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Structure of some sand piles model

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Abstract

Sand pile model (SPM) is a simple discrete dynamical system used in physics to represent granular objects. It is deeply related to integer partitions, and many other combinatorics problems, such as tilings or rewriting systems. The evolution of the system started with n stacked grains generates a lattice, denoted by $SPM(n)$. We study here the structure of this lattice. We first explain how it can be constructed, by showing its strong self-similarity property. Then, we define $SPM(\infty)$, a natural extension of SPM when one starts with an infinite number of grains. Again, we give an efficient construction algorithm and a coding of this lattice using a self-similar tree. The two approaches give different recursive formulae for $|SPM(n)|$. © 2001 Elsevier Science B.V. All rights reserved.

Keywords: SPM; Sand pile model; Lattice; Integers partitions; CFG; Discrete dynamical systems

1. Introduction

1.1. Motivations and context

In 1987, Bak et al. [4] introduced the important notion of *self-organisation criticality* (SOC): when certain systems in a steady state (named critical state) are slightly perturbated, they evolve back to another steady state. This evolution implies some arbitrarily high modifications of the system.

The typical example is an avalanche on a sand pile. At first, the pile is in a steady state and the perturbation consists in adding a grain on the pile. As a consequence, the pile evolves to a new steady state, with an avalanche starting where the grain

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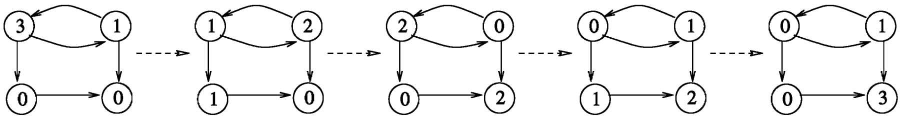

Fig. 1. An example of CFG. Each vertex load is written in this vertex.

was dropped. The fact that this avalanche size may be arbitrarily high is the main characteristic ofSOC systems.

Since the appearance of this paper, many physicists and biologists have recognized these properties in natural systems, and the SOC family still grows (see [20, 22] for example); many publications on this topic appeared recently [20, 11, 1, 23]. These phenomena are of particular interest in surface grow studies [2], in geophysics [3], in plasma confinement, in astrophysics, and many other, including, of course, studies of granular systems like dunes [11] and molecule aggregation [2].

The essence of these phenomena is captured by a well-known model in game theory and combinatorics, the *chip firing game* (CFG). The most general notion of CFG is a directed graph $G=(V,E)$ where a threshold δ_v and a load $l(v)$ are given to each vertex v. Intuitively, $l(v)$ represents the number of chips stored at v. The game evolves with respect to the following rule: if $v \in V$ contains more than δ_v chips, then it gives δ_v of them to its neighbours, i.e. the load of the vertex v is decreased by δ_v and the load of each of its neighbours is increased by δ_v/n_v where n_v is the number of v neighbours. In general, one takes $\delta_v = n_v$, but $\delta_v = \infty$ if $n_v = 0$ (v is then called a sink). See Fig. 1 for an example of such a CFG.

Under certain conditions, the CFG converges to a steady state (see for example [13]). The addition of one chip on a vertex $v \in V$ when the system is in a steady state causes a redistribution of the chips. During this redistribution, an arbitrary number of vertexes may be concerned. To give an example, we can consider the case where $l(v)=0$ and $l(v)=\delta_v \; \forall v \in V \backslash \{v\}$. If one adds successively $\delta_v - 1$ chips on v, the only concerned vertex is v (the system remains steady). If one adds one more grain, every vertex will be concerned (if G is connected). Such a diffusion can be arbitrary large [15], depending on the initial state of the system, and is always started by addition of one grain. Such a propagation is called an *avalanche*.

A particular case of this model is widely studied: the sand pile model on a rectangular grid. [1] The graph G in this case is undirected. It is a rectangular finite lattice and the value of δ_v and of n_v is 4 for all vertex v except one singular vertex v which is linked once to any vertex on the border of the lattice and twice to the four corners, and such that $\delta_v = \infty$. The distinguished vertex acts like a sink: it never gives away any of its grains and could be considered as collecting the grains that leaves the system. If the load of a vertex inside the lattice is more than 4, then it gives one grain to each of

[1] The standard term is *lattice* but, since we will use orders theory in the following, where the word *lattice* takes another meaning, we use here the word *grid*.

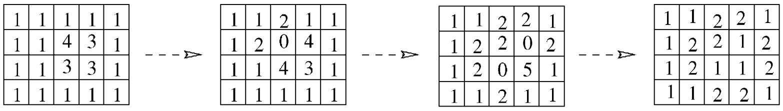

Fig. 2. Example of (parallel) sand pile.

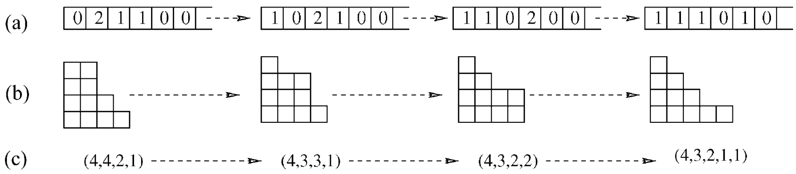

Fig. 3. Three ways to see SPM.

its four neighbours (see Fig. 2 for an example where the distinguished vertex is not represented since its load does not influence the evolution of the system). This is the model deeply studied by Dhar [9, 10]. In particular, one can show that adding a grain turns the system into an unsteady state, and that after auto-reorganisation it reaches a new steady state [21]. This confirms that we are in the SOC context. Cori and Rossin [6] generalized this notion to any rooted graph and obtained similar results.

Another special case of CFG is the *sand piles model* (SPM). The graph G in this case is an undirected chain, infinite on the right: $V=\mathbb{N}$, $E=\{(i,i+1)\ \forall i\in\mathbb{N}\}$, $\delta_v=2$ for all $v>0$ and $\delta_0=\infty$ (see Fig. 3(a)). This model is equivalent to the following. Consider an infinite chain of columns, each containing a vertical pile of grains. The height difference between the column c_i and its right neighbour column c_{i+1} is denoted by $d(i)$. If $d(i)$ is greater than or equal to 2 then a grain falls down from c_i to c_{i+1} (see Fig. 3(b)). If $i>0$, we call c_{i-1} the left neighbour of c_i and then $d(c_{i-1})$ and $d(c_{i+1})$ are increased by 1 while $d(c_i)$ is decreased by 2. We find again our initial definition of SPM, with a coding of the pile by height differences.

Notice that SPM is less general than the lattice sand pile but is not a particular case of it: the number of vertices v with $l(v)\neq 0$ is not bounded in SPM, and it effectively grows with the number of grains. Moreover, the model SPM has no sink, which is a fundamental difference. If we represent a pile by the t-uple of its columns height, each configuration of the pile represents a partition of the total number of grains (see Fig. 3(c)).

In computer science, the CFG models several problems and is applied in several algorithms (see for example [18]). SPM itself admits natural interpretations in algorithmic terms. We give here two examples about dynamical distribution of jobs on a processors network [19, 8, 16]. Each column of a sand pile represents a processor, a grain represents a job. One can imagine the processors are connected on a ring (like Token Ring): each processor can only communicate directly with its right neighbour. It corresponds to the move of a grain from one column to another. Since only neighbour processors can communicate, the communications can be processed in parallel and the parallel SPM

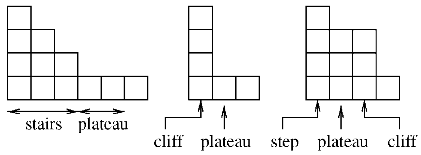

Fig. 4. Examples for the definitions.

is a good model for this problem [12]. If on the contrary the communication medium is a shared bus (like Ethernet or certain multiprocessors), we can study the evolution of sequential SPM to avoid collisions.

In the following, we are going to discuss some lattice properties of the above dynamical systems. Let us recall that a lattice can be described as a partial order such that two elements a and b admit a least upper bound (called supremum of a and b and denoted by $\sup(a,b)$) and a greatest lower bound (called infimum of a and b and denoted by $\inf(a,b)$). The element $\sup(a,b)$ is the smallest element among the elements greater than both a and b. The element $\inf(a,b)$ is defined similarly. A useful result about finite lattices is that a partial order is a lattice if and only if it admits a greatest element, and any two elements admit a greatest lower bound. For more details, see for example [7]. The fact that a dynamical system has the lattice property implies some important properties, such as convergence.

1.2. Our model: known results

Our model is the standard sequential SPM; it consists of an infinite number of ordered columns, each containing a certain number of grains. Only the first k columns are non-empty, so the state of the system is described by the k-uple $s=(s_1,s_2,\dots,s_k)$ where s_i is the number of grains in the column i for $1\leqslant i\leqslant k$.

The system is initially in the state $N=(n)$. This means that all the grains are in the first column. At each step, the system evolves with respect to the following rule: one grain can fall down from column i to column $i+1$ if and only if $s_{i+1}-s_i\geqslant 2$. This rule defines a covering relation on the set of reachable configurations. The reflexive and transitive closure of this relation is an order, called the dominance order [17]. The set of reachable configurations from the partition (n) with this order is then a lattice denoted by $SPM(n)$ [17].

Let $s=(s_1,\dots,s_k)$ be a sand pile, the *height difference of s at i*, denoted by $d_i(s)$, is the integer s_i-s_{i+1} (with the assumption that $s_{k+1}=0$). We will say that s has a *step* (resp. *plateau*, resp. *cliff*) at i if and only if its height difference at i is 1 (resp. 0, resp. $\geqslant 2$). We extend these definitions by saying that s has stairs (resp. a plateau) at the interval $[i,j]$ if and only if s has a step (resp. plateau) at k for all $i\leqslant k\leqslant j$. The integer $j-i+1$ is called the *length* of the stairs (resp. plateau). See Fig. 4 for examples.

The evolution rule of a sand pile $s=(s_1,\dots,s_i,s_{i+1},\dots,s_k)$ is then: one grain can fall from one column to the column on its right if and only if it is at the top of a cliff.

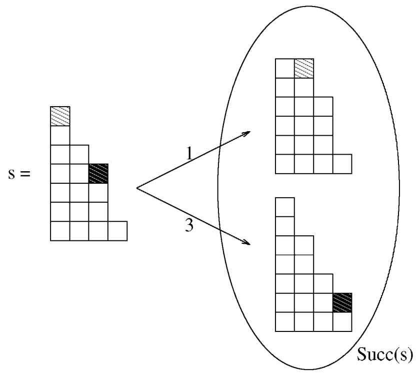

Fig. 5. Transitions, successors.

Such a transition is denoted by $\xrightarrow{i}$ where i is the number of the column from which the grain falls. The sand pile s' is called a *successor* of s, and $Succ(s)$ denotes the set of all successors of s:

$$Succ(s) = \{s' \mid s \xrightarrow{i} s',\ i \in \mathbb{N}\}.$$

See Fig. 5 for an example.

Let us now introduce a few notations. If s is a partition of n then $s^{\downarrow i}$ is the partition of $n+1$ obtained by adding a grain on the ith column of s (if it is a partition). In other words, if $s = (s_1, \ldots, s_i, \ldots, s_k)$ then $s^{\downarrow i} = (s_1, \ldots, s_i + 1, \ldots, s_k)$. We also define $S^{\downarrow i} = \{s^{\downarrow i} \mid s \in S\}$, where S is a set of partitions.

We will denote by $e(s)$ the largest integer such that s has stairs at $[1, e(s)]$. We also define P_i as the set of the sand piles in $SPM(n)$ that begins with stairs of length (at least) i. In other words, $P_i = \{s \in SPM(n) \mid e(s) \geqslant i\}$.

Characterisations of the fixed point of the system, the minimum element of the corresponding lattice, and of its elements are also known:

Theorem 1 (Goles and Kiwi [14]). *The fixed point of* $SPM(n)$ *is*

$$S_0 = (k, k-1, \ldots, p+1, p, p, p-1, \ldots, 2, 1),$$

where k *is the maximal integer such that* S_0 *is a sand pile of* n *grains*, *i.e.* k *is the integer such that* $k(k+1)/2 \leqslant n \leqslant (k+1)(k+2)/2$.

Theorem 2 (Goles et al. [17]). *A partition* s *belongs to* $SPM(n)$ *if and only if*:
- s *does not contain any sequence* $p,\ p,\ p$ *or* $p,\ p,\ (p-1),\ (p-1)$
- *there is at least one cliff between two consecutive sequences* $p,\ p$ *and* $q,\ q$.

In this paper, we study the structure of $SPM(n)$. In particular, we show in the next section how $SPM(n+1)$ can be constructed from $SPM(n)$, thus we obtain an algorithm that constructs $SPM(n)$ for any integer n. Afterwards, we define a natural infinite extension, $SPM(\infty)$, when the system is started with an infinite column of grains. The study of the structure of $SPM(\infty)$ permits more remarks on the self-similarity of the set. During this study, we obtain interesting recursive formula for $|SPM(n)|$.

2. From $SPM(n)$ to $SPM(n+1)$

The goal of this section is the construction of the lattice $SPM(n+1)$ from $SPM(n)$. We will construct the graph of the transitive reduction of the lattice, i.e. the graph of its order relation, without the reflexive edges ($x \to x$) and the transitive ones ($x \to z$ when $x \to y$ and $y \to z$). Each edge of this graph is equivalent to a transition of the SPM system. Therefore, we will label the edge $s \xrightarrow{i} s'$ with the number i of the column of s from which the grain falls in order to obtain s'. We will call the obtained labelled graph the *diagram* of the lattice. We first give some preliminary results, then we notice that $SPM(n)$ is a good starting point to construct $SPM(n+1)$, and we give a method to obtain $SPM(n+1)$ from $SPM(n)$. Finally, we inspect more deeply the construction algorithm and show a strong self-similarity in each lattice $SPM(n)$. This similarity induces a first recursive formula for the cardinality of $SPM(n)$.

2.1. Preliminaries

Let us study what happens when we add one grain on the ith column of a sand pile $s = (s_1, \ldots, s_i, \ldots, s_k)$ such that $e(s) \geqslant i-1$. We obtain the sand pile $s^{\downarrow i} = (s_1, \ldots, s_i + 1, \ldots, s_k)$. We want to determine all the possible transitions from this partition, knowing the possible ones from s. Three cases are possible (as shown in Fig. 6) corresponding to the three following propositions. Recall that we only consider sand piles s with $e(s) \geqslant i-1$, since it will be the case of interest for the rest of the paper.

Proposition 1 (plateau). *Let $s \in SPM(n)$ such that $e(s) \geqslant i-1$. If s has a plateau at i then the possible transitions from $s^{\downarrow i}$ are the same as the possible transitions from s. Moreover, if $s \xrightarrow{j} t$ then $s^{\downarrow i} \xrightarrow{j} t^{\downarrow i}$. In other words, $Succ(s^{\downarrow i}) = (Succ(s))^{\downarrow i}$ and the corresponding edges of the diagrams have the same labels.*

Proof. A transition $\xrightarrow{i}$ is only possible if there is a cliff at the column i. Now, the set of the columns where s has a cliff is equal to the set of the columns where $s^{\downarrow 1}$ has a cliff. □

Proposition 2 (cliff). *Let $s \in SPM(n)$ such that $e(s) \geqslant i-1$. If s has a cliff at i then*

1. *The possible transitions from $s^{\downarrow i}$ are the same as the possible transitions from s and if $s \xrightarrow{j} t$ then $s^{\downarrow i} \xrightarrow{j} t^{\downarrow i}$. In other words, $Succ(s^{\downarrow i}) = (Succ(s))^{\downarrow i}$ and the corresponding edges of the diagrams have the same labels.*

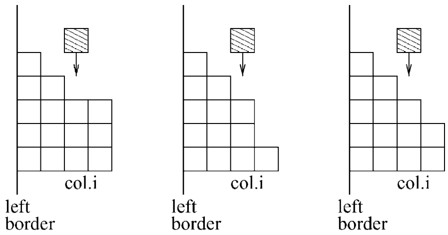

Fig. 6. The three cases considered.

2. *Moreover, if* $s^{\downarrow_i} \xrightarrow{i} s'$ *then* $s^{\downarrow_1} \xrightarrow{i \cdot i-1 \cdot \ldots \cdot 2 \cdot 1} s'$. *In other words,* s' *is reachable in* $SPM(n)^{\downarrow_1}$ *from* $s^{\downarrow_1}$ *via a path labelled* $i \cdot i-1 \cdot \ldots \cdot 2 \cdot 1$.

Proof. 1. A transition $\xrightarrow{i}$ is only possible if there is a cliff at the column i. Now, the set of the columns where s has a cliff is equal to the set of the columns where $s^{\downarrow_1}$ has a cliff.

2. The sand pile s' is equal to $s^{\downarrow_{i+1}}$ and by hypothesis s has a cliff at i and stairs at $[1, i-1]$. Therefore, in the sand pile $s^{\downarrow_1}$ a grain can fall from the column i. With this grain's fall, we create a cliff at $i-1$, therefore a new grain can now fall from column $i-2$. This process can be iterated to obtain s' at the end. To sum up, we can write

$$
\begin{aligned}
s^{\downarrow_1} &= (s_1+1, \ldots, s_{i-1}, s_j, s_{i+1}, \ldots, s_k) \\
&\xrightarrow{i} (s_1+1, \ldots, s_{i-1}, s_i - 1, s_{i+1}+1, \ldots, s_k) \\
&\xrightarrow{i-1} \cdots \xrightarrow{2} (s_1+1, s_2-1, \ldots, s_i, s_{i+1}+1, \ldots, s_k) \\
&\xrightarrow{1} (s_1+1-1, s_2, \ldots, s_i, s_{i+1}+1, \ldots, s_k) = s^{\downarrow_{i+1}} = s'.
\end{aligned}
$$

It is obvious that all the partitions on this path belong to $SPM(n)^{\downarrow_1}$. □

Proposition 3 (step). *Let* $s \in SPM(n)$ *such that* $e(s) \geqslant i-1$. *If* s *has a step at* i *then the possible transitions from* $s^{\downarrow_i}$ *are the same as from* s *with an additional transition on the column* i: $s^{\downarrow_i} \xrightarrow{i} s^{\downarrow_{i+1}}$.

Proof. The set of columns where $s^{\downarrow_i}$ has a cliff is equal to the union of $\{i\}$ and the set of columns where s has a cliff. □

2.2. Construction

Using the preliminary results from the previous section, we will here obtain an algorithm for the construction of $SPM(n+1)$ from $SPM(n)$. We first show that $SPM(n)^{\downarrow_1}$

is a good starting point for the construction of $SPM(n+1)$. Recall that $SPM(n)^{\downarrow_1}$ is the set of partitions obtained by addition of one grain on the first column of each partition in $SPM(n)$. Afterwards, we will use the previous propositions to add the missing elements and transitions in order to complete $SPM(n)^{\downarrow_1}$ into $SPM(n+1)$.

Proposition 4. *$SPM(n)^{\downarrow_1}$ is a sublattice of $SPM(n+1)$.*

Proof. Let us recall that if a and b are two partitions of $SPM(n)$ for a given n, then $\inf(a,b)$ is their first common descendant and $\sup(a,b)$ is their first common ancestor. To prove the claim, we must show that:

- If $\inf(a,b)=c$ is in $SPM(n)$ then $\inf(a^{\downarrow_1},b^{\downarrow_1})=c^{\downarrow_1}$ is in $SPM(n+1)$.
- If $\sup(a,b)=c$ is in $SPM(n)$ then $\sup(a^{\downarrow_1},b^{\downarrow_1})=c^{\downarrow_1}$ is in $SPM(n+1)$.

Recall that [17]

$$\inf(a,b)=c \quad \text{iff for all } j \text{ one has} \sum_{i=1}^{j} c_i = \min\left(\sum_{i=1}^{j} a_i, \sum_{i=1}^{j} b_i\right).$$

This implies that

$$\sum_{i=1}^{j} c_i + 1 = \min\left(\sum_{i=1}^{j} a_i + 1, \sum_{i=1}^{j} b_i + 1\right) \quad \text{for all } j,$$

i.e. $c^{\downarrow_1}$ is in $SPM(n+1)$.

Let now c be equal to $\sup(a,b)$ (in $SPM(n)$) and d be equal to $\sup(a^{\downarrow_1},b^{\downarrow_1})$ (in $SPM(n)^{\downarrow_1}$). We will show that $d=c^{\downarrow_1}$. We have $c\geqslant a$ and $c\geqslant b$, therefore $c^{\downarrow_1}\geqslant a^{\downarrow_1}$ and $c^{\downarrow_1}\geqslant b^{\downarrow_1}$, which implies that $c^{\downarrow_1} > d$. Let us begin by showing that $d_1=c_1+1$. We can suppose $a_1\geqslant b_1$. The partition $(a_1,a_1,a_1-1,a_1-2,\ldots)$ is greater than a and b, hence it is greater than c. This implies that $c_1=a_1$. Since $a^{\downarrow_1}\leqslant d\leqslant c^{\downarrow_1}$, we have $d_1=a_1+1$. Let $e=(d_1-1,d_2,d_3,\ldots)$. Since $d\leqslant c^{\downarrow_1}$, e verifies the characterisation of Theorem 2. Moreover, $d\geqslant a^{\downarrow_1}$ and $d\geqslant b^{\downarrow_1}$, hence $e\geqslant a$ and $e\geqslant b$. This implies that $e\geqslant \sup(a,b)=c$ and that $d\geqslant c^{\downarrow_1}$, and so $d=c^{\downarrow_1}$. $\square$

It is straightforward that each element s of $SPM(n+1)$ is reachable from an element of $SPM(n)^{\downarrow_1}$. Indeed, s is at least reachable from $(n)^{\downarrow_1}=(n+1)$. This shows that one can start the construction of $SPM(n+1)$ with $SPM(n)^{\downarrow_1}$ and then add the missing elements (see Fig. 7 for an example).

The construction procedure starts with the lattice $SPM(n)^{\downarrow_1}$ given by its diagram. Then, we look for those elements in $SPM(n)^{\downarrow_1}$ that have a successor out of $SPM(n)^{\downarrow_1}$. The set of these elements will be denoted by I_1, with $I_1\subseteq SPM(n)^{\downarrow_1}$. At this point, we add all the missing successors of the elements of I_1. The set of these new elements will be denoted by C_1. Now, we look for the elements in C_1 that have a successor out of the constructed set. The set of these elements is denoted by I_2. We add the new

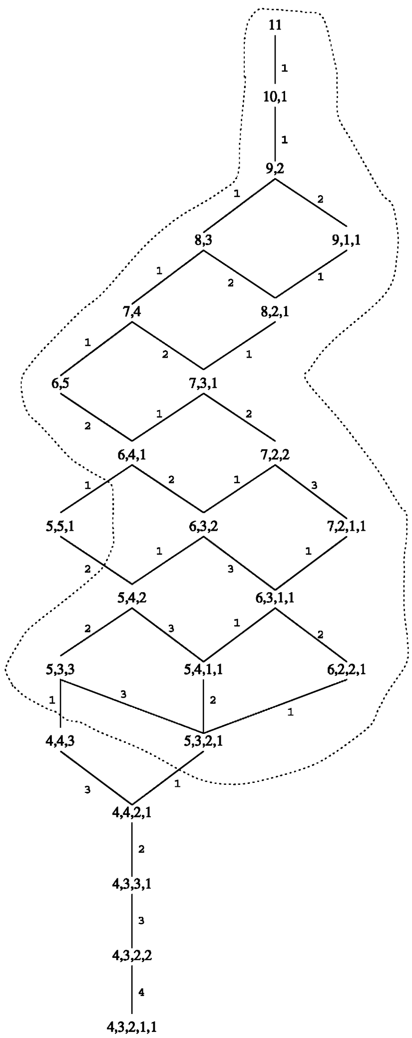

Fig. 7. $SPM(10)^{\downarrow 1}$ in $SPM(11)$.

elements (their set is denoted by C_2), and we iterate this process until the set I_i is empty.

More explicitly, in the ith step of the procedure we look for the elements in C_{i-1} with missing successors and call I_i the set of these elements. We add the new successors of the elements of I_i and call the set of these new elements C_i. At each step, when we add a new element, we also add its covering relations. $SPM(n+1)$ is a finite set, therefore this procedure terminates. At the end, we have obtained the whole set $SPM(n+1)$ with its order relation.

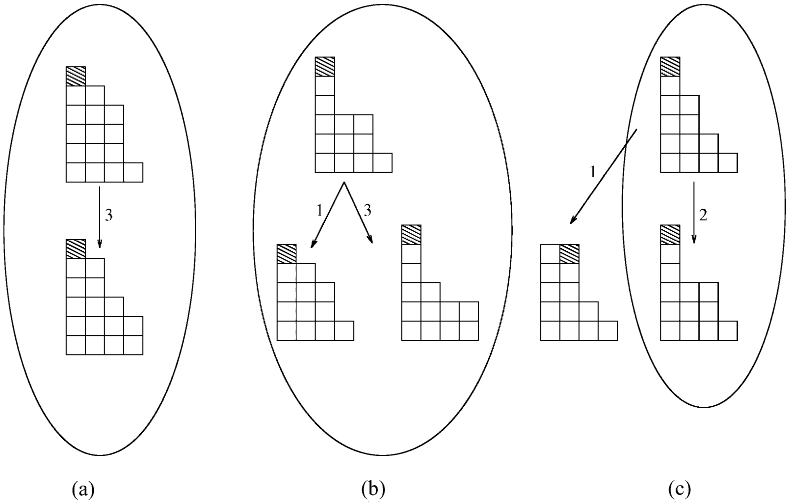

Fig. 8. Successors of $s^{\downarrow_1}$ if s begins with (a) a plateau (b) a cliff and (c) a step: there is a new element only in this last case.

Now, let us show how this completion of $SPM(n)^{\downarrow_1}$ to obtain $SPM(n+1)$ is implemented. Recall that any element t of $SPM(n)^{\downarrow_1}$ is obtained from an element s of $SPM(n)$ by adding a new grain on the first column. Three cases are possible:

- s begins with a plateau or a cliff. Then, according to Proposition 1, the possible transitions from $s^{\downarrow_1}$ are the same as the possible transitions from s, and the successors of $s^{\downarrow_1}$ are obtained by an application of $^{\downarrow_1}$ to the successors of s. Therefore they are already in $SPM(n)^{\downarrow_1}$ (see Fig. 8(a) and (b)).
- s begins with a step. In this case, s is in $P_1 = SPM(n)_{d_1=1}$. Then, according to Proposition 3, only one successor of $s^{\downarrow_1}$ is not yet in $SPM(n)^{\downarrow_1}$: the successor obtained by a transition $\xrightarrow{1}$. This element is $s^{\downarrow_2}$ (see Fig. 8(c)). It follows that $I_1 = P_1^{\downarrow_1}$ and $C_1 = P_1^{\downarrow_2}$.

This means that the first step of the construction consists in adding all the elements of $C_1 = P_1{}^{\downarrow_2}$. Notice that this set is added with a duplication of the order structure of $P_1{}^{\downarrow_1} = I_1$. Indeed, it is clear that

$$(s^{\downarrow_1} \xrightarrow{j} t^{\downarrow_1}) \quad \text{iff } (s^{\downarrow_2} \xrightarrow{j} t^{\downarrow_2}) \text{ for all } s,\ t \text{ in } P_1, \text{ and for all } j.$$

The following step consists in adding the missing successors of the elements of C_1 and the missing transitions originating from them. The analysis of the three cases (plateau, cliff, step) shows that the only elements of C_1 that do not have all their successors and transitions are

- The elements $s^{\downarrow_2} \in C_1$ such that s has a cliff at 2. Indeed, such a $s^{\downarrow_2}$ does have all its successors in the lattice, but one transition is missing: the one labelled with 2. In this case Proposition 2 shows that $s^{\downarrow_2} \xrightarrow{2} t$ where t is also obtained by $s^{\downarrow_1} \xrightarrow{2\cdot 1} t$.

Therefore, we have to add an edge $\xrightarrow{2}$ from $s^{\downarrow_2}$ to an element t which is already in the lattice. We will call *back edge* such an edge.

- The elements $s^{\downarrow_2} \in C_1$ such that s has a step at 2 (i.e. s begins with stairs of length at least 2 and hence is in $P_2^{\downarrow_2}$). According to Proposition 3, only one successor of each of these elements is not yet in the lattice: the successor obtained by the transition on the second column, i.e. the element $s^{\downarrow_3}$. Therefore, to complete the second step, we have to add the set $P_2{}^{\downarrow_3}$ (with the same order structure as $P_2{}^{\downarrow_2}$) to the existing lattice and connect the lattice to this new part by all the transitions:

$$s^{\downarrow_2} \xrightarrow{2} s^{\downarrow_3} \quad \text{for all } s^{\downarrow_2} \in P_2^{\downarrow_2}.$$

This means that $I_2 = P_2^{\downarrow_2}$ and $C_2 = P_2^{\downarrow_3}$. In general, the ith step consists in adding the missing successors of the elements added at step $i-1$ and the missing transitions originating from them. We show that the observed behaviour for the second step is general, and so the sets I_i and C_i can be characterized.

Theorem 3. *For all integer i, we have $I_i = P_i^{\downarrow_i}$ and $C_i = P_i^{\downarrow_{i+1}}$.*

Proof. By induction.

- The case $i = 1$ has already been studied. Notice that every covering relation concerning the new elements is of the following form: $s^{\downarrow_1} \xrightarrow{1} s^{\downarrow_2}$ where $s \in P_1$.
- Suppose the result is true for $i-1$. We show that it is true for i. Consider $C_{i-1} = P_{i-1}^{\downarrow_i}$. Using Propositions 1, 2 and 3, we look for the successors of $s^{\downarrow_i}$, with $s \in P_{i-1}$. Three cases are to be considered: s can have a plateau, a cliff or a step at i.

  Plateau. According to Proposition 1, $Succ(s^{\downarrow_i}) \subseteq P_{i-1}^{\downarrow_i} = C_{i-1}$. So, s has no new successor, and $s \notin I_i$. Moreover, $s \xrightarrow{j} t$ if and only if $s^{\downarrow_i} \xrightarrow{j} t^{\downarrow_i}$.

  Cliff. According to Proposition 2, $Succ(s^{\downarrow_i}) \subseteq P_{i-1}{}^{\downarrow_i} \cup SPM(n)^{\downarrow_1}$. So, $s \notin I_i$. Moreover, the edges of the covering relation originating from $s^{\downarrow_i}$ are the same than the ones originating from s plus an additional one: $s^{\downarrow_i} \xrightarrow{i} s^{\downarrow_{i+1}}$. The element $s^{\downarrow_{i+1}}$ is in $SPM(n)^{\downarrow_1}$ according to Proposition 2.

  Stair. According to Proposition 3, $s^{\downarrow_i}$ has a new successor, $s^{\downarrow_{i+1}}$, hence $s^{\downarrow_i} \in I_i$. Moreover, the edges of the covering relation originating from $s^{\downarrow_i}$ are the same as the ones originating from s plus an additional one: $s^{\downarrow_i} \xrightarrow{i} s^{\downarrow_{i+1}}$. Therefore the element $s^{\downarrow_{i+1}}$ is in C_i.

  From these three cases, we deduce the claim. □

We have obtained a characterisation of the sets I_i and C_i. It is now straightforward that Algorithm 1 constructs the lattice $SPM(n+1)$ from $SPM(n)$. Notice that we can obtain $SPM(n)$ for an arbitrary integer n by starting from $SPM(0)$ and iterating this algorithm. In the next sections, we will give more details about this construction. We will show that the complexity of Algorithm 1 is linear with respect to the number of newly added elements, and hence we have an algorithm that constructs $SPM(n)$ in linear time linear with respect to $|SPM(n)|$.

Algorithm 1. Incremental construction

```
Input: SPM(n)
Output: SPM(n+1)
begin
  i ← 1;
  I ← P_i^{↓i};
  while I ≠ ∅ do
    C ← P_i^{↓i+1};
    add C with its covering relation;
    for each s^{↓i} in I do
      add the edge: s^{↓i} --i--> s^{↓i+1} ∈ C
    for each s^{↓i+1} in C s.t. d_{i+1}(s) ≥ 2 do
      add the back edge s^{↓i+1} --i+1--> s' ∈ SPM(n)^{↓1}
    i ← i + 1;
    I ← P_i^{↓i};
end
```

2.3. *Structure of the P_i parts*

We will now study more deeply the construction procedure given above. We will obtain results on the structure of the P_i parts, which play an important role, and a recursive formula for $|SPM(n)|$. However, the results presented here are not necessary to understand the infinite extension presented in the second part of the paper. Therefore, the rest of this section can be ignored if the reader is mostly interested in the second part of the paper.

In the previous section, we characterised the sets I_i and C_i. More can be said about the structure of these sets. In fact, since $I_i = P_i^{\downarrow i}$ and $C_i = P_i^{\downarrow i+1}$, we only have to study the sets P_i. We will show that these sets are disjoint unions of lattices, and that each of these lattices is obtained from a generating partition by iteration of the SPM rule. We will give the explicit characterisation of these generating partitions, as well as their number.

Proposition 5. *P_1 is a disjoint union of lattices.*

Proof. Let $Q_{1,k}$ denote the set of all the elements of $SPM(n)$ whose first two parts are k and $k-1$. This is a non-empty subset of P_1. It is clear that if $k \neq k'$ then $Q_{1,k} \cap Q_{1,k'} = \emptyset$, so P_1 is the disjoint union of the sets $Q_{1,k}$.

Since $P_1 \subseteq SPM(n)$, the elements of $Q_{1,k}$ verify the characterisation of Theorem 2; this implies that the maximal element g of $Q_{1,k}$ has the form

$$g = (k, k-1, k-1, k-2, \dots, k-l, r)$$

with l maximal (i.e. l such that $r \leqslant k-l-1$), and $k+(k-1)+(k-1)+(k-2)+\cdots+(k-l)+r=n$. Then g belongs to $SPM(n)$ if and only if k satisfies:

$$k+k-1 \leqslant n \leqslant k+(k-1)+(k-1)+(k-2)+\cdots+1,$$

which is equivalent to $2k-1 \leqslant n \leqslant (k^2+3k-2)/2$.

Let k be such an integer. Let us study the structure of $Q_{1,k}$ by considering its maximal element g, described as above. Let s be an element of $Q_{1,k}$. It is clear that the prefix sums of s are less than or equal to the ones of g, so, according to [17], s can be obtained from g by the SPM rule. Therefore, $Q_{1,k}$ is the set of the elements of $SPM(n)$ which can be obtained from g and whose first two parts are k and $k-1$. In other words, $Q_{1,k}$ is the set of the partitions reached from g by paths without any transition labelled 1 or 2. The element g is called the *generating partition* of $Q_{1,k}$. Therefore $Q_{1,k}$ is isomorphic to the lattice of the partitions of $n-2k-1$ obtained from $(g_3,\ldots,g_n)$ by iteration of the SPM rule, and, in particular, $Q_{1,k}$ is a lattice. □

More generally, let us denote by $Q_{i,k}$ the set $Q_{1,k} \cap P_i$.

Proposition 6. *The sets $Q_{i,k}$ are lattices with all transitions labelled with integers greater than k. Moreover, for all i, P_i is the disjoint union of the lattices $Q_{i,k}$.*

Proof. Recall that P_i is the subset of P_1 containing the partitions that begin with stairs of length i, and that $Q_{1,k}$ is the subset of P_i containing the partitions that begin with k, $k-1$. So, $Q_{i,k}$ is the subset of P_1 containing the partitions that begin with the stairs k, $k-1,\ldots,k-i$. Therefore the maximal element of $Q_{i,k}$ has the form

$$g=(k,k-1,\ldots,k-i,k-i,\ldots,k-l,r),$$

where $l \geqslant i$ and l is maximal (i.e. $r \leqslant k-l-1$) (we use here the same argument as in the proof of Proposition 5). Every element s of $Q_{i,k}$ is reachable from this element g, and only transitions with labels greater than i are needed to obtain s from g. Therefore, $Q_{i,k}$ is a lattice isomorphic to the lattice of partitions obtained from $(k-i,\ldots,k-l,r)$. We have

$$P_1=\bigsqcup_k Q_{1,k},$$

where $\bigsqcup$ denotes the disjoint union, and the $Q_{1,k}$ are pairwise disjoint, then

$$P_i=P_1 \cap P_i=\bigsqcup_k (Q_{1,k} \cap P_i)=\bigsqcup_k Q_{i,k}$$

and obviously the sets $Q_{i,k}$ are also pairwise disjoint for a fixed i. □

We sum up these results in Fig. 9. An example is given for $n=10$ in Fig. 10.

We have defined the generating partition g of a set $Q_{1,k}$ as the maximal element of $Q_{1,k}$. Therefore $Q_{1,k}$ is the lattice of the sand piles reachable from g by iteration of the

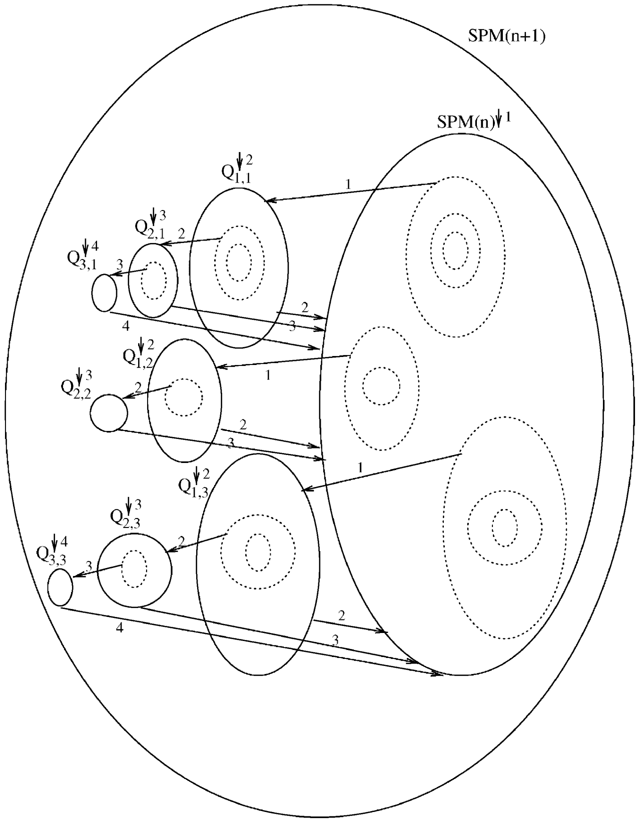

Fig. 9. Structure of $SPM(n+1)$.

SPM rule on the coulmns at the right of the second column. Using the characterisation of SPM partitions, we can now enumerate all the generating partitions for a given n.

Proposition 7 (Generating partitions). *The number of generating partitions in the set* $SPM(n)$ *is* $\lfloor n/2 + 2 - \sqrt{\frac{17}{4}2n} \rfloor$.

Proof. As seen above, the generating partitions in $SPM(n)$ have the form: $(k, k-1, k-1, k-2, \ldots, k-l, r)$ for some $l > 0$ and $r \leqslant k-l-1$. Such integers k must verify $k + (k-1) \leqslant n$ and $k - 1 + k(k+1)/2 \geqslant n$. Moreover, any of these k effectively corresponds to a generating partition. Therefore we have as many generating partitions as solutions to the system:

$$k - 1 + \frac{k(k+1)}{2} \geqslant n,$$

$$k + (k-1) \leqslant n,$$

that is $-\frac{3}{2} + \sqrt{\frac{17}{4} + 2n} \leqslant k \leqslant (n+1)/2$. $\square$

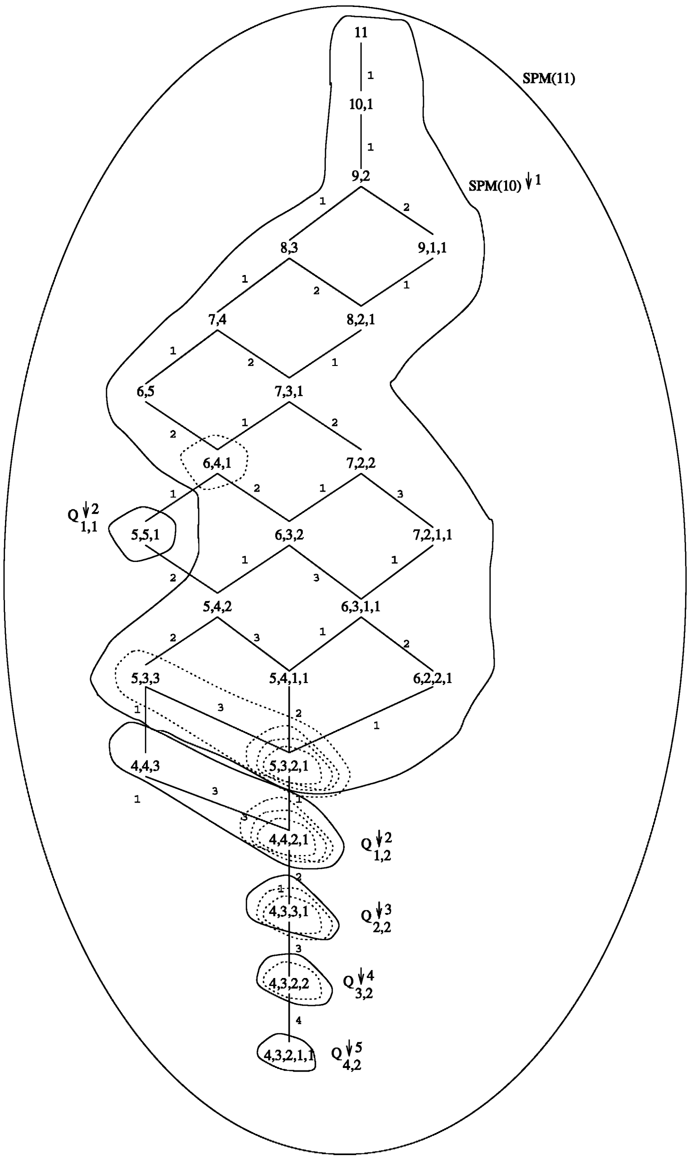

Fig. 10. Example with n=10.

We have already seen that each $Q_{1,j}$ is a lattice and contains the lattices $Q_{i,j}$ for $i > 1$. The lattices $Q_{i,j}$ also verify: $Q_{i+1,j} \subseteq Q_{i,j}$ for all j. We will show how the $Q_{1,j}$ are generated during the construction. See Fig. 11.

In order to study the parts P_i and $Q_{i,j}$ when n varies, let us extend our notations. We will denote by $P_i(n)$ the parts P_i of $SPM(n)$ ($P_i(n)$ is the set of all sand piles with

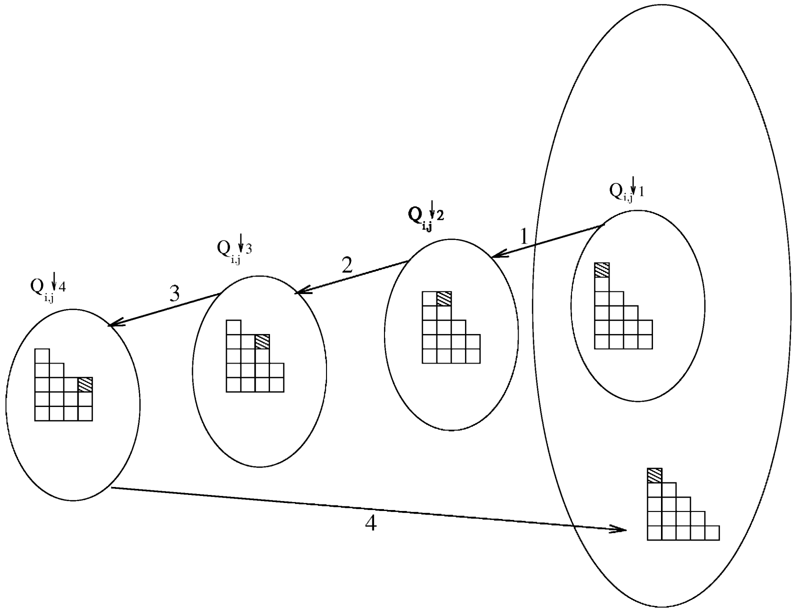

Fig. 11. Successive copies of a $Q_{i,j}$.

n grains that begins with stairs of length at least i). Likewise, $Q_{i,j}(n)$ denotes the part $Q_{i,j}$ of $SPM(n)$. We can make three remarks:

- The elements obtained from $P_i(n)$ by applying the operator $^{\downarrow_1}$ any number of times do not belong to a $P_i(m)$ with $m > n$ since they begin with a cliff.
- The elements of $(P_i(n))^{\downarrow_2}$ begin with a plateau at column 1 followed by a cliff at column 2. Then, if we apply $^{\downarrow_1}$ to these elements we obtain sand piles which begin with stairs of length exactly 1 (i.e. $(P_i(n))^{\downarrow_2\downarrow_1} \subseteq P_1(n+2)$). Likewise, $(P_i(n))^{\downarrow_k\downarrow_{k-1}\cdots\downarrow_2\downarrow_1} \subseteq P_{k-1}(n+k)$.
- The elements of $(P_i(n))^{\downarrow_k}$ $(k \leqslant i)$ begin with a stair of length exactly $k-1$ and hence are in $P_k(n+1)$.

From these remarks we deduce the result:

Theorem 4. *Let i be an integer such that $i(i+1)/2 \leqslant n+2$. Then*:

$$P_i(n+2) = T(n+2) \sqcup (P_i(n-i+1))^{\downarrow_1\downarrow_2\cdots\downarrow_i\downarrow_{i+1}} \sqcup \left(\bigsqcup_{k>i} (P_k(n+1))^{\downarrow_{k+1}} \right), \qquad (1)$$

where $T(n+2)$ is a set that contains one partition at most, namely,

$$T(n+2) = \begin{cases} \{(k, k-1, \ldots, 2, 1)\} & \textit{if } \exists k \textit{ integer s.t. } n+2 = \dfrac{k(k+1)}{2} \\ \emptyset & \textit{otherwise} \end{cases}$$

with the initial conditions:

$$P_1(1) = \{(1,0,\ldots)\},$$

$$P_i(1) = \emptyset \quad \forall i > 1,$$

$$P_i(2) = \emptyset \quad \forall i.$$

Proof. $P_i(n+2)$ contains each of the sets in the right-hand side of Eq. (1). Indeed, if we add one grain on each of the $i+1$ first columns of a sand pile with $n-i+1$ grains that begins with a stair of length i, we obtain a sand pile with n grains which also begins with a stair of length i. If we take a sand pile of $n+1$ grains which begins with a stair of length at least i, and we add a grain on column $k+1$ with $k>i$, we obtain a sand pile of $n+2$ grains which also begins with a stair of length at least i. Finally, if $n+2$ has the form $k(k+1)/2$ for some integer k and if i is smaller than or equal to k (the length of the sand pile $(k, k-1, \ldots, 1)$) then this element of $T(n+2)$ begins with a stairs of length at least i.

Likewise, each element of $P_i(n+2)$ is in one of those sets. Let s be in $P_i(n+2)$. Three cases are possible:

- s has a step at each column, i.e. $s \in T(n+2)$.
- s begins with stairs of length k with $k \geqslant i$ and s has a plateau at $k+1$. Then, it is an element of $(P_{k+1}(n+1))^{\downarrow_{k+2}}$. We know that such elements exist from the characterisation of Theorem 2.
- s begins with a stair of length k with $k \geqslant i$ and s has a cliff at $k+1$. Then, s is an element of $(P_i(n-i+1))^{\downarrow_1\downarrow_2\cdots\downarrow_{i+1}}$. We know that such elements exist from the characterisation of Theorem 2.

Now, let us show that the unions in Formula 1 are disjoints. The elements of the set $(P_k(n+1))^{\downarrow_{k+1}}$ with $k>i$ begin with stairs of length exactly k. So, the set $(P_k(n+1))^{\downarrow_{k+1}}$ and $(P_{k'}(n+1))^{\downarrow_{k'+1}}$ with $k,\ k' > i$ are pairwise disjoints. Moreover, the set $(P_i(n-i+1))^{\downarrow_1\downarrow_2\downarrow_{\ldots}\downarrow_i\downarrow_{i+1}}$ only contains elements that begin with stairs of length exactly i, so they doesn't intersect the parts $(P_k(n+1))^{\downarrow_{k+1}}$ which begin with stairs of length k with $k>i$. Finally, if $T(n+2)$ is non-empty, its element clearly does not belong to any of the other sets. □

This theorem gives a better understanding of the structure of the lattices $SPM(n)$. Since the unions are disjoints, the formula is even more interesting as it gives a way to compute the cardinality of $SPM(n)$. We first state the following corollary, immediate from Theorem 4.

Corollary 1. *Let* $p_{i,n}$ *denote* $|P_i(n)|$, *i.e.* $p_{i,n}$ *is the number of partitions in* $SPM(n)$ *that begin with stairs of length at least i. We have*

$$p_{i,n+2} = p_{i,n-i+1} + \sum_{j>i} p_{j,n+1} + \delta_{i,n+2},$$

where

$$\delta_{i,n+2} = \begin{cases} 1 & \textit{if } \exists k \in \mathbb{N} \textit{ s.t. } n = \dfrac{k(k+1)}{2}, \\ 0 & \textit{otherwise} \end{cases}$$

with the initial conditions $p_{1,1}=1$, $p_{1,j}=0$ *for all* $j>1$, $p_{j,2}=0$ *for all* $j>0$, *and* $p_{0,2}=2$.

This corollary gives a way to compute the number of elements in $SPM(n)$ since $|SPM(1)|=1$ and

$$|SPM(n)| = |SPM(n-1)| + \sum_i p_{i,n-1}.$$

Notice that this formula is nothing but the formula of the Corollary 1 specified for $i=0$. This is not surprising, since any element of $SPM(n)$ begins with stairs of length at least 0.

3. Infinite extension of SPM

Let us now present an infinite extension of SPM. Two different possible generalisations are natural to extend the notions studied until here. The first one is to consider a column with an infinite number of grains as the initial configuration, and then study the evolution of the system with respect to the SPM rule. We call this model $SPM(\infty)$. The second one is to use the construction detailed in the first part of the paper to extend the order on $\sqcup_{n\geqslant 0} SPM(n)$. It turns out that these two ideas lead to two isomorphic objects. This gives us an efficient way to construct $SPM(\infty)$, as shown below. Afterwards, we introduce the infinite tree $SPT(\infty)$, and we show a possible coding of $SPM(\infty)$ using this tree. The study of the properties of this tree gives a new recursive formula to compute $|SPM(n)|$.

3.1. The infinite lattice SPM(∞)

$SPM(n)$ is the lattice of the configurations reachable from the partition (n) by iteration of the SPM rule. We will now define $SPM(\infty)$ as the set of all configurations reachable from (∞) (this is the configuration where the first column contains infinitely many grains). The covering relation on $SPM(\infty)$ is defined by: $s \stackrel{i}{\rightarrow} t$ if and only if t is obtained from s by application of the SPM rule on the ith column. The order on $SPM(\infty)$ is the reflexive and transitive closure of this covering relation. Notice that any element s of $SPM(\infty)$ has the form $(\infty, s_2, s_3, \ldots, s_k)$. The first partitions in $SPM(\infty)$ are given in Fig. 12 along with their covering relations (the first column, which always contains an infinite number of grains, is not represented on this diagram).

Notice also that the first column does not influence the characterisation of the elements given in Theorem 2. We will now show that $SPM(\infty)$ is a lattice. To do so, we

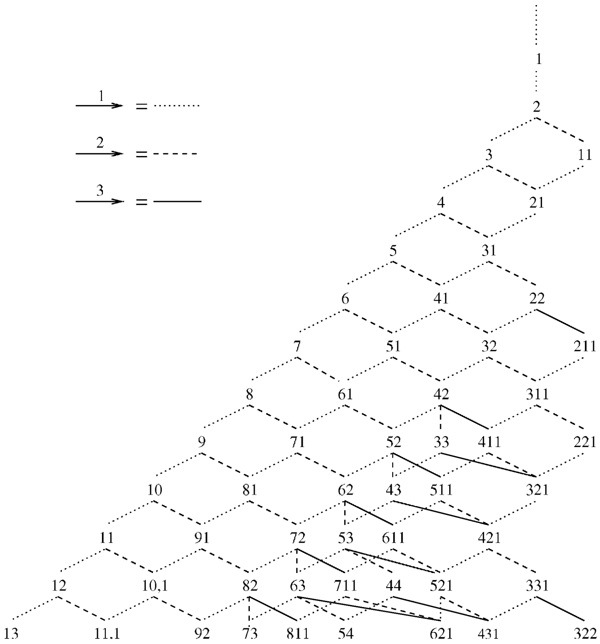

Fig. 12. First elements and transitions of $SPM(\infty)$.

will use the notion of *shot vector* (see [13, Section 5.3]). The shot vector $k(s,t)$ from the sand pile $s \in SPM(n)$ to the sand pile $t \in SPM(n)$ is defined by the following: the ith component $k_i(s,t)$ of $k(s,t)$ is the number of applications of the SPM rule on column i in order to obtain t from s.

We need here an extension of this definition: the ith component of the shot vector $k((\infty),s)$ from (∞) to $s \in SPM(\infty)$ is the number of applications of the SPM rule on column i in order to obtain s from (∞). It is straightforward to see that $k((\infty),s)$ is given by:

$$k_1((\infty),s) = s_2 + s_3 + \cdots,$$
$$k_2((\infty),s) = s_3 + s_4 + \cdots$$

and in general

$$k_i((\infty),s) = \sum_{j>i} s_j.$$

From [17], we have

Lemma 1. *Let s and t be two elements of $SPM(\infty)$. Then,*

$$s \geqslant t \quad \text{iff} \quad k((\infty),s) \leqslant k((\infty),t).$$

Moreover, if m denotes the max of $k((\infty),s)$ *and* $k((\infty),t)$ *then the partition u such that* $k((\infty),u)=m$ *is in* $SPM(\infty)$ *and* $u=\inf(s,t)$.

With this result, we can show that $SPM(\infty)$ is a lattice:

Theorem 5. *The set* $SPM(\infty)$ *is a lattice. Moreover, let* $s=(\infty,s_2,\dots,s_k)$ *and* $t=(\infty,t_2,\dots,t_l)$ *be two elements of* $SPM(\infty)$, *then*, $\inf(s,t)=u$ *in* $SPM(\infty)$, *where*

$$u_i = \max\left(\sum_{j\geqslant i} s_j,\ \sum_{j\geqslant i} t_j\right) - \sum_{j>i} u_j \quad \textit{for all } i \textit{ such that } 2\leqslant i\leqslant \max(k,l)$$

and $\sup(s,t)=\inf\{u\in SPM(\infty),\ u\geqslant s,\ u\geqslant t\}$.

Proof. From Lemma 1 and the definition of the shot vectors in $SPM(\infty)$, we have the formula for the infimum. Since (∞) is the maximal element of $SPM(\infty)$, this set is a lattice. □

From the definition, it is possible to show that $SPM(\infty)$ contains an isomorphic copy of $SPM(n)$ for any integer n.

Proposition 8. *Let n be a positive integer. The application*:

$$\pi : SPM(n) \to SPM(\infty)$$
$$s=(s_1,s_2,\dots,s_k) \to \bar{s}=(\infty,s_2,\dots,s_k)$$

is a lattice embedding, which means that it is injective and preserves the infimum and the supremum.

Proof. Again, we will use the shot vector $k(a,b)$ from a to b. Recall that $k_i(a,b)$ is nothing but the number of grains falling from column i in order to obtain b from a. Let s and t be in $SPM(n)$. Suppose $\bar{s}=\bar{t}$, i.e. $s_i=t_i$ for all $i\geqslant 2$. We have

$$s_1 = n - \sum_{i\geqslant 2} s_i = n - \sum_{i\geqslant 2} t_i = t_1,$$

hence $s=t$ and π is injective.

It is clear that $a \xrightarrow{i} b$ in $SPM(n)$ if and only if $\bar{a} \xrightarrow{i} \bar{b}$ in $SPM(\infty)$, hence $a\geqslant b$ in $SPM(n)$ if and only if $\bar{a}\geqslant\bar{b}$ in $SPM(\infty)$. So, in this case, $k(a,b)=k(\bar{a},\bar{b})$. Let c be the infimum of two elements a and b in $SPM(n)$. We show that $\bar{c}=\inf(\bar{a},\bar{b})$ in $SPM(\infty)$. Since $c=\inf(a,b)$ in $SPM(n)$, we have

$$k((n),c) = \max(k((n),a),k((n),b)).$$

Moreover, $k((n),a))=k((\infty),\bar{a})$ for all element a of $SPM(n)$, hence we can deduce $k((\infty),\bar{c})=\max(k((\infty),\bar{a}),k((\infty),\bar{b})$, and $\bar{c}=\inf(\bar{a},\bar{b})$ in $SPM(\infty)$, as expected. So, the infimum is preserved.

Now, let us prove that the supremum is preserved. Let $d=(d_1,\ldots,d_n)=\sup(a,b)$ in $SPM(n)$, and let $e=(\infty,e_2,\ldots,e_m)=\sup(\bar{a},\bar{b})$ in $SPM(\infty)$. We must show that $\bar{d}=e$. To do so, we show that an element f of $SPM(n)$ such that $e=\bar{f}$ exists. Since $d\geqslant a$ and $d\geqslant b$ in $SPM(n)$, we have $\bar{d}\geqslant\bar{a}$ and $\bar{d}\geqslant\bar{b}$ in $SPM(\infty)$, hence $\bar{d}\geqslant e$ in $SPM(\infty)$, and $k((\infty),\bar{d})\leqslant k((\infty),e)$. We first show that $k_1((\infty),\bar{d})=k_1((\infty),e)$. Without loss of generality, we can suppose that $a_1\geqslant b_1$. Notice that the partition $(a_1,a_1,a_1-1,a_1-2,\ldots)$ is greater than a and b, hence greater than d, and so $a_1\geqslant d_1$. Since $d\geqslant a$, we have $d_1=a_1$ and $k_1((\infty),\bar{d})=k_1((\infty),\bar{a})$. Moreover, $\bar{a}\leqslant e\leqslant\bar{d}$, hence $k_1((\infty),\bar{d})=k_1((\infty),e)$. Let us define $f=(n-e_2-\cdots-e_m,e_2,\ldots,e_m)$. Since $e\leqslant\bar{d}$, we have $e_2\leqslant d_2$ and since e verifies the characterisation of Theorem 2, so does f, hence $f\in SPM(n)$ and $e=\bar{f}$. Since $e\geqslant\bar{a}$ and $e\geqslant\bar{b}$, we have $f\geqslant a$ and $f\geqslant b$, hence $f\geqslant d$. This implies that $e=\bar{f}\geqslant\bar{d}$. Therefore $e=\bar{d}$. This gives the result. □

Let $\overline{SPM(n)}$ denotes the image by π of $SPM(n)$ in $SPM(\infty)$. From Proposition 8, $\overline{SPM(n)}$ is a sublattice of $\underline{SPM(\infty)}$. By Theorem 4, $SPM(n)^{\downarrow_1}$ is a sublattice of $SPM(n+1)$, hence, since $\overline{SPM(n)^{\downarrow_1}}=\overline{SPM(n)}$, we have an increasing sequence of sublattices

$$\overline{SPM(0)}\leqslant\overline{SPM(1)}\leqslant\cdots\leqslant\overline{SPM(n)}\leqslant\overline{SPM(n+1)}\leqslant\cdots\leqslant SPM(\infty),$$

where $\leqslant$ denotes the sublattice relation.

Let $s=(\infty,s_2,s_3,\ldots,s_k)$ be an element of $SPM(\infty)$, then s verifies the characterisation of Theorem 2. If one takes $s_1=s_2+1$ and $n=\sum_{i=1}^{k}s_i$, we have that $s'=(s_1,s_2,\ldots,s_k)$ is an element of $SPM(n)$. This implies that $s=\pi(s')$ and that s is an element of $\overline{SPM(n)}$, therefore

$$\bigcup_{n\geqslant 0}\overline{SPM(n)}=SPM(\infty).$$

Let us now study the disjoint union of the lattices $SPM(n)$ for $n\geqslant 0$. Let us define

$$S=\bigsqcup_{n\geqslant 0}SPM(n),$$

on which we extend the order relation of each $SPM(n)$ as follows. Let $s\in SPM(m)$ and $t\in SPM(n)$. We define $s\xrightarrow{i}t$ in S if and only if we are in one of the two following cases:

- $n=m$ and $s\xrightarrow{i}t$ in $SPM(n)$,
- $i=0$, $n=m+1$ and $b=a^{\downarrow_1}$.

In other terms, the elements of $SPM(n)$ are linked to each other as usual whereas each element a of $SPM(n)$ is linked to $a^{\downarrow_1}\in SPM(n)$ by an edge labelled 0. From this covering relation, one can define an order on the set S as the reflexive and transitive closure of this covering relation.

Theorem 6. *For all integer n, $SPM(n)$ is a sublattice of S.*

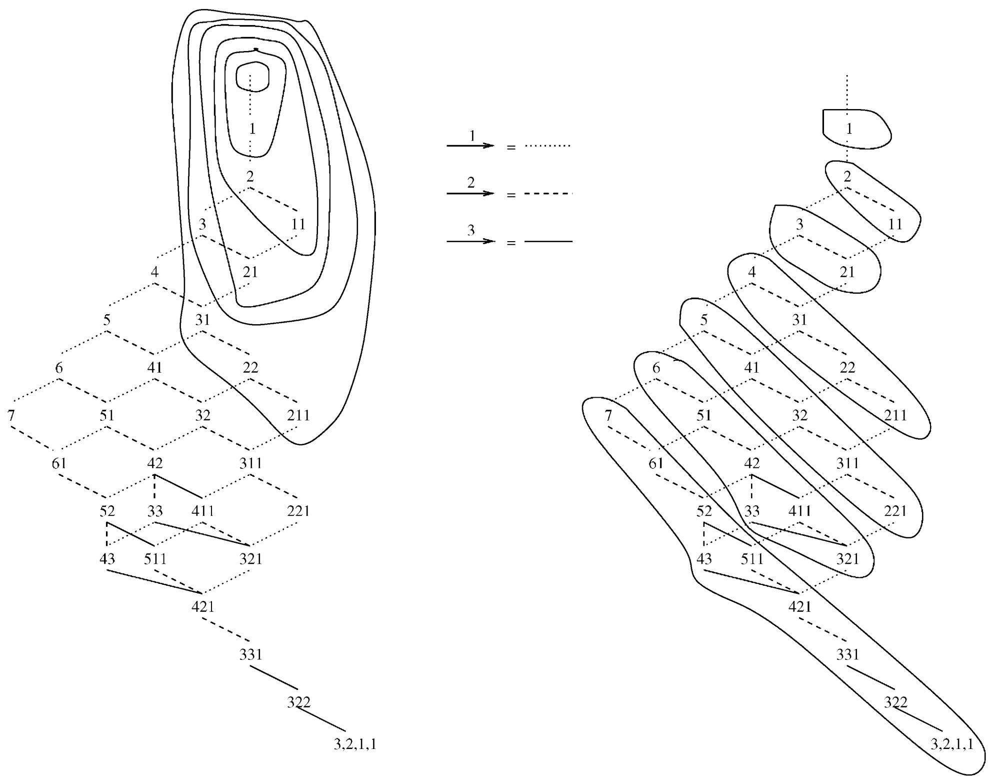

Fig. 13. The two ways to identify $SPM(n)$ in $SPM(\infty)$ for all n.

Proof. The fact that $SPM(n)$ is present in S is immediate from the definition. What we have to show is that the lattice structure of $SPM(n)$ is preserved in S. Let s and t be two elements of $SPM(n)$. We have to show that the infimum and the supremum of s and t in S are in $SPM(n)$. Let $u=\inf(s,t)$ in $SPM(n)$ and $u'=\inf(s,t)$ in S. We have that $s \geqslant u' \geqslant u$ in S, thus

$$\sum_{i \geqslant 1} s_i \leqslant \sum_{i \geqslant 1} u'_i \leqslant \sum_{i \geqslant 1} u_i,$$

and

$$\sum_{i \geqslant 1} u'_i = n.$$

Therefore u' is an element of $SPM(n)$, and we have $u'=u$. The same method can be applied for the supremum. □

This result is illustrated in Fig. 13 (right). The surprising result is that these two ways to extend the sand pile model to infinity, i.e. the first one by adding new elements to some $SPM(n)$ to extend it into $SPM(n+1)$ and infinitely iterating the process to obtain $SPM(\infty)$, and the second one by linking together all the $SPM(n)$ for all n to obtain S, lead to the same object.

Theorem 7. *The application* χ *defined by*

$$\chi : S = \bigsqcup_{n\geqslant 0} SPM(n) \to SPM(\infty),$$

$$s = (s_1, s_2, \dots, s_k) \mapsto \chi(s) = (\infty, s_1, s_2, \dots, s_k)$$

is a lattice isomorphism, which means that it is one to one and preserves the supremum and the infimum. Moreover, $s \xrightarrow{i} t$ *in* $\bigsqcup_{n\geqslant 0} SPM(n)$ *if and only if* $\chi(s) \xrightarrow{i+1} \chi(t)$ *in* $SPM(\infty)$.

Proof. The application χ is obviously injective. Let us show that χ is surjective. Let $s = (\infty, s_1, \dots, s_l)$ be an element of $SPM(\infty)$. Define $n = \sum_{i\geqslant 1} s_i$ and $s' = (s_1, \dots, s_l)$. Since s is in $SPM(\infty)$, s verifies the conditions of Theorem 2, and so does s'. Therefore s' is an element of $SPM(n)$, and since $s = \chi(s')$, the application χ is surjective.

It is clear that for all $s,\ t \in S$, one has $s \xrightarrow{i} t$ if and only if $\chi(s) \xrightarrow{i+1} \chi(t)$. Obviously, χ is an order isomorphism. Since $SPM(\infty)$ is a lattice, we can conclude that χ is a lattice isomorphism. □

This result is illustrated in Fig. 13 (left). In the following, we simplify the notations by representing the elements of $SPM(\infty)$ without their first column. Our aim is now to construct large parts of $SPM(\infty)$. A first solution is to construct $SPM(n)$ for large values of n. However, this does not lead to filters (a *filter* of a lattice is a subset of this lattice closed for the supremum) of $SPM(\infty)$. We will now define special filters of $SPM(\infty)$ and explain how we can construct them efficiently. For a given n, let us denote by $SPM(\leqslant n)$ the set

$$\bigsqcup_{0\leqslant i\leqslant n} SPM(i).$$

For example, $SPM(\leqslant 7)$ is shown in Fig. 13. It is easy to see that $SPM(\leqslant n)$ is a filter of $SPM(\infty)$ for all n. The infinite lattice $SPM(\infty)$ can be regarded as a limit of this sequence of posets. The results presented in this section give us an efficient method to construct $SPM(\leqslant n)$ for all n (see Algorithm 2 and Theorem 8). Moreover, we show another property of $SPM(\leqslant n)$.

Proposition 9. *The poset* $SPM(\leqslant n)$ *is a sublattice of* $SPM(\infty)$ *for all* n.

Proof. To show the claim, it suffices to consider $s \in SPM(k)$ and $t \in SPM(l)$, with $k \leqslant l \leqslant n$, and show that $\inf(s,t)$ and $\sup(s,t)$ (which are in $SPM(\infty)$ since $SPM(\infty)$ is a lattice) are also in $SPM(\leqslant n)$. Let $u = \sup(s,t)$ in $SPM(\infty)$. Since $u \geqslant s$ in $SPM(\infty)$, there exists an integer $m \leqslant k$ such that $u \in SPM(m)$, therefore u is in $SPM(\leqslant n)$. Let now $u' = \inf(s,t)$ in $SPM(\infty)$. Let s' be the partition obtained by addition of $l-k$ grains on the first column of s. Then, $s' \in SPM(l)$ and $\inf(s',t)$ in $SPM(\infty)$ is also in $SPM(l)$. So, $u=\inf(s,t)$ is also greater than or equal to $\inf(s',t)$ in $SPM(\infty)$, hence $u \in SPM(\leqslant l)$. □

Algorithm 2. CONSTRUCTION OF $SPM(\leqslant n)$

```
Input: an integer n
Output: SPM(≤n)
begin
  - Init Result with SPM(0);
  - for 1≤i≤n do
      - Extract SPM(i) from Result:
      Start from (0);
      Depthfirst search Result to get the connex part containing SPM(i);
      - Link SPM(i) to Result:
      for  each s ∈ SPM(i − 1) do
          Link s ∈ SPM(i − 1) to s^{↓1} in SPM(i) with an edge —1→;
  - return Result
end
```

Theorem 8. *Algorithm* 2 *computes* $SPM(\leqslant n)$ *in time linear in the number of elements and edges in* $SPM(\leqslant n)$.

Proof. Let $|S|$ denote the number of elements and edges in S for any lattice S. For all i between 1 and n, the lattice $SPM(i)$ is a connected component of *Result*. It is well known that the computation of a connected component has a cost proportional to the cardinality of this same component, so its extraction can be processed in linear time. The addition of the edges that link $SPM(i)$ to $SPM(i-1)$ can be done in $O(|SPM(i-1)|)$. Therefore, each iteration of the *for* loop is executed in $O(|SPM(i)|)$, hence the execution of the whole loop is linear with respect to the total number of elements and edges of $SPM(\leqslant n)$. This is the asymptotic cost of the whole algorithm. □

3.2. The infinite tree $SPT(\infty)$

As shown in our construction of $SPM(n+1)$ from $SPM(n)$, each element s of $SPM(n+1)$ is obtained from an element $s' \in SPM(n)$ by addition of one grain: $s = s'^{\downarrow_i}$ with i an integer between 1 and $e(s')+1$. Thus, we can define an infinite tree $SPT(\infty)$ (for Sand Pile Tree) whose nodes are the elements of $\bigsqcup_{n\geqslant 0} SPM(n)$ and in which the fatherhood relation is defined by:

$$t \text{ is the } i\text{th son of } s \text{ if and only if } t = s^{\downarrow_i} \quad \text{for some i,} \quad 1 \leqslant i \leqslant e(s)+1.$$

The edge $s \longrightarrow s^{\downarrow_i}$ is labelled with i. The root of this tree is (0). The eight first levels of $SPT(\infty)$ are shown in Fig. 14 (we call the set of elements of depth n the “level

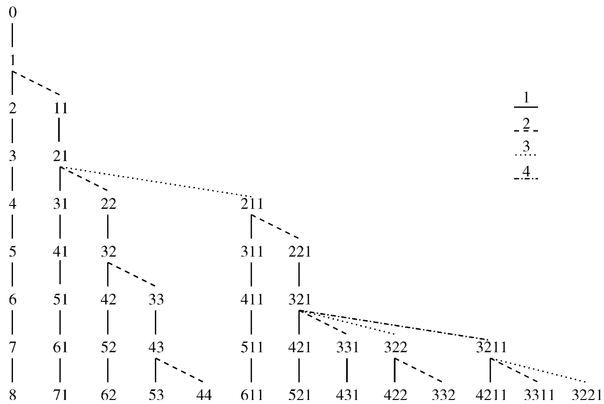

Fig. 14. The first levels of $SPT(\infty)$.

n" of the tree). Each node s of $SPT(\infty)$ has $e(s)+1$ sons linked to s with edges labelled $\xrightarrow{1}, \xrightarrow{2}, \ldots, \xrightarrow{e(s)+1}$.

Notice that, although the notation is the same as the one used for the SPM transitions in the lattice, an edge $s \xrightarrow{i} t$ in the tree means that t is obtained from s by addition of one grain on its ith column ($t = s^{\downarrow i}$), and not that t is obtained from s by having the top grain of the ith column fall onto the $(i+1)$th. Therefore, if $s \in SPM(n)$ then $t \in SPM(n+1)$. So, the structure of the lattices is not directly visible in $SPT(\infty)$. One goal in the following part of the section will be to explore the possibility of the construction of the lattices from the tree. The first results will be to state, as we could have guess from Theorem 7, that there are two ways to find $SPM(n)$ in $SPT(\infty)$.

Proposition 10. *The level n of $SPT(\infty)$ contains exactly the elements of $SPM(n)$.*

Proof. Straightforward from the construction of $SPM(n+1)$ from $SPM(n)$ given above. □

Proposition 11. *For all integer n, the set $\overline{SPM(n)}$ is a subtree of $SPT(\infty)$ that contains its root.*

Proof. The proposition is obviously true for $n = 1$. Let us suppose it is true for n, and let us verify it is true for $n+1$. By construction, the elements of $\overline{SPM(n+1)} \backslash \overline{SPM(n)}$ are sons of elements of $\overline{SPM(n)}$. The result follows. □

We will now show that $SPT(\infty)$ can be described recursively, which allows us to give a new recursive formula for $|SPM(n)|$. Let us first consider one element s of $SPT(\infty)$ and let $k = e(s)$. By definition of $SPT(\infty)$, s has exactly $k+1$ sons. Notice that the k first sons of s all verify the same following property: for all integer i between

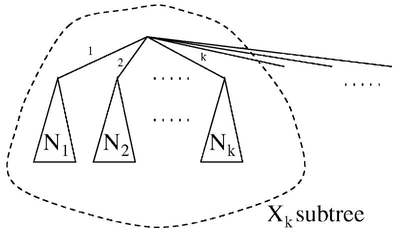

Fig. 15. Definition of the X_k subtrees.

1 and k, the son $s^{\downarrow i}$ begins with a stair of length $i-2$, has a plateau at $i-1$, and a cliff at i. From this remark, we introduce certain types of subtrees of $SPT(\infty)$.

Definition 1. We call N_k subtree, with $k \geqslant 1$, any subtree T of $SPT(\infty)$ which is rooted at an element s that begins with a stair of length $k-2$, has a plateau at $k-1$ and a cliff at k, and contains all the descendants of s.

Definition 2. We call X_k subtree, with $k \geqslant 1$, any subtree T of $SPT(\infty)$ which is rooted at a node that has at least k sons and such that the ith son is the root of a N_i subtree for all i between 1 and k. The structure of X_k subtrees is shown in Fig. 15. Moreover, we define $X_0 = \emptyset$.

Notice that $SPT(\infty)$ contains some X_k subtrees. Indeed, if one takes an element s which begins with a stair of length k, then its ith son is the root of a N_i subtree for all i, $1 \leqslant i \leqslant k$. So s is the root of a X_k subtree. Notice also that s might have other sons outside the X_k subtree. With this remark, we characterise in the following proposition the structure of the N_k subtrees, with $k \geqslant 1$. See Fig. 18 and 19.

Proposition 12. *We have the following statements*:

1. *A N_1 subtree is an infinite chain whose edges are all labelled* 1.
2. *A N_k subtree, with $k \geqslant 2$, is composed by a chain of k nodes whose edges are labelled $k-1$, $k-2, \ldots, 1$ and whose ith node is the root of a X_{k-1-i} subtree for all i between* 1 *and $k-2$. Moreover, the kth node is root of a X_k subtree.*

Proof. 1. By definition, the root of a N_1 subtree is an element s which begins with a cliff, $e(s)+1=1$ (the conditions on the initial stair and the plateau make no sense in this case). Therefore, the only son of s is $s \xrightarrow{1} s^{\downarrow 1}$, which also begins with a cliff. Therefore, $s^{\downarrow 1}$ is the root of a N_1 subtree and we can conclude by induction.

2. Let k be greater than 1. Let us consider a partition s such that:

- $e(s) = k-2$,
- s has a plateau at $k-1$,
- and s has a cliff at k.

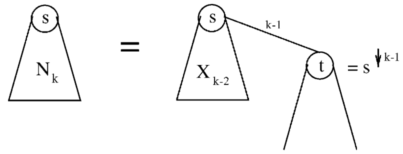

Fig. 16. First step of the structure of N_k.

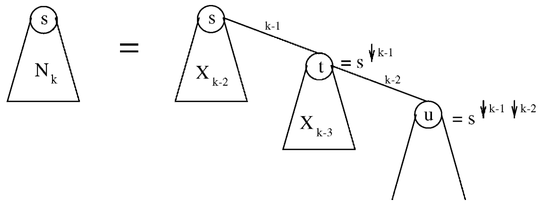

Fig. 17. Second step of the structure of N_k.

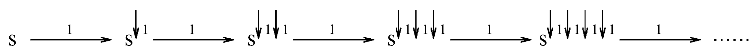

Fig. 18. Structure of the N_1 subtrees.

The node s has $k-1$ sons: $s^{\downarrow_1}$, $s^{\downarrow_2},\ldots,$ $s^{\downarrow_{k-1}}$. Since $e(s) = k-2$, and from the remarks above, the node s is the root of a X_{k-2} subtree that contains all the elements reachable from its $k-2$ first sons. Let t be the $(k-1)$th son of s (t is outside the X_{k-2} subtree rooted at s). The subtree rooted at s is then the union of a X_{k-2} subtree and a subtree with root t (see Fig. 16). Look now at the subtree with root t. We have that $e(t)=k-3$, hence t is the root of a X_{k-3} subtree. Let u be the $(k-2)$th son of t. We then obtain Fig. 17. When this process is iterated, we obtain $x=s^{\downarrow_{k-1}\downarrow_{k-2}\cdots\downarrow_2}$. This element x begins with a plateau of length 1 followed by a stair of length $k-2$ and a cliff (Fig. 18). Therefore x has only one son $x \xrightarrow{1} y$. This element y begins with a stair of length $k-1$ followed by a cliff. As noticed above, y is the root of a X_k subtree and this subtree contains all the elements reachable from y. Then we obtain the announced structure of N_k subtrees, see Fig. 19. □

Using now the fact that a X_k subtree is defined in terms of N_k subtrees, we can describe the structure of a N_k subtree only in terms of other N_i subtrees with $i \leqslant k$ as shown Fig. 20.

Notice that we can deduce directly by induction from the structure of the N_k subtrees shown in Fig. 20 that all the edges in a N_k subtree are labelled with integer smaller than or equal to k. The recursive structures we have defined, and the propositions given above allow a compact representation of the tree $SPT(\infty)$ as a chain:

Theorem 9. *The tree $SPT(\infty)$ can be represented by the infinite chain shown in Fig. 21. The nodes of this chain are the fixed points of $SPM(n)$ for $n \geqslant 0$.*

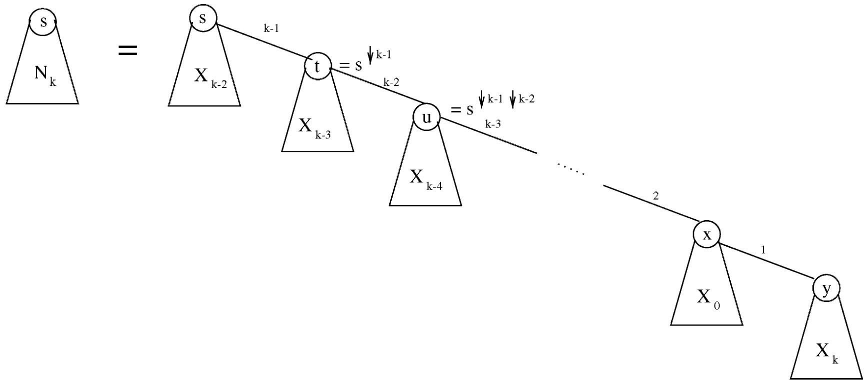

Fig. 19. Structure of N_k subtrees.

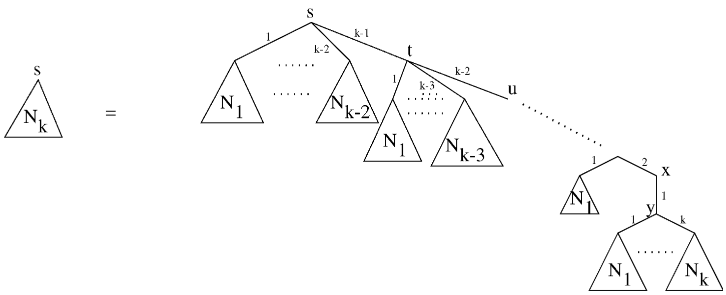

Fig. 20. Self-referencing structure of N_k.

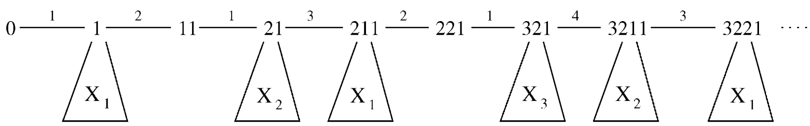

Fig. 21. The tree $SPT(\infty)$ represented as an infinite chain.

The chain is defined as follows: let k a positive integer and let $P_k = (k, k-1, k-2, \ldots, 2, 1)$ and $P_{k+1} = (k+1, k, k-1, k-2, \ldots, 2, 1)$; the subchain between P_k and P_{k+1} contains $k+1$ nodes:

$$P_k \xrightarrow{k+1} P_k^{\downarrow k+1} \xrightarrow{k} P_k^{\downarrow k+1 \downarrow k} \xrightarrow{k-1} \cdots \xrightarrow{2} P_k^{\downarrow k+1 \cdots \downarrow 2} \xrightarrow{1} P_{k+1},$$

where each node $P_k^{\downarrow k+1 \cdots \downarrow i}$ with i between 3 and $k+1$ is the root of a X_{i-2} subtree, and P_k is the root of a X_k subtree.

Proof. Let us consider the rightmost chain in $SPT(\infty)$. This chain is composed by the fixed points of $SPM(n)$ for $n \geqslant 0$. Let k be a positive integer. Let us consider the subchain of this chain that begins with $P_k = (k, k-1, \dots, 1)$ and terminates with $P_{k+1} = (k+1, k, \dots, 1)$:

$$P_k \stackrel{k+1}{\longrightarrow} P_k^{\downarrow k+1} \stackrel{k}{\longrightarrow} P_k^{\downarrow k+1 \downarrow k} \stackrel{k-1}{\longrightarrow} \cdots \stackrel{2}{\longrightarrow} P_k^{\downarrow k+1 \cdots \downarrow 2} \stackrel{1}{\longrightarrow} P_{k+1}.$$

The node $s = P_k^{\downarrow k+1 \cdots \downarrow i}$ with $i \geqslant 2$, begins with stairs of length $i-2$ followed by a plateau at $i-1$, hence s is the root of a X_{i-2} subtree and its last son is obtained by $P_k^{\downarrow k+1 \cdots \downarrow i} \stackrel{i-1}{\longrightarrow} P_k^{\downarrow k+1 \cdots \downarrow i-1}$. This is the next node in the chain. Therefore $SPT(\infty)$ can be described as indicated. □

As seen above, the level n of $SPT(\infty)$ contains exactly $SPM(n)$. Therefore, it suffices to count the number of paths of length n from the root of $SPT(\infty)$ to obtain $|SPM(n)|$. The recursive structure of the tree, detailed above, gives us a way to achieve this.

Theorem 10. *Let $c(l,k)$ denote the number of paths in a X_k subtree originating from the root and of length l, then we have*

$$c(l,k) = \begin{cases} 0 & \text{if } l \leqslant 0 \text{ and } k \leqslant 0, \\ 1 & \text{if } l > 0 \text{ and } k = 1, \\ k & \text{if } l = 1 \text{ and } k > 0, \\ c(l-k,k) + \sum_{i=1}^{k-1} c(l-i+1, k-i) + \varepsilon(l,k) & \text{otherwise}, \end{cases}$$

where $\varepsilon(l,k) = 0$ if $k > l$ and $\varepsilon(l,k) = 1$ otherwise.

Proof. The proof follows from the recursive structure of X_k detailed above. There is no path of length 0 or less, and the X_k are empty for $k \leqslant 0$, hence the first case. A X_1 subtree is a simple chain, hence there is exactly 1 path of any length, hence the second case. The third case is immediately deduced from the fact that the root of a X_k subtree has exactly k sons. Finally, the recursive formula in the fourth case comes from the fact that the structure of X_k subtrees shown in Fig. 15 allows us to consider a X_k subtree as a node s where s is the root of a X_{k-1} subtree and has one more son which is the root of a N_k subtree. Then, from the structure of N_k subtrees in terms of X_k ones shown in Fig. 19, we deduce a description of X_k subtrees in terms of X_i subtrees with $0 \leqslant i \leqslant k$, from which the formula is straightforward. □

Corollary 2. *The cardinality of $SPM(n)$ is given by*

$$|SPM(n)| = 1 + \sum_{i=1}^{k} \sum_{j=1}^{i} c\left(n - \frac{i(i-1)}{2} - j + 1, i - j + 1\right),$$

where k is the integer such that $k(k+1)/2 \leqslant n < (k+1)(k+2)/2$.

Proof. This formula is deduced from the chain structure of the tree, shown in Fig. 21. The quantity 1 corresponds to the path of length n that follows the chain without entering in a X_i subtree. The double sum corresponds to the repartition of the X_i subtrees along the chain. □

We will now show how information on $SPM(\infty)$ can be deduced from $SPT(\infty)$. The lattice structure $SPM(\infty)$ and the infinite tree $SPT(\infty)$ are defined over the same underlying set: $\bigsqcup_{n\geqslant 0} SPM(n)$. Therefore we can easily give a bijection from one to the other. We now show how the ordered structure of $SPM(\infty)$ can be deduced from $SPT(\infty)$.

Proposition 13. *Every element of $SPT(\infty)$ has an outgoing edge with label* 1 *in $SPM(\infty)$. Moreover, for $i\geqslant 1$, a partition s of $SPT(\infty)$ has an outgoing edge $\stackrel{i+1}{\rightarrow}$ in $SPM(\infty)$ if and only if s belongs to a N_i subtree of $SPT(\infty)$. It should be noticed that s may belong to several N_i subtrees for distinct values of i. In this case, s is the origin of an edge $\stackrel{i+1}{\rightarrow}$ in $SPM(\infty)$ for each such i. In other words, a is in a N_i subtree of $SPT(\infty)$ if and only if there is an outgoing edge from a labelled with $i+1$ in $SPM(\infty)$.*

Proof. By definition of $SPM(\infty)$, every partition in $SPM(\infty)$ obviously has a successor by $\stackrel{1}{\rightarrow}$. Suppose now $i\geqslant 1$ and suppose that a is in a N_i subtree of $SPT(\infty)$, and denote by s the root of this N_i subtree. By definition of the N_i subtrees, we have that $s_i - s_{i+1}\geqslant 2$, hence $a_i - a_{i+1}\geqslant 2$. Therefore, when adding the first infinite column to obtain the corresponding element of $SPM(\infty)$, we obtain that there is an outgoing edge from a labelled $i+1$ in $SPM(\infty)$.

Let us now consider a partition a having an outgoing edge labelled with $i+1$ in $SPM(\infty)$. From the structure of the tree shown in Fig. 21, we know that a is in a X_k subtree of $SPT(\infty)$. Indeed, the elements of the chain are fixed points of SPM, hence they have no outgoing edge in $SPM(\infty)$ except the ones labelled 1, as said above.

Let s be the root of such a X_k subtree of $SPT(\infty)$, that is, s is the root of a X_k subtree that contains a. Consider a path from s to a in this tree:

$$s = s_0 \stackrel{i_1}{\rightarrow} s_1 \stackrel{i_2}{\rightarrow} s_2 \stackrel{i_3}{\rightarrow} \cdots \stackrel{i_{l-1}}{\rightarrow} s_{l-1} \stackrel{i_l}{\rightarrow} s_l = a.$$

We have that $s_i - s_{i+1}\leqslant 1$ by definition and $a_i - a_{i+1}\geqslant 2$ since there is a transition from a labelled $i+1$ in $SPM(\infty)$. Therefore, there exists an integer j such that $i_j = i$. In fact, we have an even stronger condition on the path: it must verify $|i+1|<|i|$ where $|x|$ denotes the number of edges labelled with x on the path. But it is easy to see from the structure of the N_i subtrees shown in Fig. 20 that the only case where this happends is when $s_j \stackrel{i}{\rightarrow} s_{j+1}$ such that s_{j+1} is the root of a N_i subtree, hence a is in a N_i subtree, as announced. □

It follows from this proposition that we can find all the (immediate) successors of a partition s in $SPM(\infty)$: it is sufficient to go from the root of $SPT(\infty)$ to $s \in SPT(\infty)$ and so determinate the integers i such that s is in a N_i subtree.

4. Conclusion and perspectives

Through the study of the construction of $SPM(n+1)$ from $SPM(n)$, we obtained much information about this set. First, it is strongly self-similar and can be constructed using this property. Moreover, this construction procedure gives a formula for the cardinal of $SPM(n)$, where no formula was known before. In a second part, we gave a natural way to extend $SPM(n)$ to infinity, and again self-similarity of this infinite lattice appeared. Finally, we gave a tree structure to the sets $SPM(n)$ and $SPM(\infty)$, which allows efficient enumeration of $SPM(n)$, as well as another formula for the cardinal of $SPM(n)$.

The duplication process that appears during the construction of the lattices $SPM(n)$ may be much more general, and could be extended to other kinds of lattices, maybe leading to the definition of a special class of lattices, which contains the lattices $SPM(n)$. Moreover, the ideas developed in this paper could be applied to others dynamical systems, such as the Brylawski dynamical system [5], chip firing games, or tilings with flips, with some benefit.

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