial that isn't invariant under K is $x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2$ since

$$(e^{\frac{4\pi i}{4}}x_1)^2(e^{\frac{2\pi i}{4}}x_2)^2 + (e^{\frac{4\pi i}{4}}x_1)^2(e^{\frac{2\pi i}{4}}x_3)^2 + (e^{\frac{2\pi i}{4}}x_2)^2(e^{\frac{2\pi i}{4}}x_3)^2 \neq x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2.$$

This $G^\star$-invariant subspace has dimension 3 spanned by $1, x_1x_2x_3x_4$, and $x_1^2x_2^2x_3^2x_4^2$ (again suppressing the volume form).

Case 2: $g = (123)$ **or** $g = (132)$

Much like in the previous case, we know that the polynomials in either of these sectors will be a subset of those found in the A-model. Recall there were the same three polynomials for both choices of g. One can check that all three are invariant under K and L too, meaning that this case yields the same polynomials as on the A-side.

Case 3: $g \in G_W^{diag}$ **and has a trivial fixed locus**

This case means $g = (\frac{a_1}{4}, \frac{a_2}{4}, \frac{a_3}{4}, \frac{a_4}{4})$, for $1 \leq a_i \leq 3$. Any sector where a_1, a_2, a_3, and a_4 are all nonzero will fix nothing nontrivial, so it will be narrow. Since the sum of a_1, a_2, a_3, and a_4 must be a multiple of four, then (a_1, a_2, a_3, a_4) will need to be an ordering of one the following:

$$(3,3,3,3)$$

$$(3,3,1,1)$$

$$(3,2,2,1)$$

$$(2,2,2,2)$$

$$(1,1,1,1)$$

The 3 choices from the above where the components are all equal are powers of j_W. In any of those 3 cases, the conjugacy class is trivial since they will commute with $(123), j_W, K$, and L.

There are 12 different orderings of (1, 2, 2, 3). One can easily check that $x \in \text{Fix}(g)$ if and only if $h^{-1} \cdot x \in \text{Fix}(h^{-1}gh)$, so the conjugates of narrow group elements remain narrow. Conjugation by j_W, K, or L does nothing, but conjugation by (123) creates a conjugacy class of size 3, implying there will be $12/3 = 4$ conjugacy classes of this type. There are 6 orderings of (1, 1, 3, 3), so this choice gives $6/3 = 2$ additional conjugacy classes.

The powers of j_W give three more classes. Thus in this case we found a total of 9 conjugacy classes. The sums of the elements in each conjugacy class form a basis vector for a narrow sector. A few examples of these are the following:

$$\lfloor 1, j_W \rceil$$

$$\lfloor 1, (\tfrac{3}{4}, \tfrac{3}{4}, \tfrac{1}{4}, \tfrac{1}{4}) \rceil + \lfloor 1, (\tfrac{3}{4}, \tfrac{1}{4}, \tfrac{3}{4}, \tfrac{1}{4}) \rceil + \lfloor 1, (\tfrac{1}{4}, \tfrac{3}{4}, \tfrac{3}{4}, \tfrac{1}{4}) \rceil$$

$$\lfloor 1, (\tfrac{2}{4}, \tfrac{2}{4}, \tfrac{3}{4}, \tfrac{1}{4}) \rceil + \lfloor 1, (\tfrac{3}{4}, \tfrac{2}{4}, \tfrac{2}{4}, \tfrac{1}{4}) \rceil + \lfloor 1, (\tfrac{2}{4}, \tfrac{3}{4}, \tfrac{2}{4}, \tfrac{1}{4}) \rceil$$

The rest are listed in a table at the end of this chapter. In chapter 5, we will show that these 9 sums of narrow sectors correspond to the 9 untwisted broad sectors from the A-model.

Case 4: $g \in G_W^{diag}$ and has non-trivial fixed locus

Again, we have $g = (\frac{a_1}{4}, \frac{a_2}{4}, \frac{a_3}{4}, \frac{a_4}{4})$, but with $0 \leq a_i \leq 3$ and at least one of the $a_i's$ is 0.

The sectors where exactly one of the a_i's is 0 fix those coordinates. We need to check for invariance under the centralizer (recall $(*)$ under Definition 3.4), which includes powers of j_W, K, and L but not (123). If exactly one a_i is 0, then our resulting polynomial has only one variable (with degree less than 3) and a dx_i volume form, so it cannot be invariant under j_W.

If exactly two of the a_i values are zero, two standard basis vectors are included in the fixed locus. There are six ways to choose these two, and there are three in each conjugacy class, so we only need to consider two classes. Assume these are x_1 and x_2. Then the Milnor ring of W_g is $\mathbb{C}[x_1, x_2]/\langle x_1^3, x_2^3 \rangle$, with the volume form being $(dx_1 \wedge dx_2)$. Invariance under j_W tells us that each of these must be of the form x_1^2, x_1x_2, or x_2^2. However, these polynomials are not invariant under both K and L, so we get no additional basis elements from this subcase. The other cases similarly provide no contribution.

If three a_i values are 0, the fourth must be as well or else g would not be in SL_W^{diag}. This implies that $g = (0, 0, 0, 0)$, which would be a repeat of case 1. Thus case 4 yields no contribution to the state space.

Case 5: g is a narrow sector with permuations

Finally, we move on to sectors with factors of (123) and (132). In particular, $(123)j_W$, $(123)(j_W)^2$, $(123)(j_W)^3$, $(132)j_W$, $(132)(j_W)^2$, and $(132)(j_W)^3$ all have trivial fixed locus as we have seen in Case 3 of Example 3.5, and they still don't on this side, so they are narrow. However, on the B-side, these elements have nontrivial conjugacy classes. Let's consider a specific example, say $(123)j_W$. Conjugating $(123)j_W$ by K yields the following:

$$K[(123)j_W]K^{-1} = \begin{pmatrix} e^{\frac{4\pi i}{4}} & 0 & 0 & 0 \\ 0 & e^{\frac{2\pi i}{4}} & 0 & 0 \\ 0 & 0 & e^{\frac{2\pi i}{4}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & e^{\frac{2\pi i}{4}} & 0 & 0 \\ 0 & 0 & e^{\frac{2\pi i}{4}} & 0 \\ e^{\frac{2\pi i}{4}} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{\frac{2\pi i}{4}} \end{pmatrix} \begin{pmatrix} e^{\frac{2\pi i}{4}} & 0 & 0 & 0 \\ 0 & e^{\frac{6\pi i}{4}} & 0 & 0 \\ 0 & 0 & e^{\frac{6\pi i}{4}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & e^{\frac{4\pi i}{4}} & 0 & 0 \\ 0 & 0 & e^{\frac{2\pi i}{4}} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{\frac{2\pi i}{4}} \end{pmatrix} = (123)(0, \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{4})$$

This conjugation is equivalent to multiplying on the left of $(123)j_W$ by K^2L. Similarly, conjugating $(123)j_W$ by L is equivalent to multiplying on the left by K^3L. Together the conjugacy class of $(123)j_W$ reaches $(123)K^iL^j$ for any $i, j \in \{0, 1, 2, 3\}$. The same is true for the other five classes, producing 6 more narrow sectors.

In conclusion, the B-model state space contains 3 basis elements from the unwisted broad sector $(0, 0, 0, 0)$, 6 basis elements from the two twisted broad sectors (123) and (132), 9 narrow sectors from case 3, and 6 more narrow sectors from case 5, for a total of 24 basis elements. Recall that there were 24 basis elements in the A-model as well, which is sufficient for showing that the A- and B-models are isomorphic as vector spaces.

4.2 B-model Bigrading

Just like with the A-model, there is also a bigrading on the B-model state space.

Definition 4.3. The B-model bigrading is defined for an element $\lfloor P, g \rceil$ to be

$$(\deg P + \text{age } g - \text{age } j_W,\ \deg P + \text{age } g^{-1} - \text{age } j_W).$$

Example 4.4. The A-model had 24 basis elements, with 20 of them having a bidegree of $(1,1)$ and 1 of each of the following: $(0,0)$, $(2,0)$, $(0,2)$, and $(2,2)$. For the A- and B-models to be isomorphic as bigraded vector spaces, we should see the same breakdown of elements for the B-model too.

As with the A-model, we know that $\text{age}(j_W) = 1$. Hence the bidegree for all of the elements in the B-model can be reduced to

$$(\deg P + \text{age } g - 1,\ \deg P + \text{age } g^{-1} - 1).$$

As with the A-model, we will handle this by cases. Many of the basis elements on this side will be sums of elements, but we only need to look at one term in the sum since they will all have the same degree.

Case 1: $g = (0,0,0,0)$

As with the A-model, we will have age $g = 0$, and thus age $g^{-1} = 0$ as well since $(0,0,0,0)$ is its own inverse. There are only three polynomials to check in this sector, shown below.

$$\deg(1 \cdot (dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4)) = 1$$
$$\deg(x_1x_2x_3x_4 \cdot (dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4)) = 2$$
$$\deg(x_1^2x_2^2x_3^2x_4^2 \cdot (dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4)) = 3$$

Hence the bidegree for the elements in this sector is

$$(\deg P - 1,\ \deg P - 1),$$

where deg P is found above.

Case 2: $g = (123)$ **or** $g = (132)$

Recall from the A-model that age((123)) = 1 and age((132)) = 1. Also, the polynomials in this case are exactly the same as those from the A-model, where we found $\deg P = 1$ for all such polynomials. Thus the bidegree for the basis elements in these sectors is $(1, 1)$.

Case 3: $g \in G_W^{diag}$ **and has trivial fixed locus**

There are a total of 9 sectors in this case, with 3 having a basis elements with trivial conjugacy class and the other 6 having a conjugacy class of size 3. All of the polynomials in these sectors will have degree 0, so the bidegree depends just on age g and age g^{-1}. When $g = (j_W)^i$ where $1 \leq i \leq 3$, then age $g = i$ and age $g^{-1} = 4 - i$. Hence the bidegree for these 3 elements is

$$(\text{age } g - 1, \text{age } g^{-1} - 1),$$

where age g and age g^{-1} are given above.

An example of one of the 6 basis elements with non trivial conjugacy class is

$$\lfloor 1, (\tfrac{3}{4}, \tfrac{3}{4}, \tfrac{1}{4}, \tfrac{1}{4}) \rceil + \lfloor 1, (\tfrac{3}{4}, \tfrac{1}{4}, \tfrac{3}{4}, \tfrac{1}{4}) \rceil + \lfloor 1, (\tfrac{1}{4}, \tfrac{3}{4}, \tfrac{3}{4}, \tfrac{1}{4}) \rceil.$$

To find the degree of this element, we need to only look at one term in the summation. Since all of the group elements are diagonal, we can simply sum up the components of g to find age g, which is 2 in this case. Looking at g^{-1} will also yield an age of 2. In fact, the associated g and g^{-1} for all 6 of these basis elements have an age of 2. Thus the bidegree for all of them will be $(2 - 1, 2 - 1) = (1, 1)$.

Case 4: $g \in G_W^{diag}$ **and has non-trivial fixed locus**

This case yields 6 such basis elements, each with a conjugacy class of size 16. These conjugacy classes are found through conjugating $(123)j_W, (123)(j_W)^2, (123)(j_W)^3, (132)j_W, (132)(j_W)^2, (132)(j_W)^3$ by $(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, 0)$ and $(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}, 0)$. Since all of these sectors are narrow, we have $\deg P = 0$, so once again the bidegree depends solely on age g and age g^{-1}. All of these elements appeared in the A-model as well, where we found that they all have an age of 2. The inverse of $(123)^i(j_W)^j$ is $(123)^{2-i}(j_W)^{4-j}$, which also has an age of 2. One can check that each conjugate will also have age 2. Thus the bidegree for all of the elements in this case is $(2-1, 2-1) = (1,1)$.

As with the A-model, we now present of the basis elements in the B-model with their bigrading, seen in the following table:

B-model basis element	Bidegree
$\lfloor 1, (0,0,0,0) \rceil$	$(0,0)$
$\lfloor x_1x_2x_3x_4, (0,0,0,0) \rceil$	$(1,1)$
$\lfloor x_1^2x_2^2x_3^2x_4^2, (0,0,0,0) \rceil$	$(2,2)$
$\lfloor (x_1+x_2+x_3)^2, (123) \rceil$	$(1,1)$
$\lfloor (x_1+x_2+x_3)x_4, (123) \rceil$	$(1,1)$
$\lfloor x_4^2, (123) \rceil$	$(1,1)$
$\lfloor (x_1+x_2+x_3)^2, (132) \rceil$	$(1,1)$
$\lfloor (x_1+x_2+x_3)x_4, (132) \rceil$	$(1,1)$
$\lfloor x_4^2, (132) \rceil$	$(1,1)$
$\lfloor 1, j_W \rceil$	$(0,2)$
$\lfloor 1, (j_W)^2 \rceil$	$(1,1)$
$\lfloor 1, (j_W)^3 \rceil$	$(2,0)$
$\lfloor 1, (\frac{3}{4},\frac{3}{4},\frac{1}{4},\frac{1}{4}) \rceil + \lfloor 1, (\frac{3}{4},\frac{1}{4},\frac{3}{4},\frac{1}{4}) \rceil + \lfloor 1, (\frac{1}{4},\frac{3}{4},\frac{3}{4},\frac{1}{4}) \rceil$	$(1,1)$
$\lfloor 1, (\frac{2}{4},\frac{2}{4},\frac{3}{4},\frac{1}{4}) \rceil + \lfloor 1, (\frac{3}{4},\frac{2}{4},\frac{2}{4},\frac{1}{4}) \rceil + \lfloor 1, (\frac{2}{4},\frac{3}{4},\frac{2}{4},\frac{1}{4}) \rceil$	$(1,1)$
$\lfloor 1, (\frac{2}{4},\frac{2}{4},\frac{1}{4},\frac{3}{4}) \rceil + \lfloor 1, (\frac{2}{4},\frac{1}{4},\frac{2}{4},\frac{3}{4}) \rceil + \lfloor 1, (\frac{1}{4},\frac{2}{4},\frac{2}{4},\frac{3}{4}) \rceil$	$(1,1)$
$\lfloor 1, (\frac{2}{4},\frac{3}{4},\frac{1}{4},\frac{2}{4}) \rceil + \lfloor 1, (\frac{1}{4},\frac{2}{4},\frac{3}{4},\frac{2}{4}) \rceil + \lfloor 1, (\frac{3}{4},\frac{1}{4},\frac{2}{4},\frac{3}{4}) \rceil$	$(1,1)$
$\lfloor 1, (\frac{3}{4},\frac{2}{4},\frac{1}{4},\frac{2}{4}) \rceil + \lfloor 1, (\frac{1}{4},\frac{3}{4},\frac{2}{4},\frac{2}{4}) \rceil + \lfloor 1, (\frac{2}{4},\frac{3}{4},\frac{1}{4},\frac{3}{4}) \rceil$	$(1,1)$
$\lfloor 1, (\frac{3}{4},\frac{1}{4},\frac{1}{4},\frac{3}{4}) \rceil + \lfloor 1, (\frac{1}{4},\frac{3}{4},\frac{1}{4},\frac{3}{4}) \rceil + \lfloor 1, (\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4}) \rceil$	$(1,1)$
$\lfloor 1, (123)j_W \rceil$ + (15 elements in conj. class)	$(1,1)$
$\lfloor 1, (123)(j_W)^2 \rceil$ + (15 elements in conj. class)	$(1,1)$
$\lfloor 1, (123)(j_W)^3 \rceil$ + (15 elements in conj. class)	$(1,1)$
$\lfloor 1, (132)j_W \rceil$ + (15 elements in conj. class)	$(1,1)$

$$\lfloor 1, (132)(j_W)^2 \rceil + (15 \text{ elements in conj. class}) \qquad (1,1)$$

$$\lfloor 1, (132)(j_W)^3 \rceil + (15 \text{ elements in conj. class}) \qquad (1,1)$$

If we arrange these as a Hodge diamond, we have

$$\begin{array}{ccc} & 1 & \\ 1 & 20 & 1 \\ & 1 & \end{array}$$

Notice this is the same diamond as with our A-model example. This is enough to prove that the given A- and B-models are isomorphic, but we will give a more explicit isomorphism in Chapter 5.

As with Lemma 3.10, we again want to know that the bigrading of an element is unchanged when acted upon by a symmetry in $G^\star$, so we prove the same fact for B-models.

Lemma 4.5. *Given $h \in G^{max}_{W^T}$, and $\lfloor P, g \rceil \in \mathcal{B}_{W^T, G^\star}$, the element $h \cdot \lfloor P, g \rceil$ has the same bidegree as $\lfloor P, g \rceil$.*

Proof. Recall the B-model bigrading from Definition 4.3:

$$(\deg P + \operatorname{age} g - \operatorname{age} j_W, \deg P + \operatorname{age} g^{-1} - \operatorname{age} j_W).$$

This proof follows the same as the proof of Lemma 3.10. We already proved that $\deg(h \cdot P) = \deg P$ and $\operatorname{age}(h^{-1}gh) = \operatorname{age} g$ in Lemma 3.10. The work to show that $\operatorname{age}(h^{-1}g^{-1}h) = \operatorname{age} g^{-1}$ is the same, since $h^{-1}g^{-1}h$ and g^{-1} are similar matrices, implying that they too have the same eigenvalues.

Thus $h \cdot \lfloor P, g \rceil$ has the same bidegree as $\lfloor P, g \rceil$ in $\mathcal{B}_{W^T, G^\star}$. □

Thus far in our example from Chapters 3 and 4, we have shown that the specified A- and B-models have 24 basis elements with the same number of elements for each bidegree. While this in itself would be sufficient for claiming that they are isomorphic, we aim to create a canonical map which will better demonstrate which elements on one side correspond to elements on the other. In particular, we expect that this map will exchange narrow and broad sectors. This follows the map given by Krawitz [7] for A- and B-models built from abelian groups. This isomorphism between A- and B-models is known as the *mirror map*.

Example 5.1. We will continue with the same A- and B-models, and begin constructing the mirror map with the part of the map that is already laid out for us by matching the 4 elements on either side with unique bidegree.

Bidegree	**A-model**	**B-model**
$(0,0)$	$\lfloor 1, j_W \rceil$	$\lfloor 1, (0,0,0,0) \rceil$
$(2,2)$	$\lfloor 1, (j_W)^3 \rceil$	$\lfloor x_1^2 x_2^2 x_3^2 x_4^2, (0,0,0,0) \rceil$
$(0,2)$	$\lfloor 1, (0,0,0,0) \rceil$	$\lfloor 1, j_W \rceil$
$(2,0)$	$\lfloor x_1^2 x_2^2 x_3^2 x_4^2, (0,0,0,0) \rceil$	$\lfloor 1, (j_W)^3 \rceil$

This illuminates 2 more corresponding elements:

Bidegree	**A-model**	**B-model**
$(1,1)$	$\lfloor 1, (j_W)^2 \rceil$	$\lfloor x_1 x_2 x_3 x_4, (0,0,0,0) \rceil$
$(1,1)$	$\lfloor x_1 x_2 x_3 x_4, (0,0,0,0) \rceil$	$\lfloor 1, (j_W)^2 \rceil$

A nice generalization of the six element maps above can be seen by

$$\lfloor 1, (j_W)^i \rceil \leftrightarrow \lfloor x_1^{i-1} x_2^{i-1} x_3^{i-1} x_4^{i-1}, (0,0,0,0) \rceil.$$

Recall that the dimension of the untwisted broad sector in the A-model was 9, and since three of those are seen above, there are still 6 others to account for. These 6 basis elements map to the 6 narrow sectors in the B-model which have a conjugacy class of size 3. Specifically, we map the elements on the A-side whose polynomial has the same permutation structure as the group elements on the B-side. One explicit example is given by mapping the A-model element

$$\lfloor x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2, (0,0,0,0)\rceil$$

to the B-model element

$$\lfloor 1, (\tfrac{3}{4}, \tfrac{3}{4}, \tfrac{1}{4}, \tfrac{1}{4})\rceil + \lfloor 1, (\tfrac{3}{4}, \tfrac{1}{4}, \tfrac{3}{4}, \tfrac{1}{4})\rceil + \lfloor 1, (\tfrac{1}{4}, \tfrac{3}{4}, \tfrac{3}{4}, \tfrac{1}{4})\rceil.$$

Notice that the term $x_1^2x_2^2$ has a power of 2 for x_1 and x_2, and this corresponds to the first two components of the group element of $\lfloor 1, (\frac{3}{4}, \frac{3}{4}, \frac{1}{4}, \frac{1}{4})\rceil$ having a larger value by $\frac{2}{4}$. The same correspondence can be noticed between $x_1^2x_3^2$ and $\lfloor 1, (\frac{3}{4}, \frac{1}{4}, \frac{3}{4}, \frac{1}{4})\rceil$, as well as $x_2^2x_3^2$ and $\lfloor 1, (\frac{1}{4}, \frac{3}{4}, \frac{3}{4}, \frac{1}{4})\rceil$.

All 6 elements of this type are given below. The bidegree is left out, but all of the following elements have a bidegree of $(1, 1)$.

A-model	**B-model**
$\lfloor x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2, (0,0,0,0)\rceil$	$\lfloor 1, (\frac{3}{4}, \frac{3}{4}, \frac{1}{4}, \frac{1}{4})\rceil + \lfloor 1, (\frac{3}{4}, \frac{1}{4}, \frac{3}{4}, \frac{1}{4})\rceil + \lfloor 1, (\frac{1}{4}, \frac{3}{4}, \frac{3}{4}, \frac{1}{4})\rceil$
$\lfloor x_1x_2x_3^2 + x_1^2x_2x_3 + x_1x_2^2x_3, (0,0,0,0)\rceil$	$\lfloor 1, (\frac{2}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{4})\rceil + \lfloor 1, (\frac{3}{4}, \frac{2}{4}, \frac{2}{4}, \frac{1}{4})\rceil + \lfloor 1, (\frac{2}{4}, \frac{3}{4}, \frac{2}{4}, \frac{1}{4})\rceil$
$\lfloor x_1x_2x_4^2 + x_1x_3x_4^2 + x_2x_3x_4^2, (0,0,0,0)\rceil$	$\lfloor 1, (\frac{2}{4}, \frac{2}{4}, \frac{1}{4}, \frac{3}{4})\rceil + \lfloor 1, (\frac{2}{4}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4})\rceil + \lfloor 1, (\frac{1}{4}, \frac{2}{4}, \frac{2}{4}, \frac{3}{4})\rceil$
$\lfloor x_1x_2^2x_4 + x_2x_3^2x_4 + x_1^2x_3x_4, (0,0,0,0)\rceil$	$\lfloor 1, (\frac{2}{4}, \frac{3}{4}, \frac{1}{4}, \frac{2}{4})\rceil + \lfloor 1, (\frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{2}{4})\rceil + \lfloor 1, (\frac{3}{4}, \frac{1}{4}, \frac{2}{4}, \frac{2}{4})\rceil$
$\lfloor x_1^2x_2x_4 + x_2^2x_3x_4 + x_1x_3^2x_4, (0,0,0,0)\rceil$	$\lfloor 1, (\frac{3}{4}, \frac{2}{4}, \frac{1}{4}, \frac{2}{4})\rceil + \lfloor 1, (\frac{1}{4}, \frac{3}{4}, \frac{2}{4}, \frac{2}{4})\rceil + \lfloor 1, (\frac{2}{4}, \frac{1}{4}, \frac{3}{4}, \frac{2}{4})\rceil$
$\lfloor x_1^2x_4^2 + x_2^2x_4^2 + x_3^2x_4^2, (0,0,0,0)\rceil$	$\lfloor 1, (\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{3}{4})\rceil + \lfloor 1, (\frac{1}{4}, \frac{3}{4}, \frac{1}{4}, \frac{3}{4})\rceil + \lfloor 1, (\frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{3}{4})\rceil$

Again, notice that the polynomials of the A-model elements have the same permutation structure of the group elements in the B-model.

There are now 12 basis elements left to be mapped in both models, with 6 being twisted broad sectors from $g = (123)$ or $g = (132)$ and 6 being narrow sectors, where g is a product of a permutation and a power of j_W. While they all have the same bidegree, we expect that the mirror map will map broad sectors to narrow sectors and narrow sectors to broad sectors, so we will do the same here. Unlike the previous element mappings, it is not as clear exactly which A-model and B-model elements below should map to each other, since there's no permuation structure to be observed in these elements.

A-model	**B-model**
$\lfloor (x_1+x_2+x_3)^2, (123) \rceil$	$\lfloor 1, (123)j_W \rceil$ + (15 others)
$\lfloor (x_1+x_2+x_3)x_4, (123) \rceil$	$\lfloor 1, (123)(j_W)^2 \rceil$ + (15 others)
$\lfloor (x_4)^2, (123) \rceil$	$\lfloor 1, (123)(j_W)^3 \rceil$ + (15 others)
$\lfloor (x_1+x_2+x_3)^2, (132) \rceil$	$\lfloor 1, (132)j_W \rceil$ + (15 others)
$\lfloor (x_1+x_2+x_3)x_4, (132) \rceil$	$\lfloor 1, (132)(j_W)^2 \rceil$ + (15 others)
$\lfloor (x_4)^2, (132) \rceil$	$\lfloor 1, (132)(j_W)^3 \rceil$ + (15 others)
$\lfloor 1, (123)j_W \rceil$	$\lfloor (x_1+x_2+x_3)^2, (123) \rceil$
$\lfloor 1, (123)(j_W)^2 \rceil$	$\lfloor (x_1+x_2+x_3)x_4, (123) \rceil$
$\lfloor 1, (123)(j_W)^3 \rceil$	$\lfloor (x_4)^2, (123) \rceil$
$\lfloor 1, (132)j_W \rceil$	$\lfloor (x_1+x_2+x_3)^2, (132) \rceil$
$\lfloor 1, (132)(j_W)^2 \rceil$	$\lfloor (x_1+x_2+x_3)x_4, (132) \rceil$
$\lfloor 1, (132)(j_W)^3 \rceil$	$\lfloor (x_4)^2, (132) \rceil$

This completes the mirror map, and we have explicitly shown in this example that as bigraded vector spaces,

$$\mathcal{A}_{W,G} \cong \mathcal{B}_{W^T,G^\star}.$$

5.1 Proof of the Restricted Mirror Map

Later we will find that under certain conditions, that surprisingly the A- and B-models are not isomorphic as bigraded vector spaces. However, we can show that part of the mirror map always holds if we restrict to certain natural subspaces. In particular, it works if we restrict to the untwisted broad sectors of the A-model and B-model and their corresponding images as seen in the previous example. We prove that here in the following theorem.

Theorem 5.2. *Let W be an invertible Fermat polynomial and $G \leq G_W^{max}$ be an admissible group of the form $K \cdot H$, where $K \leq G$ is the subgroup of pure even permutations and $H \leq G$ is the subgroup of diagonal symmetries. Define $\mathcal{A}_0 \subseteq \mathcal{A}_{W,G}$ and $\mathcal{B}_0 \subseteq \mathcal{B}_{W^T,G^\star}$ to be the untwisted broad sectors for the A- and B-side, respectively. Let $nar' \leq G$ be the set of narrow diagonal symmetries. We will also denote $nar' \leq G^\star$ to be the corresponding set on the B-side. Then there exist bigraded vector space isomorphisms*

$$\mathcal{A}_0 \xrightarrow{\sim} \mathcal{B}_{nar'} \text{ and } \mathcal{A}_{nar'} \xrightarrow{\sim} \mathcal{B}_0.$$

Proof. Let $W = x_1^{d_1} + \cdots + x_n^{d_n}$, so G_W^{diag} is generated by the set

$$\{(\tfrac{1}{d_1}, 0, \ldots, 0), (0, \tfrac{1}{d_2}, \ldots, 0), \ldots, (0, 0, \ldots, \tfrac{1}{d_n})\}$$

Define $I_g = \{i \in \{1, \ldots, n\} | a_i \neq 0\}$ and consider the map

$$\bigoplus_{g \in G \cap G_W^{diag}} \mathcal{Q}_{W_g} \cdot \omega_g \to \bigoplus_{g' \in G^\star \cap G_W^{diag}} \mathcal{Q}_{W_{g'}} \cdot \omega_{g'}$$

given by

$$\Big\lfloor \prod_{j\notin I_g} x_j^{b_i} dx_j, (\tfrac{a_1}{d_1}, \dots, \tfrac{a_n}{d_n}) \Big\rceil \mapsto \Big\lfloor \prod_{i\notin I_g} y_i^{a_i-1} dx_i, (\tfrac{b_1'}{d_1}, \dots, \tfrac{b_n'}{d_n}) \Big\rceil,$$

for $g = (\frac{a_1}{d_1}, \dots, \frac{a_n}{d_n})$, where $b_i' = b_i + 1$ if $i \notin I_g$, and $b_i' = 0$ otherwise.

The above map is known as the *map on the unprojected state spaces*t, where invariance has not yet been considered. Notice that $G^\star \cap G_W^{diag} = G^T$ in this case. This map was proven to be a bijection by Krawitz [7]. For completeness, we will reprove the relevant part here. We will show that

$$\mathcal{Q}_W \to \bigoplus_{\substack{g'\in G_W^{diag}\\ \mathrm{Fix}(g')=\{0\}}} \mathcal{Q}_{W_{g'}} \cdot \omega_{g'}$$

is a bijection. This is the case for when $a_i = 0$ for all i and $(\frac{b_1'}{d_1}, \dots, \frac{b_n'}{d_n})$ is a diagonal symmetry with nonzero entries. We will also need that fact that

$$\bigoplus_{\substack{g\in G_W^{diag}\\ \mathrm{Fix}(g)=0}} \mathcal{Q}_{W_g} \cdot \omega_g \to \mathcal{Q}_W$$

is a bijection, considering it this time as a map from the A-model to the B-model. However, the proof of this exactly mirrors the first one, so we will exclude it here. Before proceeding, note that since W is Fermat, we know that the Milnor ring of W is

$$\mathcal{Q}_W = \frac{\mathbb{C}[x_1, \dots, x_n]}{(x_1^{d_1-1}, \dots, x_n^{d_n-1})},$$

which has a basis of elements of the form $\prod_{i=1}^n x_i^{b_i}$, where $0 \le b_i \le d_i - 2$.

First, to prove surjectivity, let

$$\lfloor 1, (\tfrac{b_1'}{d_1}, \dots, \tfrac{b_n'}{d_n}) \rceil \in \bigoplus_{\substack{g'\in G_W^{diag}\\ \mathrm{Fix}(g')=\{0\}}} \mathcal{Q}_{W_{g'}} \cdot \omega_{g'},$$

so $1 \leq b_i' \leq d_i - 1$. Since each $b_i' \neq 0$, let $b_i = b_i' - 1$, and notice $0 \leq b_i \leq d_i - 2$. The preimage of $\lfloor 1, (\frac{b_1'}{d_1}, \ldots, \frac{b_n'}{d_n}) \rceil$ is of the form $\lfloor \prod_{j=1}^n x_j^{b_i} dx_j, (0, \ldots, 0) \rceil$. Notice $W|_{\text{Fix}((0,\ldots,0))} = W$, so $\prod_{j=1}^n x_j^{b_i} dx_j \in \mathcal{Q}_W$, as desired.

To prove injectivity, let

$$\lfloor 1, (\tfrac{b_1'}{d_1}, \ldots, \tfrac{b_n'}{d_n}) \rceil = \lfloor 1, (\tfrac{c_1'}{d_1}, \ldots, \tfrac{c_n'}{d_n}) \rceil$$

be two elements in the image of the given map. This would imply that $b_i' = c_i' \pmod{d_i}$ for all i. So $b_i = c_i \pmod{d_i}$. Recall that

$$\mathcal{Q}_W = \frac{\mathbb{C}[x_1, \ldots, x_n]}{(x_1^{d_1 - 1}, \ldots, x_n^{d_n - 1})},$$

so $\prod_{j=1}^n x_j^{b_i} dx_j = \prod_{j=1}^n x_j^{c_i} dx_j$ in $\mathcal{Q}_W$ since b_i and c_i are between 1 and $d_i - 2$. Thus

$$\lfloor \prod_{j=1}^n x_j^{b_i} dx_j, (0, \ldots, 0) \rceil = \lfloor \prod_{j=1}^n x_j^{c_i} dx_j, (0, \ldots, 0) \rceil,$$

completing the bijection.

Next, we look at the invariant subspaces of the preimage and image of the above map. Specifically, we aim to show that

$$\mathcal{A}_0 \to \mathcal{B}_{nar'}$$

is a bijection with the given map, where $\mathcal{A}_0 = \left(\mathcal{Q}_W\right)^G$ and $\mathcal{B}_{nar'} = \Big(\bigoplus_{\substack{g' \in G_W^{diag} \\ \text{Fix}(g') = \{0\}}} \mathcal{Q}_{W_{g'}} \cdot \omega_{g'} \Big)^{G^\star}$.

We will need to show that if $\lfloor \sum_{r=1}^m (\prod_{j \notin I_g} x_j^{b_{jr}} dx_j), (0, \ldots, 0) \rceil$ is invariant under G, meaning it is fixed by all the elements of G, then its image is fixed by the elements of $G^\star$. Since G is generated by K and H, we need to show that if $\sum_{r=1}^m (\prod_{j \notin I_g} x_j^{b_{jr}} dx_j)$ is fixed for all $\sigma \in K$ and $h \in H$, then its image is fixed by $G^\star$.

Case 1: Let $h \in H$, so h is a diagonal symmetry of the form $(\frac{h_1}{d_1}, \dots, \frac{h_n}{d_n})$. For this case, recall that h acts on the element $\lfloor P, g \rceil$ by $h \cdot \lfloor P, g \rceil = \lfloor h \cdot P, h^{-1}gh \rceil$. Note that h fixes terms of polynomials independently since h acts diagonally, so we only need to consider P as a monomial. We consider $h \cdot \lfloor P, (0, \dots, 0) \rceil$, where $P = \prod_{j=1}^{n} x_j^{b_i} dx_j$. Then

$$h \cdot \lfloor P, (0, \dots, 0) \rceil = \lfloor h \cdot P, (0, \dots, 0) \rceil = \lfloor e^{2\pi i \sum_{i=1}^{n} \frac{h_i b_i'}{d_i}} P, (0, \dots, 0) \rceil.$$

Note that since P is fixed by h, then $\sum_{i=1}^{n} \frac{h_i b_i'}{d_i} \in \mathbb{Z}$. But notice that $\sum_{i=1}^{n} \frac{h_i b_i'}{d_i} = hA_W(\frac{b_1'}{d_1}, \dots, \frac{b_n'}{d_n}) \in \mathbb{Z}$. Thus $(\frac{b_1'}{d_1}, \dots, \frac{b_n'}{d_n}) \in H^T$, so $\lfloor 1, (\frac{b_1'}{d_1}, \dots, \frac{b_n'}{d_n}) \rceil \in (\bigoplus_{\substack{g \in G_{W^T}^{diag} \\ \mathrm{Fix}(g) = \{0\}}} \mathcal{Q}_{W^T|_{\mathrm{Fix}(g)}} \cdot \omega_g)^{H^T}$.

Case 2: Let $\sigma \in K$, so that σ fixes $\lfloor \sum_{r=1}^{m} (\Pi_{i=1}^{n} x_i^{b_{i_r}} dx_i), (0, \dots, 0) \rceil \in \mathcal{A}_0$. That is, if $\prod_{j=1}^{n} x_j^{b_i} dx_i$ is a single term of the sum $\sum_{r=1}^{m} (\Pi_{i=1}^{n} x_i^{b_{i_r}} dx_i)$, then $\sigma(\prod_{j=1}^{n} x_j^{b_i} dx_i) = \prod_{j=1}^{n} x_j^{b_{\sigma(i)}} dx_i$ must be another term in the sum. Note that σ fixes the volume form because σ is an even permutation. Now consider $\sum_{r=1}^{m} \lfloor 1, (\frac{b_{1r}'}{d_1}, \dots, \frac{b_{nr}'}{d_n}) \rceil \in \mathcal{B}_{nar'}$, which is the image of $\lfloor \sum_{r=1}^{m} (\Pi_{i=1}^{n} x_i^{b_{i_r}} dx_i), (0, \dots, 0) \rceil \in \mathcal{A}_0$. Since $\sigma(\prod_{j=1}^{n} x_j^{b_i} dx_j) = \prod_{j=1}^{n} x_j^{b_{\sigma}(i)} dx_j$ is a term of the sum $\sum_{r=1}^{m} (\Pi_{j=1}^{n} x_j^{b_{i_r}} dx_j)$, then $\sigma(\lfloor 1, (\frac{b_1'}{d_1}, \dots, \frac{b_n'}{d_n}) \rceil) = \lfloor 1, (\frac{b_{\sigma(1)}'}{d_{\sigma(1)}}, \dots, \frac{b_{\sigma(n)}'}{d_{\sigma(n)}}) \rceil$ is another term of $\sum_{r=1}^{m} \lfloor 1, (\frac{b_{1r}'}{d_1}, \dots, \frac{b_{nr}'}{d_n}) \rceil$. Thus σ fixes $\sum_{r=1}^{m} \lfloor 1, (\frac{b_{1r}'}{d_1}, \dots, \frac{b_{nr}'}{d_n}) \rceil \in \mathcal{B}_{nar'}$ as well, so it is invariant under K.

The work to show that $\mathcal{A}_{nar'} \to \mathcal{B}_0$ is a bijection follows similarly. Thus we have proved that the restricted A- and B-models are isomorphic as vector spaces. However, we claimed that they were also isomorphic as *bigraded* vector spaces, so it remains to show that the corresponding elements from either side have the same bidegree.

Recall that the A-model bigrading from Definition 3.8 is

$$(\deg P + \operatorname{age} g - \operatorname{age} j_W, N_g - \deg P + \operatorname{age} g - \operatorname{age} j_W).$$

If we restrict to $\mathcal{A}_0$, then $\operatorname{age} g = 0$ and $N_g = n$, so the above definition reduces to

$$(\deg P - \operatorname{age} j_W, n - \deg P - \operatorname{age} j_W).$$

The B-model bigrading from Definition 4.3 was

$$(\deg P' + \operatorname{age} g' - \operatorname{age} j_W,\ \deg P' + \operatorname{age} {g'}^{-1} - \operatorname{age} j_W).$$

When we consider elements of $\mathcal{B}_{nar'}$, we find that $\deg P' = 0$, so the bigrading becomes

$$(\operatorname{age} g' - \operatorname{age} j_W, \operatorname{age} {g'}^{-1} - \operatorname{age} j_W).$$

Consider the corresponding elements

$$\lfloor \sum_{r=1}^{m} (\prod_{j=1}^{n} x_j^{b_{i_r}} dx_j), (0, \dots, 0) \rceil \in \mathcal{A}_0 \text{ and } \sum_{r=1}^{m} \lfloor 1, (\tfrac{b'_{1r}}{d_1}, \dots, \tfrac{b'_{nr}}{d_n}) \rceil \in \mathcal{B}_{nar'}.$$

By Lemma 3.10 and Lemma 4.5, we only need to focus on one term in each sum. Thus to show that the mirror map preserves bidegree, we must prove that $\deg(\prod_{j=1}^{n} x_j^{b_{i_r}} dx_j) = \operatorname{age}(\frac{b'_{1r}}{d_1}, \dots, \frac{b'_{nr}}{d_n})$ and $n - \deg(\prod_{j=1}^{n} x_j^{b_{i_r}} dx_j) = \operatorname{age}(\frac{b'_{1r}}{d_1}, \dots, \frac{b'_{nr}}{d_n})^{-1}$. Observe that

$$\deg(\prod_{j=1}^{n} x_j^{b_i} dx_j) = \sum_{i=1}^{n} \frac{b_i + 1}{d_i} \text{ and } \operatorname{age}(\tfrac{b'_1}{d_1}, \dots, \tfrac{b'_n}{d_n}) = \tfrac{b'_1}{d_1} + \dots + \tfrac{b'_n}{d_n} = \sum_{i=1}^{n} \frac{b'_i}{d_i}.$$

Since $b'_i = b_i + 1$, then $\deg(\prod_{j=1}^{n} x_j^{b_{i_r}} dx_j) = \operatorname{age}(\frac{b'_{1r}}{d_1}, \dots, \frac{b'_{nr}}{d_n})$.

It follows that $n - \deg(\prod_{j=1}^{n} x_j^{b_{i_r}} dx_j) = \operatorname{age}(\frac{b'_{1r}}{d_1}, \dots, \frac{b'_{nr}}{d_n})^{-1}$ since if g is narrow, then it is known that $\operatorname{age} g = n - \operatorname{age} g^{-1}$ (Mukai [9]). This establishes that the first isomorphism $\mathcal{A}_0 \to \mathcal{B}_{nar'}$ preserves bidegree.

For the other iso, $\mathcal{A}_{nar'} \to \mathcal{B}_0$, we need to show is that the corresponding elements from $\mathcal{A}_{nar'}$ and $\mathcal{B}_0$ also have the same bidegree. The bigrading of elements from these sectors is

$$(\text{age } g - \text{age } j_W, \text{age } g - \text{age } j_W) \text{ and } (\deg P' - \text{age } j_W,\ \deg P' - \text{age } j_W),$$

respectively, where $g \in G$ and $P' \in (\mathcal{Q}_W)^{G^\star}$. This means that all we need to show is that age $g = \deg P'$, which follows the exact same work as above.

Thus we have shown that the maps

$$\mathcal{A}_0 \to \mathcal{B}_{nar'} \text{ and } \mathcal{B}_0 \to \mathcal{A}_{nar'}$$

are bigraded vector space isomorphisms. This gives us the partial mirror map. □

CHAPTER 6. ANOTHER EXAMPLE

While the example we began in Chapter 3 was a great starting place, the mirror map left a bit to be desired given that 20 of the 24 basis elements had the same bidegree of $(1,1)$. Moving up to a higher degree polynomial will create A- and B-models with larger bases and more variety in their bigrading, illuminating a clearer picture of the mirror map. With Theorem 5.2, we know what most of the map will look like, but mapping the sectors built from non-diagonal matrices is still a bit unclear.

Example 6.1. Let $W = x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5$ and $G = \langle j_W, (12)(34) \rangle$, where

$$j_W = (\tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}) \text{ and } (12)(34) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then $W^T = W$ and the non-abelian dual group of G is

$$G^\star = \langle (12)(34) \rangle \cdot \mathrm{SL}_W^{diag},$$

where $\mathrm{SL}_W^{diag} = \langle j_W, (\frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, 0), (\frac{1}{5}, \frac{2}{5}, \frac{1}{5}, \frac{1}{5}, 0), (\frac{1}{5}, \frac{1}{5}, \frac{2}{5}, \frac{1}{5}, 0) \rangle$. We will denote $K = (\frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, 0)$, $L = (\frac{1}{5}, \frac{2}{5}, \frac{1}{5}, \frac{1}{5}, 0)$, and $M = (\frac{1}{5}, \frac{1}{5}, \frac{2}{5}, \frac{1}{5}, 0)$. As with the previous example, the goal is to show that

$$\mathcal{A}_{W,G} \cong \mathcal{B}_{W^T,G^\star}$$

as bigraded vector spaces. This example follows the same recipe as Example 5.1, so we leave the details to the reader and simply provide the mirror map.

The first eight elements listed below follow the mirror map described by and Theorem 5.2 between $\mathcal{A}_0$ and $\mathcal{B}_{nar'}$ as well as $\mathcal{A}_{nar'}$ and $\mathcal{B}_0$.

Bidegree	**A-model**	**B-model**
$(0,0)$	$\lfloor 1, j_W \rceil$	$\lfloor 1, (0,0,0,0,0) \rceil$
$(1,1)$	$\lfloor 1, (j_W)^2 \rceil$	$\lfloor x_1x_2x_3x_4, (0,0,0,0,0) \rceil$
$(2,2)$	$\lfloor 1, (j_W)^3 \rceil$	$\lfloor x_1^2x_2^2x_3^2x_4^2, (0,0,0,0,0) \rceil$
$(3,3)$	$\lfloor 1, (j_W)^4 \rceil$	$\lfloor x_1^3x_2^3x_3^3x_4^3, (0,0,0,0,0) \rceil$
$(0,3)$	$\lfloor 1, (0,0,0,0,0) \rceil$	$\lfloor 1, j_W \rceil$
$(1,2)$	$\lfloor x_1x_2x_3x_4, (0,0,0,0,0) \rceil$	$\lfloor 1, (j_W)^2 \rceil$
$(2,1)$	$\lfloor x_1^2x_2^2x_3^2x_4^2, (0,0,0,0,0) \rceil$	$\lfloor 1, (j_W)^3 \rceil$
$(3,0)$	$\lfloor x_1^3x_2^3x_3^3x_4^3, (0,0,0,0,0) \rceil$	$\lfloor 1, (j_W)^4 \rceil$

The next four pages contain the basis elements in $\mathcal{A}_0$ and the corresponding narrow sectors in $\mathcal{B}_{nar'}$ as in Theorem 5.2. The first two pages all have a bidegree of $(1,2)$, while the next two after that have a bidegree of $(2,1)$. These are all described by Theorem 5.2.

A-model	**B-model**
$\lfloor x_1^3x_2^2 + x_1^2x_2^3, (0,0,0,0,0)\rceil$	$\lfloor 1, (\tfrac{4}{5}, \tfrac{3}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5})\rceil + \lfloor 1, (\tfrac{3}{5}, \tfrac{4}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5})\rceil$
$\lfloor x_3^3x_4^2 + x_3^2x_4^3, (0,0,0,0,0)\rceil$	$\lfloor 1, (\tfrac{1}{5}, \tfrac{1}{5}, \tfrac{4}{5}, \tfrac{3}{5}, \tfrac{1}{5})\rceil + \lfloor 1, (\tfrac{1}{5}, \tfrac{1}{5}, \tfrac{3}{5}, \tfrac{4}{5}, \tfrac{1}{5})\rceil$
$\lfloor x_1^2x_3^3 + x_2^2x_4^3, (0,0,0,0,0)\rceil$	$\lfloor 1, (\tfrac{3}{5}, \tfrac{1}{5}, \tfrac{4}{5}, \tfrac{1}{5}, \tfrac{1}{5})\rceil + \lfloor 1, (\tfrac{1}{5}, \tfrac{3}{5}, \tfrac{1}{5}, \tfrac{4}{5}, \tfrac{1}{5})\rceil$
$\lfloor x_1^3x_3^2 + x_2^3x_4^2, (0,0,0,0,0)\rceil$	$\lfloor 1, (\tfrac{4}{5}, \tfrac{1}{5}, \tfrac{3}{5}, \tfrac{1}{5}, \tfrac{1}{5})\rceil + \lfloor 1, (\tfrac{1}{5}, \tfrac{4}{5}, \tfrac{1}{5}, \tfrac{3}{5}, \tfrac{1}{5})\rceil$
$\lfloor x_1^3x_4^2 + x_2^3x_3^2, (0,0,0,0,0)\rceil$	$\lfloor 1, (\tfrac{4}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{3}{5}, \tfrac{1}{5})\rceil + \lfloor 1, (\tfrac{1}{5}, \tfrac{4}{5}, \tfrac{3}{5}, \tfrac{1}{5}, \tfrac{1}{5})\rceil$
$\lfloor x_1^2x_4^3 + x_2^2x_3^3, (0,0,0,0,0)\rceil$	$\lfloor 1, (\tfrac{3}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{4}{5}, \tfrac{1}{5})\rceil + \lfloor 1, (\tfrac{1}{5}, \tfrac{3}{5}, \tfrac{4}{5}, \tfrac{1}{5}, \tfrac{1}{5})\rceil$
$\lfloor x_1^3x_5^2 + x_2^3x_5^2, (0,0,0,0,0)\rceil$	$\lfloor 1, (\tfrac{4}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{3}{5})\rceil + \lfloor 1, (\tfrac{1}{5}, \tfrac{4}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{3}{5})\rceil$
$\lfloor x_3^3x_5^2 + x_4^3x_5^2, (0,0,0,0,0)\rceil$	$\lfloor 1, (\tfrac{1}{5}, \tfrac{1}{5}, \tfrac{4}{5}, \tfrac{1}{5}, \tfrac{3}{5})\rceil + \lfloor 1, (\tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{4}{5}, \tfrac{3}{5})\rceil$
$\lfloor x_1^2x_5^3 + x_2^2x_5^3, (0,0,0,0,0)\rceil$	$\lfloor 1, (\tfrac{3}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{4}{5})\rceil + \lfloor 1, (\tfrac{1}{5}, \tfrac{3}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{4}{5})\rceil$
$\lfloor x_2^2x_5^3 + x_4^2x_5^3, (0,0,0,0,0)\rceil$	$\lfloor 1, (\tfrac{1}{5}, \tfrac{3}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{4}{5})\rceil + \lfloor 1, (\tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{3}{5}, \tfrac{4}{5})\rceil$
$\lfloor x_1^3x_2x_3 + x_1x_2^3x_4, (0,0,0,0,0)\rceil$	$\lfloor 1, (\tfrac{4}{5}, \tfrac{2}{5}, \tfrac{2}{5}, \tfrac{1}{5}, \tfrac{1}{5})\rceil + \lfloor 1, (\tfrac{2}{5}, \tfrac{4}{5}, \tfrac{1}{5}, \tfrac{2}{5}, \tfrac{1}{5})\rceil$
$\lfloor x_1x_3^3x_4 + x_2x_3x_4^3, (0,0,0,0,0)\rceil$	$\lfloor 1, (\tfrac{2}{5}, \tfrac{1}{5}, \tfrac{4}{5}, \tfrac{2}{5}, \tfrac{1}{5})\rceil + \lfloor 1, (\tfrac{1}{5}, \tfrac{2}{5}, \tfrac{2}{5}, \tfrac{4}{5}, \tfrac{1}{5})\rceil$
$\lfloor x_1x_2^3x_3 + x_1^3x_2x_4, (0,0,0,0,0)\rceil$	$\lfloor 1, (\tfrac{2}{5}, \tfrac{4}{5}, \tfrac{2}{5}, \tfrac{1}{5}, \tfrac{1}{5})\rceil + \lfloor 1, (\tfrac{4}{5}, \tfrac{2}{5}, \tfrac{1}{5}, \tfrac{2}{5}, \tfrac{1}{5})\rceil$
$\lfloor x_1x_3x_4^3 + x_2x_3^3x_4, (0,0,0,0,0)\rceil$	$\lfloor 1, (\tfrac{2}{5}, \tfrac{1}{5}, \tfrac{2}{5}, \tfrac{4}{5}, \tfrac{1}{5})\rceil + \lfloor 1, (\tfrac{1}{5}, \tfrac{2}{5}, \tfrac{4}{5}, \tfrac{2}{5}, \tfrac{1}{5})\rceil$
$\lfloor x_1x_2x_3^3 + x_1x_2x_4^3, (0,0,0,0,0)\rceil$	$\lfloor 1, (\tfrac{2}{5}, \tfrac{2}{5}, \tfrac{4}{5}, \tfrac{1}{5}, \tfrac{1}{5})\rceil + \lfloor 1, (\tfrac{2}{5}, \tfrac{2}{5}, \tfrac{1}{5}, \tfrac{4}{5}, \tfrac{1}{5})\rceil$
$\lfloor x_1^3x_3x_4 + x_2^3x_3x_4, (0,0,0,0,0)\rceil$	$\lfloor 1, (\tfrac{4}{5}, \tfrac{1}{5}, \tfrac{2}{5}, \tfrac{2}{5}, \tfrac{1}{5})\rceil + \lfloor 1, (\tfrac{1}{5}, \tfrac{4}{5}, \tfrac{2}{5}, \tfrac{2}{5}, \tfrac{1}{5})\rceil$
$\lfloor x_1x_2^3x_5 + x_1^3x_2x_5, (0,0,0,0,0)\rceil$	$\lfloor 1, (\tfrac{2}{5}, \tfrac{4}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{2}{5})\rceil + \lfloor 1, (\tfrac{4}{5}, \tfrac{2}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{2}{5})\rceil$
$\lfloor x_3x_4^3x_5 + x_3^3x_4x_5, (0,0,0,0,0)\rceil$	$\lfloor 1, (\tfrac{1}{5}, \tfrac{1}{5}, \tfrac{2}{5}, \tfrac{4}{5}, \tfrac{2}{5})\rceil + \lfloor 1, (\tfrac{1}{5}, \tfrac{1}{5}, \tfrac{4}{5}, \tfrac{2}{5}, \tfrac{2}{5})\rceil$
$\lfloor x_1x_3^3x_5 + x_2x_4^3x_5, (0,0,0,0,0)\rceil$	$\lfloor 1, (\tfrac{2}{5}, \tfrac{1}{5}, \tfrac{4}{5}, \tfrac{1}{5}, \tfrac{2}{5})\rceil + \lfloor 1, (\tfrac{1}{5}, \tfrac{2}{5}, \tfrac{1}{5}, \tfrac{4}{5}, \tfrac{2}{5})\rceil$
$\lfloor x_1^3x_3x_5 + x_2^3x_4x_5, (0,0,0,0,0)\rceil$	$\lfloor 1, (\tfrac{4}{5}, \tfrac{1}{5}, \tfrac{2}{5}, \tfrac{1}{5}, \tfrac{2}{5})\rceil + \lfloor 1, (\tfrac{1}{5}, \tfrac{4}{5}, \tfrac{1}{5}, \tfrac{2}{5}, \tfrac{2}{5})\rceil$
$\lfloor x_1x_4^3x_5 + x_2x_3^3x_5, (0,0,0,0,0)\rceil$	$\lfloor 1, (\tfrac{2}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{4}{5}, \tfrac{2}{5})\rceil + \lfloor 1, (\tfrac{1}{5}, \tfrac{2}{5}, \tfrac{4}{5}, \tfrac{1}{5}, \tfrac{2}{5})\rceil$
$\lfloor x_1^3x_4x_5 + x_2^3x_3x_5, (0,0,0,0,0)\rceil$	$\lfloor 1, (\tfrac{4}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{2}{5}, \tfrac{2}{5})\rceil + \lfloor 1, (\tfrac{1}{5}, \tfrac{4}{5}, \tfrac{2}{5}, \tfrac{1}{5}, \tfrac{2}{5})\rceil$
$\lfloor x_1x_2x_5^3, (0,0,0,0,0)\rceil$	$\lfloor 1, (\tfrac{2}{5}, \tfrac{2}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{4}{5})\rceil$
$\lfloor x_3x_4x_5^3, (0,0,0,0,0)\rceil$	$\lfloor 1, (\tfrac{1}{5}, \tfrac{1}{5}, \tfrac{2}{5}, \tfrac{2}{5}, \tfrac{4}{5})\rceil$
$\lfloor x_1x_3x_5^3 + x_2x_4x_5^3, (0,0,0,0,0)\rceil$	$\lfloor 1, (\tfrac{2}{5}, \tfrac{1}{5}, \tfrac{2}{5}, \tfrac{1}{5}, \tfrac{4}{5})\rceil + \lfloor 1, (\tfrac{1}{5}, \tfrac{2}{5}, \tfrac{1}{5}, \tfrac{2}{5}, \tfrac{4}{5})\rceil$
$\lfloor x_1x_4x_5^3 + x_2x_3x_5^3, (0,0,0,0,0)\rceil$	$\lfloor 1, (\tfrac{2}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{2}{5}, \tfrac{4}{5})\rceil + \lfloor 1, (\tfrac{1}{5}, \tfrac{2}{5}, \tfrac{2}{5}, \tfrac{1}{5}, \tfrac{4}{5})\rceil$

$\lfloor x_1^2x_2^2x_3 + x_1^2x_2^2x_4, (0,0,0,0,0)\rceil$	$\lfloor 1, (\tfrac{3}{5}, \tfrac{3}{5}, \tfrac{2}{5}, \tfrac{1}{5}, \tfrac{1}{5})\rceil + \lfloor 1, (\tfrac{3}{5}, \tfrac{3}{5}, \tfrac{1}{5}, \tfrac{2}{5}, \tfrac{1}{5})\rceil$
$\lfloor x_1x_2^2x_3^2 + x_1^2x_2x_4^2, (0,0,0,0,0)\rceil$	$\lfloor 1, (\tfrac{2}{5}, \tfrac{3}{5}, \tfrac{3}{5}, \tfrac{1}{5}, \tfrac{1}{5})\rceil + \lfloor 1, (\tfrac{3}{5}, \tfrac{2}{5}, \tfrac{1}{5}, \tfrac{3}{5}, \tfrac{1}{5})\rceil$
$\lfloor x_1^2x_2x_3^2 + x_1x_2^2x_4^2, (0,0,0,0,0)\rceil$	$\lfloor 1, (\tfrac{3}{5}, \tfrac{2}{5}, \tfrac{3}{5}, \tfrac{1}{5}, \tfrac{1}{5})\rceil + \lfloor 1, (\tfrac{2}{5}, \tfrac{3}{5}, \tfrac{1}{5}, \tfrac{3}{5}, \tfrac{1}{5})\rceil$
$\lfloor x_1^2x_3^2x_4 + x_2^2x_3x_4^2, (0,0,0,0,0)\rceil$	$\lfloor 1, (\tfrac{3}{5}, \tfrac{1}{5}, \tfrac{3}{5}, \tfrac{2}{5}, \tfrac{1}{5})\rceil + \lfloor 1, (\tfrac{1}{5}, \tfrac{3}{5}, \tfrac{2}{5}, \tfrac{3}{5}, \tfrac{1}{5})\rceil$
$\lfloor x_1x_3^2x_4^2 + x_2x_3^2x_4^2, (0,0,0,0,0)\rceil$	$\lfloor 1, (\tfrac{2}{5}, \tfrac{1}{5}, \tfrac{3}{5}, \tfrac{3}{5}, \tfrac{1}{5})\rceil + \lfloor 1, (\tfrac{1}{5}, \tfrac{2}{5}, \tfrac{3}{5}, \tfrac{3}{5}, \tfrac{1}{5})\rceil$
$\lfloor x_1^2x_3x_4^2 + x_2^2x_3^2x_4, (0,0,0,0,0)\rceil$	$\lfloor 1, (\tfrac{3}{5}, \tfrac{1}{5}, \tfrac{2}{5}, \tfrac{3}{5}, \tfrac{1}{5})\rceil + \lfloor 1, (\tfrac{1}{5}, \tfrac{3}{5}, \tfrac{3}{5}, \tfrac{2}{5}, \tfrac{1}{5})\rceil$
$\lfloor x_1^2x_2^2x_5, (0,0,0,0,0)\rceil$	$\lfloor 1, (\tfrac{3}{5}, \tfrac{3}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{2}{5})\rceil$
$\lfloor x_3^2x_4^2x_5, (0,0,0,0,0)\rceil$	$\lfloor 1, (\tfrac{1}{5}, \tfrac{1}{5}, \tfrac{3}{5}, \tfrac{3}{5}, \tfrac{2}{5})\rceil$
$\lfloor x_1^2x_3^2x_5 + x_2^2x_4^2x_5, (0,0,0,0,0)\rceil$	$\lfloor 1, (\tfrac{3}{5}, \tfrac{1}{5}, \tfrac{3}{5}, \tfrac{1}{5}, \tfrac{2}{5})\rceil + \lfloor 1, (\tfrac{1}{5}, \tfrac{3}{5}, \tfrac{1}{5}, \tfrac{3}{5}, \tfrac{2}{5})\rceil$
$\lfloor x_1^2x_4^2x_5 + x_2^2x_3^2x_5, (0,0,0,0,0)\rceil$	$\lfloor 1, (\tfrac{3}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{3}{5}, \tfrac{2}{5})\rceil + \lfloor 1, (\tfrac{1}{5}, \tfrac{3}{5}, \tfrac{3}{5}, \tfrac{1}{5}, \tfrac{2}{5})\rceil$
$\lfloor x_1x_2^2x_5^2 + x_1^2x_2x_5^2, (0,0,0,0,0)\rceil$	$\lfloor 1, (\tfrac{2}{5}, \tfrac{3}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{3}{5})\rceil + \lfloor 1, (\tfrac{3}{5}, \tfrac{2}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{3}{5})\rceil$
$\lfloor x_3x_4^2x_5^2 + x_3^2x_4x_5^2, (0,0,0,0,0)\rceil$	$\lfloor 1, (\tfrac{1}{5}, \tfrac{1}{5}, \tfrac{2}{5}, \tfrac{3}{5}, \tfrac{3}{5})\rceil + \lfloor 1, (\tfrac{1}{5}, \tfrac{1}{5}, \tfrac{3}{5}, \tfrac{2}{5}, \tfrac{3}{5})\rceil$
$\lfloor x_1x_3^2x_5^2 + x_2x_4^2x_5^2, (0,0,0,0,0)\rceil$	$\lfloor 1, (\tfrac{2}{5}, \tfrac{1}{5}, \tfrac{3}{5}, \tfrac{1}{5}, \tfrac{3}{5})\rceil + \lfloor 1, (\tfrac{1}{5}, \tfrac{2}{5}, \tfrac{1}{5}, \tfrac{3}{5}, \tfrac{3}{5})\rceil$
$\lfloor x_1^2x_3x_5^2 + x_2^2x_4x_5^2, (0,0,0,0,0)\rceil$	$\lfloor 1, (\tfrac{3}{5}, \tfrac{1}{5}, \tfrac{2}{5}, \tfrac{1}{5}, \tfrac{3}{5})\rceil + \lfloor 1, (\tfrac{1}{5}, \tfrac{3}{5}, \tfrac{1}{5}, \tfrac{2}{5}, \tfrac{3}{5})\rceil$
$\lfloor x_1x_4^2x_5^2 + x_2x_3^2x_5^2, (0,0,0,0,0)\rceil$	$\lfloor 1, (\tfrac{2}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{3}{5}, \tfrac{3}{5})\rceil + \lfloor 1, (\tfrac{1}{5}, \tfrac{2}{5}, \tfrac{3}{5}, \tfrac{1}{5}, \tfrac{3}{5})\rceil$
$\lfloor x_1^2x_4x_5^2 + x_2^2x_3x_5^2, (0,0,0,0,0)\rceil$	$\lfloor 1, (\tfrac{3}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{2}{5}, \tfrac{3}{5})\rceil + \lfloor 1, (\tfrac{1}{5}, \tfrac{3}{5}, \tfrac{2}{5}, \tfrac{1}{5}, \tfrac{3}{5})\rceil$
$\lfloor x_1^2x_2x_3x_4 + x_1x_2^2x_3x_4, (0,0,0,0,0)\rceil$	$\lfloor 1, (\tfrac{3}{5}, \tfrac{2}{5}, \tfrac{2}{5}, \tfrac{2}{5}, \tfrac{1}{5})\rceil + \lfloor 1, (\tfrac{2}{5}, \tfrac{3}{5}, \tfrac{2}{5}, \tfrac{2}{5}, \tfrac{1}{5})\rceil$
$\lfloor x_1x_2x_3^2x_4 + x_1x_2x_3x_4^2, (0,0,0,0,0)\rceil$	$\lfloor 1, (\tfrac{2}{5}, \tfrac{2}{5}, \tfrac{3}{5}, \tfrac{2}{5}, \tfrac{1}{5})\rceil + \lfloor 1, (\tfrac{2}{5}, \tfrac{2}{5}, \tfrac{2}{5}, \tfrac{3}{5}, \tfrac{1}{5})\rceil$
$\lfloor x_1^2x_2x_3x_5 + x_1x_2^2x_4x_5, (0,0,0,0,0)\rceil$	$\lfloor 1, (\tfrac{3}{5}, \tfrac{2}{5}, \tfrac{2}{5}, \tfrac{1}{5}, \tfrac{2}{5})\rceil + \lfloor 1, (\tfrac{2}{5}, \tfrac{3}{5}, \tfrac{1}{5}, \tfrac{2}{5}, \tfrac{2}{5})\rceil$
$\lfloor x_1x_3^2x_4x_5 + x_2x_3x_4^2x_5, (0,0,0,0,0)\rceil$	$\lfloor 1, (\tfrac{2}{5}, \tfrac{1}{5}, \tfrac{3}{5}, \tfrac{2}{5}, \tfrac{2}{5})\rceil + \lfloor 1, (\tfrac{1}{5}, \tfrac{2}{5}, \tfrac{2}{5}, \tfrac{3}{5}, \tfrac{2}{5})\rceil$
$\lfloor x_1x_2^2x_3x_5 + x_1^2x_2x_4x_5, (0,0,0,0,0)\rceil$	$\lfloor 1, (\tfrac{2}{5}, \tfrac{3}{5}, \tfrac{2}{5}, \tfrac{1}{5}, \tfrac{2}{5})\rceil + \lfloor 1, (\tfrac{3}{5}, \tfrac{2}{5}, \tfrac{1}{5}, \tfrac{2}{5}, \tfrac{2}{5})\rceil$
$\lfloor x_1x_3x_4^2x_5 + x_2x_3^2x_4x_5, (0,0,0,0,0)\rceil$	$\lfloor 1, (\tfrac{2}{5}, \tfrac{1}{5}, \tfrac{2}{5}, \tfrac{3}{5}, \tfrac{2}{5})\rceil + \lfloor 1, (\tfrac{1}{5}, \tfrac{2}{5}, \tfrac{3}{5}, \tfrac{2}{5}, \tfrac{2}{5})\rceil$
$\lfloor x_1x_2x_3^2x_5 + x_1x_2x_4^2x_5, (0,0,0,0,0)\rceil$	$\lfloor 1, (\tfrac{2}{5}, \tfrac{2}{5}, \tfrac{3}{5}, \tfrac{1}{5}, \tfrac{2}{5})\rceil + \lfloor 1, (\tfrac{2}{5}, \tfrac{2}{5}, \tfrac{1}{5}, \tfrac{3}{5}, \tfrac{2}{5})\rceil$
$\lfloor x_1^2x_3x_4x_5 + x_2^2x_3x_4x_5, (0,0,0,0,0)\rceil$	$\lfloor 1, (\tfrac{3}{5}, \tfrac{1}{5}, \tfrac{2}{5}, \tfrac{2}{5}, \tfrac{2}{5})\rceil + \lfloor 1, (\tfrac{1}{5}, \tfrac{3}{5}, \tfrac{2}{5}, \tfrac{2}{5}, \tfrac{2}{5})\rceil$
$\lfloor x_1x_2x_3x_5^2 + x_1x_2x_4x_5^2, (0,0,0,0,0)\rceil$	$\lfloor 1, (\tfrac{2}{5}, \tfrac{2}{5}, \tfrac{2}{5}, \tfrac{1}{5}, \tfrac{3}{5})\rceil + \lfloor 1, (\tfrac{2}{5}, \tfrac{2}{5}, \tfrac{1}{5}, \tfrac{2}{5}, \tfrac{3}{5})\rceil$
$\lfloor x_1x_3x_4x_5^2 + x_2x_3x_4x_5^2, (0,0,0,0,0)\rceil$	$\lfloor 1, (\tfrac{2}{5}, \tfrac{1}{5}, \tfrac{2}{5}, \tfrac{2}{5}, \tfrac{3}{5})\rceil + \lfloor 1, (\tfrac{1}{5}, \tfrac{2}{5}, \tfrac{2}{5}, \tfrac{2}{5}, \tfrac{3}{5})\rceil$

A-model	B-model
$\lfloor x_1^3x_2^3x_3^3x_4 + x_1^3x_2^3x_3x_4^3, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{4}{5}, \frac{4}{5}, \frac{4}{5}, \frac{2}{5}, \frac{1}{5})\rceil + \lfloor 1, (\frac{4}{5}, \frac{4}{5}, \frac{2}{5}, \frac{4}{5}, \frac{1}{5})\rceil$
$\lfloor x_1^3x_2x_3^3x_4^3 + x_1x_2^3x_3^3x_4^3, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{4}{5}, \frac{2}{5}, \frac{4}{5}, \frac{4}{5}, \frac{1}{5})\rceil + \lfloor 1, (\frac{2}{5}, \frac{4}{5}, \frac{4}{5}, \frac{4}{5}, \frac{1}{5})\rceil$
$\lfloor x_1^3x_2^3x_3^3x_5 + x_1^3x_2^3x_4^3x_5, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{4}{5}, \frac{4}{5}, \frac{4}{5}, \frac{1}{5}, \frac{2}{5})\rceil + \lfloor 1, (\frac{4}{5}, \frac{4}{5}, \frac{1}{5}, \frac{4}{5}, \frac{2}{5})\rceil$
$\lfloor x_1^3x_2^3x_4^3x_5 + x_2^3x_3^3x_4^3x_5, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{4}{5}, \frac{4}{5}, \frac{1}{5}, \frac{4}{5}, \frac{2}{5})\rceil + \lfloor 1, (\frac{1}{5}, \frac{4}{5}, \frac{4}{5}, \frac{4}{5}, \frac{2}{5})\rceil$
$\lfloor x_1^3x_2^3x_3x_5^3 + x_1^3x_2^3x_4x_5^3, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{4}{5}, \frac{4}{5}, \frac{2}{5}, \frac{1}{5}, \frac{4}{5})\rceil + \lfloor 1, (\frac{4}{5}, \frac{4}{5}, \frac{1}{5}, \frac{2}{5}, \frac{4}{5})\rceil$
$\lfloor x_1x_3^3x_4^3x_5^3 + x_2x_3^3x_4^3x_5^3, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{2}{5}, \frac{1}{5}, \frac{4}{5}, \frac{4}{5}, \frac{4}{5})\rceil + \lfloor 1, (\frac{1}{5}, \frac{2}{5}, \frac{4}{5}, \frac{4}{5}, \frac{4}{5})\rceil$
$\lfloor x_1^3x_2x_3^3x_5^3 + x_1x_2^3x_4^3x_5^3, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{4}{5}, \frac{2}{5}, \frac{4}{5}, \frac{1}{5}, \frac{4}{5})\rceil + \lfloor 1, (\frac{2}{5}, \frac{4}{5}, \frac{1}{5}, \frac{4}{5}, \frac{4}{5})\rceil$
$\lfloor x_1^3x_3^3x_4x_5^3 + x_2^3x_3x_4^3x_5^3, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{4}{5}, \frac{1}{5}, \frac{4}{5}, \frac{2}{5}, \frac{4}{5})\rceil + \lfloor 1, (\frac{1}{5}, \frac{4}{5}, \frac{2}{5}, \frac{4}{5}, \frac{4}{5})\rceil$
$\lfloor x_1x_2^3x_3^3x_5^3 + x_1^3x_2x_4^3x_5^3, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{2}{5}, \frac{4}{5}, \frac{4}{5}, \frac{1}{5}, \frac{4}{5})\rceil + \lfloor 1, (\frac{4}{5}, \frac{2}{5}, \frac{1}{5}, \frac{4}{5}, \frac{4}{5})\rceil$
$\lfloor x_1^3x_3x_4^3x_5^3 + x_2^3x_3^3x_4x_5^3, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{4}{5}, \frac{1}{5}, \frac{2}{5}, \frac{4}{5}, \frac{4}{5})\rceil + \lfloor 1, (\frac{1}{5}, \frac{4}{5}, \frac{4}{5}, \frac{2}{5}, \frac{4}{5})\rceil$
$\lfloor x_1^3x_2^3x_3^2x_4^2, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{4}{5}, \frac{4}{5}, \frac{3}{5}, \frac{3}{5}, \frac{1}{5})\rceil$
$\lfloor x_1^2x_2^2x_3^3x_4^3, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{3}{5}, \frac{3}{5}, \frac{4}{5}, \frac{4}{5}, \frac{1}{5})\rceil$
$\lfloor x_1^3x_2^2x_3^3x_4^2 + x_1^2x_2^3x_3^2x_4^3, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{4}{5}, \frac{3}{5}, \frac{4}{5}, \frac{3}{5}, \frac{1}{5})\rceil + \lfloor 1, (\frac{3}{5}, \frac{4}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{5})\rceil$
$\lfloor x_1^3x_2^2x_3^2x_4^3 + x_1^2x_2^3x_3^3x_4^2, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{4}{5}, \frac{3}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{5})\rceil + \lfloor 1, (\frac{3}{5}, \frac{4}{5}, \frac{4}{5}, \frac{3}{5}, \frac{1}{5})\rceil$
$\lfloor x_1^3x_2^3x_3^2x_5^2 + x_1^3x_2^3x_4^2x_5^2, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{4}{5}, \frac{3}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{5})\rceil + \lfloor 1, (\frac{4}{5}, \frac{4}{5}, \frac{1}{5}, \frac{3}{5}, \frac{3}{5})\rceil$
$\lfloor x_1^2x_3^3x_4^3x_5^2 + x_2^2x_3^3x_4^3x_5^2, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{3}{5}, \frac{1}{5}, \frac{4}{5}, \frac{4}{5}, \frac{3}{5})\rceil + \lfloor 1, (\frac{1}{5}, \frac{3}{5}, \frac{4}{5}, \frac{4}{5}, \frac{3}{5})\rceil$
$\lfloor x_1^3x_2^2x_3^3x_5^2 + x_1^2x_2^3x_4^3x_5^2, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{4}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{5}, \frac{3}{5})\rceil + \lfloor 1, (\frac{3}{5}, \frac{4}{5}, \frac{1}{5}, \frac{4}{5}, \frac{3}{5})\rceil$
$\lfloor x_1^3x_3^3x_4^2x_5^2 + x_2^3x_3^2x_4^3x_5^2, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{4}{5}, \frac{1}{5}, \frac{4}{5}, \frac{3}{5}, \frac{3}{5})\rceil + \lfloor 1, (\frac{1}{5}, \frac{4}{5}, \frac{3}{5}, \frac{4}{5}, \frac{3}{5})\rceil$
$\lfloor x_1^2x_2^3x_3^3x_5^2 + x_1^3x_2^2x_4^3x_5^2, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{3}{5}, \frac{4}{5}, \frac{4}{5}, \frac{1}{5}, \frac{3}{5})\rceil + \lfloor 1, (\frac{4}{5}, \frac{3}{5}, \frac{1}{5}, \frac{4}{5}, \frac{3}{5})\rceil$
$\lfloor x_1^3x_3^2x_4^3x_5^2 + x_2^3x_3^3x_4^2x_5^2, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{4}{5}, \frac{1}{5}, \frac{3}{5}, \frac{4}{5}, \frac{3}{5})\rceil + \lfloor 1, (\frac{1}{5}, \frac{4}{5}, \frac{4}{5}, \frac{3}{5}, \frac{3}{5})\rceil$
$\lfloor x_1^3x_2^2x_3^2x_5^3 + x_1^2x_2^3x_4^2x_5^3, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{4}{5}, \frac{3}{5}, \frac{3}{5}, \frac{1}{5}, \frac{4}{5})\rceil + \lfloor 1, (\frac{3}{5}, \frac{4}{5}, \frac{1}{5}, \frac{3}{5}, \frac{4}{5})\rceil$
$\lfloor x_1^2x_3^3x_4^2x_5^3 + x_2^2x_3^2x_4^3x_5^3, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{3}{5}, \frac{1}{5}, \frac{4}{5}, \frac{3}{5}, \frac{4}{5})\rceil + \lfloor 1, (\frac{1}{5}, \frac{3}{5}, \frac{3}{5}, \frac{4}{5}, \frac{4}{5})\rceil$
$\lfloor x_1^2x_2^3x_3^2x_5^3 + x_1^3x_2^2x_4^2x_5^3, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{3}{5}, \frac{4}{5}, \frac{3}{5}, \frac{1}{5}, \frac{4}{5})\rceil + \lfloor 1, (\frac{4}{5}, \frac{3}{5}, \frac{1}{5}, \frac{3}{5}, \frac{4}{5})\rceil$
$\lfloor x_1^2x_3^2x_4^3x_5^3 + x_2^2x_3^3x_4^2x_5^3, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{3}{5}, \frac{1}{5}, \frac{3}{5}, \frac{4}{5}, \frac{4}{5})\rceil + \lfloor 1, (\frac{1}{5}, \frac{3}{5}, \frac{4}{5}, \frac{3}{5}, \frac{4}{5})\rceil$
$\lfloor x_1^2x_2^2x_3^3x_5^3 + x_1^2x_2^2x_4^3x_5^3, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{3}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{5}, \frac{4}{5})\rceil + \lfloor 1, (\frac{3}{5}, \frac{3}{5}, \frac{1}{5}, \frac{4}{5}, \frac{4}{5})\rceil$
$\lfloor x_1^3x_3^2x_4^2x_5^3 + x_2^3x_3^2x_4^2x_5^3, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{4}{5}, \frac{1}{5}, \frac{3}{5}, \frac{3}{5}, \frac{4}{5})\rceil + \lfloor 1, (\frac{1}{5}, \frac{4}{5}, \frac{3}{5}, \frac{3}{5}, \frac{4}{5})\rceil$

$\lfloor x_1^3x_2^2x_3^2x_4^2x_5 + x_1^2x_2^3x_3^2x_4^2x_5, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{4}{5}, \frac{3}{5}, \frac{3}{5}, \frac{3}{5}, \frac{2}{5})\rceil + \lfloor 1, (\frac{3}{5}, \frac{4}{5}, \frac{3}{5}, \frac{3}{5}, \frac{2}{5})\rceil$
$\lfloor x_1^2x_2^2x_3^3x_4^2x_5 + x_1^2x_2^2x_3^2x_4^3x_5, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{3}{5}, \frac{3}{5}, \frac{4}{5}, \frac{3}{5}, \frac{2}{5})\rceil + \lfloor 1, (\frac{3}{5}, \frac{3}{5}, \frac{3}{5}, \frac{4}{5}, \frac{2}{5})\rceil$
$\lfloor x_1^3x_2^2x_3^2x_4x_5^2 + x_1^2x_2^3x_3x_4^2x_5^2, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{4}{5}, \frac{3}{5}, \frac{3}{5}, \frac{2}{5}, \frac{3}{5})\rceil + \lfloor 1, (\frac{3}{5}, \frac{4}{5}, \frac{2}{5}, \frac{3}{5}, \frac{3}{5})\rceil$
$\lfloor x_1^2x_2x_3^3x_4^2x_5^2 + x_1x_2^2x_3^2x_4^3x_5^2, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{3}{5}, \frac{2}{5}, \frac{4}{5}, \frac{3}{5}, \frac{3}{5})\rceil + \lfloor 1, (\frac{2}{5}, \frac{3}{5}, \frac{3}{5}, \frac{4}{5}, \frac{3}{5})\rceil$
$\lfloor x_1^3x_2^2x_3x_4^2x_5^2 + x_1^2x_2^3x_3^2x_4x_5^2, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{4}{5}, \frac{3}{5}, \frac{2}{5}, \frac{3}{5}, \frac{3}{5})\rceil + \lfloor 1, (\frac{3}{5}, \frac{4}{5}, \frac{3}{5}, \frac{2}{5}, \frac{3}{5})\rceil$
$\lfloor x_1x_2^2x_3^3x_4^2x_5^2 + x_1^2x_2x_3^2x_4^3x_5^2, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{3}{5}, \frac{3}{5})\rceil + \lfloor 1, (\frac{3}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{3}{5})\rceil$
$\lfloor x_1^3x_2x_3^2x_4^2x_5^2 + x_1x_2^3x_3^2x_4^2x_5^2, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{4}{5}, \frac{2}{5}, \frac{3}{5}, \frac{3}{5}, \frac{3}{5})\rceil + \lfloor 1, (\frac{2}{5}, \frac{4}{5}, \frac{3}{5}, \frac{3}{5}, \frac{3}{5})\rceil$
$\lfloor x_1^2x_2^2x_3^3x_4x_5^2 + x_1^2x_2^2x_3x_4^3x_5^2, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{3}{5}, \frac{3}{5}, \frac{4}{5}, \frac{2}{5}, \frac{3}{5})\rceil + \lfloor 1, (\frac{3}{5}, \frac{3}{5}, \frac{2}{5}, \frac{4}{5}, \frac{3}{5})\rceil$
$\lfloor x_1^2x_2^2x_3^2x_4x_5^3 + x_1^2x_2^2x_3x_4^2x_5^3, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{3}{5}, \frac{3}{5}, \frac{3}{5}, \frac{2}{5}, \frac{4}{5})\rceil + \lfloor 1, (\frac{3}{5}, \frac{3}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5})\rceil$
$\lfloor x_1^2x_2x_3^2x_4^2x_5^3 + x_1x_2^2x_3^2x_4^2x_5^3, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{3}{5}, \frac{2}{5}, \frac{3}{5}, \frac{3}{5}, \frac{4}{5})\rceil + \lfloor 1, (\frac{2}{5}, \frac{3}{5}, \frac{3}{5}, \frac{3}{5}, \frac{4}{5})\rceil$
$\lfloor x_1^3x_2^3x_3^2x_4x_5 + x_1^3x_2^3x_3x_4^2x_5, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{4}{5}, \frac{4}{5}, \frac{3}{5}, \frac{2}{5}, \frac{2}{5})\rceil + \lfloor 1, (\frac{4}{5}, \frac{4}{5}, \frac{2}{5}, \frac{3}{5}, \frac{2}{5})\rceil$
$\lfloor x_1^2x_2x_3^3x_4^3x_5 + x_1x_2^2x_3^3x_4^3x_5, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{3}{5}, \frac{2}{5}, \frac{4}{5}, \frac{4}{5}, \frac{2}{5})\rceil + \lfloor 1, (\frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{4}{5}, \frac{2}{5})\rceil$
$\lfloor x_1^3x_2^2x_3^3x_4x_5 + x_1^2x_2^3x_3x_4^3x_5, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{4}{5}, \frac{3}{5}, \frac{4}{5}, \frac{2}{5}, \frac{2}{5})\rceil + \lfloor 1, (\frac{3}{5}, \frac{4}{5}, \frac{2}{5}, \frac{4}{5}, \frac{2}{5})\rceil$
$\lfloor x_1^3x_2x_3^3x_4^2x_5 + x_1x_2^3x_3^2x_4^3x_5, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{4}{5}, \frac{2}{5}, \frac{4}{5}, \frac{3}{5}, \frac{2}{5})\rceil + \lfloor 1, (\frac{2}{5}, \frac{4}{5}, \frac{3}{5}, \frac{4}{5}, \frac{2}{5})\rceil$
$\lfloor x_1^3x_2^2x_3x_4^3x_5 + x_1^2x_2^3x_3^3x_4x_5, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{4}{5}, \frac{3}{5}, \frac{2}{5}, \frac{4}{5}, \frac{2}{5})\rceil + \lfloor 1, (\frac{3}{5}, \frac{4}{5}, \frac{4}{5}, \frac{2}{5}, \frac{2}{5})\rceil$
$\lfloor x_1x_2^3x_3^3x_4^2x_5 + x_1^3x_2x_3^2x_4^3x_5, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{2}{5}, \frac{4}{5}, \frac{4}{5}, \frac{3}{5}, \frac{2}{5})\rceil + \lfloor 1, (\frac{4}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{2}{5})\rceil$
$\lfloor x_1^3x_2^3x_3x_4x_5^2, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{4}{5}, \frac{4}{5}, \frac{2}{5}, \frac{2}{5}, \frac{3}{5})\rceil$
$\lfloor x_1x_2x_3^3x_4^3x_5^2, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{2}{5}, \frac{2}{5}, \frac{4}{5}, \frac{4}{5}, \frac{3}{5})\rceil$
$\lfloor x_1^3x_2x_3^3x_4x_5^2 + x_1x_2^3x_3x_4^3x_5^2, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{4}{5}, \frac{2}{5}, \frac{4}{5}, \frac{2}{5}, \frac{3}{5})\rceil + \lfloor 1, (\frac{2}{5}, \frac{4}{5}, \frac{2}{5}, \frac{4}{5}, \frac{3}{5})\rceil$
$\lfloor x_1^3x_2x_3x_4^3x_5^2 + x_1x_2^3x_3^3x_4x_5^2, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{4}{5}, \frac{2}{5}, \frac{2}{5}, \frac{4}{5}, \frac{3}{5})\rceil + \lfloor 1, (\frac{2}{5}, \frac{4}{5}, \frac{4}{5}, \frac{2}{5}, \frac{3}{5})\rceil$
$\lfloor x_1^3x_2^2x_3x_4x_5^3 + x_1^2x_2^3x_3x_4x_5^3, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{4}{5}, \frac{3}{5}, \frac{2}{5}, \frac{2}{5}, \frac{4}{5})\rceil + \lfloor 1, (\frac{3}{5}, \frac{4}{5}, \frac{2}{5}, \frac{2}{5}, \frac{4}{5})\rceil$
$\lfloor x_1x_2x_3^3x_4^2x_5^3 + x_1x_2x_3^2x_4^3x_5^3, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{2}{5}, \frac{2}{5}, \frac{4}{5}, \frac{3}{5}, \frac{4}{5})\rceil + \lfloor 1, (\frac{2}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{4}{5})\rceil$
$\lfloor x_1^3x_2x_3^2x_4x_5^3 + x_1x_2^3x_3x_4^2x_5^3, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{4}{5}, \frac{2}{5}, \frac{3}{5}, \frac{2}{5}, \frac{4}{5})\rceil + \lfloor 1, (\frac{2}{5}, \frac{4}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5})\rceil$
$\lfloor x_1^2x_2x_3^3x_4x_5^3 + x_1x_2^2x_3x_4^3x_5^3, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{3}{5}, \frac{2}{5}, \frac{4}{5}, \frac{2}{5}, \frac{4}{5})\rceil + \lfloor 1, (\frac{2}{5}, \frac{3}{5}, \frac{2}{5}, \frac{4}{5}, \frac{4}{5})\rceil$
$\lfloor x_1^3x_2x_3x_4^2x_5^3 + x_1x_2^3x_3^2x_4x_5^3, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{4}{5}, \frac{2}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5})\rceil + \lfloor 1, (\frac{2}{5}, \frac{4}{5}, \frac{3}{5}, \frac{2}{5}, \frac{4}{5})\rceil$
$\lfloor x_1x_2^2x_3^3x_4x_5^3 + x_1^2x_2x_3x_4^3x_5^3, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{2}{5}, \frac{4}{5})\rceil + \lfloor 1, (\frac{3}{5}, \frac{2}{5}, \frac{2}{5}, \frac{4}{5}, \frac{4}{5})\rceil$

The following basis elements are not described by Theorem 5.2, yet we are still able to find the same number of each bidegree on either side.

Bidegree	**A-model**	**B-model**
$(1,1)$	$\lfloor 1, ((12)(34))j_W \rceil$	$\lfloor (x_1+x_2)(x_3+x_4), (12)(34) \rceil$
$(2,2)$	$\lfloor 1, ((12)(34))(j_W)^2 \rceil$	$\lfloor (x_1+x_2)^2(x_3+x_4)^2 x_5^3, (12)(34) \rceil$
$(1,1)$	$\lfloor 1, ((12)(34))(j_W)^3 \rceil$	$\lfloor x_5^2, (12)(34) \rceil$
$(2,2)$	$\lfloor 1, ((12)(34))(j_W)^4 \rceil$	$\lfloor (x_1+x_2)^3(x_3+x_4)^3 x_5, (12)(34) \rceil$
$(1,2)$	$\lfloor (x_1+x_2)^2, (12)(34) \rceil$	$\lfloor 1, ((12)(34))j_W \rceil +$ (24 others)
$(1,2)$	$\lfloor (x_3+x_4)^2, (12)(34) \rceil$	$\lfloor 1, ((12)(34))j_W^3 \rceil +$ (24 others)
$(1,2)$	$\lfloor (x_1+x_2)(x_3+x_4), (12)(34) \rceil$	$\lfloor 1, ((12)(34))j_W^2 K \rceil +$ (24 others)
$(1,2)$	$\lfloor x_5^2, (12)(34) \rceil$	$\lfloor 1, ((12)(34))j_W^2 L \rceil +$ (24 others)
$(1,2)$	$\lfloor (x_1+x_2)x_5, (12)(34) \rceil$	$\lfloor 1, ((12)(34))j_W^2 K^4 \rceil +$ (24 others)
$(1,2)$	$\lfloor (x_3+x_4)x_5, (12)(34) \rceil$	$\lfloor 1, ((12)(34))j_W^2 L^4 \rceil +$ (24 others)
$(2,1)$	$\lfloor (x_1+x_2)^3(x_3+x_4)^3 x_5, (12)(34) \rceil$	$\lfloor 1, ((12)(34))j_W^2 \rceil +$ (24 others)
$(2,1)$	$\lfloor (x_1+x_2)^3(x_3+x_4)^2 x_5^2, (12)(34) \rceil$	$\lfloor 1, ((12)(34))j_W^4 \rceil +$ (24 others)
$(2,1)$	$\lfloor (x_1+x_2)^2(x_3+x_4)^3 x_5^2, (12)(34) \rceil$	$\lfloor 1, ((12)(34))j_W^3 K \rceil +$ (24 others)
$(2,1)$	$\lfloor (x_1+x_2)^3(x_3+x_4) x_5^3, (12)(34) \rceil$	$\lfloor 1, ((12)(34))j_W^3 L \rceil +$ (24 others)
$(2,1)$	$\lfloor (x_1+x_2)(x_3+x_4)^3 x_5^3, (12)(34) \rceil$	$\lfloor 1, ((12)(34))j_W^3 K^4 \rceil +$ (24 others)
$(2,1)$	$\lfloor (x_1+x_2)^2(x_3+x_4)^2 x_5^3, (12)(34) \rceil$	$\lfloor 1, ((12)(34))j_W^3 L^4 \rceil +$ (24 others)

CHAPTER 7. FAILED EXAMPLE

Unfortunately, the correspondence previously shown does not hold for all examples. In Example 7.3 we will see that even if two A- and B-models are isomorphic as vector spaces, the given bidegrees don't match. Wolfgang Ebeling and Sabir Gusein-Zade describe the *parity condition* ([5]), which is the condition a subgroup $K \leq G$ of pure permutations must have in order for $\mathcal{A}_{W,G}$ and $\mathcal{B}_{W^T,G^\star}$ to be isomorphic as bigraded vector spaces.

Definition 7.1. (Ebeling/Gusein-Zade [5]) Let K be the subgroup of pure permuations in a group $G \leq G_W^{max}$. We say that K satisfies the *parity condition* ("PC" for short) if for each subgroup $T \leq K$ one has

$$\dim(\mathbb{C}^n)^T \equiv n \pmod 2,$$

where $(\mathbb{C}^n)^T = \{x \in \mathbb{C}^n : \sigma x = x \text{ for all } \sigma \in T\}$.

Example 7.2. Consider $K = \langle(12)(34)\rangle$ from Example 6.1. Then $(\mathbb{C}^4)^K$ has dimension 2 and $(\mathbb{C}^4)^{\{(1)\}}$ has dimension 4, which are both equal to 4 (mod 2). Thus K satisfies the parity condition, implying that the bigraded state spaces of $\mathcal{A}_{W,G}$ and $\mathcal{B}_{W^T,G^\star}$ are isomorphic, which we proved explicitly in Example 6.1.

Example 7.3. Let $W = x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5$ and $G = \langle j_W, (12)(34), (13)(24)\rangle$, where

$$j_W = (\tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}), (12)(34) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \text{ and } (13)(24) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then $W^T = W$ and the non-abelian dual group of G is

$$G^\star = \langle(12)(34), (13)(24)\rangle \cdot \mathrm{SL}_W^{diag},$$

where $\mathrm{SL}_W^{diag} = \langle j_W, (0, \frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}), (0, \frac{1}{5}, \frac{2}{5}, \frac{1}{5}, \frac{1}{5}), (0, \frac{1}{5}, \frac{1}{5}, \frac{2}{5}, \frac{1}{5})\rangle$. While the mirror map works for the sectors described in Theorem 5.2, it is not an isomorphism when considering the entire A- and B-models. We can verify this with the PC condition above. If $K =$

$\langle(12)(34),(13)(24)\rangle$, then $(\mathbb{C}^4)^K = 1 \neq 4 \pmod 2$. We will show this explictly be computing the basis elements of $\mathcal{A}_{W,G}$ and $\mathcal{B}_{W^T,G^\star}$, and then computing their bidegree. However, as proved in Theorem 5.2, the restricted mirror map is still an isomorphism, which we will list first. The eight elements listed below are the exact same as those from Example 6.1.

Bidegree	**A-model**	**B-model**
$(0,0)$	$\lfloor 1, j_W \rceil$	$\lfloor 1, (0,0,0,0,0) \rceil$
$(1,1)$	$\lfloor 1, (j_W)^2 \rceil$	$\lfloor x_1x_2x_3x_4, (0,0,0,0,0) \rceil$
$(2,2)$	$\lfloor 1, (j_W)^3 \rceil$	$\lfloor x_1^2x_2^2x_3^2x_4^2, (0,0,0,0,0) \rceil$
$(3,3)$	$\lfloor 1, (j_W)^4 \rceil$	$\lfloor x_1^3x_2^3x_3^3x_4^3, (0,0,0,0,0) \rceil$
$(0,3)$	$\lfloor 1, (0,0,0,0,0) \rceil$	$\lfloor 1, j_W \rceil$
$(1,2)$	$\lfloor x_1x_2x_3x_4, (0,0,0,0,0) \rceil$	$\lfloor 1, (j_W)^2 \rceil$
$(2,1)$	$\lfloor x_1^2x_2^2x_3^2x_4^2, (0,0,0,0,0) \rceil$	$\lfloor 1, (j_W)^3 \rceil$
$(3,0)$	$\lfloor x_1^3x_2^3x_3^3x_4^3, (0,0,0,0,0) \rceil$	$\lfloor 1, (j_W)^4 \rceil$

The following 28 corresponding elements have a bidegree of $(1,2)$. On the A side these come from untwisted sectors, and on the B side these are from the narrow sectors, again following the outline from Theorem 5.2.

<u>**A-model**</u>	<u>**B-model**</u>
$\lfloor x_1^3x_2^2 + x_1^2x_2^3 + x_3^3x_4^2 + x_3^2x_4^3, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{4}{5}, \frac{3}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})\rceil + (3 \text{ others})$
$\lfloor x_1^3x_3^2 + x_1^2x_3^3 + x_2^3x_4^2 + x_2^2x_4^3, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{4}{5}, \frac{1}{5}, \frac{3}{5}, \frac{1}{5}, \frac{1}{5})\rceil + (3 \text{ others})$
$\lfloor x_1^3x_4^2 + x_1^2x_4^3 + x_2^3x_3^2 + x_2^2x_3^3, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{4}{5}, \frac{1}{5}, \frac{1}{5}, \frac{3}{5}, \frac{1}{5})\rceil + (3 \text{ others})$
$\lfloor x_1^3x_5^2 + x_2^3x_5^2 + x_3^3x_5^2 + x_4^3x_5^2, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{4}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{3}{5})\rceil + (3 \text{ others})$
$\lfloor x_1^2x_5^3 + x_2^2x_5^3 + x_3^2x_5^3 + x_4^2x_5^3, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{3}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{4}{5})\rceil + (3 \text{ others})$
$\lfloor x_1^3x_2x_3 + x_1x_2^3x_4 + x_1x_3^3x_4 + x_2x_3x_4^3, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{4}{5}, \frac{2}{5}, \frac{2}{5}, \frac{1}{5}, \frac{1}{5})\rceil + (3 \text{ others})$
$\lfloor x_1x_2^3x_3 + x_1^3x_2x_4 + x_1x_3x_4^3 + x_2x_3^3x_4, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{2}{5}, \frac{4}{5}, \frac{2}{5}, \frac{1}{5}, \frac{1}{5})\rceil + (3 \text{ others})$
$\lfloor x_1x_2x_3^3 + x_1x_2x_4^3 + x_1^3x_3x_4 + x_2^3x_3x_4, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{2}{5}, \frac{2}{5}, \frac{4}{5}, \frac{1}{5}, \frac{1}{5})\rceil + (3 \text{ others})$
$\lfloor x_1x_2^3x_5 + x_1^3x_2x_5 + x_3x_4^3x_5 + x_3^3x_4x_5, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{2}{5}, \frac{4}{5}, \frac{1}{5}, \frac{1}{5}, \frac{2}{5})\rceil + (3 \text{ others})$
$\lfloor x_1x_3^3x_5 + x_1^3x_3x_5 + x_2x_4^3x_5 + x_2^3x_4x_5, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{2}{5}, \frac{1}{5}, \frac{4}{5}, \frac{1}{5}, \frac{2}{5})\rceil + (3 \text{ others})$
$\lfloor x_1x_4^3x_5 + x_1^3x_4x_5 + x_2x_3^3x_5 + x_2^3x_3x_5, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{4}{5}, \frac{2}{5})\rceil + (3 \text{ others})$
$\lfloor x_1x_2x_5^3 + x_3x_4x_5^3, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{2}{5}, \frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{4}{5})\rceil + \lfloor 1, (\frac{1}{5}, \frac{1}{5}, \frac{2}{5}, \frac{2}{5}, \frac{4}{5})\rceil$
$\lfloor x_1x_3x_5^3 + x_2x_4x_5^3, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{2}{5}, \frac{1}{5}, \frac{2}{5}, \frac{1}{5}, \frac{4}{5})\rceil + \lfloor 1, (\frac{1}{5}, \frac{2}{5}, \frac{1}{5}, \frac{2}{5}, \frac{4}{5})\rceil$
$\lfloor x_1x_4x_5^3 + x_2x_3x_5^3, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{2}{5}, \frac{4}{5})\rceil + \lfloor 1, (\frac{1}{5}, \frac{2}{5}, \frac{2}{5}, \frac{1}{5}, \frac{4}{5})\rceil$
$\lfloor x_1x_2^2x_3^2 + x_1^2x_2x_4^2 + x_1^2x_3x_4^2 + x_2^2x_3^2x_4, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{2}{5}, \frac{3}{5}, \frac{3}{5}, \frac{1}{5}, \frac{1}{5})\rceil + (3 \text{ others})$
$\lfloor x_1^2x_2x_3^2 + x_1x_2^2x_4^2 + x_1^2x_3^2x_4 + x_2^2x_3x_4^2, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{3}{5}, \frac{2}{5}, \frac{3}{5}, \frac{1}{5}, \frac{1}{5})\rceil + (3 \text{ others})$
$\lfloor x_1^2x_2^2x_3 + x_1^2x_2^2x_4 + x_1x_3^2x_4^2 + x_2x_3^2x_4^2, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{3}{5}, \frac{3}{5}, \frac{2}{5}, \frac{1}{5}, \frac{1}{5})\rceil + (3 \text{ others})$
$\lfloor x_1^2x_2^2x_5 + x_3^2x_4^2x_5, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{3}{5}, \frac{3}{5}, \frac{1}{5}, \frac{1}{5}, \frac{2}{5})\rceil + \lfloor 1, (\frac{1}{5}, \frac{1}{5}, \frac{3}{5}, \frac{3}{5}, \frac{2}{5})\rceil$
$\lfloor x_1^2x_3^2x_5 + x_2^2x_4^2x_5, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{3}{5}, \frac{1}{5}, \frac{3}{5}, \frac{1}{5}, \frac{2}{5})\rceil + \lfloor 1, (\frac{1}{5}, \frac{3}{5}, \frac{1}{5}, \frac{3}{5}, \frac{2}{5})\rceil$
$\lfloor x_1^2x_4^2x_5 + x_2^2x_3^2x_5, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{3}{5}, \frac{1}{5}, \frac{1}{5}, \frac{3}{5}, \frac{2}{5})\rceil + \lfloor 1, (\frac{1}{5}, \frac{3}{5}, \frac{3}{5}, \frac{1}{5}, \frac{2}{5})\rceil$
$\lfloor x_1x_2^2x_5^2 + x_1^2x_2x_5^2 + x_3x_4^2x_5^2 + x_3^2x_4x_5^2, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{2}{5}, \frac{3}{5}, \frac{1}{5}, \frac{1}{5}, \frac{3}{5})\rceil + (3 \text{ others})$
$\lfloor x_1x_3^2x_5^2 + x_1^2x_3x_5^2 + x_2x_4^2x_5^2 + x_2^2x_4x_5^2, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{2}{5}, \frac{1}{5}, \frac{3}{5}, \frac{1}{5}, \frac{3}{5})\rceil + (3 \text{ others})$
$\lfloor x_1x_4^2x_5^2 + x_1^2x_4x_5^2 + x_2x_3^2x_5^2 + x_2^2x_3x_5^2, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{3}{5}, \frac{3}{5})\rceil + (3 \text{ others})$
$\lfloor x_1^2x_2x_3x_4 + x_1x_2^2x_3x_4 + x_1x_2x_3^2x_4 + x_1x_2x_3x_4^2, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{3}{5}, \frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{1}{5})\rceil + (3 \text{ others})$
$\lfloor x_1^2x_2x_3x_5 + x_1x_2^2x_4x_5 + x_1x_3^2x_4x_5 + x_2x_3x_4^2x_5, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{3}{5}, \frac{2}{5}, \frac{2}{5}, \frac{1}{5}, \frac{2}{5})\rceil + (3 \text{ others})$
$\lfloor x_1x_2^2x_3x_5 + x_1^2x_2x_4x_5 + x_1x_3x_4^2x_5 + x_2x_3^2x_4x_5, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{2}{5}, \frac{3}{5}, \frac{1}{5}, \frac{2}{5}, \frac{2}{5})\rceil + (3 \text{ others})$
$\lfloor x_1x_2x_3^2x_5 + x_1x_2x_4^2x_5 + x_1^2x_3x_4x_5 + x_2^2x_3x_4x_5, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{2}{5}, \frac{2}{5}, \frac{3}{5}, \frac{1}{5}, \frac{2}{5})\rceil + (3 \text{ others})$
$\lfloor x_1x_2x_3x_5^2 + x_1x_2x_4x_5^2 + x_1x_3x_4x_5^2 + x_2x_3x_4x_5^2, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{1}{5}, \frac{3}{5})\rceil + (3 \text{ others})$

The following 28 corresponding elements have a bidegree of $(2,1)$. As with the previous page, the basis elements on the A-side come from the untwisted broad sector and the elements on the B side are narrow.

A-model	**B-model**
$\lfloor x_1^3x_2^3x_3^3x_4 + x_1^3x_2^3x_3x_4^3 + x_1^3x_2x_3^3x_4^3 + x_1x_2^3x_3^3x_4^3, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{4}{5}, \frac{4}{5}, \frac{4}{5}, \frac{2}{5}, \frac{1}{5})\rceil + (3\text{ others})$
$\lfloor x_1^3x_2^3x_3^3x_5 + x_1^3x_2^3x_4^3x_5 + x_1^3x_2^3x_4^3x_5 + x_2^3x_3^3x_4^3x_5, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{4}{5}, \frac{4}{5}, \frac{4}{5}, \frac{1}{5}, \frac{2}{5})\rceil + (3\text{ others})$
$\lfloor x_1^3x_2^3x_3x_5^3 + x_1^3x_2^3x_4x_5^3 + x_1x_3^3x_4^3x_5^3 + x_2x_3^3x_4^3x_5^3, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{4}{5}, \frac{4}{5}, \frac{2}{5}, \frac{1}{5}, \frac{4}{5})\rceil + (3\text{ others})$
$\lfloor x_1^3x_2x_3^3x_5^3 + x_1x_2^3x_4^3x_5^3 + x_1^3x_3^3x_4x_5^3 + x_2^3x_3x_4^3x_5^3, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{4}{5}, \frac{2}{5}, \frac{4}{5}, \frac{1}{5}, \frac{4}{5})\rceil + (3\text{ others})$
$\lfloor x_1x_2^3x_3^3x_5^3 + x_1^3x_2x_4^3x_5^3 + x_1^3x_3x_4^3x_5^3 + x_2^3x_3^3x_4x_5^3, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{2}{5}, \frac{4}{5}, \frac{4}{5}, \frac{1}{5}, \frac{4}{5})\rceil + (3\text{ others})$
$\lfloor x_1^3x_2^3x_3^2x_4^2 + x_1^2x_2^2x_3^3x_4^3, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{4}{5}, \frac{4}{5}, \frac{3}{5}, \frac{3}{5}, \frac{1}{5})\rceil + \lfloor 1, (\frac{3}{5}, \frac{3}{5}, \frac{4}{5}, \frac{4}{5}, \frac{1}{5})\rceil$
$\lfloor x_1^3x_2^2x_3^3x_4^2 + x_1^2x_2^3x_3^2x_4^3, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{4}{5}, \frac{3}{5}, \frac{4}{5}, \frac{3}{5}, \frac{1}{5})\rceil + \lfloor 1, (\frac{3}{5}, \frac{4}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{5})\rceil$
$\lfloor x_1^3x_2^2x_3^2x_4^3 + x_1^2x_2^3x_3^3x_4^2, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{4}{5}, \frac{3}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{5})\rceil + \lfloor 1, (\frac{3}{5}, \frac{4}{5}, \frac{4}{5}, \frac{3}{5}, \frac{1}{5})\rceil$
$\lfloor x_1^3x_2^3x_3^2x_5^2 + x_1^3x_2^3x_4^2x_5^2 + x_1^2x_3^3x_4^3x_5^2 + x_2^2x_3^3x_4^3x_5^2, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{4}{5}, \frac{4}{5}, \frac{3}{5}, \frac{1}{5}, \frac{3}{5})\rceil + (3\text{ others})$
$\lfloor x_1^3x_2^2x_3^3x_5^2 + x_1^2x_2^3x_4^3x_5^2 + x_1^3x_3^3x_4^2x_5^2 + x_2^3x_3^2x_4^3x_5^2, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{4}{5}, \frac{2}{5}, \frac{4}{5}, \frac{1}{5}, \frac{3}{5})\rceil + (3\text{ others})$
$\lfloor x_1^2x_2^3x_3^3x_5^2 + x_1^3x_2^2x_4^3x_5^2 + x_1^3x_3^2x_4^3x_5^2 + x_2^3x_3^3x_4^2x_5^2, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{3}{5}, \frac{4}{5}, \frac{4}{5}, \frac{1}{5}, \frac{3}{5})\rceil + (3\text{ others})$
$\lfloor x_1^3x_2^2x_3^2x_5^3 + x_1^2x_2^3x_4^2x_5^3 + x_1^2x_3^3x_4^2x_5^3 + x_2^2x_3^2x_4^3x_5^3, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{4}{5}, \frac{4}{5}, \frac{3}{5}, \frac{1}{5}, \frac{4}{5})\rceil + (3\text{ others})$
$\lfloor x_1^2x_2^3x_3^2x_5^3 + x_1^3x_2^2x_4^2x_5^3 + x_1^2x_3^2x_4^3x_5^3 + x_2^2x_3^3x_4^2x_5^3, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{3}{5}, \frac{4}{5}, \frac{3}{5}, \frac{1}{5}, \frac{4}{5})\rceil + (3\text{ others})$
$\lfloor x_1^2x_2^2x_3^3x_5^3 + x_1^2x_2^2x_4^3x_5^3 + x_1^3x_3^2x_4^2x_5^3 + x_2^3x_3^2x_4^2x_5^3, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{3}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{5}, \frac{4}{5})\rceil + (3\text{ others})$
$\lfloor x_1^3x_2^2x_3^2x_4^2x_5 + x_1^2x_2^3x_3^2x_4^2x_5 + x_1^2x_2^2x_3^3x_4^2x_5 + x_1^2x_2^2x_3^2x_4^3x_5, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{4}{5}, \frac{3}{5}, \frac{3}{5}, \frac{3}{5}, \frac{2}{5})\rceil + (3\text{ others})$
$\lfloor x_1^3x_2^2x_3^2x_4x_5^2 + x_1^2x_2^3x_3x_4^2x_5^2 + x_1^2x_2x_3^3x_4^2x_5^2 + x_1x_2^2x_3^2x_4^3x_5^2, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{4}{5}, \frac{3}{5}, \frac{3}{5}, \frac{2}{5}, \frac{3}{5})\rceil + (3\text{ others})$
$\lfloor x_1^3x_2^2x_3x_4^2x_5^2 + x_1^2x_2^3x_3^2x_4x_5^2 + x_1x_2^2x_3^3x_4^2x_5^2 + x_1^2x_2x_3^2x_4^3x_5^2, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{4}{5}, \frac{3}{5}, \frac{2}{5}, \frac{3}{5}, \frac{3}{5})\rceil + (3\text{ others})$
$\lfloor x_1^3x_2x_3^2x_4^2x_5^2 + x_1x_2^3x_3^2x_4^2x_5^2 + x_1^2x_2^2x_3^3x_4x_5^2 + x_1^2x_2^2x_3x_4^3x_5^2, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{4}{5}, \frac{2}{5}, \frac{3}{5}, \frac{3}{5}, \frac{3}{5})\rceil + (3\text{ others})$
$\lfloor x_1^2x_2^2x_3^2x_4x_5^3 + x_1^2x_2^2x_3^1x_4^2x_5^3 + x_1^2x_2x_3^2x_4^2x_5^3 + x_1x_2^2x_3^2x_4^2x_5^3, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{3}{5}, \frac{3}{5}, \frac{3}{5}, \frac{2}{5}, \frac{4}{5})\rceil + (3\text{ others})$
$\lfloor x_1^3x_2^3x_3^2x_4x_5 + x_1^3x_2^3x_3x_4^2x_5 + x_1^2x_2x_3^3x_4^3x_5 + x_1x_2^2x_3^3x_4^3x_5, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{4}{5}, \frac{4}{5}, \frac{3}{5}, \frac{2}{5}, \frac{2}{5})\rceil + (3\text{ others})$
$\lfloor x_1^3x_2^2x_3^3x_4x_5 + x_1^2x_2^3x_3x_4^3x_5 + x_1^3x_2x_3^3x_4^2x_5 + x_1x_2^3x_3^2x_4^3x_5, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{4}{5}, \frac{3}{5}, \frac{4}{5}, \frac{2}{5}, \frac{2}{5})\rceil + (3\text{ others})$
$\lfloor x_1^3x_2^2x_3x_4^3x_5 + x_1^2x_2^3x_3^3x_4x_5 + x_1x_2^3x_3^3x_4^2x_5 + x_1^3x_2x_3^2x_4^3x_5, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{4}{5}, \frac{3}{5}, \frac{2}{5}, \frac{4}{5}, \frac{2}{5})\rceil + (3\text{ others})$
$\lfloor x_1^3x_2^3x_3x_4x_5^2 + x_1x_2x_3^3x_4^3x_5^2, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{4}{5}, \frac{4}{5}, \frac{2}{5}, \frac{2}{5}, \frac{3}{5})\rceil + \lfloor 1, (\frac{2}{5}, \frac{2}{5}, \frac{4}{5}, \frac{4}{5}, \frac{3}{5})\rceil$
$\lfloor x_1^3x_2x_3^3x_4x_5^2 + x_1x_2^3x_3x_4^3x_5^2, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{4}{5}, \frac{2}{5}, \frac{4}{5}, \frac{2}{5}, \frac{3}{5})\rceil + \lfloor 1, (\frac{2}{5}, \frac{4}{5}, \frac{2}{5}, \frac{4}{5}, \frac{3}{5})\rceil$
$\lfloor x_1^3x_2x_3x_4^3x_5^2 + x_1x_2^3x_3^3x_4x_5^2, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{4}{5}, \frac{2}{5}, \frac{2}{5}, \frac{4}{5}, \frac{3}{5})\rceil + \lfloor 1, (\frac{2}{5}, \frac{4}{5}, \frac{4}{5}, \frac{2}{5}, \frac{3}{5})\rceil$
$\lfloor x_1^3x_2^2x_3x_4x_5^3 + x_1^2x_2^3x_3x_4x_5^3 + x_1x_2x_3^3x_4^2x_5^3 + x_1x_2x_3^2x_4^3x_5^3, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{4}{5}, \frac{3}{5}, \frac{2}{5}, \frac{2}{5}, \frac{4}{5})\rceil + (3\text{ others})$
$\lfloor x_1^3x_2x_3^2x_4x_5^3 + x_1x_2^3x_3x_4^2x_5^3 + x_1^2x_2x_3^3x_4x_5^3 + x_1x_2^2x_3x_4^3x_5^3, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{4}{5}, \frac{2}{5}, \frac{3}{5}, \frac{2}{5}, \frac{4}{5})\rceil + (3\text{ others})$
$\lfloor x_1^3x_2x_3x_4^2x_5^3 + x_1x_2^3x_3^2x_4x_5^3 + x_1x_2^2x_3^3x_4x_5^3 + x_1^2x_2x_3x_4^3x_5^3, (0,0,0,0,0)\rceil$	$\lfloor 1, (\frac{4}{5}, \frac{2}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5})\rceil + (3\text{ others})$

Bidegree	A-model	Bidegree	B-model
$(1,1)$	$\lfloor 1,((12)(34))j_W \rceil$	$(1,2)$	$\lfloor 1,((12)(34))j_W \rceil$ + (24 others)
$(2,2)$	$\lfloor 1,((12)(34))(j_W)^2 \rceil$	$(2,1)$	$\lfloor 1,((12)(34))(j_W)^2 \rceil$ + (24 others)
$(1,1)$	$\lfloor 1,((12)(34))(j_W)^3 \rceil$	$(1,2)$	$\lfloor 1,((12)(34))(j_W)^3 \rceil$ + (24 others)
$(2,2)$	$\lfloor 1,((12)(34))(j_W)^4 \rceil$	$(2,1)$	$\lfloor 1,((12)(34))(j_W)^4 \rceil$ + (24 others)
$(1,1)$	$\lfloor 1,((13)(24))j_W \rceil$	$(1,2)$	$\lfloor 1,((13)(24))j_W \rceil$ + (24 others)
$(2,2)$	$\lfloor 1,((13)(24))(j_W)^2 \rceil$	$(2,1)$	$\lfloor 1,((13)(24))(j_W)^2 \rceil$ + (24 others)
$(1,1)$	$\lfloor 1,((13)(24))(j_W)^3 \rceil$	$(1,2)$	$\lfloor 1,((13)(24))(j_W)^3 \rceil$ + (24 others)
$(2,2)$	$\lfloor 1,((13)(24))(j_W)^4 \rceil$	$(2,1)$	$\lfloor 1,((13)(24))(j_W)^4 \rceil$ + (24 others)
$(1,1)$	$\lfloor 1,((14)(23))j_W \rceil$	$(1,2)$	$\lfloor 1,((14)(23))j_W \rceil$ + (24 others)
$(2,2)$	$\lfloor 1,((14)(23))(j_W)^2 \rceil$	$(2,1)$	$\lfloor 1,((14)(23))(j_W)^2 \rceil$ + (24 others)
$(1,1)$	$\lfloor 1,((14)(23))(j_W)^3 \rceil$	$(1,2)$	$\lfloor 1,((14)(23))(j_W)^3 \rceil$ + (24 others)
$(2,2)$	$\lfloor 1,((14)(23))(j_W)^4 \rceil$	$(2,1)$	$\lfloor 1,((14)(23))(j_W)^4 \rceil$ + (24 others)
$(1,2)$	$\lfloor (x_1+x_2)(x_3+x_4),(12)(34) \rceil$	$(1,2)$	$\lfloor 1,((12)(34))j_W^2K \rceil$ + (99 others)
$(1,2)$	$\lfloor x_5^2,(12)(34) \rceil$	$(1,2)$	$\lfloor 1,((12)(34))j_W^2L \rceil$ + (99 others)
$(2,1)$	$\lfloor (x_1+x_2)^2(x_3+x_4)^2x_5^3,(12)(34) \rceil$	$(2,1)$	$\lfloor 1,((12)(34))j_W^3K \rceil$ + (99 others)
$(2,1)$	$\lfloor (x_1+x_2)^3(x_3+x_4)^3x_5,(12)(34) \rceil$	$(2,1)$	$\lfloor 1,((12)(34))j_W^3L \rceil$ + (99 others)
$(1,2)$	$\lfloor (x_1+x_3)(x_3+x_4),(13)(24) \rceil$	$(1,2)$	$\lfloor 1,((13)(24))j_W^2K \rceil$ + (99 others)
$(1,2)$	$\lfloor x_5^2,(13)(24) \rceil$	$(1,2)$	$\lfloor 1,((13)(24))j_W^2L \rceil$ + (99 others)
$(2,1)$	$\lfloor (x_1+x_3)^2(x_2+x_4)^2x_5^3,(13)(24) \rceil$	$(2,1)$	$\lfloor 1,((13)(24))j_W^3K \rceil$ + (99 others)
$(2,1)$	$\lfloor (x_1+x_3)^3(x_2+x_4)^3x_5,(13)(24) \rceil$	$(2,1)$	$\lfloor 1,((13)(24))j_W^3L \rceil$ + (99 others)
$(1,2)$	$\lfloor (x_1+x_4)(x_2+x_3),(14)(23) \rceil$	$(1,2)$	$\lfloor 1,((14)(23))j_W^2K \rceil$ + (99 others)
$(1,2)$	$\lfloor x_5^2,(14)(23) \rceil$	$(1,2)$	$\lfloor 1,((14)(23))j_W^2L \rceil$ + (99 others)
$(2,1)$	$\lfloor (x_1+x_4)^2(x_2+x_3)^2x_5^3,(14)(23) \rceil$	$(2,1)$	$\lfloor 1,((14)(23))j_W^3K \rceil$ + (99 others)
$(2,1)$	$\lfloor (x_1+x_4)^3(x_2+x_3)^3x_5,(14)(23) \rceil$	$(2,1)$	$\lfloor 1,((14)(23))j_W^3L \rceil$ + (99 others)

As we can see above, there is not the same number of $(1,1),(2,2),(1,2)$, and $(2,1)$ bigraded elements on either side. So although the restricted mirror map is always an isomorphism, the entire state spaces are not always isomorphic as bigraded vector spaces.

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