Theoretical Computer Science 262 (2001) 133–144

Theoretical Computer Science

www.elsevier.com/locate/tcs

Priority queues with binary priorities

K. Kalorkoti[a, *], D.H. Tulley[b]

[a] School of Computer Science, University of Edinburgh, Kings Building, Mayfield Road, Edinburgh EH9 3JZ, Scotland, UK

[b] School of Mathematical and Computational Sciences, North Haugh St Andrews KY16 9SS, Scotland, UK

Received September 1997; revised November 1999; accepted March 2000

Abstract

We consider a priority queue of unbounded capacity whose input is the sequence $1, 2, \ldots, n$ where each i is given a binary priority. We prove a previously conjectured recurrence for the number of allowable input–output pairs achievable by such a queue with z items of priority 0; the proof provides a new application of inseparable permutations. We then give upper and lower bounds for this and deduce that for fixed z the growth rate is $\Theta(n!\log^z(n))$. We also study the total number of allowable input–output pairs where the number of items of priority 0 is not fixed and provide very tight upper and lower bounds which imply that the growth rate is $\Theta(nn!\log(n))$.

1. Introduction

Certain aspects of the combinatorics of abstract data types were studied by Knuth [10], Tarjan [13] and Pratt [12]. Priority queues have long been studied by queuing theorists, e.g., by Jaiswal [8], and King [9]. In such studies it is natural to consider the special case of two priorities, see [8] or Leemans [11] (where a two-priority, two-server model is considered). The study of combinatorial aspects of priority queues has received attention more recently, e.g., see Atkinson and Thiyagarajah [1], where a priority queue processing elements with distinct priorities is considered and Atkinson [2] where a priority queue processing binary elements is considered. Tulley [14] considers several abstract data types, including priority queues, of both bounded and unbounded capacity with a variety of input types, including binary and distinct. Work initially inspired from other areas has also led to the study of priority queues (e.g., [5]). In many cases links to other combinatorial objects are revealed (e.g., [6, 3]). The discovery of these

* Corresponding author.

E-mail addresses: kk@dcs.ed.ac.uk (K. Kalorkoti), dominic@dcs.st-and.ac.uk (D.H. Tulley).

PII: S0304-3975(00)00186-9

correspondences such as with labelled trees in [6] and ordered trees in [3] is both pleasing and gives insight into the structure behind the subject. This paper augments these with the provision of a correspondence between inseparable permutations and binary priority queues.

We consider a priority queue of unbounded capacity whose input is the sequence $1, 2, \ldots, n$ where each i is given binary priority π_i (with 0 having higher precedence than 1). If τ is a possible output of the queue then we say that (π, τ) is allowable (where π denotes the sequence of priorities of the input items). We consider the cardinality of the set of input–output pairs that are allowable over all priority assignments, i.e.,

$$|\{(\pi, \tau) \mid \pi \text{ is a binary priority, } \tau \text{ is an output sequence and } (\pi, \tau) \text{ is allowable}\}|.$$

This problem has been studied by Tulley [14]. As in [14] we focus on the situation in which z of the n inputs have priority 0 and denote the number of allowable permutations by $x_{n,z}$. We use a notation that switches attention to priorities; the input is viewed as a word in the alphabet $\{0_1, 0_2, \ldots, 0_z, 1_1, 1_2, \ldots, 1_{n-z}\}$ in which the subscripts of the ones occur in increasing order and, for notational convenience in Section 2, those of the zeros occur in decreasing order. Thus $1_1 0_3 0_2 1_2 0_1$ denotes a sequence of five distinct elements, the first and fourth of which have priority 1 while the others have priority 0. One possible outcome from this sequence is $1_1 0_2 0_1 0_3 1_2$. The sequence $0_2 1_1 0_3 1_2 0_1$ is not possible since the queue must contain both 1_1 and 0_3 before it can read in the 0_2 to output it, in which case it cannot output 1_1 before 0_3. We use this convention of indexing zeros from the right to associate a permutation with any rearrangement of $0_i, 0_{i-1}, \ldots, 0_1$, e.g., the rearrangement $0_1 0_2 0_3$ of $0_3 0_2 0_1$ defines the permutation 321. If σ is an input word and τ is a possible output, we call (σ, τ) an allowable pair; thus $x_{n,z}$ is the number of such pairs.

Tulley [14] gives exact formulae and recurrences for $x_{n,z}$ provided $z \leqslant 3$. It follows from these that (with z fixed and n varying)

$$x_{n,z} = \Theta(n! \log^z n)$$

for $z \leqslant 3$ and it is conjectured, with further supporting evidence, that the estimate holds for all z. We establish this conjecture by showing that

$$\frac{n!}{z!} \log^z(n+1) \leqslant x_{n,z} \leqslant n!(1 + \log^z n)$$

for all $n \geqslant z \geqslant 0$. In fact, the evidence provided in [14] was used to conjecture a recurrence for $x_{n,z}$ and we prove the correctness of this in Section 2. Bounds on $x_{n,z}$ are obtained in Section 3. In Section 4 we study the quantity

$$s_n = \sum_{z=0}^{n} x_{n,z}$$

and provide bounds for it which are much tighter than those for $x_{n,z}$. In particular the bounds imply that the growth rate of s_n is $\Theta(nn! \log(n))$. Finally, in Section 5 we describe briefly an open question relating to the behaviour of $x_{n,z}$ for fixed n and varying z.

2. A recurrence for $x_{n,z}$

Let $\rho = \rho_1, \rho_2, \ldots, \rho_n$ be a permutation of $1, 2, \ldots, n$. We say that ρ is *inseparable* if for each i, with $1 \leqslant i < n$, the prefix $\rho_1, \rho_2, \ldots, \rho_i$ of ρ is not a permutation of $1, 2, \ldots, i$. We denote the number of such permutations by a_n. A simple argument shows that

$$a_1 = 1,$$
$$a_n = n! - \sum_{i=1}^{n-1} i! a_{n-i} \quad \text{for } n > 1,$$

e.g., see [4] or [7]. We define the *inseparability index* of ρ to be the largest j, with $1 \leqslant j \leqslant n$, such that $\rho_1, \rho_2, \ldots, \rho_j$ is an inseparable permutation of $1, 2, \ldots, j$.

Lemma 1. *Let (σ, τ) be an allowable pair. Then either τ ends with some 1_r or an inseparable permutation associated with $0_i, 0_{i-1}, \ldots, 0_1$ for some i.*

Proof. Suppose that τ does not end with an element of priority 1 and let 0_j be in the sequence of 0's forming the tail of τ. Any elements of priority 1 which follow 0_j in σ must follow 0_j in τ and so σ must have the form $\sigma' 0_j 0_{j-1} \ldots 0_1$ since no elements of priority 1 follow 0_j in τ. Further, any element of priority 0 that follows 0_j in σ must be in the same block of 0's as 0_j in τ. Therefore $0_j, 0_{j-1}, \ldots, 0_1$ must all be in the sequence forming the tail of τ. Consequently, the tail of τ is a permutation of $0_k, 0_{k-1}, \ldots, 0_1$ for some $k \geqslant j$ and we simply take i to be the index of inseparability of this permutation. □

Lemma 2. *Suppose that $n > z$. There are $n x_{n-1,z}$ allowable pairs of the form $(\sigma, \tau 1_i)$ for some i.*

Proof. Let (σ, τ) be an allowable pair where σ has $n - 1$ symbols, z of which have priority 0. Then (σ, τ) can be converted to an allowable pair of the stated form by inserting a priority 1 element into any of the n possible positions in σ, relabelling any subsequent priority 1 elements in σ and τ, and appending the new priority 1 element at the end of τ. This is still an allowable pair since the new priority 1 element can be inserted into the priority queue during the computation of τ from σ and output once the computation of τ has been completed.

On the other hand, any allowable pair of the form $(\sigma, \tau 1_i)$, for some i, can be converted to an allowable pair (σ', τ) by removing the 1_i from σ and relabelling each 1_j by 1_{j-1} for all $j > i$. It is easily seen that the new pair is allowable. □

Lemma 3. *There are* $a_i x_{n-i,z-i}$ *allowable pairs of the form* $(\sigma, \tau\rho)$ *where* ρ *is an inseparable permutation of* $0_i, 0_{i-1}, \ldots, 0_1$ *where* $1 \leqslant i \leqslant z$.

Proof. Let (σ, τ) be an allowable pair where σ consists of $n-i$ symbols, $z-i$ of which have priority 0. Then (σ, τ) can be converted to a pair of the above form by appending $0_i, 0_{i-1}, \ldots, 0_1$ to σ, relabelling each 0_j in σ by 0_{j+i} and appending any inseparable permutation of $0_i, 0_{i-1}, \ldots, 0_1$ to τ.

Conversely any allowable pair of the above form can be converted to an allowable pair (σ', τ) where σ' consists of $n-i$ symbols, $z-i$ of which have priority 0, by removing ρ from the output sequence, removing the members of ρ from σ and relabelling the remaining elements of priority 0 as appropriate. □

Theorem 4. *For all* $n \geqslant z \geqslant 0$ *we have*

$$x_{n,0} = n!,$$
$$x_{n,n} = n!,$$
$$x_{n,z} = n x_{n-1,z} + \sum_{i=1}^{z} a_i x_{n-i,z-i} \quad \textit{for } n > z > 0.$$

Proof. The first two cases are clear. The final case follows from Lemmas 2 and 3 by summing over all possible lengths of the inseparable permutation that terminates the output. □

Lemma 5. *For all* $n > z > 0$ *we have* $x_{n,z} \leqslant n x_{n-1,z} + z x_{n-1,z-1}$.

Proof. It follows from Lemma 2 that it suffices to prove that there are at most $z x_{n-1,z-1}$ allowable pairs of the form $(\sigma, \tau 0_i)$ for some i. According to the proof of Lemma 1, in any such pair the 0_i must be in a block of 0's at the tail of σ, thus there are at most z positions that 0_i can occupy in σ. If we delete 0_i and relabel the remaining 0's appropriately we obtain an allowable pair with $n-1$ symbols, $z-1$ of which have priority 0. □

3. Bounds on $x_{n,z}$

Throughout we interpret $\log^0(n)$ as 1 for all n. We shall use the following bounds, for all $n \geqslant 2$:

$$\log(n-1) + \frac{1}{n} < \log n < \log(n-1) + \frac{1}{n-1}. \tag{1}$$

These follow from

$$\log\left(\frac{n}{n-1}\right) = \log\left(\frac{1}{1-1/n}\right) = \frac{1}{n} + \frac{1}{2n^2} + \frac{1}{3n^3} + \cdots \ .$$

Lemma 6. *The following upper bounds hold*:

$$\begin{aligned} x_{n,0} &= n!, \\ x_{n,n} &= n!, \\ x_{n,1} &\leqslant n!(1+\log n), \\ x_{n,z} &\leqslant n!\log^z(n) \quad \textit{for } n\geqslant 6 \textit{ and } z\geqslant 2. \end{aligned}$$

Proof. The first two bounds are immediate. The third bound follows by induction on n. The base case is immediate while for the induction step we have

$$\begin{aligned} x_{n,1} &= nx_{n-1,1} + x_{n-1,0} \\ &\leqslant n!(1+\log(n-1)) + (n-1)! \\ &= n!\left(1+\log(n-1)+\frac{1}{n}\right) \\ &\leqslant n!(1+\log n), \end{aligned}$$

where the last inequality follows from the lower bound in (1).

We prove the final inequality of the lemma by induction on n as well. The base cases (i.e., $n=6$ and $z=2,3,4,5,6$) follow by direct calculation. The case $z=n$ is also immediate. Assume now that $n>z\geqslant 2$ with $n>6$. Then

$$\begin{aligned} x_{n,z} &\leqslant nx_{n-1,z} + zx_{n-1,z-1} \quad \text{by Lemma 5} \\ &\leqslant n!\log^z(n-1) + z(n-1)!\log^{z-1}(n-1) \quad \text{by induction.} \end{aligned}$$

It therefore suffices to prove that

$$\log^z n \geqslant \log^z(n-1) + \frac{z}{n}\log^{z-1}(n-1)$$

for all $n>z>0$. Now, using the lower bound in (1), we have

$$\begin{aligned} \log^z(n) &\geqslant \log^z(n-1)\left(1+\frac{1}{n\log(n-1)}\right)^z \\ &\geqslant \log^z(n-1)\left(1+\frac{z}{n\log(n-1)}\right). \qquad \square \end{aligned}$$

Lemma 7. $x_{n,z} \leqslant n!(1+\log^z n)$.

Proof. The claim follows from Lemma 6 for all values of n, z except for $1<z<n<6$ and these cases can be checked by direct calculation. □

Lemma 8. $i!/2 \leqslant a_i \leqslant (i-1)(i-1)!$ *for all* $i \geqslant 2$.

Proof. For the lower bound consider a separable permutation π of $1,2,\ldots,i$ with inseparability index j. We can turn π into an inseparable permutation by swapping i with π_j. It is easy to see that this map is invertible and so $a_i \geqslant i!/2$ as claimed.

For the upper bound we have

$$\begin{aligned} a_i &= i! - \sum_{j=1}^{i-1} j! a_{i-j} \\ &\leqslant i! - \frac{1}{2}\sum_{j=1}^{i-1} j!(i-j)! \\ &\leqslant i! - (i-1)!. \qquad \square \end{aligned}$$

Lemma 9. $x_{n,z} \leqslant n!(n-z+1)$.

Proof. The claim is clearly true for the cases $n=0$, $z=n$ and $z=0$ since $x_{n,n}=x_{n,0}=n!$. Otherwise we use induction on n. By Theorem 4 and the induction hypothesis we have

$$x_{n,z} \leqslant n!(n-z) + (n-z+1)\sum_{i=1}^{z} a_i(n-i)!.$$

It therefore suffices to prove that

$$\frac{n!}{n-z+1} \geqslant \sum_{i=1}^{z} a_i(n-i)!$$

for $1 \leqslant z \leqslant n-1$. We use induction on z. The base case is immediate since $a_1=1$. For the induction step we have

$$\begin{aligned} \sum_{i=1}^{z} a_i(n-i)! &= \sum_{i=1}^{z-1} a_i(n-i)! + a_z(n-z)! \\ &\leqslant \frac{n!}{n-z+2} + (z-1)(z-1)!(n-z)! \end{aligned}$$

by the induction hypothesis and Lemma 8. Finally it suffices to show that

$$\frac{n!}{n-z+1} \geqslant \frac{n!}{n-z+2} + (z-1)(z-1)!(n-z)!$$

which is equivalent to

$$n! \geqslant (n-z+2)(z-1)(z-1)!(n-z+1)!.$$

This can be rewritten as

$$\binom{n}{z-1} \geqslant (n-z+2)(z-1).$$

This inequality can be verified directly for $z=n-1$. For $1 \leqslant z \leqslant n-2$ we prove the inequality by producing $(n-z+2)(z-1)$ distinct subsets of $\{1,2,\ldots,n\}$ each of size $z-1$. For $i=1,2,\ldots,n-z+1$ we consider the set $S_i=\{i,i+1,\ldots,i+z-2\}$. From each S_i we can build a further $z-2$ distinct sets of size $z-1$ by replacing each element greater than i by $i+z-1$ in turn. All the sets obtained in this way are clearly distinct (consider minimal elements for sets built from distinct S_i, S_j) and so far we have a total of $(n-z+1)(z-1)$ sets. Finally we consider the set $S_{n-z+2}=\{n-z+2,n-z+3,\ldots,n\}$ and build a further $z-2$ sets by replacing each element other than $n-z+2$ with 1. The sets built from S_{n-z+2} are clearly distinct from any of the sets built from $S_2,S_3,\ldots,S_{n-z+1}$. In fact, the sets are also distinct from those built from S_1 since the maximal element of each of the latter is $z-1$ or z while the maximal element of each set built from S_{n-z+2} is $n-1$ or n and $z \leqslant n-2$. This brings the total number of sets to $(n-z+2)(z-1)$ as required. □

Theorem 10. $x_{n,z} \leqslant n! \min(1+\log^z(n), n-z+1)$.

Proof. The claim follows from Lemmas 7 and 9. □

Theorem 11. $x_{n,z} \geqslant (n!/z!)\log^z(n+1)$.

Proof. The claim is trivially true for $z=0$ and also holds for $z=n$ since $n! \geqslant \log^n(n+1)$; for $1 \leqslant n \leqslant 4$ the inequality can be checked directly while for $n \geqslant 5$ it follows from the fact that $n! \geqslant (n/e)^n$. We now use induction on n; the base case $n=0$ being immediate. For the induction step we assume that $n>z>0$ and then

$$\begin{aligned} x_{n,z} &= nx_{n-1,z} + \sum_{i=1}^{z} a_i x_{n-i,z-i} \\ &\geqslant \frac{n!}{z!}\log^z(n) + \sum_{i=1}^{z} a_i \frac{(n-i)!}{(z-i)!} \log^{z-i}(n-i+1). \end{aligned}$$

We show that the final expression is at least as large as $(n!/z!)\log^z(n+1)$. Now, using the upper bound of (1), we have

$$\begin{aligned} \log^z(n+1) &\leqslant \left(\log(n)+\frac{1}{n}\right)^z \\ &= \log^z(n) + \sum_{i=1}^{z} \binom{z}{i} \frac{1}{n^i} \log^{z-i}(n). \end{aligned}$$

It is therefore sufficient to prove that

$$a_i \frac{(n-i)!}{(z-i)!} \log^{z-i}(n-i+1) \geqslant \frac{n!}{z!} \binom{z}{i} \frac{1}{n^i} \log^{z-i}(n)$$

for $1 \leqslant i \leqslant z$. However, the inequality is equivalent to

$$\begin{aligned} a_i &\geqslant \frac{n!}{i!(n-i)!n^i} \left(\frac{\log n}{\log(n-i+1)} \right)^{z-i} \\ &= \frac{(n-i+1)(n-i+2)\cdots n}{i!n^i} \left(\frac{\log n}{\log(n-i+1)} \right)^{z-i}. \end{aligned}$$

It therefore suffices to prove that

$$a_i \geqslant \frac{1}{i!} \left(\frac{\log n}{\log(n-i+1)} \right)^{n-i-1}.$$

Consider n to be fixed and denote the right-hand side by $f(i)$ where i is a natural number in the range $1 \leqslant i \leqslant n-2$. The proof will be complete provided we show that $f(i)$ is a non-increasing function of i since $f(1) = 1$ and of course $a_i \geqslant 1$ for all i. Now

$$\begin{aligned} f(i) - f(i+1) = ((i+1)! \log^{n-i-1}(n-i+1))^{-1} \left(\frac{\log(n)}{\log(n-i)} \right)^{n-i-2} \\ ((i+1) \log(n) \log^{n-i-2}(n-i) - \log^{n-i-1}(n-i+1)). \end{aligned}$$

It therefore suffices to show that the last factor on the right-hand side is non-negative. In fact, we consider the function

$$g(n,m) = (n-m+1) \log(n) \log^{m-2}(m) - \log^{m-1}(m+1)$$

for $2 \leqslant m < n$ and show that $g(n,m) \geqslant 0$; the required result will then follow by setting $m = n - i$. Since $g(n,m)$ is an increasing function of n we simply need to show that $g(m+1,m) \geqslant 0$ which is equivalent to

$$\left(\frac{\log(m+1)}{\log m} \right)^{m-2} \leqslant 2.$$

Now using the upper bound of (1) we have

$$\begin{aligned} \left(\frac{\log(m+1)}{\log m} \right)^{m-2} &\leqslant \left(1 + \frac{1}{m \log m} \right)^{m-2} \\ &= \left(1 + \frac{m-2}{(m-2)m \log m} \right)^{m-2} \\ &\leqslant e^{(m-2)/m \log m}. \end{aligned}$$

To finish the proof we show that

$$\frac{m-2}{m \log m} \leqslant \log 2.$$

The derivative of the left-hand side is

$$\frac{1}{m\log m}\left(\frac{2}{m}-\frac{m-2}{m\log m}\right),$$

so that there is a unique turning point when $(m-2)/m\log m = 2/m$ which must be a maximum since $(m-2)/m\log m$ is non-negative for $m\geqslant 2$ and is zero at $m=2$. Thus we must show that $\alpha\geqslant 2/\log 2$ where α is the root of $m-2\log m-2$. We note that the last expression is an increasing function of m (bearing in mind that $m\geqslant 2$). First of all $\alpha>3$ since $3-2\log 3-2<0$. Now we can use the series

$$\log y = 2\left(\frac{y-1}{y+1}+\frac{1}{3}\left(\frac{y-1}{y+1}\right)^2+\cdots\right)$$

to deduce that $\log 2>\frac{2}{3}$ so that $2/\log 2<3$ and thus $\alpha>2/\log 2$. This completes the proof. □

Corollary 12. *For fixed z we have $x_{n,z}=\Theta(n!\,\log^z n)$.*

Proof. The claim follows from Lemma 7 and Theorem 11. □

4. Bounds for s_n

Recall that $s_n=\sum_{z=0}^{n} x_{n,z}$. Using Theorem 4 it is easy to see that

$$s_0 = 1,$$
$$s_n = n! + ns_{n-1} + \sum_{i=1}^{n-1} a_i(s_{n-i}-(n-i)!) \quad \text{for } n>0.$$

We set

$$t_n = \frac{s_n - n!}{n!},$$

so that

$$t_0 = 0,$$
$$t_n = 1 + t_{n-1} + \frac{1}{n!}\sum_{i=1}^{n-1} a_i(n-i)!t_{n-i} \quad \text{for } n>0.$$

Lemma 13. $(1/n!)\sum_{i=2}^{n-1} a_i(n-i)!t_{n-i}\leqslant 0.531$.

Proof. For $n\leqslant 98$ the claim follows by direct calculation. For $n\geqslant 99$ we proceed as follows. First note that by Lemma 9

$$t_n\leqslant\left(\sum_{z=0}^{n} n-z+1\right)-1=\frac{1}{2}n(n+3).$$

Thus, using this and Lemma 8,

$$\begin{aligned}\frac{1}{n!}\sum_{i=2}^{n-1} a_i(n-i)!t_{n-i} &\leqslant \frac{a_2(n-2)!}{n!}t_{n-2} + \frac{a_3(n-3)!}{n!}t_{n-3} + \frac{a_4(n-4)!}{n!}t_{n-4} \\ &\quad + \frac{a_{n-2}}{n!}t_2 + \frac{a_{n-1}}{n!}t_1 + \sum_{i=5}^{n-3}\frac{a_i(n-i)!}{n!}t_{n-i} \\ &\leqslant \frac{1}{2} + \frac{3}{2n} + \frac{13}{2n(n-1)} + \frac{5(n-3)}{2n(n-1)(n-2)} + \frac{n-2}{n(n-1)} \\ &\quad + \sum_{i=5}^{n-3}\frac{(i-1)(i-1)!(n-i)!}{2n!}(n-i)(n+3-i).\end{aligned}$$

We claim that

$$\binom{m}{j} \geqslant (m+1)j(m-j)(m+3-j)$$

for $m \geqslant 98$ and $4 \leqslant j \leqslant m-4$. We prove this by induction on m. For $m = 98$ we can check by direct calculation. Suppose now that $m > 98$. If $j = 4$ then the claim is equivalent to $m(m-1)(m-2)(m-3)/24 \geqslant 4(m+1)(m-4)(m-1)$, so that it suffices to have $m^2 - 98m - 96 \geqslant 0$ and this holds for all $m \geqslant 49 + \sqrt{2497}$. For $j > 4$ we have

$$\begin{aligned}\binom{m}{j} &= \frac{m}{j}\binom{m-1}{j-1} \\ &\geqslant \frac{m}{j}m(j-1)(m-j)(m+3-j).\end{aligned}$$

It therefore suffices to show that

$$\frac{m^2}{m+1} \geqslant \frac{j^2}{j-1}.$$

The right-hand side is an increasing function of j for $j > 2$ and so we need only ensure that the inequality holds for $j = m-4$. This is equivalent to $2(m^2 - 4m - 8) \geqslant 0$ which holds for $m \geqslant 2 + 2\sqrt{3}$. This establishes the claim.

Now, using the inequality of the claim with $m = n-1$ and $j = i-1$ in the bound preceding the claim, we have

$$\begin{aligned}\frac{1}{n!}\sum_{i=2}^{n-1} a_i(n-i)!t_{n-i} &\leqslant \frac{1}{2} + \frac{3}{2n} + \frac{13}{2n(n-1)} + \frac{5(n-3)}{2n(n-1)(n-2)} \\ &\quad + \frac{n-2}{n(n-1)} + \frac{n-7}{2n^2}.\end{aligned}$$

This is a decreasing function of n and so its maximum for $n \geqslant 99$ is attained at $n = 99$ which is less than 0.531. □

Theorem 14. $(n+1)(H_{n+1}-1) \leqslant t_n \leqslant 1.531(n+1)(H_{n+1}-1)$ *where* $H_m = \sum_{i=1}^{m} 1/i$ *is the mth Harmonic number.*

Proof. For the lower bound we have $t_0 = 0 = H_1 - 1$ and, for $n > 0$,

$$\begin{aligned} t_n &\geqslant 1 + t_{n-1} + \frac{a_1(n-1)!}{n!} t_{n-1} \\ &= 1 + \frac{n+1}{n} t_{n-1}. \end{aligned}$$

Setting $h_n = t_n/(n+1)$ we have $h_0 = 0$ and

$$h_n \leqslant \frac{1}{n+1} + h_{n-1}.$$

Clearly $h_n \leqslant H_{n+1} - 1$ and the lower bound for t_n follows.

The upper bound clearly holds for $n = 0$ while for $n > 0$ we have, by Lemma 13

$$t_n \leqslant 1.531 + \frac{n+1}{n} t_{n-1}$$

and the claim follows by the same argument as for the lower bound. □

Corollary 15. $(n+1)(\log(n+1) - 0.5) \leqslant t_n \leqslant 1.531(n+1)(\log(n+1) - 0.4)$.

Proof. The result follows from the preceding theorem and the estimate

$$H_n = \log(n) + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} - \varepsilon, \quad 0 < \varepsilon < \frac{1}{252n^6},$$

where $\gamma = 0.57721\ldots$ is Euler's constant (see [10]). The upper bound must be verified by direct calculation for $n \leqslant 21$. □

It is worth noting here that it seems quite hard to improve on the constants involved in the preceding corollary. Finally, from the corollary and the definition of t_n we see that $s_n = \Theta(nn! \log(n))$.

5. Concluding remarks

We have obtained reasonable estimates of $x_{n,z}$ for fixed n and varying z as well as very close estimates for s_n. The situation where we fix n and vary z is very intriguing. Experiments show that $x_{n,z}$ has a unique maximum (for $n > 2$) and this occurs very early on. The upper bound of Theorem 10 and the lower bound of Theorem 11 both exhibit this type of behaviour but they are too far apart to imply this for $x_{n,z}$. Proving that $x_{n,z}$ has a unique maximum seems difficult.

Acknowledgements

We are grateful to Mike Atkinson for introducing us to this area and for fruitful discussions. We also thank an anonymous referee whose comments led to a revision of the paper.

References

[1] M.D. Atkinson, M. Thiyagarajah, The permutational power of a priority queue, BIT 33 (1993) 2–6.
[2] M.D. Atkinson, Transforming binary sequences with priority queues, Order 10 (1993) 31–36.
[3] M.D. Atkinson, D.H. Tulley, Bounded capacity priority queues, Theoret. Comput. Sci. 205 (1997) 145–157.
[4] L. Comtet, Advanced Combinatorics, Reidel, Dordrecht, Holland, 1974.
[5] J.D. Gilbey, L.H. Kalikow, Parking functions, valet functions and priority queues, Discrete Math. 197/198 (1999) 351–373.
[6] M. Golin, S. Zaks, Labelled trees and input–output permutations in priority queues, Theoret. Comput. Sci. 205 (1997) 99–114.
[7] I.P. Goulden, D.M. Jackson, Combinatorial Enumeration, Wiley, New York, 1983.
[8] N.K. Jaiswal, Priority Queues, Academic Press, London, 1968.
[9] P.J.B. King, Computer and Communication Systems Performance Modelling, Prentice-Hall, New York, 1990.
[10] D.E. Knuth, Fundamental Algorithms, The Art of Computer Programming vol. 1, 1 ed., Addison-Wesley, Reading, MA, 1968 (see also 2nd ed., 1973).
[11] H. Leemans, Queue lengths and waiting times in the two-class two-server queue with nonpreemptive heterogeneous priority structures, 14th UK Computer and Telecommunications Performance Engineering Workshop, Department of Computer Science, The University of Edinburgh, 1998, pp. 150–165.
[12] V.R. Pratt, Computing permutations with double-ended queues, 5th ACM Symp. on Theory of Computing, 1973, pp. 268–277.
[13] R.E. Tarjan, Sorting using networks of queues and stacks, J. Assoc. Comput. Mach. 19 (1972) 341–346.
[14] D.H. Tulley, The combinatorics of abstract container data types, Ph.D. Thesis, School of Mathematical and Computational Sciences, University of St. Andrews, Scotland, 1996.