2018-04-01

A New Family of Topological Invariants

Nicholas Guy Larsen
Brigham Young University

A New Family of Topological Invariants

Nicholas Guy Larsen

A thesis submitted to the faculty of
Brigham Young University
in partial fulfillment of the requirements for the degree of

Master of Science

Gregory Conner, Chair
Curtis Kent
Eric Swenson

Department of Mathematics

Brigham Young University

ABSTRACT

A New Family of Topological Invariants

Nicholas Guy Larsen
Department of Mathematics, BYU
Master of Science

We define an extension of the nth homotopy group π_n which can distinguish a larger class of spaces. (E.g., a converging sequence of disjoint circles and the disjoint union of countably many circles, which have isomorphic fundamental groups, regardless of choice of basepoint.) We do this by introducing a generalization of homotopies, called component-homotopies, and defining the nth extended homotopy group to be the set of component-homotopy classes of maps from compact subsets of $(0, 1)^n$ into a space, with a concatenation operation.

We also introduce a method of tree-adjoinment for "connecting" disconnected metric spaces and show how this method can be used to calculate the extended homotopy groups of an arbitrary metric space.

Keywords: algebraic topology, homotopy, fundamental group

Acknowledgments

I am indebted to many people for assistance with this thesis, mathematical or otherwise. First I would like to express gratitude to my advisor, Dr. Greg Conner, for his encouragement and his sometimes inordinate confidence in me, as well as for convincing me to pursue a graduate degree in the first place. I would also like to thank Lonette Stoddard, the Mathematics Department Secretary, without whose knowledge and patience I would be totally helpless. I am grateful for the assistance of my committee, Drs. Greg Conner, Curt Kent, and Eric Swenson, in finding solutions to some unexpected difficulties encountered during the writing of this thesis. I would never have been able to get to where I am today without the support of my family, especially my parents and grandparents; I will be trying to repay them for the rest of my life. Finally, I am eternally grateful to my wife for her patience during the whole process of writing this thesis, and for pretending to be interested when I talked to her about homotopies.

Contents

Chapter 1. The Extended Homotopy Groups

1.1 Introduction

Consider the following subspaces of $\mathbb{R}^2$:

$$X = \bigcup_{i \in \mathbb{N}} \left\{ (x,y) \in \mathbb{R}^2 : d\left((x,y), \left(\frac{1}{2^{(i-1)}}, \frac{1}{2^{(i+1)}} \right) \right) = \frac{1}{2^{(i+1)}} \right\} \text{ and}$$

$$Y = X \cup \{(x,y) \in \mathbb{R}^2 : d((x,y),(-1,0)) = 1\}.$$

So X is a countable collection of circles and Y is the same collection of circles, converging to a point contained in another circle. It is easy to see that for any choice of x', we have $\pi_1(X,x') = \pi_1(Y,x') \cong \mathbb{Z}$.

Imagine that instead of finding the path-homotopy classes of maps $(I, \partial I) \to (X, x')$ or $(I, \partial I) \to (Y, x')$, we were to take a compact subset of the unit interval I with possibly infinitely many components and map each of these as a loop into either X or Y. It is clear with some consideration that we could construct such a map onto Y that is surjective, while this is impossible for X.

The motivation for this thesis is to formalize this idea in order to define a family of topological invariants which extend the homotopy groups (i.e., contain subgroups that are isomorphic to the homotopy groups) while distinguishing a larger class of spaces.

1.2 Component-homotopies

Let X be a topological space and let $\{X_i\}_{i \in J}$ be the set of path components of X, indexed by some set J. For each $i \in J$, fix $x_i \in X_i$. For $n \geq 2$, let $I_t^{n-1} = \{\overline{x} \in I^n : x_n = t\}$, where I is the unit interval $[0,1]$.

For $n \in \mathbb{N}$, let $\mathcal{U}_n$ be the collection of nonempty open subsets of $(0,1)^n$. For each $U \in \mathcal{U}_n$, let ∂U denote the boundary of U in I^n. Now define $R_n(X, \{x_i\})$ to be the set of continuous maps from elements U of $\mathcal{U}_n$ into X mapping ∂U into $\{x_i\}$:

$$R_n(X, \{x_i\}) = \{\phi : (U, \partial U) \to (X, \{x_i\}) : U \in \mathcal{U}_n\}.$$

Notice that an element of R_n can be thought of as many "simultaneous" maps of I^n into X, each sending ∂I^n to some element of $\{x_i\}$.

For simplicity, we may refer to $R_n(X, \{x_i\})$ as R_n or $R_n(X)$ when the choices of X and/or $\{x_i\}$ are clear from context.

Let b be the self-homeomorphism of I^n defined by

$$(x_1, \dots, x_n) \mapsto (1 - x_1, x_2, \dots, x_n).$$

For each $U \in \mathcal{U}_n$, let U_- denote $b(U)$. By Lemma A.1.1, $\partial(U_-) = (\partial U)_-$, so we will use the notation ∂U_- to refer to both. For each $\phi : (U, \partial U) \to (X, \{x_i\}) \in R_n$, define $\overline{\phi}$ as $\phi \circ b$, which we will call the reverse of ϕ. We will show that reverses of elements of R_n are also elements of R_n.

Claim 1.2.1. For each $\phi \in R_n(X, \{x_i\})$, we have $\overline{\phi} \in R_n(X, \{x_i\})$ as well.

Proof. Let $\phi : (U, \partial U) \to (X, \{x_i\}) \in R_n$ be arbitrary. We will show that $\overline{\phi} : (U_-, \partial U_-) \to (X, \{x_i\}) \in R_n$.

Since b is a reflection across $\{\overline{x} \in \mathbb{R}^n : x_1 = 1/2\}$, we have that $U_- = b(U) \in \mathcal{U}_n$. Also since $\overline{\phi} = \phi \circ b$ and ϕ is assumed to be continuous, $\overline{\phi}$ must be continuous.

Let $\overline{x} \in \partial(U_-)$. As we have seen, $\partial(U_-) = (\partial U)_-$, so $b(\overline{x}) \in \partial U$. Then

$$\phi(b(\overline{x})) = \overline{\phi}(\overline{x}) \in \{x_i\},$$

which shows that $\overline{\phi} : (U_-, \partial U_-) \to (X, \{x_i\}) \in R_n(X, \{x_i\})$, proving the claim. $\square$

Consider the following homeomorphisms of $\mathbb{R}^n$:

$$\begin{aligned} \ell(x_1, \ldots, x_n) &= \left(\frac{x_1}{3}, x_2, \ldots, x_n\right) \\ r(x_1, \ldots, x_n) &= \left(\frac{x_1 + 2}{3}, x_2, \ldots, x_n\right). \end{aligned}$$

Notice that ℓ and r have the following properties:

1. $\partial\ell(U) = \ell(\partial U)$ and $\partial r(U) = r(\partial U)$ (by Lemma A.1.1),

2. for $U \in \mathcal{U}_n$, $\ell(U), r(U) \in \mathcal{U}_n$,

3. for any $U, V \in \mathcal{U}_n$, $\ell(U) \cap r(V) = \emptyset$ and $\partial(\ell(U) \sqcup r(V)) = \partial\ell(U) \sqcup \partial r(V)$. Also $\ell(U) \cup \partial\ell(U)$ and $r(V) \cup \partial r(V)$ are disjoint closed sets.

We define a binary operation $*$ on $R_n(X, \{x_i\})$. Let $\phi : (U, \partial U) \to X$ and $\psi : (V, \partial V) \to (X, \{x_i\})$ in R_n be arbitrary. Let $W = \ell(U) \cup r(V)$. Define $\phi * \psi : (W, \partial W) \to (X, \{x_i\})$ by

$$(\phi * \psi)(x_1, \ldots, x_n) = \begin{cases} \phi(3x_1, x_2, \ldots, x_n) & (x_1, \ldots, x_n) \in \ell(U) \cup \partial\ell(U) \\ \psi(3x_1 - 2, x_2, \ldots, x_n) & (x_1, \ldots, x_n) \in r(V) \cup \partial r(V). \end{cases}$$

Equivalently,

$$(\phi * \psi)(\overline{x}) = \begin{cases} \phi(\ell^{-1}(\overline{x})) & (x_1, \ldots, x_n) \in \ell(U) \cup \partial\ell(U) \\ \psi(r^{-1}(\overline{x})) & (x_1, \ldots, x_n) \in r(V) \cup \partial r(V). \end{cases}$$

Now we will show that R_n is closed under $*$.

Claim 1.2.2. If $\phi, \psi \in R_n(X, \{x_i\})$, then $\phi * \psi \in R_n(X, \{x_i\})$.

Proof. Let $\phi : (U, \partial U) \to (X, \{x_i\})$ and $\psi : (V, \partial V) \to (X, \{x_i\})$ in R_n be arbitrary, and set $W = \ell(U) \cup r(V)$.

By property 2 of ℓ and r, $W = \ell(U) \cup r(V) \in \mathcal{U}_n$.

Recall that ϕ is continuous by assumption and that ℓ is a homeomorphism. Then since $(\phi * \psi) = \phi \circ \ell^{-1}$ on $\ell(U) \cup \partial\ell(U)$, $\phi * \psi$ is continuous on $\ell(U) \cup \partial\ell(U)$. Similarly, $\phi * \psi$ is continuous on $r(V) \cup \partial r(V)$. Therefore $\phi * \psi$ is continuous by the pasting lemma, noting that $\ell(U) \cup \partial\ell(U)$ and $r(V) \cup \partial r(V)$ are disjoint and closed.

Let $\overline{x} \in \partial W$. By property 3 of ℓ and r, $\overline{x}$ is contained in exactly one of $\partial\ell(U)$ and $\partial r(V)$. Suppose that $\overline{x} \in \partial\ell(U) = \ell(\partial U)$. Then $\overline{x} = \ell(\overline{y})$ for some $\overline{y} \in \partial U$, and

$$(\phi * \psi)(\overline{x}) = (\phi * \psi)(\ell(\overline{y})) = \phi(\ell^{-1}(\ell(\overline{y}))) = \phi(\overline{y}) \in \{x_i\}.$$

Similarly, if $\overline{x} \in \partial r(V)$, $(\phi * \psi)(\overline{x}) \in \{x_i\}$. Then $(\phi * \psi)(\partial W) \subseteq \{x_i\}$, proving the claim. $\square$

With the reverses of elements of R_n and the operation $*$, we are close to showing that R_n is a group. However, it is easy to see that $*$ fails to be associative. In order for this to be the case, we must define an equivalence relation $\sim$ on R_n. For $\phi : (U, \partial U) \to (X, \{x_i\}), \psi : (V, \partial V) \to (X, \{x_i\}) \in R_n$, we say that $\phi \sim \psi$ if there exists an open subset O of I^{n+1} and a continuous function $h : (O, \partial O) \to (X, \{x_i\})$ satisfying the following:

1. $O \cap I_0^n = U \times \{0\}$,
2. $O \cap I_1^n = V \times \{1\}$,
3. $h(\cdot, 0)|_{U \times \{0\}} = \phi$, and
4. $h(\cdot, 1)|_{V \times \{1\}} = \psi$,

where $I_t^n = \{\overline{x} \in I^{n+1} : x_{n+1} = t\} \cong I^n$ for $t \in I$.

Notice that in the special case where $U = V = (0,1)^n$, $\phi \sim \psi$ exactly when ϕ and ψ are homotopic, with h as a homotopy. For this reason, when $\phi \sim \psi$ we will say that ϕ and ψ are component-homotopic and that h is a component-homotopy between ϕ and ψ.

Claim 1.2.3. The relation $\sim$ is an equivalence relation.

Proof. We must show that $\sim$ is reflexive, symmetric, and transitive.

Claim 1.2.4. $\sim$ is reflexive.

Subproof. Let $\phi : (U, \partial U) \to (X, \{x_i\})$ be arbitrary. Let $O = U \times I$ and define $h : (O, \partial O)$ by

$$h(\overline{x}, t) = \phi(\overline{x}).$$

Then clearly $\phi \sim \phi$, proving the claim. $\square$

Claim 1.2.5. $\sim$ is symmetric.

Subproof. Suppose that $\phi \sim \psi$, where $\phi : (U, \partial U) \to (X, \{x_i\})$ and $\psi : (V, \partial V) \to (X, \{x_i\})$ are elements of R_n. Then by definition, there exists an open subset O of I^{n+1} and a component-homotopy $h : (O, \partial O) \to (X, \{x_i\})$ between ϕ and ψ. Define $O' = b_{n+1}(O)$ and $h' = h \circ b_{n+1}$. Then it is easy to see that h' is a component-homotopy between ψ and ϕ, proving the claim. $\square$

Claim 1.2.6. $\sim$ is transitive.

Subproof. Suppose that $\phi \sim \psi$ and $\psi \sim \theta$, where $\phi : (U, \partial U) \to (X, \{x_i\})$, $\psi : (V, \partial V) \to (X, \{x_i\})$, and $\theta : (W, \partial W) \to (X, \{x_i\})$ are elements of R_n. Then there exist $O_1 \subseteq I^{n+1}$ and $h_1 : (O_1, \partial O_1) \to (X, \{x_i\})$ satisfying the conditions of $\sim$ to show that $\phi \sim \psi$ and $O_2 \subseteq I^{n+1}$ and $h_2 : (O_2, \partial O_2) \to (X, \{x_i\})$ showing that $\psi \sim \theta$.

Let $O = \{(x_1, \ldots, x_n, x_{n+1}/2) : \overline{x} \in O_1\} \cup \{(x_1, \ldots, x_n, x_{n+1}/2 + 1/2) : \overline{x} \in O_2\}$ and define $h : (O, \partial O) \to (X, \{x_i\})$ by

$$h(\overline{x}) = \begin{cases} h_1(x_1, \ldots, x_n, 2x_{n+1}) & x_{n+1} \in [0, 1/2] \\ h_2(x_1, \ldots, x_n, 2x_{n+1} - 1) & x_{n+1} \in [1/2, 1]. \end{cases}$$

Then h is a component-homotopy between ϕ and θ, proving the claim. $\square$

Since $\sim$ is reflexive, symmetric, and transitive, it is an equivalence relation. $\square$

Now we define

$$\rho_n(X, \{x_i\}) = R_n(X, \{x_i\})/\sim,$$

so $\rho_n(X, \{x_i\})$ is the set of component-homotopy classes of maps $(U, \partial U) \to (X, \{x_i\})$ for $U \in \mathcal{U}_n$. For simplicity we may refer to $\rho_n(X, \{x_i\})$ as ρ_n or $\rho_n(X)$ when the choice of X and/or $\{x_i\}$ is clear from context, as we did with R_n.

1.3 The extended homotopy groups

Recall that the obstacle to showing that R_n is a group under $*$ was that $*$ is not associative on elements of R_n. However, we can show that $*$ is associative on component-homotopy classes, and therefore that ρ_n is in fact a group. It is worth noting that this proof is very similar to the proof that π_n is a group, with the only significant differences coming from the fact that the operation $*$ compresses the first coordinate by a factor of three, while the operation on π_n compresses the first coordinate by a factor of two.

Claim 1.3.1. The set $\rho_n(X, \{x_i\})$ has a group structure under $*$, defined by $[\phi]*[\psi] = [\phi*\psi]$ for $[\phi], [\psi] \in \rho_n(X, \{x_i\})$.

Proof. First we must show that $*$ is well-defined with respect to $\sim$.

Claim 1.3.2. The operation $*$ on $\rho_n(X, \{x_i\})$ is well-defined with respect to $\sim$.

Subproof. Let $[\phi], [\psi] \in \rho_n(X, \{x_i\})$ be arbitrary. Let $\phi_1 : (U_1, \partial U_1) \to (X, \{x_i\})$ and $\phi_2 : (U_2, \partial U_2) \to (X, \{x_i\})$ be representatives of $[\phi]$ and let $\psi_1 : (V_1, \partial V_1) \to (X, \{x_i\})$ and $\psi_2 : (V_2, \partial V_2) \to (X, \{x_i\})$ be representatives of $[\psi]$. To show that $*$ is well-defined, we must show that $[\phi_1 * \psi_1] = [\phi_2 * \psi_2]$, or equivalently,

$$(\phi_1 * \psi_1) \sim (\phi_2 * \psi_2).$$

Since $\phi_1, \phi_2 \in [\phi]$ and $\psi_1, \psi_2 \in [\psi]$, $\phi_1 \sim \phi_2$ and $\psi_1 \sim \psi_2$. Then there exists some open $O_\phi \subseteq I^{n+1}$ and $h_\phi : (O_\phi, \partial O_\phi) \to (X, \{x_i\})$, and some open $O_\psi \subseteq I^{n+1}$ and $h_\psi : (O_\psi, \partial O_\psi) \to$

$(X, \{x_i\})$ satisfying the properties of $\sim$ to show that $\phi_1 \sim \phi_2$ and $\psi_1 \sim \psi_2$, respectively.

Let $O = \ell(O_\phi) \cup r(O_\psi)$ and define $h : (O, \partial O) \to (X, \{x_i\})$ by

$$h(\overline{x}) = \begin{cases} h_\phi(\ell^{-1}(\overline{x})) & \overline{x} \in \ell(O_\phi) \\ h_\psi(r^{-1}(\overline{x})) & \overline{x} \in r(O_\psi). \end{cases}$$

Then h is a component-homotopy, proving the claim. $\square$

Claim 1.3.3. $\rho_n(X, \{x_i\})$ is closed under $*$.

Subproof. This follows immediately from Claim 2. $\square$

Claim 1.3.4. $\rho_n(X, \{x_i\})$ has an identity element under $*$.

Subproof. Fix $x' \in \{x_i\}$. Let $e_{x'} : (I^n, \partial I^n) \to (X, \{x_i\})$ be the constant map to x'. Then clearly $[e_{x'}] \in \rho_n$. We will show that $[e_{x'}]$ is an identity element of ρ_n. It suffices to show that for each $\phi \in R_n$, $(\phi * e_{x'}) \sim \phi$.

Let $\phi : (U, \partial U) \to (X, \{x_i\})$ be arbitrary. Then since $\sim$ is reflexive, there exists an open set $O_\phi \subseteq I^{n+1}$ and a component-homotopy $h_\phi : (O_\phi, \partial O_\phi) \to (X, \{x_i\})$ between ϕ and itself. Let k be the homeomorphism of I^{n+1} defined by

$$k(x_1, \ldots, x_n, t) = \left(\frac{2t+1}{3}(x_1), x_2, \ldots, x_n, t \right).$$

(Notice that $k(x_1, \ldots, x_n, 0) = \ell(x_1, \ldots, x_n, 0)$ and $k(x_1, \ldots, x_n, 1) = (x_1, \ldots, x_n, 1)$.) Let $O_{x'}$ be an open subset of I^{n+1} such that $O_{x'} \cap I_0^n = r(I^n)$, and $x_1 > 2/3$ and $x_{n+1} < 1/9$ for all $\overline{x} \in O_{x'}$. Define $O = k(O_\phi) \cup O_{x'}$ and $h : (O, \partial O) \to (X, \{x_i\})$ by

$$h(\overline{x}) = \begin{cases} \phi(k^{-1}(\overline{x})) & \overline{x} \in k(O_\phi) \\ x' & \overline{x} \in O_{x'}. \end{cases}$$

Then h is a component-homotopy between $(\phi * e_{x'})$ and ϕ, which proves the claim. $\square$

Claim 1.3.5. Each element of $\rho_n(X,\{x_i\})$ has a two-sided inverse under $*$.

Subproof. Let $[\phi] \in \rho_n(X,\{x_i\})$ be arbitrary. Let $\phi : (U,\partial U) \to (X,\{x_i\})$ be a representative of $[\phi]$. By Claim 1, $\overline{\phi} : (U_-,\partial U_-) \to (X,\{x_i\}) \in R_n$. We will show that $[\overline{\phi}]$ is a two-sided inverse for $[\phi]$. It suffices to show that $(\phi * \overline{\phi}) \sim e_{x'}$ for some $x' \in \{x_i\}$, since $\overline{(\overline{\phi})} = \phi$.

First notice that since $\phi \sim \phi$, there exists an open set $O_\phi \subseteq I^{n+1}$ and a component-homotopy $h_\phi : (O_\phi, \partial O_\phi) \to (X,\{x_i\})$ between ϕ and itself. Let $O_{x'} = \{\overline{x} \in I^{n+1} : 0 < x_1 < 1, x_{n+1} \geq 2/3\}$. Let σ be a homeomorphism from I^{n+1} to $\{\overline{x} \in I^{n+1} : x_{n+1} \leq 1/3\}$ such that $\sigma(\text{int}(I_0^n)) = \ell(\text{int}(I_0^n)$ and $\sigma(\text{int}(I_1^n)) = r(\text{int}(I_0^n))$. Let $O = \sigma(O_\phi) \cup O_{x'}$ and define $h : (O,\partial O) \to (X,\{x_i\})$ by

$$h(\overline{x}) = \begin{cases} h_\phi(\sigma^{-1}(\overline{x})) & \overline{x} \in \sigma(O_\phi) \\ x' & \overline{x} \in O_{x'}. \end{cases}$$

Then h is a component-homotopy between $(\phi * \overline{\phi})$ and $e_{x'}$, proving the claim. $\square$

Claim 1.3.6. The operation $*$ on $\rho_n(X,\{x_i\})$ is associative.

Subproof. Let $[\phi], [\psi], [\theta] \in \rho_n(X,\{x_i\})$. Let $\phi : (U,\partial U) \to (X,\{x_i\})$, $\psi : (V,\partial V) \to (X,\{x_i\})$, $\theta : (W,\partial W) \to (X,\{x_i\})$ be representatives of these classes respectively. We must show that $([\phi] * [\psi]) * [\theta] = [\phi] * ([\psi] * [\theta])$, or equivalently, that $(\phi * \psi) * \theta \sim \phi * (\psi * \theta)$. This means that we must show that there exists some open $O \subseteq I^{n+1}$ and a component-homotopy $h : (O,\partial O) \to (X,\{x_i\})$ between $(\phi * \psi) * \theta$ and $\phi * (\psi * \theta)$.

Since $\sim$ is reflexive, there exist open sets O_ϕ, O_ψ, O_θ and functions h_ϕ, h_ψ, h_θ satisfying the conditions of $\sim$ to show that $\phi \sim \phi$, $\psi \sim \psi$, and $\theta \sim \theta$, respectively. Define σ_ϕ, σ_ψ, and

σ_θ to be homeomorphisms of $U \times I$, $V \times I$, and $W \times I$, respectively, such that

$$\sigma_\phi(x_1, \ldots, x_n, t) = \left(\frac{2t+1}{9}(x_1), x_2, \ldots, x_n, t\right)$$
$$\sigma_\psi(x_1, \ldots, x_n, t) = \left(\frac{x_1+4t+2}{9}, x_2, \ldots, x_n, t\right)$$
$$\sigma_\theta(x_1, \ldots, x_n, t) = \left(\frac{(3-t)x_1+2t}{9}, x_2, \ldots, x_n, t\right).$$

Then notice that $\sigma_\phi(U \times I) \cap I_0^n = \ell^2(U)$, $\sigma_\phi(U \times I) \cap I_1^n = \ell(U)$, $\sigma_\psi(V \times I) \cap I_0^n = \ell(r(V))$, $\sigma_\psi(V \times I) \cap I_1^n = r(\ell(V))$, $\sigma_\theta(W \times I) \cap I_0^n = r(W)$, and $\sigma_\theta(W \times I) \cap I_1^n = r^2(W)$. Let $O = \sigma_\phi(U \times I) \cup \sigma_\psi(V \times I) \cup \sigma_\theta(W \times I)$ and define $h : (O, \partial O) \to (X, \{x_i\})$ by

$$h(\overline{x}) = \begin{cases} h_\phi(\sigma_\phi^{-1}(\overline{x})) & \overline{x} \in \sigma_\phi(U \times I) \\ h_\psi(\sigma_\psi^{-1}(\overline{x})) & \overline{x} \in \sigma_\psi(V \times I) \\ h_\theta(\sigma_\theta^{-1}(\overline{x})) & \overline{x} \in \sigma_\theta(W \times I). \end{cases}$$

Then h is a component-homotopy between $(\phi * \psi) * \theta$ and $\phi * (\psi * \theta)$, proving the claim. □

Since $*$ is a well-defined, associative operation under which ρ_n is closed and has two-sided inverses, ρ_n is a group under $*$. □

Chapter 2. Properties of the Extended Homotopy Groups

2.1 Relationship to homotopy groups

Now we show that ρ_n is in fact an extension of π_n by showing that π_n can be isomorphically embedded in ρ_n.

Claim 2.1.1. The homotopy group $\pi_n(X, x')$ is isomorphic to a subgroup of $\rho_n(X, \{x_i\})$.

Proof. Notice that since x' is contained in the same path component as some element of $\{x_i\}$, we can assume without loss of generality that $x' \in \{x_i\}$.

For an arbitrary $\phi : (I^n, \partial I^n) \to (X, x')$, let $[\phi]_{\pi_n}$ denote the homotopy class of ϕ (which is an element of π_n) and $[\phi]_{\rho_n}$ the component-homotopy class of ϕ (which is an element of ρ_n).

Choose $[\phi] \in \pi_n(X, x')$ and let $\phi \in [\phi]$. Then ϕ is a continuous map $(I^n, \partial I^n) \to (X, x')$, which means that $\phi \in R_n(X, \{x_i\})$. Define $f : \pi_n(X, x') \to \rho_n(X, \{x_i\})$ by $f([\phi]_{\pi_n}) = [\phi]_{\rho_n}$.

Suppose that $[\phi]_{\pi_n}, [\psi]_{\pi_n} \in \pi_n(X, x')$. To avoid confusion, we will use $\star$ to denote the operation in $\pi_n(X, x')$ and $*$ to denote the operation that we have defined previously on $\rho_n(X, \{x_i\})$. We will use the standard definition of $\star$:

$$(\phi \star \psi)(\overline{x}) = \begin{cases} \phi(2x_1, x_2, \ldots, x_n) & x_1 \in [0, 1/2] \\ \psi(2x_1 - 1, x_2, \ldots, x_n) & x_1 \in [1/2, 1]. \end{cases}$$

We claim that f is a homomorphism. We must show that

$$f([\phi \star \psi]_{\pi_n}) = f([\phi]_{\pi_n}) * f([\psi]_{\pi_n})$$
$$[\phi \star \psi]_{\rho_n} = [\phi]_{\rho_n} * [\psi]_{\rho_n}$$
$$(\phi \star \psi) \sim (\phi * \psi).$$

Since $\sim$ is reflexive, there exists an open $O_\phi \subseteq I^{n+1}$ and a component-homotopy $h_\phi : (O_\phi, \partial O_\phi) \to (X, x')$ between ϕ and itself. Similarly, there exists an open $O_\psi \subseteq I^{n+1}$ and a component-homotopy $h_\psi : (O_\psi, \partial O_\psi) \to (X, x')$ between ψ and itself. Let σ_ϕ and σ_ψ be homeomorphisms of I^{n+1} such that

$$\sigma_\phi(x_1, \ldots, x_n, t) = \left(\frac{(3-t)x_1}{6}, x_2, \ldots, x_n, t\right)$$
$$\sigma_\psi(x_1, \ldots, x_n, t) = \left(\frac{(3-t)x_1 + (3+t)}{6}, x_2, \ldots, x_n, t\right).$$

Then notice that $\sigma_\phi(I^{n+1}) \cap I_0^n = \{(x_1/2, x_2, \ldots, x_n) : \overline{x} \in I^n\}$, $\sigma_\phi(I^{n+1}) \cap I_1^n = \ell(I^n)$, $\sigma_\psi(I^{n+1}) \cap I_0^n = \{x_1/2 + 1/2, x_2, \ldots, x_n) : \overline{x} \in I^n\}$, and $\sigma_\psi(I^{n+1}) \cap I_1^n = r(I^n)$. Let $O = \sigma_\phi(O_\phi) \cup \sigma_\psi(O_\psi)$ and define $h : (O, \partial O) \to (X, \{x_i\})$ by

$$h(\overline{x}) = \begin{cases} h_\phi(\sigma_\phi^{-1}(\overline{x})) & \overline{x} \in \sigma_\phi(O_\phi) \\ h_\psi(\sigma_\psi^{-1}(\overline{x})) & \overline{x} \in \sigma_\psi(O_\psi). \end{cases}$$

Then h is a component-homotopy between $\phi \star \psi$ and $\phi * \psi$, proving that f is a homomorphism.

Now suppose that $f([\phi])$ is the identity element of $\rho_n(X, \{x_i\})$ for some $[\phi] \in \pi_n(X, x')$, $\phi : (I^n, \partial I^n) \to (X, x')$. Then $\phi \sim e_{x'}$, where $e_{x'} : (I^n, \partial I^n) \to (X, x')$ is the constant map to x'. By definition of $\sim$, there exists an open $O \subseteq I^{n+1}$ and a component-homotopy $h : (O, \partial O) \to (X, x')$ between ϕ and $e_{x'}$. Notice that h is an extension of $\phi : (I^n, \partial I^n) \to (X, x')$ to a map $(\text{int}(I^{n+1}), \partial I^{n+1}) \to (X, x')$, which means that $[\phi]$ is trivial in π_n and therefore f is injective.

Since f is an injective homomorphism, we have that

$$\pi_n(X, x') \cong f(\pi_n(X, x')) \leq \rho_n(X, \{x_i\}). \qquad \square$$

It follows easily from this proof that when X is path-connected, $\rho_n(X) \cong \pi_n(X)$.

2.2 Tree-connected spaces

Now we introduce a method for calculating $\rho_n(X, \{x_i\})$ when X is a metric space. Suppose that X is a metric space, with metric d_X. Further suppose that the set of basepoints $\{x_i\}_{i \in J}$

is a closed subset of X. We define $T(X,\{x_i\})$ to be the quotient space of

$$\left(\bigsqcup_{i\in J}[0,1]_i\right)\cup X$$

formed by

1. $0_j \sim 0_k$ for all $j,k\in J$

2. $1_i \sim x_i$ for all $i\in J$

3. $t_j \sim t_k$ whenever $t\le 1-\frac{1}{2}d_X(x_j,x_k)$.

Let K denote the "tree" part of $T(X,\{x_i\})$; i.e., $K=T(X,\{x_i\})\setminus(X\setminus\{x_i\})$. Then K is a metric space under the shortest-path metric d_K. Also notice that we can replace d_X with the topologically equivalent metric d'_X defined by

$$d'_X(p_i,p_j)=\min\{d_X(p_i,p_j),d_X(p_i,x_i)+2+d_X(x_j,p_j)\}$$

for $p_i\in X_i, p_j\in X_j$. Then it is easy to see that $T(X,\{x_i\})$ is a path-connected metric space, with a metric d which agrees exactly with d'_X on X and d_K on K. We claim that we can calculate the nth extended homotopy group of X by finding the nth homotopy group of $T(X)$.

Claim 2.2.1. $\pi_n(T(X,\{x_i\}),x')\cong\rho_n(X,\{x_i\})$.

Proof. As before, we will let $[\phi]_{\pi_n}$ denote an element of π_n and $[\psi]_{\rho_n}$ denote an element of ρ_n, where ϕ and ψ satisfy the necessary conditions. Also since $T(X,\{x_i\})$ is path-connected, $\pi_n(T(X,\{x_i\}),x')$ is independent (up to isomorphism) of the choice of basepoint, so we may suppose without loss of generality that $x'\in\{x_i\}$.

Let $[\phi]_{\pi_n}\in\pi_n$. Choose $\phi:(I^n,\partial I^n)\to(X,x')\in[\phi]_{\pi_n}$. Notice that X is closed in $T(X,\{x_i\})$, since $T(X)\backslash X=\bigcup_{i\in J}[0,1)_i$ is open. Also $X\backslash\{x_i\}$ is open. Let $U=\phi^{-1}(X\backslash\{x_i\})$

and define $\phi' = \phi|_{(U \cup \partial U)}$. Then we claim that $\phi' : (U, \partial U) \to (X, \{x_i\}) \in R_1$ (and so $[\phi']_{\rho_n} \in \rho_n$).

We have that $\phi'(U) \subseteq X$ by definition. Since ϕ is continuous and $X - \{x_i\}$ is open, so is U, so $U \in \mathcal{U}_1$. Also since ϕ is continuous, so is ϕ'. To see that $\phi'(\partial U) \subseteq \{x_i\}$, let $x \in \partial U$. Then by definition there exists a sequence $\{s_\alpha\}$ in U that converges to x. Since U is open and $x \in \partial U$, then $x \notin U$. Since X is closed and $\phi'(\{x_i\}) \subseteq X$, then $\phi'(x) \in X$, since ϕ' is continuous. Since $U = \phi^{-1}(X \setminus \{x_i\})$ and $x \notin U$, then $\phi'(x) \notin X \setminus \{x_i\}$. Then it must be the case that $\phi'(x) \in \{x_i\}$ and therefore $\phi'(\partial U) \subseteq \{x_i\}$.

Now define $f : \pi_n(T(X, \{x_i\}), x') \to \rho_n(X, \{x_i\})$ by $f([\phi]_{\pi_n}) = [\phi']_{\rho_n}$. We claim that f is an isomorphism.

Claim 2.2.2. f is well-defined.

Subproof. Suppose that $[\phi_1]_{\pi_n} = [\phi_2]_{\pi_n}$, where ϕ_1 and ϕ_2 are maps $(I^n, \partial I^n) \to (T(X, \{x_i\}), x')$. Then there exists a homotopy $h : (I^{n+1}, \partial I^{n+1}) \to (X, x')$ between ϕ_1 and ϕ_2. Let $O = h^{-1}(X \setminus \{x_i\})$, which is an open subset of I^{n+1}. It is easy to see that $h' = h|_{(O \cup \partial O)}$ is a component-homotopy between ϕ_1' and ϕ_2', so $f([\phi_1]_{\pi_n}) = [\phi_1']_{\rho_n} = [\phi_2']_{\rho_n} = f([\phi_2]_{\pi_n})$ and f is well-defined. □

Claim 2.2.3. f is injective.

Subproof. Suppose that $[f(\phi)]_{\rho_n} = [f(\psi)]_{\rho_n}$, where ϕ and ψ are maps $(I^n, \partial I^n) \to (T(X, \{x_i\}), x')$. Then there exists a component-homotopy $h : (O, \partial O) \to (X, \{x_i\})$ between $f(\phi)$ and $f(\psi)$. Since our goal is to show that $[\phi]_{\pi_n} = [\psi]_{\pi_n}$, if we find a homotopy between ϕ and ψ then we are done.

Let $W = \mathrm{cl}(O) \cup \partial I^{n+1}$. Notice that we can extend h continuously to a map h_1 on W by defining $h_1 = \phi(x)$ on I_0^n, $h_1 = \psi(x)$ on I_1^n, and $h_1 = x'$ otherwise. Now we must show that we can extend h_1 continuously to the rest of I^{n+1}.

Since it is a finite union of closed sets, W is closed, so $\partial W \subseteq W$. Consider $W^c = I^{n+1} \backslash W$. By definition, $\partial W = \partial W^c$, so $h_1(\partial W^c)$ is defined. We wish to show that $h_1(\partial W^c) = h_1(\partial W) \subseteq K$.

Recall that $\partial W^c = \partial W \subseteq \partial \mathrm{cl}(O) \cup \partial I^2$ and let $x \in W^c$ be arbitrary. If $x \in \partial \mathrm{cl}(O)$, then by definition $h_1(x) \in \{x_i\}$, so $h_1(x) \in K$. Then suppose that $x \in \partial I^{n+1} \cap W^c$. If $x \notin I_0^n \cup I_1^n$, then $h_1(x) = x' \in \{x_i\}$ by construction, so suppose that $x \in I_0^n \cup I_1^n$, so either $h_1(x) = \phi(x)$ or $h_1(x) = \psi(x)$. Also notice that clearly $\partial W^c \cap O = \emptyset$, since $O \subseteq W$, so we know that $x \notin O$. Let $U_\phi = \phi^{-1}(X \setminus \{x_i\})$, so that we have $f(\phi) : (U_\phi, \partial U_\phi) \to (T(X, \{x_i\}), x')$. Define U_ψ similarly. Recall that by definition, $O \cap \partial I_0^n = U_\phi$ and $O \cap \partial I_1^n = U_\psi$, so since $x \notin O$, then $x \notin U_\phi$ and $x \notin U_\psi$. Since by definition, $U_\phi = \phi^{-1}(X \setminus \{x_i\})$ and $U_\psi = \psi^{-1}(X \setminus \{x_i\})$, $h_1(x) \notin (X \setminus \{x_i\})$. But since $K = T(X, \{x_i\}) \setminus (X \setminus \{x_i\})$, this means that $h_1(x) \in K$.

Let $h_2 : \partial W^c \to K$ be defined as $h_1|_{\partial W^c}$. Notice that ∂W^c is a closed subset of the normal space W^c (since $W^c \subseteq I^{n+1}$). Since K is an absolute retract for normal spaces, h_2 can be extended continuously to a map $h_3 : \mathrm{cl}(W^c) \to K$. [1] Define $h' : I^{n+1} \to T(X, \{x_i\})$ by

$$h'(x) = \begin{cases} h_1(x) & x \in \mathrm{cl}(W) \\ h_3(x) & x \in \mathrm{cl}(W^c). \end{cases}$$

By construction, h_1 and h_3 agree on the intersection of their domains, $\partial W = \partial W^c$. Then by the pasting lemma, h' is continuous and therefore h' is a homotopy between ϕ and ψ, proving the claim. $\square$

Claim 2.2.4. f is surjective.

Subproof. Let $[\phi]_{\rho_n}$ be arbitrary. Choose $\phi : (U, \partial U) \to (X, \{x_i\}) \in [\phi]_{\rho_n}$ We want to show that there exists $[\psi]_{\pi_n}$ and $\psi \in [\psi]_{\pi_n}$ such that $\psi|_U = \phi$, so that $f([\psi]_{\pi_n}) = [\phi]_{\rho_n}$.

Let $V = I^n \setminus \mathrm{cl}(U)$ and let V_α be a component of V. Notice that ϕ is defined on $\partial V = \partial U$ and that $\phi(\partial V) \subseteq \{x_i\}$.

Define $\phi' : \partial V \cup \partial I^n \to K$ by $\phi'(0, a_2, \ldots, a_n) = \phi'(1, b_2, \ldots, b_n) = x'$ and $\phi'(x) = \phi(x)$ otherwise. Then since $\partial V \cup \partial I^n$ is a closed subspace of V and K is an absolute retract for normal spaces, ϕ' extends continuously to a map $\phi'' : V \to K$. Define $\psi : (I^n, \partial I^n) \to (T(X, \{x_i\}), x')$ by

$$\psi(x) = \begin{cases} \phi(x) & x \in \mathrm{cl}(U) \\ \phi''(x) & x \in \mathrm{cl}(V). \end{cases}$$

Since $\mathrm{cl}(U) \cap \mathrm{cl}(V) = \partial U = \partial V$ and $\phi(\partial U) = \phi''(\partial U)$ by construction, ψ is continuous by the pasting lemma. Then $[\psi] \in \pi_n(T(X, \{x_i\}), x')$ and $f([\psi]_{\pi_n}) = [\phi]_{\rho_n}$, which proves the claim. □

Claim 2.2.5. *f is a homomorphism.*

Subproof. Let $[\phi]_{\pi_n}$ and $[\psi]_{\pi_n}$ be arbitrary. Letting $\star$ denote the usual operation in π_n, it suffices to show that

$$\begin{aligned} f([\phi]_{\pi_n} \star [\psi]_{\pi_n}) &= f([\phi]_{\pi_n}) * f([\psi]_{\pi_n}) \\ f([\phi \star \psi]_{\pi_n}) &= f([\phi]_{\pi_n}) * f([\psi]_{\pi_n}) \\ [(\phi \star \psi)']_{\rho_n} &= [\phi']_{\rho_n} * [\psi']_{\rho_n} \\ (\phi \star \psi)' &\sim \phi' * \psi'. \end{aligned}$$

Define $\theta_0, \theta_1 : (I^n, \partial I^n) \to (T(X, \{x_i\}), x')$ as follows:

$$\theta_0(a_1, \ldots, a_n) = (\phi \star \psi)(a_1, \ldots, a_n)$$

$$\theta_1(a_1, \ldots, a_n) = \begin{cases} \phi(3a_1, a_2, \ldots, a_n) & a_1 \in [0, 1/3] \\ x' & a_1 \in [1/3, 2/3] \\ \psi(3a_1 - 2, a_2, \ldots, a_n) & a_1 \in [2/3, 1]. \end{cases}$$

It is easy to see that $[\theta_0]_{\pi_n} = [\theta_1]_{\pi_n}$, so there exists a homotopy $h : (I^{n+1}, \partial I^{n+1}) \to$

$(T(X,\{x_i\}),x')$ such that $h|_{I_0^n} = \theta_0$ and $h|_{I_1^n} = \theta_1$. Define $O = h^{-1}(X)$ and $h' = h|_O$. Then h' is a component-homotopy between $(\phi \star \psi)'$ and $\phi' * \psi'$, since $\theta_1|_O = \phi' * \psi'$. This shows that $(\phi \star \psi)' \sim \phi' * \psi'$ and that f is a homomorphism. $\square$

Since f is a bijective homomorphism, we can conclude that

$$\pi_n(T(X,\{x_i\}),x') \cong \rho_n(X,\{x_i\}). \qquad \square$$

2.3 Examples

Example 2.3.1. For $i \in \mathbb{N}$, let

$$X_i = \left\{(x,y) \in \mathbb{R}^2 : d\left((x,y), \left(\frac{1}{2^{(i-1)}}, \frac{1}{2^{(i+1)}}\right)\right) = \frac{1}{2^{(i+1)}}\right\},$$

and let $X_0 = \{(0,0)\}$. Let $x_i = \left(\frac{1}{2^{(i-1)}}, 0\right)$ for $i \in \mathbb{N}$ and let $x_0 = (0,0)$. Let $X = \bigcup_{i=0}^{\infty} X_i$. It is an easy corollary of Claim 6 that $\rho_1(X,\{x_i\})$ is isomorphic to the Hawaiian Earring Group $\mathbb{H}$.

We will see from the next example the importance of the inclusion of the limit point $(0,0)$ in the previous example.

Example 2.3.2. For $i \in \mathbb{N}$ define X_i and x_i as in the previous example. Let $X' = \bigcup_{i\in\mathbb{N}} X_i$. Then $(X',\{x_i\}) \cong (Y,\{y_i\}) = \bigcup_{i\in\mathbb{N}}(Y_i,y_i)$, where

$$Y_i = \{(x,y) \in \mathbb{R}^2 : d((x,y),(i,1)) = 1\}$$

and $y_i = (i,0)$. Letting $Z = (Y,\{y_i\}) \cup \{(x,0) : x \in \mathbb{R}\}$, it is easy to see that

$$\rho_1(X',\{x_i\}) \cong \rho_1(Y,\{y_i\}) \cong \pi_1(T(Y,\{y_i\}),y_1) \cong \pi_1(Z) \cong \bigoplus_{i\in\mathbb{N}} \mathbb{Z}.$$

Appendix A. Lemmas

A.1 Lemmas

Lemma A.1.1. Suppose $\phi : X \to Y$ is a homeomorphism and $A \subseteq X$. Then $\partial\phi(A) = \phi(\partial A)$.

Proof. If $\partial\phi(A) = \emptyset$, then by definition, $\phi(A)$ is both open and closed. Since ϕ is a homeomorphism, A is both open and closed, and therefore $\phi(\partial A) = \phi(\emptyset) = \emptyset$ and we are done.

Now suppose that $\partial\phi(A) \neq \emptyset$. Choose $p \in \partial\phi(A)$. Let U be a neighborhood of $\phi^{-1}(p)$. It suffices to show that $U \cap (X \setminus A) \neq \emptyset$. Since ϕ is a homeomorphism, $\phi(U)$ is a neighborhood of p, and by definition, $\phi(U) \cap (Y \setminus \phi(A)) \neq \emptyset$. Choose $q \in \phi(U) \cap (Y \setminus \phi(A))$. Then $\phi^{-1}(q) \in U$ and $\phi^{-1}(q) \notin A$, so $\phi^{-1}(q) \in U \cap (X \setminus A) \neq \emptyset$ and we are done.

Notice that the reverse inclusion follows from a similar argument by considering the homeomorphism $\phi^{-1} : Y \to X$. □

Lemma A.1.2. For $U \in \mathcal{U}_n$, $(\ell(U))_- = r(U_-)$ and $(r(U))_- = \ell(U_-)$.

Proof. Recall that we used the notation U_- to signify $b(U)$, where $b(x_1, \ldots, x_n, x_{n+1}) = (x_1, \ldots, x_n, -x_{n+1})$. Then to show that $(\ell(U))_- = r(U_-)$, we must show that $b(\ell(U)) = r(b(U))$.

Let $\overline{x} \in U \in \mathcal{U}_n$. Then

$$\begin{aligned}
b(\ell(\overline{x})) = b(x_1/3, x_2, \ldots, x_n) &= (1 - x_1/3, x_2, \ldots, x_n) \text{ and} \\
r(b(\overline{x})) = r(1 - x_1, x_2, \ldots, x_n) &= ((1 - x_1 + 2)/3, x_2, \ldots, x_n) \\
&= ((3 - x_1)/3, x_2, \ldots, x_n) \\
&= (1 - x_1/3, x_2, \ldots, x_n),
\end{aligned}$$

so $b(\ell(\overline{x})) = r(b(\overline{x}))$ and therefore $(\ell(U))_- = r(U_-)$.

The proof that $(r(U))_- = \ell(U_-)$ is similar. □

Bibliography

[1] Karol Borsuk. *Theory of Retracts*. Polish Scientific, 1967.

[2] J. W. Cannon and G. R. Conner. On the fundamental groups of one-dimensional spaces, 1998.

[3] Wlodzimierz J. Charatonik and Janusz R. Prajs. Maps of absolute retracts for tree-like continua, 2011.

[4] G. Conner, M. Meilstrup, D. Repovs, A. Zastrow, and M. Zeljko. On small homotopies of loops, 2007.

[5] Allen Hatcher. *Algebraic Topology*. Cambridge University Press, 2002.

[6] Wolfram Hojka. Travelling around the harmonic archipelago, 2013.

[7] James Munkres. *Topology*. Pearson, 1974.