Paths and trails in edge-colored graphs ☆

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This paper deals with the existence and search for properly edge-colored paths/trails between two, not necessarily distinct, vertices s and t in an edge-colored graph from an algorithmic perspective. First we show that several versions of the $s - t$ path/trail problem have polynomial solutions including the shortest path/trail case. We give polynomial algorithms for finding a longest properly edge-colored path/trail between s and t for a particular class of graphs and characterize edge-colored graphs without properly edge-colored closed trails. Next, we prove that deciding whether there exist k pairwise vertex/edge disjoint properly edge-colored $s - t$ paths/trails in a c-edge-colored graph G^c is NP-complete even for $k = 2$ and $c = \Omega(n^2)$, where n denotes the number of vertices in G^c. Moreover, we prove that these problems remain NP-complete for c-edge-colored graphs containing no properly edge-colored cycles and $c = \Omega(n)$. We obtain some approximation results for those maximization problems together with polynomial results for some particular classes of edge-colored graphs.

1. Introduction, Notation and Terminology

In the last few years a great number of problems have been dealt with in terms of edge-colored graphs for modeling purposes, as well as for theoretical investigation [3,7–9,18,23]. Previous work on the subject has focused on the determination of particular properly edge-colored subgraphs, such as Hamiltonian or Eulerian configurations, colored in a specified pattern [1,2,4–6,10,21,22,25,27], that is, subgraphs such that adjacent edges have different colors.

Our first aim in that respect was to extend the graph-theoretic concept of connectivity to colored graphs with a view to gaining some insight into our problem from Menger's Theorem in particular. In other words, we intended to define some sort of local alternating connectivity for edge-colored graphs. Informally speaking, local connectivity in general (non-colored) graphs is a local parameter. For two given vertices x and y, it is the maximum number of (edge-disjoint or vertex-disjoint) paths between them. By contrast, connectivity is a global parameter defined to be the minimum number over all x, y of their local connectivity's. Difficulties arose, however, from local connectivity being not polynomially characterizable in edge-colored graphs, as can easily be seen. Thus, there can be no counterpart to Menger's Theorem as such, and even the notion of a connected component as an equivalence class does not carry over to edge-colored graphs since the concatenation of two

properly edge-colored paths is not necessarily properly edge-colored. We settled then for some practical and theoretical results, herein presented, which deal with the existence of vertex-disjoint paths/trails between given vertices in c-edge-colored graphs. Most of those path/trail problems happen to be NP-complete, which thwarts all attempts at systematization.

Formally, let $I_c = \{1, 2, \ldots, c\}$ be a set of given colors, $c \geq 2$. Throughout the paper, G^c will denote an edge-colored simple graph such that each edge is in some color $i \in I_c$ and no parallel edges linking the same pair of vertices occur. The vertex and edge-sets of G^c are denoted by $V(G^c)$ and $E(G^c)$, respectively. The *order* of G^c is the number n of its vertices. The *size* of G^c is the number m of its edges. For a given color i, $E^i(G^c)$ denotes the set of edges of G^c colored i. For edge-colored complete graphs, we write K_n^c instead of G^c. If H^c is a subgraph of G^c, then $N_{H^c}^i(x)$ denotes the set of vertices of H^c, linked to x by an edge colored i. The colored i-degree of x in H^c, denoted by $d_{H^c}^i(x)$, is $|N_{H^c}^i(x)|$, *i.e.*, the *cardinality* of $N_{H^c}^i(x)$. An edge between two vertices x and y is denoted by xy, its color by $c(xy)$ and its cost (if any) by $cost(xy)$. The cost of a subgraph is the sum of its edge costs. A subgraph of G^c containing at least two edges is said to be *properly edge-colored* if any two adjacent edges in this subgraph differ in color. A *properly edge-colored path* does not allow vertex repetitions and any two successive edges on this path differ in color. A *properly edge-colored trail* does not allow edge repetitions and any two successive edges on this trail differ in color. However, note that the edges on this trail need not form a properly edge-colored subgraph since we can have adjacent and not successive edges with the same color. The *length* of a path/trail is the number of its edges. Given two vertices s and t in G^c, we define a properly edge-colored $s - t$ path/trail (or just, $s - t$ path/trail for short) to be a path/trail with end-vertices s and t. Sometimes s will be called the *source*, and t the *destination* of the path/trail. A properly edge-colored path/trail is said to be *closed* if its endpoints coincide, and its first and last edges differ in color. A closed properly edge-colored path (trail) is usually called a *properly edge-colored cycle* (*closed trail*).

Given a digraph $D(V, A)$, we denote by $\vec{uv}$ an arc of A, where $u, v \in V$. In addition, we define $N_D^+(x) = \{y \in V : \vec{xy} \in A\}$ the *out-neighborhood* of x in D, and by $N_D^-(x) = \{y \in V : \vec{yx} \in A\}$ the *in-neighborhood* of x in D. Finally, we represent by $N_D(x) = N_D^+(x) \cup N_D^-(x)$ the *in-out-neighborhood* of $x \in V$ (or just *neighborhood* for short). Also, given an induced subgraph Q of a non colored graph G, a *contraction* of Q in G consists of replacing Q by a new vertex, say z_Q, so that each vertex x in $G - Q$ is connected to z_Q by an edge, if and only if, there exists an edge xy in G for some vertex y in Q.

This paper is concerned with algorithmic issues regarding various trail/path problems between two given vertices s and t in G^c. First, we consider the $s - t$ path/trail version problem whose objective is to determine the existence or not of an arbitrary properly edge-colored $s - t$ path/trail in G^c. Polynomial algorithms are established for such problems as the *Shortest properly edge-colored path/trail*, the *Shortest properly edge-colored path/trail with forbidden pairs*, the *Shortest properly edge-colored cycles/closed trails* and the *Longest properly edge-colored path/trail* for a particular class of instances. Actually, we show that all these results may be derived from the Szeider's Algorithm for the properly edge-colored $s - t$ paths. We also characterize edge-colored graphs without properly edge-colored closed trails. Next, we deal with the *Maximum Properly Vertex Disjoint Path* and *Maximum Properly Edge Disjoint Trail* problems (respectively, MPVDP and MPEDT for short), whose objective is to find the maximum number of properly edge-colored vertex-disjoint paths (respectively, edge-disjoint trails) between s and t. Although these problems can be solved in polynomial time in general non-colored graphs, most of their instances are proved to be NP-complete in the case of edge-colored graphs. In particular we prove that, given an integer $k \geq 2$, deciding whether there exist k properly edge-colored vertex/edge disjoint $s - t$ paths/trails in G^c is NP-complete even for $k = 2$ and $c = \Omega(n^2)$. Moreover, for an arbitrary k we prove that these problems remain NP-complete for c-edge-colored graphs containing no properly edge-colored cycles/closed trails and $c = \Omega(n)$. We show a greedy procedure for these maximization problems, through the successive construction of properly edge-colored shortest $s - t$ paths/trails. This is a straightforward generalization of the greedy procedure to maximize the number of edge or vertex disjoint paths between k pair of vertices in non-colored graphs (see [20,17] for details). Similarly, we obtain an approximation performance ratio. We finish the paper by exhibiting a polynomially solvable class of instances for the related maximization problems.

The following two results will be used in this paper. The first result, initially proved by Grossman and Häggkvist [16] for 2-edge-colored graphs and generalized by Yeo [27], characterizes c-edge-colored graphs without properly edge-colored cycles.

Theorem 1.1 (*Yeo*). *Let G^c be a c-edge-colored graph, $c \geq 2$, such that every vertex of G^c is incident with at least two edges colored differently. Then either G^c has a properly edge-colored cycle or for some vertex v, no component of $G^c - v$ is joined to v by at least two edges in different colors.*

In terms of edge-colored graphs, Szeider's main result [24] on graphs with prescribed general transition systems may be formulated as follows:

Theorem 1.2 (*Szeider*). *Let s and t be two vertices in a c-edge-colored graph G^c, $c \geq 2$. Then, either we can find a properly edge-colored $s - t$ path or else decide that such a path does not exist in G^c in linear time on the size of the graph.*

Given G^c, the main idea of the proof is based on earlier work by Edmonds (see for instance Lemma 1.1 in [21]) and amounts to reducing the properly edge-colored path problem in G^c to a perfect matching problem in a non-colored graph defined appropriately. The latter graph will be called henceforth *the Edmonds–Szeider graph* and is defined as follows. Given two vertices s and t in G^c, set $W = V(G^c) \setminus \{s, t\}$. Now, for each $x \in W$, we first define a subgraph G_x with vertex- and edge-sets, respectively:

$$V(G_x) = \bigcup_{i \in I_c} \{x_i, x_i' | N_{G^c}^i(x) \neq \emptyset\} \cup \{x_a'', x_b''\} \quad \text{and}$$

$$E(G_x) = \{x_a'' x_b''\} \cup \left(\bigcup_{\{i \in I_c \mid x_i' \in V(G_x)\}} \left(\{x_i x_i'\} \cup \left(\bigcup_{j=a,b} \{x_i' x_j''\} \right) \right) \right).$$

Now, the Edmonds–Szeider non-colored graph $G(V, E)$ is constructed as follows:

$$V(G) = \{s, t\} \cup \left(\bigcup_{x \in W} V(G_x) \right), \quad \text{and}$$

$$E(G) = \bigcup_{i \in I_c} \left(\{sx_i \mid sx \in E^i(G^c)\} \cup \{x_i t \mid xt \in E^i(G^c)\} \cup \{x_i y_i \mid xy \in E^i(G^c)\} \right) \cup \left(\bigcup_{x \in W} E(G_x) \right).$$

The interesting point in the construction is that, given a particular (trivial) perfect matching M in $G - \{s, t\}$, a properly edge-colored $s - t$ path exists in G^c if and only if there exists an augmenting path P relative to M between s and t in G. Recall that a path P is augmenting with respect to a given matching M if, for any pair of adjacent edges in P, exactly one of them is in M, with the further condition that the first and last edges of P are not in M. Since augmenting paths in G can be found in $O(|E(G)|)$ linear time (see [26], p.122), the same execution time holds for finding properly edge-colored paths in G^c as well, since $O(|E(G)|) = O(|E(G^c)|)$.

2. The $s - t$ path/trail problem

Given two, not necessarily distinct, vertices s and t in G^c, the main question of this section is to give polynomial algorithms for finding (if any) a properly edge-colored $s - t$ path or trail in G^c. The properly edge-colored $s - t$ path problem was first solved by Edmonds for two colors (see Lemma 1.1 in [21]) and then extended by Szeider [24] to include any number of colors. Here we deal with variations of the properly edge-colored path/trail problem, *i.e.*, the problem of finding an $s - t$ trail, closed trails, the shortest $s - t$ path/trail, the longest $s - t$ path (trail) in graphs with no properly edge-colored cycles (closed trails) and $s - t$ paths/trails with forbidden pairs.

2.1. Properly edge-colored $s - t$ trails and the characterization of graphs without properly edge-colored closed trails

This section is devoted to the properly edge-colored $s - t$ trail problem. Among other results, we prove that the $s - t$ trail problem reduces to the $s - t$ path problem over a new c-edge-colored graph. As the latter problem has been proved polynomial [24], it follows that our problem is polynomial as well. We conclude the section with some results on closed trails in edge-colored graphs. Let us start with the following simple, though important, result.

Lemma 2.1. *Given two vertices s, t of G^c, assume that there exists a properly edge-colored $s - t$ trail T in G^c. Further, suppose that at least one internal vertex on this trail is visited three times or more. Then, there exists another properly edge-colored $s - t$ trail T' in G^c such that no vertex is visited more than twice on T'.*

Proof. Set $T = e_1 e_2 \ldots e_k$, where e_i are the edges of the trail. Let $\{a_1, a_2, \ldots, a_r\}$ denote the set of distinct vertices of T. Let now λ_i denote the number of times vertex a_i is visited on T, for each $i = 1, 2, \ldots, r$. Set $\lambda = \max(\lambda_1, \lambda_2, \ldots, \lambda_r)$. Let us choose T to be the shortest such trail so that λ is the smallest possible, as is therefore the number of vertices a_i with $\lambda_i = \lambda$. If $\lambda \leq 2$ we are done. Assume therefore $\lambda \geq 3$. Thus, there exist some vertex, say a_p, $1 \leq p \leq r$, visited at least three times on T. Assume $\lambda = 3$, the proof being almost identical for higher values. Let us rewrite $T = e_1 e_2 \ldots e_i e_{i+1} \ldots e_j e_{j+1} \ldots e_f e_{f+1} e_{f+2} \ldots e_k$ so that : (i) a_p is the vertex common to the pair of edges e_i, e_{i+1}, (respectively to e_j, e_{j+1} and to e_f, e_{f+1}) and (ii) a_p is not a member of the vertex set of the graph induced by the edges of the segment $e_{f+2} \ldots e_k$. Notice that edges e_i and e_{j+1} have the same color, for otherwise, the trail $e_1 e_2 \ldots e_i e_{j+1} \ldots e_f e_{f+1} e_{f+2} \ldots e_k$ violates the choice of T, since a_p is visited fewer times on this trail than on T. Similarly, edges e_i and e_{f+1} have the same color. But then the trail $e_1 e_2 \ldots e_i e_{i+1} \ldots e_{j-1} e_j e_{f+1} e_{f+2} \ldots e_k$ violates the choice of T. Finally, we update every λ_i in this trail and repeat the process until no more vertices with $\lambda \geq 3$ are found. This completes the argument and the proof of lemma. $\square$

Thus, as will be discussed later, for checking the existence of $s - t$ trails, it suffices to take into account only those trails where no vertices are visited more than twice.

Now, we show how to transform the trail- to the path-problem over a new c-edge-colored graph. Given G^c and an integer $p \geq 2$, let us consider an edge-colored graph denoted by $p - H^c$ (henceforth called *the trail-path graph*) obtained from G^c as follows. Replace each vertex x of G^c by p new vertices $x_1, x_2, \ldots, x_p$. Furthermore for any edge xy of G^c colored, say j, add two new vertices v_{xy} and u_{xy}, add the edges $x_i v_{xy}$, $u_{xy} y_i$ for $i = 1, 2, \ldots, p$ all of them colored j, and finally add the edge $v_{xy} u_{xy}$ with color $j' \in \{1, 2, \ldots, c\}$ and $j' \neq j$. For convenience of notation, the edge-colored subgraph of $p - H^c$ induced by the vertices x_i, v_{xy}, u_{xy}, y_i (for $i = 1, \ldots, p$) and associated with the edge xy of G^c will be denoted throughout by H_{xy}^c. Moreover for $p = 2$, we just write H^c instead of $p - H^c$.

Therefore, as a consequence of Lemma 2.1, we have the following relation between G^c and trail-path graph H^c:

Theorem 2.2. *Given two vertices s and t in G^c, there exists a properly edge-colored $s - t$ trail in G^c, if and only if, there exists a properly edge-colored $s_1 - t_1$ path in H^c.*

Proof. Let s, t be two vertices in G^c. Assume first that there exists a properly edge-colored trail, say, $T = e_1, e_2, \ldots, e_k$ between s and t in G^c, where e_i are the edges of the trail and s is the left endpoint of e_1 while t is the right endpoint of e_k. By Lemma 2.1, we may choose T so that no vertex is visited more than twice on T. Given H^c as defined above, we show how to construct a properly edge-colored path P between s_1 and t_1 in H^c. For any edge $e_i = xy$ of T, we consider the associated subgraph $H^c_{e_i}$ in H^c, and then replace the edge e_i by one of the segments $x_1 v_{xy}, v_{xy} u_{xy}, u_{xy} y_1$ or $x_1 v_{xy}, v_{xy} u_{xy}, u_{xy} y_2$ or $x_2 v_{xy}, v_{xy} u_{xy}, u_{xy} y_1$ or $x_2 v_{xy}, v_{xy} u_{xy}, u_{xy} y_2$ in H^c.

Conversely, any properly edge-colored $s_1 - t_1$ path in H^c uses precisely one of the sub-paths $x_1 v_{xy}, v_{xy} u_{xy}, u_{xy} y_1$ or $x_1 v_{xy}, v_{xy} u_{xy}, u_{xy} y_2$ or $x_2 v_{xy}, v_{xy} u_{xy}, u_{xy} y_1$ or $x_2 v_{xy}, v_{xy} u_{xy}, u_{xy} y_2$ in each subgraph H^c_{xy} of H^c. Now it is easy to see that a properly edge-colored $s_1 - t_1$ path in H^c will correspond to a properly edge-colored $s - t$ trail T in G^c where no vertices are visited more than twice on T. □

The following corollary is a straightforward consequence of Theorems 1.2 and 2.2. The proof is left to the reader.

Corollary 2.3. *Consider two distinct vertices s and t in a c-edge-colored graph G^c. Then we can either find a properly edge-colored $s - t$ trail or else decide correctly that such a trail does not exist in G^c in linear time on the size of G^c.*

Another possibility, is to deal with a based BFS procedure to solve the properly edge-colored $s - t$ trail problem. In our case, however, we are particularly concerned with the Szeider's algorithm and its consequences.

Now, we intend to characterize edge-colored graphs without properly edge-colored closed trails. Recall that the problem of checking whether G^c contains no properly edge-colored cycle was initially solved by Grossman and Häggkvist [16] for 2-edge-colored graphs and then by Yeo [27] for an arbitrary number of colors (see Theorem 1.1). In both studies, the authors used the concept of a cut-vertex separating colors, *i.e.*, a vertex x such that all the edges between each component of $G^c - x$ and x are colored alike. Thus, from the definition of trail-path graph and the Theorem 1.1 we obtain the following:

Theorem 2.4. *Let G^c be a c-edge-colored graph, such that every vertex of G^c is incident with at least two edges colored differently. Then either G^c has a bridge or G^c has a properly edge-colored closed trail.*

Proof. Given G^c, consider the trail-path graph H^c associated with G^c as in the foregoing. Observe that if a vertex x of G^c is incident with two edges colored differently in G^c, then both x_1 and x_2 will be incident with edges of different colors in H^c. In addition, for every edge xy of G^c, we have by the definition of H^c that both v_{xy} and u_{xy} are incident with edges of two different colors. Therefore, we conclude that if every vertex of G^c is incident with at least two edges in different colors in G^c, than every vertex of H^c will be incident with at least two edges of different colors in H^c. Then, it follows by Theorem 1.1 that H^c has either a cut-vertex separating colors or a properly edge-colored cycle.

Now, suppose first that H^c has a cut-vertex separating colors. Notice that, since every vertex x of G^c is incident with at least 2 edges of different colors, we cannot have a vertex x_i separating colors in H^c (even if x is a cut-vertex separating colors in G^c). Thus, if this cut-vertex is one of $v_{xy} \in H^c_{xy}$, it is easy to see that u_{xy} is another cut-vertex of H^c separating colors. Therefore, the edge $v_{xy} u_{xy}$ is a bridge in H^c. This implies that the edge xy of G^c associated with H^c_{xy} is also a bridge in G^c. In addition, note that if H^c contains no cut-vertex separating colors, G^c contains no bridges.

Assume now that H^c has a properly edge-colored cycle. Then we conclude that G^c has a properly edge-colored trail if and only if we have a properly edge-colored cycle in H^c.

From the above, it follows that if each vertex of G^c is incident with at least two edges colored differently, then G^c has either a bridge or a properly edge-colored trail, as required. □

As for the algorithmic aspects of this problem, it suffices to delete all bridges (if any) and all vertices adjacent to edges of the same color in G^c, to test for the existence of a properly edge-colored closed trail in polynomial time. Notice that all such edges and vertices may be deleted without any properly edge-colored closed trail being destroyed. Thus, if the resulting graph is non-empty, it will contain a properly edge-colored closed trail.

2.2. Shortest properly edge-colored paths/trails

In this section we consider shortest properly edge-colored $s - t$ paths and trails. By associating appropriate costs with the edges of the Edmonds–Szeider non-colored graph $G(V, E)$ defined in Section 1, we first show how to find, if any, a shortest properly edge-colored path between (not necessarily distinct) s and t in G^c. As a consequence, this procedure may be used to find a shortest properly edge-colored trail between s and t in G^c. At the end of the section, we will show how adapt these ideas to find a shortest properly edge-colored cycle and closed trail. For the shortest properly edge-colored path problem, let us consider the following algorithm:

Algorithm 1: Shortest properly edge-colored path
Input: A c-edge-colored graph G^c, vertices $s, t \in V(G^c)$.
Output: If any, a shortest properly edge-colored $s - t$ path P in G^c.
Begin
  1. Define: $W = V(G^c) \setminus \{s, t\}$;
  2. For every $x \in W$, construct G_x as defined in Section 1;
  3. Construct the Edmonds–Szeider graph G associated with G^c;

4. Define: $E' = \cup_{x \in W} E(G_x)$;

5. **For every** $pq \in E(G) \setminus E'$ **do** $cost(pq) \leftarrow 1$;

6. **For every** $pq \in E'$ **do** $cost(pq) \leftarrow 0$;

7. Find a minimum weighted perfect matching M in G (if any);

8. Use M to build a path P in G^c and return P, or say that P does not exist;

End.

Henceforth, we define the weighted non-colored graph G above as the *weighted Edmonds–Szeider graph*. Intuitively, the idea in Algorithm 1 is to penalize all edges of G associated with edges in the original graph G^c. In this way, we ensure that a minimum perfect M will maximize the number of edges of $E(G_x)$ (with cost 0) associated with $x \in V(G^c) \setminus \{s, t\}$. To obtain P from M in Step 8, we contract all subgraphs G_x of G to a single vertex x. The remaining edges of M in this resulting non-colored graph, say G', will define a $s - t$ path in G' which is associated to a properly edge-colored $s - t$ path in G^c. Notice that all the vertices not in this $s - t$ path in G^c are isolated, *i.e* we cannot have properly edge-colored cycles containing these vertices (otherwise, M would not be a minimum weighted perfect matching in G).

In addition, observe that the overall complexity of Algorithm 1 is dominated by the complexity of a minimum weighted perfect matching (Step 7). Several matching algorithms exist in the literature. Gabow's bound [12] in $O(n(m + n \log n))$, is one of the best in terms of n and m, but other bounds are possible when the edge weights are integers. Note that Algorithm 1 may be easily adapted if we deal with arbitrary positive costs associated with colored edges. Gabow and Tarjan [14] proposed an ingenious approach to obtain a bound in $O(m \log(nN) \sqrt{n \alpha(n, m) \log n})$, where $\alpha(n, m)$ is the Tarjan's "inverse" of Ackerman's function and N is the maximum weight of an edge. See Gerards [15] for a good reference on general matchings.

Formally, we have established the following result:

Theorem 2.5. *Algorithm* 1 *always find in polynomial time, a shortest properly edge-colored $s - t$ path in G^c, if any.*

Proof. Let M be a minimum weighted perfect matching in G and P the associated path in G^c (obtained after Step 8). For a contradiction, suppose that P is not a properly edge-colored shortest path in G^c. Then, there exists another properly edge-colored $s - t$ path P' in G^c with $cost(P') < cost(P)$. In addition, suppose that all the remaining vertices not in P' are isolated. Now, observe that $cost(pq) = 1$ for every $pq \in E(G) \setminus E'$ and $cost(pq) = 0$ for every $pq \in E'$. Thus, we can easily construct a new matching M' in G such that all edges with unit costs are associated with edges in the $s - t$ path P'. The remaining edges of M' will have cost *zero*. In this way, since $cost(P') < cost(P)$, we obtain $cost(M') < cost(M)$ resulting in a contradiction. Therefore, P is a shortest properly edge-colored path in G^c. Obviously, Algorithm 1 runs in polynomial time. □

Now, to solve the shortest trail problem, it suffices to use the above algorithm as follows: Given s and t in G^c, construct the trail-path graph H^c associated with G^c. Next, we find a shortest properly edge-colored $s_1 - t_1$ path P in H^c by the previous algorithm. Then, by using path P of H^c, come back and construct a shortest properly edge-colored $s - t$ trail T in G^c. Remember that each subgraph H_{xy}^c of H^c is associated with some edge xy of G^c. Furthermore, observe that a properly edge-colored path between x_i and y_j in H_{xy}^c contains exactly 3 edges. Thus, in order to obtain T in G^c from P in H^c, it suffices to replace each $x_i - y_j$ path of P in H_{xy}^c with the corresponding edge xy in G^c. Therefore, we obtain a shortest $s - t$ trail with $cost(T) = cost(P)/3$. The correctness of this algorithm is guaranteed by Theorems 2.2 and 2.5.

We conclude this section with some algorithmic results on shortest properly edge-colored cycles and closed trails. Firstly, we adapt the ideas described above to construct such shortest cycles in G^c (if any), as follows: For an arbitrary vertex x of G^c, construct a graph $G_{sc}^{c+1}(x)$ (with $c + 1$ colors) associated with x by appropriately splitting x into vertices, say s_x and t_x, and c auxiliary vertices $x_1, \ldots, x_c$. Vertices s_x and t_x will correspond to temporary source and destination of $G_{sc}^{c+1}(x)$, and vertices $x_1, \ldots, x_c$ are defined in such a way that properly edge-colored $s_x - t_x$ paths in $G_{sc}^{c+1}(x)$ will correspond to properly edge-colored cycles in G^c passing through vertex $x \in V(G^c)$. Therefore, to find a shortest properly edge-colored cycle, it suffices to repeat this process for every vertex x of G^c while saving the minimum cost solution at each iteration. Formally, we define:

$$V(G_{sc}^{c+1}(x)) = (V(G^c) \setminus \{x\}) \cup \{s_x, t_x, x_1, \ldots, x_c\} \quad \text{and}$$

$$E(G_{sc}^{c+1}(x)) = (E(G^c) \setminus \{xy : y \in N_{G^c}(x)\}) \cup \left(\bigcup_{i \in I_c} \{x_i y : y \in N_{G^c}^i(x)\} \right) \cup (\{s_x, t_x\} \times \{x_1, \ldots, x_c\}).$$

In the construction of $E(G_{sc}^{c+1}(x))$ above we set $c(x_i y) = i$ for every color $i \in I_c$. In addition we color every edge of $\{s_x, t_x\} \times \{x_1, \ldots, x_c\}$ with a new color $c + 1$. After this construction, we find (if any) a shortest properly edge-colored path between s_x and t_x in $G_{sc}^{c+1}(x)$. This process is repeated for the remaining vertices of G^c. Note that a properly edge-colored $s_x - t_x$ path P_x in $G_{sc}^{c+1}(x)$ of length $|P_x|$ is associated with a properly edge-colored cycle C_x in G^c passing through x of length $|C_x| = |P_x| - 2$. We denote this algorithm by *PSC* (Properly Shortest Cycle), for short.

Formally we have established the following result:

Theorem 2.6. *Given G^c, Algorithm PSC above always finds in polynomial time a shortest properly edge-colored cycle in G^c or else decides correctly that G^c has no properly edge-colored cycles at all.*

The correctness and polynomial complexity of Algorithm *PSC* is guaranteed by Theorem 2.5.

As for shortest properly edge-colored cycles, to exhibit a shortest properly edge-colored closed trail, we define a graph $G_{sct}^{c+1}(x)$ associated to $x \in V(G)$ in the following way. Let $G_{aux}^{c_x}$ be an auxiliary edge-colored graph with $c_x \leq c$ colors obtained from G^c after deleting $x \in V(G^c)$. Now, as described in the Section 2.1, construct the trail-path graph $H_{aux}^{c_x}$ associated to $G_{aux}^{c_x}$. Thus, $G_{sct}^{c+1}(x)$ is defined by:

$$V(G_{sct}^{c+1}(x)) = V(H_{aux}^{c_x}) \cup \{s_x, t_x, x_1, \ldots, x_c\} \quad \text{and}$$

$$E(G_{sct}^{c+1}(x)) = E(H_{aux}^{c_x}) \cup \left(\bigcup_{i \in I_c} (\cup_{j \in \{1,2\}} \{x_i y_j : y \in N_{G^c}^i(x)\}) \right) \cup (\{s_x, t_x\} \times \{x_1, \ldots, x_c\}).$$

We define $color(pq) = c + 1$ for every $pq \in \{s_x, t_x\} \times \{x_1, \ldots, x_c\}$. Finally, for every $i \in I_c$ and $y \in N_{G^c}^i(x)$, color $c(x_i y_j) = i$ for $j \in \{1, 2\}$. After this construction, we find a shortest properly edge-colored path between s_x and t_x in $G_{sct}^{c+1}(x)$ using Algorithm 1. The overall process is repeated for the remaining vertices of G^c. Note that a shortest properly edge-colored $s_x - t_x$ path P_x in $G_{sct}^{c+1}(x)$ of length $|P_x|$ is associated with a shortest properly edge-colored closed trail CT_x in G^c passing through x of length $|CT_x| = \frac{|P_x| - 4}{3} + 2$. We denote this algorithm by *PSCT* (Properly Shortest Closed Trail), for short. The correctness of *PSCT* is guaranteed by Lemma 2.1, Theorems 2.2 and 2.5.

2.3. The longest properly edge-colored $s - t$ path/trail problem

The problem of finding the longest properly edge-colored $s - t$ path in arbitrary c-edge-colored graphs is obviously NP-complete since it generalizes the Hamiltonian Path problem in non-colored graphs. Based on the maximum weighted perfect matching problem (see [12,14] for further details), we propose a polynomial time procedure for finding a longest properly edge-colored $s - t$ path (trail) in graphs with no properly edge-colored cycles (closed trails).

Theorem 2.7. *Assume that G^c has no properly edge-colored cycles. Then, we can always find in polynomial time a longest properly edge-colored $s - t$ path or else decide that such a path does not exist in G^c.*

Proof. Construct the weighted Edmonds–Szeider graph G (associated to G^c) and compute the maximum weighted perfect matching M in G (if any), otherwise, we would not have a properly edge-colored path between s and t in G^c (see [12,14] for the complexity of the maximum weighted perfect matching problem). Now, given M, to determine the associated $s - t$ path P in G^c, we construct a new non-colored graph G' by just contracting subgraphs G_x to a single vertex x. It is easy to see that G' will contains a $s - t$ path, cycles and isolated vertices, associated respectively to a properly edge-colored $s - t$ path, properly edge-colored cycles and isolated vertices in G^c. However, by hypothesis G^c does not contains properly edge-colored cycles. Therefore, each edge with unitary cost in M it will be associated to an edge of P and vice-versa. Then, since M is a maximum weighted perfect matching, the associated path P will be the longest properly edge-colored $s - t$ path in G^c. □

Observe in the problem above that, since every vertex is visited at most once and we do not have properly edge-colored cycles, all the vertices not in the longest $s - t$ path will be isolated. However, to find a longest properly edge-colored $s - t$ trail we do not know how many times a given vertex $x \in V(G^c) \setminus \{s, t\}$ will be visited. Note that Lemma 2.1 cannot be applied to this case. Nonetheless, constructing a trail-path graph $p - H^c$ for a convenient parameter p, we obtain the following result concerning the longest properly edge-colored $s - t$ trail.

Theorem 2.8. *Let G^c be a c-edge-colored graph with no properly edge-colored closed trails and two vertices $s, t \in V(G^c)$. Then, we can always find in polynomial time, a longest properly edge-colored $s - t$ trail in G^c, provided that one exists.*

Proof. Given G^c, construct the associated trail-path graph $p - H^c$ for $p = \lfloor (n - 1)/2 \rfloor$ (as described in Section 2.1). Note that, no vertices may be visited more than p times in G^c. To see that, consider a properly edge-colored $s - t$ trail T passing by $x \in V(G^c)$ with the maximum possible number of cycles through x of length 3.

Now, using the same arguments as in Theorem 2.2, we can easily prove that each properly edge-colored closed trail in G^c is associated with a properly edge-colored cycle in $p - H^c$. Therefore, since G^c does not contain properly edge-colored closed trails (by hypothesis), it follows that $p - H^c$ has no properly edge-colored cycles. In addition, note that $p - H^c$ has $O(n^2)$ vertices. Thus, by Theorem 2.7 we can find (if any) a longest properly edge-colored path, say P between s_1 and t_1 in $p - H^c$ in polynomial time. Therefore, the associated trail, say T in G^c will be a longest properly edge-colored $s - t$ trail with $cost(T) = \frac{cost(P)}{3}$. □

2.4. The forbidden-pair version of the one $s - t$ path/trail problem

Consider a c-edge-colored graph G^c for an arbitrary $c \geq 2$, a pair of vertices s, t and a set $S = \{\{s_1, t_1\}, \{s_2, t_2\}, \ldots, \{s_k, t_k\}\}$ of k pairs of vertices of G^c. In the Properly edge-colored $s - t$ Path with k Forbidden Pairs problem (PPKFP for short), the objective is to find a properly edge-colored $s - t$ path containing at most one vertex from each pair in S. Using a simple transformation attributed to Häggkvist [21], we prove the following result concerning c-edge-colored graphs:

Theorem 2.9. *The PPKFP problem is NP-complete even for graphs without properly edge-colored cycles.*

Proof. The PPKFP obviously belongs to NP. To prove that PPKFP is NP-hard, we construct a reduction from the *Path with Forbidden Pairs* problem − PFP. Given a digraph $D(V, A)$, a pair a vertices s, t and a set $S = \{\{s_1, t_1\}, \{s_2, t_2\}, \ldots, \{s_k, t_k\}\}$ of k pair of vertices, the objective in the PFP problem is to find a $s − t$ directed path in D that contains at most one vertex from each pair in S or else decide that such a path does not exist in D. As discussed in [13], this problem is NP-complete even on acyclic digraphs. In the present reduction, we construct a c-edge-colored graph $G^c(V', E)$ with $V' = V \cup \{P^1_{\vec{xy}}, \ldots, P^{c-1}_{\vec{xy}} : \vec{xy} \in A\}$. To simplify the notation, for every $\vec{xy} \in A$ consider $x = P^0_{\vec{xy}}$ and $y = P^c_{\vec{xy}}$. Now, the edge set E is constructed in the following way: every arc $\vec{xy} \in A$ is changed by edges $P^j_{\vec{xy}} P^{j+1}_{\vec{xy}}$ for $j = 0, \ldots, c − 1$ with $c(P^j_{\vec{xy}} P^{j+1}_{\vec{xy}}) = j + 1$. The set S of forbidden pairs in G^c remains the same. Notice that the new edge-colored graph does not contains properly edge-colored cycles. After that, it is easy to see that feasible $s − t$ paths in D corresponds to feasible $s − t$ paths in G^c and vice-versa. $\square$

In addition, notice that if $k = O(log n)$, the PPKFP problem can be easily solved in polynomial time. Basically, at each step i of this algorithm, we construct a new graph $G^{c_i}_i(V_i, E_i)$ with $c_i \leq c$ colors and $V_i = V(G^c) \setminus P_i$ where $P_i = \{p^i_1, \ldots, p^i_k\}$ and $p^i_j = s_j$ or t_j (for $j = 1, \ldots, k$), and $E_i = E(G^{c_i}_i)$. For each subgraph $G^{c_i}_i$ for $i = 1, \ldots, O(n)$; we polynomially find a properly edge-colored $s − t$ path (provided that one exists) using its associated Edmonds–Szeider graph. Finally, the $s − t$ trail case is analogous and will be omitted here.

3. The k-path/trail problem

Let k-PVDP and k-PEDT be the decision versions associated respectively with Maximum Properly Vertex Disjoint Path (MPVDP) and the Maximum Properly Edge Disjoint Trail (MPEDT) problems, *i.e.*, given a c-edge-colored graph G^c, two vertices s, $t \in V(G^c)$ and an integer $k \geq 2$, we want to determine if G^c contains at least k properly edge-colored vertex disjoint paths (respectively, edge disjoint trails) between s and t. Initially, in next section we show that both k-PVDP and k-PEDT are NP-complete even for $k = 2$ and $c = \Omega(n^2)$. In particular, in graphs with no properly edge-colored cycles (respectively, closed trails) and $c = \Omega(n)$ colors, we prove that k-PVDP (respectively, k-PEDT) is NP-complete for an arbitrary $k \geq 2$. Next, at the end of the section, we establish some approximation results and polynomial algorithms for special cases for both MPVDP and MPEDT problems.

3.1. NP-complete results for general graphs

In Theorem 3.2 we will prove that both 2-PVDP and 2-PEDT are NP-complete for 2-edge-colored graphs. In view of that theorem, let us first consider 2 auxiliary results concerning directed cycles and closed trails in (non-colored) digraphs. Let u and v be two fixed vertices in a digraph D. Deciding if D contains or not a directed cycle containing both u and v is known to be NP-complete [11]. In next theorem we prove that deciding if D contains or not a directed closed trail containing both u and v is also NP-complete. We will denote these 2 problems, respectively, by *Directed Cycle* (DC) and *Directed Closed Trail* (DCT).

Theorem 3.1. *The DCT problem is NP-complete.*

Proof. The DCT problem obviously belongs to NP. To prove that DCT is NP-hard, we define a reduction from the following problem. Given four vertices p_1, q_1, p_2, q_2 belonging to a digraph D, we wish to determine if there exist 2 arc-disjoint directed trails connecting $p_1 − q_1$ and $p_2 − q_2$ in D. Here, this problem will be named 2-*Arc Disjoint Trail* (2-ADT) problem. As proved in [11] the 2-ADT is NP-complete.

In particular, given a digraph D, we show how to construct in polynomial time another directed graph D' with a pair of vertices u, v in D' such that there are 2 arc-disjoint trails $p_1 − q_1$ and $p_2 − q_2$ in D, if and only if there exists a directed closed trail containing both u and v in D'.

Before constructing D' let us set $S = \{p_1, p_2, q_1, q_2\}$, $S' = \{p'_1, p'_2, q'_1, q'_2\}$ and $S'' = \{p''_1, p''_2, q''_1, q''_2\}$. The idea is to split appropriately each vertex p_i (vertex q_i) in S into two new vertices p'_i and p''_i (q'_i and q''_i) belonging to S' and S'', respectively. Thus, we have:

$$V(D') = (V(D) \setminus S) \cup S' \cup S'' \cup \{u, v\},$$

and

$$A(D') = \left(A(D) \setminus \left\{ \bigcup_{x \in S} \{\vec{xy}, \vec{yx} : y \in N_D(x)\} \right\} \right) \cup \left(\bigcup_{x'' \in S''} \{\vec{x''w} : w \in N^+_D(x)\} \right)$$

$$\cup \left(\bigcup_{x' \in S'} \{\vec{wx'} : w \in N^-_D(x)\} \right) \cup \{\vec{up'_1}, \vec{p'_1 p''_1}, \vec{p'_2 p''_2}, \vec{q'_1 q''_1}, \vec{q''_1 v}, \vec{vp'_2}, \vec{q'_2 q''_2}, \vec{q''_2 u}\}.$$

Given the definitions above, consider two arc-disjoint trails $p_1 − q_1$ and $p_2 − q_2$, say T_1 and T_2 respectively, in D. Then, it is easy to see that the sequence:

$$T = (u, p'_1, p''_1, T_1, q'_1, q''_1, v, p'_2, p''_2, T_2, q'_2, q''_2, u)$$

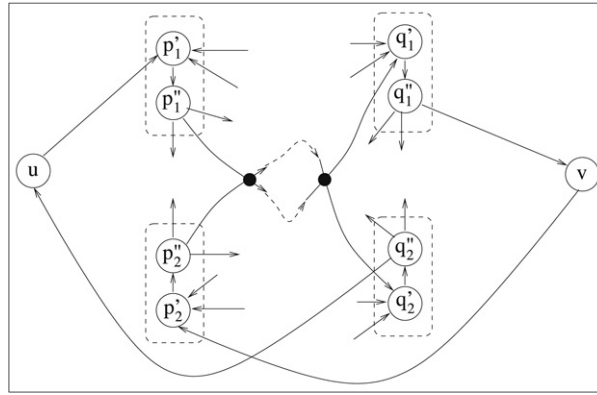defines a closed trail containing both u and v in D' (see Fig. 1).

Fig. 1. Reduction 2-ADT α DCT.

Conversely, consider a directed closed trail containing both vertices u and v in D'. Note that, we have exactly one outcoming and one incoming arc incident to u and v. It follows that, all closed trails containing u and v, also contain all vertices in S' and S'' and each pair (p'_i, p''_i) and (q'_i, q''_i), for $i = 1, 2$, must be visited exactly once. This is possible, if and only if we have a trail between p'_1 and q''_1, and p'_2 and q''_2 in D'. If we delete $u, v \in D'$, and contract all pairs (p'_i, p''_i) to obtain p_i, and (q'_i, q''_i) to obtain q_i, $i = 1, 2$, we obtain 2 arc-disjoint trails $p_1 - q_1$ and $p_2 - q_2$ in D. $\square$

Now, using both DC and DCT problems we prove the following result:

Theorem 3.2. *Both* 2-PVDP *and* 2-PEDT *problems are NP-complete for* 2*-edge-colored graphs.*

Proof. We can easily check in polynomial time that both 2-PVDP and 2-PEDT problems are in NP. To show they are NP-hard, we propose polynomial time reductions from the DC and DCT problems, respectively. Consider two vertices u and v in a digraph D . We show how to construct in polynomial time, a 2-edge-colored graph G^c and a pair of vertices $a, b \in V(G^c)$, such that there is a cycle (respectively, closed trail) containing u and v in D, if and only if there are 2 vertex-disjoint properly edge-colored $a - b$ paths (respectively, 2 edge-disjoint properly edge-colored $a - b$ trails) in G^c. Let us first define from D another digraph D' by replacing u by two new vertices s_1, s_2 with $N_{D'}^-(s_2) = N_D^-(u), N_{D'}^+(s_1) = N_D^+(u)$. Similarly replace t_1, t_2 and $N_{D'}^-(t_2) = N_D^-(v), N_{D'}^+(t_1) = N_D^+(v)$. Finally, add the arcs (s_2, s_1) and (t_2, t_1) in D'. Now in order to define G^c replace each arc $\vec{xy}$ of D' by a colored segment xzy where z is a new vertex and edges xz, zy are on colors red and blue, respectively. Finally, we define $z = a$ for z between s_1 and s_2, and $z = b$ for z between t_1 and t_2. Observe now that there is a vertex-disjoint cycle (respectively, arc-disjoint closed trail) containing u and v in D if and only if there are two vertex-disjoint properly edge-colored $a - b$ paths (respectively, properly edge-colored edge-disjoint $a - b$ trails) in G^c. $\square$

Intuitively speaking, note that both 2-PVDP and 2-PEDT problems become easier when 3 colors or more are considered (an extreme case is when all edges of G^c have different colors). As a consequence of that, an interesting question is to study the NP-completeness of these problems for graphs with many colors. Thus, we prove the following result:

Theorem 3.3. *Both* 2-PVDP *and* 2-PEDT *problems remain NP-complete even for graphs with* $\Omega(n^2)$ *colors.*

Proof. Both 2-PVDP and 2-PEDT problems restricted to graphs with $\Omega(n^2)$ colors obviously belong to NP. Now, given a 2-edge-colored graph G^c with n vertices, define a complete graph $K_n^{c'}$ with $I_{c'} \supseteq I_c$ and an additional edge xy with $x \in V(K_n^{c'})$, $y \in V(G^c)$ and some color $c(xy) \in I_{c'}$. In this way, the new resulting graph $G_\alpha^{c'}$ with vertices $V(G_\alpha^{c'}) = V(G^c) \cup V(K_n^{c'})$ and edges $E(G_\alpha^{c'}) = E(G^c) \cup E(K_n^{c'}) \cup \{xy\}$ will have, respectively, $2n$ vertices and at most $\frac{n(n-1)}{2}$ different edge colors. Therefore, 2 properly edge-colored $s - t$ paths/trails in G^c (with 2 colors) will correspond to 2 properly edge-colored paths/trails in $G_\alpha^{c'}$ with $c' = \Omega(n^2)$ colors and vice-versa. Thus, from the preceding theorem (restricted to 2-edge-colored graphs), we conclude that both 2-PVDP and 2-PEDT problems in graphs with $\Omega(n^2)$ colors are NP-complete. $\square$

3.2. NP-complete results for graphs with no properly edge-colored cycles (closed trails)

Now, we prove that k-PVDP (respectively, k-PEDT) for $k \geq 2$, remains NP-complete even for 2-edge-colored graphs with no properly edge-colored cycles (respectively, closed trails). We conclude this section generalizing these results for graphs with $\Omega(n)$ colors.

Recall that, as discussed in previous sections, the existence or not of properly edge-colored cycles or closed trails in edge-colored graphs may be checked in polynomial time. Our proof is based on some ideas similar to those used by Karp [19] for the Discrete Multicommodity Flow problem for non-oriented (and non-colored) graphs (usually known in the literature as the Vertex Disjoint Path problem).
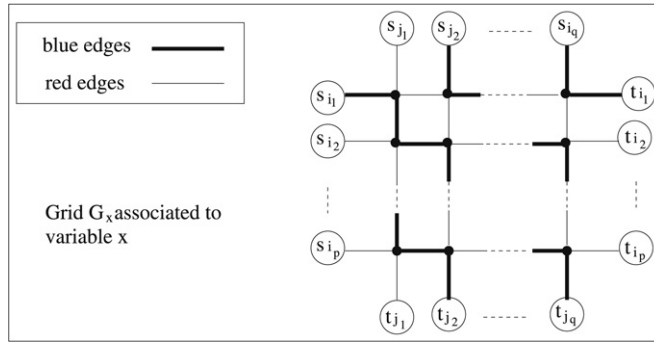
Fig. 2. Blocking property.

Theorem 3.4. *Let G^c be a 2-edge-colored graph without properly edge-colored cycles (respectively, closed trails). Given two vertices s, t in G^c and an integer k, to decide if there exist k properly edge-colored vertex-disjoint $s - t$ paths (respectively, edge-disjoint $s - t$ trails) in G^c is NP-complete.*

Proof. Let us first consider the vertex-disjoint case. The k-PVDP problem obviously belongs to NP. To show that k-PVDP is NP-hard we construct a reduction using the Satisfiability problem. Consider a boolean expression $B = \wedge_{l=1}^{k} C_l$ in the Conjunctive Normal Formula with k clauses and n variables $x_1, \ldots, x_n$. We show how to construct a 2-edge-colored graph G^c with two vertices $s, t \in V(G^c)$ and with no properly edge-colored cycles, such that a truth assignment for B corresponds to k properly edge-colored vertex disjoint $s - t$ paths in G^c, and reciprocally, k properly edge-colored vertex-disjoint $s - t$ paths in G^c define a truth assignment for B. Basically, the idea is to construct a set of k auxiliary source-sink pairs s_l, t_l of vertices, each pair corresponding a to clause C_l. Each variable x_j is associated to a 2-edge-colored *grid* graph G_j. Then graph G^c is obtained by appropriately joining all together these grid graphs and then adding two new vertices s and t. As described in the sequel, the construction of G^c will be done in 4 steps.

Given B, consider a boolean variable x occurring in the positive form in clauses $i_1, i_2, \ldots, i_p$ and in the negative form in clauses $j_1, j_2, \ldots, j_q$. Each occurrence of x in the positive (negative) form is associated to a horizontal path $s_{i_a} - t_{i_a}$ (vertical path $s_{j_b} - t_{j_b}$) in the grid G_x such that all consecutive edges between vertices s_{i_a} and t_{i_a} for $a = 1, \ldots, p$ (respectively, between s_{j_b} and t_{j_b} for $b = 1, \ldots, q$) differ in one color. Every properly edge-colored path $s_{i_a} - t_{i_a}$ has a vertex in common with every properly edge-colored path $s_{j_b} - t_{j_b}$. We say that grid G_x satisfy the *blocking property* if there are no properly edge-colored paths between s_{i_a} and t_{j_b}, or respectively, between s_{j_b} and t_{i_a} for every $a \in \{1, \ldots, p\}$ and $b \in \{1, \ldots, q\}$ (see the example of Fig. 2). In the first step, all grids G_{x_j}, for $j = 1, \ldots, n$, are constructed in order to satisfy the blocking property. Note that, different colorings of G_x satisfying the blocking property are possible. In this case, we can choose any one at random among them. This finish the first step.

Now, we say that a set of grids satisfies the *color constraint* if all edges incident to s_l and t_l, $l = 1, \ldots, k$, in all occurrences of s_l and t_l in the various grids, have the same color. All grids G_{x_j} for $j = 1, \ldots, n$, must be constructed in order to satisfy both blocking property and color constraint. However, note that the color constraint may be not verified after the first step. To solve this problem, suppose w.l.o.g., that all edges incident to s_l in the various grids must be red if l is odd, and blue if l is even. Similarly, all edges incident to t_l (in the various grids) must be blue if l is odd, and red if l is even.

Therefore, suppose that edge $s_l w$ (for $w \in N_{G_{x_j}}(s_l)$) must be *blue*. If $c(s_l w) = blue$, we are done. Otherwise, we add a new vertex p between s_l and w and fix $c(s_l p) = blue$ and $c(pw) = red$. We apply this procedure for every edge incident to s_l (for $l = 1, \ldots, k$) in the various subgraphs G_{x_j} for $j = 1, \ldots, n$. Finally, we repeat the same transformation for every t_l and G_{x_j} for $l = 1, \ldots, k$ and $j = 1, \ldots, n$. Note that, at the end of the second step, we have all grids satisfying both blocking property and color constraint (see Fig. 3(a)).

Now, in the third step, we identify all occurrences of s_l (respectively, t_l) belonging to the various grids G_{x_j}, as a single vertex s_l' (respectively, t_l'). We repeat this process for each $l = 1, \ldots, k$. Let G' be this new 2-edge-colored graph. Note that, due to the color constraint, all edges incident to s_l' (respectively t_l') in G' must have the same color. Finally, in the third step, we add a source s and destination t, and new edges ss_l' and $t_l't$ for $l = 1, \ldots, k$. Therefore, to construct k properly edge-colored paths between s and t in this new graph, all edges ss_l' (respectively $t_l't$) must be colored with a different color, other than those incident to s_l or t_l in G' (see Fig. 3(b)). Let G'' this new 2-edge-colored graph.

In the last step, note that we can have $c(ss_a') \neq c(ss_b')$ (analogously $c(t_a't) \neq c(t_b't)$) for some $a, b \in \{1, \ldots, k\}$ and $a \neq b$. In addition, by construction of our grids, we can have a properly edge-colored path between s_a' and s_b' in some grid G_{x_j} for some $j \in \{1, \ldots, n\}$. Therefore, in this case, we can have a properly edge-colored cycle through s (or t) in G'' (what is not allowed by hypothesis). To avoid that in the construction of G^c, it suffices to add auxiliary vertices p_i between s and s_i' (respectively, auxiliary vertices q_i between t and t_i') and conveniently change the colors of edges sp_i (respectively q_it) such that all edges incident to s (respectively t), have the same color. In this way, the new resulting graph G^c (obtained from G'') will contains no properly edge-colored cycles.

Thus, given a truth assignment for B, we obtain a set of k properly edge-colored vertex disjoint $s - t$ paths in the following manner. If variable x_j is true, we select the horizontal paths in the grid G_{x_j} between vertices s_{i_a} and t_{i_a} (for $a = 1, \ldots, p$); if x_j
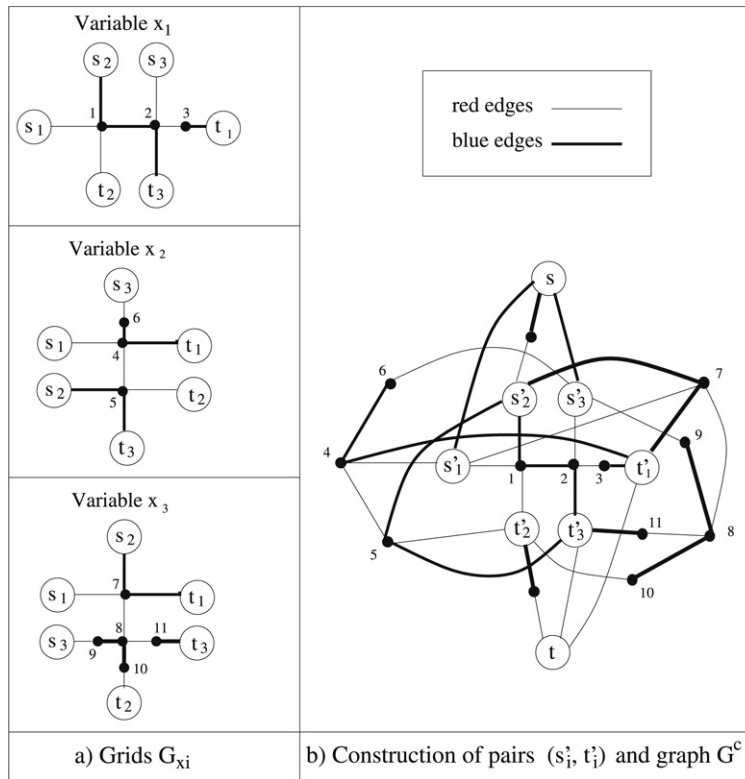
Fig. 3. Reduction using $B = (x_1 \vee x_2 \vee x_3) \wedge (\bar{x_1} \vee x_2 \vee \bar{x_3}) \wedge (\bar{x_1} \vee \bar{x_2} \vee x_3)$. (a) To satisfy the color constraint, we colored all edges incident to s_1, s_3, t_2 and s_2, t_1, t_3, respectively, with *red* and *blue* colors. (b) To construct G^c we add s and t. All edges incident to s and t must be *blue* and *red*, respectively.
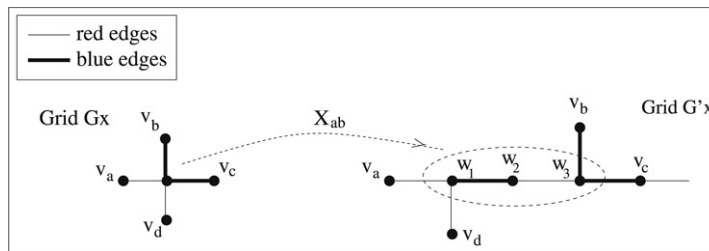

Fig. 4. Transformation from the k-PVDP to the k-PEDT problem.

is false, we select the vertical paths between s_{j_b} and t_{j_b} (for $b = 1, \ldots, q$). Note that, if either x_j or $\bar{x}_j$ occurs in clause C_l, and is true in the assignment, we have a path between vertices s'_l and t'_l in G' and consequently, between s and t in G^c. Therefore, if B is true, we will have k properly edge-colored vertex-disjoint paths between s and t in G^c, each of them passing by s'_l and t'_l for $l = 1, \ldots, k$.

Conversely, consider a set of k properly edge-colored vertex disjoint $s - t$ paths in G^c. Observe in the grid G_{x_j} that, if we have a properly edge-colored path between vertices s_{i_a} and $t_{i_{a'}}$ for $a \in \{1, \ldots, p\}$ and $a' \leq a$, the clause C_{i_a} and variable x_j will be true. Analogously, if we have a path between s_{j_b} and $t_{j_{b'}}$ for $b \in \{1, \ldots, q\}$ and $b' \leq b$, the clause C_{j_b} will be true and variable x_j will be false. Thus, k properly edge-colored vertex disjoint $s - t$ paths will correspond to k true clauses in B. Therefore, for an arbitrary $k \geq 2$, we proved that k-PVDP problem is NP-complete in 2-edge-colored graphs with no properly edge-colored cycles.

We turn now to the edge-disjoint version (k-PEDT) of this problem. We will first consider properly edge-colored $s - t$ paths and finally conclude with properly edge-colored $s - t$ trails. We can use similar arguments in the construction of G^c as above. However, we can have 2-edge-disjoint paths between s and t in G^c corresponding to vertical and horizontal paths in some grid G_x. In another words, we can have a vertex in the intersection of two $s - t$ paths. If this happens, we cannot determine the value of x in B. To solve this problem, we add a fifth step in the construction of a new 2-edge-colored graph, say G_α^c, obtained from G^c as follows. We change each vertex of G_x (represented by X_{ab}) in the intersection of paths $s_{i_a} - t_{i_a}$ for $a = 1, \ldots, p$ (horizontal path) and $s_{j_b} - t_{j_b}$ for $b = 1, \ldots, q$ (vertical path) by 3 new vertices w_1, w_2 and w_3 obtaining a new grid G'_x as described in Fig. 4. In addition, for every grid G_x, suppose that vertices v_a, X_{ab} and v_c belong to path $s_{i_a} - t_{i_a}$, vertices

v_b, X_{ab} and v_d belong to path $s_{j_b} - t_{j_b}$, and $c(v_a X_{ab}) = c(X_{ab} v_d) = red$ and $c(v_b X_{ab}) = c(X_{ab} v_c) = blue$ in G^c. Finally, set $c(w_1 w_2) = blue$ and $c(w_2 w_3) = red$ in the grids G'_x (see Fig. 4). Note that G^c_α with these new grid graphs G'_x also satisfy both blocking property and color constraint. Therefore, if we have a path between s_{i_a} and t_{i_a} (for some $a \in \{1, \ldots, p\}$) passing by v_a and v_c, we cannot have a path between s_{j_b} and t_{j_b} (for some $b \in \{1, \ldots, q\}$) passing by v_b and v_d in G^c_α (otherwise, both $s-t$ paths would not be edge-disjoint). Now, to deal with properly edge-colored $s - t$ trails we can replace one or more arbitrary edges xy of G^c_α with some color $i \in \{red, blue\}$ by a colored segment xyz where z is a new vertex between x and y, and 2 additional vertices p, q with edges zp, pq and qz. These edges are colored in the following way: $c(xz) = c(zy) = c(pq) = i$ and $c(zp) = c(qz) \neq i$. If we repeat this construction for every grid G'_x, we conclude that k-PEDT problem is NP-complete in 2-edge-colored graphs with no properly edge-colored cycles.

Finally, to apply this result to 2-edge-colored graphs with no properly edge-colored closed trails (represented by G^c_β), it suffices to repeat the construction of steps 1, 2, 3 and 5 as above. Note that, since the forth step was ommited, we can have properly edge-colored cycles passing by s or t in G^c_β, but no properly edge-colored closed trails. In this way, k properly edge-colored edge-disjoint $s - t$ trails in G^c_β will be associated to a true assignment for B and vice versa. □

Theorem 3.5. *The k-PVDP (respectively, k-PEDT) problem remains NP-complete even for graphs with $\Omega(n)$ colors and no properly edge-colored cycles (respectively, closed trails).*

Proof. Here, we only deal with the k-PVDP problem, the k-PEDT will be analogous. The k-PVDP problem in graphs with n colors and with no properly edge-colored cycles is obviously in NP. Let G^c be a 2-edge-colored graph with no properly edge-colored cycles and 2 vertices $s, t \in V(G^c)$. Using G^c, we construct a new graph $G^{c'}_\gamma$ with no properly edge-colored cycles and $c' \leq n$, such that k properly edge-colored vertex-disjoint $s - t$ paths in G^c, corresponds to k properly edge-colored vertex-disjoint $s - t$ paths in $G^{c'}_\gamma$ and vice versa.

Firstly, consider a non-colored complete graph $G_1 = K_n$. For every non-colored graph G_i for $i = 1, \ldots, n - 1$, choose $x \in V(G_i)$ and color $c(xy) = j$ for every $y \in N_{G_i}(x)$ with some color $j \in \{1, 2, \ldots, i\}$. Let $G_i = G_{i-1} \setminus \{x\}$ (for $i \geq 2$) be the resulting non-colored complete graph. Obviously, the final edge-colored $K_n^{c'}$ (with $c' \leq n - 1$) obtained in this way contains no properly edge-colored cycles. Finally, add a new edge pq with $p \in V(G^c)$, $q \in V(K_n^{c'})$ and a new color $c(pq) = c' + 1$. Note that the new graph $G^{c'}_\gamma$ with vertices $V(G^{c'}_\gamma) = V(K_n^{c'}) \cup V(G^c)$ and edges $E(G^{c'}_\gamma) = E(G^c) \cup E(K_n^{c'}) \cup \{xy\}$ contains no alternating cycles and will have at most n different colors. Therefore, it follows from the preceding theorem (restricted to 2-edge-colored graphs) that both k-PVDP and k-PEDT problems in graphs with $\Omega(n)$ colors and no properly edge-colored cycles/closed trails is NP-complete. □

3.3. Some approximation and polynomial results

In this section, we describe greedy procedures for both MPEDT and MPVDP, based in the determination of shortest properly edge-colored $s - t$ trails (respectively $s - t$ paths). Their performance ratio are based on the same arguments used for the Edge/Vertex Disjoint Path problem between k pairs of vertices in non-directed graphs [17,20]. We conclude this section by presenting some polynomial results for some particular instances of both problems.

At each steep of the greedy procedure for the MPEDT problem, we find a shortest properly edge-colored $s - t$ trail T in G^c. We then delete all edges in this trail and repeat the process until no $s - t$ trails are found. We denote this procedure by GREEDY-ED, for short.

In this section, $s - t$ trails (or paths), means properly edge-colored $s - t$ trails (or paths) for short. Now consider the following definitions: we say that a $s - t$ trail T_1 *hits* a $s - t$ trail T_2, or equivalently, that T_2 is *hitted* by T_1, if and only if T_1 and T_2 share a common edge. If Γ denotes the set of all properly edge-colored $s - t$ trails, we define $I \subseteq \Gamma$ as the subset of trails obtained by the greedy procedure and $J \subseteq \Gamma$ the subset of $s - t$ trails associated to an optimal solution. Then, we have the following:

Theorem 3.6. *Algorithm GREEDY-ED has performance ratio equal to $O(1/\sqrt{m})$.*

Proof. Let $T \in \Gamma$ be an arbitrary properly edge-colored $s-t$ trail in G^c. We say that a $s-t$ trail $T \in \Gamma$ is *short* if $|E(T)| \leq \sqrt{m}$, and *long* otherwise. Therefore, for a trail $T \in J_{long}$ we have $|E(T)| \geq (\sqrt{m} + 1)$ and $|J_{long}|(\sqrt{m} + 1) \leq m$. Thus, w.l.o.g., if we consider $|I| \geq 1$, it follows that $|J_{long}| < \sqrt{m} < |I|\sqrt{m}$.

Additionally, we can say that every $s - t$ trail $T_j \in J_{short} \setminus I$ is hitted by a $s - t$ trail $T_i \in I_{short}$, otherwise (if $T_i \in I_{long}$) at the point when T_i was picked, T_j was available and shorter than T_i and should have been taken by the greedy procedure. Thus, if T_i is the shortest $s - t$ trail that hits T_j we have $|E(T_i)| \leq |E(T_j)| \leq \sqrt{m}$.

Now, observe that all $s - t$ trails in I_{short} have at most $|I_{short}|\sqrt{m}$ edges and each $P_j \in J_{short} \setminus I$ is hitted by at least one edge of I_{short}. Furthermore, since all $s - t$ trails T_j are edge-disjoint it follows that one edge in I_{short} cannot hit more then one $s - t$ trail T_j. Thus, $|J_{short} \setminus I| \leq |I_{short}|\sqrt{m} \leq |I|\sqrt{m}$.

Finally, we have $|J| = |J_{short}| + |J_{long}| < |(J_{short} \setminus I) \cup I| + |I|\sqrt{m} \leq (2\sqrt{m} + 1)|I|$ which guarantees a $O(1/\sqrt{m})$ performance ratio for the MPEDT problem. □

To give some idea about the determination of the value $\sqrt{m}$ above, suppose that a $s - t$ trail T_1 hits k $s - t$ trails of $J \setminus I_1$ at the first step of the Greedy-ED. Note that, one edge of T_1 can hit at most one other trail of J and therefore T_1 has length at
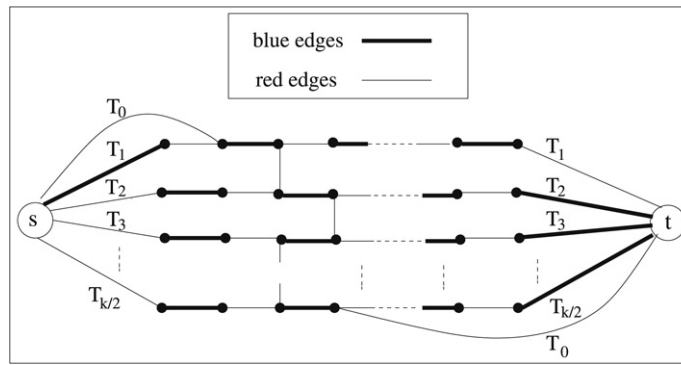
Fig. 5. Let G^c be a 2-edge-colored graph. Suppose $|E(T_i)| = k + 2$ for $i = 1, \ldots, k/2$. The ratio between GREEDY-ED and the optimal solution is $2/k$.

least k. Since T_1 is a shortest $s - t$ trail, all other trails in $J \setminus I_1$ also have at least k edges. Therefore, $k^2 \leq m$, so $k = \sqrt{m}$. This idea may be inductively applied for the remaining steps of the greedy procedure.

In the Fig. 5, we consider a 2-edge-colored graph G^c with $|E(T_i)| = k + 2$ for $i = 1, \ldots, k/2$. In this case, since $|E(T_0)| = k + 1$ (the shortest $s - t$ trail), the Greedy-ED procedure first select T_0, hitting $k/2$ properly edge-colored $s - t$ trails. Clearly an optimal solution is obtained by choosing trails $T_1, \ldots, T_{k/2}$. Thus $2/k$ is ratio between the greedy and an optimal solution where $k \leq \sqrt{m}$.

We turn now to the vertex-disjoint version of the above problem, namely, the Maximum number of Properly Vertex-Disjoint $s - t$ paths in G^c. We can easily modify the GREEDY-VD procedure to solve the MPVDP problem. In this case, after the determination of a shortest $s - t$ path P (instead of a $s - t$ trail T), it suffices to remove all vertices belonging to $P \setminus \{s, t\}$. We repeat this process until no more properly edge-colored $s - t$ paths are found. We call this new procedure GREEDY-VD. Using the same ideas as described in Theorem 3.6, we prove the following result:

Theorem 3.7. *The Greedy-VD procedure has performance ratio equal to $O(1/\sqrt{n})$ for the MPVDP problem.*

We end this section with some polynomial results for some specific families of graphs. To begin with, we introduce the following definition: given an c-edge-colored graph G^c, we say that a cycle $C_x = xa_1 \ldots a_j x$ with $x \neq a_i$ for $i = 1, \ldots, j$ is an *almost properly edge-colored cycle (closed trail) through x in G^c*, if and only if $c(xa_1) = c(xa_j)$ and both paths (respectively trails) $x - a_1$ and $x - a_j$ are properly edge-colored . If $c(xa_1) \neq c(xa_j)$, then C_x define a *properly edge-colored cycle (closed trail) through x*. In the sequel, we show how to solve the MPVDP (respectively, MPEDT) problem in polynomial time for graphs containing no properly or almost properly edge-colored cycles (respectively, closed trails) through s or t. Notice that to check if an edge-colored graph G^c contains or not a properly edge-colored or an almost properly edge-colored cycle (closed trail) through x, it suffices to define an auxiliary graph G_x^c obtained from G^c by replacing x with two new vertices x_a and x_b and setting $N_{G_x^c}(x_a) = N_{G^c}(x)$ and $N_{G_x^c}(x_b) = N_{G^c}(x)$. Now, using Theorem 1.2 (respectively, Corollary 2.3) we compute, if any, a properly edge-colored $x_a - x_b$ path (trail) in G_x^c. Clearly if no such $x_a - x_b$ path (trail) exists in G_x^c, then there exists no properly or almost properly edge-colored cycle (closed trail) through x in G^c.

Initially, consider the following decision version associated with MPVDP problem. Given an integer $k \geq 1$, we show how to construct a polynomial time procedure for the k-PVDP in graphs with no (almost) properly edge-colored cycles through s or t.

Theorem 3.8. *Consider an integer $k \geq 1$ and a c-edge-colored graph G^c with no (almost) properly colored cycles through s or t. Then, the k-PVDP problem may be solved in polynomial time.*

Proof. Suppose, *w.l.o.g.*, that we do not have (almost) properly edge-colored cycles through vertex s in G^c. Observe in this case that (almost) properly edge-colored closed trails through s are allowed.

For $k = 1$, the problem is polynomially solved by Edmonds–Szeider's Algorithm. For $k \geq 2$, we construct an auxiliary non-colored graph G' in the following way. As discussed in Section 1, we first define $W = V(G^c) \setminus \{s, t\}$, and non-colored graphs G_x for every $x \in W$ (see the first part in the definition of the Edmonds–Szeider's graph). Now, define $S_k = \{s_1, \ldots, s_k\}$, $T_k = \{t_1, \ldots, t_k\}$ and proceed as follows:

$$V(G') = S_k \cup T_k \cup \left(\bigcup_{x \in W} V(G_x) \right), \quad \text{and}$$

$$E(G') = \bigcup_{j=1,\ldots,k} \left(\bigcup_{i \in I_c} \left(\{s_j x_i | sx \in E^i(G^c)\} \cup \{x_i t_j | xt \in E^i(G^c)\} \right) \right)$$

$$\cup \left(\bigcup_{i \in I_c} \{x_i y_i | xy \in E^i(G^c)\} \right) \cup \left(\bigcup_{x \in W} E(G_x) \right).$$

Now, find a perfect matching M (if any) in G' and contract each subgraph G_x into a single vertex x. Let G'' this new non-colored graph. Observe that all $s_i - t_j$ paths in G'' are defined by edges belonging to $M \cap E(G'')$. In addition, we cannot have a path between s_i and s_j in G'' (otherwise, we would have a (an almost) properly edge-colored cycle through s in G^c). In this way, all paths in G'' begins at vertex $s_i \in S_k$ and finish at some vertex $t_j \in T_k$. Finally, we construct a non-colored graph G''' by contracting subsets S_k and T_k respectively to vertices s and t. In this way, note that non-colored $s - t$ paths in G''' are associated to properly edge-colored $s - t$ paths in G^c and vice-versa. Therefore, if the construction of a perfect matching M in G' is possible (what is done in polynomial time), we obtain k properly edge-colored $s - t$ paths in G^c. □

Since the perfect matching problem is solved in polynomial time, we can easily construct a polynomial time procedure for the MPVDP in graphs with no (almost) properly colored cycles through s or t. To do that, it suffices to repeat all the steps described in Theorem 3.8 for $k = 1, \ldots, n - 2$ until some non-colored graph G' containing no perfect matchings is found.

The ideas above may be generalized for the MPEDT in graphs with no (almost) properly colored closed trails through s or t. Firstly, we deal with its associated decision version.

Theorem 3.9. *Consider a constant $k \geq 1$ and a c-edge-colored graph G^c with no (almost) properly edge-colored closed trails through s or t. Then, the k-PEDT problem can be solved in polynomial time.*

Proof. Given G^c, construct the associated trail-graph $p - H^c$ (as described in Section 2) for $p = \lfloor (n - 1)/2 \rfloor$. Note that, no vertices may be visited more than p times in G^c even if they are shared by different properly edge-colored $s - t$ trails. To see that, consider a vertex $x \in V(G^c)$ and a properly edge-colored $s - t$ trail of length 2 through x, all other properly edge-colored trails through x will have at least 4 edges, each of them containing at least 2 new vertices in G^c.

Thus, suppose *w.l.o.g.*, that we do not have (almost) properly colored closed trails through vertex s in G^c. Now, using Theorem 2.2, we can easily prove that G^c contains a (an almost) properly colored closed trail through s, if and only if, H^c contains a (an almost) properly colored cycle through s_1. As a consequence of that, we have no (almost) properly edge-colored cycles through s_1 in $p - H^c$. Thus, by Theorem 3.8 we can find in polynomial time (if any) k properly edge-colored paths between s_1 and t_1 in the graph $p - H^c$. Now, substituting every subgraph H^c_{xy} in $p - H^c$ by edge xy in G^c we obtain k properly edge-colored $s - t$ trails in G^c in polynomial time. □

Similarly to the MPVDP problem, to construct a polynomial procedure for the MPEDT, it suffices to repeat all the steps above (in Theorem 3.9) for $k = 1, \ldots, n - 2$ until some non-colored graph associated to $p - H^c$ and containing no perfect matching is found.

4. Conclusions and open problems

In this work, we have considered path and trail problems in edge-colored graphs. We generalized some previous results concerning properly edge-colored paths and cycles in edge-colored graphs, which allowed us to devise efficient algorithms for finding them. On the negative side, we proved that finding k properly vertex/edge disjoint $s-t$ paths/trails is NP-complete even for $k = 2$ and $c = \Omega(n^2)$. In addition, we showed that both problems remain NP-complete for arbitrary $k \geq 2$ in graphs with no properly edge-colored cycles (closed trails) and $c = \Omega(n)$, which led us to investigate approximation. For that purpose, a procedure for MPEDT, which greedily builds shortest properly edge-colored $s - t$ trails, was shown to have a respectable $O(1/\sqrt{m})$ performance ratio. Similarly, we obtained an approximation ratio in $O(1/\sqrt{n})$ for the MPVDP. Finally, we showed that both MPVDP (MPEDT) are solved in polynomial time when restricted to graphs with no (almost) properly edge-colored cycles (closed trails) through s or t. However, the following questions are left open.

Is the following problem NP-complete?

Problem 4.1. *Input*: Given a 2-edge-colored graph G^c with no properly edge-colored cycles, two vertices $s, t \in V(G^c)$ and a fixed constant $k \geq 2$.
Question: Does G^c contains k properly edge-colored vertex/edge disjoint paths between s and t?

As a future direction, another important question is to consider improved approximation performance ratios (as well as inapproximability results) for both MPVDP and MPEDT for general edge-colored graphs, or for graphs with no properly edge-colored cycles (closed trails). We conclude our paper by recalling the following open problem from [21].

Problem 4.2. *Input*: Given a 2-edge-colored complete graph K_n^c and two vertices $s, t \in V$ *Question*: Does there exist a polynomial algorithm for finding the maximum number of properly edge-colored edge-disjoint $s - t$ trails in K_n^c?

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