Theoretical Computer Science 262 (2001) 349–375

Theoretical Computer Science

www.elsevier.com/locate/tcs

Multiplicities of covers for sofic shifts

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Received 15 December 1999; revised 15 April 2000; accepted 22 June 2000

Communicated by D. Perrin

Abstract

We consider a transitive sofic shift T and a SFT cover $f : S \to T$. We define the *multiplicity of the cover* (S, f) to be the largest number of preimages of a point. The *intrinsic multiplicity of* T is the minimum of the multiplicities over all covers of T, denoted by $m(T)$. Is $m(T)$ computable? We do not answer this question. However the attempt to solve this problem led us to find sharp estimates for the intrinsic multiplicity, sharpen a result of Williams, and solve a problem posed by Trow.

1. Introduction

We assume that the reader is familiar with the basic theory of shifts of finite type and sofic shifts as presented in [5] or [4].

Let T be a sofic shift, S a transitive shift of finite type (SFT) and $f : S \to T$ a factor map, i.e., f is continuous, shift commuting and onto. Then either there is a point $y \in T$ such that $f^{-1}y$ is an uncountable set of points or there is some integer M such that $\#f^{-1}y \leqslant M$ for all $y \in T$. We define the *multiplicity of* (S, f) to be $m(S, f) = \max\{\#f^{-1}y \mid y \in T\}$ if all $f^{-1}y$ are finite and $m(S, f) = \infty$ otherwise. The *intrinsic multiplicity of* T, denoted by $m(T)$, is the minimum of all multiplicities, that is $m(T) := \min\{m(S, f) \mid f : S \to T$ is a factor map, S is a transitive SFT$\}$. Then $m(T) < \infty$ for all sofic T. Since $m(S, f) = 1$ iff f is a conjugacy, $m(T) = 1$ iff T is a SFT. The intrinsic multiplicity is a conjugacy invariant, i.e., if R, T are sofic shifts and $g : R \to T$ is a conjugacy (a shift commuting homeomorphism) then $m(R) = m(T)$.

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0304-3975/01/$ - see front matter
PII: S0304-3975(00)00278-4

To compute $m(S,f)$ it is often possible to just consider periodic points in T, that is points in T which are fixed under some power of the shift. We define the periodic multiplicity of (S,f) to be $pm(S,f)=\max\{\#f^{-1}y \mid y\in T \text{ periodic}\}$ if for all y periodic the set $f^{-1}y$ is finite, and $pm(S,f)=\infty$ otherwise. The intrinsic periodic multiplicity of T is $pm(T)=\min\{pm(S,f)\mid f:S\to T$ is a factor map, S is a transitive SFT$\}$. Since $pm(S,f)\leqslant m(S,f)$, we have $pm(T)\leqslant m(T)$. Furthermore $pm(S,f)<\infty$ iff $m(S,f)<\infty$. The periodic multiplicity of T is also a conjugacy invariant. We show that $pm(T)<m(T)$ is possible (Example 2.5).

Let A be a finite set. Let G be a finite irreducible graph with a set E of directed edges and vertex set V and let $\Lambda:E\to A$ be a map. Then (G,Λ) defines a transitive SFT S_G whose points are the bi-infinite paths along edges in G and Λ defines a continuous shift commuting map $f_\Lambda:S_G\to A^{\mathbb{Z}}$ by $f_\Lambda(x)_i:=\Lambda(x_i)$, $i\in\mathbb{Z}$. Then $T:=f_\Lambda(S_G)$ is a sofic shift and $f_\Lambda:S_G\to T$ is a factor map. We call (G,Λ) a cover for T. On the other hand, if S is a transitive SFT and $f:S\to T$ is a factor map, then there is a cover (G,Λ) for T and a conjugacy $g:S\to S_G$ such that $f_\Lambda\circ g=f$ [5]. Since g is a conjugacy, we thus have $m(S,f)=m(S_G,f_\Lambda)$. We call $m(S_G,f_\Lambda)$ the multiplicity of the cover (G,Λ) and denote it by $m(G,\Lambda)$. Thus, $m(T)=\min\{m(G,\Lambda)\mid (G,\Lambda)$ is a cover for $T\}$.

Observe that either $pm(G,\Lambda)=\infty$ or $pm(G,\Lambda)\leqslant$ number of vertices of G, since a bounded-to-1 map has no diamonds, [5].

Given any labeled graph (G,Λ) there is a decision procedure to see if (G,Λ) is a cover for T, [5]. We show how to compute the multiplicity of a cover (G,Λ) for T (Lemma 2.1).

Since there are only countably many covers of a sofic shift and the multiplicity of a cover (G,Λ) is computable, it is natural to ask:

"Is the intrinsic multiplicity of a sofic shift computable?"

We do not answer this question. Though the attempt to solve this problem led us to find sharp estimates for the intrinsic multiplicity (Theorems 2.6 and 2.7), sharpen a result of Williams (Theorem 5.1), and solve a problem of Trow (Example 4.2).

Before we proceed we give a rough outline of the content of the paper.

Every sofic shift has two distinguished covers, the right and the left Fischer cover [5]. These covers are conjugacy invariants for the sofic shift. Every right closing cover factors through the right Fischer cover [5]. Thus the multiplicity of the right Fischer cover is the least multiplicity under all right closing covers, and the multiplicity of the left Fischer cover is the least multiplicity under all left closing covers. Furthermore, if T is AFT, that means that the right Fischer cover is conjugate to the left Fischer cover, then every cover factors through one of the Fischer covers [2]. Thus, if T is AFT, then $m(T)$ is the multiplicity of the right and of the left Fischer cover. In general however, the intrinsic multiplicity can be strictly smaller than the minimum over the multiplicities of the Fischer covers (Examples 3.2, and 4.1). We show that if one of the Fischer covers of a sofic shift T has multiplicity $\geqslant 2^n-1$ then $m(T)\geqslant n$ (Theorem 2.6) and that this lower bound for the intrinsic multiplicity is sharp (Theorem 2.7).

There are several finite subsets of covers of a sofic shift which can be computed. Since the multiplicity of a given cover can be computed, if one could show that for every sofic shift there is a cover of least multiplicity in a finite computable subset of covers, then one would have a computing procedure for the intrinsic multiplicity. Examples for those finite subsets of covers which can be computed are

$$C_1(T) := \{(G, \Lambda) \mid \text{the number of vertices in } G \text{ is at most the number of vertices in either of the Fischer covers}\}.$$

$$C_2(T) := \{(G, \Lambda) \mid (S_G, f_\Lambda) \text{ is a factor of the fibre product of the Fischer covers}\}.$$

$$C_3(T) := \{(G, \Lambda) \mid (G, \Lambda) \text{ is a minimal lifting cover}\}.$$

The set $C_2(T)$ is computable [1], and the set $C_3(T)$ is computable [6]. The set $C_3(T)$ is closely related to $C_2(T)$, since a degree 1 cover is lifting iff it is a factor of the fiber product of the Fischer covers [6]. We do not know if multiplicity considered only for degree 1 covers could be strictly bigger than multiplicity of all covers.

We show however, that for every $i \in \{1, 2\}$ there is a sofic shift T such that no cover with least multiplicity is in $C_i(T)$. A variation of the last of these examples solves a problem of Trow [6]. One can easily use those examples to construct a sofic shift T such that no cover of least multiplicity lies in $C_1(T) \cup C_2(T)$, (but we do not here). Thus, there is no easy way to compute the intrinsic multiplicity.

We show that for every positive integer k there is a positive integer $N(k)$ such that if T is a sofic shift which has a cover with k vertices, then there is a cover with least multiplicity having at most $N(k)$ vertices. However, we do not know how to compute $N(k)$.

Finally we investigate how many covers with least multiplicity a sofic shift can have. We show that for every sofic non-AFT shift there is some N and infinitely many non-conjugate covers having multiplicity N and having no proper factors (Theorem 5.1). This sharpens a result of S. Williams [7, 8]. As an application of Theorem 5.1 we obtain a sofic shift having infinitely many non-conjugate covers having least multiplicity (Example 5.2).

2. A sharp lower bound for the intrinsic multiplicity

Throughout this paper all sofic shifts considered will be transitive. We start with the simple observation that for a given cover the multiplicity is computable. Let T be a transitive sofic shift. For a point $y \in T$ and $n \leqslant m$ we denote by $y[n, m]$ the subblock $y_n y_{n+1} \ldots y_{m-1} y_m$ of y. We denote by $B_n(T) = \{y[1, n] \mid y \in T\}$ the T-blocks of length n. Let (G, Λ) be a cover for T. Let $p = p_1 \ldots p_n$ be a path of length n in G. Let $a_i = \Lambda(p_i)$ and $w = a_1 \ldots a_n$. Then we say the path p is labeled by the block w. We say that two distinct paths p, q in G with the same length, say n, both starting at the same vertex and ending at the same vertex and such that p and q are labeled by the same block w form a diamond of length n. Then $m(G,\Lambda) < \infty$ iff (G,Λ) has no diamonds [5].

Lemma 2.1. *The multiplicity* $m(G,\Lambda)$ *of a cover* (G,Λ) *of* T *is computable.*

Proof. Let n be the number of edges in G. First we check if the cover has a diamond. If p, q form a diamond of length $N > n^2$ then there are some $1 \leqslant i < j \leqslant n^2$ such that $p_i = p_j$ and $q_i = q_j$. Thus $p' = p_1 \dots p_i p_{j+1} \dots p_N$ and $q' = q_1 \dots q_i q_{j+1} \dots q_N$ form a diamond of length $< N$. Thus, if the cover has a diamond then it has a diamond of length $\leqslant n^2$. Thus, we can decide if $m(G,\Lambda) < \infty$ by considering all paths of length $\leqslant n^2$.

Now let (G,Λ) be a cover without diamonds. Let $S = S_G$ and $f = f_\Lambda$. We compute $d := m(S,f)$. Since the cover has no diamonds, $d \leqslant n^2$. Let $y \in T$ such that $f^{-1}y = \{x^1,\dots,x^d\}$. There is some $M \leqslant 0 \leqslant K$ such that $\#\{x[M,K] \mid x \in f^{-1}y\} = d$. For $i \in \mathbb{Z}$ let $E_i = ((x^1)_i,\dots,(x^d)_i)$. Let $N = n^{(n^2)} + 1$. If $M + K + 1 > N$ then there is some $M \leqslant i < j \leqslant K$ with $E_i = E_j$. Let u^k be the points in S with $u^k(-\infty,i] = x^k(-\infty,i]$ and $u^k[i,\infty) = x^k[j,\infty)$, $1 \leqslant k \leqslant d$. Then, since there are no diamonds, $\#\{u^k[M,K-(j-i)] \mid 1 \leqslant k \leqslant d\} = d$ and, since $f = f_\Lambda$, it holds $f(u^k) = f(u^{k'})$, $1 \leqslant k,k' \leqslant d$. Thus, repeating the argument at most $M + K + 1 - N$ times shows, there is a point $z \in T$ with $\#\{v[a,b] \mid v \in f^{-1}z\} = d$ and $b - a + 1 \leqslant N$.

Thus we can compute $m(S,f)$ as follows: For a T-block m of length $3N$ let $D(m) := \{v \mid |v| = N$ and there are u, w with $|u| = |w| = N$ such that uvw is a path in G with $\Lambda(uvw) = m\}$. Let m be a T-block such that $\#D(m)$ is maximal. Then by the above $m(S,f) \leqslant \#D(m)$. For $v \in D(m)$ let u^v, w^v be paths in G of length N such that $u^v v w^v$ is a path in G with $\Lambda(u^v v w^v) = m$. Write $u^v = u_1^v \dots u_N^v$ and $w^v = w_1^v \dots w_N^v$. Since $N = n^{(n^2)} + 1$ and $\#D(m) \leqslant n^2$, there are $1 \leqslant i < j \leqslant N$, $1 \leqslant k < l \leqslant N$ such that $u_i^v = u_j^v$ and $w_k^v = w_l^v$ for each $v \in D(m)$. Let $a^v = u_i^v \dots u_{j-1}^v$, $b^v = w_k^v \dots w_{l-1}^v$. Then $x^v := (a^v)^\infty u_j^v \dots u_N^v v w_1^v \dots w_{k-1}^v (b^v)^\infty$ are $\#D(m)$ periodic points in S with $f(x^v) = f(x^{v'})$ for all $v, v' \in D(m)$. Thus $f(x^v)$ is a point with $\#D(m)$ preimages and thus $m(S,f) = \#D(m)$. Thus, we can compute $m(S,f)$ by checking all paths in G of length $3N$. $\square$

We need a compact way to describe a labeled graph (G,Λ). We introduce a matrix $M = M(G,\Lambda)$ as follows. Say, G is a graph with vertex set V and edge set E and $\Lambda : E \to A$ is a map assigning labels to the edges in E. For $i,j \in V$ and $a \in A$ let $n_{i,j}(a) := \#\{e \in E \mid e$ is an edge from vertex i to vertex j and $\Lambda(e) = a\}$. Then we define $M_{i,j} = \sum_{a\in A} n_{i,j}(a)a$ for $i,j \in V$ and M is a $\#V \times \#V$ matrix with entries in the free abelian group with generators A. In particular $M_{i,j} = 0$ iff there is no edge in G from vertex i to vertex j. On the other hand, given an $n \times n$ matrix M with entries $M_{i,j} = \sum_{a\in A} n_{i,j}(a)a$, $n_{i,j}(a) \geqslant 0$, then M defines a labeled graph (G_M,Λ_M) where G_M is a graph with vertex set $V = \{0,\dots,n-1\}$ and there are $\sum_{a\in A} n_{i,j}(a)$ edges from

i to j if $M_{i,j} = \sum_{a\in A} n_{i,j}(a)a$. Then Λ_M is defined so that for all i, j we have $\Lambda_M(e) = a$ for exactly $n_{i,j}(a)$ edges from i to j. The graphs G and G_M define conjugate shifts and Λ and Λ_M induce factor maps with the same multiplicity.

We now start with an easy example showing that every integer occurs as the intrinsic multiplicity of some sofic shift. For that recall that a labeled graph (G, Λ) is called 1-step right-closing (resp. 1-step left-closing), if e, e' are distinct edges in G with the same initial vertex (resp. terminal vertex) then $\Lambda(e) \neq \Lambda(e')$. Recall further that the follower set of a vertex α is the set of T-blocks w such that there is a path p in G starting at α with label w. Similarly predecessor sets are defined.

Example 2.2. Let $n \geqslant 1$. We define $T \subset \{a, b\}^{\mathbb{Z}}$. Let M be the $n \times n$ matrix with $M_{i,j} = a$ if $j = i + 1 \bmod n$, $M_{0,0} = b$ and $M_{i,j} = 0$ otherwise. The labeled graph (G, Λ) defined by M is 1-step right- and left-closing and has all follower sets distinct, and is thus the Fischer cover of the AFT system $T = f_\Lambda(S_G)$ [5]. Thus $m(T) = m(G, \Lambda) = pm(T) = n$ [2]. $\square$

Lemma 2.3. *Let (G, Λ) be a cover for T with Λ 1-step right- or 1-step left-closing. Then $pm(G, \Lambda) = m(G, \Lambda)$.*

Proof. Let Λ be 1-step left closing. Let $(S, f) = (S_G, f_\Lambda)$ and $y \in T$ with $\#f^{-1}y = m(G, \Lambda)$. Since Λ is 1-step left closing, there is some K such that for all $k \geqslant K$ the $\#\{x_k \mid x \in f^{-1}y\} = m(G, \Lambda)$. Thus there are $K \leqslant i < j < \infty$ such that $x_i = x_j$ for all $x \in f^{-1}y$. Hence, $y[i, j-1]^\infty$ is a periodic point in T with $m(G, \Lambda)$ preimages. The case Λ 1-step right closing is symmetric. $\square$

Recall that the right Fischer cover (F^+, π^+) of a transitive sofic shift T is the unique transitive cover which has all follower sets of vertices distinct and is 1-step right-closing. A cover is right-closing (left-closing) if there is some N such that any two distinct paths of length N starting (ending) with the same edge have distinct labels. Every right-closing cover (G, Λ) factors through the right Fischer cover, that means there is a (surjective) factor map $g: S_G \to S_{F^+}$ such that $f_\Lambda = f_{\pi^+} \circ g$ [2] and thus $m(G, \Lambda) \geqslant m(F^+, \pi^+)$ for any right-closing cover (G, Λ).

Let (G, Λ) be a cover for T. If (G, Λ) is not right-closing there is a vertex α and two distinct right infinite paths x^1, x^2 starting at α with the same labels. Similarly if the cover is not left-closing there is a vertex β and two distinct left infinite paths y^1, y^2 ending at β with the same labels. Thus if a cover is neither right- nor left-closing then we can choose a path p from β to α, and then the four distinct points $y^i p x^j \in S_G$ with the zero coordinate at the first symbol of x^j say, $i, j \in \{1, 2\}$ all have the same image. Thus a cover which is neither right- nor left-closing has multiplicity at least 4, thus

Lemma 2.4. *Let (F^+, π^+) and (F^-, π^-) be the right and left Fischer cover of a sofic shift T. If $\min(m(F^+, \pi^+),\ m(F^-, \pi^-)) \leqslant 4$, then $m(T) = \min(m(F^+, \pi^+),\ m(F^-, \pi^-))$.*

Example 2.5. A sofic shift T with $pm(T) < m(T)$. Let (G, Λ) be the labeled graph defined by

$$M = \begin{pmatrix} a & b & b \\ 0 & a+b & e \\ c & d & a+b \end{pmatrix},$$

and $T = f_\Lambda(S_G)$. Then G has only 3 vertices, (G, Λ) has no diamonds and thus $pm(G, \Lambda) = 3$, since $f_\Lambda^{-1}(a^\infty) = 3$. It is easy to check that both Fischer covers (for an easy way to construct these, see the paragraph below) have multiplicity 4, thus by Lemma 2.4, $m(T) = 4$. And thus $pm(T) < m(T)$.

Example 2.9 below provides an example with $pm(T) = 2$ and $m(T) = 3$.

The multiplicity of a Fischer cover is trivially an upper bound for the intrinsic multiplicity of the sofic shift. Now we shall give a lower bound for the intrinsic multiplicity of a sofic shift in terms of the multiplicity of a Fischer cover. We shall show that this lower bound is sharp. For these purposes we recall the construction of the "subset follower cover" of a given cover [5, 6].

Let (G, Λ) be a cover for a sofic shift T. We first construct the complete subset follower graph as follows: The vertices are the non-empty subsets of the set of vertices of G. Let E, F be non-empty subsets of the set of vertices of G. Draw an edge labeled a from E to F iff F is exactly the set of terminal vertices of all the edges starting in E, which are labeled a. This defines a 1-step right-closing labeled graph, which is in general not irreducible. But it always has an irreducible component factoring onto T: For any T-block w let $E(w)$ be the set of vertices α in G for which there is a path in G ending at α labeled w. Let w be a T-block such that $\#E(w)$ is minimal. Then for all T-blocks wuw it holds that $E(w) = E(wuw)$, and thus $E(w)$ is contained in an irreducible component of the subset graph which factors onto T. Let G^+ be the irreducible component containing the vertex $E(w)$ and Λ^+ be the restriction of Λ to the edges of G^+. Then (G^+, Λ^+) is called the subset follower cover of (G, Λ) and it is a 1-step right-closing cover for T. Recall that (G^+, Λ^+) factors through the right Fischer cover by identifying the vertices with the same follower sets [5].

The subset predecessor cover (G^-, Λ^-) is constructed similarly using predecessors instead of followers.

Theorem 2.6. *Let T be a transitive sofic shift and (G, Λ) be a cover of T with $pm(G, \Lambda) = n$. Then both Fischer covers have multiplicity at most $2^n - 1$. Thus, if one of the Fischer covers has multiplicity $\geqslant 2^n - 1$ then $m(T) \geqslant pm(T) \geqslant n$.*

Proof. We shall show that the subset follower cover (G^+, Λ^+) of (G, Λ) has multiplicity at most $2^n - 1$. The argument for the subset predecessor cover is symmetric. Since the Fischer covers are factors of the subset covers, the result follows.

Let E, F be vertices in G^+ such that there is an edge from E to F labeled a, say. Then for every $v \in F$ there is some $u \in E$ and an edge e in G from u to v with

$\Lambda(e)=a$. Thus, if there is a path in G^+ of length n from vertex E to vertex F with labels $a=a_1\dots a_n$ then for all $v\in F$ there is some $u\in E$ and a path of length n in G from u to v with label a. We apply this to periodic points as follows.

Let $y\in T$ be periodic, with least period p, say. Let $a=y[1,p]$ and let $x\in(f_{\Lambda+})^{-1}(y)$. Then there is some $k\geqslant 1$ and distinct vertices $E_0,E_1,E_2,\dots,E_{k-1}$ in G^+ such that for each $0\leqslant i\leqslant k-1$ the block $x[1+ip,p+ip]$ is a path labeled a from E_i to $E_{(i+1)\,\mathrm{mod}\,k}$. Thus, for every $v\in E_0$ there is $u(v)\in E_0$ such that in G there are paths from $u(v)$ to v labeled a^k. Since E_0 is a finite set, there is some m and a vertex $u\in E_0$ such that there is a path in G from u to u labeled a^{km}.

Let $F=\{v$ a vertex in $G\,|\,$there is some $m\geqslant 1$ and a path from v to v in G labeled $a^m\}$ and let $E=F\cap E_0$. Since E is a non-empty subset of E_0 and there is a path in G^+ from E_0 to E_0 labeled a^k, we get that every path in G that starts in a vertex of E and has label a^{km} for some $m\geqslant 1$, ends in a vertex of E_0. But, for $v\in E_0-E$ there is some $w\in E$ and some large $m\geqslant 1$ and a path in G from w to v labeled a^{km}. Therefore, E_0 is the set of terminal vertices of paths in G which start in E and have a label a^{km} for some $m\geqslant 1$.

Thus

$$\begin{aligned}\#(f_{\Lambda+})^{-1}(y) &= \#\{E_0\,|\,E_0 \text{ is initial vertex of } x_1 \text{ for some } x\in(f_{\Lambda+})^{-1}(y)\}\\ &= \#\{F\cap E_0\,|\,E_0 \text{ is initial vertex of } x_1 \text{ for some } x\in(f_{\Lambda+})^{-1}(y)\}\\ &\leqslant \#\{E\,|\,E\subset F,\ E\neq\emptyset\}\\ &= 2^{\#F}-1\\ &\leqslant 2^n-1,\end{aligned}$$

since $\#F=\#f_\Lambda^{-1}(y)\leqslant pm(G,\Lambda)=n$. This holds for all periodic points $y\in T$ and thus $pm(G^+,\Lambda^+)\leqslant 2^n-1$. Since Λ^+ is right closing, by Lemma 2.3, this proves the theorem. □

This theorem gives thus a lower bound for the intrinsic multiplicity: If T is sofic and one of the Fischer covers has multiplicity $\geqslant 2^n-1$ then $m(T)\geqslant n$. The next theorem shows that this lower bound is sharp. Furthermore it provides an example where the Fischer covers have different multiplicities.

Theorem 2.7. *Let $n\geqslant 2$. Then there is a transitive sofic shift T where the left Fischer cover has multiplicity n and the right Fischer cover has multiplicity 2^n-1. And thus $m(T)=n$.*

Proof. We define a sofic shift T with symbols $\{a\}\cup\{b_1,\dots,b_n\}\cup\{a_E\,|\,\emptyset\neq E\subset\{1,\dots,n\}\}$. Let $M=A+B+C$ be the $n\times n$ matrix where $A=a\cdot Id$, $B_{i,1}=b_i$, $1\leqslant i\leqslant n$ and $B_{i,j}=0$ for $j\neq 1$ and $C_{1,j}=\sum_{\{E|j\in E\}}a_E$ and $C_{i,j}=0$ if $i\neq 1$.

Let (G,Λ) be the labeled graph defined by M and let T be the sofic shift defined by (G,Λ). Then the fix point $a^\infty\in T$ has exactly n preimages in S_G. Since every T-symbol occurs at most once in each column of M, the labeling Λ is 1-step left closing. Thus,

$m(G,\Lambda) = pm(G,\Lambda) = n$, by Lemma 2.3. The T-symbol $a_{\{j\}}$ occurs in the matrix M only in column j. Thus the vertices in G have distinct predecessor sets, and thus (G,Λ) is the left Fischer cover of T.

Consider the complete follower subset graph (G',Λ') of (G,Λ). Let E,F be vertices in G', that is E,F are non empty subsets of $\{1,\dots,n\}$. Say $i \in E$. Then in (G',Λ') there is an edge from E to $\{1\}$ labeled b_i and there is an edge from vertex $\{1\}$ to F labeled a_F. Thus G' is irreducible, and thus $(G',\Lambda') = (G^+,\Lambda^+)$. Let E be a vertex in (G^+,Λ^+). Then there is an edge with label b_i in (G^+,Λ^+) starting from E iff $i \in E$. Thus in (G^+,Λ^+) the follower sets are distinct, and thus (G^+,Λ^+) is the right Fischer cover of T.

Since for each vertex i of G there is an edge from i to i labeled a, for each vertex E in G^+ there is an edge from E to E labeled a. Thus $m(G^+,\Lambda^+) =$ number of vertices of $G^+ = 2^n - 1$. □

We conclude this section with the consideration of a special class of covers, the bi-synchronizing covers. We shall see that a bi-synchronizing cover with finite multiplicity is always a cover with least periodic multiplicity. Thus a bi-synchronizing cover is useful for determining the intrinsic periodic multiplicity. However bi-synchronizing covers not always exist, so this concept is not always applicable.

Let (G,Λ) be a cover for a sofic shift T. We say that a vertex v is (*cosynchronizing*) *synchronizing* if there is a T-block x_v, such that every path in G with label x_v (starts) ends at vertex v. We say that (G,Λ) is (*cosynchronizing*) *synchronizing* if every vertex is (cosynchronizing) synchronizing. Thus, the right Fischer cover is synchronizing and the left Fischer cover is cosynchronizing. A cover (G,Λ) is *bi-synchronizing* if it is both synchronizing and cosynchronizing.

A bi-synchronizing cover (G,Λ) has the least number of vertices in any possible cover ([3, Theorem 10.1]). Thus, it is easy to compute if a sofic shift has a bi-synchronizing cover or not. There is a SFT that has no bi-synchronizing cover, for example the SFT given by the labeled graph defined by the matrix

$$\begin{pmatrix} 0 & a & c \\ 0 & b & c \\ b & a & 0 \end{pmatrix},$$

but there are non-AFT shifts with bi-synchronizing right Fischer cover, for example the sofic shift given by the matrix

$$\begin{pmatrix} a & c & b \\ 0 & a & b+e \\ d & 0 & 0 \end{pmatrix}.$$

Theorem 2.8. *Let (G,Λ) be a bi-synchronizing cover of a sofic shift T. Then $m(G,\Lambda) < \infty$ implies $pm(T) = pm(G,\Lambda)$.*

Proof. Since $m(G,\Lambda)<\infty$, there is a periodic point $p^\infty \in T$ with least period $|p|$ such that $\# f_\Lambda^{-1}(p^\infty) = pm(G,\Lambda)$. The preimages of p^∞ in S_G can be partitioned into say n orbits, the i'th orbit of length, say $|p|\cdot k(i)$. There are thus n vertices, say $v_1,\dots,v_n$ in G such that there is a loop at v_i labeled $p^{k(i)}$, for $1\leqslant i\leqslant n$ and for $i\neq j$ there is no path from v_i to v_j labeled p^r, for some $r\geqslant 1$. Since the cover (G,Λ) is bi-synchronizing, we can fix for each i a T-block $x_i y_i$ such that x_i is synchronizing for v_i and y_i is co-synchronizing for v_i. Now consider a cover (G',Λ') with $pm(G',\Lambda') = pm(T)$. Let $s>$ number of vertices in G' and fix $1\leqslant i\leqslant n$. Then there is a path in G' labeled $x_i p^{sk(i)} y_i$. By the choice of s, there is a vertex u_i in G' and integers $a(i)$, $b(i)$, $c(i)\geqslant 0$, $b(i)\geqslant 1$ such that there is a path labeled $x_i p^{a(i)}$ which ends in vertex u_i, there is a loop labeled $p^{b(i)}$ at u_i and there is a path labeled $p^{c(i)} y_i$ starting from u_i. Now let $r\geqslant 0$ and $1\leqslant i$, $j\leqslant n$ such that there is a path from u_i to u_j in G' labeled p^r. Then $x_i\, p^{a(i)+r+c(j)} y_j$ is a T-block. Since x_i is synchronizing for the vertex v_i and y_j is cosynchronizing for the vertex v_j in G, it follows that there is a path from v_i to v_j in G labeled $p^{a(i)+r+c(j)}$. But G is a finite-to-1 cover and by the choice of the vertices v_i it follows $b(i) = 0 \bmod k(i)$. Thus, $\#(f_{\Lambda'})^{-1}(p^\infty) \geqslant k(1)+k(2)+\cdots+k(n) = pm(G,\Lambda)$. Thus $pm(G,\Lambda)\leqslant pm(G',\Lambda') = pm(T)$ and the theorem is proved. □

The last theorem thus shows that a finite-to-1 bi-synchronizing cover always has least periodic multiplicity. However it need not be a cover of least multiplicity:

Example 2.9. Let $T\subset\{a,b,c,d,u,v,x,y,r\}^{\mathbb{Z}}$ be given by the matrix

$$M = \begin{pmatrix} b & u & d & 0 & 0 \\ v & b & d & 0 & 0 \\ 0 & 0 & 0 & c & c \\ 0 & 0 & 0 & a & x \\ 0 & r & 0 & y & a \end{pmatrix}.$$

Let (G,Λ) be the labeled graph defined by M. In each row (column) of M there is a symbol which occurs only in this row (column). Thus each vertex in G is bi-synchronizing and thus, since the cover has no diamonds $m(G,\Lambda)<\infty$, $pm(T) = pm(G,\Lambda) = 2$. Considering the T-orbit $b^\infty dca^\infty$ shows $m(G,\Lambda)\geqslant 4$. Constructing the Fischer covers shows that $m(F^+,\pi^+) = m(F^-,\pi^-) = 3$ and thus, by Lemma 2.4, $m(T) = 3$.

Thus this is also an example for a sofic shift with $pm(T) = 2 < 3 = m(T)$.

3. Sofic shifts for which every cover with least multiplicity has more vertices than the Fischer covers

In the last section we have seen that a bi-synchronizing cover with finite multiplicity always has least periodic multiplicity. Bi-synchronizing covers always have the least number of vertices, [3, Theorem 10.1]. So one might ask if there is always a cover with least multiplicity also having the least number of vertices. We shall see however that there are sofic shifts where no cover with least multiplicity has least number of vertices (Example 3.1). In this example the right Fischer cover is a cover with least multiplicity. However, we shall use Example 3.1 to show in fact stronger, that there is a sofic shift such that every cover with least multiplicity has more vertices than both of the Fischer covers (Example 3.2).

Example 3.1. A sofic shift where no cover with least multiplicity has the least number of vertices.

The following matrix represents the right Fischer cover of a sofic shift. It has 14 vertices and (periodic) multiplicity 9. The left Fischer cover of this sofic shift has multiplicity $\geqslant 11$ and 13 vertices. Any cover of this sofic shift with $\leqslant 13$ vertices has multiplicity >9. The intrinsic multiplicity is 9.

In the following we prove our claims. Let M be the matrix

$$\begin{array}{cc}
 & \begin{array}{ccccccccccccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & u_1 & u_2 & u_3 & u_4 & u_5 \end{array} \\
\begin{array}{c} b_1 \\ b_1+b \\ b_2 \\ b_2+b \\ b_3 \\ b_3+b \\ b_4 \\ b_4+b \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} &
\begin{pmatrix}
a & c & . & . & . & . & . & . & . & . & . & . & . \\
. & a+c & . & . & . & . & . & . & . & . & . & . & . \\
. & . & a & c & . & . & . & . & . & . & . & . & . \\
. & . & . & a+c & . & . & . & . & . & . & . & . & . \\
. & . & . & . & a & c & . & . & . & . & . & . & . \\
. & . & . & . & . & a+c & . & . & . & . & . & . & . \\
. & . & . & . & . & . & a & c & . & . & . & . & . \\
. & . & . & . & . & . & . & a+c & . & . & . & . & . \\
. & . & . & . & . & . & . & . & a+c & . & . & . & . \\
. & . & . & . & . & . & . & . & a & c & . & . & . \\
. & . & . & . & . & . & . & . & a & . & c & . & . \\
. & . & . & . & . & . & . & . & a & . & . & c & . \\
. & . & . & . & . & . & . & . & a & . & . & . & c
\end{pmatrix}
\end{array}.$$

Here $M_{i,j} = .$ means $M_{i,j} = 0$. Let T be the sofic shift given by the cover defined by M. Thus the T-symbols are

$$\{a,c\} \cup \{a_i \mid 1 \leqslant i \leqslant 8\} \cup \{b_i \mid 1 \leqslant i \leqslant 4\} \cup \{b\}$$
$$\cup \{u_i \mid 1 \leqslant i \leqslant 5\} \cup \{v_i \mid 1 \leqslant i \leqslant 5\}.$$

Since in every row of M a T-symbol occurs at most once, the labeling is 1-step right closing. Using the first column of M one gets that in the labeled graph defined by M all follower sets are distinct. Thus the labeled graph defined by M is the right Fischer cover (F^+, π^+) of T. Recall that a T-block m is called synchronizing, if it holds that um, mv are T-blocks then umv is a T-block. All symbols except $\{a,c\}$ are synchronizing. Thus, if $y \in T$ periodic with $y_i \notin \{a,c\}$ for some $i \in \mathbb{Z}$ then y has a unique preimage in S_{F^+}. If $y \in T$ periodic and $y_i \in \{a,c\}$ for all i, then y has at most 9 preimages and if y is a fix point then it has 9 preimages in S_{F^+}. Thus, the right Fischer cover of T has 14 vertices and multiplicity 9.

We want to determine the number of vertices of the left Fischer cover of T. For this we use the fact that (F^-,π^-) is the subset predecessor cover of (F^+,π^+) [6]. Let the rows of M, and thus the vertices of F^+ also, be enumerated with $0,1,2,\dots,13$. For a T-block m let $E(m)$ be the set of vertices α in F^+ such that there is a path labeled m starting in α.

Then

$$\begin{aligned}
&E(a_i)=E(u_j)=\{0\}, && 1\leqslant i\leqslant 8,\ 1\leqslant j\leqslant 5,\\
&E(v_i)=E(cv_i)=\{8+i\}, && 1\leqslant i\leqslant 5,\\
&E(av_1)=E(wav_1)=\{9,10,11,12,13\}, && w\in\{a,c\},\\
&E(b_i)=E(wb_i)=\{2i-1,2i\},\ 1\leqslant i\leqslant 4, && w\in\{a,c\},\\
&E(b)=E(ab)=\{2,4,6,8\}, && \\
&E(cb)=E(wcb)=\{1,2,3,4,5,6,7,8\}, && w\in\{a,c\}.
\end{aligned}$$

These are all vertices of the subset predecessor cover of (F^+,π^+). Thus the left Fischer cover has 13 vertices. Furthermore, the above list shows that each of the 11 vertices $E(v_i)$, $E(av_1)$, $E(b_i)$, $E(cb)$ has a loop labeled c and thus the fixed point $y=c^\infty$ has at least 11 preimages in the left Fischer cover, thus the multiplicity of the left Fischer cover is $\geqslant 11$.

Let (G,Λ) be any cover for T with at most 13 vertices. We want to show that $m(G,\Lambda)>9$. Assume $m(G,\Lambda)\leqslant 9$. Let V be the vertex set of G. Let α_0 be the terminal vertex of an edge labeled b_1. Since b_1y being a 2-block of T implies $y\notin\{a,c\}$, there is no edge in G labeled with a or c which begins in α_0.

We shall now use frequently the following trivial fact:

$(*)$ If there is a path labeled m_1 ending at vertex α and a path labeled m_2 starting at α then m_1m_2 is a T-block.

For $1\leqslant i\leqslant 5$ the block $u_ic^{13}v_i$ is a T-block, thus, since G has at most 13 vertices, we can fix a vertex δ_i such that for some $m,p\geqslant 0$, $n\geqslant 1$ (m,p,n depend on i) with $m+n+p=13$ there is a path labeled u_ic^m ending at δ_i, a loop labeled c^n at δ_i and a path labeled c^pv_i starting at vertex δ_i.

Similarly for $1\leqslant j\leqslant 4$ the block $a_{2j-1}c^{13}b_j$ is a T-block and thus there is a vertex γ_j and some $m,p\geqslant 0$, $n\geqslant 1$ (m,p,n depend on j) with $m+n+p=13$ and there is a path labeled $a_{2j-1}c^m$ ending at γ_j, a loop labeled c^n at γ_j and a path labeled c^pb_j starting at vertex γ_j.

By $(*)$ the set $V(c):=\{\delta_i\mid 1\leqslant i\leqslant 5\}\cup\{\gamma_j\mid 1\leqslant j\leqslant 4\}$ has cardinality 9. Furthermore from $(*)$ it follows that if for α, $\beta\in V(c)$ there is a path from α to β labeled c^k then $\alpha=\beta$. Since by assumption $m(G,\Lambda)\leqslant 9$, it follows that $m(G,\Lambda)=9$ and at every vertex from $V(c)$ there is a loop labeled c. (This shows that $m(T)=9$, too.)

Thus if α is a vertex and $1\leqslant j\leqslant 4$ such that there is a path labeled $a_{2j-1}c^{13}$ ending at α and a path labeled c^{13} starting at α, then $\alpha=\gamma_j$. Similarly if α is a vertex and $1\leqslant i\leqslant 5$ such that there is a path labeled u_ic^{13} ending at α and a path labeled c^{13} starting at α

then $\alpha=\delta_i$. Since $a_{2j-1}c^{26}a^{13}$, $1\leqslant j\leqslant 4$, and $u_ic^{26}a^{13}$, $1\leqslant i\leqslant 5$ are T-blocks, this also shows that the points $c^\infty a^\infty$, say with zero coordinate at the first a, have exactly 9 preimages. Therefore, in particular for $1\leqslant j\leqslant 4$ there is a unique vertex α_j such that if a path labeled $c^{13}a^{13}$ starting at vertex γ_j and ends at vertex α and there is a path labeled a^{13} starting at α then $\alpha=\alpha_j$. Thus, since $a_{2j-1}c^{26}a^{26}b_j$ and $a_{2j-1}c^{26}a^{26}b$ are T-blocks we obtain that there is a path starting at α_j which is labeled $a^{13}b_j$ and also a path labeled $a^{13}b$.

Since an edge labeled a starts at α_j, we get that $\alpha_j\neq\alpha_0$. Since $a_{2j-1}c^{26}a^{13}$ is a path ending at α_j and $a^{13}b_j$ is a path starting at α_j we obtain from $(*)$ that $\alpha_j\neq\delta_i$ for all $1\leqslant i\leqslant 5$ and that the set $\{\alpha_j\mid 1\leqslant j\leqslant 4\}$ has cardinality 4.

Now since for $1\leqslant j\leqslant 4$ the block $a_{2j-1}ab_j$ is a T-block, there is a vertex $\beta_j\neq\alpha_0$ such that there is an edge labeled a_{2j-1} ending at β_j and a path labeled ab_j beginning at β_j. Since $a_{2j-1}a^{13}b$ is not a T-block, by $(*)$ $\{\alpha_j\mid 1\leqslant j\leqslant 4\}\cap\{\beta_j\mid, 1\leqslant j\leqslant 4\}=\emptyset$ and $\#\{\beta_j\mid 1\leqslant j\leqslant 4\}=4$. But $\{\alpha_j\mid 1\leqslant j\leqslant 4\}\cup\{\beta_j\mid 1\leqslant j\leqslant 4\}$ is contained in $V-(\{\alpha_0\}\cup\{\delta_i\mid 1\leqslant i\leqslant 5\})$, a set of cardinality at most 7. This contradiction proves that the assumption $m(G,\Lambda)\leqslant 9$ was wrong and thus $m(G,\Lambda)>9$. $\square$

Example 3.2. A sofic shift such that every cover of least multiplicity has strictly more vertices than the Fischer covers.

We start with the matrix M from the last example. Let $0,1,2,\ldots,13$ be the vertices of the graph defined by M. We first add 55 vertices, named $14,15,\ldots,68$. For each new vertex $i\in\{14,\ldots,68\}$ we add an edge from 0 to i with label q_i, two loops at i one with label a, the other with label c and we add an edge from i to 0 labeled r_i. Call the obtained labeled graph (G,Λ) and let S denote the sofic shift defined by (G,Λ). The symbol set of S is

$$\begin{aligned}A=&\{a,c\}\cup\{a_i\mid 1\leqslant i\leqslant 8\}\cup\{b_i\mid 1\leqslant i\leqslant 4\}\cup\{b\}\\&\cup\{u_i,v_i\mid 1\leqslant i\leqslant 5\}\cup\{q_i,r_i\mid 14\leqslant i\leqslant 68\}.\end{aligned}$$

Then (G,Λ) is the right Fischer cover of S, has 69 vertices, $m(G,\Lambda)=pm(G,\Lambda)\leqslant 64$ and a^∞ has $9+55=64$ preimages. Arguments as in the last example show that every cover of S with $\leqslant 68$ vertices has multiplicity $\geqslant 65$.

Let (G',Λ') be the labeled graph obtained from (G,Λ) by reversing the direction of the edges and replacing a label x by the label x'. Let S' denote the sofic shift defined by (G',Λ'). Then (G',Λ') is the left Fischer cover of S', has 69 vertices and multiplicity 64. A cover of S' with $\leqslant 68$ vertices has multiplicity $\geqslant 65$.

The shift S' has symbol set $\{x'\mid x\in A\}$ which is disjoint from A and we denote by $0'$ the unique vertex of G' which is the initial vertex of any edge labeled b'. Finally add an edge from vertex 0 to vertex $0'$ labeled u and an edge from $0'$ to 0 labeled u'. Call this labeled graph (R,ρ) and the sofic shift defined by it T. The right Fischer cover of T is obtained from (R,ρ) by replacing the subgraph (G',Λ') by the right Fischer cover of S' and has thus $69+(11+55)=135$ vertices. Similarly the left Fischer cover of T has also 135 vertices.

The graph R has $2 \times 69 = 138$ vertices and the symbol set of T is $A \cup \{u\} \cup \{x' \mid x \in A \cup \{u\}\}$. Observe that S and S' are disjoint subsets of T. If $y \in S$ then $f_\rho^{-1} y$ is contained in S_G and thus $\# f_\rho^{-1} y \leqslant 64$. Similarly for $y' \in S'$. If $y \in T - (S \cup S')$ then there is some $i \in \mathbb{Z}$ such that $y_i \in \{u, u'\}$. Let $x \in f_\rho^{-1}(y)$. If $y[i, i+n]$ is a T-block beginning and ending with a symbol from $\{u, u'\}$, then $x[i, i+n]$ is uniquely determined by $y[i, i+n]$. If $y[i+1, \infty)$ does not see a symbol from $\{u, u'\}$ then $y[i+1, \infty)$ is a ray in S or in S' and $x[i, \infty)$ starts in vertex 0 resp $0'$. Thus there are at most 8 possibilities for $x[i, \infty)$ (namely if $y[i+1, \infty) = b'c'a'a' \ldots$). Similarly for the past. Thus $\# f_\rho^{-1}(y) \leqslant 8 \times 8 = 64$. Since $\# f_\rho^{-1}(a^\infty) = 64$, we obtain thus $m(R, \rho) = 64$.

Now let (H, η) be a cover of T with $\leqslant 137$ vertices. Let α be a vertex of H. Assume there is an edge e starting in α with a label in $A \cup \{u\}$ and there is an edge e' starting at α with a label in $A' \cup \{u'\}$. Let f be an edge with terminal vertex α. Since $\eta(f)\eta(e)$ is a T-block, $\eta(f) \in A \cup \{u'\}$. But, since $\eta(f)\eta(e')$ is a T-block, $\eta(f) \in A' \cup \{u\}$, a contradiction. Thus either all edges starting in a vertex have their label in $A \cup \{u\}$ or all have their label in $A' \cup \{u'\}$. Thus we can partition the vertex set of H into two sets V and V' according to the labels of the outgoing edges. The subgraph of H consisting only of edges with initial and terminal vertices in V resp V' contains a cover for S resp S'. Since $\#V + \#V' \leqslant 137$, one of the sets has $\leqslant 68$ vertices. Thus by the above this has already multiplicity $\geqslant 65$. Thus $m(R, \rho) \geqslant 65$.

Thus the cover (R, ρ) of T has 138 vertices and multiplicity 64 and every cover of T with $\leqslant 137$ vertices has multiplicity $\geqslant 65$.

Thus there are sofic shifts where every cover of least multiplicity has more vertices than the Fischer covers. But, as we shall see in the next lemma, there is a bound on the number of vertices for a cover with minimal multiplicity given the Fischer cover has k vertices. However we do not know how to compute or estimate this bound.

Lemma 3.3. *For each $k \in \mathbb{N}$ there is some $N(k)$ such that if T is a sofic shift which has a cover with k vertices then T has a cover with least multiplicity having at most $N(k)$ vertices.*

Proof. For a sofic shift T let $d(T)$ denote the minimal number of vertices in a cover (G, Λ) of T with $m(G, \Lambda) = m(T)$. We shall show that for every sofic shift T that has a cover with k vertices, there is a sofic shift T' with at most 2^{k^2} symbols, with cover of k vertices, with $m(T) = m(T')$ and $d(T) = d(T')$.

The lemma then follows from the fact that the number of labeled graphs with k vertices, having no parallel edges with the same label, and at most 2^{k^2} symbols is finite. We can take $N(k) = \max\{d(T) \mid T$ is a sofic shift with at most 2^{k^2} symbols and has a cover with k vertices$\}$.

Let (G, Λ) be a cover for T with k vertices. To each T-symbol a assign the set $L(a) = \{(i,j) \mid$ there is an edge labeled with a from vertex i to vertex $j\}$. Let $L(G, \Lambda) = \{L(a) \mid a$ is a T-symbol$\}$. For $L \in L(G, \Lambda)$ let M_L denote the $k \times k$ matrix with entries 0 or L which has $(M_L)_{i,j} = L$ iff $(i,j) \in L$. Let $M = \sum_{L \in L(G, \Lambda)} M_L$ and let $T' := T_{L(G,\Lambda)}$ denote the sofic shift with symbol set $L(G, \Lambda)$ which is defined by M.

Let (R',γ') be a cover for T'. Let $A = M_{(R',\gamma')}$ be the matrix describing the cover (R',γ'). If $A_{i,j} = \sum_{L\in L(G,\Lambda)} n_{i,j}(L)L$, then let

$$B_{i,j} = \sum_{L\in L(G,\Lambda)} n_{i,j}(L) \sum_{\{a \mid L(a)=L\}} a.$$

Let (R,γ) be the labeled graph with labels being the T-symbols a, defined by the matrix B. By the definition of the sets $L(a)$, (R,γ) is a cover for T with the same number of vertices as R'. There is a factor map $g : T \to T'$ given by $(gx)_0 = L(x_0)$. Let $t' \in T'$ and let $t \in T$ such that $gt = t'$. Then $\#\gamma^{-1}t \leqslant \#(\gamma')^{-1}t'$, since a preimage $s' \in (\gamma')^{-1}t'$ consists of edges s_i' which have label $\gamma'(s_i') = L(t_i)$ and thus there is $s \in S$ with $\gamma(s_i) = t_i$. This assignment $s' \to s$ is onto $\gamma^{-1}(t)$, thus $m(R,\gamma) \leqslant m(R',\gamma')$ and therefore $m(T) \leqslant m(T')$, since (R',γ') was an arbitrary cover for T'.

Now fix for each T$'$-symbol L a T-symbol a_L with $L(a_L) = L$. Then define a shift commuting map $i : T' \to T$ by $(ix)_0 = a_L$ iff $x_0 = L$. Observe that i embeds T' into T as an isolated subsystem. Thus, if (R,γ) is a cover for T, then R contains an irreducible subgraph R' such that $(R',\gamma|_{R'})$ is a cover for $i(T')$. Since $m(R',\gamma|_{R'}) \leqslant m(R,\gamma)$, we get $m(T') \leqslant m(T)$.

Thus we have shown that $m(T) = m(T_{L(G,\Lambda)})$ and T has a cover with least multiplicity and d vertices iff $T_{L(G,\Lambda)}$ has a cover with least multiplicity and d vertices. Since $L(G,\Lambda)$ can be considered as a subset of the set of $k \times k$ matrices with entries 0 or 1, there are only 2^{k^2} possible sets for $L(G,\Lambda)$. This concludes the proof. □

4. Sofic shifts with no cover of least multiplicity being a factor of the fiber product of the Fischer covers

Let (F^+,π^+) and (F^-,π^-) be the right and left Fischer cover of a sofic shift T. Then the SFT $\{(x,y) \in F^+ \times F^- \mid \pi^+(x) = \pi^-(y)\}$ contains an unique irreducible component F factoring onto F^+ and F^- by coordinate projections. We call the cover (F,π) of T, where $\pi(x,y) := \pi^+(x)$, $(x,y) \in F$, the fiber product of the right and left Fischer cover. Since (F,π) is a bounded-to-1 cover of T, it has only finitely many factors, i.e. there are only finitely many covers (S,f) of T such that there is a factor map $\phi : F \to S$ with $\pi = f \circ \phi$ [1]. Furthermore there is an algorithm to compute these factors [1]. Thus if there would be always a cover of least multiplicity which is a factor of the fiber product of the Fischer covers, then one would have a computing algorithm for the intrinsic multiplicity of the sofic shift. In this section however we shall give an example for a sofic shift such that no cover with least multiplicity is a factor of the fiber product of the right and left Fischer covers.

A more general concept is that of lifting covers [6]. A cover which is 1–1a.e. is lifting iff it is a factor of the fiber product of the Fischer covers [6]. Trow asked whether a right lifting cover which has no proper factors is already lifting. We shall answer this question into the negative.

Example 4.1. A sofic shift such that no cover of least multiplicity is a factor of the fiber product of the right and left Fischer cover. Let

$$M=\begin{pmatrix} . & e & f & g & h & u_1 & u_2 & c+d & . & . \\ x+y & a+c & . & . & . & . & . & . & . & . \\ z+w & . & a+c & d & d & . & . & . & . & . \\ x+w & c & c & a+d & . & . & . & . & . & . \\ z+y & . & . & . & a+d & . & . & . & . & . \\ v_1+b & . & . & . & . & a & . & . & . & . \\ v_2+b & . & . & . & . & . & a & . & . & . \\ . & . & . & . & . & . & . & . & c+d & . \\ . & . & . & . & . & . & . & . & . & c+d \\ c+d & . & . & . & . & . & . & . & . & . \end{pmatrix}.$$

Let T be the sofic shift defined by the labeled graph (G,Λ) which is given by the matrix M. We enumerate the rows of M from 0 to 9. Thus G has vertices $\{0,1,\dots,8,9\}$. We shall show that T has no cover with least multiplicity that is a factor of the fiber product of the Fischer covers.

We first study the Fischer covers. The symbols v_1, v_2, x, y, z, w, b are all synchronizing symbols for T and they have pairwise distinct predecessor sets. For any symbol s in this list the T-block as has the same predecessor set as s. Thus $(f_{\pi-})^{-1}(a^\infty)\geqslant 7$ and thus $m(F^-,\pi^-)\geqslant 7$. The blocks u_1, u_2, e, f, g, h, gc are synchronizing for T, they do have pairwise distinct follower sets and for each block m in this list the follower set of ma is the same as that of m. Thus $(f_{\pi+})^{-1}(a^\infty)\geqslant 7$ and $m(F^+,\pi^+)\geqslant 7$.

For later use also observe that $(f_{\pi+})^{-1}(a^\infty)=7$, since if m is a synchronizing T-block such that ma is a T-block, then the follower set of m coincides with the follower set of one of the blocks u_1, u_2, e, f, g, h, gc, since the follower set of the block gc coincides with the follower set of the block fd.

We shall now see that $m(G,\Lambda)=6$. There are only two symbols which occur twice in a row of the matrix M, namely the symbols c and d in row 3 (resp. 2): $M_{3,1}=M_{3,2}=c$ and $M_{2,3}=M_{2,4}=d$. The only common symbols in row 3 and 4 are a and d: $M_{3,3}=M_{4,4}=a+d$. Similarly, the only common symbols in row 1 and 2 are a and c: $M_{1,1}=M_{2,2}=a+c$. Thus, starting from any fixed vertex of the graph, there are at most two paths with the same label and the labeling has no diamonds. Now consider two distinct edges in G with the same label and the same terminal vertex. Let $\{i,j\}$ be the initial vertices of these edges. Then $\{i,j\}\neq\{1,2\}$ and $\{i,j\}\neq\{3,4\}$ by inspection of M. Thus the only common symbol in column i and column j of M is a, namely $M_{i,i}=M_{j,j}=a$. Thus there are at most two paths with the same label ending in the same vertex. Thus, since all symbols $\notin\{a,c,d\}$ determine a vertex in G, if $y\in T$, $y\notin\{a,c,d\}^{\mathbb{Z}}$ then $\#f_\Lambda^{-1}y\leqslant 4$. Now let $y\in T\cap\{a,c,d\}^{\mathbb{Z}}$. If $y=a^\infty$ then $\#f_\Lambda^{-1}y=6$. If $y\in\{c,d\}^{\mathbb{Z}}$ then $\#f_\Lambda^{-1}y\leqslant 6$. And finally if $y\neq a^\infty$ and $y\notin\{c,d\}^{\mathbb{Z}}$ then every preimage of y has all its vertices in $\{1,2,3,4\}$. Let $x\in f_\Lambda^{-1}y$. If $y[i,i+n+1]=ca^nd$ for some $n\geqslant 0$

and some $i \in \mathbb{Z}$ then x_i ends in vertex 2, similarly if $y[i, i+n+1] = da^n c$ then x_i ends in vertex 3 and by inspection of the subgraph with vertices $\{1,2,3,4\}$ and edges with labels $\{a,c,d\}$ we see that $\# f_\Lambda^{-1} y \leqslant 4$. Thus, we get $m(G,\Lambda) = 6$. In particular $m(T) \leqslant 6$.

Now let (R,ρ) be a cover for T with $m(R,\rho) \leqslant 6$. We shall show that (R,ρ) is not a factor of the fiber product of the left and right Fischer cover. Recall that $(f_{\pi+})^{-1}(a^\infty) = 7$. Thus if $\#(f_{\rho+})^{-1}(a^\infty) \geqslant 8$ then (R^+, ρ^+) is not conjugate to (F^+, π^+) and thus (R,ρ) is not a factor of the fiber product of the left and right Fischer cover, [6]. Thus we only need to show that $m(R,\rho) \leqslant 6$ implies $\#(f_{\rho+})^{-1}(a^\infty) \geqslant 8$.

Let K be the number of vertices of the cover (R,ρ) and fix $N > 2^K$. Since $ea^N x$ is a T-block, there is a vertex, $\alpha(e)$ say, so that there is a path labeled ea^i leading to $\alpha(e)$, a loop labeled a^j at $\alpha(e)$ and a path labeled $a^k x$ starting at $\alpha(e)$, with $i + j + k = N$ and $j \geqslant 1$. Similarly the blocks $fa^N z$, $ga^N w$, $ha^N y$, $u_1 a^N v_1$ and $u_2 a^N v_2$ yield vertices $\alpha(f)$, $\alpha(g)$, $\alpha(h)$, $\alpha(u_1)$ and $\alpha(u_2)$. Since for no m the block $ea^m z$ is a T-block, $\alpha(e) \neq \alpha(f)$. Analoguosly, the vertices $\alpha(e)$, $\alpha(f)$, $\alpha(g)$, $\alpha(h)$, $\alpha(u_1)$ and $\alpha(u_2)$ are pairwise distinct. Thus $f_\rho^{-1}(a^\infty) \geqslant 6$. (In particular, $m(T) = 6$.) Since $m(R,\rho) \leqslant 6$, it follows that $f_\rho^{-1}(a^\infty) = 6$. By the arguments as above it follows also that no two of the six vertices lie in the same orbit labeled a^n for some $n \geqslant 2$ and thus each of the six vertices $\alpha(e)$, $\alpha(f)$, $\alpha(g)$, $\alpha(h)$, $\alpha(u_1)$ and $\alpha(u_2)$ has a loop labeled a and there is no other vertex with a loop labeled a^k for some $k \geqslant 1$.

For a T-block m let $E(m)$ denote the set of vertices α in R such that there is a path ending at α labeled m. Let W be a T-block such that xWe is a T-block and such that $\#E(W)$ is minimal. Let $X = \{Wea^N, Wfa^N, Wga^N, Wha^N, Wu_1 a^N, Wu_2 a^N, Wgc^N a^N, Wfd^N a^N\}$. We want to use the vertices $\alpha(e)$, $\alpha(f)$, $\alpha(g)$, $\alpha(h)$, $\alpha(u_1)$ and $\alpha(u_2)$ to show that

$$\#\{E(m) \mid m \in X\} = 8,$$

which then implies $\#(f_{\rho+})^{-1}(a^\infty) \geqslant 8$, since $N > 2^k > \#V_{R+}$ and thus $E(m) = E(ma^i)$ for some $1 \leqslant i \leqslant N$ since $\#E(W)$ is minimal.

First observe that if $m, m' \in X$, $m \neq m'$ and $\{m, m'\} \neq \{Wgc^N a^N, Wfd^N a^N\}$ then the follower sets of m and m' are distinct and thus $E(m) \neq E(m')$. Thus all we need to show is that $E(Wgc^N a^N) \neq E(Wfd^N a^N)$.

For that, as before, considering the T-blocks $ec^N x$, $fc^N z$ and $v_1 c^{4N} u_1$ shows there are six vertices $\gamma(e)$, $\gamma(f)$ and $\gamma_1, \ldots, \gamma_4$ with every loop labeled c^j starting at one of these. Since $gc^{N+m} u_1$ is not a T-block, there is no path labeled gc^N ending in one of the $\gamma_1, \ldots, \gamma_4$. By the choice of N thus every path labeled $gc^N a^N$ has a subpath labeled $c^i a^j$ from a vertex in $\{\gamma(e), \gamma(f)\}$ to a vertex in $\{\alpha(e), \alpha(f), \alpha(g), \alpha(h)\}$.

Consider now a subpath p labeled $c^i a^j$ of a path which is labeled $gc^N a^N x$. Then, since $fc^N a^N x$ is not a T-block, p begins in vertex $\gamma(e)$. Since $fa^N x$ and $ha^N x$ are not T-blocks, p does not end at $\alpha(f)$ and not at $\alpha(h)$. Since $ec^N a^N W$ is not a T-block and p begins in $\gamma(e)$, p does not end at $\alpha(g)$. Thus p ends in $\alpha(e)$ and we have shown that every path P which is labeled Wgc^Na^Nx has a subpath from $\gamma(e)$ to $\alpha(e)$ labeled c^ia^j, this subpath being a subpath of the suffix path of P labeled gc^Na^Nx, and thus $\alpha(e)\in E(Wgc^Na^N)$.

Consider now a subpath q labeled $c^{i'}a^{j'}$ of a path which is labeled Wgc^Na^Nz. Since ec^Na^Nz is not a T-block, q begins in $\gamma(f)$. Since ea^Nz and ga^Nz are not T-blocks, q does not end in $\alpha(e)$ and not in $\alpha(g)$. Since fc^Na^Ny is not a T-block, q does not end in $\alpha(h)$. Thus q begins in $\gamma(f)$ and ends in $\alpha(f)$. Thus every path which is labeled Wgc^Na^Nz has a subpath labeled $c^{i'}a^{j'}$ from vertex $\gamma(f)$ to $\alpha(f)$ (of the suffix path with label gc^Na^Nz) and thus $\alpha(f)\in E(Wgc^Na^N)$.

Thus $\{\alpha(e),\ \alpha(f)\}\subset E(Wgc^Na^N)$. In the same way (or by the observation that exchanging the symbols e with h, f with g, x with z and c with d keeps the cover (R,ρ) fixed) one shows $\{\alpha(g),\ \alpha(h)\}\subset E(Wfd^Na^N)$.

Recall that we want to show that $E(Wgc^Na^N)\neq E(Wfd^Na^N)$. Assume that $\{\alpha(g),\ \alpha(h)\}\subset E(Wgc^Na^N)$. Every path labeled $Wea^NxWgc^Na^Nx$ has a subpath labeled a^kxWgc^m from vertex $\alpha(e)$ to $\gamma(e)$. Thus there is a path labeled a^NxWgc^N from vertex $\alpha(e)$ to $\gamma(e)$. Considering the block $Wea^NxWgc^Na^Nz$ yields that there is a path also labeled a^NxWgc^N from vertex $\alpha(e)$ to a vertex γ' with a loop labeled c^j starting and ending at γ'. Note that $\gamma'=\gamma(f)$, since otherwise $\#(f_\rho)^{-1}(c^\infty)>6$. Since $\{\alpha(e),\ \alpha(f),\ \alpha(g),\ \alpha(h)\}\subset E(Wgc^Na^N)$ and for each $\alpha\in\{\alpha(e),\ \alpha(f),\ \alpha(g),\ \alpha(h)\}$ there is a path labeled c^Na^N from a vertex in $\{\gamma(e),\ \gamma(f)\}$ to α, there are thus four distinct paths starting from $\alpha(e)$ and being labeled $a^NxWgc^Nc^Na^\infty$. Since $6=m(R,\rho)<m(F^-,\pi^-)$, the map ρ is not left closing, thus there are two distinct paths ending at $\alpha(e)$ having the same label x_-. Thus $(f_\rho)^{-1}(x_-a^NxWgc^Nc^Na^\infty)\geqslant 2\times 4=8$, a contradiction. Thus the assumption $\{\alpha(g),\ \alpha(h)\}\subset E(Wgc^Na^N)$ was wrong.

Since $\{\alpha(g),\ \alpha(h)\}\subset E(Wfd^Na^N)$, we have $E(Wgc^Na^N)\neq E(Wfd^Na^N)$ and this finishes the proof that (R,ρ) is not a factor of (F,π). $\square$

We conclude this section with an example that a right lifting cover, which has no proper factors need not be left lifting, answering a question of Trow [6].

Example 4.2. Let

$$M=\begin{pmatrix} 0 & e+c & f+c & g+d & h+d\\ x+y & a & 0 & 0 & 0\\ z+w & 0 & a & 0 & 0\\ x+w & 0 & 0 & a & 0\\ z+y & 0 & 0 & 0 & a \end{pmatrix}$$

and let (G,Λ) be the labeled graph defined by M and let T denote the sofic shift obtained from (G,Λ).

We shall show that (G^+,Λ^+) factors properly onto the right Fischer cover, that (G^-,Λ^-) is the left Fischer cover and finally that (G,Λ) has no proper factors. Then in Trow's terminology, (G,Λ) is a minimal right lifing cover which is not left lifting.

One checks that (G^-, Λ^-) is given by the matrix

$$M^- = \begin{pmatrix} 0 & e+g+c+d & e+h+c+d & f+h+c+d & f+g+c+d \\ x & a & 0 & 0 & 0 \\ y & 0 & a & 0 & 0 \\ z & 0 & 0 & a & 0 \\ w & 0 & 0 & 0 & a \end{pmatrix}.$$

Since for any two distinct columns of M^- there is a symbol occurring only in one of those columns, all predecessor sets in (G^-, Λ^-) are distinct and (G^-, Λ^-) is the left Fischer cover of T. (G^+, Λ^+) is given by the matrix

$$M^+ = \begin{pmatrix} . & e & f & g & h & c & d \\ x+y & a & . & . & . & . & . \\ z+w & . & a & . & . & . & . \\ x+w & . & . & a & . & . & . \\ z+y & . & . & . & a & . & . \\ x+y+z+w & . & . & . & . & a & . \\ x+y+z+w & . & . & . & . & . & a \end{pmatrix}.$$

The last two rows of M^+ show that the corresponding vertices in G^+ do have the same follower sets both of them having a fixed point loop labeled a. Thus, the corresponding fixed points will be identified when factoring through the right Fischer cover and thus $\#(f_{\pi+})^{-1}(a^\infty) \leqslant 5$, while $\#(f_{\Lambda+})^{-1}(a^\infty) = 6$. Thus (G^+, Λ^+) is not conjugate to the right Fischer cover.

As in Example 4.1, consider the 8 blocks of the form ea^nx, $ea^ny, \ldots, ha^nz$, ha^ny for very large n. Assume that there are 3 of those blocks, say $u_ia^nv_i$, $1 \leqslant i \leqslant 3$, and $u_i \in \{e, f, g, h\}$, $v_i \in \{x, y, z, w\}$, which are realized at the same vertex in some cover of T. If $u_1 = u_2$ then $\{v_1, v_2\}$ is one of the sets $\{x, y\}$, $\{z, w\}$, $\{x, w\}$, $\{z, y\}$, since otherwise $\#\{u_ia^nv_i, 1 \leqslant i \leqslant 3\} \leqslant 2$. But since $u_3a^nv_1$ and $u_3a^nv_2$ are also T-blocks, this implies $u_3 = u_1$, and $v_3 \in \{v_1, v_2\}$ and thus $\#\{u_ia^nv_i, 1 \leqslant i \leqslant 3\} \leqslant 2$, a contradiction. This shows $u_i \neq u_j$ for $i \neq j$. But since for $v \in \{x, y, z, w\}$ the set $\{u \in \{e, f, g, h\} \mid ua^nv$ is a T-block$\}$ has cardinality 2, we get again a contradiction, and thus the assumption that there are 3 blocks $u_ia^nv_i$, $1 \leqslant i \leqslant 3$, with $u_i \in \{e, f, g, h\}$, $v_i \in \{x, y, z, w\}$, which are realized at the same vertex in some cover of T was wrong. Since the $u_ia^nv_i$ are 8 blocks, thus in every cover a^∞ has at least 4 preimages.

Now (G, Λ) has no proper factor maps: Assume we have a factor, say given by the labeled graph (H, ρ). Let $f : S_G \to S_H$ be a factor map from (S_G, f_Λ) onto (S_H, f_ρ). Then, since $\#(f_\Lambda)^{-1}(a^\infty) = 4$, also $\#(f_\Lambda)^{-1}(a^\infty) = 4$ and thus, there have to be four vertices, each of which has a fixpoint loop labeled a. Thus f restricts to a bijection between the four fixpoints of S_G and the four fixpoints in the factor S_H. Assume that f is not injective. Then there are $s, t \in S_G$ with $s_0 \neq t_0$ and $fs = ft$. Since $f_\Lambda = f_\rho f$, the points s and t have the same label. Since $s \neq t$, they are not fixpoints. If s_0 does not end in vertex 0 then $s_0 \neq t_0$ and $f_\Lambda(s) = f_\Lambda(t)$ implies $\Lambda(s_i) = \Lambda(t_i) = a$ for all $i \geqslant 1$ and thus $s[1, \infty)$ and $t[1, \infty)$ are rays of distinct fixpoints in S_G, a contradiction to the fact that f is injective on the fixpoints. If s_0 does end in vertex 0 then $s_0 \neq t_0$ implies that $\Lambda(s_i) = \Lambda(t_i) = a$ for all $i \leqslant -1$ and thus $s(-\infty, -1]$ and $t(-\infty, -1]$ are rays of distinct fixpoints in S_G, a contradiction. Thus the assumption that f is not injective is wrong and so, f is a conjugacy. $\square$

5. Sofic shifts with infinitely many non-conjugate minimal covers all of which have least multiplicity

In this section we shall first strengthen a result of Williams in showing that for every sofic shift which is not AFT there is an integer N and infinitely many covers of multiplicity N, no two of them having a common factor. From this we construct a sofic shift having infinitely many covers of least multiplicity all of which have no proper factors.

In the proof of the next theorem we need to split graphs with respect to subgraphs. Consider a bounded-to-1 cover (R, ρ) of T and let H be an irreducible subgraph of R. We want to define an incoming splitting with respect to H. For an edge $e \in E_R$ let $i(e)$ denote the initial vertex of e and $t(e)$ the terminal vertex of e. Let $E_1 := \{e \in E_R - E_H \mid t(e) \in V_H\}$. If $\#E_1 \leqslant 1$ then we cannot income split. Now assume $\#E_1 > 1$ and choose a partition of E_1 into two non-empty sets, say A and $E_1 - A$. We create from (R, ρ) a new labeled graph as follows: The vertex set is $V := \{(i, 1) \mid i \in \mathrm{V}_R\} \cup \{(i, 0) \mid i \in V_H\}$. For each edge $e \in E_R - A$ there is an edge from $(i(e), 1)$ to $(t(e), 1)$ labeled $\rho(e)$. For each edge $e \in A$ there is an edge from $(i(e), 1)$ to $(t(e), 0)$ labeled $\rho(e)$. For each $e \in E_H$ there is an edge from $(i(e), 0)$ to $(t(e), 0)$ with label $\rho(e)$. And for each edge $e \in E_R - E_H$ with $i(e) \in V_H$ there is an edge from $(i(e), 0)$ to $(t(e), 1)$ with label $\rho(e)$. This defines the incoming splitted graph with respect to the subgraph H and the partition $\{A,\ E_1 - A\}$, which we call (R', ρ').

Then (R', ρ') is a cover for T, and since H is irreducible, R' is irreducible. Furthermore $m(R', \rho') \leqslant 2m(R, \rho) < \infty$ and if f_ρ is a 1–1ae map then $f_{\rho'}$ too.

The outgoing splitting with respect to H is defined analogously using a partition of the set $E_2 := \{e \in E_R - E_H \mid i(e) \in V_H\}$ and producing a copy of H also.

Theorem 5.1. *Let T be a transitive sofic shift which is not AFT. Then there are infinitely many minimal pairwise non-conjugate covers for T of the same finite multiplicity.*

Proof. The outline of the proof is as follows: The first step is to construct a bounded-to-1 cover (G, Λ) of T such that G contains a subgraph H such that $(H, \Lambda|_H)$ is of a certain type, see the first picture below or the description at the end of step 1. Then by repeated splittings we obtain a graph G in which the subgraph H is "isolated" in the sense that there is a unique edge which does not belong to H but has terminal vertex in H and there is another unique edge which does not belong to H but has initial vertex in H and this distinguished terminal and initial vertex are the same. Then we shall replace H by a graph H_n, see the second picture below, obtaining a family of covers (G_n, Λ_n). The constructions are made that all these graphs (G_n, Λ_n) have the same finite multiplicity and the structure of their subgraphs H_n ensures that no two of them have a common factor. By [1, Corollary 2.8] every cover (G_n, Λ_n) has a factor (G_n', Λ_n') which is minimal, in the sense that it has no proper factors. Since no two of the covers (G_n, Λ_n) have a common factor, the covers (G_n', Λ_n') are pairwise not conjugate. Being factors of covers with the same finite multiplicity, there are thus infinitely many of the covers (G_n', Λ_n') having all the same finite multiplicity.

Step 1: Constructing a cover (G, Λ) which contains a subgraph H of a certain type.

For that consider the right Fischer cover (F^+, π^+) of T. Since T is not AFT, the labeling π^+ is not left-closing. Thus, there is a vertex α' and two paths with distinct terminal edges leading to α' and having the same labels. Since the cover is finite, there is a pair of vertices, say (β', γ') which is visited by these paths at the same times twice. Since the labeling is 1-step right-closing, $\beta' \neq \gamma'$. Thus, there are loops at β' resp γ', both labeled with a block, say a'. And there are paths from β' resp γ' to α' both labeled b'. By extending these paths to the right we may assume that b' is a synchronizing block. Since π^+ is 1-step right closing we may assume that the first symbol of a' is distinct from the first symbol of b'. Since the follower sets of β' and γ' are distinct, we can find a block c' which is in the follower set of β', say, but not in the follower set of γ'. By extending the block c' to the right we may assume that c' is a synchronizing block and that there is a path from β' to γ' labeled c'. Finally choose a path from α' to β' labeled with a synchronizing block d'. Notice that $\alpha' = \beta'$ or $\alpha' = \gamma'$ is possible. Fix $N > 2.(|a'| + |b'| + |c'| + |d'|)$ such that N is a multiple of $|a'|$ and of $|d'b'|$. Pass to the higher block system, say (G, Λ) of (F^+, π^+) where now the vertices are F^+ – blocks of the form $e_{-N} \dots .e_{N-1}$ and the edges are F^+– blocks of the form $e_{-N} \dots e_N$ with label $\pi^+(e_0)$ which starts at vertex $e_{-N} \dots e_{N-1}$ and ends at vertex $e_{-N+1} \dots e_N$.

Let $x' \in S_{F+}$ be the periodic point which has label $(d'b')^\infty$ and x_0' starts in vertex α'. Let y', $z' \in S_{F+}$ be the periodic points which have label $(a')^\infty$ and y_0' starts in vertex β' and z_0' starts in vertex γ'. Then let $\alpha := x'[-N, N-1]$, $\beta := y'[-N, N-1]$ and $\gamma = z'[-N, N-1]$. Then α, β, γ are three distinct vertices in G. There is a simple loop $p_{\beta,\beta}$ at β with label say a'' such that for some $i(\beta) \geqslant 1$ it holds that $a' = (a'')^{i(\beta)}$, and a simple loop $p_{\gamma,\gamma}$ at γ with label a''' such that for some $i(\gamma) \geqslant 1$ it holds $a' = (a''')^{i(\gamma)}$. Note however that it may be that the orbit of $(p_{\beta,\beta})^\infty$ in S_G is the same as of $(p_{\gamma,\gamma})^\infty$.

Let $i, j \geqslant 1$ be such that $|(a')^i| = |(d'b')^j| = N$. Since in (F^+, π^+) there is a path p with label $(a')^{2i} b' (d'b')^{2j}$ starting with the path $y'[-N, N-1]$ and ending with the path $x'[-N, N-1]$, there is a path $p_{\beta,\alpha}$ in the higher block system (G, Λ) from β to α labeled $b := (a')^i b' (d'b')^j$. Since the first symbol of a' is distinct from the first symbol of b', all the subpaths of length $2N$ of p in F^+ are pairwise distinct, and thus the path $p_{\beta,\alpha}$ in G has $|p_{\beta,\alpha}|$ distinct edges and $|p_{\beta,\alpha}| + 1$ distinct vertices.

Similarly, there is a path $p_{\gamma,\alpha}$ from γ to α labeled b, there is a path $p_{\alpha,\beta}$ from α to β labeled $d := (d'b')^j d' (a')^i$ and there is a path $p_{\beta,\gamma}$ from β to γ labeled $c := (a')^i c' (a')^i$.

Let H' be the subgraph of G consisting of the vertices and edges used in the paths $p_{\beta,\alpha}$, $p_{\gamma,\alpha}$, $p_{\alpha,\beta}$ and $p_{\beta,\gamma}$. By the choice of N and since the first symbol of the block a' is distinct from the first symbol of the block b', the paths $p_{\beta,\alpha}$, $p_{\gamma,\alpha}$ and $p_{\alpha,\beta}$ consist of $|p_{\beta,\alpha}|+|p_{\gamma,\alpha}|+|p_{\alpha,\beta}|$ distinct edges, being also distinct from the edges in the loops $p_{\beta,\beta}$ and $p_{\gamma,\gamma}$. The paths $p_{\beta,\alpha}$ and $p_{\beta,\gamma}$ share the first k edges if c' and b' begin with the same prefix of length k. Since c' and b' are synchronizing T-blocks with distinct follower sets, however $p_{\beta,\alpha}\neq p_{\beta,\gamma}$, $p_{\beta,\alpha}$ is not a prefix path of $p_{\beta,\gamma}$ and $p_{\beta,\gamma}$ is not a prefix path of $p_{\beta,\alpha}$. In particular, the paths $p_{\beta,\alpha}$, $p_{\gamma,\alpha}$, $p_{\alpha,\beta}$ and $p_{\beta,\gamma}$ visit the vertices α, β and γ only at their initial and at their terminal vertex.

So far we have chosen a distinguished subgraph H' of G and two simple loops $p_{\beta,\beta}$ and $p_{\gamma,\gamma}$ in G. If in S_G the orbit of $(p_{\beta,\beta})^\infty$ is the same as of $(p_{\gamma,\gamma})^\infty$ then $p_{\beta,\beta}=p_1p_2$ and $p_{\gamma,\gamma}=p_2p_1$ where p_1 is a path from β to γ and p_2 a path from γ to β. Let a_i be the label of p_i, $i=1,2$. Then $a_1a_2=a_2a_1$. We now perform an incoming splitting of the graph (G,Λ) according to the subgraph K consisting of the simple loop $p_{\beta,\beta}$, where we choose $A=\{\text{last edge of the path } p_{\beta,\gamma}\}$. Call the splitted graph (G',Λ'). In G' there are paths $p'_{\beta,\alpha}$ and $p'_{\gamma,\alpha}$ from $(\beta,1)$ (resp. $(\gamma,0)$) to $(\alpha,1)$ both labeled b, there is a path $p'_{\alpha,\beta}$ from $(\alpha,1)$ to $(\beta,1)$ labeled d, there is a path $p'_{\beta,\gamma}$ from $(\beta,1)$ to $(\gamma,0)$ labeled c. Furthermore at vertices $(\gamma,0)$ and $(\beta,1)$ there are simple loops labeled $a_1a_2=a_2a_1$, which do not share a vertex.

Thus, by performing a splitting if necessary, we may assume that in (G,Λ) the orbits of $(p_{\beta,\beta})^\infty$ and of $(p_{\gamma,\gamma})^\infty$ are distinct. Finally, we want to see that we can assume that the simple loops $p_{\beta,\beta}$ and $p_{\gamma,\gamma}$ have the same length. If $|p_{\beta,\beta}|\neq|p_{\gamma,\gamma}|$, then we proceed as follows: Let $n=|p_{\beta,\beta}|$ and $e_0,e_1,\dots,e_{n-1}$ denote the edges of the simple loop $p_{\beta,\beta}$, that is $p_{\beta,\beta}=e_0e_1\dots e_{n-1}$. Recall further that $a'=(a'')^{i(\beta)}$ and thus $|a'|=ni(\beta)$. We delete the edge e_{n-1} from G and add $(i(\beta)-1)n+1$ edges, named f_{n-1+k}, with label $\pi^+(e_{n-1+k \bmod n})$, $0\leqslant k\leqslant(i(\beta)-1)n$, such that $i(f_{n-1})=i(e_{n-1})$, $i(f_{n-1+k+1})=t(f_{n-1+k})$ for $0\leqslant k<(i(\beta)-1)n$ and $t(f_{n-1+(i(\beta)-1).n})=\beta$. Furthermore if in G for some $0\leqslant m<n$ there is an edge e with $i(e)=i(e_m)$ and $e\neq e_m$ then we add an edge from $i(f_{n-1+k})$ to $t(e)$ with label $\pi^+(e)$ for each $0\leqslant k\leqslant(i(\beta)-1)n$ such that $n-1+k=m \bmod n$. The new labeled graph is again a bounded-to-1 cover for T, and at the vertex β there is now a simple loop labeled a' which has no vertices with H' in common except β. We do the same procedure with the simple loop $p_{\gamma,\gamma}$ and obtain a bounded-to-1 cover of T, which we call again (G,Λ), with the following properties:

There is a T-block a, there are synchronizing T-blocks b,c,d, there are three distinct vertices $\alpha,\beta,\gamma\in V_G$ such that:

- there is a simple loop $p_{\beta,\beta}$ at β labeled a,
- there is a simple loop $p_{\gamma,\gamma}$ at γ labeled a,
- there is a path $p_{\beta,\alpha}$ from β to α labeled b,
- there is a path $p_{\gamma,\alpha}$ from γ to α labeled b,
- there is a path $p_{\beta,\gamma}$ from β to γ labeled c,
- there is a path $p_{\alpha,\beta}$ from α to β labeled d.

We call $(H,\Lambda_{|H})$ the subgraph of (G,Λ) consisting of the edges and vertices of those 6 paths $p_{\beta,\beta}$, $p_{\gamma,\gamma}$, $p_{\beta,\alpha}$, $p_{\gamma,\alpha}$, $p_{\beta,\gamma}$ and $p_{\alpha,\beta}$. Note that if N was large enough then $\#E_G-\#E_H\geqslant 2$.

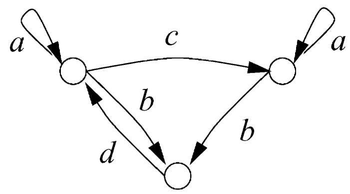

Fig. 1. Cover $(H, \Lambda_{|H})$.

For the special case that $|a| = |b| = |c| = |d| = 1$ the labeled graph $(H, \Lambda_{|H})$ is depicted in Fig. 1.

Step 2: Making the subgraph H "isolated" in G, that is we shall perform splittings on G to obtain a cover (R, ρ) which contains a copy of $(H, \Lambda|_H)$ and such that there is a unique edge $e \in E_R - E_H$ with $t(e) \in V_H$ and this edge has $t(e) = \alpha$, and that there is a unique edge $f \in E_R - E_H$ with $i(f) \in V_H$ and this edge has $i(f) = \alpha$ and furthermore $e \neq f$.

Since $\#(E_G - E_H) \geqslant 2$ we can fix edges $e', f' \in E_G - E_H$, $e' \neq f'$ with $t(e') \in V_H$ and $i(f') \in V_H$. If $t(e') = \alpha$ then let $e = e'$. If $t(e') \neq \alpha$, then we do the following "soft splitting" on G: Fix $k \geqslant 1$ such that there is a path of length k in H from $t(e)$ to $\alpha \in V_H$. Let $Q := \{q = q_1 \dots q_k | q$ is a path of length k in G starting at $t(e)\}$. Now change G as follows: Erase the edge e and then add for each $q = q_1 \dots q_k \in Q$ a new path of length $k+1$ from $i(e)$ to $t(q_k)$ with label $\Lambda(e)\Lambda(q_1) \dots \Lambda(q_k)$. The new graph is conjugate to (G, Λ), it contains all the edges from $E_G - \{e\}$, thus has also a copy of H and there is now an edge e which does not belong to H but has terminal vertex α and $e \neq f'$. If $i(f') = \alpha$, then we let $f = f'$. Otherwise we perform again a soft splitting by erasing the edge f' and use paths which end in initial vertex of f'. We obtain a cover, which we call again (G, Λ), which contains a copy of H and there is an edge $e \in E_G - E_H$ with $t(e) = \alpha$ and an edge $f \in E_G - E_H$ with $i(f) = \alpha$ and $e \neq f$. Since $e, f \notin E_H$, $i(e), t(f) \notin V_H$.

Now let $E_1 := \{e' \in E_G - E_H \mid t(e') \in V_H\}$.

Then $e \in E_1$. If $E_1 = \{e\}$ then let $(R, \rho) = (G, \Lambda)$. If $\#E_1 \geqslant 2$ then let $A = \{e\}$. Perform an incoming splitting of (G, Λ) with respect to H and partition $\{A, E_1 - A\}$ to obtain the splitted graph (R, ρ). By this splitting the edges of R corresponding to H which have initial and terminal vertex in $\{(i, 0) \mid i \in V_H\}$ give a copy of H, call it H'. Since $A = \{e\}$, there is a unique edge in $E_R - E_{H'}$ which has terminal vertex in $V_{H'} = \{(i, 0) \mid i \in V_H\}$, namely the edge from $(i(e), 1)$ to $(t(e), 0) = (\alpha, 0)$. Observe that there is an edge in $E_R - E_{H'}$ which has initial vertex $(i(f), 0) = (\alpha, 0)$, call this edge f'.

Now we want to do an outgoing splitting of (R, ρ). For that let $E_2 := \{f \in E_R - E_{H'} \mid i(f) \in V_{H'}\}$. Thus $f' \in E_2$. If $\#E_2 = 1$ then let $R' = R$, $H'' = H'$. If $\#E_2 \geqslant 2$ let $A = \{f'\}$. Do an outgoing splitting of (R, ρ) with respect to H' and partition $\{A, E_2 - A\}$ to obtain the splitted graph (R', ρ'). Let H'' be the subgraph of R' which consists of the edges which do have initial and terminal vertex in $\{(i, 0) \mid i \in V_{H'}\} = \{((i, 0), 0) \mid i \in V_H\}$ and correspond to H'. Then H'' is a copy of H'. There is a unique edge f'' in $E_{R'} - E_{H''}$

with initial vertex in $V_{H''}$, and this edge begins in $(i(f'),0) = ((\alpha,0),0)$. Also, there is a unique edge e'' in $E_{R'} - E_{H''}$ with terminal vertex in $V_{H''}$, and this edge ends in $((t(e),0),0) = ((\alpha,0),0)$ and furthermore $e'' \neq f''$. For convenience we call (R',ρ') again (G,Λ) and the subgraph $(H'',\rho'|_{H''})$ again $(H,\Lambda|_H)$, the edge e'' again e, f'' again f and the vertices $((\alpha,0),0),((\beta,0),0)$ and $((\gamma,0),0)$ of H'' again α,β and γ resp. We thus have the desired cover (G,Λ) which contains a copy of H sitting "isolated" in G.

Step 3: Construction of the labeled graphs H_n which shall replace H.

For that let $X = f_\Lambda(S_H)$. Then $(H,\Lambda|_H)$ is a cover for X. For each $n \geqslant 2$ we define a cover (H_n,Λ_n) as follows: The vertex set of H_n is V_n and it contains

$$V_n^* = \{i \in \mathbb{Z} \mid -(n+2) \leqslant i \leqslant 2\} \cup \{(i,j) \mid 1 \leqslant i \leqslant n+4, j \in \{0,1\}\}.$$

To give a compact description of the labeled edges of H_n we use the following notion: For a block $w \in \{a,b,c,d\}$ and $v_1,v_2 \in V_n^*$ we mean by "there is a path labeled w from v_1 to v_2" that if $|w| = k$ and $w = w_1 \ldots w_k$ then there are vertices $(v_1,v_2,w,i) \in V_n$, $1 \leqslant i \leqslant k-1$ and an edge labeled w_1 from v_1 to $(v_1,v_2,w,1)$, an edge labeled w_i from $(v_1,v_2,w,i-1)$ to (v_1,v_2,w,i) for $2 \leqslant i \leqslant k-1$ and finally an edge labeled w_k from $(v_1,v_2,w,k-1)$ to v_2.

With this notion we can describe the edges of H_n: There are loops labeled a at the vertices 1, 2 and $-(n+2)$. There is a path labeled d from 0 to 1, a path labeled c from 1 to 2 and a path labeled b from 2 to 0. There is a path labeled d from 0 to -1, a path labeled a from i to $i-1$ for $-(n+2) < i \leqslant -1$. There is a path labeled b from i to 0 for $-(n+2) \leqslant i \leqslant -1$, $i \neq -(n+1)$. There is a path labeled d from 0 to $(1,j)$, j $\in \{0,1\}$. There is a path labeled a from (i,j) to $(i+1,j)$, $1 \leqslant i < n+1$, $j \in \{0,1\}$. There is a path labeled b from $(n+1,j)$ to $(n+2,j)$ and a path labeled d from $(n+2,j)$ to $(n+3,j)$, $j \in \{0,1\}$. There is a path labeled a from $(n+3,j)$ to $(n+4,j)$ and from $(n+4,j)$ to $(n+3,j)$, $j \in \{0,1\}$. There is a path labeled c from $(n+3+i,0)$ to $(n+3+j,1)$ for $i,j \in \{0,1\}$. Finally there is a path labeled b from $(n+4,0)$ to 0 and from $(n+3,1)$ to 0. We call the vertices V_n^* the essential vertices of H_n.

For the special case that $|a| = |b| = |c| = |d| = 1$ we have $V_n^* = V_n$ and the labeled graph (H_n,Λ_n) is presented in Fig. 2.

Now we check that (H_n,Λ_n) is a cover for X. Let $S_n := S_{H_n}$ and $f_n := f_{\Lambda_n}$. Observe that the first return loops at vertex $0 \in V_n$ in H_n are labeled by X-blocks beginning with d and ending with b. Since b, d and bd are synchronizing X-blocks, $f_n(S_n) \subset X$. For every X-block $m = dwb$ where w is a finite concatenation from blocks from $\{a,c\}$ and $w \neq a^n$ there is a unique first return loop at vertex $0 \in V_n$ with label m. It uses only essential vertices from $\{i \in \mathbb{Z} \mid -(n+2) \leqslant i \leqslant 2\}$. For every X-block $m = da^n bda^k b$, $k \geqslant 0$ there is a unique first return loop at vertex $0 \in V_n$ with label m. It uses essential vertices from $\{(i,1) \mid 1 \leqslant i \leqslant n+4\}$ if k is even and from $\{(i,0) \mid 1 \leqslant i \leqslant n+4\}$ if k is odd. And finally for a X-block $m = da^n bda^k ca^m b$, $k,m \geqslant 0$ there is a unique first return loop at vertex $0 \in V_n$ with label m. It uses first essential vertices from the set $\{(i,0) \mid 1 \leqslant i \leqslant n+4\}$ and then from $\{(n+3,1),(n+4,1),0\}$. Thus $f_n(S_n) = X$ and (S_n,f_n) is a cover for X.

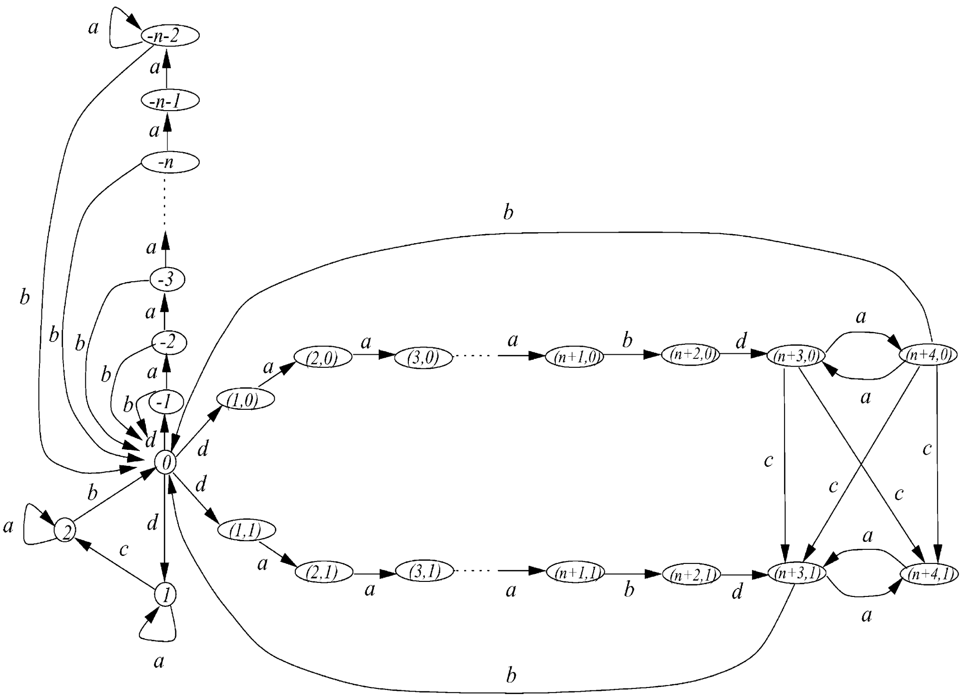

Fig. 2. Cover (S_n, f_n).

Furthermore $\#(f_n)^{-1}(a^\infty) = 7 \cdot \text{period } (a^\infty)$ and the above argument shows that $\#(f_n)^{-1}(p) = 1$ for every periodic point $p \in X - \{a^\infty, (da^n b)^\infty\}$ and $\#(f_n)^{-1}((da^n b)^\infty) = 2$. Thus $pm(S_n, f_n) = 7 \cdot \text{period}(a^\infty)$.

Now let $x \in X - \{a^\infty, (da^n b)^\infty\}$ and let $y \in S_n$ with $f_n(y) = x$. If there is some i such that $x(-\infty, i-1]$ ends with a block from $\{ca^k b \mid k \geqslant 0\} \cup \{da^k b \mid k \geqslant 0,\ k \neq n\}$ or $x(-\infty, i-1] = a^\infty b$ then x_i starts in vertex 0. By inspection, the number of paths starting in vertex 0 having label $x[i, \infty)$ is at most 2 and the number of paths ending in 0 having label $x(-\infty, i-1]$ is at most 4. Thus $\#(f_n)^{-1}(x) \leqslant 8$ in this case. Now consider the case that no such i exists. If there is some i such that $x_i = c$ then this i is unique and $x[i, \infty) = ca^\infty$. Then either x is in the orbit $x(-\infty, i-1] = a^\infty$ and, by inspection, x has 5 preimages or $x(-\infty, i-1]$ determines a unique vertex from $\{1, (n+3, 0), (n+4, 0)\}$ in which y_i starts. Thus $\#(f_n)^{-1}(x) \leqslant 8$ in this case, too. If $x_i \neq c$ for all i then x is from the orbit $(da^n b)^\infty da^\infty$. By inspection of the cover, $\#(f_n)^{-1}(x) = 3$. Thus $m(S_n, f_n) = \max\ (8, 7\text{period } (a^\infty))$. Every point in the orbit of $a^\infty bda^\infty$ has 8 preimages. Thus all the covers (S_n, f_n) of X have the same finite multiplicity.

Step 4: Replacing the subgraph H of G by H_n.

Consider the cover (G, Λ) constructed in step 2 which contains $(H, \Lambda|_H)$. Recall that there is a unique vertex $\alpha \in V_H$, there is a unique edge $e \in E_G - E_H$ with $t(e) \in V_H$ and this edge e has terminal vertex $t(e) = \alpha$ and there is a unique edge $f \in E_G - E_H$ with $i(f) \in V_H$ and this has $i(f) = \alpha$ and furthermore $e \neq f$.

For $n \geqslant 2$ we define (G_n, Λ_n) to be the disjoint union of $(G - H, \Lambda)$ and (H_n, Λ_n) where the vertices $\alpha \in V_G$ and $0 \in V_n$ are identified and furthermore an edge f' is added starting in the essential vertex $(n+2, 0) \in V_n$ and ending in $t(f) \in V_G - V_H$ having label $\Lambda_n(f') = \Lambda(f)$. Then (G_n, Λ_n) is a cover of T: The loops in H (resp. H_n) at vertex α resp 0 begin with the synchronizing block d and end with the synchronizing block b. For every first return loop at α in H which is not labeled da^nb there is a first return loop at α in H_n with the same labels. A loop in G which starts with the first return loop at α having label da^nb has as its $n+2'$th edge either the distinguished edge f or the edge in H starting at vertex α which is the initial vertex of the path $p_{\alpha,\beta}$ leading from α to β and having label d. In the first case by construction there is a path in G_n from α to $t(f)$ having label $da^nb\Lambda(f)$ and in the second case there is a first return loop at α in H_n with the same label. Thus (G_n, Λ_n) is a cover of T.

We compare the multiplicity of (G, Λ) with that of (G_n, Λ_n). Let $x \in S_G$ and $y \in S_{G_n}$. We say $x \approx y$ if $f_\Lambda x = f_{\Lambda_n} y$ and $x_i = y_i$ whenever $x_i \in E_G - (E_H \cup \{f\})$. Let $\pi(x) := \#\{y \in S_{G_n} \mid x \approx y\}$. Now fix $x \in S_G$ and let $y \in S_{G_n}$ with $x \approx y$.

First we consider the case that for some i the edges x_i and y_i both begin in vertex α. If for some $i < k$ the block $x[i,k]$ is a first return loop at vertex α, then since y_i begins in α too, $y[i,k]$ is uniquely determined by $x[i,\infty)$ except if $(f_\Lambda x)[i,\infty] = da^nbda^\infty$. In this latter case $y[i,\infty]$ has at most two possibilities. If x_j does not begin in α for $j > i$ then either $x_i = f$ and $x_j \in E_G - (E_H \cup \{f\})$ for all $j > i$ and $y[i,\infty)$ is determined by $x[i-n-2,\infty)$ or $x_j \in E_H$ for all $j \geqslant i$ and $f_\Lambda x[i,\infty) \in \{da^\infty, da^kca^\infty \mid k \geqslant 0\}$ and in this case $y[i,\infty)$ has at most two possibilities.

Thus if x_i begins in vertex α then $\#\{y[i,\infty) \mid y \in S_{G_n}, x \approx y \text{ and } y_i \text{ begins in } \alpha\} \leqslant 2$ and a similar argument shows that $\#\{y(-\infty, i-1] \mid y \in S_{G_n},\ x \approx y \text{ and } y_i \text{ begins in } \alpha\} \leqslant 4$.

If $x_i = e$ then $x \approx y$ implies $y_i = e$. If $x_i = f$ then $x \approx y$ implies $y_i = f'$ if $f_\Lambda(x)(-\infty, i-1]$ ends with the block da^nb and otherwise $y_i = f$, since $x_{i+1} = y_{i+1} \in E_G - (E_H \cup \{f\})$. Thus if x sees the edge e or the edge f then there is some i such that x_i starts in α and $x \approx y$ implies y_i starts in α too and thus $\pi(x) \leqslant 8$ in this case. Now consider the case that x never sees e and never sees f. Then either $x_i \in E_G - (E_H \cup \{f\})$ for all i and thus $\pi(x) = 1$ or $x_i \in E_H$ for all i and then $f_\Lambda(x) \in X$ and thus $x \approx y$ implies $y_i \in E_{H_n}$ for all i and thus by the argument in step 3 $\pi(x) \leqslant 8 \cdot \text{period } (a^\infty)$. Thus $\pi(x) \leqslant 8 \cdot \text{period } (a^\infty)$ in any case and thus $m(G_n, \Lambda_n) \leqslant 8m(G, \Lambda)\text{period } (a^\infty)$.

Note that the argument also showed that f_{Λ_n} is 1–1a.e., since f_Λ is 1–1a.e. Let m be a T-block of length $2k+1$, say, such that whenever $s \in S_G$ with $f_\Lambda(s)[-k,k] = m$ then s_0 begins in α. By prolongating m if necessary, we may assume that m begins and ends with the block dcb and there is a loop in G at vertex α with label m. By the argument above, there is also a loop in G_n at vertex α labeled m and if $s \in S_{G_n}$ with $f_{\Lambda_n}(s)[-k,k] = m$ then s_0 begins in vertex α. We call m a strong magic T-block.

Step 5: Verification that no two of the covers (G_n, Λ_n) have a common factor.

Now let for notational convenience $S_n := S_{G_n}$ and $f_n = f_{\Lambda_n}$. Let (K, κ) be a cover for T, $S = S_K$, $f = f_\kappa$ and let $g: S_n \to S$ be a factor map such that $f_n = f \circ g$. (Thus, (S, f) is a factor of (S_n, f_n).)

Claim. *Let $x \in S_n$ be the point of least period $2|a|$ which has $f_n(x) = a^\infty$ and x_0 begins in vertex $(n+3, 0) \in V_n$. Then gx is a point of least period $2|a|$.*

Proof of the Claim. Assume that gx has period less than $2|a|$. Let $y \in S_n$ be the point of least period $2|a|$ which has $f_n(y) = a^\infty$ and y_0 begins in vertex $(n+3, 1)$. Let M be so large that $s, t \in S_n$ with $s[-M, M] = t[-M, M]$ implies $(gs)_0 = (gt)_0$. Let m be a strong magic T-block. Let $s \in S_n$ such that $f_n s(-\infty, (M+1)|a| + |bm| - 1]$ ends with $mda^n bda^{2M+1}bm$. Then $s[-M, |a| - 1 + M] = x[-M, |a| - 1 + M]$ and thus $gs[0, |a| - 1] = gx[0, |a| - 1]$. By assumption gx has period less than $2|a|$ and the period of gx divides $|a|$, thus $gx[0, |a| - 1]$ is a loop in K at some vertex with label a, there is thus a point $z \in S$ with $z[0, |a| - 1] = gx[0, |a| - 1]$ and $fz(-\infty, (M+2)|a| + |bm| - 1]$ ends with $mda^n bda^{2M+2}bm$. Let $t \in g^{-1}z$. Then, since $f_n t(-\infty, (M+2)|a| + |bm| - 1]$ ends with $mda^n bda^{2M+2}bm$, and m is a strong magic block, we get that $t[-M, 2|a| + M] = y[-M, 2|a| + M]$. Thus $gy[0, |a| - 1] = gy[|a|, 2|a| - 1] = gx[0, |a| - 1]$ and thus $gy = gx$. Now let $r \in S_n$ with $r(-\infty, -1] = x(-\infty, -1]$ and $r[|c|, \infty) = y[0, \infty)$ and $f_n r = a^\infty ca^\infty$. Then, since $gx = gy$, there is thus a loop in S labeled $a^M ca^M$, thus $ca^{2M}c$ is a T-block, a contradiction. Thus the assumption that the period of gx is less than $2|a|$ was wrong, and thus gx is a point of least period $2|a|$. The claim is thus proved. $\square$

Now let (S, f) be a common factor of (S_n, f_n) and (S_k, f_k), say g is a factor map from (S_n, f_n) onto (S, f) and h is a factor map from (S_k, f_k) onto (S, f). Let m be a strong magic T-block. Let $M \geqslant 1$ be greater than the coding lengths of g and h. Let $s \in S_n$ with $f_n(s)(-\infty, -1] = m^\infty da^n bd$ and $f_n(s)[0, \infty) = a^{4M+1}bm^\infty$. Then s_0 begins in vertex $(n+3, 0)$. Let $t \in S_k$ such that $ht = gs$. Since in (G_k, Λ_k) there are a loops at vertex α labeled m and $da^n da^{4M+1}b$, the latter being a loop in the subgraph H_k of G_k, and since m is a strong magic T-block, $t(-\infty, -|da^n bd| - 1]$ ends in vertex α and $t[-|da^n bd|, |a^{4M+1}b| - 1]$ is a loop at α. If $n \neq k$ then $t[0, |a^{4M+1}| - 1]$ is a loop at vertex $-(k+2)$, and thus $t' := (t[0, |a^{4M+1}| - 1])^\infty$ is a point in S_k of period $|a|$ with label a^∞. Thus ht' has period $\leqslant |a|$, and thus by the choice of M, for $s' = (s[0, |a^{4M+1}| - 1])^\infty$ it holds $gs' = ht'$, a contradiction to the last claim. Thus the assumption $n \neq k$ was wrong, and thus $n = k$. Thus we have shown that the covers (S_n, f_n) of T are covers with multiplicity $\leqslant m(G, \Lambda) \cdot 8 \cdot$period (a^∞) and no two of them have a common factor.

By [1, Corollary 2.8] this proves the theorem. $\square$

Example 5.2. There is a sofic system such that there are infinitely many non-conjugate minimal covers with least multiplicity. Consider the labeled graph (H, Λ) given by the matrix

$$\begin{pmatrix} 0 & d & 0 \\ b & a & c \\ b & 0 & a \end{pmatrix}.$$

Call the vertex where the edge labeled d begins α. Now add a loop of length 8 with label e^8 at vertex α, to obtain a labeled graph (G, Λ). Let T be the sofic shift defined by (G, Λ). It is easy to see that $m(T) = 8$. Then T is not AFT and (G, Λ) has the subgraph H as constructed in step 2 of the proof of Theorem 5.1. Now replace H by H_n and add the special edge f' as described in step 4 of the proof of Theorem 5.1. Then by step 3–5 of the proof of Theorem 5.1 these are covers with no common factor and all have multiplicity 8.

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