



Quasiperiodic and Lyndon episturmian words[☆]

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ABSTRACT

Recently the second two authors characterized *quasiperiodic* Sturmian words, proving that a Sturmian word is non-quasiperiodic if and only if, it is an *infinite Lyndon word*. Here we extend this study to *episturmian words* (a natural generalization of Sturmian words) by describing all the quasiperiods of an episturmian word, which yields a characterization of quasiperiodic episturmian words in terms of their *directive words*. Even further, we establish a complete characterization of all episturmian words that are Lyndon words. Our main results show that, unlike the Sturmian case, there is a much wider class of episturmian words that are non-quasiperiodic, besides those that are infinite Lyndon words. Our key tools are morphisms and directive words, in particular *normalized* directive words, which we introduced in an earlier paper. Also of importance is the use of *return words* to characterize quasiperiodic episturmian words, since such a method could be useful in other contexts.

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1. Introduction

Sturmian words are a fascinating family of infinite words defined on a 2-letter alphabet which has been extensively studied since the pioneering work of Morse and Hedlund in 1940 (see [36]). Over the years, these infinite words have been shown to have numerous equivalent definitions and characterizations, and their beautiful properties are related to many fields such as Number Theory, Geometry, Dynamical Systems, and Combinatorics on Words (see [1,31,38,6] for recent surveys).

Many recent works have been devoted to generalizations of Sturmian words to arbitrary finite alphabets. An especially interesting generalization is the family of *episturmian words*, introduced by Droubay, Justin, and Pirillo in 2001 [12] (see also [23,25] for example). Episturmian words include not only the Sturmian words, but also the well-known *Arnoux–Rauzy sequences* (e.g., see [5,24,38,46]). More precisely, the family of episturmian words is composed of the Arnoux–Rauzy sequences, images of the Arnoux–Rauzy sequences by *episturmian morphisms*, and certain periodic infinite words. In the binary case, Arnoux–Rauzy sequences are exactly the Sturmian words whereas episturmian words include all recurrent *balanced* words, that is, periodic balanced words and Sturmian words (see [18,37,44] for recent results relating episturmian words to the balanced property). See also [17] for a recent survey on episturmian theory.

[☆] This work combines and extends two conference papers, one by the first author [A. Glen, Order and quasiperiodicity in episturmian words, in: Proceedings of the 6th International Conference on Words, Marseille, France, September 17–21, 2007, pp. 144–158] and the other by the second two authors [F. Levé, G. Richomme, Quasiperiodic episturmian words, in: Proceedings of the 6th International Conference on Words, Marseille, France, September 17–21, 2007, pp. 201–211] presented at the Sixth International Conference on Words, Marseille, France, September 17–21, 2007.

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Episturmian morphisms play a central role in the study of episturmian words (Section 2.3 recalls the definition of these morphisms). Introduced first as a generalization of Sturmian morphisms, Justin and Pirillo [23] showed that they are exactly the morphisms that preserve the aperiodic episturmian words. They also proved that any episturmian word is the image of another episturmian word by some so-called *pure episturmian morphism*. Even more, any episturmian word can be infinitely decomposed over the set of pure episturmian morphisms. This last property allows an episturmian word to be defined by one of its morphic decompositions or, equivalently, by a certain *directive word*, which is an infinite sequence of rules for decomposing the given episturmian word by morphisms. In consequence, many properties of episturmian words can be deduced from properties of episturmian morphisms. This approach is used for instance in [7,16,27,43,44,46] and, of course, in the papers of Justin et al.

Morphic decompositions of Sturmian words have been used in [29] to characterize quasiperiodic Sturmian words. Quasiperiodicity of finite words was first introduced by Apostolico and Ehrenfeucht [3] in the following way: “a word w is quasiperiodic if there exists a second word $u \neq w$ such that every position of w falls within some occurrence of u in w ”. The word u is then called a quasiperiod of w . In the last fifteen years, quasiperiodicity and covering of finite words has been extensively studied (see [2,21] for some surveys). In [33], Marcus extended this notion to infinite words and opened some questions, particularly concerning quasiperiodicity of Sturmian words. After a brief answer to some of these questions in [27], the Sturmian case was fully studied in [29] where it was proved that a Sturmian word is non-quasiperiodic if and only if, it is an infinite Lyndon word. Here we extend this study to episturmian words.

In Sections 2 and 3, we recall useful results on episturmian words and their directive words. A particularly important tool is the *normalization* of words directing the same episturmian word, which we recently introduced in [28,19]. This idea allows an episturmian word to be defined uniquely by its so-called *normalized directive word*, defined by some factor avoidance. It can be seen as a generalization of a previous result by Berthé, Holton, and Zamboni [7], which was used in [29] to show that the directive word of a non-quasiperiodic Sturmian word can take only two possible (similar) forms. For non-binary episturmian words, even those defined on a ternary alphabet, this simplicity does not hold since a combinatorial explosion of the number of cases occurs. In particular there exist ternary non-quasiperiodic episturmian words that have infinitely many directive words. As such, the method we use to characterize quasiperiodic episturmian words greatly differs from the one in the Sturmian case.

In Section 4, we characterize quasiperiodic episturmian words. To prove it, we introduce a new way to tackle quasiperiodicity by stating an equivalent definition that is related to the notion of *return words*. From this, we show that any *standard* episturmian word (or *epistandard word*) is quasiperiodic; in particular, sufficiently long palindromic prefixes of an epistandard word are *quasiperiods* of it. We then extend this result by describing the quasiperiods of any (quasiperiodic) episturmian word (see Theorem 4.19). This yields a characterization of quasiperiodic episturmian words in terms of their directive words (Theorem 4.28). Note that the set of quasiperiods of an episturmian word was previously described only for the Fibonacci word [27]. (In [29] this set was not described for quasiperiodic Sturmian words). At the end of Section 4, using the normalization aspect, we give a second characterization of quasiperiodic episturmian words which itself provides an effective way to decide whether or not a given episturmian word is quasiperiodic (see Theorem 4.29).

Section 5 is concerned with the study of the action of episturmian morphisms in relation to quasiperiodicity. This study leads to non-trivial extensions of results in [29]. Using this approach, we provide a completely different proof of our main characterization of quasiperiodic episturmian words. We also characterize episturmian morphisms that map any word onto a quasiperiodic one (see Section 5.4). This result naturally allows us to consider quasiperiodicity of words defined using episturmian morphisms.

Lastly, in Section 6, we characterize episturmian Lyndon words in terms of their directive words. This result shows that, unlike the Sturmian case, there exist non-quasiperiodic episturmian words that are not infinite Lyndon words.

2. Episturmian words and morphisms

We assume the reader is familiar with combinatorics on words and morphisms (e.g., see [30,31]). In this section, we recall some basic definitions and properties relating to episturmian words which are needed throughout the paper. For the most part, we follow the notation and terminology of [12,23,25,18].

2.1. Notation and terminology

Let \mathcal{A} denote a finite non-empty *alphabet*. A finite word over \mathcal{A} is a finite sequence of letters from \mathcal{A} . The *empty word* ε is the empty sequence. Under the operation of concatenation, the set \mathcal{A}^* of all finite words over \mathcal{A} is a *free monoid* with identity element ε and set of generators \mathcal{A} . The set of *non-empty* words over \mathcal{A} is the *free semigroup* $\mathcal{A}^+ = \mathcal{A}^* \setminus \{\varepsilon\}$.

Given a finite word $w = x_1x_2 \cdots x_m \in \mathcal{A}^+$ with each $x_i \in \mathcal{A}$, the *length* of w , denoted by $|w|$, is equal to m . By convention, the empty word is the unique word of length 0. We denote by $|w|_a$ the number of occurrences of the letter a in the word w . If $|w|_a = 0$, then w is said to be *a-free*. The *reversal* of w , denoted by \tilde{w} , is its mirror image: $\tilde{w} = x_mx_{m-1} \cdots x_1$, and if $w = \tilde{w}$, then w is called a *palindrome*.

A (right) *infinite word* (or simply *sequence*) \mathbf{x} is a sequence indexed by \mathbb{N}^+ with values in \mathcal{A} , i.e., $\mathbf{x} = x_1x_2x_3 \cdots$ with each $x_i \in \mathcal{A}$. The set of all infinite words over \mathcal{A} is denoted by \mathcal{A}^ω . An *ultimately periodic* infinite word can be written as

$uv^\omega = uvvv \dots$, for some $u, v \in \mathcal{A}^*$, $v \neq \varepsilon$. If $u = \varepsilon$, then such a word is (purely) *periodic*. An infinite word that is not ultimately periodic is said to be *aperiodic*. For easier reading, infinite words are hereafter typically typed in boldface to distinguish them from finite words.

Given a set X of finite words, X^* (resp. X^ω) is the set of all finite (resp. infinite) words that can be obtained by concatenating words of X . The empty word ε belongs to X^* .

A finite word w is a *factor* of a finite or infinite word z if $z = uvw$ for some words u, v (where v is infinite iff, z is infinite). In the special case $u = \varepsilon$ (resp. $v = \varepsilon$), we call w a *prefix* (resp. *suffix*) of z . We use the notation $p^{-1}w$ (resp. ws^{-1}) to indicate the removal of a prefix p (resp. suffix s) of the word w . An infinite word $\mathbf{x} \in \mathcal{A}^\omega$ is called a *suffix* of $\mathbf{z} \in \mathcal{A}^\omega$ if there exists a word $w \in \mathcal{A}^*$ such that $\mathbf{z} = w\mathbf{x}$. That is, \mathbf{x} is a shift of \mathbf{z} , given by $\mathbf{x} = T^{|\mathbf{z}|}(\mathbf{z}) = w^{-1}\mathbf{z}$, where T denotes the *shift map*: $T((x_n)_{n \geq 1}) = (x_{n+1})_{n \geq 1}$. Note that a prefix or suffix u of a finite or infinite word w is said to be *proper* if $u \neq w$. For finite words $w \in \mathcal{A}^*$, the shift map T acts circularly, i.e., if $w = xv$ where $x \in \mathcal{A}$, then $T(w) = vx$.

The *alphabet* of a finite or infinite word w , denoted by $\text{Alph}(w)$ is the set of letters occurring in w , and if w is infinite, we denote by $\text{Ult}(w)$ the set of all letters occurring infinitely often in w .

A factor of an infinite word \mathbf{x} is *recurrent* in \mathbf{x} if it occurs infinitely often in \mathbf{x} , and \mathbf{x} itself is said to be *recurrent* if all of its factors are recurrent in it. Furthermore, \mathbf{x} is *uniformly recurrent* if for each n there exists a positive integer $K(n)$ such that any factor of \mathbf{x} of length at least $K(n)$ contains all factors of \mathbf{x} of length n . Equivalently, \mathbf{x} is uniformly recurrent if any factor of \mathbf{x} occurs infinitely many times in \mathbf{x} with bounded gaps [9].

2.2. Episturmian words

In this paper, our vision of episturmian words will be the characteristic property stated in [Theorem 2.1](#). However, we first give one of their equivalent definitions to aid in understanding. For this, we recall that a factor u of a finite or infinite word $w \in \mathcal{A}^\omega$ is *right* (resp. *left*) *special* if ua, ub (resp. au, bu) are factors of w for some letters $a, b \in \mathcal{A}$, $a \neq b$.

An infinite word $\mathbf{t} \in \mathcal{A}^\omega$ is *episturmian* if its set of factors is closed under reversal and \mathbf{t} has at most one right (or equivalently left) special factor of each length. Moreover, an episturmian word is *standard* if all of its left special factors are prefixes of it.

In the initiating paper [12], episturmian words were defined as an extension of standard episturmian words, which were themselves first introduced and studied as a generalization of standard Sturmian words using *palindromic closure* (see [Theorem 4.5](#)). Specifically, an infinite word was said to be episturmian if it has exactly the same set of factors as some standard episturmian word [12]. This definition is equivalent to the aforementioned one by [Theorem 5](#) in [12]. Moreover, it was proved in [12] that episturmian words are uniformly recurrent. Hence ultimately periodic episturmian words are (purely) periodic.

Note. Hereafter, we refer to a standard episturmian word as an *epistandard word*, for simplicity.

To study episturmian words, Justin and Pirillo [23] introduced *episturmian morphisms*. In particular they proved that these morphisms, which we recall below, are precisely the morphisms that preserve the set of aperiodic episturmian words.

2.3. Episturmian morphisms

Let us recall that given an alphabet \mathcal{A} , a *morphism* f on \mathcal{A} is a map from \mathcal{A}^* to \mathcal{A}^* such that $f(uv) = f(u)f(v)$ for any words u, v over \mathcal{A} . A morphism on \mathcal{A} is entirely defined by the images of letters in \mathcal{A} . All morphisms considered in this paper will be non-erasing: the image of any non-empty word is never empty. Hence the action of a morphism f on \mathcal{A}^* can be naturally extended to infinite words; that is, if $\mathbf{x} = x_1x_2x_3 \dots \in \mathcal{A}^\omega$, then $f(\mathbf{x}) = f(x_1)f(x_2)f(x_3) \dots$.

In what follows, we will denote the composition of morphisms by juxtaposition as for concatenation of words.

Episturmian morphisms are the compositions of the permutation morphisms (i.e., the morphisms f such that $f(\mathcal{A}) = \mathcal{A}$) and the morphisms L_a and R_a where, for all $a \in \mathcal{A}$:

$$L_a : \begin{cases} a \mapsto a \\ b \mapsto ab \end{cases}, \quad R_a : \begin{cases} a \mapsto a \\ b \mapsto ba \end{cases} \quad \text{for all } b \neq a \text{ in } \mathcal{A}.$$

Here, we will work only on *pure* episturmian morphisms, i.e., morphisms obtained by composition of elements of the sets:

$$\mathcal{L}_{\mathcal{A}} = \{L_a \mid a \in \mathcal{A}\} \quad \text{and} \quad \mathcal{R}_{\mathcal{A}} = \{R_a \mid a \in \mathcal{A}\}.$$

Note. In [23], the morphism L_a (resp. R_a) is denoted by ψ_a (resp. $\bar{\psi}_a$). We adopt the current notation to emphasize the action of L_a (resp. R_a) when applied to a word, which consists of placing an occurrence of the letter a on the left (resp. right) of each occurrence of any letter different from a .

Epistandard morphisms (resp. *pure episturmian morphisms*, *pure epistandard morphisms*) are the morphisms obtained by concatenation of morphisms in $\mathcal{L}_{\mathcal{A}}$ and permutations on \mathcal{A} (resp. in $\mathcal{L}_{\mathcal{A}} \cup \mathcal{R}_{\mathcal{A}}$, in $\mathcal{L}_{\mathcal{A}}$). Note that the episturmian morphisms are exactly the *Sturmian morphisms* when \mathcal{A} is a 2-letter alphabet.

2.4. Morphic decomposition of episturmian words

Justin and Pirillo [23] proved the following insightful characterizations of epistandard and episturmian words (see Theorem 2.1), which show that any episturmian word can be *infinitely decomposed* over the set of pure episturmian morphisms.

The statement of Theorem 2.1 needs some extra definitions and notation. First, we define the following new alphabet, $\bar{\mathcal{A}} = \{\bar{x} \mid x \in \mathcal{A}\}$. A letter \bar{x} is considered to be x with *spin* R , whilst x itself has spin L . A finite or infinite word over $\mathcal{A} \cup \bar{\mathcal{A}}$ is called a *spinned* word. To ease the reading, we sometimes call a letter with spin L (resp. spin R) an L -spinned (resp. R -spinned) letter. By extension, an L -spinned (resp. R -spinned) word is a word having only letters with spin L (resp. spin R).

The *opposite* \bar{w} of a finite or infinite spinned word w is obtained from w by exchanging all spins in w . For instance, if $v = ab\bar{a}$, then $\bar{v} = \bar{a}\bar{b}a$. When $v \in \mathcal{A}^+$, then its opposite $\bar{v} \in \bar{\mathcal{A}}^+$ is an R -spinned word and we set $\bar{\varepsilon} = \varepsilon$. Note that, given a finite or infinite word $w = w_1w_2 \dots$ over \mathcal{A} , we sometimes denote $\check{w} = \check{w}_1\check{w}_2 \dots$ any spinned word such that $\check{w}_i = w_i$ if \check{w}_i has spin L and $\check{w}_i = \bar{w}_i$ if \check{w}_i has spin R . Such a word \check{w} is called a *spinned version* of w .

Note. In Justin and Pirillo’s original papers, spins are 0 and 1 instead of L and R . It is convenient here to change this vision of the spins because of the relationship with episturmian morphisms, which we now recall.

For $a \in \mathcal{A}$, let $\mu_a = L_a$ and $\mu_{\bar{a}} = R_a$. This operator μ can be naturally extended (as done in [23]) to a morphism mapping any word over $(\mathcal{A} \cup \bar{\mathcal{A}})$ into a pure episturmian morphism: for a spinned finite word $\check{w} = \check{w}_1 \dots \check{w}_n$ over $\mathcal{A} \cup \bar{\mathcal{A}}$, $\mu_{\check{w}} = \mu_{\check{w}_1} \dots \mu_{\check{w}_n}$ (μ_ε is the identity morphism). We will say that the word w *directs* or is a *directive word* of the morphism μ_w . The following result extends the notion of directive words to infinite episturmian words.

Theorem 2.1 ([23]). (i) An infinite word $\mathbf{s} \in \mathcal{A}^\omega$ is epistandard if and only if, there exists an infinite word $\Delta = x_1x_2x_3 \dots$ over \mathcal{A} and an infinite sequence $(\mathbf{s}^{(n)})_{n \geq 0}$ of infinite words such that $\mathbf{s}^{(0)} = \mathbf{s}$ and for all $n \geq 1$, $\mathbf{s}^{(n-1)} = \mu_{x_n}(\mathbf{s}^{(n)}) (=L_{x_n}(\mathbf{s}^{(n)}))$.
 (ii) An infinite word $\mathbf{t} \in \mathcal{A}^\omega$ is episturmian if and only if, there exists a spinned infinite word $\check{\Delta} = \check{x}_1\check{x}_2\check{x}_3 \dots$ over $\mathcal{A} \cup \bar{\mathcal{A}}$ and an infinite sequence $(\mathbf{t}^{(n)})_{n \geq 0}$ of recurrent infinite words such that $\mathbf{t}^{(0)} = \mathbf{t}$ and for all $n \geq 1$, $\mathbf{t}^{(n-1)} = \mu_{\check{x}_n}(\mathbf{t}^{(n)})$.

For any epistandard word (resp. episturmian word) \mathbf{t} and L -spinned (resp. spinned) infinite word Δ (resp. $\check{\Delta}$) satisfying the conditions of the above theorem, we say that Δ (resp. $\check{\Delta}$) is a (spinned) *directive word* for \mathbf{t} or that \mathbf{t} is *directed* by Δ (resp. $\check{\Delta}$).

Remark 2.2. It follows immediately from Theorem 2.1 that if \mathbf{t} is an episturmian word directed by a spinned infinite word $\check{\Delta}$, then each $\mathbf{t}^{(n)}$ (as defined in part (ii)) is an episturmian word directed by $T^n(\check{\Delta}) = \check{x}_{n+1}\check{x}_{n+2}\check{x}_{n+3} \dots$.

By Theorem 1 in [12] (see also Theorem 4.5), any epistandard word has a unique L -spinned directive word, but also has infinitely many other directive words (see [23,25,19]). For example, the *Tribonacci word* (or *Rauzy word* [39]) is directed by $(abc)^\omega$ and also by $(abc)^n\bar{a}\bar{b}\bar{c}(abc)^\omega$ for each $n \geq 0$, as well as infinitely many other spinned words. More generally, by Proposition 3.11 in [23], any spinned infinite word $\check{\Delta}$ having infinitely many L -spinned letters directs a unique episturmian word \mathbf{t} beginning with the left-most L -spinned letter in $\check{\Delta}$. Moreover, by one of the main results in [19] (see Theorem 3.2), \mathbf{t} has infinitely many other directive words.

The following important fact links the two parts of Theorem 2.1.

Fact 2.3 ([23]). If \mathbf{t} is an episturmian word directed by a spinned version $\check{\Delta}$ of an L -spinned infinite word Δ , then \mathbf{t} has exactly the same set of factors as the (unique) epistandard word \mathbf{s} directed by Δ .

Moreover, with the same notation as in the above remark, the episturmian word \mathbf{t} is periodic if and only if, the epistandard word \mathbf{s} is periodic, and this holds if and only if $|\text{Ult}(\Delta)| = 1$ (see [23, Prop. 2.9]). More precisely, a periodic episturmian word takes the form $(\mu_{\check{w}}(x))^\omega$ for some finite spinned word \check{w} and letter x .

Note. Sturmian words are precisely the aperiodic episturmian words on a 2-letter alphabet.

When an episturmian word is aperiodic, we have the following fundamental link between the words $(\mathbf{t}^{(n)})_{n \geq 0}$ and the spinned infinite word $\check{\Delta}$ occurring in Theorem 2.1: if a_n is the first letter of $\mathbf{t}^{(n)}$, then $\mu_{\check{x}_1 \dots \check{x}_n}(a_n)$ is a prefix of \mathbf{t} and the sequence $(\mu_{\check{x}_1 \dots \check{x}_n}(a_n))_{n \geq 1}$ is not ultimately constant (since $\check{\Delta}$ is not ultimately constant), then $\mathbf{t} = \lim_{n \rightarrow \infty} \mu_{\check{x}_1 \dots \check{x}_n}(a_n)$. This fact is a slight generalization of a result of Risley and Zamboni [46, Prop. III.7] on *S-adic representations* for characteristic Arnoux–Rauzy sequences. See also the recent paper [7] for *S-adic representations* of Sturmian words. Note that *S-adic dynamical systems* were introduced by Ferenczi [14] as *minimal dynamical systems* (e.g., see [38]) generated by a finite number of substitutions. In the case of episturmian words, the notion itself is actually a reformulation of the well-known *Rauzy rules*, as studied in [40]. In fact, it is well-known that the *subshift* of an aperiodic episturmian word \mathbf{t} (i.e., the topological closure of the shift orbit of \mathbf{t}) is a minimal dynamical system, i.e., it consists of all the episturmian words with the same set of factors as \mathbf{t} .

3. Useful results on directive words

Notions concerning directive words of episturmian words and morphisms have been recalled in the previous section. Two natural questions concerning these words are: when do two distinct finite spinned words direct the same episturmian morphism? When do two distinct spinned infinite words direct the same unique episturmian word? In this section we recall existing answers to these questions. We also present a way to uniquely define any episturmian word through a *normalization* of its directive words. This powerful tool was recently introduced in our papers [19,28].

3.1. Presentation versus block-equivalence

Generalizing a study of the monoid of Sturmian morphisms by Séébold [47], the third author [41] answered the question: “When do two distinct finite spinned words direct the same episturmian morphism?” by giving a presentation of the monoid of episturmian morphisms. This result was reformulated in [42] using another set of generators and it was independently and differently treated in [25]. As a direct consequence, one can see that the monoid of pure epistandard morphisms is a free monoid and one can obtain the following presentation of the monoid of pure episturmian morphisms:

Theorem 3.1 (Direct Consequence of [42, Prop. 6.5]; Reformulation of [25, Th. 2.2]). *The monoid of pure episturmian morphisms with $\{L_\alpha, R_\alpha \mid \alpha \in \mathcal{A}\}$ as a set of generators has the following presentation:*

$$R_{a_1}R_{a_2}\cdots R_{a_k}L_{a_1} = L_{a_1}L_{a_2}\cdots L_{a_k}R_{a_1}$$

where $k \geq 1$ is an integer and $a_1, \dots, a_k \in \mathcal{A}$ with $a_1 \neq a_i$ for all $i, 2 \leq i \leq k$.

This result means that two different compositions of morphisms in $\mathcal{L}_{\mathcal{A}} \cup \mathcal{R}_{\mathcal{A}}$ yield the same pure episturmian morphism if and only if, one composition can be deduced from the other in a rewriting system, called the *block-equivalence* in [25]. Although **Theorem 3.1** allows us to show that many properties of episturmian words are linked to properties of episturmian morphisms, it will be convenient for us to have in mind the block-equivalence that we now recall.

A word of the form xvx , where $x \in \mathcal{A}$ and $v \in (\mathcal{A} \setminus \{x\})^*$, is called a (x -based) *block*. A (x -based) *block-transformation* is the replacement in a spinned word of an occurrence of $xv\bar{x}$ (where xvx is a block) by $\bar{x}v\bar{x}$ or vice-versa. Two finite spinned words w, w' are said to be *block-equivalent* if we can pass from one to the other by a (possibly empty) chain of block-transformations, in which case we write $w \equiv w'$. For example, $\bar{b}\bar{a}\bar{b}\bar{c}\bar{b}\bar{a}\bar{c}$ and $b\bar{a}b\bar{c}b\bar{a}c$ are block-equivalent because $\bar{b}\bar{a}\bar{b}\bar{c}\bar{b}\bar{a}\bar{c} \rightarrow b\bar{a}\bar{b}\bar{c}\bar{b}\bar{a}\bar{c} \rightarrow b\bar{a}b\bar{c}b\bar{a}c$ and vice-versa. The block-equivalence is an equivalence relation over spinned words, and moreover one can observe that if $w \equiv w'$ then w and w' are spinned versions of the same word over \mathcal{A} .

Theorem 3.1 can be reformulated in terms of block-equivalence:

Theorem 3.1. *Let w, w' be two spinned words over $\mathcal{A} \cup \bar{\mathcal{A}}$. Then $\mu_w = \mu_{w'}$ if and only if $w \equiv w'$.*

3.2. Words directing the same episturmian word

Using the block-equivalence notion, the question: “When do two distinct spinned infinite words direct the same unique episturmian word?” was almost completely solved by Justin and Pirillo in [25] for *bi-infinite episturmian words*, i.e., episturmian words with letters indexed by \mathbb{Z} (and not by \mathbb{N} as we consider here). More recently, in [19], we showed that Justin and Pirillo’s results on directive words of bi-infinite episturmian words are still valid for words directing (right-infinite) episturmian words. We also established the following complete characterization of pairs of spinned infinite words directing the same unique episturmian word. Not only does our characterization provide the relative forms of two spinned infinite words directing the same episturmian word, but it also fully solves the periodic case, which was only partially solved in [25].

Theorem 3.2 ([19]). *Given two spinned infinite words Δ_1 and Δ_2 , the following assertions are equivalent.*

- (i) Δ_1 and Δ_2 direct the same right-infinite episturmian word;
- (ii) Δ_1 and Δ_2 direct the same bi-infinite episturmian word;
- (iii) One of the following cases holds for some i, j such that $\{i, j\} = \{1, 2\}$:
 1. $\Delta_i = \prod_{n \geq 1} v_n, \Delta_j = \prod_{n \geq 1} z_n$ where $(v_n)_{n \geq 1}, (z_n)_{n \geq 1}$ are spinned words such that $\mu_{v_n} = \mu_{z_n}$ for all $n \geq 1$;
 2. $\Delta_i = wx \prod_{n \geq 1} v_n \bar{x}_n, \Delta_j = w' \bar{x} \prod_{n \geq 1} \bar{v}_n \hat{x}_n$ where w, w' are spinned words such that $\mu_w = \mu_{w'}, x$ is an L -spinned letter, $(v_n)_{n \geq 1}$ is a sequence of non-empty x -free L -spinned words, and $(\bar{x}_n)_{n \geq 1}, (\hat{x}_n)_{n \geq 1}$ are sequences of non-empty spinned words over $\{x, \bar{x}\}$ such that, for all $n \geq 1, |\bar{x}_n| = |\hat{x}_n|$ and $|\bar{x}_n|_x = |\hat{x}_n|_x$;
 3. $\Delta_1 = w\mathbf{x}$ and $\Delta_2 = w'\mathbf{y}$ where w, w' are spinned words and $\mathbf{x} \in \{x, \bar{x}\}^\omega, \mathbf{y} \in \{y, \bar{y}\}^\omega$ are spinned infinite words for some letters x, y such that $\mu_w(x) = \mu_{w'}(y)$.

In items 1 and 2 of **Theorem 3.2**, the two considered directive words are spinned versions of the same L -spinned word. This does not hold in item 3, which concerns only periodic episturmian words. In particular, we make the following observation:

Fact 3.3. *If an aperiodic episturmian word is directed by two spinned words Δ_1 and Δ_2 , then Δ_1 and Δ_2 are spinned versions of the same L -spinned word Δ .*

As an example of item 3, one can consider the periodic episturmian word $(bcba)^\omega$ which is directed by both bca^ω and $\bar{b}\bar{a}c^\omega$. Note also that $(bcba)^\omega$ is epistandard and has the same set of factors as the epistandard word $(abc)^\omega$ directed by bac^ω . Actually, in view of Fact 2.3, we observe the following:

Fact 3.4. The subshift of any aperiodic episturmian word contains a unique (aperiodic) epistandard word, whereas the subshift of a periodic episturmian word contains exactly two (periodic) epistandard words, except if this word is a^ω with a a letter.

3.3. Normalized directive word of an episturmian word

Items 2 and 3 of Theorem 3.2 show that any episturmian word is directed by a spinned word having infinitely many L -spinned letters, but also by a spinned infinite word having both infinitely many L -spinned letters and infinitely many R -spinned letters. To emphasize the importance of these facts, let us recall from Proposition 3.11 in [23] that if $\check{\Delta}$ is a spinned infinite word over $\mathcal{A} \cup \bar{\mathcal{A}}$ with infinitely many L -spinned letters, then there exists a unique episturmian word \mathbf{t} directed by $\check{\Delta}$. Uniqueness comes from the fact that the first letter of \mathbf{t} is fixed by the first L -spinned letter in $\check{\Delta}$.

To work on Sturmian words, Berthé, Holton and Zamboni [7] proved that any Sturmian word has a unique directive word over $\{a, b, \bar{a}, \bar{b}\}$ containing infinitely many L -spinned letters but no factor of the form $\bar{a}b^n a$ or $\bar{b}a^n b$ with n an integer. Using Theorems 3.1 and 3.2, we recently generalized this result to episturmian words:

Theorem 3.5 ([19,28]). Any episturmian word \mathbf{t} has a spinned directive word containing infinitely many L -spinned letters, but no factor in $\bigcup_{a \in \mathcal{A}} \bar{a}\bar{\mathcal{A}}^*a$. Such a directive word is unique if \mathbf{t} is aperiodic.

Note that uniqueness does not necessarily hold for periodic episturmian words. For example, the periodic episturmian word $(ab)^\omega = L_a(b^\omega) = R_b(a^\omega)$ is directed both by ab^ω and by $\bar{b}a^\omega$ ($L_a(b) = ab = R_b(a)$).

A directive word of an aperiodic episturmian word \mathbf{t} with the above property is called the *normalized directive word* of \mathbf{t} . We extend this definition to morphisms: a finite spinned word w is said to be a *normalized directive word* of the morphism μ_w if w has no factor in $\bigcup_{a \in \mathcal{A}} \bar{a}\bar{\mathcal{A}}^*a$.

One can observe from Theorem 3.1 that for any morphism in $L_a\mathcal{L}_{\mathcal{A}}^*R_a$, we can find another decomposition of the morphism in the set $R_a\mathcal{R}_{\mathcal{A}}^*L_a$. Equivalently, for any spinned word in $a\bar{\mathcal{A}}^*\bar{a}$, there exists a word w' in $\bar{a}\bar{\mathcal{A}}^*a$ such that $\mu_w = \mu_{w'}$. This was the main idea used in the proof of Theorem 3.5.

Example 3.6. Let f be the pure episturmian morphism with directive word $\bar{a}\bar{b}c\bar{b}a\bar{b}\bar{a}c\bar{b}\bar{a}c\bar{a}$. By Theorem 3.1, $\mu_{\bar{a}\bar{b}c\bar{b}a\bar{b}\bar{a}c\bar{b}\bar{a}c\bar{a}} = \mu_{\bar{a}c\bar{b}a\bar{c}\bar{a}}$ and hence $f = \mu_{\bar{a}\bar{b}c\bar{b}a\bar{b}\bar{a}c\bar{b}\bar{a}c\bar{a}} = \mu_{\bar{a}\bar{b}c\bar{b}a\bar{b}a\bar{c}\bar{b}\bar{a}c\bar{a}}$ and $\bar{a}\bar{b}c\bar{b}a\bar{b}a\bar{c}\bar{b}\bar{a}c\bar{a}$ is the normalized directive word of f .

3.4. Episturmian words having a unique directive word

Using our characterization of pairs of words directing the same episturmian word (Theorem 3.2) together with normalization (Theorem 3.5), we recently characterized episturmian words having a unique directive word.

Theorem 3.7 ([19]). An episturmian word has a unique directive word if and only if its (normalized) directive word contains 1) infinitely many L -spinned letters, 2) infinitely many R -spinned letters, 3) no factor in $\bigcup_{a \in \mathcal{A}} \bar{a}\bar{\mathcal{A}}^*a$, 4) no factor in $\bigcup_{a \in \mathcal{A}} a\bar{\mathcal{A}}^*\bar{a}$. Such an episturmian word is necessarily aperiodic.

As an example, a particular family of episturmian words having unique directive words consists of those directed by *regular wavy words*, i.e., spinned infinite words having both infinitely many L -spinned letters and infinitely many R -spinned letters such that each letter occurs with the same spin everywhere in the directive word. More formally, a spinned version \check{w} of a finite or infinite word w is said to be *regular* if, for each letter $x \in \text{Alph}(w)$, all occurrences of \check{x} in \check{w} have the same spin (L or R). For example, $aba\bar{a}c\bar{b}$ and $(abc)^\omega$ are regular, whereas $\bar{a}b\bar{a}c\bar{b}$ and $(ab\bar{a})^\omega$ are not regular. In [25], a spinned infinite word is said to be *wavy* if it contains infinitely many L -spinned letters and infinitely many R -spinned letters. For example, the two preceding infinite words are wavy.

In the Sturmian case, we have:

Proposition 3.8 ([19]). Any Sturmian word has a unique spinned directive word or infinitely many spinned directive words. Moreover, a Sturmian word has a unique directive word if and only if its (normalized) directive word is regular wavy.

In the next section, we shall see that any episturmian word having a unique directive word is necessarily non-quasiperiodic. This will follow from Theorem 3.7 and our characterizations of quasiperiodic episturmian words (Theorems 4.19, 4.28 and 4.29).

4. Quasiperiodicity of episturmian words

4.1. Quasiperiodicity

Recall (from [4,21,33]) that a finite or infinite word w is quasiperiodic if it can be constructed by concatenations and superpositions of one of its proper factors u , which is called a *quasiperiod* of w (or the *smallest quasiperiod* of w when it is of minimal length). We also say that u covers w or that w is u -quasiperiodic. For example, the word $w = abaababaabaababaaba$ has aba , $abaaba$, $abaababaaba$ as quasiperiods, and the smallest quasiperiod of w is aba . Words that are not quasiperiodic are naturally called *non-quasiperiodic* words.

When defining infinite quasiperiodic words, for convenience, we consider the words preceding the occurrences of a quasiperiod: an infinite word \mathbf{w} is *quasiperiodic* if and only if, there exist a finite word u and words $(p_n)_{n \geq 0}$ such that $p_0 = \varepsilon$, $|p_n| < |p_{n+1}| \leq |p_n u|$, and $p_n u$ is a prefix of \mathbf{w} for all $n \geq 0$. Then u is a *quasiperiod* of \mathbf{w} and we say that the sequence $(p_n u)_{n \geq 0}$ is a *covering sequence of prefixes of the word \mathbf{w}* . Necessarily, any quasiperiod of a quasiperiodic word must be a prefix of it.

Readers will find several examples of infinite quasiperiodic words in [27,32,33]. Let us mention for instance that the *Fibonacci word*, directed by $(ab)^\omega$, is *aba*-quasiperiodic (see [27]).

Let us now recall a simple, yet important, fact about quasiperiodic words.

Fact 4.1. If w is a (finite or infinite) u -quasiperiodic word and f is a non-erasing morphism, then $f(w)$ is $f(u)$ -quasiperiodic.

Note that the converse of this fact is not true. For example, let $f = R_a$ and $\mathbf{w} = ab^\omega$, then $R_a(\mathbf{w}) = a(ba)^\omega$. The word $R_a(\mathbf{w})$ is covered by $R_a(ab)$, but \mathbf{w} is not covered by ab .

4.2. Return words and quasiperiodicity

We now use the notion of a ‘return word’ to give an equivalent definition of quasiperiodicity (see Lemma 4.3), which proves to be a useful tool for studying quasiperiodicity in episturmian words.

Return words were introduced independently by Durand [13] and by Holton and Zamboni [20] when studying primitive substitutive sequences. Such words can be defined in the following way.

Definition 4.2. Let v be a recurrent factor of an infinite word $\mathbf{w} = w_1 w_2 w_3 \dots$, starting at positions $n_1 < n_2 < n_3 \dots$ in \mathbf{w} . Then each word $r_i = w_{n_i} w_{n_i+1} \dots w_{n_{i+1}-1}$ is called a *return to v in \mathbf{w}* .

That is, a return to v in \mathbf{w} is a non-empty factor of \mathbf{w} beginning at an occurrence of v and ending exactly before the next occurrence of v in \mathbf{w} . Thus, if r is a return to v in \mathbf{w} , then rv is a factor of \mathbf{w} that contains exactly two occurrences of v , one as a prefix and one as a suffix. As any episturmian word \mathbf{t} is uniformly recurrent [12], each factor of \mathbf{t} has only a finite number of different returns (for more details see Theorem 4.7).

Note. A return to v in \mathbf{w} always has v as a prefix or is a prefix of v . In particular, we observe that a return to v is not necessarily longer than v , in which case v has *overlapping* occurrences in \mathbf{w} (i.e., $vz^{-1}v$ is a factor of \mathbf{w} for some non-empty word z). We say that v has *adjacent* occurrences in \mathbf{w} if vv is a factor of \mathbf{w} . In this case, if v is *primitive* (i.e., not an integer power of a shorter word), then v is a return to itself; otherwise, the corresponding return to v is the *primitive root* of v .

In terms of return words, we have the following equivalent definition of a quasiperiodic infinite word.

Lemma 4.3. A finite word v is a quasiperiod of an infinite word \mathbf{w} if and only if, v is a recurrent prefix of \mathbf{w} such that any return to v in \mathbf{w} has length at most $|v|$.

Proof. If v is a quasiperiod of \mathbf{w} , then v is a prefix of \mathbf{w} and its occurrences entirely cover \mathbf{w} . That is, v is recurrent in \mathbf{w} and successive occurrences of v in \mathbf{w} are either adjacent or overlap, and hence any return to v has length at most $|v|$. Conversely, if v is a recurrent prefix of \mathbf{w} such that any return to v has length at most $|v|$, then successive occurrences of v in \mathbf{w} are either adjacent or overlap, and hence entirely cover \mathbf{w} . Thus \mathbf{w} is v -quasiperiodic. \square

Immediately:

Corollary 4.4. An infinite word \mathbf{w} is quasiperiodic if and only if, there exists a recurrent prefix v of \mathbf{w} such that any return to v in \mathbf{w} has length at most $|v|$, in which case v is a quasiperiod of \mathbf{w} . Moreover, the shortest such prefix v is the smallest quasiperiod of \mathbf{w} .

A noteworthy fact is that a quasiperiodic infinite word is not necessarily recurrent [33], although it must have a prefix that is recurrent in it.

4.3. Return words and palindromic closure in episturmian words

In this section, we recall a characterization of return words in episturmian words, given by Justin and Vuillon in [26]. For this, we first recall the construction of epistandard words using *palindromic right-closure* as well as some related properties from [12,23] that will be used throughout Section 4.4–5.4.

The *palindromic right-closure* $w^{(+)}$ of a finite word w is the (unique) shortest palindrome having w as a prefix (see [11]). That is, $w^{(+)} = wv^{-1}\tilde{w}$ where v is the longest palindromic suffix of w . The *iterated palindromic closure* function [22], denoted by Pal , is defined recursively as follows. Set $Pal(\varepsilon) = \varepsilon$ and, for any word w and letter x , define $Pal(wx) = (Pal(w)x)^{+}$. For instance, $Pal(abc) = (Pal(ab)c)^{+} = (abac)^{+} = abacaba$.

Generalizing a construction given in [11] for *standard Sturmian words*, Droubay, Justin and Pirillo established the following characterization of epistandard words.

Theorem 4.5 ([12]). *An infinite word $\mathbf{s} \in \mathcal{A}^\omega$ is epistandard if and only if, there exists an infinite word $\Delta = x_1x_2x_3 \cdots$ ($x_i \in \mathcal{A}$) such that $\mathbf{s} = \lim_{n \rightarrow \infty} Pal(x_1 \cdots x_n)$.*

Note that the palindromes $Pal(x_1 \cdots x_n)$ are very often denoted by u_{n+1} in the literature.

In [23], Justin and Pirillo showed that the word Δ is exactly the directive word of \mathbf{s} as it occurs in Theorem 2.1. Moreover, by construction, Δ uniquely determines the epistandard word \mathbf{s} . Notice also that by construction, the words $(Pal(x_1 \cdots x_i))_{i \geq 0}$ are exactly the palindromic prefixes of \mathbf{s} .

There exist many relations between palindromes and episturmian morphisms. The following ones will be useful in the next few sections. First recall from [22,23] that we have

$$Pal(wv) = \mu_w(Pal(v))Pal(w) \quad \text{for any words } w, v. \tag{4.1}$$

In particular, for any x a letter, $Pal(xv) = L_x(Pal(v))x$ and $Pal(wx) = \mu_w(x)Pal(w)$.

For letters $(x_j)_{1 \leq j \leq i}$, formula (4.1) inductively leads to:

$$Pal(x_1 \cdots x_i) = \mu_{x_1 \cdots x_{i-1}}(x_i) \cdots \mu_{x_1}(x_2)x_1 = \prod_{1 \leq j \leq i} \mu_{x_1 \cdots x_{j-1}}(x_j). \tag{4.2}$$

Note that by convention, $x_1 \cdots x_0 = \varepsilon$ in the above product.

Now let $\check{w} = \check{x}_1\check{x}_2 \cdots \check{x}_n$ be a spinned version of $w = x_1x_2 \cdots x_n$ (viewed as a prefix of a spinned version $\check{\Delta}$ of Δ). Then, for any finite word v , we have

$$\mu_{\check{w}}(v) = S_{\check{w}}^{-1} \mu_w(v) S_{\check{w}} \quad \text{where } S_{\check{w}} = \prod_{\substack{i=n, \dots, 1 \\ \check{x}_i = \check{x}_i}} \mu_{x_1 \cdots x_{i-1}}(x_i). \tag{4.3}$$

The word $S_{\check{w}}$ is called the *shifting factor* of $\mu_{\check{w}}$ [25]. Observe that $S_{\check{w}}$ is a prefix of $Pal(w)$; in particular $S_{\check{w}} = Pal(w)$ by equation (4.2). Note also that $\mu_{\check{w}}(v) = T^{S_{\check{w}}^{-1}}(\mu_w(v))$.

For example, for $\check{w} = \check{a}\check{b}\check{c}\check{a}$, we have $S_{\check{w}} = \mu_{abc}(a)\mu_a(b) = abacabaab$. Thus, since $\mu_{abca}(ca) = abacabaab.acabacaba$, $\mu_{\check{a}\check{b}\check{c}\check{a}}(ca) = T^9(\mu_{abca}(ca)) = acabacaba.abacabaab$.

Likewise, for any infinite word $\mathbf{y} \in \mathcal{A}^\omega$,

$$\mu_{\check{w}}(\mathbf{y}) = S_{\check{w}}^{-1} \mu_w(\mathbf{y}). \tag{4.4}$$

This formula used with $\check{w} = \bar{w}$ shows that:

Fact 4.6. Any word of the form $\mu_w(\mathbf{y})$ with \mathbf{y} infinite begins with $Pal(w)$.

Amongst the numerous interests of the palindromes $(Pal(x_1 \cdots x_n))_{n \geq 0}$, we have the following explicit characterization of the returns to any factor of an epistandard word.

Theorem 4.7. [26] *Suppose \mathbf{s} is an epistandard word directed by $\Delta = x_1x_2x_3 \cdots$ with $x_i \in \mathcal{A}$, and consider any factor v of \mathbf{s} . If the word $u_{n+1} = Pal(x_1 \cdots x_n)$ is the shortest palindromic prefix of \mathbf{s} containing v with $u_{n+1} = fvg$, then the returns to v are given by $f^{-1}\mu_{x_1 \cdots x_n}(x)f$ where $x \in \text{Alph}(x_{n+1}x_{n+2} \cdots)$.*

Because of the uniform recurrence of episturmian words, the following simple but important fact about return words holds.

Lemma 4.8. *Suppose \mathbf{s} is an epistandard word and let \mathbf{t} be any episturmian word in the subshift of \mathbf{s} . Then, for any factor v of \mathbf{s} , r is a return to v in \mathbf{s} if and only if, r is a return to v in \mathbf{t} .*

That is, the returns to any factor v of an epistandard word \mathbf{s} are the same as the returns to v as a factor of any episturmian word \mathbf{t} with the same set of factors as \mathbf{s} . Hereafter, we often use the above result without reference to it.

The following result is particularly useful in Sections 4.3–4.6.

Proposition 4.9. *Suppose \mathbf{s} is an epistandard word directed by $\Delta = x_1x_2x_3 \cdots$ with $x_i \in \mathcal{A}$, and let \mathbf{t} be an episturmian word directed by a spinned version of Δ . Then, \mathbf{t} begins with $Pal(x_1 \cdots x_n)$ for some non-negative integer n if and only if \mathbf{t} has a directive word of the form $\Delta = x_1 \cdots x_n \check{x}_{n+1} \check{x}_{n+2} \cdots$ where the prefix $x_1 \cdots x_n$ is L -spinned.*

Proof. When \mathbf{t} is directed by a spinned version of Δ of the form $\check{\Delta} = x_1x_2 \cdots x_n \check{x}_{n+1} \check{x}_{n+2} \cdots$ where the prefix $x_1x_2 \cdots x_n$ is L -spinned, \mathbf{t} begins with $Pal(x_1 \cdots x_n)$ by Fact 4.6.

Conversely, suppose $Pal(x_1 \cdots x_n)$ is a prefix of \mathbf{t} , and suppose that \mathbf{t} is directed by $\check{\Delta} = (\check{x}_i)_{i \geq 1}$ (a spinned version of $\Delta = (x_i)_{i \geq 1}$). From Theorem 3.2 we can suppose that $\check{\Delta}$ contains infinitely many L -spinned letters. If $n = 0$, there is nothing to prove. Else let $x_1 = a$. Assume first that $\check{x}_1 = \bar{a}$. Let k be the smallest positive integer such that $\check{x}_k \in \mathcal{A}$. Since \mathbf{t} begins with the letter a (which is the first letter of $Pal(x_1 \cdots x_n)$), we have $x_k = a$. Then $\Delta = \bar{a}\check{x}_2 \cdots \check{x}_{k-1} a \check{x}_{k+1} \cdots$, and hence by Theorem 3.1 \mathbf{t} is also directed by the spinned infinite word beginning with $a x_2 \cdots x_{j-1} \bar{a}$, with $j \leq k$. Therefore, we may assume from now on that $\check{x}_1 = a$. Let \mathbf{t}' be the episturmian word directed by $T(\check{\Delta}) = (\check{x}_i)_{i \geq 2}$. It is easily seen from the equality $Pal(x_1 \cdots x_n) = L_a(Pal(x_2 \cdots x_n))a$ that $Pal(x_2 \cdots x_n)$ is a prefix of \mathbf{t}' . Hence by induction \mathbf{t}' is directed by a spinned version of $T(\Delta)$ with $x_2 \cdots x_n$ as a prefix. And so \mathbf{t} is directed by a spinned version of Δ of the form $x_1x_2 \cdots x_n \check{x}_n \check{x}_{n+1} \cdots$ where the prefix $x_1x_2 \cdots x_n$ is L -spinned. \square

4.4. All epistandard words are quasiperiodic

A first consequence of Theorem 4.7 is that any epistandard word is quasiperiodic. More precisely:

Theorem 4.10. *Suppose \mathbf{s} is an epistandard word with directive word $\Delta = x_1x_2x_3 \cdots$ with $x_i \in \mathcal{A}$, and let m be the smallest positive integer such that $Alph(x_1x_2 \cdots x_m) = Alph(\mathbf{s})$. Then, for all $n \geq m$, $Pal(x_1 \cdots x_n)$ is a quasiperiod of \mathbf{s} .*

Proof of Theorem 4.10. We suppose that \mathbf{s} is an epistandard word with directive word $\Delta = x_1x_2x_3 \cdots$, $x_i \in \mathcal{A}$. Let m be the smallest positive integer such that $Alph(x_1x_2 \cdots x_m) = Alph(\mathbf{s})$. Clearly, for $n < m$, $Pal(x_1 \cdots x_n)$ cannot be a quasiperiod of \mathbf{s} since it does not contain all of the letters in $Alph(\mathbf{s})$.

Now let $n \geq m$. We know that $Pal(x_1 \cdots x_n)$ is a prefix of \mathbf{s} . Suppose that for some $k \geq n$, $Pal(x_1 \cdots x_k)$ is covered by $Pal(x_1 \cdots x_n)$. Since by choice of m , x_{k+1} belongs to $\{x_1, \dots, x_k\}$, we have $|(Pal(x_1 \cdots x_k)_{x_{k+1}})^{(+)}| \leq 2|Pal(x_1 \cdots x_k)|$. Hence, since $Pal(x_1 \cdots x_k)$ is a palindrome, $Pal(x_1 \cdots x_{k+1})$ is covered by $Pal(x_1 \cdots x_k)$, and so by $Pal(x_1 \cdots x_n)$. The result follows from Theorem 4.5 by induction. \square

Example 4.11. Recall the Tribonacci word:

$$\mathbf{r} = abacabaabacababacabaabacabacabaabaca \cdots,$$

which is the epistandard word directed by $(abc)^\omega$. Observe that $Pal(abc) = abacaba$ is the shortest palindromic prefix of \mathbf{r} such that $Alph(Pal(abc)) = Alph(\mathbf{r}) = \{a, b, c\}$. By Theorem 4.7, the returns to $Pal(abc)$ in \mathbf{r} are: $\mu_{abc}(a) = abacaba$, $\mu_{abc}(b) = abacab$, $\mu_{abc}(c) = abac$, none of which are longer than $Pal(abc)$. Hence \mathbf{r} is *abacaba*-quasiperiodic; in fact $Pal(abc) = \mu_{abc}(a) = abacaba$ is the smallest quasiperiod of \mathbf{r} since its prefixes *abac*, *abaca*, *abacab* have returns longer than themselves. This latter fact is also evident from our description of quasiperiods of a (quasiperiodic) episturmian word (Theorem 4.19, to follow).

More generally, the *k*-bonacci word, which is directed by $(a_1a_2 \cdots a_k)^\omega$, is quasiperiodic with smallest quasiperiod $Pal(x_1 \cdots x_k)$. This fact was also observed in [27] by noting that the *k*-bonacci word is generated by the morphism φ_k on $\{a_1, a_2, \dots, a_k\}$ defined by $\varphi_k(a_i) = a_1a_{i+1}$ for all $i \neq k$, and $\varphi_k(a_k) = a_1$.

Remark 4.12. From Fact 4.1 and Theorem 4.10, we immediately deduce that $\varphi(\mathbf{s})$ is a quasiperiodic infinite word for any epistandard word \mathbf{s} . Moreover, if φ is a pure episturmian morphism, then $\varphi(\mathbf{s})$ is a *quasiperiodic episturmian word*. More precisely, $\mu_{\check{w}}(\mathbf{s})$ is a quasiperiodic episturmian word for any epistandard word \mathbf{s} and spinned word \check{w} . Such an episturmian word is directed by a spinned infinite word of the form $\check{w}\Delta$ where Δ is the L -spinned directive word of \mathbf{s} . Hence, if an episturmian word \mathbf{t} is directed by a spinned infinite word with all spins ultimately L , then \mathbf{t} is quasiperiodic.

More generally, we have the following consequence of Theorem 4.10 (a converse of this result is stated in Theorem 4.19, and a generalization is provided by Theorem 5.1).

Corollary 4.13. *If an episturmian word \mathbf{t} is directed by $\check{\Delta} = \check{w}v\check{y}$ for some spinned words \check{w} , \check{y} and L -spinned word v such that $Alph(v) = Alph(v\mathbf{y})$, then \mathbf{t} is quasiperiodic.*

Moreover, any word of the form $\mu_{\check{w}}(Pal(v))p$ with p a prefix of $S_{\check{w}}^{-1}Pal(w)$ is a quasiperiod of \mathbf{t} .

Proof. Let $(\mathbf{t}^{(n)})_{n \geq 0}$ be the infinite sequence of episturmian words associated to \mathbf{t} and $\check{\Delta}$ in Theorem 2.1. Then by Proposition 4.9, the episturmian word $\mathbf{t}^{(l|w)}$, which is directed by $v\check{y}$, begins with the palindromic prefix $Pal(v)$ of the epistandard word $\mathbf{s}^{(l|w)}$ directed by the L -spinned version of $v\check{y}$. Moreover, since $Alph(v) = Alph(v\mathbf{y})$, $Pal(v)$ is a quasiperiod of $\mathbf{s}^{(l|w)}$ by Theorem 4.10. By Fact 2.3, $\mathbf{s}^{(l|w)}$ is in the subshift of $\mathbf{t}^{(l|w)}$. It follows from Lemma 4.8 that $Pal(v)$ has the same returns in $\mathbf{t}^{(l|w)}$ as it does in $\mathbf{s}^{(l|w)}$, and since it is a prefix of $\mathbf{t}^{(l|w)}$, it is also a quasiperiod of $\mathbf{t}^{(l|w)}$. Therefore by Fact 4.1 $\mathbf{t} = \mu_{\check{w}}(\mathbf{t}^{(l|w)})$ is quasiperiodic and $\mu_{\check{w}}(Pal(v))$ is a quasiperiod of \mathbf{t} .

Now by formula (4.4) and by Fact 4.6, each occurrence of $\mu_{\check{w}}(Pal(v))$ is followed by $S_{\check{w}}^{-1}Pal(w)$ in \mathbf{t} , and so is followed by p for any prefix p of $S_{\check{w}}^{-1}Pal(w)$. This shows that $\mu_{\check{w}}(Pal(v))p$ is a quasiperiod of \mathbf{t} . \square

4.5. Ultimate quasiperiods

Theorem 4.10 shows that long enough palindromic prefixes of an epistandard word \mathbf{s} are quasiperiods of \mathbf{s} . Our next goal is to extend Theorem 4.10 by describing all the quasiperiods of any (quasiperiodic) episturmian word. Several lemmas are required, using the notion of ultimate quasiperiods that we now define.

A factor v of an infinite word \mathbf{w} is said to be an *ultimate quasiperiod* of \mathbf{w} if it is a quasiperiod of a suffix of \mathbf{w} . In particular, when \mathbf{w} is a uniformly recurrent word (which is the case for episturmian words), a recurrent factor v in \mathbf{w} is an ultimate quasiperiod of \mathbf{w} if any return to v in \mathbf{w} has length at most $|v|$.

Clearly, if an ultimate quasiperiod is a prefix of \mathbf{w} , then \mathbf{w} is quasiperiodic by Lemma 4.3. Also note that the set of quasiperiods of a (quasiperiodic) infinite word \mathbf{w} consists of all its ultimate quasiperiods that are prefixes of it.

Remark that we have

Proposition 4.14. *All episturmian words are ultimately quasiperiodic.*

Proof. From Theorem 4.10, all epistandard words are quasiperiodic. Let \mathbf{s} be an epistandard word and let q be a quasiperiod of \mathbf{s} . Then any episturmian word \mathbf{t} in the subshift of \mathbf{s} has a suffix \mathbf{t}' beginning with q . Since \mathbf{t} has the same set of factors than \mathbf{s} , we can find infinitely many prefixes of \mathbf{t}' which are covered by q , i.e., \mathbf{t}' is quasiperiodic. Hence \mathbf{t} is ultimately quasiperiodic. \square

Now, we recall some insight about palindromic closure that will be useful later. As previously, let \mathbf{s} denote an epistandard word with directive word $\Delta = x_1x_2x_3 \cdots$ with $x_i \in \mathcal{A}$. For $n \geq 0$ let $u_{n+1} = \text{Pal}(x_1 \cdots x_n)$. Note in particular, that $u_1 = \varepsilon$ and by Theorem 4.5, $\mathbf{s} = \lim_{n \rightarrow \infty} u_n$.

As in [26,23], let us define $P(i) = \sup\{j < i \mid x_j = x_i\}$ if this number exists, undefined otherwise. That is, if $x_i = a$, then $P(i)$ is the position of the right-most occurrence of the letter a in the prefix $x_1x_2 \cdots x_{i-1}$ of the directive word Δ . For instance, if $\Delta = (abc)^\omega$, $P(i) = i - 3$ for any $i \geq 4$ and $P(1), P(2)$, and $P(3)$ are undefined.

From the definitions of palindromic closure and the palindromes $(u_i)_{i \geq 1}$, it follows that, for all $i \geq 1$,

$$u_{i+1} = \begin{cases} u_i x_i u_i & \text{if } x_i \notin \text{Alph}(u_i), \\ u_i u_{P(i)}^{-1} u_i & \text{otherwise.} \end{cases} \tag{4.5}$$

Therefore, using Theorem 4.7 with $f = g = \varepsilon$, we deduce that for $n \geq 0$, the length of the longest return r_{n+1} to u_{n+1} in \mathbf{s} satisfies

$$|r_{n+1}| = \begin{cases} |u_{n+1}| + 1 & \text{if some } x \in \text{Alph}(\mathbf{s}) \text{ does not occur in } u_{n+1}, \\ |u_{n+1}| - |u_{p_n}| & \text{otherwise,} \end{cases}$$

where $p_n = \inf\{P(i) \mid i \geq n + 1\}$ (see also [26, Lem. 5.6]).

In other words, $p_n = \sup\{i \leq n \mid \text{Alph}(x_i \cdots x_n) = \text{Alph}(x_i \cdots x_n \cdots)\}$. For instance, if $\Delta = (abc)^\omega$, then $p_4 = 1$.

The next lemma gives the set of all ultimate quasiperiods of any episturmian word \mathbf{t} . It simply amounts to determining all of the factors of \mathbf{t} that have no returns longer than themselves.

Notation. Hereafter, we denote by $F(w)$ the set of factors of a finite or infinite word w .

Lemma 4.15. *Suppose \mathbf{s} is an epistandard word directed by $\Delta = x_1x_2x_3 \cdots$, $x_i \in \mathcal{A}$. Let m be the smallest positive integer such that $\text{Alph}(x_1x_2 \cdots x_m) = \text{Alph}(\mathbf{s})$ and let $u_{n+1} = \text{Pal}(x_1 \cdots x_n)$ for all $n \geq 0$. Then, the set of all ultimate quasiperiods of any episturmian word \mathbf{t} in the subshift of \mathbf{s} is given by*

$$\mathcal{Q} = \bigcup_{n \geq m} \mathcal{Q}_n \quad \text{with } \mathcal{Q}_n = \{q \in F(u_{n+1}) \mid |q| \geq |u_{n+1}| - |u_{p_n}|\},$$

where $p_n = \sup\{i \leq n \mid \text{Alph}(x_i \cdots x_n) = \text{Alph}(x_i \cdots x_n \cdots)\}$.

Proof. First observe that the number p_n exists for all $n \geq m$. Indeed, the set $\{i \leq n \mid \text{Alph}(x_i \cdots x_n) = \text{Alph}(x_i \cdots x_n \cdots)\}$ is not empty, as by the definition of m it contains $i = 1$.

Clearly, if $n < m$, then no factor of u_{n+1} can be an ultimate quasiperiod of \mathbf{t} since u_{n+1} does not contain all of the letters in $\text{Alph}(\mathbf{t}) = \text{Alph}(\Delta)$. So let us now fix $n \geq m$. For any $q \in \mathcal{Q}_n$, $|q| \geq |u_{n+1}| - |u_{p_n}| > |u_n|$. Indeed from formula (4.5), if $n = m$ then $u_{n+1} = u_n x_n u_n$, and if $n \geq m + 1$, then $u_{n+1} = u_n u_{P(n)}^{-1} u_n$ where $P(n) \geq p_n$. Hence $q \in F(u_{n+1}) \setminus F(u_n)$, i.e., u_{n+1} is the shortest palindromic prefix of \mathbf{s} containing q . Therefore, by Theorem 4.7, the returns to $q \in \mathcal{Q}_n$ are a certain circular shift of the returns to u_{n+1} and the longest of these return words has length $|u_{n+1}| - |u_{p_n}|$. Thus, any return to q has length at most $|q|$; whence q is an ultimate quasiperiod of \mathbf{t} . It remains to show that any other factor $w \in F(u_{n+1}) \setminus F(u_n)$ with $|w| < |u_{n+1}| - |u_{p_n}|$ is not an ultimate quasiperiod. This is clearly true since the longest return to any such w has length $|u_{n+1}| - |u_{p_n}| > |w|$. That is, at least one of the returns to w is longer than it, which implies that w is not an ultimate quasiperiod of \mathbf{t} . \square

Example 4.16. Let us consider the Fibonacci case. As observed in [10], $|u_n| = F_{n+1} - 2$ for all $n \geq 1$ where F_k is the k -th Fibonacci number ($F_1 = 1, F_2 = 2, F_k = F_{k-1} + F_{k-2}$ for $k \geq 2$). Since for $n \geq 2, p_n = n - 1$ and $|u_{n+1}| - |u_{p_n}| = F_{n+1}$, the ultimate quasiperiods of the Fibonacci word are the factors of u_k of length between F_k and $F_{k+1} - 2$ for all $k \geq 3$. The first few ultimate quasiperiods of the Fibonacci word (in order of increasing length) are: aba (u_3), $abaab$, $baaba$, $abaaba$ (u_4), $abaababa$, $baababaa$, $aababaab$, $ababaaba$, $abaababaa$, $baababaab$, $aababaaba$, $abaababaab$, $baababaaba$, $abaababaaba$ (u_5), . . .

Lemma 4.15 yields the following trivial characterization of quasiperiodic episturmian words.

Corollary 4.17. *Suppose \mathbf{s} is an epistandard word with set of ultimate quasiperiods \mathcal{Q} . Then an episturmian word \mathbf{t} in the subshift of \mathbf{s} is quasiperiodic if and only if some $v \in \mathcal{Q}$ is a prefix of \mathbf{t} .*

Moreover, Lemma 4.15 can be reformulated (more nicely) using episturmian morphisms, together with the iterated palindromic closure function.

Lemma 4.18. *Suppose \mathbf{s} is an epistandard word directed by $\Delta \in \mathcal{A}^\omega$. Then the set of ultimate quasiperiods of any episturmian word \mathbf{t} in the subshift of \mathbf{s} is the set of all words*

$$q \in F(\text{Pal}(wv)), \quad \text{with } |q| \geq |\mu_w(\text{Pal}(v))|,$$

where w, v are words such that $\Delta = wv\mathbf{y}$ with $\text{Alph}(v) = \text{Alph}(v\mathbf{y})$.

Proof. Let $\Delta = x_1x_2x_3 \cdots$ and let m be the smallest positive integer such that $\text{Alph}(x_1x_2 \cdots x_m) = \text{Alph}(\mathbf{s})$. Then, by Lemma 4.15, the set of all ultimate quasiperiods of \mathbf{s} (and hence of \mathbf{t}) is given by $\mathcal{Q} = \bigcup_{n \geq m} \mathcal{Q}_n$ with

$$\mathcal{Q}_n = \{q \in F(u_{n+1}) \mid |q| \geq |u_{n+1}| - |u_{p_n}|\},$$

where $p_n = \sup\{i \leq n \mid \text{Alph}(x_i \cdots x_n) = \text{Alph}(x_i \cdots x_n \cdots)\}$. So, for fixed $n \geq m$, Δ can be written as $\Delta = wv\mathbf{y}$ where $w = x_1 \cdots x_{p_n-1}$, $v = x_{p_n} \cdots x_n$, $\mathbf{y} = x_{n+1}x_{n+2} \cdots$, and $\text{Alph}(v) = \text{Alph}(v\mathbf{y})$ (by the definition of p_n). Then $u_{n+1} = \text{Pal}(x_1 \cdots x_n) = \text{Pal}(wv)$ and $u_{p_n} = \text{Pal}(x_1 \cdots x_{p_n}) = \text{Pal}(w)$. So now, using formula (4.1), we have $\text{Pal}(wv) = \mu_w(\text{Pal}(v))\text{Pal}(w)$; in particular, $|u_{n+1}| - |u_{p_n}| = |\text{Pal}(wv)| - |\text{Pal}(w)| = |\mu_w(\text{Pal}(v))|$. Thus, Lemma 4.15 tells us that for $q \in \mathcal{Q}_n$, there exist words w, v such that $q \in F(\text{Pal}(wv))$ and $|q| \geq |\mu_w(\text{Pal}(v))|$.

Conversely, assume $\Delta = wv\mathbf{y}$ with $\text{Alph}(v) = \text{Alph}(v\mathbf{y})$. Let v_1, v_2 be such that $v = v_1v_2$ with v_2 the smallest suffix of v such that $\text{Alph}(v_2) = \text{Alph}(v_2\mathbf{y})$. Let $n = |wv_1|$. It follows from the choice of v_2 that $p_n = |wv_1| + 1$. By Theorem 4.10, $\text{Pal}(v_2)$ is a quasiperiod of the epistandard word directed by $v_2\mathbf{y}$. By Fact 4.1 $\mu_{wv_1}(\text{Pal}(v_2))$ is a quasiperiod of \mathbf{s} . Since any occurrence of $\mu_{wv_1}(\text{Pal}(v_2))$ in \mathbf{s} is followed by $\text{Pal}(wv_1)$, we deduce that any factor of $\mu_{wv_1}(\text{Pal}(v_2))\text{Pal}(wv_1)$ of length greater than $|\mu_{wv_1}(\text{Pal}(v_2))|$ is an ultimate quasiperiod of \mathbf{t} . Using formula (4.1), we see that $\text{Pal}(wv_1) = \mu_w(\text{Pal}(v_1))\text{Pal}(w)$ so that $\mu_{wv_1}(\text{Pal}(v_2)) = \mu_w(\mu_{v_1}(\text{Pal}(v_2))\text{Pal}(v_1)) = \mu_w(\text{Pal}(v_1v_2)) = \mu_w(\text{Pal}(v))$. Hence, any factor of $\mu_w(\text{Pal}(v))\text{Pal}(w) = \text{Pal}(wv)$ of length greater than $|\mu_w(\text{Pal}(v))|$ ($\geq |\mu_{wv_1}(\text{Pal}(v_2))|$) is an ultimate quasiperiod of \mathbf{t} . \square

4.6. Quasiperiods of episturmian words

We are now ready to state the main theorem of this section, which describes all of the quasiperiods of an episturmian word.

Theorem 4.19. *The set of quasiperiods of an episturmian word \mathbf{t} is the set of all words*

$$\mu_{\tilde{w}}(\text{Pal}(v))p, \quad \text{with } p \text{ a prefix of } S_{\tilde{w}}^{-1}\text{Pal}(w), \tag{4.6}$$

where w, v are L -spinned words such that \mathbf{t} is directed by $\tilde{w}v\tilde{\mathbf{y}}$ for some spinned version \tilde{w} of w and some spinned version $\tilde{\mathbf{y}}$ of an L -spinned infinite word \mathbf{y} with $\text{Alph}(v) = \text{Alph}(v\tilde{\mathbf{y}})$.

Moreover, the smallest quasiperiod of \mathbf{t} is the word $\mu_{\tilde{w}}(\text{Pal}(v))$ where wv is of minimal length for the property $\text{Alph}(wv) = \text{Alph}(wv\tilde{\mathbf{y}})$, and amongst all decompositions of wv into w and v , the word v is the shortest suffix of wv such that $\text{Alph}(v) = \text{Alph}(v\tilde{\mathbf{y}})$.

The above theorem shows that if there do not exist words $\tilde{w}, \mathbf{y}, \tilde{\mathbf{y}}$, and v (as defined above) such that \mathbf{t} is directed by $\tilde{w}v\tilde{\mathbf{y}}$ with $\text{Alph}(v) = \text{Alph}(v\tilde{\mathbf{y}})$, then \mathbf{t} does not have any quasiperiods, and hence \mathbf{t} is non-quasiperiodic. For instance, any regular wavy word Δ (recall the definition from Section 3.4) clearly directs an episturmian word with no quasiperiods since Δ is the only directive word for \mathbf{t} and it does not contain an L -spinned factor v containing all letters that follow it in Δ . For example, $(abc)^\omega$ directs a non-quasiperiodic episturmian word in the subshift of the Tribonacci word.

Let us now illustrate the last part of the theorem. For the epistandard word directed by $\Delta = ca(ab)^\omega$, i.e., the image of the Fibonacci word by the morphism L_cL_a , the shortest word wv such that $\Delta = wv\mathbf{y}$ for some infinite word \mathbf{y} with $\text{Alph}(wv) = \text{Alph}(wv\mathbf{y})$ is the word $caab$. There are three ways to decompose this word $caab$ into wv : (1) $w = \varepsilon$ and $v = caab$; (2) $w = c$ and $v = aab$; (3) $w = ca$ and $v = ab$. The corresponding quasiperiods of the form $\mu_w(\text{Pal}(v))$ are respectively: (1) $\text{Pal}(caab) = cacabcacac$; (2) $\mu_c(\text{Pal}(aab)) = cacabcaca$; (3) $\mu_{ca}(\text{Pal}(ab)) = cacabc$.

Proof of Theorem 4.19. Corollary 4.13 proves that any word of the form (4.6) is a quasiperiod of \mathbf{t} .

Now suppose that q is a quasiperiod of \mathbf{t} . We show that \mathbf{t} has at least one directive word of the form $\check{w}v\check{y}$ where \check{w}, \check{y} are spinned versions of some L -spinned words w, \mathbf{y} , and where v is an L -spinned word such that $\text{Alph}(v) = \text{Alph}(v\mathbf{y})$. Moreover, we show that $q = \mu_{\check{w}}(\text{Pal}(v))p$ for some prefix p of $S_{\check{w}}^{-1}\text{Pal}(w)$.

Let \mathbf{s} be an epistandard word with L -spinned directive word Δ such that \mathbf{t} is directed by a spinned version of Δ . The quasiperiod q is an ultimate quasiperiod of \mathbf{t} that occurs as a prefix of \mathbf{t} . So, by Lemma 4.18,

$$q \in F(\text{Pal}(wv)), \quad \text{with } |q| \geq |\mu_w(\text{Pal}(v))|,$$

for some L -spinned words w, v , and \mathbf{y} such that $\Delta = wv\mathbf{y}$ with $\text{Alph}(v) = \text{Alph}(v\mathbf{y})$. In particular, we have $\text{Pal}(wv) = fqg$ for some words f, g with $|f| + |g| \leq |\text{Pal}(w)|$, where $\text{Pal}(wv) = \mu_w(\text{Pal}(v))\text{Pal}(w)$. By definition of the Pal function, $\text{Pal}(w)$ is a prefix of $\text{Pal}(wv)$. Consequently f is a prefix of $\text{Pal}(w)$.

Note that

$$q = f^{-1}\mu_w(\text{Pal}(v))\text{Pal}(w)g^{-1}. \tag{4.7}$$

Thus $q = (f^{-1}\mu_w(\text{Pal}(v))f)p$ with $p := f^{-1}\text{Pal}(w)g^{-1}$ a prefix of $f^{-1}\text{Pal}(w)$.

The following result shows that $f = S_{\check{w}}$ for some spinned version \check{w} of w .

Lemma 4.20. *Given a word w and a prefix f of $\text{Pal}(w)$, there exists a spinned version \check{w} of w such that $f = S_{\check{w}}$.*

Proof. The proof proceeds by induction on $|w|$. The lemma is clearly true for $|w| = 0$ since in this case f is a prefix of $\text{Pal}(w) = \text{Pal}(\varepsilon) = \varepsilon$, and hence $f = \varepsilon = S_\varepsilon$.

Now suppose $|w| \geq 1$ and let us write $w = xw'$ where x is a letter. Since f is a prefix of $\text{Pal}(w) = \text{Pal}(xw') = \mu_x(\text{Pal}(w'))x$ (see formula (4.1)), we have $f = \mu_x(f')$ or $f = \mu_x(f')x$ for some prefix f' of $\text{Pal}(w')$. Moreover, by the induction hypothesis, $f' = S_{\check{w}'}$ for some spinned version \check{w}' of w' . Hence, using formula (4.3), we have $f = \mu_x(S_{\check{w}'}) = S_{x\check{w}'}$ or $f = \mu_x(S_{\check{w}'})x = S_{x\check{w}'}$. That is, $f = S_{\check{w}}$ for some spinned version \check{w} of $w = xw'$. \square

Now we have to prove that \mathbf{t} is directed by a spinned infinite word beginning with $\check{w}v$. For this we need some further intermediate results, as follows.

Lemma 4.21. *For any word u containing at least two different letters and for any other word w , there exists a word u_w containing at least two different letters such that $\mu_w(u) = \text{Pal}(w)u_w$.*

Proof. The proof proceeds by induction on $|w|$. The lemma is trivially true for $|w| = 0$. Now suppose $|w| \geq 1$ and let us write $w = xw'$ where x is a letter. Then $\mu_w(u) = \mu_{xw'}(u) = \mu_x(\mu_{w'}(u))$ where, by the induction hypothesis, $\mu_{w'}(u) = \text{Pal}(w')u_{w'}$ for some word $u_{w'}$ containing at least two different letters. Hence, $\mu_w(u) = \mu_x(\text{Pal}(w')u_{w'}) = \mu_x(\text{Pal}(w'))\mu_x(u_{w'})$, and therefore by formula (4.1) we have $\mu_w(u) = \text{Pal}(xw')x^{-1}\mu_x(u_{w'})$ where the word $x^{-1}\mu_x(u_{w'})$ contains at least two different letters. This completes the proof of the lemma. \square

Corollary 4.22. *For any letter x and for any words v, w such that v contains at least one letter different from x ,*

$$|\mu_w(\text{Pal}(vx))| \geq |\text{Pal}(wv)| + 2. \tag{4.8}$$

Proof. We distinguish two cases: $x \notin \text{Alph}(v), x \in \text{Alph}(v)$. If $x \notin \text{Alph}(v)$, then $\text{Pal}(vx) = \text{Pal}(v)x\text{Pal}(v)$ and the word $u = x\text{Pal}(v)$ contains at least two different letters. On the other hand, if $x \in \text{Alph}(v)$, then $v = v_1xv_2$ where v_2 is x -free, in which case $\text{Pal}(vx) = \text{Pal}(v)\text{Pal}(v_1)^{-1}\text{Pal}(v)$ and the word $u = \text{Pal}(v_1)^{-1}\text{Pal}(v)$ contains at least two different letters. Hence, in either case, $\text{Pal}(vx) = \text{Pal}(v)u$ where u is a word containing at least two different letters. Thus, it follows from Lemma 4.21 that $\mu_w(\text{Pal}(vx)) = \mu_w(\text{Pal}(v))\text{Pal}(w)u_w$ for some word u_w containing at least two different letters. Since $\text{Pal}(wv) = \mu_w(\text{Pal}(v))\text{Pal}(w)$ and $|u_w| \geq 2$, the proof is thus complete. \square

Note. Inequality (4.8) is not true in general. For instance, when $v = \varepsilon$ and $w = xx$, we have $\mu_w(\text{Pal}(vx)) = x$ and $\text{Pal}(wv) = xx$.

Let us come back to our proof of Theorem 4.19. Writing $v = v'x$, we have $|q| \geq |\mu_w(\text{Pal}(v))| > |\text{Pal}(wv')|$. Hence, as an immediate consequence of Corollary 4.22, when v contains at least two different letters, $\text{Pal}(wv)$ is the smallest palindromic prefix of \mathbf{s} of which q is a factor. Therefore, by Theorem 4.7 and Lemma 4.8, the returns to q in \mathbf{t} are the words $f^{-1}\mu_{wv}(\alpha)f$ where $\alpha \in \text{Alph}(\mathbf{y})$. Consequently, each occurrence of q is preceded by the word f . Thus, the set of factors of $f\mathbf{t}$ is exactly the same as the set of factors of \mathbf{t} ; whence, the infinite word $f\mathbf{t}$ (which is clearly recurrent) is episturmian. Moreover, returns to fq in $f\mathbf{t}$ are of the form $\mu_{wv}(\alpha)f$ for letters $\alpha \in \text{Alph}(\mathbf{y})$. Hence, we deduce that there exists an infinite word \mathbf{t}' such that $f\mathbf{t} = \mu_{wv}(\mathbf{t}')$. Moreover, we deduce from the following lemma that \mathbf{t}' is episturmian.

Lemma 4.23. *For any letter α , an infinite word \mathbf{w} is episturmian if and only if $\mu_\alpha(\mathbf{w})$ is episturmian.*

Proof. (\Rightarrow): Immediately follows from Theorem 2.1 (see also Corollary 3.12 in [23] which shows more generally that if \mathbf{w} is episturmian and φ is an episturmian morphism, then $\varphi(\mathbf{w})$ is episturmian).

(\Leftarrow): Conversely, suppose $\mathbf{z} := \mu_\alpha(\mathbf{w})$ is an episturmian word. Then, \mathbf{z} begins with the letter α . Hence, by Proposition 4.9, \mathbf{z} is directed by a spinned infinite word Δ beginning with α , say $\Delta = \alpha\check{y}$ for some spinned infinite word \check{y} . So by Theorem 2.1 and Remark 2.2, $\mathbf{z} = \mu_\alpha(\mathbf{z}')$ where \mathbf{z}' is an episturmian word directed by \check{y} . By the injectivity of μ_α , $\mathbf{z}' = \mathbf{w}$; whence \mathbf{w} is episturmian. \square

Now, since $f = S_{\check{w}}$, we have $\mathbf{t} = \mu_{\check{w}v}(\mathbf{t}')$ by formula (4.4), and so \mathbf{t} is directed by a spinned infinite word beginning with $\check{w}v$.

It remains to consider the case when v is a power of a letter x . In this case, the condition $\text{Alph}(v) = \text{Alph}(v\mathbf{y})$ implies that $\mathbf{y} = x^\omega$; thus \mathbf{s} (and hence \mathbf{t}) is periodic. More precisely, $\mathbf{s} = (\mu_w(x))^\omega$ and $q = f^{-1}\mu_w(x)f$. Since $f = S_{\check{w}}$, it follows from formula (4.3) that $q = \mu_{\check{w}}(x)$ and $\mathbf{t} = q^\omega$ showing that \mathbf{t} is directed by $\check{w}x^\omega$.

The proof of Theorem 4.19 is thus complete, except for the last part concerning the smallest quasiperiod which we prove below.

As previously, let \mathbf{s} be an epistandard word such that \mathbf{t} belongs to the subshift of \mathbf{s} and let Δ be the L -spinned directive word of \mathbf{s} . First of all, we observe that the smallest quasiperiod of \mathbf{t} is of the form $q := \mu_{\check{w}}(\text{Pal}(v))$, where \check{w} is a spinned version of a word w such that $\Delta = wv\mathbf{y}$ and $\text{Alph}(v) = \text{Alph}(v\mathbf{y})$. Assume first that v has a proper suffix s such that $\text{Alph}(s) = \text{Alph}(s\mathbf{y})$, so that writing $v = ps$ for a non-empty word p , $\mu_{\check{w}p}(\text{Pal}(s))$ is also a quasiperiod of \mathbf{t} . By formula (4.1), $\mu_p(\text{Pal}(s))$ is a proper prefix of $\text{Pal}(ps) = \text{Pal}(v)$. Thus $\mu_{\check{w}p}(\text{Pal}(s))$ is a proper prefix of $\mu_{\check{w}}(\text{Pal}(v))$ showing that the second of these words is not the smallest quasiperiod of \mathbf{t} .

From now on, we consider L -spinned words w_1, w_2, v_1, v_2 , spinned versions \check{w}_1 and \check{w}_2 of w_1 and w_2 respectively, spinned words \mathbf{y}_1 and \mathbf{y}_2 such that $\mu_{\check{w}_1}(\text{Pal}(v_1))$ and $\mu_{\check{w}_2}(\text{Pal}(v_2))$ are quasiperiods of \mathbf{t} , $\Delta = w_1v_1\mathbf{y}_1 = w_2v_2\mathbf{y}_2$, $\text{Alph}(v_1) = \text{Alph}(v_1\mathbf{y}_1)$, and $\text{Alph}(v_2) = \text{Alph}(v_2\mathbf{y}_2)$. Moreover we assume that v_1 and v_2 verify the following hypothesis:

(H) for $i = 1, 2$, v_i has no proper suffix s_i with $\text{Alph}(s_i) = \text{Alph}(s_i\mathbf{y}_i)$

Note that $|\mu_{\check{w}_1}(\text{Pal}(v_1))| = |\mu_{w_1}(\text{Pal}(v_1))|$ and $|\mu_{\check{w}_2}(\text{Pal}(v_2))| = |\mu_{w_2}(\text{Pal}(v_2))|$, so that to determine whether $\mu_{\check{w}_1}(\text{Pal}(v_1))$ or $\mu_{\check{w}_2}(\text{Pal}(v_2))$ is the smallest quasiperiod, we just have to determine which of the two words $\mu_{w_1}(\text{Pal}(v_1))$ and $\mu_{w_2}(\text{Pal}(v_2))$ is the shortest word. But, then we can use the fact that w_1v_1 and w_2v_2 are both prefixes of Δ . When $|w_1| = |w_2|$, Hypothesis (H) implies that $w_1 = w_2$ and $v_1 = v_2$.

Before considering the case where $|w_1v_1| < |w_2v_2|$, let us recall that if α is a letter and if x is an α -free word, then $\mu_x(\alpha) = \text{Pal}(x)\alpha$ (see formula (4.1)). Moreover, we observe that for any word x and distinct letters α and β , $\text{Pal}(x\alpha)$ is a prefix of $\mu_x(\text{Pal}(\alpha\beta))$. Indeed, $\mu_x(\text{Pal}(\alpha\beta)) = \mu_x(\mu_\alpha(\beta)\alpha) = \mu_{x\alpha}(\beta\alpha)$ and this word contains $\text{Pal}(x\alpha)$ as a prefix by Lemma 4.21.

Assume $|w_1v_1| < |w_2v_2|$ (the case $|w_1v_1| > |w_2v_2|$ is symmetric) and let us show that $|\mu_{\check{w}_1}(\text{Pal}(v_1))|$ is less than (or equal to) $|\mu_{\check{w}_2}(\text{Pal}(v_2))|$. In this case w_1v_1 is a proper prefix of w_2v_2 . Let x be the non-empty word such that $w_1v_1x = w_2v_2$. Hypothesis (H) implies that v_1 cannot be a factor of v_2 except as a prefix. Thus, there exists a possibly empty word y such that $v_1x = yv_2$. Assume x has length at least 2 and is a suffix of v_2 . Then Hypothesis (H) implies that the two last letters of v_2 and so of x are different. Let α and β be the two last letters of x , and let x' be the word such that $x = x'\alpha\beta$. Then $\text{Pal}(v_1)$ is a prefix of $\text{Pal}(v_1x')$ which is a prefix of $\mu_{v_1x'}(\alpha\beta)$ by Lemma 4.21. Finally we can see that $\mu_{v_1x'}(\alpha\beta)$ is a prefix of $\mu_y(\text{Pal}(v_2))$. This shows that $\text{Pal}(v_1)$ is a proper prefix of $\mu_y(\text{Pal}(v_2))$ and so $\mu_{\check{w}_1}(\text{Pal}(v_1))$ is a shorter quasiperiod than $\mu_{\check{w}_2}(\text{Pal}(v_2))$.

Now consider $|x| = 1$ and $|v_2| \geq 2$. Let α be the last letter of v_1 and let v'_1 and v'_2 be the words such that $v_1 = v'_1\alpha$ and $v_2 = v'_2\alpha x$. Then $\text{Pal}(v_1) = \text{Pal}(v'_1\alpha)$ is a proper prefix of $\mu_{v'_1}(\text{Pal}(\alpha x)) = \mu_{yv'_2}(\text{Pal}(\alpha x))$ (see the paragraph before last, taking $\alpha\beta = \alpha x$), which is a prefix of $\mu_y(\text{Pal}(v_2))$. So again we find that $\mu_{\check{w}_1}(\text{Pal}(v_1))$ is a shorter quasiperiod than $\mu_{\check{w}_2}(\text{Pal}(v_2))$.

Now we come to the case when $v_2 = \alpha$ for a letter α . If x contains a letter different from α , then $\text{Pal}(v_1)$ is a proper prefix of $\mu_{v_1}(\mu_x(\alpha)) = \mu_y(\alpha) = \mu_y(\text{Pal}(v_2))$. We still conclude as previously.

Lastly, assume that x is a power of α . Hypothesis (H) implies that α is the first letter of v_1 and more precisely $v_1 = \alpha v'_1$ with v'_1 an α -free word. Thus $\text{Pal}(v_1) = \mu_\alpha(\text{Pal}(v'_1))\alpha = \mu_\alpha(\text{Pal}(v'_1)\alpha) = \mu_\alpha(\mu_{v'_1}(\alpha)) = \mu_{\alpha v'_1}(\alpha) = \mu_{v_1}(x) = \mu_y(v_2)$. So in this particular case $\mu_{\check{w}_1}(\text{Pal}(v_1)) = \mu_{\check{w}_2}(\text{Pal}(v_2))$, and once again $\mu_{\check{w}_1}(\text{Pal}(v_1))$ is the smallest quasiperiod. \square

Remark 4.24. Theorem 4.19 shows in particular that if an episturmian word \mathbf{t} is directed by a spinned word $\check{\Delta}$ with all spins ultimately L , then \mathbf{t} has infinitely many quasiperiods since there are infinitely many factorizations of $\check{\Delta}$ into the given form (i.e., any positive shift of an epistandard word is quasiperiodic).

Let us demonstrate Theorem 4.19 with some examples. First, we provide an example of an episturmian word having infinitely many quasiperiods, but which is not epistandard and all of its directive words are wavy (recall that a wavy word is a spinned infinite word containing infinitely many L -spinned letters and infinitely many R -spinned letters).

Example 4.25. Consider the episturmian word \mathbf{t} with normalized directive word $\check{\Delta} = abc(\bar{b}a)^\omega$. From Theorem 3.2, we observe that \mathbf{t} has infinitely many directive words:

$$\check{\Delta} = abc(\bar{b}a)^\omega \quad \text{and} \quad \check{\Delta}_i = a\bar{b}\bar{c}(\bar{b}\bar{a})^i ba(\bar{b}a)^\omega \quad \text{for each } i \geq 0,$$

all of which are wavy. Hence, by Theorem 4.19, the set of quasiperiods of \mathbf{t} consists of $\text{Pal}(abc) = abacaba$ and, for each $i \geq 0$, the words:

$$\mu_{\check{w}}(bab) \quad \text{and} \quad \mu_{\check{w}}(bab)a \quad \text{where } \check{w} = a\bar{b}\bar{c}(\bar{b}\bar{a})^i.$$

Note that $\text{Pal}(ba) = bab$ and $S_{\check{w}}^{-1}\text{Pal}(w) = (\text{Pal}(w)a^{-1})^{-1}\text{Pal}(w) = a$.

Next we give some examples of quasiperiodic episturmian words having only finitely many quasiperiods.

Example 4.26. Consider the quasiperiodic episturmian word $\mu_{adb\bar{c}\bar{a}}(\mathbf{ar})$ where \mathbf{r} is the Tribonacci word; it has only two directive words: $adb\bar{c}\bar{a}(ab\bar{c})^\omega$ and $adb\bar{c}\bar{d}(ab\bar{c})^\omega$ (the first one being its normalized directive word). So we see that \mathbf{t} has only one quasiperiod, namely $Pal(adb\bar{c}) = adabadacadabada$. Similarly, the episturmian word $\mu_{cbaa}(\mathbf{cr})$, which is directed by exactly three different spinned infinite words $(cbaa(\bar{a}\bar{b}\bar{c}))^\omega$, $(cbaa(\bar{b}\bar{c}\bar{a}))^\omega$, and $(cbaa(\bar{c}\bar{a}\bar{b}))^\omega$, has only two quasiperiods: $Pal(cba) = cbcacbc$ and $Pal(cbaa) = cbcacbcacbc$.

We can also construct episturmian words having exactly k quasiperiods for any fixed integer $k \geq 1$, as shown by the following example.

Example 4.27. Consider the episturmian word with normalized directive word $\check{\Delta} = abc\bar{a}(b\bar{c}\bar{a})^\omega$. By Theorem 3.2, \mathbf{t} has exactly two directive words: $\check{\Delta}$ and $\bar{a}\bar{b}\bar{c}\bar{a}(b\bar{c}\bar{a})^\omega$. Hence, by Theorem 4.19, \mathbf{t} has only one quasiperiod, namely $Pal(abc)$. Now, to construct an episturmian word having exactly k quasiperiods for a fixed integer $k \geq 1$, we consider $\check{w}\check{\Delta} = \check{w}abc\bar{a}(b\bar{c}\bar{a})^\omega$ where \check{w} is a spinned version of any L -spinned word w with $Alph(w) \cap \{a, b, c\} = \emptyset$ and $|Pal(w)| = k - 1$. Then, by Theorem 4.19, the set of quasiperiods of the episturmian word directed by $\check{w}\check{\Delta}$ consists of the k words:

$$\mu_{\check{w}}(Pal(abc))p \quad \text{where } p \text{ is a prefix of } Pal(w).$$

For example, $d^{k-1}\check{\Delta} = d^{k-1}abc\bar{a}(b\bar{c}\bar{a})^\omega$ directs an episturmian word with exactly k quasiperiods: $\mu_d^{k-1}(Pal(abc))p = d^{k-1}ad^{k-1}bd^{k-1}ad^{k-1}cd^{k-1}ad^{k-1}bd^{k-1}ap$ where p is a prefix of d^{k-1} .

4.7. Characterizations of quasiperiodic episturmian words

From Theorem 4.19, we immediately obtain the following characterization of quasiperiodic episturmian words.

Theorem 4.28. An episturmian word is quasiperiodic if and only if, there exists a spinned word \check{w} , an L -spinned word v and a spinned version $\check{\mathbf{y}}$ of an L -spinned word \mathbf{y} such that \mathbf{t} is directed by $\check{w}v\check{\mathbf{y}}$ with $Alph(v) = Alph(v\check{\mathbf{y}})$.

The above characterization is clearly not useful when one wants to decide whether or not a given episturmian word is quasiperiodic. In this regard, our normalized directive word plays an important role as it provides a more effective way to decide.

Theorem 4.29. An episturmian word \mathbf{t} is quasiperiodic if and only if, the (unique) normalized directive word of \mathbf{t} takes the form $\check{w}av_1\bar{a}v_2\bar{a} \cdots v_k\bar{a}v\check{\mathbf{y}}$ for some L -spinned letter a , spinned word \check{w} , a -free L -spinned words v, v_1, \dots, v_k ($k \geq 0$), and a spinned version $\check{\mathbf{y}}$ of an L -spinned word \mathbf{y} such that $Alph(av) = Alph(av\check{\mathbf{y}})$.

Proof. First, assume that an episturmian word \mathbf{t} is directed by $\check{w}av_1\bar{a}v_2\bar{a} \cdots v_k\bar{a}v\check{\mathbf{y}}$ (as in the hypotheses). Then, by Theorem 3.1, \mathbf{t} is also directed by $\check{w}\bar{a}\bar{v}_1\bar{a}\bar{v}_2\bar{a} \cdots \bar{v}_k\bar{a}v\check{\mathbf{y}}$. By Theorem 4.28, \mathbf{t} is quasiperiodic.

Now, assume that \mathbf{t} is quasiperiodic. By Theorem 4.28, one of its directive words is $\check{w}_1v\check{\mathbf{y}}$ for spinned words $\check{w}_1, \check{\mathbf{y}}$ and an L -spinned word v' with $Alph(v') = Alph(v'\check{\mathbf{y}})$. If v' contains only one letter a , $\check{\mathbf{y}} = a^\omega$ and the normalized directive word of \mathbf{t} ends with ad^ω , then the condition is verified.

Now, assume that v' contains at least two (different) letters: let us write $v' = av$. Without loss of generality, we can assume that $\check{\mathbf{y}}$ and \check{w}_1 are normalized. If \check{w}_1 has no suffix in $\bar{a}\bar{\mathcal{A}}^*$, then

$\check{w}_1av\check{\mathbf{y}}$ is normalized and of the required form (with $k = 0$). Otherwise $\check{w}_1 = \check{w}\bar{a}\bar{v}_1\bar{a}\bar{v}_2\bar{a} \cdots \bar{v}_k$ for some spinned words $\check{w}, \check{\mathbf{y}}$ and some a -free L -spinned words v_1, \dots, v_k ($k \geq 0$) such that \check{w} has no suffix in $\bar{a}\bar{\mathcal{A}}^*$. Then the normalized directive word of \mathbf{t} is $\check{w}av_1\bar{a}v_2\bar{a} \cdots v_k\bar{a}v\check{\mathbf{y}}$. \square

Example 4.30. The episturmian words with directive words $(ab\bar{c})^\omega$, $(ab\bar{c}ab\bar{c})^\omega$ or $(ab\bar{a}ca\bar{b}cb\bar{c})^\omega$ are not quasiperiodic whereas the one with normalized directive word $ab\bar{a}bc\bar{c}(ab\bar{c})^\omega$ is quasiperiodic (it is also directed by $\bar{a}\bar{b}ab\bar{c}\bar{c}(ab\bar{c})^\omega$).

Remark 4.31. It follows from Theorem 4.28 that any quasiperiodic episturmian word has at least two directive words. Indeed, if \mathbf{t} is a quasiperiodic episturmian word, then \mathbf{t} has a directive word of the form $\check{\Delta} = \check{w}v\check{\mathbf{y}}$ where the words $\check{w}, \check{\mathbf{y}}$ are spinned versions of some L -spinned words w, \mathbf{y} and v is an L -spinned word such that $Alph(v) = Alph(v\check{\mathbf{y}})$. If $\check{\mathbf{y}}$ (and hence $\check{\Delta}$) has all spins ultimately L , then item 2 in part iii) of Theorem 3.2 shows that \mathbf{t} also has a wavy directive word. Now, suppose $\check{\mathbf{y}}$ does not have all spins ultimately L . Then $\check{\mathbf{y}}$ must be wavy, and hence contains infinitely many R -spinned letters. Choose x to be the left-most R -spinned letter in $\check{\mathbf{y}}$. Then, $\check{\mathbf{y}}$ begins with $u\bar{x}$ for some L -spinned word u (possibly empty). Hence, since x occurs in v , we see that $\check{\Delta} = \check{w}v\check{\mathbf{y}}$ contains the factor $v\bar{u}\bar{x} = v'xv''u\bar{x}$. Thus, $\check{\Delta}$ does not satisfy the conditions of Theorem 3.7 as it contains a factor in $x\bar{\mathcal{A}}^*\bar{x}$, and therefore \mathbf{t} does not have a unique directive word. Moreover, we easily deduce from Theorems 4.28 and 3.7 that any episturmian word having a unique directive word is necessarily non-quasiperiodic.

In view of the above remark, one might suspect that an episturmian word is non-quasiperiodic if and only if, it has a unique directive word. But this is not true. For example, both $ba(\bar{b}\bar{c}\bar{a})^\omega$ and $\bar{b}\bar{a}\bar{b}(\bar{c}\bar{a}\bar{b})^\omega$ direct the same non-quasiperiodic episturmian word \mathbf{t} by Theorems 3.2 and 4.28. These two spinned infinite words are the only directive words for \mathbf{t} , which might lead one to guess that an episturmian word is non-quasiperiodic if and only if, it has finitely many directive words. But again, this is not true. For example, as stated in Example 4.26, $adb\bar{c}\bar{d}(ab\bar{c})^\omega$ and $adb\bar{c}\bar{d}(ab\bar{c})^\omega$ are the only two directive words of the quasiperiodic episturmian word $\mu_{adb\bar{c}\bar{a}}(\mathbf{ar})$ where \mathbf{r} is the Tribonacci word. Moreover, there exist non-quasiperiodic episturmian words, such as the one directed by $(ab\bar{b}\bar{c})^\omega$, that have infinitely many directive words (the words $(ab\bar{b}\bar{c})^n ab\bar{c}(ab\bar{b}\bar{c})^\omega$ are pairwise different directive words).

Nevertheless, in the Sturmian case we have:

Proposition 4.32. *A Sturmian word is non-quasiperiodic if and only if it has a unique directive word.*

Proof. First, let us suppose by way of contradiction that \mathbf{t} is a non-quasiperiodic Sturmian word, but has more than one directive word. Then, by Proposition 3.8, \mathbf{t} has infinitely many directive words. Moreover, as \mathbf{t} is aperiodic, all of the directive words of \mathbf{t} are spinned versions of the same $\Delta \in \{a, b\}^\omega \setminus (\mathcal{A}^*a^\omega \cup \mathcal{A}^*b^\omega)$ by Fact 3.3 and none of these directive words are regular wavy. Hence, the normalized directive word of \mathbf{s} contains ab or ba . But then \mathbf{t} is quasiperiodic by Theorem 4.28, a contradiction.

Conversely, by Proposition 3.8, if a Sturmian word \mathbf{t} has a unique directive word, then its (normalized) directive word is regular wavy. Such a word clearly does not fulfill Theorem 4.28 and so \mathbf{t} is not quasiperiodic. \square

Let us recall that a Sturmian word is non-quasiperiodic if and only if, it is a Lyndon word [27]. The generalization of this aspect will be discussed in Section 6.

5. Quasiperiodicity and episturmian morphisms

In this section, we draw connections between the results of the previous section and *strongly quasiperiodic* morphisms. As in [29], a morphism f on \mathcal{A} is called *strongly quasiperiodic* (on \mathcal{A}) if for any (possibly non-quasiperiodic) infinite word \mathbf{w} , $f(\mathbf{w})$ is quasiperiodic.

5.1. Strongly quasiperiodic epistandard morphisms

Quasiperiodicity of epistandard words can also be explained by the strong quasiperiodicity of epistandard morphisms.

Theorem 5.1. *Let v be a word over an alphabet \mathcal{A} containing at least two letters. The epistandard morphism μ_v is strongly quasiperiodic if and only if $\text{Alph}(v) = \mathcal{A}$. Moreover $\text{Pal}(v)$ is a quasiperiod of $\mu_v(\mathbf{w})$ for any infinite word \mathbf{w} .*

To prove this result, which is a direct consequence of Lemma 5.2, we need to consider infinite words covered by several words. We say that a set X of words covers an infinite word \mathbf{w} if and only if, there exist two sequences of words $(p_n)_{n \geq 0}$ and $(z_n)_{n \geq 0}$ such that, for all $n \geq 0$, $p_n z_n$ is a prefix of \mathbf{w} , $z_n \in X$, $p_0 = \varepsilon$ and $|p_n| < |p_{n+1}| \leq |p_n z_n|$. The last inequalities mean that p_n is a prefix of p_{n+1} which is a prefix of $p_n z_n$, itself a prefix of \mathbf{w} . Once again, the sequence $(p_n z_n)_{n \geq 0}$ is called a covering sequence of prefixes of the word \mathbf{w} . Observe that a word is covered by X if and only if, it is covered by $X \cup \{\varepsilon\}$.

Lemma 5.2. *For \mathbf{w} an infinite word over \mathcal{A} , y a letter, X a subset of \mathcal{A} and u a word, if \mathbf{w} is covered by $\{u\} \cup \{ux \mid x \in X \cup \{y\}\}$ then $L_y(\mathbf{w})$ is covered by $\{L_y(u)y\} \cup \{L_y(u)yx \mid x \in X \setminus \{y\}\}$.*

Proof. If $y \in X$, then $L_y(\mathbf{w})$ is covered by $\{L_y(u)\} \cup \{L_y(u)y\} \cup \{L_y(u)yx \mid x \in X \setminus \{y\}\}$, and each occurrence of $L_y(u)$ is followed by the letter y . So $L_y(\mathbf{w})$ is covered by $\{L_y(u)y\} \cup \{L_y(u)yx \mid x \in X \setminus \{y\}\}$. If $y \notin X$, thus $X = X \setminus \{y\}$ and then $L_y(\mathbf{w})$ is covered by $\{L_y(u)\} \cup \{L_y(u)yx \mid x \in X\}$, and similarly each occurrence of $L_y(u)$ is followed by the letter y . So $L_y(\mathbf{w})$ is covered by $\{L_y(u)y\} \cup \{L_y(u)yx \mid x \in X\}$. \square

Proof of Theorem 5.1. Let $v = v_1 \cdots v_{|v|}$ be a word (with each v_i a letter) such that $\text{Alph}(v) = \mathcal{A}$. Let $u_{|v|} = \varepsilon$ and, for any i from $|v|$ to 1, let $u_{i-1} = L_{v_i}(u_i)v_i$: observe that $u_{i-1} = \text{Pal}(v_i \cdots v_{|v|})$. It is immediate that any infinite word \mathbf{w} is covered by the set \mathcal{A} , which can be expressed as $\mathcal{A} = \{\varepsilon\} \cup \{\varepsilon x \mid x \in \mathcal{A}\} = \{u_{|v|}\} \cup \{u_{|v|}x \mid x \in \mathcal{A}\}$.

By induction using Lemma 5.2, we can state that, for $1 \leq i \leq |v|$, $\mu_{v_i \cdots v_{|v|}}(\mathbf{w})$ is covered by $\{u_{i-1}\} \cup \{u_{i-1}x \mid x \in \mathcal{A} \setminus \{v_i, \dots, v_{|v|}\}\}$. So, since $u_0 = \text{Pal}(v)$, $\mu_v(\mathbf{w})$ is covered by $\{\text{Pal}(v)\} \cup \{\text{Pal}(v)x \mid x \in \mathcal{A} \setminus \{v_1, \dots, v_{|v|}\}\}$. Therefore, since $\text{Alph}(v) = \mathcal{A}$, $\mu_v(\mathbf{w})$ is covered by $\text{Pal}(v)$, that is, \mathbf{w} is $\text{Pal}(v)$ -quasiperiodic. Hence μ_v is strongly quasiperiodic.

Now let us consider the case where $\text{Alph}(v) \neq \mathcal{A}$. First suppose that $v = \varepsilon$. Then, since \mathcal{A} contains at least two letters (and so there exists at least one non-quasiperiodic word over \mathcal{A}), the morphism μ_v , which is the identity, is not strongly quasiperiodic. Assume now that $v \neq \varepsilon$. Let a be a letter in $\mathcal{A} \setminus \text{Alph}(v)$ and let b be a letter in $\text{Alph}(v)$. The word $\mu_v(ab^\omega)$ contains only one occurrence of the letter a , and so it is non-quasiperiodic. Thus the morphism μ_v is not strongly quasiperiodic. \square

Remark 5.3. Note that any infinite word Δ can be written $\Delta = v\mathbf{y}$ with $\text{Alph}(v) = \text{Alph}(v\mathbf{y})$. So if \mathbf{t} (resp. \mathbf{t}') is the epistandard word directed by Δ (resp. \mathbf{y}), then $\mathbf{t} = \mu_v(\mathbf{t}')$ and \mathbf{t} is quasiperiodic by Theorem 5.1. The same approach allows us to show that if an episturmian word is directed by a spinned infinite word with all spins ultimately L , then it is quasiperiodic, as deduced previously (see Remark 4.12). Note also that Corollary 4.13 is a direct consequence of Theorem 5.1.

5.2. Useful lemmas

We now introduce useful material for a second proof of [Theorem 4.19](#) using morphisms (see [Section 5.3](#)). The next results (that maybe some readers will read when necessary to understand the proof of [Theorem 4.19](#)) show how to obtain some quasiperiods (or a covering set) of a word of the form $L_x(\mathbf{w})$ or $R_x(\mathbf{w})$ for an infinite word \mathbf{w} (not necessarily episturmian).

Lemma 5.4 (Converse of [Lemma 5.2](#)). *For an infinite word \mathbf{w} over \mathcal{A} , y a letter, X a subset of \mathcal{A} and u a word, if $L_y(\mathbf{w})$ is covered by $\{L_y(u)y\} \cup \{L_y(u)yx \mid x \in X \setminus \{y\}\}$, then \mathbf{w} is covered by $\{u\} \cup \{ux \mid x \in X \cup \{y\}\}$.*

Proof. By hypothesis, there exist $(p_n)_{n \geq 0}$, $(u_n)_{n \geq 0}$ such that $p_0 = \varepsilon$, $u_n \in \{L_y(u)y\} \cup \{L_y(u)yx \mid x \in X \setminus \{y\}\}$ and for all $n \geq 0$, both $|p_n| < |p_{n+1}| \leq |p_n u_n|$ and $p_n u_n$ is a prefix of \mathbf{w} .

For all $n \geq 0$, $p_n L_y(u)y$ is a prefix of $L_y(\mathbf{w})$ so there exists p'_n such that $p_n = L_y(p'_n)$.

We consider the following two complementary cases:

1. Case $u_n = L_y(u)y$.

The word $p_n L_y(u)y = L_y(p'_n u)$ is a prefix of $L_y(\mathbf{w})$ so p'_n is a prefix of \mathbf{w} . For the prefixes p_{n+1} and $p_n u_n$ of \mathbf{w} , we have $|p_{n+1}| \leq |p_n u_n|$, thus p_{n+1} is a prefix of $p_n u_n$.

If $|p_{n+1}| < |p_n u_n|$, let $u'_n = u$: $p_{n+1} = L_y(p'_{n+1})$ is a prefix of $p_n L_y(u) = L_y(p'_n u)$. Moreover, $p_{n+1} y$ and $p_n L_y(u)y$ are prefixes of $L_y(\mathbf{w})$. So p'_{n+1} is a prefix of $p'_n u$ which itself is a prefix of \mathbf{w} .

If $|p_{n+1}| = |p_n u_n|$, then $p_{n+1} = p_n u_n$ and from $p_{n+1} u_{n+1}$ prefix of \mathbf{w} , we know that $p_{n+1} y$ is a prefix of \mathbf{w} , hence $p'_n u y = p'_{n+1}$ is a prefix of \mathbf{w} . In this case, let $u'_n = u y$.

2. Case $u_n = L_y(u)yx$ for some $x \in X \setminus \{y\}$. In this case (as previously), we can see that with $u'_n = ux$, p'_{n+1} is a prefix of $p'_n u'_n$ itself a prefix of \mathbf{w} .

We have proved that the sequence $(p'_n u'_n)_{n \geq 0}$ is a covering sequence of prefixes of \mathbf{w} . Thus, the word \mathbf{w} is covered by $\{u\} \cup \{ux \mid x \in X \cup \{y\}\}$. \square

Lemma 5.5 (Generalization of [[29, Lem. 5.5](#)]). *Let \mathbf{w} be an infinite word starting with a letter x . For any letter $y \neq x$, the word $R_y(\mathbf{w})$ is quasiperiodic if and only if, \mathbf{w} is quasiperiodic.*

Moreover, if q is a quasiperiod of $R_y(\mathbf{w})$ then two cases are possible:

1. $q = R_y(q')$ with q' a quasiperiod of \mathbf{w} ;
2. $qy = R_y(q'z)$ with $z \neq a$ a letter such that both q' and $q'z$ are quasiperiods of \mathbf{w} .

Proof. Assume first that \mathbf{w} is quasiperiodic. By [Fact 4.1](#), $R_y(\mathbf{w})$ is quasiperiodic. More precisely, if v is a quasiperiod of \mathbf{w} , then $R_y(v)$ is a quasiperiod of $R_y(\mathbf{w})$.

Assume now that $R_y(\mathbf{w})$ is quasiperiodic and let q be one of its quasiperiods. By hypothesis, we know that q starts with x (as \mathbf{w}).

A first case is that $q = R_y(v)$ (for a word v) which is equivalent to the fact that q ends with the letter y . Let $(p_n)_{n \geq 0}$ be a sequence of words such that $(p_n q)_{n \geq 0}$ is a covering sequence of prefixes of $R_y(\mathbf{w})$. Let $n \geq 0$. From the fact that q starts with x , we deduce that there exists a word p'_n such that $p_n = R_y(p'_n)$. Then we can see that $p'_n v$ is a prefix of \mathbf{w} . Moreover the inequality $|p_n| < |p_{n+1}| \leq |p_n q|$ implies $|p'_n| < |p'_{n+1}| \leq |p'_n v|$. So $(p'_n v)_{n \geq 0}$ is a covering sequence of prefixes of \mathbf{w} : v is a quasiperiod of \mathbf{w} .

Now assume that q ends with a letter z different from y . Then $q = R_y(v)z$ for a word v and each occurrence of q is followed by the letter y . Let $(p_n)_{n \geq 0}$ be a sequence of words such that $(p_n q)_{n \geq 0}$ is a covering sequence of prefixes of $R_y(\mathbf{w})$. Let $n \geq 0$. From the fact that q starts with x , we deduce that there exists a word p'_n such that $p_n = R_y(p'_n)$. Then, we can see that $p'_n v$ is a prefix of \mathbf{w} . Now since q does not end with y , the inequality $|p_n| < |p_{n+1}| \leq |p_n q|$ is actually $|p_n| < |p_{n+1}| \leq |p_n R_y(v)|$. And so $(p'_n v)_{n \geq 0}$ is a covering sequence of prefixes of \mathbf{w} ; thus v is a quasiperiod of \mathbf{w} . Since each occurrence of $R_y(v)$ is followed by y , $(p'_n v z)_{n \geq 0}$ is also a covering sequence of prefixes of \mathbf{w} ; whence vz is also a quasiperiod of \mathbf{w} . \square

Lemma 5.6. *Let x, y be two different letters, let \mathbf{w} be an infinite word, and let u be a finite word such that $L_x(\mathbf{w})$ is uy -quasiperiodic. Then there exists a word v such that \mathbf{w} is vy -quasiperiodic, $uy = L_x(vy)$, and $|uy| > |vy|$.*

Proof. Let x, y, u, \mathbf{w} be as in the lemma. Since uy is a prefix of $L_x(\mathbf{w})$ and since $x \neq y$, there exists a word v such that $uy = L_x(vy)$. We have $|uy| > |vy|$.

Let $(p_n)_{n \geq 0}$ be such that $(p_n uy)_{n \geq 0}$ is a covering sequence of prefixes of $L_x(\mathbf{w})$. Since uy is a prefix of $L_x(\mathbf{w})$, x is the first letter of u , and so from $p_n x$ prefix of $L_x(\mathbf{w})$, there exists for any $n \geq 0$ a word p'_n such that $p_n = L_x(p'_n)$. From the prefix $L_x(p'_n vy)$ of $L_x(\mathbf{w})$, we deduce that $p'_n vy$ is a prefix of \mathbf{w} . The inequality $|p_n| < |p_{n+1}| \leq |p_n uy|$, that is $|L_x(p'_n)| < |L_x(p'_{n+1})| \leq |L_x(p'_n vy)|$, implies that $|p'_n| < |p'_{n+1}| \leq |p'_n vy|$. Hence vy covers \mathbf{w} . \square

Remark 5.7. [Lemmas 5.5](#) and [5.6](#) are not true when $x = y$. For instance ab^ω is not quasiperiodic whereas $R_a(ab^\omega) = a(ba)^\omega$ is quasiperiodic. Moreover the word $L_a(b^\omega) = (ab)^\omega$ is aba -quasiperiodic whereas b^ω has no quasiperiod ending with a .

Lemma 5.8. *Suppose \mathbf{t} is an episturmian word starting with a letter z . Let x be a letter different from z and let u be a non-empty word. Then there exists a set \mathcal{B} of letters such that the word $R_x(\mathbf{t})$ is covered by $\{u\} \cup \{u\mathcal{B}$ if and only if, this word is covered by $\{u, ux\}$.*

Proof. The if part is immediate. Assume that $R_x(\mathbf{t})$ is covered by $\{u\} \cup \{u\}\mathcal{B}$. There is nothing to prove if $\mathcal{B} \subseteq \{x\}$; hence we assume that \mathcal{B} contains a letter y different from x . Since \mathbf{t} starts with z and $z \neq x$, the word $R_x(\mathbf{t})$ and its prefix u start with z . If the word uy is a factor of $R_x(\mathbf{t})$, the word u ends with the letter x . Let $(p_n)_{n \geq 0}$ and $(u_n)_{n \geq 0}$ such that $(p_n u_n)_{n \geq 0}$ is a covering sequence of prefixes of $R_x(\mathbf{t})$ and for all $n \geq 0$, $u_n \in \{u\} \cup \{u\}\mathcal{B}$. Let n be any integer such that $u_n = uy$. Since $y \neq x$ and $z \neq x$, the word yz is not a factor of $R_x(\mathbf{t})$. Since $|p_{n+1}| \leq |p_n u_n|$, since p_{n+1} and $p_n u_n$ are prefixes of $R_x(\mathbf{t})$, and since u_{n+1} starts with z , necessarily $|p_{n+1}| < |p_n u_n| = |p_n u y|$. We deduce from what precedes that $R_x(\mathbf{t})$ is covered by $\{x\} \cup \{u\}(\mathcal{B} \setminus \{y\})$. Hence, considering successively all letters of \mathcal{B} different from x , it follows that $R_x(\mathbf{t})$ is covered by $\{u, ux\}$. \square

Lemma 5.9. *Let \mathbf{t} be an episturmian word starting with a letter z . Let x be a letter different from z and let q be a word. If the word $R_x(\mathbf{t})$ is covered by $\{q, qx\}$ then $L_x(\mathbf{t})$ is covered by qx .*

Proof. If q is empty, the result is immediate since in this case $\mathbf{t} = x^\omega$. If q ends with a letter different from x , then each occurrence of q is followed by an x and so $R_x(\mathbf{t})$ is covered by qx . Hence, we can assume, without loss of generality, that q ends with x . Then $q = R_x(v)$ and $qx = R_x(vx)$ for some word v . By the same technique used in the proof of Lemma 5.5, we can see that \mathbf{t} is covered by $\{v, vx\}$. This implies that $L_x(\mathbf{t})$ is covered by $\{L_x(v), L_x(v)x\}$ with all the occurrences of $L_x(v)$ followed by an x . So $L_x(\mathbf{t})$ is covered by $L_x(v)x$. It is well-known that $L_x(v)x = xR_x(v) = xq$. \square

Lemma 5.10. *Suppose \mathbf{t} is an infinite word (not necessarily episturmian) covered by the set $\{q, qa\}$ and by the set $\{q, qc\}$ for some word q and two different letters a and c . Then \mathbf{t} is q -quasiperiodic.*

Proof. Let $(p_n)_{n \geq 0}$ be prefixes of \mathbf{t} and $(q_n)_{n \geq 0}$ be words belonging to $\{q, qa\}$ such that $(p_n q_n)_{n \geq 0}$ is a covering sequence of \mathbf{t} . Moreover, let us assume that there is no word p except the elements of $(p_n)_{n \geq 0}$ such that pq is a prefix of \mathbf{t} . Let n be any integer such that $q_n = qa$. Since \mathbf{t} is also covered by $\{q, qc\}$ and since all the words p such that pq is a prefix of \mathbf{t} belong to $\{p_n \mid n \geq 0\}$, we necessarily have $|p_{n+1}| \leq |pq|$ (that is $|p_{n+1}| \neq |p_n qc|$). The sequence $(p_n q)$ is a covering sequence of \mathbf{t} ; hence \mathbf{t} is q -quasiperiodic. \square

5.3. A second proof of Theorem 4.19

We now give a second proof of Theorem 4.19 which provides further insight into the connection between quasiperiodicity and morphisms.

For any spinned word \check{w} , let us denote by $q_{\check{w}}$ the word $S_{\check{w}}^{-1}Pal(w)$ appearing in Theorem 4.19. Then, for any spinned word \check{v} and letter a , we have:

- $q_\varepsilon = \varepsilon$;
- $q_{a\check{v}} = L_a(q_{\check{v}})a$;
- $q_{\check{a}v} = R_a(q_{\check{v}})$.

Indeed by formula (4.3), we have $S_{a\check{v}} = L_a(S_{\check{v}})$ and $S_{\check{a}v} = L_a(S_{\check{v}})a = aR_a(S_{\check{v}})$. Hence $q_{a\check{v}} = S_{a\check{v}}^{-1}L_a(Pal(v))a = (L_a(S_{\check{v}}))^{-1}L_a(Pal(v))a = L_a(S_{\check{v}}^{-1}Pal(v))a = L_a(q_{\check{v}})a$ and, since $L_a(Pal(v))a = aR_a(Pal(v))$, $q_{\check{a}v} = S_{\check{a}v}^{-1}(aR_a(Pal(v))) = (aR_a(S_{\check{v}}))^{-1}(aR_a(Pal(\check{v}))) = R_a(S_{\check{v}}^{-1}(Pal(\check{v})))$.

These formulae, which define recursively the word $q_{\check{w}}$, will be helpful in the following proof.

First, assume that \mathbf{t} is an episturmian word directed by $\check{w}v\check{y}$ where \check{w} is a spinned word, \check{y} is a spinned version of an L -spinned infinite word \mathbf{y} and v is an L -spinned word such that $\text{Alph}(v) = \text{Alph}(v\check{y})$. Then, by Theorem 5.1, $Pal(v)$ is a quasiperiod of $\mu_{\check{w}}(\mathbf{t}')$ where \mathbf{t}' is the episturmian word directed by \check{y} . Now we prove by induction on $|\check{w}|$ that $\mu_{\check{w}}(Pal(v))p$ is a quasiperiod of \mathbf{t} for any prefix p of $q_{\check{w}}$. More precisely, we prove that $\mu_{\check{w}}(Pal(v))$ is a quasiperiod of $\mathbf{t} = \mu_{\check{w}}(\mu_{\check{v}}(\mathbf{t}'))$ (this is also a consequence of Fact 4.1) and each occurrence of $\mu_{\check{w}}(Pal(v))$ in \mathbf{t} is followed by the word $q_{\check{w}}$ (which is a direct consequence of the above inductive formulae). Observe that the previous fact obviously holds if $\check{w} = \varepsilon$. When it holds, let $(p_n)_{n \geq 0}$ be a sequence of prefixes such that $|p_n| \leq |p_{n+1}| \leq |p_n \mu_{\check{w}}(Pal(v))|$ and $p_n \mu_{\check{w}}(Pal(v))q_{\check{w}}$ is a prefix of \mathbf{t} . Then $L_a(p_n \mu_{\check{w}}(Pal(v))q_{\check{w}}) = L_a(p_n) \mu_{a\check{w}}(Pal(v))L_a(q_{\check{w}})$ and is followed as a prefix of $L_a(\mathbf{t})$ by a . Note that the word $L_a(p_n) \mu_{a\check{w}}(Pal(v))L_a(q_{\check{w}})a = L_a(p_n) \mu_{a\check{w}}(Pal(v))q_{a\check{w}}$ is a prefix of $L_a(\mathbf{t})$. Moreover, we can verify that $|L_a(p_n)| \leq |L_a(p_{n+1})| \leq |L_a(p_n) \mu_{a\check{w}}(Pal(v))|$ since p_n is a prefix of p_{n+1} itself a prefix of $p_n \mu_{\check{w}}(Pal(v))$. Similarly, $R_a(p_n \mu_{\check{w}}(Pal(v))q_{\check{w}}) = R_a(p_n) \mu_{\check{a}v}(Pal(v))R_a(q_{\check{w}}) = R_a(p_n) \mu_{\check{a}v}(Pal(v))q_{\check{a}v}$ is a prefix of $R_a(\mathbf{t})$ and $|R_a(p_n)| \leq |R_a(p_{n+1})| \leq |R_a(p_n) \mu_{\check{a}v}(Pal(v))|$.

To end the proof of Theorem 4.19, we prove by induction on $|q|$ that if q is the quasiperiod of an episturmian word \mathbf{t} then (at least) one directive word of \mathbf{t} can be written $\Delta = \check{w}v\check{y}$ where \check{w} is a spinned word, \check{y} is a spinned version of an L -spinned infinite word \mathbf{y} and v is an L -spinned word such that $\text{Alph}(v) = \text{Alph}(v\check{y})$. Moreover we have $q = \mu_{\check{w}}(Pal(v))p$ with p a prefix of $q_{\check{w}} = S_{\check{w}}^{-1}Pal(w)$.

We observe that $|q| \geq 1$ since q covers the infinite word \mathbf{t} .

When $|q| = 1$, for a letter a , $q = a$ and the word $\mathbf{t} = a^\omega$ is directed by a^ω which is of the form $\check{w}va^\omega$ with $\check{w} = \varepsilon$ and $v = a$. Then $q_{\check{w}} = \varepsilon$, and taking $p = \varepsilon$, $q = \mu_{\check{w}}(Pal(v))p$.

Assume from now on that $|q| \geq 2$. Without loss of generality, we assume that $\Delta = \check{x}_1 \check{x}_2 \cdots$ is the normalized directive word of \mathbf{t} , and let $(\mathbf{t}^{(n)})_{n \geq 0}$ be the infinite sequence of episturmian words associated with \mathbf{t} and Δ in Theorem 2.1.

Let us first consider:

Case $\mathbf{t} = L_a(\mathbf{t}^{(1)})$ (that is, $\check{x}_1 = a$ for a letter a).

Assume that q does not end with the letter a . By Lemma 5.6, $q = L_a(q_1)$ for a quasiperiod q_1 of $\mathbf{t}^{(1)}$ such that $|q_1| < |q|$. By the induction hypothesis, $\mathbf{t}^{(1)}$ has a directive word of the form $\check{w}v\check{\mathbf{y}}$ with $\check{w}, v, \check{\mathbf{y}}$ as in the theorem, and $q_1 = \mu_{\check{w}}(Pal(v))p$ for a prefix p of $q_{\check{w}}$. We have $q = L_a(\mu_{\check{w}}(Pal(v))p) = \mu_{a\check{w}}(Pal(v))L_a(p)$. Observing that \mathbf{t} is directed by $(a\check{w})v\check{\mathbf{y}}$ and that $L_a(p)$ is a prefix of $L_a(q_{\check{w}})a = q_{a\check{w}}$, we get the result for q and \mathbf{t} .

Now assume that q ends with the letter a , that is $q = L_a(q_1)a$ for a word q_1 . By Lemma 5.4, $\mathbf{t}^{(1)}$ is covered by the set $\{q_1, q_1a\}$.

Possibly q_1a is a quasiperiod of $\mathbf{t}^{(1)}$. In this case, $q = L_a(q_1a)$. Observe that $|q| = |L_a(q_1a)| \geq |q_1a|$. The equality holds only when a is the only letter occurring in q_1 so in q . In this case, taking $\check{w} = a^{|q_1|-1}$ and $v = a$, the word \mathbf{t} is directed by $\check{w}v\check{\mathbf{a}}$ and we have $q = a^{|q_1|-1} = Pal(a)q_{a^{|q_1|-1}} = \mu_{a^{|q_1|-1}}(Pal(a))q_{\check{w}} = \mu_{\check{w}}(Pal(v))q_{\check{w}}$: the induction result holds here. When $|q| = |L_a(q_1a)| > |q_1a|$, the proof ends, using the induction hypothesis, as in the previous case where q did not end by a (the only difference is that q ends with the word $L_a(p)a$ but $L_a(p)a$ is still a prefix of $q_{a\check{w}}$).

It is also possible that q_1 is a quasiperiod of $\mathbf{t}^{(1)}$, but in this case we can once again conclude using the induction hypothesis.

Now we assume that neither q_1 nor q_1a is a quasiperiod of $\mathbf{t}^{(1)}$. This implies in particular that $\mathbf{t}^{(1)}$ has a factor q_1b for a letter b different from a . So the following property holds for $n = 1$ (and $a_1 = a$):

Prop(n): $\mathbf{t} = L_{a_1}L_{a_2} \cdots L_{a_n}(\mathbf{t}^{(n)})$ (that is, $\check{x}_i = a_i$, for $1 \leq i \leq n$), $\mathbf{t}^{(n)}$ is covered by the set $\{q_n\} \cup \{q_n a_i \mid i = 1, \dots, n\}$ with $q_0 = q$, and for $i = 1, \dots, n$, $q_{i-1} = L_{a_i}(q_i)a_i$. Moreover $q_n a$ and $q_n b$ are both factors of $\mathbf{t}^{(n)}$.

Let us assume that Property *Prop(n)* holds for some $n \geq 1$ with $q_n \neq \varepsilon$ and $\mathbf{t}^{(n)} = L_{a_{n+1}}(\mathbf{t}^{(n+1)})$ (that is $\check{x}_{n+1} = a_{n+1}$). From $a \neq b$ (so $a \neq a_{n+1}$ or $b \neq a_{n+1}$), we deduce that $q_n c$ is a factor of $\mathbf{t}^{(n)}$ for a letter $c \neq a_{n+1}$ and so q_n must end with the letter a_{n+1} , that is, $q_n = L_{a_{n+1}}(q_{n+1})a_{n+1}$ for some word q_{n+1} . By Lemma 5.4, q_{n+1} is covered by the set $\{q_{n+1}\} \cup \{q_{n+1}a_i \mid i = 1, \dots, n, n+1\}$. Moreover, it is quite immediate that $q_{n+1}a$ and $q_{n+1}b$ are both factors of $\mathbf{t}^{(n+1)}$. Hence *Prop(n+1)* holds.

Observing that for any $i \geq 1$ we have $|q_{i+1}| < |q_i|$, we deduce that one of the two following cases holds:

1. *There exists an integer $n \geq 1$ such that Property Prop(n) holds with $q_n = \varepsilon$.* In this case, we verify that $q = Pal(a_1 \cdots a_n)$. Let $\check{w} = \varepsilon, v = a_1 \cdots a_n$ and let $\check{\mathbf{y}}$ be any directive word of the episturmian word $\mathbf{t}^{(n)}$. Then, \mathbf{t} is directed by $\check{w}v\check{\mathbf{y}}$ and $q = \mu_{\check{w}}(Pal(v))$. Since $\mathbf{t}^{(n)}$ is covered by the letters of v , we deduce that $Alph(v) = Alph(v\check{\mathbf{y}})$. Hence, the induction result holds.
2. *There exists an integer $n \geq 1$ and a letter c such that Property Prop(n) holds with $q_n \neq \varepsilon$ and $\mathbf{t}^{(n)} = R_c(\mathbf{t}^{(n+1)})$ (that is $\check{x}_{n+1} = \bar{c}$).* By Theorem 3.5, since Δ is normalized, the word $\mathbf{t}^{(n+1)}$ does not start with the letter c (otherwise the infinite word $\check{x}_{n+1}\check{x}_{n+2} \cdots$ starts with a factor in $\bar{c}\bar{A}c$). Hence, the first letter of q_n is not c whereas its last letter is c since $a \neq b$ and both $q_n a$ and $q_n b$ are factors of $R_c(\mathbf{t}^{(n+1)})$. By Lemma 5.8, we deduce that $R_c(\mathbf{t}^{(n+1)})$ is covered by $\{q_n, q_n c\}$. Since for $i = 1, \dots, n$, $q_{i-1} = L_{a_i}(q_i)a_i$, we can inductively verify that the words $\mathbf{t}^{(i)}$ ($i = n, \dots, 1$) are covered by $\{q_i, q_i c\}$. In particular $\mathbf{t}^{(1)}$ is covered by $\{q_1, q_1 c\}$. By Lemma 5.10, since we have assumed that $\mathbf{t}^{(1)}$ is not q_1 -quasiperiodic, we have $c = a$. So $R_a(\mathbf{t}^{(n+1)})$ is covered by $\{q_n, q_n a\}$. By Lemma 5.9, $L_a(\mathbf{t}^{(n+1)})$ is aq_n -quasiperiodic. Observe that $\mathbf{t} = L_{a_1}L_{a_2} \cdots L_{a_n}R_{a_1}(\mathbf{t}^{(n+1)}) = R_{a_1}R_{a_2} \cdots R_{a_n}L_a(\mathbf{t}^{(n+1)})$, and that $|q_n| < |q|$. By induction hypothesis, $L_a(\mathbf{t}^{(n+1)})$ is directed by a word $\check{w}'v\check{\mathbf{y}}$ for a finite spinned word \check{w}' , a finite L -spinned word v and a spinned version $\check{\mathbf{y}}$ of an infinite L -spinned word \mathbf{y} such that $Alph(v) = Alph(v\mathbf{y})$, and $aq_n = \mu_{\check{w}'}(Pal(v))p'$ with p' a prefix of $q_{\check{w}'}$. Thus, we have proved that the word \mathbf{t} is directed by $\check{w}v\check{\mathbf{y}}$ where $\check{w} = \bar{a}_1\bar{a}_2 \cdots \bar{a}_n\check{w}'$, and $\mu_{\bar{a}_1\bar{a}_2 \cdots \bar{a}_n}(qa_n) = \mu_{\check{w}}(Pal(v))p'$. To end the proof of the current case we have to prove that $q = \mu_{\check{w}}(Pal(v))p'$. First, assume that $a_i = a$ for some i between 2 and n . We know that the word $\mathbf{t}^{(i)}$ is covered by $\{q_i, q_i a\}$ and $\mathbf{t}^{(i-1)} = L_{a_i}(\mathbf{t}^{(i)}) = L_a(\mathbf{t}^{(i)})$. Hence, by Lemma 5.2, $\mathbf{t}^{(i-1)}$ is covered by $L_a(q_i)a = q_{i-1}$. Using Fact 4.1, we can deduce that $\mathbf{t}^{(1)}$ is $q_1 a$ -quasiperiodic, a contradiction. Hence for $i = 2, \dots, n$, $a_i \neq a$. By induction we can prove that $\mu_{\bar{a}_1\bar{a}_{i+1} \cdots \bar{a}_n}(aq_n) = aq_{i-1}$ for $i = n+1, \dots, 2$. Indeed if $\mu_{\bar{a}_{i+1} \cdots \bar{a}_n}(aq_n) = aq_i$ then $\mu_{\bar{a}_1\bar{a}_{i+1} \cdots \bar{a}_n}(aq_n) = \mu_{\bar{a}_1}(aq_i) = R_{a_i}(aq_i) = aR_{a_i}(q_i) = aL_{a_i}(q_i)a_i = aq_{i-1}$. Now $\mu_{\bar{a}_1\bar{a}_2 \cdots \bar{a}_n}(aq_n) = R_{a_1}(aq_1) = aR_{a_1}(q_1) = L_a(q_1)a = q$.

Now we come to:

Case $\mathbf{t} = R_a(\mathbf{t}^{(1)})$. By Theorem 3.5, since Δ is the normalized directive word of \mathbf{t} , the word \mathbf{t} does not start with the letter a . By Lemma 5.5, the word $\mathbf{t}^{(1)}$ is quasiperiodic and more precisely there exists a quasiperiod q_1 of $\mathbf{t}^{(1)}$ such that $q = R_a(q_1)$ or $q = R_a(q_1)z$ with $z \neq a$: in this last case, $q_1 z$ is also a quasiperiod of $\mathbf{t}^{(1)}$. Since q does not start with a , $|q_1| < |q|$. By induction hypothesis, $\mathbf{t}^{(1)}$ has a directed word of the form $\check{w}v\check{\mathbf{y}}$ with $\check{w}, v, \check{\mathbf{y}}$ as in the theorem, and $q_1 = \mu_{\check{w}}(Pal(v))p$ for a prefix p of $q_{\check{w}}$. When $q = R_a(q_1)$, since $R_a(\mu_{\check{w}}) = \mu_{\bar{a}\check{w}}$ and $R_a(p)$ is a prefix of $R_a(q_{\check{w}}) = q_{\bar{a}\check{w}}$, $q = \mu_{\bar{a}\check{w}}(Pal(v))R_a(p)$ and \mathbf{t} (which is directed by $\bar{a}\check{w}v\check{\mathbf{y}}$) verify the induction result. When $q = R_a(q_1)z$ with $z \neq a$, since $q_1 z$ is also a quasiperiod of $\mathbf{t}^{(1)}$, we deduce that $p z$ is a prefix of $q_{\check{w}}$ and so $R_a(p)z$ is a prefix of $R_a(p)z a$ a prefix of $q_{\bar{a}\check{w}}$, which allows us to conclude once again that $q = \mu_{\bar{a}\check{w}}(Pal(v))R_a(p)z$ and \mathbf{t} (which is directed by $\bar{a}\check{w}v\check{\mathbf{y}}$) verify the induction result.

This ends the second proof of Theorem 4.19.

5.4. Strongly quasiperiodic episturmian morphisms

The aim of this section is to characterize all the episturmian morphisms that are strongly quasiperiodic, i.e., the episturmian morphisms that map any infinite word onto a quasiperiodic word. Our characterization (Theorem 5.14) generalizes Theorem 5.1 to all episturmian morphisms.

Looking at [Theorems 5.1](#) and [4.19](#), one might guess that an episturmian morphism $\mu_{\check{u}}$ is strongly quasiperiodic if and only if, \check{u} has an L -spinned factor v . We will see that it is not the case. In particular, there exist R -spinned words \check{u} such that $\mu_{\check{u}}$ is strongly quasiperiodic, as shown by the following result.

Lemma 5.11. *Let \check{u} be a finite spinned word over $\mathcal{A} \cup \bar{\mathcal{A}}$ and let a be a letter in \mathcal{A} . If there exist L -spinned words v, y and a spinned word \check{w} such that $\check{u} = \check{w}\bar{a}\check{v}\check{y}$ and $\text{Alph}(v) = \mathcal{A} \setminus \{a\}$, then for any infinite word \mathbf{t} , $\mu_{\check{u}}(\mathbf{t})$ is quasiperiodic.*

Let us mention that for instance the word $\bar{a}\bar{b}\bar{c}\bar{a}\bar{b}$ verifies the condition of the previous lemma for each of the letters a, b and c , so that $\mu_{\bar{a}\bar{b}\bar{c}\bar{a}\bar{b}}$ is strongly quasiperiodic over $\{a, b, c\}$.

[Lemma 5.11](#) is a consequence of the following one.

Lemma 5.12. *Suppose a is an L -spinned letter, x is an L -spinned word, v is an a -free L -spinned word, and \mathbf{w} is an infinite word over \mathcal{A} . Then the words $\mu_{av}(xa)$, $\mu_{\bar{a}\bar{v}}(ax)$, and $\mu_{\bar{a}\bar{v}}(\mathbf{aw})$ are $\text{Pal}(av)$ -quasiperiodic.*

Proof. (1) By [Theorem 5.1](#), the word $\mu_{av}(xa^\omega)$ is $\text{Pal}(av)$ -quasiperiodic where $\text{Pal}(av) = \mu_{av}(a)$ since $\text{Pal}(ava) = \mu_{av}(a)\text{Pal}(av) = \text{Pal}(av)\text{Pal}(av)$ by formulae (4.1) and (4.5). Consequently, since $\mu_{av}(xa)$ ends with $\mu_{av}(a)$, it is $\text{Pal}(av)$ -quasiperiodic.

(2) By formulae (4.3) and (4.2), $\mu_{\bar{a}\bar{v}}(xa) = S_{\bar{a}\bar{v}}^{-1}\mu_{av}(xa)S_{\bar{a}\bar{v}}$ and $S_{\bar{a}\bar{v}} = \text{Pal}(av)$. So $\text{Pal}(av)\mu_{\bar{a}\bar{v}}(ax) = \mu_{av}(ax)\mu_{av}(a) = \mu_{av}(axa) = \mu_{av}(a)\mu_{av}(xa) = \text{Pal}(av)\mu_{av}(xa)$, that is $\mu_{\bar{a}\bar{v}}(ax) = \mu_{av}(xa)$. It follows from (1) that $\mu_{\bar{a}\bar{v}}(ax)$ is $\text{Pal}(av)$ -quasiperiodic.

(3) From (2), we deduce that $\mu_{\bar{a}\bar{v}}(\mathbf{aw})$ has infinitely many quasiperiodic prefixes with quasiperiod $\text{Pal}(av)$. Hence $\mu_{\bar{a}\bar{v}}(\mathbf{aw})$ is $\text{Pal}(av)$ -quasiperiodic. \square

Proof of Lemma 5.11. Let $\check{u}, a, \check{w}, v, y$ be as in the hypotheses of [Lemma 5.11](#). Let \mathbf{t} be an infinite word over \mathcal{A} . The word $\mu_{\check{y}}(\mathbf{t})$ starts with a . By [Lemma 5.12](#), $\mu_{\bar{a}\bar{v}}(\mu_{\check{y}}(\mathbf{t}))$ is quasiperiodic and so $\mu_{\check{u}}(\mathbf{t}) = \mu_{\check{w}\bar{a}\check{v}\check{y}}(\mathbf{t})$ is quasiperiodic. \square

The following remark will be useful several times (see the proof of [Theorem 4.29](#) for more details):

Remark 5.13. An infinite word over $\mathcal{A} \cup \bar{\mathcal{A}}$ has a decomposition $\check{w}v\check{y}$ with $\text{Alph}(v) = \text{Alph}(v\check{y})$ if and only if, its normalized decomposition can be written in the form $\check{w}'av_1\bar{a}v_2 \cdots \bar{a}v_k\check{y}'$ with $\text{Alph}(av_i) = \text{Alph}(av_1av_{i+1} \cdots av_k\check{y}')$ for some $1 \leq i \leq k$.

Now we state our characterization of strongly quasiperiodic episturmian morphisms.

Theorem 5.14. *Let \mathcal{A} be an alphabet containing at least three letters. An episturmian morphism is strongly quasiperiodic on \mathcal{A} if and only if, its normalized directive word \check{u} verifies one of the following three conditions:*

- (i) $\check{u} = \check{w}av_1\bar{a}v_2\bar{a} \cdots v_k\bar{a}v\check{y}$ where a is a letter of \mathcal{A} (with spin L), \check{w}, \check{y} are spinned words, v_1, \dots, v_k ($k \geq 0$) are a -free L -spinned words, and v is an L -spinned word such that $\text{Alph}(av) = \text{Alph}(av\check{y})$.
- (ii) For any letter a in \mathcal{A} , $\check{u} = \check{w}\bar{a}\check{v}\check{y}$ for some spinned word \check{w} and L -spinned words v, y such that $\text{Alph}(v) = \mathcal{A} \setminus \{a\}$.
- (iii) \check{u} verifies case (ii) for all letters in \mathcal{A} except for one letter $a \in \mathcal{A}$ such that $\check{u} = \check{w}v\bar{a}\check{y}$ where v and y are L -spinned words verifying $\text{Alph}(v) = \mathcal{A} \setminus \{a\}$.

Before proving this theorem, let us observe that we do not include in this result the Sturmian case. Indeed, [Theorem 5.14](#) is no longer valid when $n \leq 2$. For instance the morphism $\mu_{\bar{a}\bar{b}}(a \mapsto aba, b \mapsto ba)$ is strongly quasiperiodic but its normalized directive word fulfills none of the above conditions. A complete description of strongly quasiperiodic Sturmian morphisms is provided in [\[29\]](#).

Proof. Let \check{u} be the normalized directive word (of the morphism $\mu_{\check{u}}$).

If \check{u} verifies (i), then by [Theorem 3.1](#), $av_1\bar{a}v_2\bar{a} \cdots v_k\bar{a}v \equiv \bar{a}\bar{v}_1\bar{a}\bar{v}_2\bar{a} \cdots \bar{v}_kav$. Moreover, from [Theorem 5.1](#), the epistandard morphism μ_{av} is strongly quasiperiodic. Thus $\mu_{\check{u}}$ is strongly quasiperiodic on \mathcal{A} .

If \check{u} verifies (ii), then by [Lemma 5.11](#), $\mu_{\check{u}}$ is strongly quasiperiodic over \mathcal{A} .

If \check{u} verifies (iii), then [Lemma 5.11](#), $\mu_{\check{u}}(\mathbf{t})$ is quasiperiodic for any word \mathbf{t} that does not start with a .

Let us decompose $y = y_0ay_1 \cdots ay_k$ where k is the number of occurrences of a in y . In the proof of [Lemma 5.12](#), we have seen that $\mu_{\bar{a}\bar{y}_k}(ax) = \mu_{ay_k}(xa)$ for any finite word x . This formula naturally extends to any infinite word \mathbf{t} , $\mu_{\bar{a}\bar{y}_k}(\mathbf{t}) = \mu_{ay_k}(\mathbf{t})$. Hence, $\mu_{\check{u}}(\mathbf{t}) = \mu_{\check{w}\bar{a}\bar{y}_0\bar{a}\bar{y}_1 \cdots \bar{y}_{k-1}ay_k}(\mathbf{t})$. By [Theorem 3.1](#), $\mu_{\bar{a}\bar{y}_i a} = \mu_{ay_i\bar{a}}$ for each i , so that $\mu_{\check{u}}(\mathbf{t}) = \mu_{\check{w}vay_0\bar{a}\bar{y}_1 \cdots \bar{y}_{k-1}\bar{a}y_k}(\mathbf{t})$. From [Theorem 5.1](#), the epistandard morphism μ_{va} is strongly quasiperiodic. Thus $\mu_{\check{u}}(\mathbf{t})$ is quasiperiodic. Consequently, $\mu_{\check{u}}$ is strongly quasiperiodic on \mathcal{A} .

To end, we prove that if \check{u} verifies none of the conditions (i)–(iii), then there exists (at least) one word \mathbf{t} such that $\mu_{\check{u}}(\mathbf{t})$ is not quasiperiodic (and so $\mu_{\check{u}}$ is not strongly quasiperiodic). This is immediate if $\check{u} = \varepsilon$.

- Let us first consider the case where \check{u} ends with an L -spinned letter a , that is $\check{u} = \check{w}a$ for some spinned word \check{w} . Since $|\mathcal{A}| \geq 2$, there exist $m \geq 2$ pairwise different letters $a_1 = a, a_2, \dots, a_m$ such that $\mathcal{A} = \{a_1, a_2, \dots, a_m\}$. Let \mathbf{t} be the episturmian word with normalized directive word $(\bar{a}_2 \cdots \bar{a}_m a)^\omega$. Since $\check{w}a$ is normalized, $\check{w}a(\bar{a}_2 \cdots \bar{a}_m a)^\omega$ is also normalized. Moreover, since \check{u} does not verify condition (i), the word $\check{w}a(\bar{a}_2 \cdots \bar{a}_m a)^\omega$ cannot be decomposed into the form $\check{w}'bv_1\bar{b}v_2 \cdots \bar{b}v_k\check{y}' \cdots$ where b is an L -spinned letter, \check{w}' is a spinned word, \check{y}' is a spinned version of an L -spinned word y' , and v_1, \dots, v_k ($k \geq 0$) are b -free L -spinned words such that $\text{Alph}(bv') = \text{Alph}(bv'y')$. By [Remark 5.13](#) and [Theorem 4.29](#), the word $\mu_{\check{u}}(\mathbf{t})$ is not quasiperiodic.

- Now, we consider the case when \check{u} ends with an R -spinned letter, that is $\check{u} = \check{w}\bar{v}$ for a non-empty L -spinned word v and a spinned word \check{w} such that $\check{w} = \varepsilon$ or \check{w} ends with an L -spinned letter. Two cases can hold:

Case 1: $\mathcal{A} \neq \text{Alph}(v)$.

Let a be a letter in $\mathcal{A} \setminus \text{Alph}(v)$, and let b be any other letter (remember $|\mathcal{A}| \geq 2$). Let \mathbf{t} be the episturmian word with normalized directive word $(\bar{a}\bar{b})^\omega$ and let $\mathbf{t}' = \mu_{\check{u}}(\mathbf{t})$. Then, \mathbf{t}' is directed by $\check{\mathbf{u}}' = \check{w}\bar{v}(\bar{a}\bar{b})^\omega$ and, since $\check{w}\bar{v}$ is normalized and $\bar{a} \notin \text{Alph}(\bar{v})$, $\check{\mathbf{u}}'$ is normalized. Moreover, since $\check{w}\bar{v}$ does not verify (i), $\check{\mathbf{u}}'$ cannot be decomposed into the form $\check{w}'c v_1 \bar{c} v_2 \cdots \bar{c} v_k \check{y}' \cdots$ with an L -spinned letter c , a spinned word \check{w}' , a spinned version \check{y}' of an L -spinned word y' and some c -free L -spinned words v_1, \dots, v_k ($k \geq 0$) such that $\text{Alph}(c v_k) = \text{Alph}(c v_k y')$. By Remark 5.13 and Theorem 4.29, the word $\mu_{\check{u}}(\mathbf{t})$ is not quasiperiodic.

Case 2: $\mathcal{A} = \text{Alph}(v)$.

Since \check{u} does not verify (ii), there exists a letter a and a -free L -spinned words $v_0, \dots, v_k \in \mathcal{A}^*$ such that $\check{u} = \check{w}\bar{v}_0 \bar{a} \bar{v}_1 \cdots \bar{a} \bar{v}_k$ and for all $i, 1 \leq i \leq k$, $\text{Alph}(v_i) \neq \mathcal{A} \setminus \{a\}$. Moreover, since \check{u} does not verify (iii), then either $v_0 \neq \varepsilon$, or \check{w} cannot be written in the form $\check{w} = \check{w}'v'$ for a spinned word \check{w} and an L -spinned word v' such that $\text{Alph}(v') = \mathcal{A} \setminus \{a\}$.

Since \mathcal{A} contains at least three letters, there exist $m \geq 3$ pairwise different letters $a_1 = a, a_2, \dots, a_m$ such that $\mathcal{A} = \{a_1, a_2, \dots, a_m\}$.

Let \mathbf{t} be the episturmian word with normalized directive word $(aa_2 \bar{a}_3 \cdots \bar{a}_m)^\omega$ and let $\mathbf{t}' = \mu_{\check{u}}(\mathbf{t})$. Then \mathbf{t}' is directed by $\check{w}\bar{v}_0 \bar{a} \bar{v}_1 \cdots \bar{a} \bar{v}_k aa_2 \bar{a}_3 \cdots \bar{a}_m (aa_2 \cdots \bar{a}_m)^\omega \equiv \check{w}\bar{v}_0 a v_1 \bar{a} \cdots \bar{a} v_k \bar{a} a_2 \bar{a}_3 \cdots \bar{a}_m (aa_2 \bar{a}_3 \cdots \bar{a}_m)^\omega$. Since \check{w} ends with an L -spinned letter and each v_i ($0 \leq i \leq k$) is a -free, this word is normalized.

Since $\check{w}\bar{v}$ does not verify (i) and from the previous observations, the word $\check{w}\bar{v}_0 a v_1 \bar{a} \cdots \bar{a} v_k \bar{a} a_2 \bar{a}_3 \cdots \bar{a}_m (aa_2 \cdots \bar{a}_m)^\omega$ cannot be decomposed into the form $\check{w}'b v'_1 \bar{b} v'_2 \cdots \bar{b} v'_k \check{y}' \cdots$ with an L -spinned letter b , a spinned word \check{w}' , a spinned version \check{y}' of an L -spinned word y' and some b -free L -spinned words v'_1, \dots, v'_k ($k \geq 0$) such that with $\text{Alph}(b v'_k) = \text{Alph}(b v'_k \check{y}')$. By Remark 5.13 and Theorem 4.29, the word $\mu_{\check{u}}(\mathbf{t})$ is not quasiperiodic. \square

6. Episturmian Lyndon words

Theorem 4.28 provides a characterization of quasiperiodic episturmian words. In the binary case, it was proved in [29] that a Sturmian word is quasiperiodic if and only if, it is not an infinite Lyndon word. A natural question to ask is then: “does this result still hold for episturmian words on a larger alphabet?” By a result in [29], one can see that any infinite Lyndon word is non-quasiperiodic. In this section, we show that there is a much wider class of episturmian words that are non-quasiperiodic, besides those that are infinite Lyndon words. This follows from our characterization of episturmian Lyndon words (Theorem 6.1, to follow).

Let us first recall the notion of lexicographic order and the definition of Lyndon words (see [30] for instance).

Suppose the alphabet \mathcal{A} is totally ordered by the relation $<$. Then we can totally order \mathcal{A}^* by the *lexicographic order* \leq defined as follows. Given two words $u, v \in \mathcal{A}^*$, we have $u \leq v$ if and only if either u is a prefix of v or $u = xau'$ and $v = xbv'$, for some $x, u', v' \in \mathcal{A}^*$ and letters a, b with $a < b$. This is the usual alphabetic ordering in a dictionary. We write $u < v$ when $u \leq v$ and $u \neq v$, in which case we say that u is (strictly) *lexicographically smaller* than v . The notion of lexicographic order naturally extends to infinite words in \mathcal{A}^ω . We denote by $\min(\mathcal{A})$ the smallest letter with respect to the lexicographic order.

A non-empty finite word w over \mathcal{A} is a *Lyndon word* if it is lexicographically smaller than all of its proper suffixes for the given order $<$ on \mathcal{A} . Equivalently, w is the lexicographically smallest primitive word in its conjugacy class; that is, $w < vu$ for all non-empty words u, v such that $w = uv$. The first of these definitions extends to infinite words: an infinite word over \mathcal{A} is an *infinite Lyndon word* if and only if it is (strictly) lexicographically smaller than all of its proper suffixes for the given order on \mathcal{A} . That is, a finite or infinite word w is a Lyndon word if and only if $w < T^i(w)$ for all $i > 0$.

In this section, we assume that $|\mathcal{A}| > 1$ since on a 1-letter alphabet there are no infinite Lyndon words. Also note that an infinite Lyndon word cannot be periodic. Therefore we consider only aperiodic episturmian words (i.e., those with $|\text{Ult}(\Delta)| > 1$).

6.1. A complete characterization

In this section, generalizing previous results in [29] (Sturmian case) and [15] (Arnoux–Rauzy sequences or strict episturmian words), we prove:

Theorem 6.1. *Let $\mathcal{A} = \{a_1, \dots, a_m\}$ be an alphabet ordered by $a_1 < a_2 < \dots < a_m$ and, for $1 \leq i \leq m$, let $\mathcal{B}_i = \{a_i, \dots, a_m\}$. An episturmian word \mathbf{w} is an infinite Lyndon word if and only if, there exists an integer j such that $1 \leq j < m$ and the (normalized) directive word of \mathbf{w} belongs to:*

$$(\bar{\mathcal{B}}_2^* a_1)^* \cdots (\bar{\mathcal{B}}_j^* a_{j-1})^* (\bar{\mathcal{B}}_{j+1}^* a_j)^* (\bar{\mathcal{B}}_{j+1}^+ \{a_j\}^+)^{\omega}.$$

Note. In the above theorem, we have put the word *normalized* between brackets since one can easily verify from Theorem 3.7 that a spinned infinite word of the given form is the unique directive word of exactly one episturmian word.

Example 6.2. Let $\mathcal{A} = \{a, b, c, d\}$. Then, the word $(\bar{b}\bar{c}a)(\bar{d}\bar{c}b)^2(\bar{d}cc)^\omega$ directs a Lyndon episturmian word, so does $aa(\bar{d}c)^\omega$, but $\bar{c}ab\bar{a}\bar{d}c\bar{d}^\omega$ does not (this spinned word directs a periodic word).

Remark 6.3. The “if and only if” condition can be reformulated as follows:

The (normalized) directive word of \mathbf{w} takes the form $v_1 \cdots v_j \mathbf{y}$ where:

- for $1 \leq k \leq j$, v_k is a spinned word in $(\tilde{\mathcal{B}}_{k+1}^* a_k)^*$;
- \mathbf{y} is a spinned infinite word belonging to $(\tilde{\mathcal{B}}_{j+1}^+ \{a_j\}^+)^{\omega}$.

Remark 6.4. Theorems 3.7 and 6.1 show that any episturmian Lyndon word has a unique spinned directive word, but the converse is not true. Certainly, there exist episturmian words with a unique directive word which are not infinite Lyndon words. For example, the regular wavy word $(abc)^{\omega}$ is the unique directive word of the strict episturmian word:

$$\lim_{n \rightarrow \infty} \mu_{abc}^n(a) = acabaabacabacabaabaca \cdots$$

which is clearly not an infinite Lyndon word by Theorem 6.1 and also by the fact that $acabaaw$ is not a Lyndon word for any order on $\{a, b, c\}$ and for any word w .

In order to prove the Theorem 6.1, we recall two useful results and state a new one (Lemma 6.7).

Lemma 6.5. [34] An infinite word is a Lyndon word if and only if it has infinitely many different Lyndon words as prefixes.

A morphism f is said to preserve finite (resp. infinite) Lyndon words if for each finite (resp. infinite) Lyndon word w , $f(w)$ is a finite (resp. infinite) Lyndon word. For episturmian morphisms, we have:

Proposition 6.6 ([42,45]). Let $\mathcal{A} = \{a_1, \dots, a_m\}$ be an alphabet ordered by $a_1 < a_2 < \cdots < a_m$. Then the following assertions are equivalent for an episturmian morphism:

- f preserves finite Lyndon words;
- f preserves infinite Lyndon words;
- $f \in (\mathcal{R}_{\{a_2, \dots, a_m\}}^* L_{a_1})^* \{R_{a_m}\}^*$.

Now we prove a lemma concerning the action of morphisms in $\mathcal{L}_{\mathcal{A}} \cup \mathcal{R}_{\mathcal{A}}$ on infinite Lyndon words.

Lemma 6.7. Suppose \mathbf{w} is an infinite word over an ordered alphabet \mathcal{A} and let $a, b \in \mathcal{A}$ with $a < b$. Then, the following properties hold.

- (i) $L_b f L_a(\mathbf{w})$ is not an infinite Lyndon word for any non-erasing morphism f .
- (ii) $R_a f L_b(\mathbf{w})$ is not an infinite Lyndon word for any morphism f in $\mathcal{R}_{\mathcal{A}}^*$.
- (iii) If \mathbf{w} is recurrent, then $R_x f L_x(\mathbf{w})$ is not an infinite Lyndon word for any letter x and morphism f in $\mathcal{R}_{\mathcal{A}}^*$.

Proof. (i) The infinite word $L_b f L_a(\mathbf{w})$ starts with b and contains an occurrence of the letter a ; thus, since $a < b$, it cannot be an infinite Lyndon word.

(ii) As $f \in \mathcal{R}_{\mathcal{A}}^*$, the infinite word $R_a f L_b(\mathbf{w})$ starts with b and contains an occurrence of the letter a ; thus, since $a < b$, it cannot be an infinite Lyndon word.

(iii) To be an infinite Lyndon word, $R_x f L_x(\mathbf{w})$ must be aperiodic, in which case a letter different from x occurs in it; in particular this letter occurs in $f L_x(\mathbf{w})$. Moreover, as $f \in \mathcal{R}_{\mathcal{A}}^*$, the infinite word $f L_x(\mathbf{w})$ begins with x and hence with a prefix $x^n y$ for some integer $n \geq 1$ and a letter $y \neq x$. The recurrence of the infinite word \mathbf{w} implies the recurrence of $f L_x(\mathbf{w})$, and so $f L_x(\mathbf{w})$ contains a factor $z x^{n+r} y$ for some letter $z \neq x$ and integer $r \geq 0$. Now $R_x f L_x(\mathbf{w})$ begins with $x^n y$ and contains $z x^{n+r+1} y$, and so contains x^{n+1} . To be an infinite Lyndon word, it needs $x = \min(\text{Alph}(R_x f L_x(\mathbf{w})))$, but then $x^{n+1} < x^n y$. Thus $R_x f L_x(\mathbf{w})$ is not an infinite Lyndon word. \square

Last, we need an important easy fact:

Fact 6.8 ([45]). Any morphism f in $\mathcal{L}_{\mathcal{A}} \cup \mathcal{R}_{\mathcal{A}}$ preserves the lexicographic order for infinite words. More precisely, for any infinite words \mathbf{w} and \mathbf{w}' , $\mathbf{w} < \mathbf{w}'$ if and only if $f(\mathbf{w}) < f(\mathbf{w}')$.

A consequence of the above fact is that for any word \mathbf{w} and for any morphism f in $\mathcal{L}_{\mathcal{A}} \cup \mathcal{R}_{\mathcal{A}}$, if $f(\mathbf{w})$ is a Lyndon word then necessarily \mathbf{w} is also a Lyndon word.

Proof of Theorem 6.1. Assume that Δ is the normalized directive word of a Lyndon episturmian word. Then Δ contains no factor of the form $\bar{x}v\bar{x}$ for any letter x and $v \in \mathcal{A}^*$. By Lemma 6.7, it does not contain any factor of the form bva or $\bar{a}v\bar{b}$ with $v \in \mathcal{A}^*$ and $a < b$. Thus, Δ takes the form given in the statement of the theorem. Indeed, by item (i) of Lemma 6.7 and by Fact 6.8, only one letter (namely a_j) can have all spins ultimately L . Since a Lyndon word is not periodic, at least one other letter in \mathcal{A} should occur infinitely often. By items (ii)–(iii) of Lemma 6.7, such a letter should belong to $\tilde{\mathcal{B}}_{j+1}$. Moreover, the sequence of letters with spin L must be order-increasing and items (ii)–(iii) of Lemma 6.7 determine the conditions on letters with spin R .

Conversely, suppose that the (normalized) directive word Δ of the episturmian word \mathbf{w} takes the form given in the statement of the theorem. Write $\Delta = v_1 \cdots v_j \mathbf{y}$ where, for $1 \leq k \leq j$, v_k is a spinned word in $(\tilde{\mathcal{B}}_{k+1}^* a_k)^*$ and \mathbf{y} is a spinned infinite word belonging to $(\tilde{\mathcal{B}}_{j+1}^+ \{a_j\}^+)^{\omega}$. Then, because of the recurrence of the letter a_j and of at least one other letter in $\tilde{\mathcal{B}}_{j+1}$ in \mathbf{y} , there exists a sequence of spinned words $(v_n)_{n \geq j}$, with each v_n in $\tilde{\mathcal{B}}_{j+1}^+ \{a_j\}^+$, such that $\mathbf{y} = \prod_{n \geq j} v_n$. Now, for each $k \geq 1$, μ_{v_k} is a Lyndon morphism on \mathcal{B}_k by Proposition 6.6. Hence, for each $k \geq 1$, the word $\mu_{v_1 \cdots v_k}(a_j)$ is a Lyndon word. From $\mathbf{w} = \lim_{k \rightarrow \infty} \mu_{v_1 \cdots v_k}(a_j)$, we deduce from Lemma 6.5 that the episturmian word \mathbf{w} is an infinite Lyndon word. \square

6.2. Strict episturmian Lyndon words

Let us recall from [23] that an epistandard word \mathbf{s} , or any episturmian word in the subshift of \mathbf{s} , is said to be \mathcal{A} -strict if its L -spinned directive word Δ verifies $\text{Ult}(\Delta) = \mathcal{A}$. For these words, also called Arnoux–Rauzy sequences [5], Theorem 6.1 gives:

Corollary 6.9. *Let $\mathcal{A} = \{a_1, \dots, a_m\}$ be an alphabet ordered by $a_1 < a_2 < \dots < a_m$. An \mathcal{A} -strict episturmian word \mathbf{w} is an infinite Lyndon word if and only if the (normalized) directive word of \mathbf{w} belongs to $\{a_1, \bar{a}_2, \dots, \bar{a}_m\}^\omega$.*

This can be reformulated as a generalization of Proposition 6.4 in [29]:

Corollary 6.10 ([15]). *An \mathcal{A} -strict episturmian word \mathbf{t} is an infinite Lyndon word if and only if, it can be infinitely decomposed over the set of morphisms $\{L_a, R_x \mid x \in \mathcal{A} \setminus \{a\}\}$ where $a = \min(\mathcal{A})$ for the given order on \mathcal{A} .*

The above result also follows from the following generalization of a result on Sturmian words given by Borel and Laubie [8] (see also [43]).

Theorem 6.11. *An \mathcal{A} -strict episturmian word \mathbf{t} is an infinite Lyndon word if and only if, $\mathbf{t} = \mathbf{as}$ where $a = \min(\mathcal{A})$ for the given order on \mathcal{A} and \mathbf{s} is an (aperiodic) \mathcal{A} -strict epistandard word. Moreover, if Δ is the L -spinned directive word of \mathbf{s} , then $\mathbf{t} = \mathbf{as}$ is the unique episturmian word in the subshift of \mathbf{s} directed by the spinned version of Δ having all spins R , except when $x_i = a$.*

The proof of the above theorem requires the following result that is essentially Theorem 3.17 from [23], apart from the fact that \mathbf{as} is in the subshift of \mathbf{s} , which follows from Fact 2.3.

Theorem 6.12. *Suppose \mathbf{s} is an epistandard word directed by $\Delta = x_1x_2x_3\dots$ and let a be a letter. Then, \mathbf{as} is an episturmian word if and only if, $a \in \text{Ult}(\Delta)$, in which case \mathbf{as} is the unique episturmian word in the subshift of \mathbf{s} directed by the spinned version of Δ having all spins R , except when $x_i = a$.*

Proof of Theorem 6.11. Let $\mathcal{A} = \{a_1, a_2, \dots, a_m\}$ with $a_1 < a_2 < \dots < a_m$. By Corollary 6.9, an \mathcal{A} -strict episturmian word \mathbf{t} is an infinite Lyndon word if and only if, the (normalized) directive word Δ of \mathbf{t} belongs to $\{a_1, \bar{a}_2, \dots, \bar{a}_m\}^\omega$, i.e., if and only if, $\mathbf{t} = a_1\mathbf{s}$ where \mathbf{s} is the unique epistandard word directed by the L -spinned version of Δ , by Theorem 6.12. \square

Note. If \mathbf{s} is an epistandard word over \mathcal{A} , then \mathbf{as} is an infinite Lyndon word for any order such that $a = \min(\mathcal{A})$.

Let us point out that completely different proofs of Corollary 6.9 and Theorem 6.11, using a characterization of episturmian words via lexicographic orderings, were given in [15] by the first author. A refinement of one of the main results in [18] is also given in [15].

To end, let us observe that, contrary to the fact that there exists $|\mathcal{A}|!$ possible orders of a finite alphabet \mathcal{A} , Theorem 6.11 shows that there exist exactly $|\mathcal{A}|$ infinite Lyndon words in the subshift of a given \mathcal{A} -strict epistandard word \mathbf{s} , when $|\mathcal{A}| > 1$ (since there are no Lyndon words when $|\mathcal{A}| = 1$). That is, for any order with $\min(\mathcal{A}) = a$, the subshift of \mathbf{s} contains a unique infinite Lyndon word beginning with a , namely \mathbf{as} .

Example 6.13. With $\Delta = (abcd)^\omega$, the spinned versions $(\bar{a}\bar{b}\bar{c}\bar{d})^\omega, (\bar{a}\bar{b}\bar{c}\bar{d})^\omega, (\bar{a}\bar{b}\bar{c}\bar{d})^\omega, (\bar{a}\bar{b}\bar{c}\bar{d})^\omega, (\bar{a}\bar{b}\bar{c}\bar{d})^\omega, (\bar{a}\bar{b}\bar{c}\bar{d})^\omega, (\bar{a}\bar{b}\bar{c}\bar{d})^\omega$ and their opposites direct non-quasiperiodic episturmian words in the subshift of the 4-bonacci word \mathbf{z} . Only the first four of these words direct Lyndon episturmian words: $\mathbf{az}, \mathbf{bz}, \mathbf{cz}, \mathbf{dz}$, respectively.

7. Concluding remarks

In [35], Monteil proved that any Sturmian subshift contains a *multi-scale quasiperiodic word*, i.e., an infinite word having infinitely many quasiperiods. A shorter proof of this fact was provided in [29]. This can be easily extended to episturmian words. Certainly, by Fact 3.4, any episturmian subshift contains at most two epistandard words (one in the aperiodic case and two in the periodic case) and any epistandard word has infinitely many quasiperiods (by Theorem 4.10).

Actually, the characterization of quasiperiodic Sturmian words in [29] shows that in any Sturmian subshift there are only two non-quasiperiodic Sturmian words and all other (Sturmian) words in the subshift have infinitely many quasiperiods. It is easy to see that the same result does not hold for episturmian words defined over an alphabet containing more than two letters. For instance, any episturmian word having a spinned directive word in $\{ab\bar{c}, \bar{a}b\bar{c}\}^\omega$ is non-quasiperiodic: all of these non-quasiperiodic episturmian words belong to the subshift of the Tribonacci word \mathbf{r} , directed by $(abc)^\omega$. Moreover, one can verify (using Theorem 3.2) that the quasiperiodic episturmian word \mathbf{t} directed by $(abc)^n(ab\bar{c})(\bar{a}b\bar{c})^\omega$ for some $n \geq 1$ (which is in the subshift of \mathbf{r}) has exactly $n + 1$ directive words:

$$(abc)^i ab\bar{c}(\bar{a}b\bar{c})^{n-i}(\bar{a}b\bar{c})^\omega, \quad 0 \leq i \leq n.$$

Hence, it is clear from Theorem 4.19 that \mathbf{t} has only finitely many quasiperiods. (See also Examples 4.25–4.27.)

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References

- [1] J.-P. Allouche, J. Shallit, *Automatic Sequences: Theory, Applications, Generalizations*, Cambridge University Press, 2003.
- [2] A. Apostolico, M. Crochemore, String pattern matching for a deluge survival kit, in: J. Abello, P.M. Pardalos, M.G.C. Resende (Eds.), *Handbook of Massive Data Sets*, Massive Comput., vol. 4, Kluwer Academic Publishers, 2001.
- [3] A. Apostolico, A. Ehrenfeucht, Efficient detection of quasiperiodicities in strings, *Theoret. Comput. Sci.* 119 (1993) 247–265.
- [4] A. Apostolico, M. Farach, C.S. Iliopoulos, Optimal superprimitivity testing for strings, *Inform. Process. Lett.* 39 (1) (1991) 17–20.
- [5] P. Arnoux, G. Rauzy, Représentation géométrique de suites de complexités $2n + 1$, *Bull. Soc. Math. France* 119 (1991) 199–215.
- [6] J. Berstel, Sturmian and episturmian words (a survey of some recent results), in: *Proceedings of CAI 2007*, in: *Lecture Notes in Computer Science*, vol. 4728, Springer-Verlag, 2007, pp. 23–47.
- [7] V. Berthé, C. Holton, L.Q. Zamboni, Initial powers of Sturmian sequences, *Acta Arith.* 122 (2006) 315–347.
- [8] J.P. Borel, F. Laubie, Quelques mots sur la droite projective réelle, *J. Théorie Nomb. Bordeaux* 5 (1993) 23–51.
- [9] E.M. Coven, G.A. Hedlund, Sequences with minimal block growth, *Math. Syst. Theory* 7 (1973) 138–153.
- [10] A. de Luca, A combinatorial property of the Fibonacci words, *Inform. Process. Lett.* 12 (4) (1981) 193–195.
- [11] A. de Luca, Sturmian words: Structure, combinatorics and their arithmetics, *Theoret. Comput. Sci.* 183 (1997) 45–82.
- [12] X. Droubay, J. Justin, G. Pirillo, Episturmian words and some constructions of de Luca and Rauzy, *Theoret. Comput. Sci.* 255 (1–2) (2001) 539–553.
- [13] F. Durand, A characterization of substitutive sequences using return words, *Discrete Math.* 179 (1998) 89–101.
- [14] S Ferenczi, Complexity of sequences and dynamical systems, *Discrete Math.* 206 (1999) 145–154.
- [15] A. Glen, Order and quasiperiodicity in episturmian words, in: *Proceedings of the 6th International Conference on Words*, Marseille, France, September 17–21, 2007, pp. 144–158.
- [16] A. Glen, A characterization of fine words over a finite alphabet, *Theoret. Comput. Sci.* 391 (2008) 51–60.
- [17] A. Glen, J. Justin, Episturmian words: A survey, preprint, 2007.
- [18] A. Glen, J. Justin, G. Pirillo, Characterizations of finite and infinite episturmian words via lexicographic orderings, *European J. Combin.* 29 (2008) 45–58.
- [19] A. Glen, F. Levé, G. Richomme, Directive words of episturmian words: Equivalence and normalization, *RAIRO - Theoret. Inform. Appl.* 2008 (in press).
- [20] C. Holton, L.Q. Zamboni, Descendants of primitive substitutions, *Theory Comput. Syst.* 32 (1999) 133–157.
- [21] C.S. Iliopoulos, L. Mouchard, Quasiperiodicity and string covering, *Theoret. Comput. Sci.* 218 (1) (1999) 205–216.
- [22] J. Justin, Episturmian morphisms and a Galois theorem on continued fractions, *Theoret. Inform. Appl.* 39 (1) (2005) 207–215.
- [23] J. Justin, G. Pirillo, Episturmian words and episturmian morphisms, *Theoret. Comput. Sci.* 276 (1–2) (2002) 281–313.
- [24] J. Justin, G. Pirillo, On a characteristic property of Arnoux-Rauzy sequences, *Theoret. Inform. Appl.* 36 (4) (2003) 385–388.
- [25] J. Justin, G. Pirillo, Episturmian words: Shifts, morphisms and numeration systems, *Internat. J. Found. Comput. Sci.* 15 (2) (2004) 329–348.
- [26] J. Justin, L. Vuillon, Return words in Sturmian and episturmian words, *Theoret. Inform. Appl.* 34 (2000) 343–356.
- [27] F. Levé, G. Richomme, Quasiperiodic infinite words: some answers, *Bull. Eur. Assoc. Theoret. Comput. Sci.* 84 (2004) 128–138.
- [28] F. Levé, G. Richomme, Quasiperiodic episturmian words, in: *Proceedings of the 6th International Conference on Words*, Marseille, France, September 17–21, 2007, pp. 201–211.
- [29] F. Levé, G. Richomme, Quasiperiodic Sturmian words and morphisms, *Theoret. Comput. Sci.* 372 (2007) 15–25.
- [30] M. Lothaire, *Combinatorics on Words*, in: *Encyclopedia of Mathematics and its Applications*, vol. 17, Addison-Wesley, 1983.
- [31] M. Lothaire, *Algebraic Combinatorics on Words*, in: *Encyclopedia of Mathematics and its Applications*, vol. 90, Cambridge University Press, 2002.
- [32] S. Marcus, Bridging two hierarchies of infinite words, *J. UCS* 8 (2002) 292–296.
- [33] S. Marcus, Quasiperiodic infinite words, *Bull. Eur. Assoc. Theoret. Comput. Sci.* 82 (2004) 170–174.
- [34] G. Melançon, Lyndon factorization of Sturmian words, *Discrete Math.* 210 (2000) 137–149.
- [35] T. Monteil, *Illumination dans les billards polygonaux et dynamique symbolique*, Ph.D. Thesis, Université de la Méditerranée, Faculté des Sciences de Luminy, December 2005.
- [36] M. Morse, G.A. Hedlund, Symbolic Dynamics II. Sturmian trajectories, *Amer. J. Math.* 61 (1940) 1–42.
- [37] G. Paquin, L. Vuillon, A characterization of balanced episturmian sequences, *Electron. J. Combin.* 14 (2007) #R33, pp. 12.
- [38] N. Pytheas Fogg, Substitutions in Dynamics, Arithmetics and Combinatorics, in: *Lecture Notes in Mathematics*, vol. 1794, Springer, 2002.
- [39] G. Rauzy, Nombres algébriques et substitutions, *Bull. Soc. Math. France* 110 (1982) 147–178.
- [40] G. Rauzy, Mots infinis en arithmétique, in: M. Nivat, D. Perrin (Eds.), *Automata on Infinite Words*, in: *Lecture Notes in Computer Science*, vol. 192, Springer-Verlag, Berlin, 1985, pp. 165–171.
- [41] G. Richomme, Conjugacy and episturmian morphisms, *Theoret. Comput. Sci.* 302 (1–3) (2003) 1–34.
- [42] G. Richomme, Lyndon morphisms, *Bull. Belg. Math. Soc. Simon Stevin* 10 (2003) 761–785.
- [43] G. Richomme, Conjugacy of morphisms and Lyndon decomposition of standard Sturmian words, *Theoret. Comput. Sci.* 380 (3) (2007) 393–400.
- [44] G. Richomme, A local balance property of episturmian words, in: *Proc. DLT'07*, in: *Lecture Notes in Computer Science*, vol. 4588, Springer, Berlin, 2007, pp. 371–381.
- [45] G. Richomme, On morphisms preserving infinite Lyndon words, *Discrete Math. Theoret. Comput. Sci.* 9 (2007) 89–108.
- [46] R.N. Risley, L.Q. Zamboni, A generalization of Sturmian sequences: combinatorial structure and transcendence, *Acta Arith.* 95 (2000) 167–184.
- [47] P. Séébold, Fibonacci morphisms and Sturmian words, *Theoret. Comput. Sci.* 88 (2) (1991) 365–384.