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Longest fault-free paths in star graphs with vertex faults☆

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Abstract

The star graph S_n has been recognized as an attractive alternative to the hypercube. Since S_1, S_2, and S_3 have trivial structures, we focus our attention on S_n with $n \geqslant 4$ in this paper. Let F_v denote the set of faulty vertices in S_n. We show that when $|F_v| \leqslant n-5$, S_n with $n \geqslant 6$ contains a fault-free path of length $n! - 2|F_v| - 2$ ($n! - 2|F_v| - 1$) between arbitrary two vertices of even (odd) distance. Since S_n is bipartite with two partite sets of equal size, the path is longest for the worst-case scenario. The situation of $n \geqslant 4$ and $|F_v| > n-5$ is also discussed.

Keywords: Bipartite graph; Cayley graph; Fault-tolerant embedding; Graph–theoretic interconnection network; Hamiltonian path; Longest path; Star graph

1. Introduction

The star graph [1], which belongs to the class of Cayley graphs [2], has been recognized as an attractive alternative to the hypercube. It possesses many nice topological properties such as recursiveness, symmetry, maximal fault tolerance, sublogarithmic degree and diameter, and strong resilience, which are all desirable when we are designing the interconnection topology for a parallel and distributed system. Besides, the star graph can embed rings [16], meshes [17], trees [4], and hypercubes [15]. Many efficient algorithms [3] have been designed on the star graph.

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Since processor and/or link faults may happen when a network is put in use, it is practically meaningful and important to consider faulty networks. The problems of diameter [18], routing [7], multicasting [14], broadcasting [21], gossiping [6], embedding [19], and fault-tolerant graphs [5, 12] have been solved on a variety of faulty networks. This paper is concerned with the problem of fault-tolerant embedding. Throughout this paper, we use network and graph, processor and vertex, and link and edge, interchangeably.

Previously, the problem of fault-tolerant embedding on a faulty star graph has been studied in [8, 13, 20]. Let F_v and F_e denote the sets of faulty vertices and faulty edges, respectively. In [13], Latifi and Bagherzadeh showed that an n-dimensional star graph with faulty vertices contains a fault-free ring and a fault-free path of each length $n!-m!$ if all faulty vertices belong to an m-dimensional star graph and $m \leqslant n$ is minimal. In [20], Tseng et al. showed that an n-dimensional star graph with $|F_e| \leqslant n-3$ faulty edges contains a fault-free longest ring of length $n!$, and an n-dimensional star graph with $|F_v| \leqslant n-3$ faulty vertices contains a fault-free ring of length at least $n! - 4|F_v|$, where $n \geqslant 4$. In [8], the authors showed that an n-dimensional star graph with $|F_e| \leqslant n-4$ faulty edges contains a longest fault-free path of length $n!-2$ $(n!-1)$ between arbitrary two vertices of even (odd) distance, where $n \geqslant 6$. The situation of $|F_e| = n-3$ was also discussed in [8].

In this paper, we show that an n-dimensional star graph with $|F_v| \leqslant n-5$ faulty vertices contains a fault-free path of length $n! - 2|F_v| - 2$ $(n! - 2|F_v| - 1)$ between arbitrary two vertices of even (odd) distance, where $n \geqslant 6$. Since the star graph is bipartite with two partite sets of equal size [11], the path is longest for the worst-case scenario. The situation of $n \geqslant 4$ and $|F_v| > n-5$ is also discussed.

The rest of this paper is organized as follows. In the next section, the star graph is reviewed and some basic operations are defined. In Section 3, a longest fault-free path is constructed between arbitrary two vertices of an n-dimensional star graph with $|F_v| \leqslant n-5$ vertex faults. The situations of $|F_v| = n-4$ and $n-3$ are also discussed. Finally, some concluding remarks are given in Section 4.

2. Preliminaries

In this section, some necessary definitions, notations, and prerequisite results are introduced. First, the n-dimensional star graph, denoted by S_n, is defined as follows.

Definition 1. The vertex set of S_n is denoted by $\{a_1a_2\ldots a_n \mid a_1a_2\ldots a_n$ is a permutation of $\{1,2,\ldots,n\}\}$. Vertex adjacency is defined as follows: $a_1a_2\ldots a_n$ is adjacent to $a_ia_2\ldots a_{i-1}a_1a_{i+1}\ldots a_n$ for all $2 \leqslant i \leqslant n$.

The vertices of S_n are $n!$ permutations of $\{1,2,\ldots,n\}$, and there is an edge between two vertices of S_n if and only if they can be obtained from each other by swapping the leftmost number with one of the other $n-1$ numbers. For convenience we refer to the position of a_i in $a_1a_2\dots a_n$ as the ith *dimension*, and $(a_1a_2\dots a_n,\ a_ia_2\dots a_{i-1}a_1a_{i+1}\dots a_n)$ as the ith-*dimensional edge*. It is easy to see that S_n is regular of degree $n-1$. Since S_1 is a vertex, S_2 is an edge, and S_3 is a cycle of length six, we focus our attention on S_n with $n\geqslant 4$ in this paper.

There are embedded S_r's contained in S_n, where $1\leqslant r\leqslant n$. An embedded S_r can be conveniently represented by $\langle s_1s_2\dots s_n\rangle_r$, where $s_1=*$, $s_i\in\{*,1,2,\dots,n\}$ for all $2\leqslant i\leqslant n$, and exactly r of $s_1,s_2,\dots,s_n$ are $*$ ($*$ denotes a "don't care" symbol). For example, $\langle ***3\rangle_3$, which represents an embedded S_3 in S_4, contains six nodes 1243, 1423, 2143, 2413, 4123, and 4213. In terms of graph, $\langle ***3\rangle_3$ is a subgraph of S_4 induced by {1243, 1423, 2143, 2413, 4123, 4213}. When $r=n$, $\langle s_1s_2\dots s_n\rangle_n$ represents S_n. Two basic operations on S_n are defined as follows.

Definition 2. An *i-partition on* $\langle s_1s_2\dots s_n\rangle_r$ partitions $\langle s_1s_2\dots s_n\rangle_r$ into r embedded S_{r-1}'s, denoted by $\langle s_1s_2\dots s_{i-1}qs_{i+1}\dots s_n\rangle_{r-1}$, where $2\leqslant i\leqslant n$, $s_i=*$, and $q\in\{1,2,\dots,n\}-\{s_1,s_2,\dots,s_n\}$.

Definition 3. An $(i_1,i_2,\dots,i_m)$*-partition on* $\langle s_1s_2\dots s_n\rangle_r$ performs an i_1-partition, an i_2-partition,$\dots$, an i_m-partition, sequentially, on $\langle s_1s_2\dots s_n\rangle_r$, where $1\leqslant m\leqslant r-1$ and $i_1i_2\dots i_m$ is a permutation of m numbers from $\{2,3,\dots,n\}$. After executing an $(i_1,i_2,\dots,i_m)$-partition, $\langle s_1s_2\dots s_n\rangle_r$ is partitioned into $r(r-1)\cdots(r-m+1)$ embedded S_{r-m}'s.

For example, when a $(3,2)$-partition is applied to $\langle ***15\rangle_3$, a 3-partition is first executed on $\langle ***15\rangle_3$ to produce three embedded S_2's, i.e., $\langle **215\rangle_2$, $\langle **315\rangle_2$, and $\langle **415\rangle_2$. Then, a 2-partition is executed on each S_2 to produce six embedded S_1's, i.e., $\langle *3215\rangle_1$, $\langle *4215\rangle_1$, $\langle *2315\rangle_1$, $\langle *4315\rangle_1$, $\langle *2415\rangle_1$, and $\langle *3415\rangle_1$.

Two embedded S_r's $\langle s_1s_2\dots s_n\rangle_r$ and $\langle t_1t_2\dots t_n\rangle_r$ are said to be *adjacent* if $s_j\neq *$, $t_j\neq *$, and $s_j\neq t_j$ for some $2\leqslant j\leqslant n$, and $s_i=t_i$ for all $1\leqslant i\leqslant n$ and $i\neq j$. The position j is denoted by $dif(\langle s_1s_2\dots s_n\rangle_r,\langle t_1t_2\dots t_n\rangle_r)$. For example, $\langle **23\rangle_2$ is adjacent to $\langle **13\rangle_2$, and $dif(\langle **23\rangle_2,\langle **13\rangle_2)=3$.

Definition 4. Let $A_1,A_2,\dots,A_{n(n-1)\cdots(r+1)}$ represent those embedded S_r's that are obtained by executing an $(i_1,i_2,\dots,i_{n-r})$-partition on S_n, where $1\leqslant r\leqslant n-1$. They form an *r-path*, denoted by $P_r=[A_1,A_2,\dots,A_{n(n-1)\cdots(r+1)}]$, if A_i is adjacent to A_{i+1} for all $1\leqslant i\leqslant n(n-1)\cdots(r+1)-1$. Each vertex in P_r, i.e., A_i, is called an *r-vertex*, and each edge in P_r, i.e., (A_i,A_{i+1}), is called an *r-edge*.

For example, there is a $P_4=(\langle ****1\rangle_4,\ \langle ****2\rangle_4,\ \langle ****3\rangle_4,\ \langle ****4\rangle_4,\ \langle ****5\rangle_4)$ in S_5, where $\langle ****1\rangle_4$, $\langle ****2\rangle_4$, $\langle ****3\rangle_4$, $\langle ****4\rangle_4$, and $\langle ****5\rangle_4$, are all 4-vertices and $(\langle ****1\rangle_4,\ \langle ****2\rangle_4)$, $(\langle ****2\rangle_4,\ \langle ****3\rangle_4)$, $(\langle ****3\rangle_4,\ \langle ****4\rangle_4)$, and $(\langle ****4\rangle_4,\ \langle ****5\rangle_4)$ are all 4-edges. We note that an r-vertex is an embedded S_r and an r-edge comprises $(r-1)!$ edges of S_n.

Definition 5. An *i-partition on a* $P_r = [A_1, A_2, \ldots, A_{n(n-1)\cdots(r+1)}]$ performs an i-partition on $A_1, A_2, \ldots, A_{n(n-1)\cdots(r+1)}$, respectively, where $2 \leqslant i \leqslant n$ and $1 < r \leqslant n-1$. An i-partition on a P_r is abbreviated to *a partition on a* P_r if the position i is "don't care".

After executing an i-partition on a $P_r = [A_1, A_2, \ldots, A_{n(n-1)\cdots(r+1)}]$, each A_j is partitioned into r $(r-1)$-vertices, where $1 \leqslant j \leqslant n(n-1)\cdots(r+1)$. Since every two of the r $(r-1)$-vertices are joined with an $(r-1)$-edge, each A_j can be viewed as a complete graph of r $(r-1)$-vertices. Throughout this paper, we use K_r^{r-1} to denote the complete graph. We note that each vertex in K_r^{r-1} is an $(r-1)$-vertex and each edge in K_r^{r-1} is an $(r-1)$-edge.

Suppose $A_j = \langle s_1 \ldots s_{i-1} s_i s_{i+1} \ldots s_{k-1} x s_{k+1} \ldots s_n \rangle_r$ and $A_{j+1} = \langle s_1 \ldots s_{i-1} s_i s_{i+1} \ldots s_{k-1} y s_{k+1} \ldots s_n \rangle_r$ are two neighboring r-vertices in a P_r, where $1 < r \leqslant n-1$, $x \neq y$, and $s_i = *$ (without loss of generality, we assume $i < k$). After executing an i-partition, they each are partitioned into r $(r-1)$-vertices, and the r-edge between them is split into $r-1$ $(r-1)$-edges connecting $r-1$ pairs of $(r-1)$-vertices that belong to them, respectively. The $(r-1)$-vertex belonging to A_j (A_{j+1}) that is not connected to A_{j+1} (A_j) is $\langle s_1 \ldots s_{i-1} y s_{i+1} \ldots s_{k-1} x s_{k+1} \ldots s_n \rangle_{r-1}$ $(\langle s_1 \ldots s_{i-1} x s_{i+1} \ldots s_{k-1} y s_{k+1} \ldots s_n \rangle_{r-1})$. The following lemma was shown in [8].

Lemma 1 (Hsieh et al. [8]). *Suppose* $E = \langle e_1 e_2 \ldots e_n \rangle_r$, $F = \langle f_1 f_2 \ldots f_n \rangle_r$, *and* $G = \langle g_1 g_2 \ldots g_n \rangle_r$ *are arbitrary three consecutive r-vertices in a* P_r, *where* $1 < r \leqslant n-1$. *If* $e_{dif(E,F)} \neq g_{dif(F,G)}$, *then after executing a partition on the* P_r, *each* $(r-1)$-*vertex in F is connected to E or G.*

In this paper, we consider S_n with faulty vertices. An r-vertex is *faulty* if it contains one or more faulty vertices, and an r-path is *faulty* if one or more of its r-vertices are faulty, where $1 \leqslant r \leqslant n$. A vertex ($r$-vertex, r-path) is *fault-free* if it is not faulty. A path is *fault-free* if it does not contain any faulty vertex. The following lemma was shown in [20].

Lemma 2 (Tseng et al. [20]). *Suppose* $|F_v| \leqslant n-3$. *There exists a sequence* $b_1, b_2, \ldots, b_{n-4}$ *of positions so that after executing an* $(b_1, b_2, \ldots, b_{n-4})$-*partition on* S_n, *each resulting* 4-*vertex contains at most one faulty vertex.*

The positions $b_1, b_2, \ldots, b_{n-4}$ in Lemma 2 can be easily determined as follows. We let b_1 be a position where at least two faulty vertices differ. For example, if $F_v = \{123456, 123654\}$, b_1 may be set to 4 or 6. A b_1-partition is then executed on S_n to produce n $(n-1)$-vertices, and F_v is partitioned so that two faulty vertices fall into the same subset if and only if they belong to the same $(n-1)$-vertex. The position b_2 can be determined similarly. We simply let b_2 be a position where at least two faulty vertices in some subset differ. A b_2-partition is then executed on the n $(n-1)$-vertices to produce $n(n-1)$ $(n-2)$-vertices, and every non-empty subset of F_v is further partitioned accordingly. The process is repeated until every non-empty subset contains one faulty vertex, when the remaining positions are determined arbitrarily.

Lemma 3. *Suppose* $u = u_1u_2 \dots u_n$ *and* $v = v_1v_2 \dots v_n$ *are arbitrary two distinct vertices in* S_n *and* $|F_v| \leqslant n-5$. *There exists a sequence* $a_1, a_2, \dots, a_{n-4}$ *of positions so that* $u_{a_1} \neq v_{a_1}$ *and after executing an* $(a_1, a_2, \dots, a_{n-4})$*-partition on* S_n, *each resulting* 4*-vertex contains at most one vertex in* $\{u, v\} \cup F_v$.

Proof. We let $a_1 = j$, where $u_j \neq v_j$ and $j \neq 1$. The other positions $a_2, a_3, \dots, a_{n-4}$ can be determined similar to $b_2, b_3, \dots, b_{n-4}$ in Lemma 2. □

In the rest of this paper, we use u and v to denote arbitrary two distinct fault-free vertices in S_n. Our purpose is to construct a longest path from u to v (u–v path for short) in S_n. We call u and v the *beginning vertex* and *ending vertex* of the path, respectively. An r-vertex is called the *beginning r-vertex* (*ending r-vertex*) if it contains $u(v)$.

Definition 6. A $P_5 = [A_1, A_2, \dots, A_{n(n-1)\cdots 6}]$ in S_n is said to be *good* if it satisfies the following three conditions.
(Cond. 1) A_1 and $A_{n(n-1)\cdots 6}$ are the beginning and ending 5-vertices, respectively.
(Cond. 2) One of $A_1, A_2, \dots, A_{n(n-1)\cdots 6}$ contains at most two faulty vertices, and the others each contain at most one faulty vertex.
(Cond. 3) For every three consecutive 5-vertices in the P_5, say $B = \langle b_1b_2 \dots b_n \rangle_5$, $C = \langle c_1c_2 \dots c_n \rangle_5$, and $D = \langle d_1d_2 \dots d_n \rangle_5$, $b_{dif(B,C)} \neq d_{dif(C,D)}$ holds.

Definition 7. A $P_4 = [A_1, A_2, \dots, A_{n(n-1)\cdots 5}]$ in S_n is said to be *good* if it satisfies the following five conditions.
(Cond. 1) A_1 and $A_{n(n-1)\cdots 5}$ are the beginning and ending 4-vertices, respectively.
(Cond. 2) $A_1, A_2, A_{n(n-1)\cdots 5-1}$, and $A_{n(n-1)\cdots 5}$ are fault-free, and the other 4-vertices each contain at most one faulty vertex.
(Cond. 3) For every three consecutive 4-vertices in the P_4, say $E = \langle e_1e_2 \dots e_n \rangle_4$, $F = \langle f_1f_2 \dots f_n \rangle_4$, and $G = \langle g_1g_2 \dots g_n \rangle_4$, $e_{dif(E,F)} \neq g_{dif(F,G)}$ holds.
(Cond. 4) Both every two consecutive 4-vertices in the P_4 are not faulty.
(Cond. 5) After executing a k-partition on the P_4 for some $2 \leqslant k \leqslant n$, the beginning and ending 3-vertices (in A_1 and $A_{n(n-1)\cdots 5}$, respectively) are not connected to A_2 and $A_{n(n-1)\cdots 5-1}$, respectively.

The following lemma was shown in [9].

Lemma 4 (Hsieh et al. [9]). *Suppose* $|F_v| = 1$. *There is a fault-free path of maximal length* $4!-3$ *between arbitrary two adjacent fault-free vertices of* S_4.

3. Longest fault-free paths in S_n

In this section, unless particularly specified, we assume $|F_v| \leqslant n-5$, $n \geqslant 6$, and a_1, $a_2, \dots, a_{n-4}$ are a sequence of positions satisfying Lemma 3. We aim to construct a fault-free u–v path of length $n!-2|F_v|-2$ if $dist(u,v)$ is even, and of length $n!-2|F_v|-1$

if $dist(u,v)$ is odd, where $dist(u,v)$ denotes the distance between u and v. Since S_n is bipartite with two partite sets of equal size, the path is longest for the worst-case scenario.

Lemma 5. *A good P_5 can be obtained by executing an $(a_1,a_2,\dots,a_{n-5})$-partition on S_n.*

Proof. Suppose $u=u_1u_2\dots u_n$ and $v=v_1v_2\dots v_n$. A good P_5 can be obtained by the following two steps: (S1) generate a P_6 from S_n, and (S2) generate a good P_5 from the P_6. Step (S1) is explained below.

If $n=6$, then S_n is itself a P_6 with only one 6-vertex. If $n>6$, then a P_6 can be obtained by executing an $(a_1,a_2,\dots,a_{n-6})$-partition on S_n as follows. Initially, an a_1-partition is applied to S_n, and so a K_n^{n-1} can result. Since $u_{a_1}\neq v_{a_1}$, u and v belong to two different $(n-1)$-vertices. A P_{n-1} whose first and last $(n-1)$-vertices are the beginning and ending $(n-1)$-vertices, respectively, can be easily generated. For $j=2,3,\dots,n-6$, a P_{n-j} whose first and last $(n-j)$-vertices are the beginning and ending $(n-j)$-vertices, respectively, can be generated from a P_{n-j+1} as explained below.

Suppose $P_{n-j+1}=[A_{n-j+1,1},\ A_{n-j+1,2},\dots,A_{n-j+1,n(n-1)\cdots(n-j+2)}]$, where $A_{n-j+1,1}$ is the beginning $(n-j+1)$-vertex and $A_{n-j+1,n(n-1)\cdots(n-j+2)}$ is the ending $(n-j+1)$-vertex. After an a_j-partition on the P_{n-j+1}, each $A_{n-j+1,k}$ forms a K_{n-j+1}^{n-j}, where $1\leqslant k\leqslant n(n-1)\cdots(n-j+2)$. Since each $A_{n-j+1,k}$ contains at least seven $(n-j)$-vertices, there are two distinct $(n-j)$-vertices in $A_{n-j+1,k}$, say X_k and Y_k, so that X_1 is the beginning $(n-j)$-vertex, $Y_{n(n-1)\cdots(n-j+2)}$ is the ending $(n-j)$-vertex, and for $2\leqslant j\leqslant n(n-1)\cdots(n-j+2)-1$, X_j and Y_j are adjacent to Y_{j-1} and X_{j+1}, respectively. Since there is a Hamiltonian X_k-Y_k path in each K_{n-j+1}^{n-j} formed by $A_{n-j+1,k}$, a desired P_{n-j} can be generated if all Hamiltonian paths are interleaved with $(n-j)$-edges (Y_1,X_2), $(Y_2,X_3),\dots,(Y_{n(n-1)\cdots(n-j+2)-1},X_{n(n-1)\cdots(n-j+2)})$. When $j=n-6$, a $P_6=[A_{6,1},A_{6,2},\dots,A_{6,n(n-1)\cdots 7}]$ can be obtained, where $A_{6,1}$ is the beginning 6-vertex and $A_{6,n(n-1)\cdots 7}$ is the ending 6-vertex. In the rest of this paper, X_k and Y_k thus specified are referred to as the *entry* $(n-j)$-*vertex* and the *exit* $(n-j)$-*vertex* of $A_{n-j+1,k}$, respectively.

Step (S2) is explained as follows. If $n=6$, then a P_5 whose first and last 5-vertices are the beginning and ending 5-vertices, respectively, can be generated in S_6 after an a_1-partition. It is not difficult to check that the P_5 is good. If $n>6$, then each $A_{6,r}$ forms a K_6^5 after an a_{n-5}-partition on the P_6, where $1\leqslant r\leqslant n(n-1)\cdots 7$. Let $X_r\neq Y_r$ denote the entry and exit 5-vertices of each $A_{6,r}$, respectively. It is easy to establish a Hamiltonian X_1-Y_1 path in the K_6^5 formed by $A_{6,1}$ whose last two 5-vertices are connected to $A_{6,2}$, a Hamiltonian $X_{n(n-1)\cdots 7}-Y_{n(n-1)\cdots 7}$ path in the K_6^5 formed by $A_{6,n(n-1)\cdots 7}$ whose first two 5-vertices are connected to $A_{6,n(n-1)\cdots 7-1}$, and for $2\leqslant j\leqslant n(n-1)\cdots 7-1$ a Hamiltonian X_j-Y_j path in the K_6^5 formed by each $A_{6,j}$ whose first and last two 5-vertices are connected to $A_{6,j-1}$ and $A_{6,j+1}$, respectively. All Hamiltonian paths interleaved with 5-edges $(Y_1,X_2),(Y_2,X_3),\dots,(Y_{n(n-1)\cdots 7-1},X_{n(n-1)\cdots 7})$ form a P_5. The P_5 is good, as explained below.

Obviously, (Cond. 1) holds. (Cond. 2) holds as a consequence of Lemma 3. (Cond. 3) holds as a consequence of the fact that the first two 5-vertices and the last two 5-vertices in each Hamiltonian $X_i - Y_i$ path are connected to A_{i-1} (if existing) and A_{i+1} (if existing), respectively, where $1 \leqslant i \leqslant n(n-1)\cdots 7$. The details can be found in [10], which is an extended version of this paper. □

Lemma 6. *Each of the beginning and ending* 5*-vertices of a good* P_5 *that was obtained according to Lemma* 5 *contains at most one faulty vertex.*

Proof. Since the good P_5 results from executing an $(a_1, a_2, \ldots, a_{n-5})$-partition on S_n, it contains at least $min\{n-4, |F_v| + 2\}$ 5-vertices that each contain at least one vertex in $\{u, v\} \cup F_v$. If the beginning 5-vertex (or the ending 5-vertex) contains two faulty vertices, then F_v should contain $n-5$ faulty vertices. This means that there are at most $n-5$ 5-vertices in the P_5 that each contain at least one vertex in $\{u, v\} \cup F_v$, which is a contradiction. □

The following lemma can be shown similarly.

Lemma 7. *At most one of the beginning and ending* 5*-vertices of a good* P_5 *that was obtained according to Lemma* 5 *is faulty.*

Lemma 8. *A good* P_4 *can be obtained by executing an* $(a_1, a_2, \ldots, a_{n-4})$*-partition on* S_n.

Proof. Suppose $u = u_1u_2 \ldots u_n$ and $v = v_1v_2 \ldots v_n$. By the aid of Lemma 5, we only need to show that a good P_4 can be obtained after an a_{n-4}-partition on a good P_5. Suppose $[A_1, A_2, \ldots, A_{n(n-1)\cdots 6}]$ is a good P_5 that was obtained by executing an $(a_1, a_2, \ldots, a_{n-5})$-partition on S_n. A good P_4 can be constructed according to the following three cases.

Case 1. A_1 is faulty. A good P_4 can be generated by the following three steps: (S1) generate a Hamiltonian path in the K_5^4 formed by A_1, (S2) generate a Hamiltonian path in the K_5^4 formed by $A_{n(n-1)\cdots 6}$, and (S3) generate a Hamiltonian path in each K_5^4 formed by A_j for all $2 \leqslant j \leqslant n(n-1)\cdots 6 - 1$.

Step (S1) is explained as follows. Three 4-vertices $X_1 = \langle x_1x_2 \ldots x_n \rangle_4, T$, and Q are first determined from A_1 so that X_1 is the beginning 4-vertex, T is not connected to A_2, and Q is faulty. By Lemma 6, Q is unique. By Lemma 3, X_1 is fault-free. Also determine $W = \langle w_1w_2 \ldots w_n \rangle_4 \notin \{X_1, Q\}$ from A_1 with $w_{a_{n-4}} = u_k$ for some $k \in \{2, 3, \ldots, n\} - \{a_1, a_2, \ldots, a_{n-4}\}$. A Hamiltonian path in the K_5^4 formed by A_1 can be established as (X_1, W, Q, M_1, Y_1) if $T \in \{X_1, Q, W\}$, and (X_1, W, T, Q, Y_1) if $T \notin \{X_1, Q, W\}$, where M_1 and Y_1 denote the other 4-vertices in A_1. Since there are four 4-vertices in A_1 that are connected to A_2, Y_1 should be connected to A_2.

Step (S2) is explained as follows. A Hamiltonian path in the K_5^4 formed by $A_{n(n-1)\cdots 6}$ can be established as follows. By Lemma 7, $A_{n(n-1)\cdots 6}$ is fault-free. Let $Y_{n(n-1)\cdots 6}$ be the ending 4-vertex, C be the 4-vertex in $A_{n(n-1)\cdots 6}$ that is not connected to $A_{n(n-1)\cdots 6-1}$,

$D = \langle d_1 d_2 \ldots d_n \rangle_4$ be the 4-vertex in $A_{n(n-1)\cdots 6}$ with $d_{a_{n-4}} = v_k$ (here k is identical with that appearing in $w_{a_{n-4}} = u_k$ above), and $X_{n(n-1)\cdots 6} \notin \{C, D, Y_{n(n-1)\cdots 6}\}$ be a 4-vertex in $A_{n(n-1)\cdots 6}$. If $Y_{n(n-1)\cdots 6} = C$ or ($Y_{n(n-1)\cdots 6} \neq C$ and $C = D$), then a Hamiltonian path in the K_5^4 formed by $A_{n(n-1)\cdots 6}$ can be established as $P[X_{n(n-1)\cdots 6}, D] + (D, Y_{n(n-1)\cdots 6})$, where $P[X_{n(n-1)\cdots 6}, D]$ represents an $X_{n(n-1)\cdots 6} - D$ path passing all the vertices of the K_5^4 but $Y_{n(n-1)\cdots 6}$ exactly once. If $Y_{n(n-1)\cdots 6} \neq C$ and $C \neq D$, then a Hamiltonian path in the K_5^4 formed by $A_{n(n-1)\cdots 6}$ can be established as $P[X_{n(n-1)\cdots 6}, C] + (C, D) + (D, Y_{n(n-1)\cdots 6})$, where $P[X_{n(n-1)\cdots 6}, C]$ represents an $X_{n(n-1)\cdots 6} - C$ path passing all the vertices of the K_5^4 but D and $Y_{n(n-1)\cdots 6}$ exactly once.

Step (S3) is explained as follows. We first determine the entry and exit 4-vertices of A_j, denoted by X_j and Y_j, respectively, so that both Y_{j-1} and X_j are not faulty and both $Y_{n(n-1)\cdots 6-1}$ and $X_{n(n-1)\cdots 6}$ are not faulty. Since four 4-vertices in A_j are connected to A_{j-1} and four 4-vertices in A_j are connected to A_{j+1}, there exists a Hamiltonian $X_j - Y_j$ path in the K_5^4 formed by A_j whose first two and last two 4-vertices are connected to A_{j-1} and A_{j+1}, respectively. A P_4 can be obtained by joining the Hamiltonian paths that were established in the three steps above.

Case 2. $A_{n(n-1)\cdots 6}$ is faulty. A good P_4 can be constructed similar to Case 1.

Case 3. Both A_1 and $A_{n(n-1)\cdots 6}$ are fault-free. We first assume that there is a 5-vertex, say A_r, in the P_5 that contains two faulty vertices. By Lemma 6, $r \notin \{1, n(n-1)\cdots 6\}$. By Lemma 1, each 4-vertex in A_r is connected to A_{r-1} or A_{r+1}. A good P_4 can be obtained by the following two steps: (S1) generate a Hamiltonian path in the K_5^4 formed by A_r and (S2) generate a Hamiltonian path in each K_5^4 formed by A_j for all $1 \leqslant j \leqslant n(n-1)\cdots 6$ and $j \neq r$.

Step (S1) is explained as follows. Let $B \neq F$ be the two 4-vertices in A_r that are not connected to A_{r-1} and A_{r+1}, respectively. Since $a_1, a_2, \ldots, a_{n-4}$ are a sequence of positions satisfying Lemma 3, the two faulty vertices belong to two different 4-vertices in A_r, denoted by S and T. If S is connected to both A_{r-1} and A_{r+1} and T is connected to both A_{r-1} and A_{r+1}, then a Hamiltonian path in the K_5^4 formed by A_r can be established as (F, S, M_1, T, B), where M_1 denotes the other 4-vertex in A_r.

If S is connected to both A_{r-1} and A_{r+1} and T is connected to one of A_{r-1} and A_{r+1}, then a Hamiltonian path in the K_5^4 formed by A_r can be established as (M_1, T, M_2, S, B) if T is connected to A_{r-1}, and (F, S, M_1, T, M_2), if T is connected to A_{r+1}, where M_1 and M_2 denote the other two 4-vertices in A_r. If S is connected to one of A_{r-1} and A_{r+1} and T is connected to both A_{r-1} and A_{r+1}, then a Hamiltonian path in the K_5^4 formed by A_r can be established similarly.

If S is connected to only A_{r-1} and T is connected to only A_{r+1}, then a Hamiltonian path in the K_5^4 formed by A_r can be established as (M_1, S, M_2, T, M_3), where M_1, M_2, and M_3 denote the other three 4-vertices in A_r. If S is connected to only A_{r+1} and T is connected to only A_{r-1}, then the Hamiltonian path can be established as (M_1, T, M_2, S, M_3). It is impossible that both S and T are connected to only A_{r-1} or only A_{r+1}.

We note that the first and last 4-vertices in each Hamiltonian path above are the entry and exit 4-vertices of A_r, respectively. Besides, the first two and last two 4-vertices are connected to A_{r-1} and A_{r+1}, respectively.

Step (S2) is explained as follows. Let X_1 and $Y_{n(n-1)\cdots 6}$ be the beginning and ending 4-vertices, respectively, and determine $W=\langle w_1w_2\ldots w_n\rangle_4$ and $D=\langle d_1d_2\ldots d_n\rangle_4$ from A_1 and $A_{n(n-1)\cdots 6}$, respectively, with $w_{a_{n-4}}=u_l$ and $d_{a_{n-4}}=v_l$ for some $l\in\{2,3,\ldots,n\}-\{a_1,a_2,\ldots,a_{n-4}\}$. A Hamiltonian path in the K_5^4 formed by A_1 can be established as (X_1,W,M_1,M_2,M_3), where M_1, M_2, and M_3 denote the other three 4-vertices in A_1 and both M_2 and M_3 are connected to A_2. A Hamiltonian path in the K_5^4 formed by $A_{n(n-1)\cdots 6}$ can be established as $(M_1,M_2,M_3,D,Y_{n(n-1)\cdots 6})$, where M_1,M_2, and M_3 denote the other three 4-vertices in $A_{n(n-1)\cdots 6}$ and both M_1 and M_2 are connected to $A_{n(n-1)\cdots 6-1}$. Let X_t and Y_t denote the entry and exit 4-vertices of A_t, respectively, where $2\leqslant t\leqslant n(n-1)\cdots 6-1$ and $t\neq r$. A Hamiltonian X_t-Y_t path in the K_5^4 formed by A_t can be established similar to the Hamiltonian X_j-Y_j path in Case 1. Clearly a P_4 can be generated by joining the Hamiltonian paths that were established in the two steps above.

The P_4 that was obtained for the three cases above is good, as explained below. Obviously, (Cond. 1), (Cond. 2), and (Cond. 4) hold. (Cond. 3) holds as a consequence of the fact that the first two 4-vertices and the last two 4-vertices in each Hamiltonian X_i-Y_i path are connected to A_{i-1} (if existing) and A_{i+1} (if existing), respectively, where $1\leqslant i\leqslant n(n-1)\cdots 6$. The details can be found in [10]. (Cond. 5) holds with the following reason. Suppose $W=\langle w_1w_2\ldots w_n\rangle_4$ and $D=\langle d_1d_2\ldots d_n\rangle_4$ are the neighboring 4-vertices of the beginning and ending 4-vertices in the P_4, respectively. We have $w_{a_{n-4}}=u_z$ and $d_{a_{n-4}}=v_z$ for some $z\in\{2,3,\ldots,n\}-\{a_1,a_2,\ldots,a_{n-4}\}$. Let $H=\langle h_1h_2,\ldots,h_n\rangle_3$ be the beginning 3-vertex after a z-partition on the P_4. Since $h_z=u_z=w_{a_{n-4}}\neq h_{a_{n-4}}$ and $h_t=w_t$ for $t\in\{1,2,\ldots,n\}-\{z,a_{n-4}\}$, H is not connected to W. Similarly, the ending 3-vertex is not connected to D.

On the other hand, if every 5-vertex in the P_5 contains at most one faulty vertex, then a good P_4 can be obtained similarly (in fact, more easily). □

We note that S_3 forms a ring of length six. The following two lemmas were shown in [20].

Lemma 9 (Tseng et al. [20]). *Suppose E and F are two adjacent 3-vertices in S_n, and let $(c_0,c_1,\ldots,c_5)$ denote the ring formed by E. The vertices in E that are connected to F are c_j and $c_{(j+3)\bmod 6}$ for some $0\leqslant j\leqslant 5$.*

Lemma 10 (Tseng et al. [20]). *Suppose $E=\langle e_1e_2\ldots e_n\rangle_3$, $F=\langle f_1f_2\ldots f_n\rangle_3$, and $G=\langle g_1g_2\ldots g_n\rangle_3$ are three 3-vertices in S_n, and F is adjacent to both E and G. If $e_{dif(E,F)}\neq g_{dif(F,G)}$, then the two vertices in F that are connected to E are disjoint from the two vertices in F that are connected to G.*

Lemma 11. *A fault-free $u-v$ path can be generated from a good P_4 that was obtained according to Lemma 8. The path has length $n!-2|F_v|-2$ if $dist(u,v)$ is even, and $n!-2|F_v|-1$ if $dist(u,v)$ is odd.*

Proof. Suppose the P_4 is $[A_1, A_2, \ldots, A_{n(n-1)\cdots 5}]$. First, a k-partition is executed on the P_4, where k satisfies (Cond. 5) of Definition 7. As a consequence, the beginning and ending 3-vertices are not connected to A_2 and $A_{n(n-1)\cdots 5-1}$, respectively. In order to generate a fault-free $u-v$ path, we determine two 3-vertices, denoted by α_i and β_i, in A_i for all $1 \leqslant i \leqslant n(n-1)\cdots 5$ that satisfy the following. α_1 is the beginning 3-vertex and $\beta_{n(n-1)\cdots 5}$ is the ending 3-vertex. For faulty A_i, $\alpha_i = \beta_i$ is connected to both A_{i-1} and A_{i+1}, and both β_{i-1} and α_{i+1} are adjacent to α_i $(=\beta_i)$. For every two consecutive fault-free 3-vertices, say A_i and A_{i+1}, β_i and α_{i+1} are adjacent. It was shown in [10] that if $\alpha_i = \beta_i$, then $b_{dif(\beta_{i-1},\alpha_i)} \neq a_{dif(\alpha_i,\alpha_{i+1})}$, where $\beta_{i-1} = \langle b_1 b_2 \ldots b_n \rangle_3$ and $\alpha_{i+1} = \langle a_1 a_2 \ldots a_n \rangle_3$.

A fault-free $u-v$ path can be generated by the following three steps: (S1) generate a fault-free Hamiltonian path in A_1, (S2) generate a fault-free path in $A_2, A_3, \ldots, A_{n(n-1)\cdots 5-1}$, sequentially, and (S3) generate a fault-free path in $A_{n(n-1)\cdots 5}$. A desired $u-v$ path can be obtained if all the paths above are joined together.

Step (S1) is explained as follows. Assume $u = u_1 u_2 \ldots u_n$, and let $Q_1 = \langle q_1 q_2 \ldots q_n \rangle_3 \notin \{\alpha_1, \beta_1\}$ be a 3-vertex in A_1 with $q_k \neq u_1$. Since α_1 is the beginning 3-vertex which is not connected to A_2, we have $\alpha_1 \neq \beta_1$. A fault-free Hamiltonian $\alpha_1 - \beta_1$ path in the K_4^3 formed by A_1 can be established as $(\alpha_1, Q_1, M_1, \beta_1)$, where M_1 is the other 3-vertex in A_1.

Suppose $E = \langle e_1 e_2 \ldots e_n \rangle_3$, $F = \langle f_1 f_2 \ldots f_n \rangle_3$, and $G = \langle g_1 g_2 \ldots g_n \rangle_3$ are arbitrary three consecutive 3-vertices in $(\alpha_1, Q_1, M_1, \beta_1, \alpha_2)$. It was shown in [10] that $e_{dif(E,F)} \neq g_{dif(F,G)}$ holds. We note that u is not connected to Q_1 because $q_k \neq u_1$ and each edge between α_1 and Q_1 is a k-dimensional edge. Since A_1 and A_2 are fault-free, Lemmas 9 and 10 assure that there are distinct fault-free vertices x_1, $b \in \alpha_1$, $c, d \in Q_1$, $e, f \in M_1$, and $g, y_1 \in \beta_1$ so that every two consecutive vertices in $\{x_1, b, c, d, e, f, g, y_1, x_2\}$ are adjacent, where $x_1 = u$ and x_2 is a fault-free vertex in α_2. It is easy to see that there are fault-free Hamiltonian $x_1 - b$, $c - d$, $e - f$, and $g - y_1$ paths in α_1, Q_1, M_1, and β_1, respectively. These fault-free Hamiltonian paths interleaved with edges (b,c), (d,e), and (f,g) constitute a fault-free Hamiltonian $x_1 - y_1$ path in A_1.

Step (S2) is explained as follows. Suppose a fault-free $x_{j-1} - y_{j-1}$ path in A_{j-1} has been obtained and y_{j-1} is adjacent to x_j which is a fault-free vertex in α_j, where $2 \leqslant j \leqslant n(n-1)\cdots 5-1$. A fault-free $x_j - y_j$ path in A_j can be constructed as follows, where y_j is a fault-free vertex in β_j. If $\alpha_j = \beta_j$, then $b_{dif(\beta_{j-1},\alpha_j)} \neq a_{dif(\alpha_j,\alpha_{j+1})}$ holds, where $\beta_{j-1} = \langle b_1 b_2 \ldots b_n \rangle_3$ and $\alpha_{j+1} = \langle a_1 a_2 \ldots a_n \rangle_3$ are assumed. Since $\alpha_j = \beta_j$ is fault-free, Lemmas 9 and 10 assure that y_j can be determined so that it is adjacent to both x_j and a fault-free vertex in α_{j+1}. If A_j is fault-free, then it is not difficult to generate a fault-free Hamiltonian $x_j - y_j$ path in A_j. If A_j is faulty, by Lemma 4 there is a fault-free $x_j - y_j$ path of maximal length $4! - 3$ in A_j.

On the other hand, if $\alpha_j \neq \beta_j$, then two 3-vertices Q_j and L_j in A_j are first determined so that $Q_j \neq \beta_j$ is not connected to A_{j+1} and $L_j \neq \alpha_j$ is not connected to A_{j-1}. Since A_j is fault-free, a fault-free Hamiltonian α_j–β_j path in the K_4^3 formed by A_j can be established as $(\alpha_j, M_1, M_2, \beta_j)$ if $Q_j = \alpha_j$ and $L_j = \beta_j$, $(\alpha_j, Q_j, M_2, \beta_j)$ if $Q_j \neq \alpha_j$ and $L_j = \beta_j$, $(\alpha_j, M_1, L_j, \beta_j)$ if $Q_j = \alpha_j$ and $L_j \neq \beta_j$, and $(\alpha_j, Q_j, L_j, \beta_j)$ if $Q_j \neq \alpha_j$ and $L_j \neq \beta_j$, where M_1 and M_2 are the other 3-vertices in A_j. Without loss of generality, we assume that $(\alpha_j, Q_j, L_j, \beta_j)$ is established. The discussions for other possibilities are all similar. Suppose $E = \langle e_1 e_2 \dots e_n \rangle_3$, $F = \langle f_1 f_2 \dots f_n \rangle_3$, and $G = \langle g_1 g_2 \dots g_n \rangle_3$ are arbitrary three consecutive 3-vertices in $(\beta_{j-1}, \alpha_j, Q_j, L_j, \beta_j, \alpha_{j+1})$. Then, $e_{dif(E,F)} \neq g_{dif(F,G)}$ holds similar to step (S1). A fault-free Hamiltonian x_j–y_j path can be generated from $(\alpha_j, Q_j, L_j, \beta_j)$, which is similar to step (S1). After steps (S1) and (S2), there is a fault-free u–$y_{n(n-1)\cdots 5-1}$ path of length $n! - 2|F_v| - 25$.

Step (S3) is explained as follows. Assume $v = v_1 v_2 \dots v_n$, and let $D = \langle d_1 d_2 \dots d_n \rangle_3 \notin \{\alpha_{n(n-1)\cdots 5}, \beta_{n(n-1)\cdots 5}\}$ be a 3-vertex in $A_{n(n-1)\cdots 5}$ with $d_k \neq v_1$. Since $\beta_{n(n-1)\cdots 5}$ is the ending 3-vertex which is not connected to $A_{n(n-1)\cdots 5-1}$, we have $\alpha_{n(n-1)\cdots 5} \neq \beta_{n(n-1)\cdots 5}$. A fault-free Hamiltonian $\alpha_{n(n-1)\cdots 5} - \beta_{n(n-1)\cdots 5}$ path in the K_4^3 formed by $A_{n(n-1)\cdots 5}$ can be established as $(\alpha_{n(n-1)\cdots 5}, M_1, D, \beta_{n(n-1)\cdots 5})$, where M_1 is the other 3-vertex in $A_{n(n-1)\cdots 5}$.

Suppose $E = \langle e_1 e_2 \dots e_n \rangle_3$, $F = \langle f_1 f_2 \dots f_n \rangle_3$, and $G = \langle g_1 g_2 \dots g_n \rangle_3$ are arbitrary three consecutive 3-vertices in $(\beta_{n(n-1)\cdots 5-1}, \alpha_{n(n-1)\cdots 5}, M_1, D, \beta_{n(n-1)\cdots 5})$. Then, $e_{dif(E,F)} \neq g_{dif(F,G)}$ holds similarly. We note that v is not connected to D because $d_k \neq v_1$ and each edge between D and $\beta_{n(n-1)\cdots 5}$ is a k-dimensional edge. Since $A_{n(n-1)\cdots 5-1}$ and $A_{n(n-1)\cdots 5}$ are fault-free, Lemmas 9 and 10 assure that there are distinct fault-free vertices $x_{n(n-1)\cdots 5}$, $r \in \alpha_{n(n-1)\cdots 5}$, $s, t \in M_1$, and $w, z \in D$ so that every two consecutive vertices in $\{x_{n(n-1)\cdots 5}, r, s, t, w, z, f\}$ are adjacent, where $x_{n(n-1)\cdots 5}$ is the fault-free vertex in $\alpha_{n(n-1)\cdots 5}$ that is adjacent to $y_{n(n-1)\cdots 5-1}$ and f is a fault-free vertex in $\beta_{n(n-1)\cdots 5}$.

Let $(c_0, c_1, c_2, c_3, c_4, c_5)$ represent the ring formed by $\beta_{n(n-1)\cdots 5}$, where $c_0 = v$ is assumed. We let $y_{n(n-1)\cdots 5} = v = c_0$. Since v is not connected to D, we have $f \notin \{c_0, c_3\}$ by Lemma 9. If $dist(u, v)$ is even, then every u–v path has even length, because S_n is bipartite. We have $f \in \{c_2, c_4\}$, for otherwise $f \in \{c_1, c_5\}$ will cause a u–v path of length $n! - 2|F_v| - 1$, which is a contradiction. A fault-free $x_{n(n-1)\cdots 5} - y_{n(n-1)\cdots 5}$ path of length 22 can be generated in $A_{n(n-1)\cdots 5}$ accordingly. Similarly, if $dist(u, v)$ is odd, then $f \in \{c_1, c_5\}$ and a fault-free Hamiltonian $x_{n(n-1)\cdots 5} - y_{n(n-1)\cdots 5}$ path of length 23 can be generated in $A_{n(n-1)\cdots 5}$.

A fault-free u–v path can result if the fault-free $x_1 - y_1, x_2 - y_2, \dots, x_{n(n-1)\cdots 5} - y_{n(n-1)\cdots 5}$ paths in $A_1, A_2, \dots, A_{n(n-1)\cdots 5}$, respectively, interleaved with edges (y_1, x_2), $(y_2, x_3), \dots, (y_{n(n-1)\cdots 5-1}, x_{n(n-1)\cdots 5})$. The path has length $n! - 2|F_v| - 2$ if $dist(u, v)$ is even, and $n! - 2|F_v| - 1$ if $dist(u, v)$ is odd. □

Since S_n is bipartite with two partite sets of equal size, a u–v path in S_n has length at most $n! - 2|F_v| - 2$ if $dist(u, v)$ is even, and at most $n! - 2|F_v| - 1$ if $dist(u, v)$ is odd, for the worst-case scenario. As a consequence of Lemma 11, we have the following theorem.

Theorem 1. *Suppose u, v are arbitrary two distinct fault-free vertices in an S_n with $|F_v| \leqslant n-5$ faulty vertices, where $n \geqslant 6$. There is a fault-free u–v path in the S_n whose length is $n!-2|F_v|-2$ if $dist(u,v)$ is even, and $n!-2|F_v|-1$ if $dist(u,v)$ is odd. The path is longest for the worst-case scenario.*

The authors have also discussed the situation of $|F_v| = n-4$ or $n-3$, and the result appeared in [10].

Theorem 2 (Hsieh et al. [10]). *Suppose u, v are arbitrary two distinct fault-free vertices in an S_n with $|F_v| = n-4$ or $n-3$ faulty vertices, where $n \geqslant 4$. There is a fault-free u–v path of length at least $n!-4|F_v|-10$ if $dist(u,v)$ is even, and at least $n!-4|F_v|-9$ if $dist(u,v)$ is odd.*

4. Concluding remarks

In this paper, we have shown that an n-dimensional star graph with $|F_v| \leqslant n-5$ faulty vertices contains a fault-free path of length $n!-2|F_v|-2$ ($n!-2|F_v|-1$) between arbitrary two distinct vertices of even (odd) distance, where $n \geqslant 6$. Since the star graph is bipartite with two partite sets of equal size, the path is longest for the worst-case scenario. If the two end vertices are adjacent, then a fault-free ring of length $n!-2|F_v|$ can result. This improves Tseng et al.'s work [20] for $|F_v| \leqslant n-5$. In [20], a fault-free ring of length $n!-4|F_v|$ can be determined in an n-dimensional star graph with $|F_v| \leqslant n-3$ faulty vertices.

We also discussed the situation of $|F_v| = n-4$ or $n-3$ in [10], where $n \geqslant 4$. It is still unknown whether or not the path is longest for the worst-case scenario. Since an n-dimensional star graph is regular of degree $n-1$, $|F_v| = n-3$ is maximal in order to construct a longest fault-free path between arbitrary two distinct vertices.

Incidentally, our result reveals that a star graph without faulty vertices contains a path of length $n!-1$ between arbitrary two distinct vertices of odd distance. That is, this paper provides an alternative proof that the star graph is Hamiltonian. An earlier proof appeared in [11].

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