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Termination and derivational complexity of confluent one-rule string-rewriting systems

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Abstract

It is not known whether the termination problem is decidable for one-rule string-rewriting systems, though the confluence of such systems is decidable by Wrathall (in: Word Equations and Related Topics, Lecture Notes in Computer Science, vol. 572, 1992, pp. 237–246). In this paper we develop techniques to attack the termination and complexity problems of confluent one-rule string-rewriting systems. With given such a system we associate another rewriting system over another alphabet. The behaviour of the two systems is closely related and the termination problem for the new system is sometimes easier than for the original system. We apply our method to systems of the special type $\{a^p b^q \to t\}$, where t is an arbitrary word over $\{a, b\}$, and give a complete characterization for termination. We also give a complete analysis of the derivational complexity for the system $\{a^p b^q \to b^n a^m\}$.

Keywords: Termination problem; String rewriting system; Derivational complexity

1. Introduction

String-rewriting systems (semi-Thue systems) are special term-rewriting systems where all the function symbols are of arity one. One-rule string-rewriting systems are thus considered to be the most simple example of rewriting systems. Even for these simplest systems two fundamental problems still remains unsolved, the word problem and the termination problem.

It is undecidable whether given a finite string-rewriting system is terminating [2]. Actually, termination is undecidable for three-rule string-rewriting systems [4]. However, it

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is not known whether the termination problem is decidable for one-rule string-rewriting systems, though the confluence of such systems is decidable [3, 12]. In our previous paper [9], we showed that the termination of confluent one-rule string-rewriting systems is reduced to that of one-rule systems $\{s \to t\}$ such that s is self-overlap-free (sof), that is, the set $\mathrm{OVL}(s,s) = \{\alpha \mid s'\alpha = \alpha s'' = s,\ \alpha \neq 1,\ \alpha \neq s\}$ of self-overlaps of s is empty. In this paper we develop some techniques to attack the termination and complexity problems of such systems.

Let $R = \{s \to t\}$ be a one-rule system over an alphabet Σ such that s is sof. With a derivation sequence $D : x_1 \to_R x_2 \to_R \cdots \to_R x_n \to \cdots$, we associate another sequence $\phi(D) : \omega_1 \to \omega_2 \to \cdots \to \omega_n \to \cdots$, called the trace of D, which traces the seams created by applications of the rule $s \to t$ in D. If $x = x'sx''$ is rewritten to $y = x'tx''$, two positions (x', tx'') and $(x't, x'')$ of y are the seams of y created by the step $x'sx'' \to_R x'tx''$, and all the seams of x inside the subword s are destroyed or patched. For each seam σ of x we give a label $\phi(\sigma)$ which is a letter in a certain alphabet different from Σ (the details will be given in Section 6). The trace $\phi(x)$ of x is a word $\phi(\sigma_1) \cdots \phi(\sigma_k)$ spelling the labels of seams of x from left to right. If by a step $x \to_R y$, seams $\sigma_i, \ldots, \sigma_j$ are patched and a pair of seams τ_1 and τ_2 are created, then the trace $\phi(x) = \phi(\sigma_1) \cdots \phi(\sigma_k)$ of x is transformed (rewritten) to the trace $\phi(\sigma_1) \cdots \phi(\sigma_{i-1})\phi(\tau_1)\phi(\tau_2)\phi(\sigma_{j+1}) \cdots \phi(\sigma_k)$ of y. Through this mechanism we associate another rewriting system over another alphabet with the original system R. The point is that the behaviour of these two systems is closely related. The new system is not any more a one-rule system and looks complex, but analysis of termination for it is sometimes easier than for the original system.

To demonstrate that our method is powerful we apply it to systems of the special type $\{a^p b^q \to t\}$, where t is an arbitrary word over $\{a, b\}$. It was proved by Sénizergues [8] that termination of these systems is decidable. In this paper we give a complete characterization for the systems to terminate. We also give a complete analysis of the derivational complexity for the system $\{a^p b^q \to b^n a^m\}$, for which a complete characterization of termination was given by Zantema and Geser [13].

In Section 2 we give basic definitions and some simple observations on derivation sequences. From Section 3 we concentrate on a one-rule system $\{s \to t\}$ such that s is sof. The notion of seam is introduced in Section 3. In a step $x = x'sx'' \to_R y = x'tx''$ of R-derivation, a left seam (x', tx'') and a right seam $(x't, x'')$ of y are created and a seam inside of s in x, if exists, is patched. A seam of x that is not patched is inherited by a seam of y in the step. In this way a derivation sequence induces a seamed word whose seams are created in some steps of the sequence and are inherited afterwards. In Section 4 we discuss simple derivation (an application of the rule is simple if it patches at most one seam). We introduce a certain system $\mathbf{S}_R$ over $A \cup B$ which simulates simple R-derivation, where the sets $A = \mathrm{OVL}(t, s)$ and $B = \mathrm{OVL}(s, t)$ of overlappings of s and t are considered to be alphabets. We prove that there is no infinite sequence of simple R-derivation if and only if $\mathbf{S}_R$ terminates. In Section 5 we discuss gentle derivation and tame derivation which are more general than simple derivation. An application of the rule is gentle if it is simple or it patches a pair of a right seam and a left seam. Set $\Xi = A \cup B \cup A^{\#} \cup B^{\#}$ and $\bar{\Xi} = \Xi \cup \{\not{c}, \$\}$, where $A^{\#}$ and $B^{\#}$ are sets bijective to A

and B, respectively, and $\not{c}$ and \$ are symbols outside Ξ. In Section 6 we give a label $\phi(\sigma) \in \bar{\Xi}$ to every seam σ in some natural way. For a word x with seams $\sigma_1, \ldots, \sigma_m$ from left to right, we have a word $\phi(x) = \phi(\sigma_1) \cdots \phi(\sigma_m)$ over $\bar{\Xi}$, which is called the trace of x. For a derivation sequence $D : x_1 \to_R x_2 \to_R \cdots \to_R x_n$, the trace $\phi(D)$ of D is the sequence $\phi(x_1) \to \phi(x_2) \to \cdots \to \phi(x_n)$ of traces. In Section 7 using this correspondence between R-derivation sequences and traces, we construct a system $\mathbf{G}_R$ over the alphabet Ξ, which simulates gentle R-derivation. We prove that there is no infinite sequence of gentle R-derivation if and only if $\mathbf{G}_R$ terminates. We also give a result on complexity of gentle derivation in terms of $\mathbf{G}_R$. Section 8 is devoted to prove some technical lemmas on termination of finite (not necessarily one-rule) rewriting systems. In Section 9 we give a complete result on the complexity for the system $\{a^p b^q \to b^n a^m\}$, and in Section 10 we give a complete characterization for termination of systems of the type $\{a^p b^q \to t\}$, where t is an arbitrary word over $\{a, b\}$.

2. Preliminaries

Let Σ be a (finite) alphabet and Σ^* be the free monoid generated by Σ. We set $\Sigma^+ = \Sigma^* - \{1\}$, where 1 is the empty word. The length of a word $x \in \Sigma^*$ is denoted by $|x|$. A *string-rewriting system* R is a subset of $\Sigma^+ \times \Sigma^*$. An element (s, t) of R is called a *rule* and written as $s \to t$. We have a *one-step derivation* $x \to_R y$ if $x = x'sx''$, $y = x'tx''$ for some rule $s \to t \in R$. If there is no y such that $x \to_R y$, x is *irreducible*, otherwise, it is *reducible*. Let $\mathrm{Irr}(R)$ denote the set of irreducible words. We have

$$\mathrm{Irr}(R) = \Sigma^* - \Sigma^* \cdot \mathrm{Left}(R) \cdot \Sigma^*,$$

where $\mathrm{Left}(R) = \{s \mid s \to t \in R\}$. A system R is *terminating* (or *noetherian*), if there is no infinite sequence of derivation:

$$x_1 \to_R x_2 \to_R \cdots \to_R x_n \to_R \cdots .$$

If there is a derivation sequence of length n from x to y, we write as $x \to_R^n y$. In particular, $\to_R^0$ is the identity relation. By $\to_R^*$ we denote the reflexive transitive closure of the relation $\to_R$, that is, $\to_R^* = \bigcup_{n=0}^{\infty} \to_R^n$. Moreover, we set $\to_R^+ = \bigcup_{n=1}^{\infty} \to_R^n$. If $x \to_R^* y$, x is called an *ancestor* of y and y is called a *descendant* of x. R is *confluent* if any x and y have a common descendant as far as they have a common ancestor.

Let X be a subset of Σ^*. R is terminating on X if there is no infinite sequence of derivation starting with an element of X. R is *weakly terminating* on X if every word of X has an irreducible descendant. For a word $x \in \Sigma^*$, R is (weakly) terminating on x if it is so on the set $\{x\}$. The maximal length of a derivation sequence starting with x is denoted by $\delta(x) = \delta_R$, that is, $m \leqslant \delta(x)$ for any sequence $x = x_0 \to_R x_1 \to_R \cdots \to_R x_m$. For x on which R is not terminating, we set $\delta(x) = \infty$. The function d_R defined by

$$d_R(n) = \max\{\delta(x) \mid x \in \Sigma^n\}$$

is the *derivational complexity* of R, where Σ^n is the set of words over Σ of length n. R has *linear* (resp. *polynomial*) *complexity*, if there is a constant C (resp. polynomial P) such that

$$d_R(n) \leqslant C \cdot n \quad (\text{resp. } d_R(n) \leqslant P(n))$$

for all sufficiently large $n \in \mathbb{N}$. R has *exponential complexity*, if there are positive constants C_1 and C_2 such that

$$C_1^n \leqslant d_R(n) \leqslant C_2^n$$

for all sufficiently large $n \in \mathbb{N}$.

For two functions f and g from $\mathbb{N}$ to $\mathbb{N}$, we write $f \preccurlyeq g$ if there are positive constants C_1 and C_2 such that

$$f(n) \leqslant C_1 \cdot g(C_2 \cdot n)$$

holds for all sufficiently large $n \in \mathbb{N}$. With the usual O-notation it can be rephrased as $f(n) \leqslant \mathrm{O}(g(\mathrm{O}(n)))$. We say that f and g are *equivalent*, if both $f \preccurlyeq g$ and $g \preccurlyeq f$ hold. For two systems R_1 and R_2, if d_{R_1} and d_{R_2} are equivalent, we say that the systems have equivalent complexity. By definition, R has exponential complexity if and only if d_R is equivalent to an exponential function, and R has linear (resp. polynomial) complexity, if and only if $d_R \preccurlyeq f$ for a linear (resp. polynomial) function f.

A word $x \in \Sigma^*$ is *minimal right* (resp. *left*) *reducible*, if x is reducible but every proper suffix (resp. prefix) of x is irreducible. We denote by ${}_RZ$ (resp. Z_R) the set of all minimal left (resp. right) reducible words; ${}_RZ = \mathrm{Left}(R) \cdot \Sigma^* - \Sigma^+ \cdot \mathrm{Left}(R) \cdot \Sigma^*$. The following result is given in [9].

Lemma 2.1. *A length-increasing confluent system R is terminating, if and only if it weakly terminates on ${}_RZ$ (or Z_R).*

A *position* of a word $x \in \Sigma^*$ is a pair (x', x'') of a prefix x' and a suffix x'' of x such that $x = x'x''$. By the mth position of x we mean the position (x', x'') such that $|x'| = m$. The 0th position is the *left-most* position, the $|x|$th position is the *right-most* position, and the other positions are *inner* positions of x. If y is a subword of x; $x = y'yy''$, then a position (y_1, y_2) of y is naturally identified with the position $(y'y_1, y_2y'')$ of x, and is called a position of y in x, that is, the kth position of y corresponds to the $(|y'| + k)$th position of x.

Let us consider a one-step derivation $x \to_R y$ with x, $y \in \Sigma^*$, that is, $x = x'sx''$ and $y = x'tx''$ for $s \to t \in R$. A position in x that occurs as an inner position of s is *covered*, and a position of y that occurs as an inner position of t is *created* through this step. A position of x which is not covered is *inherited* by the corresponding position in y through this step. Thus, if $0 \leqslant n \leqslant |x'|$ (resp. $|x's| \leqslant n \leqslant |x|$), the nth position of x is inherited by the nth (resp. $(n + |t| - |s|)$th) position of y. Similarly, a letter in x which appears in the subword s of x is covered, and a letter outside s is inherited by the corresponding letter in y through the above step.

Let

$$D: x_1 \to_R x_2 \to_R \cdots \to_R x_n \tag{2.1}$$

be a sequence of derivation with respect to R (a sequence of R-derivation for short). A letter in x_1 or a position of x_1 is *inherited* by the corresponding letter in x_n or the corresponding position of x_n through D, if it is inherited through every step of D. It is *covered* through D if it is inherited upto some step and covered in the next step in D. All the letters and the positions in the initial word x_1 are *raw*. A letter or a position in x_n is raw, if it inherits a letter or a position in the original x_1. A nonempty subword of x_n is raw if all its letters are raw. An application of a rule is *raw*, if it covers at least one raw letter or one raw position. It is *totally raw* if all the letters it covers are raw.

Let $x, y \in \Sigma^*$. If $x = x'x_1x''$ and $y = x'x_nx''$ with $x', x'' \in \Sigma^*$ and there is a derivation sequence D as (2.1), then there is a derivation sequence from x to y which is equal to D on x_1 but leaves x' and x'' untouched. We denote this sequence by $x'Dx''$ and sometimes write it as $x \to_D y$.

The following simple observation is useful.

Lemma 2.2. *A system R is terminating if it is terminating on any word x that is induced by some derivation D such that neither a position nor a letter in x is raw through D.*

For two derivation sequences D_1 from x_1 to x_2 and D_2 from x_2 to x_3, the sequence from x_1 to x_3 composing D_1 and D_2 at x_2 is denoted by $D_1 \circ D_2$. For two derivation sequences D from x to y and D' from x' to y', we have two sequences $D \cdot x' \circ y \cdot D'$ and $x \cdot D' \circ D \cdot y'$ from xx' to yy'. They have the same length and the position (x,x') in xx' is inherited by the position (y,y') in yy' through them. From this observation we have

Lemma 2.3. *If a position (x_1',x_1'') $(x_1 = x_1'x_1'')$ of x_1 is inherited by a position (x_n',x_n'') $(x_n = x_n'x_n'')$ of x_n untouched by a sequence D in* (2.1), *then there are derivation sequences D' from x_1' to x_n' and D'' from x_1'' to x_n'' such that $D'x_1'' \circ x_n'D''$ and $x_1'D'' \circ D'x_n''$ are derivation sequences from x_1 to x_n with the same length as D.*

Let $x, y \in \Sigma^*$. We say x *overlaps* with y on the left (or y overlaps with x on the right), if there is $u \neq 1$ such that $x = x'u$ and $y = uy'$ with $x', y' \in \Sigma^*$, $x'y' \neq 1$. Let $\mathrm{OVL}(x,y)$ be the set of such words u. In particular, x overlaps with y on the left if x is a proper prefix of y. The set $\mathrm{OVL}(x,x)$ is simply denoted by $\mathrm{OVL}(x)$. If $\mathrm{OVL}(x) = \emptyset$, x is called *self-overlap-free* (*sof* for short). To include the perfect overlap, we set

$$\overline{\mathrm{OVL}}(x,y) = \begin{cases} \mathrm{OVL}(x,y) & \text{if } x \neq y, \\ \mathrm{OVL}(x,y) \cup \{x\} & \text{if } x = y. \end{cases}$$

In our previous paper [9] we showed that the termination problem for a confluent one-rule rewriting system is reduced to the same problem for a one-rule system $\{s \to t\}$ such that s is sof. In this paper we mainly concentrate on systems with a single rule whose left-hand side is sof, though some of the results hold for more general systems.

In the subsequent sections, $R = \{s \to t\}$ is always a string rewriting system consisting of a single rule $s \to t$ such that s is sof. In order to exclude trivial cases we assume s is not a subword of t, otherwise R is nonterminating, and $|s| < |t|$, otherwise R is terminating. Also we assume $|s| \geqslant 2$, because the case $|s| = 1$ is not interesting.

3. Standard decompositions and seamed words

Let R be a one-rule system stated in the last paragraph. Set $A = \mathrm{OVL}(t, s)$ and $B = \mathrm{OVL}(s, t)$. These two sets play a very important role for termination of R. For example, R is terminating if $A = \emptyset$ or $B = \emptyset$ (Kurth's Criterion D in [3]). So, we assume that both A and B are nonempty. Since s is not a subword of t, s is not contained in A nor in B, and since s is sof, $A \cap B = \emptyset$. For $\alpha \in A$ and $\beta \in B$, s_α, t_α, s_β and t_β are the words determined by $s = \alpha \cdot s_\alpha = s_\beta \cdot \beta$ and $t = t_\alpha \cdot \alpha = \beta \cdot t_\beta$.

Lemma 3.1. *We have*

$$\overline{\mathrm{OVL}}(\beta, \alpha) = \overline{\mathrm{OVL}}(s_{\alpha'}, \alpha) = \overline{\mathrm{OVL}}(\beta, s_{\beta'}) = \overline{\mathrm{OVL}}(s_\alpha, s_\beta) = \emptyset$$

for any $\alpha, \alpha' \in A$ *and* $\beta, \beta' \in B$.

Proof. If one of the sets is nonempty and contains an element γ, then $\gamma \in \mathrm{OVL}(s)$, and hence, s cannot be sof. $\square$

Lemma 3.2. *For* $\alpha, \alpha' \in A$ (*resp.* $\beta, \beta' \in B$), *if* $|\alpha'| < |\alpha|$ (*resp.* $|\beta'| < |\beta|$), *then* $\alpha' \in \mathrm{OVL}(\alpha)$ (*resp.* $\beta' \in \mathrm{OVL}(\beta)$).

Proof. Suppose $\alpha, \alpha' \in A$ and $|\alpha'| < |\alpha|$. Since both α and α' are prefixes of s, α' is a proper prefix of α. Similarly, α' is a proper suffix of α, and hence, $\alpha' \in \mathrm{OVL}(\alpha)$. $\square$

Set $S_A = \{s_\alpha \mid \alpha \in A\}$ and $S_B = \{s_\beta \mid \beta \in B\}$. A subset X of Σ^* is called a *prefix* (resp. *suffix*) *code*, if no element of X is a prefix (resp. suffix) of another element of X. A prefix (suffix) code forms a code (see [1]).

Lemma 3.3. S_A *is a prefix code and* S_B *is a suffix code.*

Proof. If s_α is a prefix of $s_{\alpha'}$ for different $\alpha, \alpha' \in A$, then $\alpha' \cdot s_\alpha$ is a proper prefix of $\alpha' \cdot s_{\alpha'} = s$. Since α' is a proper suffix of α by Lemma 3.2, $\alpha' \cdot s_\alpha$ is a proper suffix of $\alpha \cdot s_\alpha = s$. This is impossible because s is sof. Thus, S_A is a prefix code, and similarly S_B is a suffix code. $\square$

Since S_B is a suffix code, for any $\alpha \in A$, t_α is uniquely decomposed as

$$t_\alpha = w_\alpha s_{\beta_k} \cdots s_{\beta_2} s_{\beta_1} \tag{3.1}$$

with $k \geqslant 0$, $\beta_i \in B$, and $w_\alpha \in \Sigma^* - \Sigma^* S_B$. Similarly, for $\beta \in B$, t_β is uniquely decomposed as

$$t_\beta = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_\ell} w_\beta \tag{3.2}$$

with $\ell \geqslant 0$, $\alpha_i \in A$, and $w_\beta \in \Sigma^* - S_A \Sigma^*$.

We consider A and B to be (new) alphabets. For a word $\boldsymbol{\alpha} = \alpha_1 \cdots \alpha_n \in A^*$ over A, we define words $s_{\boldsymbol{\alpha}}$ and $t_{\boldsymbol{\alpha}}$ over Σ by

$$s_{\boldsymbol{\alpha}} = s_{\alpha_1} \cdots s_{\alpha_n}, \quad t_{\boldsymbol{\alpha}} = t_{\alpha_1} \cdots t_{\alpha_n}.$$

Similarly, words $s_{\boldsymbol{\beta}}$ and $t_{\boldsymbol{\beta}}$ over Σ are defined for $\boldsymbol{\beta} \in B^*$ (bold greek letters are used for words over $A \cup B$).

Let $\boldsymbol{\beta}(\alpha)$ and $\boldsymbol{\alpha}(\beta)$ denote the words $\beta_k \cdots \beta_2 \beta_1$ and $\alpha_1 \alpha_2 \cdots \alpha_\ell$ over B and A determined above in (3.1) and (3.2) for $\alpha \in A$ and $\beta \in B$, respectively.

Lemma 3.4. *For any $\alpha \in A$ and $\beta \in B$, t is decomposed uniquely as*

$$t = \beta \cdot s_{\boldsymbol{\alpha}(\beta)} \cdot w_{\beta\alpha} \cdot s_{\boldsymbol{\beta}(\alpha)} \cdot \alpha \tag{3.3}$$

with $w_{\beta\alpha} \in \Sigma^* - (S_A \Sigma^* \cup \Sigma^* S_B)$.

Proof. By (3.1) and (3.2), t is decomposed as

$$t = \beta \cdot s_{\boldsymbol{\alpha}(\beta)} \cdot w_\beta = w_\alpha \cdot s_{\boldsymbol{\beta}(\alpha)} \cdot \alpha. \tag{3.4}$$

As easily seen, $\overline{\mathrm{OVL}}(x_1 x_2 \cdots x_m, y_1 y_2 \cdots y_n) = \emptyset$ for any words $x_1, \ldots, x_m$, $y_1, \ldots, y_n$ such that $\overline{\mathrm{OVL}}(x_i, y_j) = \emptyset$ for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$. Hence, by Lemma 3.1, $\beta \cdot s_{\boldsymbol{\alpha}(\beta)}$ does not overlap with $s_{\boldsymbol{\beta}(\alpha)} \cdot \alpha$ on the left. Thus, $|w_\beta| \geqslant |s_{\boldsymbol{\beta}(\alpha)} \cdot \alpha|$ and (3.3) follows from (3.4). $\square$

Decomposition (3.3) is called the *standard decomposition* of t with respect to $\alpha \in A$ and $\beta \in B$. Since $t = \beta \cdot s_{\boldsymbol{\alpha}(\beta)} \cdot w_\beta = w_\alpha \cdot s_{\boldsymbol{\beta}(\alpha)} \cdot \alpha$, we have

$$w_\alpha = \beta \cdot s_{\boldsymbol{\alpha}(\beta)} \cdot w_{\beta\alpha}, \quad w_\beta = w_{\beta\alpha} \cdot s_{\boldsymbol{\beta}(\alpha)} \cdot \alpha \tag{3.5}$$

for any $\alpha \in A$ and $\beta \in B$. Let $\ell_1 = \max\{|\beta \cdot s_{\boldsymbol{\alpha}(\beta)}| \mid \beta \in B\}$ and $\ell_2 = \max\{|s_{\boldsymbol{\beta}(\alpha)} \cdot \alpha| \mid \alpha \in A\}$, and let t_ℓ be the prefix of t of length ℓ_1 and t_r be the suffix of t of length ℓ_2. Since $\beta \cdot s_{\boldsymbol{\alpha}(\beta)}$ does not overlap with $s_{\boldsymbol{\beta}(\alpha)} \cdot \alpha$, t is written as

$$t = t_\ell \cdot t_m \cdot t_r$$

with some $t_m \in \Sigma^*$, where t_m is a subword of $w_{\beta\alpha}$ and t_ℓ (resp. t_r) has $\beta \cdot s_{\boldsymbol{\alpha}(\beta)}$ (resp. $s_{\boldsymbol{\beta}(\alpha)} \cdot \alpha$) as prefix (resp. suffix) for $\beta \in B$ (resp. $\alpha \in A$);

$$t_\ell = \beta \cdot s_{\boldsymbol{\alpha}(\beta)} \cdot t'_\ell, \quad t_r = t'_r \cdot s_{\boldsymbol{\beta}(\alpha)} \cdot \alpha.$$

Example 3.5. Let $R=\{s=aabb\to t=bbbaaa\}$. Then we have

$$A=\{a,aa\},\quad B=\{b,bb\},$$

and

$$S_A=\{s_a=abb,s_{aa}=bb\},\quad S_B=\{s_b=aab,s_{bb}=aa\}.$$

Since

$$t=b\cdot s_{aa}\cdot aaa=bbb\cdot s_{bb}\cdot a,$$

we see

$$\boldsymbol{\alpha}(b)=aa,\quad \boldsymbol{\alpha}(bb)=1,\quad \boldsymbol{\beta}(a)=bb,\quad \boldsymbol{\beta}(aa)=1,$$

and we have four standard decompositions of t:

$$t=b\cdot s_{aa}\cdot s_{bb}\cdot a=b\cdot s_{aa}\cdot a\cdot aa=bb\cdot b\cdot s_{bb}\cdot a=bb\cdot ba\cdot aa.$$

Moreover,

$$t_\ell=bbb=b\cdot s_{aa},\quad t_r=aaa=s_{bb}\cdot a,\quad t_m=1.$$

Now, let us take a close look at the position of a word where the rule applied. Let $x=x'sx''\to_R y=x'tx''$ with x', $y'\in\Sigma^*$ and $s\to t\in R$. The left-most (resp. right-most) position of t in y is the *left* (resp. *right*) *seam* of y created by this derivation step. A left (resp. right) seam is graphically denoted by the symbol [(resp.]). Thus, the above derivation is displayed as

$$x=x'sx''\to_R y=x'[t]x''$$

(see (3.7) below). Let

$$D\colon x_1\to_R x_2\to_R\cdots\to_R x_n\to\cdots \tag{3.6}$$

be a (possibly infinite) derivation sequence. Suppose that the step $x_{n-1}\to_R x_n$ is given as $x_{n-1}=x'sx''$ and $x_n=x'tx''$ with $s\to t\in R$. No position of the initial word x_1 is a seam for D. For $n\geqslant 2$, a position of x_n is a (left) seam of x_n with respect to D, if either it is the (left) seam created by the step $x_{n-1}\to_R x_n$ or it inherits a (left) seam of x_{n-1}. A seam of x_{n-1} which is not located at a position of the subword s is inherited in the step. A seam at the left-most (resp. right-most) position of s in x_{n-1} is inherited if it is a right (resp. left) seam, but is not inherited if it is a left (resp. right) seam. So, a position can be a left and right seam at the same time, which inherits a left seam when it is created as a right seam or inherits a right seam when it is created as a left seam. Such a seam is called a *double seam* and is counted twice. If a seam of x_{n-1} is not inherited by any seam of x_n in D, it is *patched* in the derivation step $x_{n-1}\to_R x_n$. So, every seam of x_{n-1} inside the subword s is patched in the step.

For example, let us consider the system $R = \{ab \to baa\}$ over $\{a, b\}$. We have a derivation sequence

$$\underline{ab}ab \to [baa]\underline{ab} \to [ba\underline{a][b}aa] \to [b\underline{a\overset{\circ}{[}b}aa\overset{\circ}{]}aa] \to [b[baa\overset{\circ}{]}aa\overset{\circ}{]}aa]. \tag{3.7}$$

The brackets [and] denote a left seam and a right seam, respectively. The underlines show the places where the rule is applied. The marking $\circ$ will be explained later in Section 6. The pair][of seams in the third term is a double seam, which is patched by the third step. The first three steps are raw but the last one is not.

A subword u consisting of letters between two adjacent seams of x is a *piece* of x. If x has a double seam, then the empty word 1 is a piece as a subword between the right seam and the left seam that overlap. If σ is the left-most (resp. right-most) seam of x at position (x', x''), $x = x'x''$, with $x' \neq 1$ (resp. $x'' \neq 1$), then x' (resp. x'') is called the *edge piece* of x. A *seamed piece* is a piece accompanied with seams on the both sides (on one side if it is an edge piece). The fourth term in (3.7) contains three pieces ba, baa and aa, and corresponding seamed pieces $[ba[$, $[baa]$ and $]aa]$. A left (resp. right) seam with pieces u, v on the both sides are sometimes displayed as $u[v$ (resp. $u]v$).

A word x in which some positions are designated as seams is a *seamed word*. A seamed word is expressed by a word over $\Sigma \cup \{[,]\}$ as in (3.7). If it is seamed through a derivation sequence D from some raw word, it is called a *seamed word induced by D*. If moreover neither a letter nor an inner position of x is raw through D, x is *fully seamed*. In sequence (3.7), the last two terms are fully seamed but the first three are not (in the third no letter is raw but the third position (baa, baa) is raw).

For two seams σ and σ' of a seamed word x, if σ is left to σ' we write $\sigma < \sigma'$. If σ is a right seam and σ' is a left seam at the same position (that is, they form a double seam), we also write $\sigma < \sigma'$. The seams of x are thus linearly ordered form left to right.

By Lemmas 2.1 and 2.2 we have

Lemma 3.6. *R is terminating if and only if it is weakly terminating on every fully seamed word which descends from a minimal left* (*or right*) *reducible word.*

The following lemma describes the forms of nonraw seamed pieces appearing in a derivation sequence.

Lemma 3.7. *Let u be a piece of a seamed word induced by a derivation sequence. If the seamed piece corresponding to u is of the form* $[u]$, *then* $u = t$. *If it is of the form* $[u[$, *then u is a prefix of* t_α *for some* $\alpha \in A$ *containing* t_ℓ *as prefix. If it is of the form* $]u]$, *then u is a suffix of* t_β *for some* $\beta \in B$ *containing* t_r *as suffix. If it is of the form* $]u[$ *and u is not raw, then u is a subword of* $s_{\alpha(\beta)} w_{\beta\alpha} s_{\beta(\alpha)}$ *for some* $\alpha \in A$ *and* $\beta \in B$ *containing a subword of* t_m *as subword.*

Proof. We prove the assertion by induction. The seamed piece $[t]$ is created in each step of derivation. Suppose that a nonempty suffix of the piece t is covered in the next step. Then, the suffix of t that is a prefix of s is an element α of $A = \mathrm{OVL}(t,s)$. Thus, the new piece t_α between the old left seam [and the new left seam [created is born;

$$\cdots[t]\cdots = \cdots[t_\alpha \cdot \underline{\alpha]s_\alpha}\cdots \to_R \cdots[t_\alpha[t]\cdots \ .$$

In general, suppose that the step covers a right part of a seamed piece y which is of the asserted form by induction hypothesis. The step yields a new piece u' on the left of the created left seam. If $y =]u]$ with a suffix u of $t_\beta = s_{\boldsymbol{\alpha}(\beta)} w_{\beta\alpha} s_{\boldsymbol{\beta}(\alpha)}\alpha$ containing $t_r = t'_r s_{\boldsymbol{\beta}(\alpha)}\alpha$ as suffix, then the new piece u' between the seam] of y and the left seam created is a suffix of $s_{\boldsymbol{\alpha}(\beta)} w_{\beta\alpha} s_{\boldsymbol{\beta}(\alpha)}$ for some $\alpha \in A$ containing t_r;

$$\cdots]u]\cdots = \cdots]\cdots s_{\boldsymbol{\beta}(\alpha)}\underline{\alpha]s_\alpha}\cdots \to_R \cdots]\cdots s_{\boldsymbol{\beta}(\alpha)}[t]\cdots \ .$$

If $y = [u[$ with prefix u of t_α containing t_ℓ as prefix, then the new piece u' is of the same form as u because t_ℓ does not overlap with s on the left. Finally, if $y =]u[$ with a subword u of $s_{\boldsymbol{\alpha}(\beta)} w_{\beta\alpha} s_{\boldsymbol{\beta}(\alpha)}$ containing a subword of t_m as subword, then again u' is of the same form as u.

The dual argument will work in the case where a left part of y is covered in the step. We leave the proof to the reader. □

4. s-open seams and simple derivation

The derivation sequences in the following lemma are standard.

Lemma 4.1. (1) *For* $\boldsymbol{\alpha} = \alpha_1 \cdots \alpha_n \in A^+$ *we have*

$$\alpha_1 \cdot s_{\boldsymbol{\alpha}} \to_R^n t_{\boldsymbol{\alpha}'} \cdot t,$$

where $\boldsymbol{\alpha}' = \alpha_2 \cdots \alpha_n$.

(2) *For* $\boldsymbol{\beta} = \beta_n \cdots \beta_1 \in B^+$ *we have*

$$s_{\boldsymbol{\beta}} \cdot \beta_1 \to_R^n t \cdot t_{\boldsymbol{\beta}'},$$

where $\boldsymbol{\beta}' = \beta_n \cdots \beta_2$.

Proof. If $n = 1$, we have

$$\alpha_1 \cdot s_{\alpha_1} = s \to_R t.$$

If $n > 1$, by induction we may assume that $\alpha_2 \cdot s_{\boldsymbol{\alpha}'} \to_R^{n-1} t_{\boldsymbol{\alpha}''} \cdot t$, where $\boldsymbol{\alpha}'' = \alpha_3 \cdots \alpha_n$. Thus, We have a derivation sequence

$$D(\boldsymbol{\alpha}) : \alpha_1 \cdot s_{\boldsymbol{\alpha}} = s \cdot s_{\boldsymbol{\alpha}'} \to_R t \cdot s_{\boldsymbol{\alpha}'} = t_{\alpha_2} \cdot \alpha_2 s_{\boldsymbol{\alpha}'} \to_R^{n-1} t_{\alpha_2} \cdot t_{\boldsymbol{\alpha}''} \cdot t = t_{\boldsymbol{\alpha}'} \cdot t.$$

This shows (1). The derivation in (2) is given by

$$D(\boldsymbol{\beta}) : s_{\boldsymbol{\beta}} \cdot \beta_1 \to_R s_{\boldsymbol{\beta}'} \cdot t = s_{\boldsymbol{\beta}'} \cdot \beta_2 \cdot t_{\beta_2} \to_R^{n-1} t \cdot t_{\boldsymbol{\beta}'}. \quad \square$$

Example 4.2. Let $R = \{s = aabb \to t = bbbaaa\}$ be the system in Example 3.5. We have a (seamed) derivation sequence:

$$a \cdot s_a \cdot s_{aa} \cdot s_a = \underline{aabb}bbabb \to [bbba\underline{aa]bb}abb$$

$$\to [bbba[bbbaa\underline{a]abb} \to [bbba[bbbaa[bbbaaa] = [t_{aa}[t_a[t].$$

As one can observe from the above example, in any step of the sequence $D(\boldsymbol{\alpha})$ in Lemma 4.1, at most one seam is patched. Actually just one left seam is patched except for the first step (the first step is raw). In the sequence $D(\boldsymbol{\beta})$ just one right seam is patched except for the first step. In general, a derivation step is called *simple* if it patches at most one seam. A derivation sequence is *simple* if all the steps in it are simple. Thus, the sequences $D(\boldsymbol{\alpha})$ and $D(\boldsymbol{\beta})$ are simple. In sequence (3.7), the third step is not simple (because it patches a double seam), but the other steps are simple.

As far as R is fixed, $\to_R$ is abbreviated as $\to$, and if a step $x \to y$ is simple, we write as $x \to_{(s)} y$. We say that R is *s-terminating* if there is no infinite sequence of simple derivation, otherwise R is *s-nonterminating*. A seamed word is *s-reducible* if a simple application of the rule is possible, otherwise it is *s-irreducible*. If y is a subword of w such that $x \to^n w$ (resp. $x \to^n_{(s)} w$), we write as $x \rhd^n y$ (resp. $x \rhd^n_{(s)} y$). If $x \rhd^n y$ (resp. $x \rhd^n_{(s)} y$) for some $n > 0$, we just write as $x \rhd y$ (resp. $x \rhd_{(s)} y$). If $x \rhd^n x$ (resp. $x \rhd^n_{(s)} x$), we say that R has an *n-loop* (resp. a *simple n-loop*). Clearly, $\rhd$ and $\rhd_{(s)}$ are transitive relations, and if R has a loop (resp. simple loop), it is nonterminating (resp. s-nonterminating).

Let $x \in \Sigma^*$ be a seamed word and σ be a seam with the left-hand piece u and the right-hand piece v (u and v may be edge pieces). First, suppose that σ is a left seam; $\sigma = u[v$. If an application of the rule to x is simple and patches σ, then $s = s_1 s_2$ and s_1 is a suffix of u and s_2 is a prefix of v. Since v is a prefix of t containing t_ℓ as prefix by Lemma 3.7, we see $s_2 = \beta \in B$ and $s_1 = s_\beta$; $\sigma = \cdots s_\beta[\beta \cdots$. On the other hand if σ is a right seam, then α is a suffix of u and s_α is a prefix of v; $\sigma = \cdots \alpha]s_\alpha \cdots$. Clearly, a simple application patching a seam is possible in only these situations. Now, we say that the left (resp. right) seam σ is *s-open*, if β (resp. s_α) is a prefix of v and s_β (resp. α) is a suffix of u for some $\beta \in B$ (resp. $\alpha \in A$), otherwise, σ is *s-closed*. From our argument above we have the following two lemmas.

Lemma 4.3. *A left seam $u[v$ is s-open if and only if u has a suffix from S_B. A right seam $u]v$ is s-open if and only if v has a prefix from S_A.*

Lemma 4.4. *A seamed word x has an s-open seam if and only if a simple application that is not totally raw to x is possible.*

Corollary 4.5. *A fully seamed word is s-reducible if and only if it has an s-open seam.*

The following also depends on the assumption that s is sof.

Lemma 4.6. *The relations $\rightarrow$, and $\rightarrow_{(s)}$, are confluent. More precisely, if $x \rightarrow^m y$ (resp. $x \rightarrow_{(s)}^m y$) and $x \rightarrow^n z$ (resp. $x \rightarrow_{(s)}^n z$), then there is w such that $y \rightarrow^{n'} w$ (resp. $y \rightarrow_{(s)}^{n'} w$) and $z \rightarrow^{m'} w$ (resp. $z \rightarrow_{(s)}^{m'} w$) with $m + n' = n + m'$.*

Proof. Any different applications $x \rightarrow y$ (resp. $x \rightarrow_{(s)} y$) and $x \rightarrow z$ (resp. $x \rightarrow_{(s)} z$) of the rule to x are disjoint, because s is sof. Therefore, we have a word w such that $y \rightarrow w$ (resp. $y \rightarrow_{(s)} w$) and $z \rightarrow w$ (resp. $z \rightarrow_{(s)} w$). From this observation the assertion can be easily proved. □

Corollary 4.7. *If $\rightarrow$ (resp. $\rightarrow_{(s)}$) is terminating on x, x has a unique irreducible (resp. s-irreducible) descendant by $\rightarrow$ (resp. $\rightarrow_{(s)}$).*

Lemma 4.6 also tells us that the length of a derivation sequence from x to y does not depend on the choice of sequences. But this fact is straightly seen as follows. An application of the rule increases the length of a word by $|t| - |s|$, and so the length of a sequence from x to y is equal to $(|y| - |x|)/(|t| - |s|)$.

The unique irreducible (resp. s-irreducible) descendant of x in Corollary 4.7 is called the *canonical* (resp. *s-canonical*) form of x.

We define $\delta_{(s)}(x)$ to be the length of sequences of simple derivation from x to its canonical form. For an s-terminating system R, its *s-complexity* is the function $d_{(s)}$ defined by

$$d_{(s)}(n) = \max\{\delta_{(s)}(x) \mid x \in \Sigma^n\}.$$

A minimal left reducible word x is written as $x = s \cdot y$ with $y \in \mathrm{Irr}(R)$. Since S_A is a prefix code, y is uniquely decomposed as $y = s_{\boldsymbol{\alpha}} \cdot x'$ for $\boldsymbol{\alpha} \in A^*$ and $x' \in \Sigma^* - S_A\Sigma^*$. By Lemma 4.1 we have a sequence of simple derivation

$$x = s \cdot s_{\boldsymbol{\alpha}} \cdot x' \rightarrow t \cdot s_{\boldsymbol{\alpha}} \cdot x' = t_{\alpha_1} \cdot \alpha_1 \cdot s_{\boldsymbol{\alpha}} \cdot x' \rightarrow_{D(\boldsymbol{\alpha})} t_{\boldsymbol{\alpha}} \cdot t \cdot x'.$$

The right-most right seam $t]x'$ of $t_{\boldsymbol{\alpha}}tx'$ is s-closed because x' has no prefix from S_A. So, it will never be patched by a simple application of the rule and x' will be left untouched. A similar fact holds for a minimal right reducible word. Thus we have

Lemma 4.8. *A minimal left (resp. right) irreducible word x is decomposed uniquely as $x = s \cdot s_{\boldsymbol{\alpha}} \cdot x'$ (resp. $x' \cdot s_{\boldsymbol{\beta}} \cdot s$) for $\boldsymbol{\alpha} \in A^*$ (resp. $\boldsymbol{\beta} \in B^*$) and $x' \in \Sigma^* - S_A\Sigma^*$ (resp. $\Sigma^* - \Sigma^* S_B$), and the subword x' remains raw in any simple derivation starting with x.*

Since $\rightarrow_{(s)}$ is confluent, Lemmas 3.6 and 4.8 give

Corollary 4.9. *R is s-terminating if and only if it weakly terminates on the seamed words of the form*

$$[t_{\alpha_1}[t_{\alpha_2}[\cdots[t_{\alpha_n}[t] \tag{4.1}$$

with $\alpha_1\alpha_2\cdots\alpha_n \in A^+$, *or the seamed words of the form*

$$[t]t_{\beta_n}]t_{\beta_{n-1}}]\cdots]t_{\beta_1}] \tag{4.2}$$

with $\beta_n\beta_{n-1}\cdots\beta_1 \in B^+$.

We set

$$A_{(s)} = \{\alpha \in A \mid \boldsymbol{\beta}(\alpha) \neq 1\}, \quad B_{(s)} = \{\beta \in B \mid \boldsymbol{\alpha}(\beta) \neq 1\}.$$

An s-open left seam $u[v$ is *standard* if $s_{\boldsymbol{\beta}(\alpha)}$ is a suffix of u for some $\alpha \in A_{(s)}$. If x has a standard s-open left seam $\cdots s_{\boldsymbol{\beta}(\alpha)}[v$ with $\boldsymbol{\beta}(\alpha) = \beta_n\cdots\beta_1$, then by Lemma 4.1 we have a sequence of derivation

$$x = \cdots s_{\boldsymbol{\beta}(\alpha)}[\beta_1 v'\cdots \rightarrow_{D(\boldsymbol{\beta}(\alpha))} \cdots [t]t_{\beta_n}]\cdots]t_{\beta_2}]v'\cdots, \tag{4.3}$$

because $v = \beta_1 v'$ with $v' \in \Sigma^*$ by Lemma 3.7. Dually an s-open right seam $u]v$ is *standard* if v has a prefix $s_{\boldsymbol{\alpha}(\beta)}$ for some $\beta \in B_{(s)}$. If $\boldsymbol{\alpha}(\beta) = \alpha_1\cdots\alpha_n$, because u has α_1 as suffix, we have a sequence

$$x = \cdots u'\alpha_1]s_{\boldsymbol{\alpha}(\beta)}\cdots \rightarrow_{D(\boldsymbol{\alpha}(\beta))} \cdots u'[t_{\alpha_2}\cdots[t_{\alpha_n}[t]\cdots\ . \tag{4.4}$$

A sequence of simple derivation is called *standard* if it is a composition of sequences of type (4.3) or (4.4).

Lemma 4.10. *A standard s-open seam remains standard by a derivation step that does not patch it.*

Proof. Let $\sigma = \cdots s_{\boldsymbol{\beta}(\alpha)}[v$ be a standard s-open seam and suppose that it is not patched by a step. Then any part of $s_{\boldsymbol{\beta}(\alpha)}$ never be covered by the step, because $\mathrm{OVL}(s, s_{\boldsymbol{\beta}(\alpha)}) = \emptyset$. Therefore, the piece on the left of σ still has $s_{\boldsymbol{\beta}(\alpha)}$ as suffix, and thus σ remains standard. □

A sequence of type (4.3) or (4.4) does not create a nonstandard s-open seam because the pieces $t_{\beta_n},\ldots,t_{\beta_2}$ and v' have prefixes $s_{\boldsymbol{\alpha}(\beta_n)},\ldots,s_{\boldsymbol{\alpha}(\beta_2)}$ and $s_{\boldsymbol{\alpha}(\beta_1)}$, respectively. Combining this fact and Lemma 4.10, we have

Corollary 4.11. *Let y be a seamed word obtained from a seamed word x through a standard sequence of simple derivation. If every s-open seam of x is standard, every s-open seam of y is also standard.*

Let y be the seamed word in (4.1). Since the seam [between t_{α_i} and $t_{\alpha_{i+1}}$ of y is s-open if and only if $\boldsymbol{\beta}(\alpha_i) \neq 1$, every s-open seam of y is standard. If the seam $t_{\alpha_i}[t_{\alpha_{i+1}}$ is s-open and if $\boldsymbol{\beta} = \boldsymbol{\beta}(\alpha_i) = \beta_m\beta_{m-1}\cdots\beta_1 \neq 1$, we can apply the standard derivation $D(\boldsymbol{\beta})$ patching this seam;

$$y = \cdots[t_{\alpha_i}[t_{\alpha_{i+1}}[\cdots = \cdots[w_{\alpha_i}s_{\boldsymbol{\beta}}[\beta_1 t'_{\beta_1}\cdots[\cdots$$

$$\rightarrow_{D(\boldsymbol{\beta})} y_2 = \cdots[w_{\alpha_i}[t]t_{\beta_m}]t_{\beta_{m-1}}\cdots]t_{\beta_2}]t'_{\beta_1}\cdots[\cdots\ .$$

By Corollary 4.11, every s-open seam in y_2 is standard. As far as y_2 has an s-open seam, we can apply the standard derivation. In this way we have a standard sequence of simple derivation

$$y = y_1 \to_{D_1} y_2 \to_{D_2} \cdots \to_{D_k} y_k \to_{D_{k+1}} \cdots, \tag{4.5}$$

where D_i is a sequence of the form $D(\boldsymbol{\beta}(\alpha))$ or $D(\boldsymbol{\alpha}(\beta))$.

With the seamed words y_1 and y_2 we associate the word $\boldsymbol{\omega}_1 = \alpha_1\alpha_2\cdots\alpha_n$ and $\boldsymbol{\omega}_2 = \alpha_1\cdots\alpha_{i-1}\beta_m\cdots\beta_1\alpha_{i+1}\cdots\alpha_n$ over $A\cup B$, respectively. The word $\boldsymbol{\omega}_2$ is obtained from $\boldsymbol{\omega}_1$ by applying the rule $\alpha_i \to \beta_m\cdots\beta_1$ on it. Now, we define a new system $\mathbf{S}_R$ over the alphabet $A\cup B$ as

$$\mathbf{S}_R = \{\alpha \to \boldsymbol{\beta}(\alpha), \beta \to \boldsymbol{\alpha}(\beta) \mid \alpha \in A_{(\mathrm{s})}, \beta \in B_{(\mathrm{s})}\}.$$

Continuing the above argument inductively, with sequence (4.5) we can associate a sequence of words over $A\cup B$ of $\mathbf{S}_R$-derivation

$$\boldsymbol{\omega}_1 \to_{\mathbf{S}_R} \boldsymbol{\omega}_2 \to_{\mathbf{S}_R} \cdots \to_{\mathbf{S}_R} \boldsymbol{\omega}_k \to_{\mathbf{S}_R} \cdots \tag{4.6}$$

so that the following holds for each k. To every left (resp. right) s-open seam σ of y_k corresponds uniquely an occurrence of a letter $\alpha\in A_{(\mathrm{s})}$ (resp. $\beta\in B_{(\mathrm{s})}$) in $\boldsymbol{\omega}_k$, and vice versa. A derivation of the form $D(\boldsymbol{\alpha}(\beta))$ (resp. $D(\boldsymbol{\beta}(\alpha))$) is applied on y_k patching the seam σ in the kth step of (4.5), if and only if the rule $\alpha\to\boldsymbol{\beta}(\alpha)$ (resp. $\beta\to\boldsymbol{\alpha}(\beta)$) is applied on the corresponding letter α (resp. β) in $\boldsymbol{\omega}_k$ in the kth step of (4.6).

Theorem 4.12. *R is s-terminating, if and only if $\mathbf{S}_R$ is terminating.*

Proof. If R is s-nonterminating, then by Corollary 4.9, there is an infinite standard sequence (4.5) of simple derivation starting with the seamed word y in (4.1). Corresponding to it we have an infinite sequence (4.6) of $\mathbf{S}_R$-derivation, and $\mathbf{S}_R$ is nonterminating.

Conversely, suppose that there is an infinite sequence (4.6) of $\mathbf{S}_R$-derivation. We may assume that $\boldsymbol{\omega}_1 = \alpha\in A_{(\mathrm{s})}$. Let $y_1 = t_\alpha[t]$ be a seamed word. We can construct a infinite sequence (4.5) of seamed words so that ω_k corresponds to y_k for each k. $\square$

The termination problem of the system $\mathbf{S}_R$ is easily solved by looking at the following bipartite digraph $\Gamma_{\mathrm{s}}(R)$: The set of vertices of $\Gamma_{\mathrm{s}}(R)$ is $A\cup B$. For $\alpha\in A$ and $\beta\in B$, there is an edge from α to β if and only if β appears in $\boldsymbol{\beta}(\alpha)$, and there is an edge form β to α if and only if α appears in $\boldsymbol{\alpha}(\beta)$.

Because the left-hand side of a rule in $\mathbf{S}_R$ is a single letter, we see that $\mathbf{S}_R$ is nonterminating, if only if Γ_{s} has no nontrivial cycle, that is, Γ_{s} is a forest (a union of trees). If Γ_{s} has a nontrivial cycle $\alpha\to^+\alpha$ with $\alpha\in A_{(\mathrm{s})}$ (resp. $\beta\to^+\beta$ with $\beta\in B_{(\mathrm{s})}$), then corresponding to it there is a simple loop $s_{\boldsymbol{\beta}(\alpha)}\beta_1 \rhd_{(\mathrm{s})} s_{\boldsymbol{\beta}(\alpha)}\beta_1$ (resp. $\alpha_1 s_{\boldsymbol{\alpha}(\beta)} \rhd_{(\mathrm{s})} \alpha_1 s_{\boldsymbol{\alpha}(\beta)}$). Thus, we have

Theorem 4.13. *The following statements are equivalent*:
(1) *R is* s-*terminating.*
(2) *R has no simple loop.*
(3) *Γ_s is acyclic.*

Remark 4.14. Simple derivation has been studied by McNaughton [6]. By Lemma 3.7 a nonraw piece is a subword of t. Any nonraw piece of a word in a simple sequence has t_m as subword and it is not touched in the sequence. By inserting a dummy symbol ι outside Σ into t at a position of t_m, we have McNaughton's *inhibition system T* associated with R. There is a one-to-one correspondence between simple derivation and T-derivation. Thus, our simple derivation is nothing but McNaughton's well-behaved derivation, and Theorem 4.13 is essentially due to him.

Next, we study the complexity of s-terminating systems. To this end we need a simple fact on raw derivation.

Lemma 4.15. *In any derivation sequence the number of raw steps starting with a raw word $x \in \Sigma^*$ is bounded by $|x|$.*

Proof. (1) A totally raw application consumes at least two raw letters of x (because $|s| \geqslant 2$) and creates two raw seams. (2) A raw application that creates one nonraw seam consumes at least one raw letter and patches one raw seam. (3) A raw application that creates two nonraw seams patches two raw seams. Let n_1, n_2 and n_3 be the numbers of raw applications of type (1), (2) and (3) above, respectively. Since a nonraw step does not consume a raw letter nor creates a raw seam, we see that $2n_1 + n_2 \leqslant |x|$ and $n_3 \leqslant n_1$. Thus the total number $n_1 + n_2 + n_3$ of raw applications is bounded by $|x|$. □

Theorem 4.16. *If R is s-terminating, then $\to_{(s)}$ has linear derivational complexity, that is, $d_{(s)}$ is a linear function.*

Proof. If R is s-terminating, then Γ_s is acyclic. Let L be the length of the longest directed path in Γ_s and let M be the maximal positive degree of vertices of Γ_s;

$$M = \max\{|\boldsymbol{\beta}(\alpha)|, |\boldsymbol{\alpha}(\beta)| \mid \alpha \in A_{(s)}, \beta \in B_{(s)}\}.$$

Starting with $\alpha \in A$ or $\beta \in B$, the length of a sequence of $\mathbf{S}_R$-derivation is bounded by M^L.

Let $x \in \Sigma^+$ be a raw word and D_0 be a sequence of raw and simple derivation from x to the word y on which a raw and simple application is impossible. We claim that every s-open seam of y is standard. In fact, an s-open seam of y is nonraw, because a raw and simple application is impossible to y. So, it is created by a step patching a raw seam. If the step patches a raw left seam $u[v$, then u is raw and v is a prefix of t having t_ℓ as prefix by Lemma 3.7. Hence, $u = \cdots s_\beta$ and $v = \beta \cdot s_{\boldsymbol{\alpha}(\beta)} \cdots$, and the new nonraw seam created is in the form $t]s_{\boldsymbol{\alpha}(\beta)}\cdots$ and is standard. The case the step patches a right seam is similar.

By Lemma 4.15 the length of D_0 is less than or equal to $|x|$. Since a raw and simple step creates at most one nonraw s-open seam, y has at most $|x|$ s-open seams, which are standard. We have a word $\boldsymbol{\omega}$ over $A_{(\mathrm{s})}\cup B_{(\mathrm{s})}$ such that an occurrence of a letter in $\boldsymbol{\omega}$ corresponds to an s-open seam of y. If y has an s-open left seam $\sigma = \cdots s_{\boldsymbol{\beta}(\alpha)}[v$ with $\alpha\in A_{(\mathrm{s})}$ for example, the derivation sequence $D(\alpha)$ can be applied to y, and correspondingly the rule $\alpha\rightarrow\boldsymbol{\beta}(\alpha)$ can be applied to $\boldsymbol{\omega}$. Starting with y apply such derivation sequences as much as possible, then we will reach an s-irreducible word z in at most $M^L\cdot|x|$ applications of the sequences due to our correspondence between sequences (4.5) and (4.6). Since the length of each sequence $D(\boldsymbol{\beta}(\alpha))$ (resp. $D(\boldsymbol{\alpha}(\beta))$) with $\alpha\in A_{(\mathrm{s})}$ (resp. $\beta\in B_{(\mathrm{s})}$) is less than or equal to M, we get z from y making at most $M^{L+1}\cdot|x|$ applications of the rule. Hence, $\delta_{(\mathrm{s})}(x)\leqslant(M^{L+1}+1)\cdot|x|$. □

A system R is *simple*, if to any fully seamed word only a simple application of a rule is possible. A condition for R to be simple will be given in the next section.

Corollary 4.17. *If R is simple, then it terminates if and only if Γ_{s} is acyclic. If it terminates, it has linear derivational complexity.*

Proof. In a derivation sequence starting with $x\in\Sigma^+$, the number of raw applications is at most $|x|$. The number of nonraw s-open seams created by raw applications in total is at most $|x|$. Starting with one s-open seam, the length of simple sequence of derivation is bounded by M^{L+1} as we have seen in the proof of Theorem 4.16. It follows that $\delta(x)\leqslant(M^{L+1}+1)\cdot|x|$. □

Example 4.18. Let $\Sigma=\{a,b\}$. We consider two systems over Σ:

(1) Let $R_1=\{ababbb\rightarrow babbbab\}$. We have $A=\{ab\}$, $B=\{b,babbb\}$, $s_{ab}=abbb$, $s_b=ababb$, $s_{babbb}=a$, and the standard decompositions

$$babbbab = b\cdot s_{ab}\cdot ab = babbb\cdot ab.$$

R_1 is a simple system but the reason for it will be given in the next section. The graph Γ_1 associated with R_1 is shown as

$$\begin{array}{ll} ab\ \circ \leftarrow \circ\ b & \\ \qquad\quad\ \circ\ babbb. \end{array}$$

Since Γ_1 is acyclic, R_1 is terminating.

(2) Let $R_2=\{abb\rightarrow bbbaba\}$. Then, $A=\{a\}$, $B=\{b,bb\}$, $s_a=bb$, $s_b=ab$, $s_{bb}=a$ and we have the standard decompositions

$$bbbaba = b\cdot s_a\cdot s_b\cdot a = bb\cdot b\cdot s_b\cdot a.$$

The graph Γ_2 is

$$\begin{array}{l} a \circ \rightleftarrows \circ\, b \\ \qquad\ \ \circ\, bb. \end{array}$$

Since Γ_2 has a cycle a $\rightarrow b \rightarrow a$, R_2 is nonterminating. In fact, we have a simple 2-loop $abbbb \rhd abbbb$; $\underline{abb}bb \rightarrow [bbbab\underline{a]bb} \rightarrow [bbbab[bbbaba]$.

5. Gentle systems and tame systems

In this section we consider nonsimple applications of the rule and discuss systems more general than simple systems. We start with the following basic lemma.

Lemma 5.1. *Let x be a seamed word induced by a derivation sequence. Suppose that $\sigma_1 < \cdots < \sigma_n$ are the seams of x patched by an application of the rule to x. Then, there is a number m with $0 \leqslant m \leqslant n$ such that $\sigma_1, \ldots, \sigma_m$ are right seams and $\sigma_{m+1}, \ldots, \sigma_n$ are left seams.*

Proof. If there is no such m, then there is i with $1 \leqslant i < n$ such that σ_i is a left seam and σ_{i+1} is a right seam. The seamed piece of x between σ_i and σ_{i+1} is of the form $[t]$ by Lemma 3.7. This implies that t is contained in the left-hand side s of the rule. This contradiction proves our assertion. $\square$

Suppose that $\sigma_1 < \cdots < \sigma_n$ are the seams patched by a derivation step. If $n = 1$, the step is simple. If $n \geqslant 2$ and σ_1 is a right seam and σ_n is a left seam, the step is called *purely tame*. A step which is simple or purely tame is called *tame*, otherwise it is *wild*. So, a wild step patches seams $\sigma_1 < \cdots < \sigma_n$ $(n \geqslant 2)$ which are either all left seams or all right seams. A tame step such that $n = 2$ is called *purely gentle*. So, a purely gentle step patches only one left seam and only one right seam (or just one double seam). A step is *gentle* if it is simple or purely gentle. A step which is not gentle is *rude*.

The system R is *g-terminating* (resp. *t-terminating*), if there is no infinite sequence of gentle (resp. tame) derivation, otherwise we say R is *g-nonterminating* (resp. *t-nonterminating*). We write $\rightarrow_{(\mathrm{g})}$ (resp. $\rightarrow_{(\mathrm{t})}$) for a one-step of gentle (resp. tame) derivation. A seamed word x is called *g-reducible* (resp. *t-reducible*), if there is a word y such that $x \rightarrow_{(\mathrm{g})} y$ (resp. $x \rightarrow_{(\mathrm{t})} y$). A word that is not g-reducible (resp. t-reducible) is *g-irreducible* (resp. *t-irreducible*). A *gentle n-loop* $x \rhd^n_{(\mathrm{g})} x$ and a *tame n-loop* $x \rhd^n_{(\mathrm{t})} x$ are defined in the same way as before.

Example 5.2. Let $\Sigma = \{a, b\}$:

(1) Let $R_1 = \{ababb \rightarrow babbba\}$. We have a sequence of R_1-derivation:

$$aa\underline{ababb}babb \rightarrow a\underline{a[babb}ba]babb \rightarrow \underline{a[babb}ba]ba]babb \rightarrow$$

$$[babbba]ba]b\underline{a]babb} \rightarrow [babbb\underline{a]ba]b[b}abbba] \rightarrow [babbb[babbba]abbba]. \quad (5.1)$$

The first four steps above are simple, but the last step is tame but not gentle because it patches two right seams and one left seam.

(2) Let $R_2 = \{ababb \to bbabbab\}$. We have a sequence of R_2-derivation:

$$
\begin{aligned}
&abaabab\underline{ababb} \to abaa\underline{bab[b}babbab] \to \underline{aba[bb}abbab]babbab] \to \\
&[bbabb\underline{ab]abb}ab]babbab] \to [bbabb[bbabb\underline{ab]ab]b}abbab] \to \\
&[bbabb[bbabb[bbabbab]abbab]. \qquad (5.2)
\end{aligned}
$$

The first four steps above are simple, but the last step is wild because it patches two right seams only.

The following is an elaboration of Lemma 3.7 for tame derivation. Recall decompositions (3.1)–(3.3) and (3.5) in Section 3.

Lemma 5.3. *Let x be a seamed word induced by a sequence of tame derivation steps and let v be a seamed piece of x. If v is of the form $]u]$ (resp. $[u[$) with $u \in \Sigma^*$, then $u = s_{\boldsymbol{\alpha}'} w_\beta$ (resp. $u = w_\alpha s_{\boldsymbol{\beta}'}$) for some $\beta \in B$ (resp. $\alpha \in A$), and if y is nonraw and of the form $]u[$, then $u = s_{\boldsymbol{\alpha}'} w_{\beta\alpha} s_{\boldsymbol{\beta}'}$ for some β and $\alpha \in A$, where $\boldsymbol{\alpha}'$ (resp. $\boldsymbol{\beta}'$) is a suffix of $\boldsymbol{\alpha}(\beta)$ (resp. prefix of $\boldsymbol{\beta}(\alpha)$).*

Proof. Let y be a seamed word satisfying the condition on pieces in the lemma and let x be obtained from y by a tame application of the rule. It suffices to prove that x also satisfies the condition on pieces. First assume that the step is simple, patches a seam τ of y and creates new seams σ_1 and σ_2 ($\sigma_1 < \sigma_2$). Moreover we suppose that τ is a left seam (the case τ is a right seam is similar). Let v_1 (resp. v_2) be the seamed piece of y on the left (resp. right) of τ. We have to consider four cases; v_1 is of the form $[u_1[$ or $]u_1[$, v_2 is of the form $[u_2[$ or $[u_2]$. If v_1 is of the form $[u_1[$, and v_2 is of the form $[u_2[$ then, $u_1 = w_\alpha s_{\boldsymbol{\beta}'}$ and $u_2 = w_{\alpha'} s_{\boldsymbol{\beta}''}$ by assumption, where $\alpha, \alpha' \in A$, $\boldsymbol{\beta}'$ is a prefix of $\boldsymbol{\beta}(\alpha)$, and $\boldsymbol{\beta}''$ is a prefix of $\boldsymbol{\beta}(\alpha')$. Since τ is s-open, $\boldsymbol{\beta}'$ is nonempty and $\boldsymbol{\beta}' = \boldsymbol{\beta}^\dagger \beta$ with $\boldsymbol{\beta}^\dagger \in B^*$ and $\beta \in B$. Since τ is patched, the seamed word

$$y = \cdots [u_1[u_2[\cdots = \cdots [\omega_\alpha s_{\boldsymbol{\beta}^\dagger} \underline{s_\beta[\beta} s_{\boldsymbol{\alpha}(\beta)} w_{\beta\alpha'} s_{\boldsymbol{\beta}''}[\cdots$$

is transformed to

$$x = \cdots [w_\alpha s_{\boldsymbol{\beta}^\dagger} [t] s_{\boldsymbol{\alpha}(\beta)} w_{\beta\alpha'} s_{\boldsymbol{\beta}''}[\cdots$$

by the step, and the seamed piece of x on the left of σ_1 is $[w_\alpha s_{\boldsymbol{\beta}^\dagger}[$ and the seamed piece on the right of σ_2 is $]s_{\boldsymbol{\alpha}(\beta)} w_{\beta\alpha'} s_{\boldsymbol{\beta}''}[$. This implies that x also satisfies the condition on pieces. The other three cases are similar and we omit them.

Next, suppose that the step is purely tame. Let $\tau_1 < \cdots < \tau_k$ be the seams patched by the step. Since the step is purely tame, τ_1 is a right seam and τ_k is a left seam. Let

v_1 (resp. v_2) be the seamed piece of y on the left of τ_1 (resp. right of τ_k) and let w be the subword of y between τ_1 and τ_k. Again there are four cases to be considered, but here we only treat the case where v_1 is of the form $]u_1]$ and v_2 is of the form $[u_2]$. Then, $u_1 = s_{\boldsymbol{\alpha}'} w_\beta$ for some $\beta \in A$ and a suffix $\boldsymbol{\alpha}'$ of $\boldsymbol{\alpha}(\beta)$ by assumption, and $u_2 = t$ by Lemma 3.7. Since $\tau_1, \ldots, \tau_k$ are patched by the step, we see that $s = \alpha w \beta$ for some $\alpha \in A$ and $\beta \in B$, and

$$y = \cdots]s_{\boldsymbol{\alpha}'} w_\beta] w [t] \cdots = \cdots]s_{\boldsymbol{\alpha}'} w_{\beta\alpha} s_{\boldsymbol{\beta}(\alpha)} \underline{\alpha] w [\beta} t_\beta] \cdots$$

is transformed to

$$x = \cdots]s_{\boldsymbol{\alpha}'} w_{\beta\alpha} s_{\boldsymbol{\beta}(\alpha)} [t] t_\beta] \cdots \ .$$

The piece of x on the left of σ_1 is $s_{\boldsymbol{\alpha}'} w_{\beta\alpha} s_{\boldsymbol{\beta}(\alpha)}$ and the piece on the right of σ_2 is $t_\beta = s_{\boldsymbol{\alpha}(\beta)} w_\beta$. This implies x satisfies the condition on pieces. □

A system R is *gentle* (resp. *tame*), if to any fully seamed word no rude (resp. wild) application of a rule is possible. R is *very gentle*, if to any seamed (not necessarily fully seamed) word induced by a derivation sequence no rude application of a rule is possible. In the following we give a condition for a system to be tame, gentle or simple.

Lemma 5.4. (1) *R is tame if s is not written either as*

$$s = \alpha w_{\beta_1} \cdots w_{\beta_k} w, \tag{5.3}$$

where $k \geqslant 1$, $\alpha \in A$, $\beta_1, \ldots, \beta_k \in B$ and w is a nonempty prefix of $w_{\beta'\alpha'}$ for some $\alpha' \in A$ and $\beta' \in B$, nor as

$$s = w w_{\alpha_1} \cdots w_{\alpha_\ell} \beta, \tag{5.4}$$

where $\ell \geqslant 1$, $\beta \in B$, $\alpha_1, \ldots, \alpha_\ell \in A$ and w is a nonempty suffix of $w_{\beta'\alpha'}$ for some $\alpha' \in A$ and $\beta' \in B$.

(2) *R is gentle if R is tame and s is not written as*

$$s = \alpha w_{\beta_1} \cdots w_{\beta_k} w_{\beta'\alpha'} w_{\alpha_1} \cdots w_{\alpha_\ell} \beta, \tag{5.5}$$

where $k \geqslant 1$ or $\ell \geqslant 1$, and α, α', $\alpha_1, \ldots, \alpha_\ell \in A$ and β, β', $\beta_1, \ldots, \beta_k \in B$.

(3) *R is simple if and only if R is gentle and s is not written as*

$$s = \alpha' w_{\beta\alpha} \beta', \tag{5.6}$$

where α, $\alpha' \in A$ and β, $\beta' \in B$.

Proof. (1) Suppose that R is not tame. Let x be a fully seamed word to which a wild application can be made. We suppose that this is the first nonraw wild application, that is, no nonraw wild application was made before the seamed word x is induced. Suppose moreover that the right seams $\sigma_1 < \cdots < \sigma_{k+1}$ $(k \geqslant 1)$ are patched by this application and a left seam σ is created. Let σ_0 (resp. σ_{k+2}) be the neighbour seam of σ_1 (resp. σ_{k+1}) on the left (resp. right) in x and let u_i be the piece between σ_i and σ_{i+1} for $i = 0, 1, \ldots, k+1$;

$$x = \cdots u_0]u_1]\cdots]u_k]u_{k+1}\cdots .$$

By Lemma 3.7, u_0 is a suffix of t containing t_r as suffix, and by Lemma 5.3, $u_i = s_{\boldsymbol{\alpha}'_i} w_{\beta_i}$ for some $\beta_i \in B$, where $\boldsymbol{\alpha}'_i$ is a suffix of $\boldsymbol{\alpha}(\beta_i)$ for $i = 1, \ldots, k$ and u_{k+1} is a prefix of $s_{\boldsymbol{\alpha}'_{k+1}} w_{\beta_{k+1}}$ for some $\beta_{k+1} \in B$, where $\boldsymbol{\alpha}'_{k+1}$ is a prefix of $\boldsymbol{\alpha}(\beta_{k+1})$. Thus $s = \alpha \cdot s_{\boldsymbol{\alpha}'_1} w_{\beta_1} \cdots s_{\boldsymbol{\alpha}'_k} w_{\beta_k} w$, where α is a nonempty suffix of u_0 (hence, $\alpha \in A$) and w is a nonempty prefix of u_{k+1}. If $s_{\boldsymbol{\alpha}'_1} = s_{\alpha''} \cdots$ is not empty, then $u_0 s_{\alpha''}$ has s as suffix because u_0 has the suffix α''. On the other hand, because $s = \alpha u_1 \cdots$ and $|s_{\alpha''}| < |u_1|$, we see $|\alpha''| > |\alpha|$ and $\alpha s_{\alpha''} \in \mathrm{OVL}(s)$, but this is impossible because s is sof. Hence, $\boldsymbol{\alpha}'_1$ is empty. Similarly $\boldsymbol{\alpha}'_i$ is empty for each $i = 2, \ldots, k$, otherwise $w_{\beta_{i-1}} s_{\boldsymbol{\alpha}'_k}$ contains s. Letting $\beta' = \beta_{k+1}$, we see that w is a prefix of $w_{\beta'}$. Moreover, w is a prefix of $w_{\beta'\alpha'}$, because s (and w) does not overlaps with $s_{\boldsymbol{\beta}(\alpha')}\alpha'$ for any $\alpha' \in A$. Finally we find that s is written as (5.3). Similarly, if the left seams are patched, s is written as (5.4).

The proof of (2) and the 'if' part of (3) is similar. We shall prove the 'only if' part of (3). Suppose that s is written as (5.6). Decompose t as $t = \beta s_{\boldsymbol{\alpha}} w_{\beta\alpha} s_{\boldsymbol{\beta}} \alpha$, where $\boldsymbol{\alpha} = \boldsymbol{\alpha}(\beta) = \alpha_1 \cdots \alpha_k$ and $\boldsymbol{\beta} = \boldsymbol{\beta}(\alpha) = \beta_\ell \cdots \beta_1$, and let $x = s_\beta \cdot s \cdot s_\alpha$. Then, we have a derivation sequence:

$$x \to s_\beta t s_\alpha = s_\beta \beta s_{\boldsymbol{\alpha}} w_{\beta\alpha} s_{\boldsymbol{\beta}} \alpha s_\alpha \to^2 t s_{\boldsymbol{\alpha}} w_{\beta\alpha} s_{\boldsymbol{\beta}} t = t_{\alpha_1} \alpha_1 s_{\boldsymbol{\alpha}} w_{\beta\alpha} s_{\boldsymbol{\beta}} \beta_1 t_{\beta_1}$$

$$\to_{D(\boldsymbol{\alpha})} t_{\boldsymbol{\alpha}} t w_{\beta\alpha} s_{\boldsymbol{\beta}} \beta_1 t_{\beta_1} \to_{D(\boldsymbol{\beta})} t_{\boldsymbol{\alpha}} t w_{\beta\alpha} t t_{\boldsymbol{\beta}} = t_{\boldsymbol{\alpha}} t_{\alpha'} s t_{\beta'} t_{\boldsymbol{\beta}}$$

$$\to t_{\boldsymbol{\alpha}} t_{\alpha'} t t_{\beta'} t_{\boldsymbol{\beta}}.$$

The last step is purely gentle taken on the fully seamed word $t_{\boldsymbol{\alpha}} t w_{\beta\alpha} t t_{\boldsymbol{\beta}}$, and hence R is not simple. $\square$

For very gentle systems we have the following characterization.

Lemma 5.5. *R is very gentle, if and only if neither of the following holds*:
(1) *αw_β is a prefix of s for some $\alpha \in A$ and $\beta \in B$.*
(2) *$w_\alpha \beta$ is a suffix of s for some $\alpha \in A$ and $\beta \in B$.*

Proof. Assume (1) holds; $s = \alpha w_\beta s'$ with $s' \in \Sigma^*$. Let $\boldsymbol{\alpha} = \boldsymbol{\alpha}(\beta) = \alpha_1 \cdots \alpha_\ell$. Then, we have a derivation sequence:

$$s_\beta s s' \to s_\beta t s' = s t_\beta s' \to t t_\beta s' = t_{\alpha_1} \alpha_1 s_{\boldsymbol{\alpha}} w_\beta s'$$

$$\to_{D(\boldsymbol{\alpha})} t_{\boldsymbol{\alpha}} t w_\beta s' = t_{\boldsymbol{\alpha}} t_\alpha \underline{\alpha w_\beta s'} \to t_{\boldsymbol{\alpha}} t_\alpha t.$$

The last step $t_{\boldsymbol{\alpha}} t_\alpha \alpha w_\beta s' \to t_{\boldsymbol{\alpha}} t_\alpha t$ is rude (actually wild) because it patches two right seams $t]w_\beta$ and $w_\beta]s'$.

Conversely, we suppose that R is not very gentle. Let x be a seamed word to which a rude application can be made. We suppose that this is the first rude application and no rude application was made before x is induced. Suppose moreover that by this rude application $\sigma_1 < \cdots < \sigma_k$ are the seams patched. By Lemma 5.1, $\sigma_1, \ldots, \sigma_j$ are right seams and $\sigma_{j+1}, \ldots, \sigma_k$ are left seams for some $0 \leqslant j \leqslant k$. Since they are patched by a rude application, we see that $j \geqslant 2$ or $j \leqslant k-2$. If $j \geqslant 2$, then the piece of x between σ_1 and σ_2 is $s_{\boldsymbol{\alpha}'} w_\beta$ for some $\beta \in B$ by Lemma 5.3, where $\boldsymbol{\alpha}'$ is a suffix of $\boldsymbol{\alpha}(\beta)$. We can show that $s_{\boldsymbol{\alpha}'} = 1$ in the same way as the proof of (1) of Lemma 5.4. Because $\sigma_1, \ldots, \sigma_k$ are patched by the step, $s = \alpha w_\beta \cdots$ for $\alpha \in A$. When $j \leqslant k-2$, similarly we can show that s has a suffix of the form $w_\alpha \beta$ with $\alpha \in A$ and $\beta \in B$. $\square$

If (1) (resp. (2)) in Lemma 5.5 does not hold, we say that R is *left* (resp. *right*) *very gentle*.

The system $\{ab \to baa\}$ is very gentle but not simple (sequence (3.7) is not simple). The system R_1 in (1) of Example 4.18 is simple (*a fortiori*, gentle) because $s = ababbb$ can not be written as (5.3), (5.4) or (5.5) in Lemma 5.4. However, it is not left very gentle because s satisfies condition (1) of Lemma 5.5, in fact, we have the sequence

$$a\underline{ababbb}bb \to \underline{a[babbb}ab]bb \to [babbb\underline{ab]ab]bb} \to [babbb[babbbab].$$

The last step above is wild (though it is raw). It seems that there is no system that is neither left very gentle nor right very gentle, though we do not have a proof of it at hand. The system R_1 in Example 5.2 is tame but not gentle. The system R_2 in Example 5.2 is not even tame.

For the same reason as in the proof of Lemma 4.6 we have

Lemma 5.6. *The relations $\to_{(\mathrm{g})}$ and $\to_{(\mathrm{t})}$ are confluent. If $x \to_{(\mathrm{g})}^m y$ (resp. $x \to_{(\mathrm{t})}^m y$) and $x \to_{(\mathrm{g})}^n z$ (resp. $x \to_{(\mathrm{t})}^n z$), then there is w such that $y \to_{(\mathrm{g})}^{n'} w$ (resp. $y \to_{(\mathrm{t})}^{n'} w$) and $z \to_{(\mathrm{g})}^{m'} w$ (resp. $z \to_{(\mathrm{t})}^{m'} w$) with $m + n' = n + m'$.*

Due to Lemma 5.6, if $\to_{(\mathrm{g})}$ (resp. $\to_{(\mathrm{t})}$) is terminating on x, x has a unique g-irreducible (resp. t-irreducible) descendant by $\to_{(\mathrm{g})}$ (resp. $\to_{(\mathrm{t})}$), which are called the *g-canonical* (resp. *t-canonical*) form of x. For $x \in \Sigma^*$ we define $\delta_{(\mathrm{g})}(x)$ (resp. $\delta_{(\mathrm{t})}(x)$) to be the maximal length of sequences of gentle (resp. tame) derivation steps starting with x. It is equal to the length of a sequence from x to the g-canonical (resp. t-canonical) form of x. The *g-complexity* and *t-complexity* of R are the functions $d_{(\mathrm{g})}$ and $d_{(\mathrm{t})}$ defined by

$$d_{(\mathrm{g})}(n) = \max\{\delta_{(\mathrm{g})}(x) \mid x \in \Sigma^n\},$$

$$d_{(\mathrm{t})}(n) = \max\{\delta_{(\mathrm{t})}(x) \mid x \in \Sigma^n\},$$

respectively. Clearly, we have

$$d_{(\mathrm{s})}(n) \leqslant d_{(\mathrm{g})}(n) \leqslant d_{(\mathrm{t})}(n) \leqslant d_R(n).$$

We see that for a minimal left reducible word $x = s \cdot s_{\boldsymbol{\alpha}} \cdot x'$ (or a minimal right reducible word $x = x' \cdot s_{\boldsymbol{\beta}} \cdot s$) in Lemma 4.8, the subword x' remains raw not only in any simple derivation but also in any tame derivation starting with x. Thus we have

Lemma 5.7. *R is* g-*terminating* (*resp.* t-*terminating*) *if and only if it is weakly* g-*terminating* (*resp.* t-*terminating*) *on the seamed words $t_{\boldsymbol{\alpha}} \cdot t$ (or $t \cdot t_{\boldsymbol{\beta}}$) induced by the sequence*

$$s \cdot s_{\boldsymbol{\alpha}} \to t \cdot s_{\boldsymbol{\alpha}} = t_{\alpha_1} \cdot \alpha_1 \cdot s_{\boldsymbol{\alpha}} \to_{D(\boldsymbol{\alpha})} t_{\boldsymbol{\alpha}} \cdot t$$

$$(\text{or } s_{\boldsymbol{\beta}} \cdot s \to s_{\boldsymbol{\beta}} \cdot t \to_{D(\boldsymbol{\beta})} t \cdot t_{\boldsymbol{\beta}}),$$

where $\boldsymbol{\alpha} \in A^*$ (or $\boldsymbol{\beta} \in B^*$).

In the rest of section we discuss some more properties of the system which are useful to study termination.

We say R is *left* (resp. *right*) s-barren if for any $\alpha \in A$ and $\beta \in B$, s_α (resp. s_β) is not a prefix (resp. suffix) of t_β (resp. t_α), in other words, $B_{(\mathrm{s})} = \emptyset$ (resp. $A_{(\mathrm{s})} = \emptyset$).

Proposition 5.8. *A left or right* s-*barren system is* t-*terminating.*

Proof. If R is left s-barren, then any seamed word $t \cdot t_{\boldsymbol{\beta}}$ in Lemma 5.7 is t-irreducible. In fact, no simple application to $t \cdot t_{\boldsymbol{\beta}}$ is possible, because R is left s-barren, and no nonsimple tame application is possible either, because, $t \cdot t_{\boldsymbol{\beta}}$ has only right seams which are inner. □

Corollary 5.9. *A left or right* s-*barren tame system is terminating.*

The notion of left s-barren was introduced by McNaughton [5] as 'left barren'. He gave without proof a theorem [5, Theorem 2.8] that a left barren system is terminating. Later in [7], he announced that he could not prove the theorem and modified the definition of 'left barren'.

R is *left barren* if for any $\alpha \in A$ and $\beta_1 \cdots \beta_n \in B^+$, s_α is not a prefix of $t_{\beta_1} \cdots t_{\beta_n}$. Dually, R is *right barren*, if for any $\beta \in B$ and $\alpha_1 \cdots \alpha_n \in B^+$, s_β is not a suffix of $t_{\alpha_1} \cdots t_{\alpha_n}$. In particular, R is left and right barren if $A = \emptyset$ or $B = \emptyset$. Clearly, if R is left (resp. right) barren, then it is left (resp. right) s-barren.

Proposition 5.10 (McNaughton [7]). *A left or right barren system is terminating.*

Proof. Suppose that R is left-barren. Let $x = x' s_{\boldsymbol{\beta}} s \in \Sigma^*$ be the decomposition of a minimal right reducible word x in Lemma 4.8, where $\boldsymbol{\beta} \in B^*$. By Lemma 4.1, $x \to^+ x' \cdot t \cdot t_{\boldsymbol{\beta}}$, and the last word is reducible if and only if $t \cdot t_{\boldsymbol{\beta}}$ contains s as subword, but it never happens because R is left-barren. By Lemma 3.6, R is terminating. □

It is an open problem whether a left (right) s-barren system is terminating.

Lemma 5.11. *R is left (resp. right) barren if and only if it is left (resp. right)* s-*barren and* s *is not written as* (5.3) (*resp.* (5.4)).

Proof. Suppose that R is left s-barren but it is not left barren. Then, s_α is a prefix of $t_{\beta_1}\cdots t_{\beta_n}$ for some $\alpha\in A$ and $\beta_1,\ldots,\beta_n\in B$, and s_α cannot be a prefix of t_{β_1}. Hence, $s=\alpha s_\alpha=\alpha t_{\beta_1}\cdots t_{\beta_m}w$ for $1\leqslant m<n$ and w is a nonempty prefix of $t_{\beta_{m+1}}$. Since s does not overlap with $s_{\beta(\alpha')}$ for $\alpha'\in A$, w is a prefix of $w_{\beta'\alpha'}$, where $\beta'=\beta_{m+1}$. Since R is left s-barren, we see $t_{\beta_i}=w_{\beta_i}$ for all i, and s is written as (5.3). Thus, the 'if' part is proved. The proof of the 'only if' part is also easy. □

Because s is not written as (5.3) (resp. (5.4)) for a left (resp. right) very gentle system R, we have

Corollary 5.12. *R is left (resp. right) barren, if it is left (resp. right)* s-*barren and left (resp. right) very gentle.*

6. Traces and virtually seamed words

In Section 4 we introduced the system $\mathbf{S}_R$ over the alphabet $A\cup B$ associated with R, and established the correspondence between sequences of simple R-derivation and sequences of $\mathbf{S}_R$-derivation. In this section we generalize this transformation method to general (not necessarily simple) derivation. To this end the following notion of open seam is important.

Suppose that when a left (resp. right) seam σ is created in a derivation sequence, it patches seams $\tau_1<\cdots<\tau_n$. If the left-most (resp. right-most) patched seam $\tau=\tau_1$ (resp. $\tau=\tau_n$) is a right (resp. left) seam, then σ is called an *open* seam, otherwise, σ is a *closed* seam. Let σ be an open seam created by an application which patches a right seam τ of a seamed word x as the left-most patched seam (so σ is a left seam). Let u be the piece on the left of τ; $x=x'u]x''$. By Lemma 3.7, u is a suffix of t containing t_r as prefix. Hence, $u=u'\alpha$, $x''=s_\alpha x^\dagger$ for some $\alpha\in A$, and

$$x'u'\underline{\alpha]s_\alpha}x^\dagger\to x'u'[t]x^\dagger$$

is the step patching $\tau=u]\cdots$ and creating $\sigma=u'[t$. We associate the symbol α to the seam σ and call it the *label* of σ. On the other hand, if σ is a open right seam created by an application which patches a left seam τ as the right-most patched seam. The piece u on the right is a prefix of t containing t_ℓ as prefix; $x=x'[ux''$. Then, $u=\beta u'$, $x'=x^\dagger s_\beta$ for some β and

$$x^\dagger\underline{s_\beta[\beta}u'x''\to x^\dagger[t]u'x''$$

is the step. We associate β to σ and call it the *label* of σ. If the open seam σ is inherited in the next step, it remains open and retains the label.

In sequence (3.7) in Section 3, the seams marked with $\circ$ are open and the others are closed. The labels of the open seams are a, b, b and a, correspondingly from left to right.

Lemma 6.1. *A totally raw step creates two closed seams. A raw (resp. nonraw) seam created by a raw step is closed (resp. open). A purely tame step creates two open seams. A simple step or a wild step creates one open seam and one closed seam.*

Proof. Immediate from the definitions. $\square$

The following is another elaboration of Lemma 3.7 (cf. Lemma 5.3). The reasoning of it can be found in the argument above of the labelling method of open seams.

Lemma 6.2. *Let x be a seamed word induced by a derivation sequence and let $v=\sigma u\tau$ be a seamed piece of x with seams σ and τ. If σ is an open seam with label $\beta\in B$ (so, σ is a right seam), then u is a prefix of t_β. If moreover τ is a right seam ($v=]u]$), then $u=t_\beta$. On the other hand, if τ is an open seam with label α (so, τ is a left seam), then u is a suffix of t_α. If moreover σ is a left seam ($v=[u[$), then $u=t_\alpha$. In particular, if σ and τ are both open seams with labels β and α, respectively ($v=]u[$), then $u=s_{\boldsymbol{\alpha}}(\beta)w_{\beta\alpha}s_{\boldsymbol{\beta}}(\alpha)$.*

We introduce new sets $A^\#=\{\alpha^\#\mid\alpha\in A\}$ and $B^\#=\{\beta^\#\mid\beta\in B\}$ bijective to $A=\mathrm{OVL}(t,s)$ and $B=\mathrm{OVL}(s,t)$, respectively. We consider A, B, $A^\#$ and $B^\#$ to be mutually disjoint sets of symbols. Set $\Xi=A\cup B\cup A^\#\cup B^\#$, and with extra symbols $\not{c}$ and \$ outside Ξ set $\bar{\Xi}=\Xi\cup\{\not{c},\$\}$.

Let x_n be a seamed word induced by a derivation sequence

$$x_1\longrightarrow x_2\longrightarrow\cdots\longrightarrow x_n.$$

We give a *label* $\phi(\sigma)\in\bar{\Xi}$ to each seam σ of x_n by induction on n. When $n=1$, x_1 is seamless and there is nothing to do. Let $n>1$. For a seam σ of x_n which inherits a seam σ' of x_{n-1}, let $\phi(\sigma)=\phi(\sigma')$. Now, let (σ,τ) be a pair of seams of x_n created by the step $x_{n-1}\to x_n$. If the application does not patch any seam (so it is totally raw), we define $\phi(\sigma)=\not{c}$ and $\phi(\tau)=\$$. Suppose that it patches some seams and among them σ' is the left-most seam and τ' is the right-most seam. Suppose first that σ' is a right seam, that is, σ is open. Then $\phi(\sigma)$ is the label $\alpha\in A$ of σ already defined above before Lemma 6.1.

If σ' is a left seam, that is, σ is closed, we define $\phi(\sigma)$ according to the label of σ' as follows:

$$\phi(\sigma)=\begin{cases}\alpha^\# & \text{if } \phi(\sigma')=\alpha \text{ or } \alpha^\# \text{ with } \alpha\in A,\\ \not{c} & \text{if } \phi(\sigma')=\not{c}.\end{cases}$$

If τ' is a left seam, that is, τ is an open, $\phi(\tau)$ is the label $\beta \in B$ of τ defined above. If τ' is a right seam, that is τ is closed, then define $\phi(\tau)$ by

$$\phi(\tau) = \begin{cases} \beta^{\#} & \text{if } \phi(\tau') = \beta \text{ or } \beta^{\#} \text{ with } \beta \in B, \\ \$ & \text{if } \phi(\tau') = \$. \end{cases}$$

Let σ be a seam of a seamed word x induced by a derivation sequence. By definition we easily have

(1) If σ is an open left (resp. right) seam, then $\phi(\sigma) \in A$ (resp. B),
(2) If σ is a nonraw closed left (resp. right) seam, then $\phi(\sigma) \in A^{\#}$ (resp. $B^{\#}$),
(3) If σ is a raw left (resp. right) seam, then $\phi(\sigma) = ¢$ (resp. \$).

Now, we are ready to introduce a notion of trace which will play a central role in the rest of this paper.

For a seamed word x induced by a derivation sequence with seams $\sigma_1 < \sigma_2 < \cdots < \sigma_m$, $\phi(x)$ is defined to be the word $\phi(\sigma_1)\phi(\sigma_2)\cdots\phi(\sigma_m)$ over $\bar{\Xi}$. We call $\phi(x)$ the *trace* of x. For a sequence D of derivation (3.6) we have a corresponding sequence

$$\phi(D) \colon 1 = \phi(x_1) \to \phi(x_2) \to \cdots \to \phi(x_n) \to \cdots$$

of traces, which is called the *trace* of D.

Here, we give the traces of some of the sequences which we considered before.

For the system $R = \{ab \to baa\}$ we have $A = \{a\}$ and $B = \{b\}$. The trace corresponding to sequence (3.7) is

$$1 \to ¢\$ \to ¢\$\ ¢\$ \to ¢ab\$ \to ¢a^{\#}bb\$.$$

For the system R_1 in Example 5.2, $A = \{\alpha = a\}$, $B = \{\beta_1 = b, \beta_2 = babb\}$ and the trace of sequence (5.1) is

$$1 \to ¢\$ \to ¢\beta_2\$ \to ¢\beta_2\beta_2\$ \to ¢\beta_2\beta_2\alpha\$ \to ¢\alpha\beta_1\$.$$

For the system R_2 in the same example, $A = \{\alpha = ab\}$, $B = \{\beta_1 = b, \beta_2 = bb\}$ and the trace of sequence (5.2) is

$$1 \to ¢\$ \to ¢\beta_1\$ \to ¢\beta_2\beta_1\$ \to ¢\alpha\beta_2^{\#}\beta_1\$ \to ¢\alpha\alpha\beta_1^{\#}\$.$$

Next, we reformulate the results on simple derivation in terms of traces. First note that the trace of the seamed word (4.1) and (4.2) in Corollary 4.9 are $¢\alpha_1\cdots\alpha_m\$$ and $¢\beta_n\cdots\beta_1\$$, respectively. Let x be a seamed word with standard s-open left seam $\cdots s_{\boldsymbol{\beta}}[\beta_1\cdots$ with label $\alpha \in A_{(\mathrm{s})}$, where $\boldsymbol{\beta} = \boldsymbol{\beta}(\alpha) = \beta_n\cdots\beta_1 \in B^+$. Then, we have the standard sequence $x \to_{D_{\boldsymbol{\beta}}} y$ of simple derivation in (4.3). The trace of this sequence is given by

$$\alpha \to \alpha^{\#}\beta_1 \to \alpha^{\#}\beta_2\beta_1 \to \cdots \to \alpha^{\#}\beta_n\cdots\beta_2\beta_1.$$

If x has a standard s-open right seam $\cdots\alpha_1]s_{\boldsymbol{\alpha}}\cdots$ with label $\beta \in B_{(\mathrm{s})}$, where $\boldsymbol{\alpha} = \boldsymbol{\alpha}(\beta) = \alpha_1\cdots\alpha_m \in A^+$, then we have the standard sequence $x \to_{D_{\boldsymbol{\alpha}}} y$ in (4.4). The trace of the sequence is

$$\beta \to \alpha_1\beta^{\#} \to \alpha_1\alpha_2\beta^{\#} \to \cdots \to \alpha_1\alpha_2\cdots\alpha_m\beta^{\#}.$$

Now, we define a system $\mathbf{S}_R$ over Ξ by

$$\mathbf{S}_R = \{\alpha \to \alpha^{\#}\boldsymbol{\beta}(\alpha), \beta \to \boldsymbol{\alpha}(\beta)\beta^{\#} \mid \alpha \in A_{(s)}, \beta \in B_{(s)}\}.$$

The system is different from the system $\mathbf{S}_R$ defined in Section 4, but the symbols $\alpha^{\#}$ and $\beta^{\#}$ added only play a role of stopper (dummy), and thus the new system is terminating if and only if the old one is terminating. We use the same symbol $\mathbf{S}_R$ for the new system as for the old one.

The following is a reformulation of the correspondence between sequences (4.5) and (4.6) in Section 4.

Lemma 6.3. *Let x_1 be a fully seamed word with trace $\boldsymbol{\omega}_1$. There is a one-to-one correspondence between standard simple sequences of R-derivation*

$$x_1 \to_{D_1} x_2 \to_{D_2} \cdots \to_{D_{n-1}} x_n$$

and sequences of $\mathbf{S}_R$-derivation

$$\boldsymbol{\omega}_1 \to \boldsymbol{\omega}_2 \to \cdots \to \boldsymbol{\omega}_n$$

such that $\phi(x_i) = \boldsymbol{\omega}_i$ for $i = 1,\ldots,n$, and D_i is either $D(\boldsymbol{\beta}(\alpha))$ or $D(\boldsymbol{\alpha}(\beta))$ according as the step $\boldsymbol{\omega}_i \to \boldsymbol{\omega}_{i+1}$ is by a rule $\alpha \to \alpha^{\#}\boldsymbol{\beta}(\alpha)$ or by $\beta \to \boldsymbol{\alpha}(\beta)\beta^{\#}$ for $i = 1,\ldots,n-1$.

If x is a fully seamed word, then the first and the last positions are the only raw seams. Hence we see that $\phi(x) \in \not{c}\,\Xi^*\$$ for a fully seamed word x. On the other hand, a fully seamed word does not necessarily exist with given trace $\boldsymbol{\omega} \in \not{c}\,\Xi^*\$$. To fill this gap, we need to introduce a notion of virtually seamed word.

Let $\boldsymbol{\omega} = \omega_0\omega_1\cdots\omega_n$ be a word over $\bar{\Xi}$ such that $\omega_0 = \not{c}$, $\omega_n = \$$, $\omega_1\cdots\omega_{n-1} \in \Xi^*$. A *virtually seamed word* corresponding to $\boldsymbol{\omega}$ is a seamed word $x \in \Sigma^*$ whose $n+1$ positions are (formally) designated as seams as follows. Let $\sigma_0 < \sigma_1 < \cdots < \sigma_n$ be the seams of x and u_i be the piece between σ_{i-1} and σ_i for $i = 1,\ldots,n$. The edge pieces of x are empty ($x = u_1\cdots u_n$) and the ith piece u_i is equal to the (ω_{i-1}, ω_i)-entry in the following table:

	α'	β'	$\alpha'^{\#}$	$\beta'^{\#}$	$\$$
α	$t_{\alpha'}$	t	$w_{\alpha'}$	t	t
β	$s_{\boldsymbol{\alpha}(\beta)}w_{\beta\alpha'}s_{\boldsymbol{\beta}(\alpha')}$	t_β	$s_{\boldsymbol{\alpha}(\beta)}w_{\beta\alpha'}$	t_β	t_β
$\alpha^{\#}$	$t_{\alpha'}$	t	$w_{\alpha'}$	t	t
$\beta^{\#}$	$w_{\beta\alpha'}s_{\boldsymbol{\beta}(\alpha')}$	w_β	$w_{\beta\alpha'}$	w_β	w_β
$\not{c}$	$t_{\alpha'}$	t	$w_{\alpha'}$	t	t

where α, $\alpha' \in A$ and β, $\beta' \in B$. For example, if $\omega_{i-1} = \beta \in A$ and $\omega_i = \alpha^{\#} \in A^{\#}$, then $u_i = s_{\boldsymbol{\alpha}(\beta)}w_{\beta\alpha}$.

For the virtually seamed word x above, ω_i is the *label* of the seam σ_i and $\boldsymbol{\omega}$ is the *trace* of x. The seam σ_i is *open* (resp. *closed*) if the label ω_i is in $A \cup B$ (resp. $A^\# \cup B^\# \cup \{¢, \$\}$).

Remark that a virtually seamed word is not necessarily a seamed word induced by a derivation sequence starting with a raw word. By the definition of virtually seamed word we have

Lemma 6.4. *In a virtually seamed word, a closed seam is* s-*closed. An open seam is* s-*open if and only if it is a standard* s-*open if and only if the label is in* $A_{(\mathrm{s})} \cup B_{(\mathrm{s})}$.

Lemma 6.5. *For any* $\boldsymbol{\omega} \in ¢\Xi^*\$$, *there exists a unique virtually seamed word* $x \in \Sigma^*$ *with trace* $\boldsymbol{\omega}$.

Proof. We can build x by joining up appropriate pieces as indicated in the above table. $\square$

The virtually seamed word uniquely determined by $\boldsymbol{\omega}$ in Lemma 6.5 will be denoted by $\psi(\boldsymbol{\omega})$.

7. Gentle termination

In this section we introduce a system $\mathbf{G}_R$ over Ξ which simulates gentle R-derivation.

Let $\alpha \in A$ and $\beta \in B$, then $t = \beta s_{\boldsymbol{\alpha}(\beta)} w_{\beta\alpha} s_{\boldsymbol{\beta}(\alpha)} \alpha$. If $s = \alpha' w_{\beta\alpha} \beta'$ for some $\alpha' \in A$ and $\beta' \in B$, then we say that β and α are *linked*. Let $\boldsymbol{\alpha} = \boldsymbol{\alpha}(\beta) = \alpha_1 \cdots \alpha_j$ and $\boldsymbol{\beta} = \boldsymbol{\beta}(\alpha) = \beta_k \cdots \beta_1$. If the trace $\boldsymbol{\omega}$ of a virtually seamed word x contains a subword $\beta\alpha$ with linked β and α, then x contains a subword $t_r \cdot s_{\boldsymbol{\alpha}(\beta)} w_{\beta\alpha} s_{\boldsymbol{\beta}(\alpha)} \cdot t_\ell$. If $\boldsymbol{\alpha}(\beta) \neq 1$ and $\boldsymbol{\beta}(\alpha) \neq 1$, that is $\beta \in B_{(\mathrm{s})}$ and $\alpha \in A_{(\mathrm{s})}$, then x contains a subword $\alpha_1 s_{\boldsymbol{\alpha}(\beta)} w_{\beta\alpha} s_{\boldsymbol{\beta}(\alpha)} \beta_1$ and we have a derivation

$$\begin{aligned} D(\beta\alpha) : \alpha_1 s_{\boldsymbol{\alpha}} w_{\beta\alpha} s_{\boldsymbol{\beta}} \beta_1 &\rightarrow_{D(\boldsymbol{\alpha})} t_{\boldsymbol{\alpha}^\dagger} \cdot t \cdot w_{\beta\alpha} \cdot s_{\boldsymbol{\beta}} \beta_1 \\ &\rightarrow_{D(\boldsymbol{\beta})} t_{\boldsymbol{\alpha}^\dagger} \cdot t \cdot w_{\beta\alpha} \cdot t \cdot t_{\boldsymbol{\beta}^\dagger} = t_{\boldsymbol{\alpha}^\dagger} \cdot t_{\alpha'} \cdot \underline{\alpha' w_{\beta\alpha} \beta'} \cdot t_{\beta'} \cdot t_{\boldsymbol{\beta}^\dagger} \\ &\rightarrow_{D'(\beta\alpha)} t_{\boldsymbol{\alpha}^\dagger} \cdot t_{\alpha'} \cdot t \cdot t_{\beta'} \cdot t_{\boldsymbol{\beta}^\dagger}, \end{aligned}$$

where $\boldsymbol{\alpha}^\dagger = \alpha_2 \cdots \alpha_j$ and $\boldsymbol{\beta}^\dagger = \beta_k \cdots \beta_2$. The last step $D'(\beta\alpha)$ above is purely gentle. Correspondingly, the trace goes as follows:

$$\beta\alpha \rightarrow^* \boldsymbol{\alpha}\beta^\#\alpha \rightarrow^* \boldsymbol{\alpha}\beta^\#\alpha^\#\boldsymbol{\beta} \rightarrow \boldsymbol{\alpha}\alpha'\beta'\boldsymbol{\beta}.$$

If $\alpha \in A_{(\mathrm{s})}$ but $\beta \notin B_{(\mathrm{s})}$, then x contains a subword $\alpha_1 s_{\boldsymbol{\alpha}} w_{\beta\alpha} \beta'$ and we have a derivation

$$D(\beta\alpha) : \alpha_1 s_{\boldsymbol{\alpha}} w_{\beta\alpha} \beta' \rightarrow_{D(\boldsymbol{\alpha})} t_{\boldsymbol{\alpha}^\dagger} \cdot t \cdot w_{\beta\alpha} \cdot \beta' = t_{\boldsymbol{\alpha}^\dagger} \cdot t_{\alpha'} \cdot \alpha' w_{\beta\alpha} \beta' \rightarrow_{D'(\beta\alpha)} t_{\boldsymbol{\alpha}^\dagger} \cdot t_{\alpha'} \cdot t,$$

and the corresponding trace goes as

$$\beta\alpha \to \boldsymbol{\alpha}\beta^{\#}\alpha \to \boldsymbol{\alpha}\alpha'\beta'.$$

The case $\alpha \notin A_{(\mathrm{s})}$, $\beta \in B_{(\mathrm{s})}$ is dual.

Finally, if $\alpha \notin A_{(s)}$ and $\beta \notin B_{(s)}$, then x contains $\alpha' w_{\beta\alpha}\beta'$ and we have a derivation

$$D(\beta\alpha) = D'(\beta\alpha) : \alpha' w_{\beta\alpha}\beta' \to t$$

and the corresponding trace is

$$\beta\alpha \to \alpha'\beta'.$$

Lemma 7.1. *Let x be a virtually seamed word with trace $\boldsymbol{\omega}$.*

(1) *$\boldsymbol{\omega}$ contains $\alpha \in A_{(\mathrm{s})}$ (resp. $\beta \in A_{(\mathrm{s})}$), if and only if the derivation $D(\boldsymbol{\beta}(\alpha))$ (resp. $D(\boldsymbol{\alpha}(\beta))$) is applicable to x patching the seam corresponding to α (resp. β). If $\boldsymbol{\omega}'$ is obtained by replacing the α (resp. β) by $\alpha^{\#}\boldsymbol{\beta}(\alpha)$ (resp. $\boldsymbol{\alpha}(\beta)\beta^{\#}$) and x' is obtained by the application of $D(\boldsymbol{\beta}(\alpha))$ (resp. $D(\boldsymbol{\alpha}(\beta))$) to x, then x' is a virtually seamed word with trace $\boldsymbol{\omega}'$.*

(2) *If $\boldsymbol{\omega}$ contains a subword $\beta\alpha$ for linked β and α, then the derivation $D(\beta\alpha)$ is applicable to x patching the seams with the labels β and α. If $\boldsymbol{\omega}'$ is obtained by replacing the above subword $\beta\alpha$ by $\boldsymbol{\alpha}(\beta)\alpha'\beta'\boldsymbol{\beta}(\alpha)$ ($s = \alpha' w_{\beta\alpha}\beta'$) to $\boldsymbol{\omega}$ and x' is obtained by the application of $D(\beta\alpha)$ to x, then x' is a virtually seamed word with trace $\boldsymbol{\omega}'$.*

Proof. (1) The first half of the assertion follows from Lemma 6.4. Suppose that $D(\boldsymbol{\beta}(\alpha))$ is applied on a seam σ of x with label $\alpha \in A_{(\mathrm{s})}$. Let a seam σ' (resp. σ'') be the left (resp. right) neighbour of σ and u (resp. v) be the piece between σ' and σ (resp. σ and σ'') in x. We have to check all the cases corresponding to the labels of σ' and σ''. Here we only check the case where $\phi(\sigma') \in A$ and $\phi(\sigma'') \in B$, and leave the other cases to the reader. In this case, $u = t_\alpha$, $v = t$. Let $\boldsymbol{\beta}(\alpha) = \beta_n \cdots \beta_1$, then $t_\alpha = w_\alpha s_{\boldsymbol{\beta}(\alpha)}$ and $t = \beta_1 t_{\beta_1}$. Applying the derivation $D(\boldsymbol{\beta}(\alpha))$ to $x = \cdots uv \cdots = \cdots w_\alpha s_{\boldsymbol{\beta}(\alpha)}\beta_1 t_{\beta_1} \cdots$, we have $x' = \cdots w_\alpha \cdot t \cdot t_{\boldsymbol{\beta}(\alpha)} \cdots$. On the other hand, replacing α by $\alpha^{\#}\boldsymbol{\beta}(\alpha)$ in $\boldsymbol{\omega} = \cdots \alpha \cdots$, we get $\boldsymbol{\omega}' = \cdots \alpha^{\#}\boldsymbol{\beta}(\alpha) \cdots$. It is easy to see that x' is a virtually seamed word with the trace $\boldsymbol{\omega}'$.

(2) Let adjacent pieces σ and τ ($\sigma < \tau$) be labelled as β and α, respectively. The piece between them is $s_{\boldsymbol{\alpha}(\beta)} w_{\beta\alpha} s_{\boldsymbol{\beta}(\alpha)}$. Let $\boldsymbol{\alpha}(\beta) = \alpha_1 \cdots \alpha_j$ and $\boldsymbol{\beta}(\alpha) = \beta_k \cdots \beta_1$ and suppose $j, k \geqslant 1$ (the other cases can be treated similarly). The piece of x on the left of σ (resp. right of τ) has α_1 as suffix (resp. β_1 as prefix). Applying the derivation $D(\beta\alpha)$ on $x = \cdots \alpha_1 s_{\boldsymbol{\alpha}(\beta)} w_{\beta\alpha} s_{\boldsymbol{\beta}(\alpha)} \beta_1 \cdots$, we have $x' = \cdots t_{\boldsymbol{\alpha}^\dagger} t_{\alpha'} t t_{\beta'} t_{\boldsymbol{\beta}^\dagger} \cdots$, where $\boldsymbol{\alpha}^\dagger = \alpha_2 \cdots \alpha_j$ and $\boldsymbol{\beta}^\dagger = \beta_k \cdots \beta_2$. On the other hand, replacing the $\beta\alpha$ by $\boldsymbol{\alpha}(\beta)\alpha'\beta'\boldsymbol{\beta}(\alpha)$ in $\boldsymbol{\omega} = \cdots \beta\alpha \cdots$, we get $\boldsymbol{\omega}' = \cdots \boldsymbol{\alpha}(\beta)\alpha'\beta'\boldsymbol{\beta}(\alpha) \cdots$. It is easy to see that x' is a virtually seamed word with the trace $\boldsymbol{\omega}'$. $\square$

Now, for a linked pair (β, α) with $s = \alpha' w_{\beta\alpha}\beta'$, we consider the rule $\beta\alpha \to \boldsymbol{\alpha}(\beta)\alpha'\beta'$ $\boldsymbol{\beta}(\alpha)$ over Ξ, and let $\mathbf{G}'_R$ be the system of all such rules. By Lemma 5.4, (3), R

is simple if and only if R is gentle and there is no linked pair, that is, $\mathbf{G}'_R = \emptyset$. Now, we set

$$\mathbf{G} = \mathbf{G}_R = \mathbf{S}_R \cup \mathbf{G}'_R.$$

Example 7.2. Let $R = \{abb \to bbaba\}$, then $A = \{\alpha = a\}$, $B = \{\beta_1 = b, \beta_2 = bb\}$. R is very gentle by Lemma 5.5. The pairs (β_1, α) and (β_2, α) are linked, and the system $\mathbf{G}_R$ consists of the following rules:

$$\alpha \to \alpha^{\#}\beta_1, \quad \beta_1\alpha \to \alpha\beta_1\beta_1, \quad \beta_2\alpha \to \alpha\beta_2\beta_1. \tag{7.1}$$

The following is immediate from Lemma 7.1.

Corollary 7.3. *Let*

$$\mathscr{D} : \boldsymbol{\omega}_1 \to \boldsymbol{\omega}_2 \to \cdots \to \boldsymbol{\omega}_n \to \cdots$$

be a (finite or infinite) sequence of **G**-*derivation with* $\boldsymbol{\omega}_n \in \Xi^*$ *for* $n \geqslant 1$ *and let* $x_1 = \psi(\cent\boldsymbol{\omega}_1\$)$ *be the virtually seamed word with trace* $\cent\boldsymbol{\omega}_1\$$. *Then, there is a corresponding gentle sequence of R-derivation*

$$\psi(\mathscr{D}) : x_1 \to^+ x_2 \to^+ \cdots \to^+ x_n \to^+ \cdots,$$

where $x_i = \psi(\cent\boldsymbol{\omega}_i\$)$, *and each step* $x_i \to^+ x_{i+1}$ *is* $D(\boldsymbol{\beta}(\alpha))$, $D(\boldsymbol{\alpha}(\beta))$ *or* $D(\beta\alpha)$ *according as the rule applied in the step* $\boldsymbol{\omega}_i \to \boldsymbol{\omega}_{i+1}$ *is* $\alpha \to \alpha^{\#}\boldsymbol{\beta}(\alpha)$, $\beta \to \boldsymbol{\alpha}(\beta)\beta^{\#}$ *or* $\beta\alpha \to \boldsymbol{\alpha}(\beta)\alpha'\beta'\boldsymbol{\beta}(\alpha)$.

The following is a key for the proof of our main theorems on gentle derivation (Theorems 7.5 and 7.8).

Lemma 7.4. *Let* $\boldsymbol{\omega}_1 \in (A \cup B)^*$ *and*

$$\mathscr{D} : \boldsymbol{\omega}_1 \to \boldsymbol{\omega}_2 \to \cdots \to \boldsymbol{\omega}_n$$

be a sequence of **G**-*derivation. If* $x_n = \psi(\cent\boldsymbol{\omega}_n\$)$ *is* g-*reducible, then there is another sequence of* **G**-*derivation*

$$\mathscr{D}' : \boldsymbol{\omega}'_1 \to \boldsymbol{\omega}'_2 \to \cdots \to \boldsymbol{\omega}'_m$$

such that $\boldsymbol{\omega}'_1 = \boldsymbol{\omega}_1$ *and the sequence* $\psi(\mathscr{D}')$ *induced by* $\mathscr{D}'$ *is longer than the sequence* $\psi(\mathscr{D})$ *induced by* $\mathscr{D}$.

Proof. If x_n is s-reducible, then x_n has an s-open seam. Hence the trace $\boldsymbol{\omega}_n$ contains a letter from $A_{(\mathrm{s})} \cup B_{(\mathrm{s})}$ by Lemma 6.4. If $\boldsymbol{\omega}_n$ contains $\alpha \in A_{(\mathrm{s})}$ (resp. $\beta \in B_{(\mathrm{s})}$), then rule $\alpha \to \alpha^{\#}\boldsymbol{\beta}(\alpha)$ (resp. $\beta \to \boldsymbol{\alpha}(\beta)\beta^{\#}$) is applicable to $\boldsymbol{\omega}_n$ and $\mathscr{D}$ can be prolonged to $\mathscr{D}' : \boldsymbol{\omega}_1 \to_{\mathscr{D}} \boldsymbol{\omega}_n \to_{\mathbf{G}} \boldsymbol{\omega}_{n+1}$ by Lemma 7.1, (1). The sequence $\psi(\mathscr{D}')$ is longer than $\psi(\mathscr{D})$.

Next, suppose that x_n is g-reducible but s-irreducible. Then, a purely gentle application of the rule on x_n is possible. Let (λ, ρ) be a pair of the labels of the seams patched by the application. The piece between the seams must be $w_{\beta\alpha}$, and there are four possibilities: $(\lambda,\rho)=(\beta^{\#},\alpha^{\#})$ with $\beta\in B_{(s)}$, $\alpha\in A_{(s)}$, $(\lambda,\rho)=(\beta^{\#},\alpha)$ with $\beta\in B_{(s)}$, $\alpha\in A-A_{(s)}$, $(\lambda,\rho)=(\beta,\alpha^{\#})$ with $\beta\in B-B_{(s)}$, $\alpha\in A_{(s)}$, and $(\lambda,\rho)=(\beta,\alpha)$ with $\beta\in B-B_{(s)}$, $\alpha\in A-A_{(s)}$. Here, we only treat the first case where $(\lambda,\rho)=(\beta^{\#},\alpha^{\#})$ with $\beta\in B_{(s)}$, $\alpha\in A_{(s)}$. Since $\boldsymbol{\omega}_1$ has no letter from $A^{\#}\cup B^{\#}$, in some place in $\mathscr{D}$ applications of the rules $\beta\to\boldsymbol{\alpha}(\beta)\beta^{\#}$ and $\alpha\to\alpha^{\#}\boldsymbol{\beta}(\alpha)$ create $\beta^{\#}$ and $\alpha^{\#}$ respectively, and they are not touched afterward. Suppose that the application of the rule $\beta\to\boldsymbol{\alpha}(\beta)\beta^{\#}$ comes first and the application of the rule $\alpha\to\alpha^{\#}\boldsymbol{\beta}(\alpha)$ comes later, thus,

$$\mathscr{D}=\mathscr{D}_1\circ\mathscr{D}_\beta\boldsymbol{\zeta}_1\circ\mathscr{D}_2\circ\boldsymbol{\xi}_2'\mathscr{D}_\alpha\circ\mathscr{D}_3,$$

where $\mathscr{D}_1$ (resp. $\mathscr{D}_2$, $\mathscr{D}_3$) is a derivation sequence from $\boldsymbol{\omega}_1$ to $\boldsymbol{\xi}_1\boldsymbol{\zeta}_1$ (resp. $\boldsymbol{\xi}_2\boldsymbol{\zeta}_1$ to $\boldsymbol{\xi}_2'\boldsymbol{\zeta}_1'$, $\boldsymbol{\xi}_2'\boldsymbol{\zeta}_2'$ to $\boldsymbol{\omega}_n$), and $\mathscr{D}_\beta$ (resp. $\mathscr{D}_\alpha$) is a one-step derivation from $\boldsymbol{\xi}_1$ to $\boldsymbol{\xi}_2$ (resp. $\boldsymbol{\zeta}_1'$ to $\boldsymbol{\zeta}_2'$) by applying the rule $\beta\to\boldsymbol{\alpha}(\beta)\beta^{\#}$ (resp. $\alpha\to\alpha^{\#}\boldsymbol{\alpha}(\beta)$) to the β (resp. α) at the right-most (resp. left-most) position of $\boldsymbol{\xi}_1$ (resp. $\boldsymbol{\zeta}_1'$). The position $(\boldsymbol{\xi}_2,\boldsymbol{\zeta}_1)$ of $\boldsymbol{\xi}_2\boldsymbol{\zeta}_1$ is inherited in $\mathscr{D}_2$ untouched, because the $\beta^{\#}$ created by $\mathscr{D}_\beta$ is untouched. Applying Lemma 2.3 we have a derivation sequence

$$\mathscr{D}''=\mathscr{D}_1\circ\mathscr{D}_\beta\boldsymbol{\zeta}_1\circ\boldsymbol{\xi}_2\mathscr{D}_2'\circ\mathscr{D}_2''\boldsymbol{\zeta}_1'\circ\boldsymbol{\xi}_2'\mathscr{D}_\alpha\circ\mathscr{D}_3$$

from $\boldsymbol{\omega}_1$ to $\boldsymbol{\omega}_n$ with the same length as $\mathscr{D}$, where $\mathscr{D}_2'$ (resp. $\mathscr{D}_2'$) is a derivation sequence from $\boldsymbol{\zeta}_1$ to $\boldsymbol{\zeta}_1'$ (resp. $\boldsymbol{\xi}_2$ to $\boldsymbol{\xi}_2'$). Again by Lemma 2.3 we have a sequence

$$\mathscr{D}^{\dagger}=\mathscr{D}_1\circ\boldsymbol{\xi}_1\mathscr{D}_2'\circ\mathscr{D}_\beta\boldsymbol{\zeta}_1'\circ\boldsymbol{\xi}_2\mathscr{D}_\alpha\circ\mathscr{D}_2''\boldsymbol{\zeta}_2'\circ\mathscr{D}_3$$

from $\boldsymbol{\omega}_1$ to $\boldsymbol{\omega}_n$. Since the subword $\beta^{\#}\alpha^{\#}$ created by $\mathscr{D}_\beta\boldsymbol{\zeta}_1'\circ\boldsymbol{\xi}_2\mathscr{D}_\alpha$ is not touched in $\mathscr{D}_2''\boldsymbol{\zeta}_2'\circ\mathscr{D}_3$, we can replace $\mathscr{D}_\beta\boldsymbol{\zeta}_1'\circ\boldsymbol{\xi}_2\mathscr{D}_\alpha$ in $\mathscr{D}^{\dagger}$ by the derivation $\mathscr{D}_{\beta\alpha}$ that is caused by an application of the rule $\beta\alpha\to\boldsymbol{\alpha}(\beta)\alpha'\beta'\boldsymbol{\beta}(\alpha)$ on the subword $\beta\alpha$ of $\boldsymbol{\xi}_1\boldsymbol{\zeta}_1'$, and we get a new sequence

$$\mathscr{D}'=\mathscr{D}_1\circ\boldsymbol{\xi}_1\mathscr{D}_2'\circ\mathscr{D}_{\beta\alpha}\circ\mathscr{D}_4$$

starting from $\boldsymbol{\omega}_1$, where $\mathscr{D}_4$ is the derivation obtained from $\mathscr{D}_2''\boldsymbol{\zeta}_2'\circ\mathscr{D}_3$ by replacing the untouched subword $\beta^{\#}\alpha^{\#}$ by $\alpha'\beta'$ throughout. The sequence $\psi(\mathscr{D}')$ induced by $\mathscr{D}'$ is longer than $\psi(\mathscr{D})$ by 1. $\square$

Theorem 7.5. *A system R is* g*-terminating if and only if the system* **G** *is terminating.*

Proof. Suppose that **G** is not terminating and

$$\mathscr{D}:\boldsymbol{\omega}_1\to\boldsymbol{\omega}_2\to\cdots\to\boldsymbol{\omega}_n\to\cdots$$

be an infinite sequence of **G**-derivation, where $\boldsymbol{\omega}_n\in\Xi^*$. By Lemma 6.1, there is a virtually seamed word x with trace $\not{c}\boldsymbol{\omega}_1\$$. By Corollary 7.3 there is an infinite sequence $D=\psi(\mathscr{D})$ of gentle derivation starting with x. If we forget the (formally given) seams of x and consider x to be raw, D is an infinite sequence of gentle derivation starting from a raw word. Thus R is g-nonterminating.

Conversely, assume that R is not g-terminating. Then, by Lemma 5.7, there is an element $x \in \Sigma^*$ of the form $t_{\boldsymbol{\alpha}} \cdot t$ with trace $\not{c}\boldsymbol{\alpha}\$$, where $\boldsymbol{\alpha} \in A^*$ such that x has no g-irreducible descendant by gentle derivation. Using Lemma 7.4 repeatedly, we can construct a derivation sequence $\mathscr{D} : \boldsymbol{\alpha} \to^* \boldsymbol{\omega}$ of **G**-derivation such that the sequence $\psi(\mathscr{D})$ is arbitrarily long. This implies that we can make $\mathscr{D}$ arbitrarily long too. Hence, **G** is not terminating. $\square$

Corollary 7.6. *A gentle system R is terminating if and only if the system* **G** *is terminating.*

Proof. A gentle system is terminating if only if it is g-terminating on the set of fully seamed word by Lemma 2.2. Thus the result follows from Theorem 7.5. $\square$

We also see that if **G** is nonterminating and has a loop, then R has a gentle loop. Now, before getting into the complexity problem of gentle derivation we need the following lemma.

Lemma 7.7. *Let x be the seamed word induced by a sequence of raw and gentle derivation. Suppose that a raw and gentle application is impossible on x. Then x is decomposed as $x_0'x_1x_1' \cdots x_kx_k'$, where $x_0', x_1', \ldots, x_k'$ are raw and x_i is a virtually seamed word with $\phi(x_i) \in \not{c}(A \cup B)^*\$$ for $i = 1, \ldots, k$. Moreover, any sequence of gentle derivation starting from x leaves every x_i' untouched.*

Proof. Clearly x is decomposed as $x_0'x_1x_1' \cdots x_kx_k'$, where $x_0', x_1', \ldots, x_k'$ are raw and $x_1, \ldots, x_k$ are fully seamed. Due to Lemma 6.1 every nonraw seams of x is open, and hence $\phi(x_i) \in \not{c}(A \cup B)^*\$$. By Lemma 6.2 we find that x_i is a virtually seamed word. Assume that there is a sequence D from x of gentle derivation that touches some x_i'. Consider the first raw application in D. If the application is simple, it patches one raw seam with label \$ or $\not{c}$. If it is purely gentle, it patches two raw seams σ and τ labelled as \$ and $\not{c}$, respectively, and the piece between σ and τ is x_i' for some i. In either case it is possible to make this application to x, but this contradicts the assumption on x. $\square$

Theorem 7.8. *The* g-*complexity $d_{(\mathrm{g})}$ of R is equivalent to the complexity of* **G** *provided* $\mathbf{G} \neq \emptyset$.

Proof. First suppose that $\to_{(\mathrm{g})}$ is terminating with complexity function f. Let

$$\mathscr{D} : \boldsymbol{\omega}_0 \to \boldsymbol{\omega}_1 \to \cdots \to \boldsymbol{\omega}_n$$

with $\boldsymbol{\omega}_n \in \Xi^*$ be a sequence of **G**-derivation. By Corollary 7.3 we have the gentle sequence of R-derivation

$$\psi(\mathscr{D}) : x_0 \to^+ x_1 \to^+ \cdots \to^+ x_n,$$

where x_i is virtually seamed word with trace $\cent\omega_i\$$. Since x_0 consists of $|\omega_0|+1$ pieces and every piece is a subword of t, we have $|x_0| \leqslant |t| \cdot (|\omega_0|+1)$. Thus,

$$n \leqslant f(|x_0|) \leqslant f(C \cdot |\omega_0|),$$

where $C = 2 \cdot |t|$. It follows that **G** is also terminating with complexity $\preccurlyeq f$.

Conversely, suppose that **G** is terminating with complexity function g. Let $x \in \Sigma^*$ be any raw word. Let x_0 be a seamed word induced by a raw and gentle sequence D_0 from x, and suppose that a raw and gentle application on x_0 is impossible. By Lemma 4.15 the length of D_0 is bounded by $|x|$. In virtue of Lemma 7.7 we may suppose that x_0 is a virtually seamed word whose trace is in $\cent(A \cup B)^*\$$; $\phi(x_0) = \cent\omega_0\$$, $\omega_0 \in (A \cup B)^*$. By a similar argument to the latter part of the proof of Theorem 7.5 using Lemma 7.4, we have a derivation sequence of **G**-derivation $\mathscr{D}: \omega_0 \to \omega_1 \to \cdots \to \omega_n$ such that $x_n = \psi(\cent\omega_n\$)$ is a g-irreducible element in Σ^*. By assumption $n \leqslant g(|\omega_0|)$. Since the number of open seams of x_0 is bounded by $|x|$, we have $|\omega_0| \leqslant |x|$, and thus $n \leqslant g(|x|)$. Let L be the maximum of the lengths of the sequences $D(\alpha)$ for $\alpha \in A_{(\mathrm{s})}$, $D(\beta)$ for $\beta \in B_{(\mathrm{s})}$ and $D(\beta\alpha)$ for linked pairs of $\alpha \in A$ and $\beta \in B$. Then the length of the sequence $\psi(\mathscr{D})$ is bounded by $L \cdot n \leqslant L \cdot g(|x|)$. Since x_n is a g-canonical form of x, we have

$$\delta_{(\mathrm{g})}(x) \leqslant |x| + L \cdot g(|x|).$$

Therefore, if $\mathbf{G} \neq \emptyset$, then g is at least a linear function, and $\delta_{(\mathrm{g})}(x) \leqslant L' \cdot g(|x|)$ for a constant L'. Thus, $\to_{(\mathrm{g})}$ and **G** has equivalent complexity. □

Remark that if $\mathbf{G} = \emptyset$, then $\to_{(\mathrm{g})}$ always has linear complexity.

Corollary 7.9. *A very gentle system R that is terminating has complexity equivalent to* **G**, *provide* $\mathbf{G} \neq \emptyset$.

Proof. If R is very gentle, then $\to \, = \, \to_{(\mathrm{g})}$. Hence the result follows from Theorem 7.8. □

We conjecture that any terminating gentle system (even if it is not very gentle) has complexity equivalent to **G** provided $\mathbf{G} \neq \emptyset$.

Since the left-hand side of any rule from **G** does not contain a letter from $A^\# \cup B^\#$, every letter from $A^\# \cup B^\#$ only plays a role of stopper for **G**. So reducing them into one dummy symbol #, we get a simplified systems over $A \cup B \cup \{\#\}$ which we denote by the same symbols $\mathbf{S}_R$ and **G** as before:

$$\mathbf{S}_R = \{\alpha \to \#\boldsymbol{\beta}(\alpha), \beta \to \boldsymbol{\alpha}(\beta)\# \mid \alpha \in A_{(\mathrm{s})}, \beta \in B_{(\mathrm{s})}\},$$

$$\mathbf{G} = \mathbf{S}_R \cup \mathbf{G}'_R.$$

Theorems 4.12, 7.5 and 7.8 are remain valid for these simplified systems.

It is easy to see that system (7.1) in Example 7.2 is terminating, and thus $R = \{abb \to bbaba\}$ is terminating by Theorem 7.6. This also directly follows from Corollary 5.9, because R is a right s-barren gentle system.

In Sections 9 and 10 we mobilize our theorems obtained in these sections to analyse systems of the type $\{a^p b^q \to t\}$.

8. Some technical lemmas

In this section we give some technical lemmas about termination and complexity of finite string-rewriting systems (not necessarily one-rule systems), which will be used in the subsequent two sections.

A (nonnegative) *weight function* on Σ^* is a morphism $f : \Sigma^* \to \mathbb{R}$ from the monoid Σ^* to the additive group $\mathbb{R}$ of real numbers such that $f(x) \geqslant 0$. A weight function f is determined by the values $f(a)$ for $a \in \Sigma$. A rewriting system S is *f-nonincreasing* (resp. *f-decreasing*), if $f(s) \geqslant f(t)$ (resp. $f(s) > f(t)$) for every $s \to t \in R$. Let $\mathbf{f} = (f_1, \dots, f_u)$ be a sequence of weight functions on Σ^*. For a word $x \in \Sigma^*$, $\mathbf{f}(x)$ means the vector $(f_1(x), \dots, f_u(x)) \in \mathbb{R}^u$. Let $<$ be the lexicographic order on $\mathbb{R}^u$, that is, $(r_1, \dots, r_u) < (r'_1, \dots, r'_u)$ if and only if there is v such that $1 \leqslant v \leqslant u$, $r_1 = r'_1, \dots, r_{v-1} = r'_{v-1}$ and $r_v < r'_v$. A system R is $\mathbf{f}$-nonincreasing (resp. $\mathbf{f}$-decreasing), if $\mathbf{f}(s) \geqslant \mathbf{f}(t)$ (resp. $\mathbf{f}(s) > \mathbf{f}(t)$) for every $s \to t \in R$. For an $\mathbf{f}$-nonincreasing system R, set

$$S = \mathrm{Stab}_{\mathbf{f}}(R) = \{s \to t \mid \mathbf{f}(s) = \mathbf{f}(t)\}.$$

Clearly, $R - S$ is $\mathbf{f}$-decreasing.

Lemma 8.1. *Let $\mathbf{f} = (f_1, \dots, f_u)$ be a sequence of weight functions and let R be a finite system that is $\mathbf{f}$-nonincreasing. Let $S = \mathrm{Stab}_{\mathbf{f}}(R)$ and $R' = R - S$. Then, there is a constant C such that for any $x \in \Sigma^*$ and for any sequence of R-derivation starting with x, the number of steps by rules from R' is bounded by $C \cdot |x|$.*

Proof. We proceed by induction on u. If $u = 1$, then R' is f_1-decreasing. Let $F_1 = \max\{f_1(a) \mid a \in \Sigma\}$ and $m_1 = \min\{f_1(s) - f_1(t) \mid s \to t \in R'\}$. Then, as easily seen, every R-derivation sequence starting with $x \in \Sigma^*$ contains at most $F_1/m_1 \cdot |x|$ steps of derivation by rules from R'.

Suppose that $u > 1$. Let $\mathbf{f}' = (f_1, \dots, f_{u-1})$ and $S' = \mathrm{Stab}'_{\mathbf{f}}(R)$. Then $S' \supset S$ and $S' - S$ is f_u-decreasing. Set $R_1 = R - S'$ and $R_2 = S' - S$. Then, $R' = R_1 \cup R_2$ and R_1 is $\mathbf{f}'$-decreasing. Let $x = x_0 \to x_1 \to \cdots \to x_n$ be a sequence of derivation by R. By induction hypothesis there is a constant C_1 such that there exist at most $C_1 \cdot |x|$ steps of derivations by R_1. Let $F = \max\{f_u(a) \mid a \in \Sigma\}$, $m = \min\{f_u(s) - f_u(t) \mid s \to t \in R_2\}$ and $M = \max\{f_u(t) - f_u(s) \mid s \to t \in R_1\}$. Then, $f_u(x_{j+1}) - f_u(x_j) \leqslant M$ if $x_j \to_{R_1} x_{j+1}$, $f_u(x_{j+1}) - f_u(x_j) \leqslant -m$ if $x_j \to_{R_2} x_{j+1}$, and $f_u(x_{j+1}) - f_u(x_j) = 0$ otherwise. It follows that the number of steps of derivation by R_2 in the sequence is bounded by

$$\frac{f_u(x) + C_1|x| \cdot M}{m} \leqslant \frac{F + C_1 \cdot M}{m} \cdot |x|.$$

Consequently, the number of steps of derivation by R' is bounded by

$$\frac{F + C_1 \cdot (M+m)}{m} \cdot |x|. \qquad \square$$

Corollary 8.2. *A finite system that decreases a sequence* $\mathbf{f}$ *of weight functions terminates and has linear derivational complexity.*

Let Σ_1 be another alphabet and let $\Phi : \Sigma^* \to \Sigma_1^*$ be a morphism. For a rewriting system R over Σ, $\Phi(R) = \{\Phi(s) \to \Phi(t) \mid s \to t \in R\}$ is a system over Σ_1.

Lemma 8.3. *Let* R *be a rewriting system over* Σ *and* $\Phi : \Sigma^* \to \Sigma_1^*$ *be a morphism. If the system* $\Phi(R)$ *over* Σ_1 *is terminating, then so is* R.

Proof. If there is an infinite sequence $x_1 \to_R x_2 \to_R \cdots$ of derivation by R, we have an infinite sequence $\Phi(x_1) \to_{\Phi(R)} \Phi(x_2) \to_{\Phi(R)} \cdots$ of derivation by $\Phi(R)$. $\square$

The following less trivial lemma (already used in [11]) is also useful.

Lemma 8.4. *Let* R *be a rewriting system over* Σ *and* $\Phi : \Sigma^* \to \Sigma_1^*$ *be a morphism. Suppose that* $R = R_1 \cup R_2$, $R_1 \cap R_2 = \emptyset$, *and* $\Phi(s) = \Phi(t)$ *for all* $s \to t \in R_2$. *If both the systems* R_2 *and* $\Phi(R_1)$ *is terminating, then so is* R.

Proof. If there is an infinite sequence $D : x_1 \to_R x_2 \to_R \cdots$, we have an infinite sequence $\Phi(D) : \Phi(x_1) \to_{\Phi(R)} \Phi(x_2) \to_{\Phi(R)} \cdots$. Since R_2 is terminating, an infinite number of steps of D are applications of rules from R_1. If a step $x_i \to_R x_{i+1}$ is an application of a rule from R_2, then $\Phi(x_i) = \Phi(x_{i+1})$ by assumption, and we can shorten the sequence $\Phi(D)$ at this place. After shortening every such step, we have an infinite sequence of derivation by rules from $\Phi(R_1)$. $\square$

Let Σ_1 be a subset of Σ. For $x \in \Sigma^*$, $|x|_{\Sigma_1}$ denotes the number of occurrences in x of letters from Σ_1. $|.|_{\Sigma_1}$ is a typical weight function. When $\Sigma_1 = \{a\}$ we write $|.|_a$ for $|.|_{\Sigma_1}$.

Let $a \in \Sigma$ and f be a weight function on Σ_1^*, where $\Sigma_1 = \Sigma - \{a\}$. For a real number $\theta > 1$ we define a function $f_{a,\theta}$ on Σ^* by

$$f_{a,\theta}(x) = f(u_0) + f(u_1)\theta + \cdots + f(u_m)\theta^m \tag{8.1}$$

for $x \in \Sigma^*$ that is uniquely written as

$$x = u_0 a u_1 a \cdots a u_m,$$

where $m = |x|_a$ and $u_i \in \Sigma_1^*$ for $i = 0, 1, \ldots, m$. With this function we define a relation $\preccurlyeq$ on Σ^* as follows: for $x, y \in \Sigma^*$, $x \preccurlyeq y$ if and only if

(1) $|x|_a < |y|_a$, or

(2) $|x|_a = |y|_a$ and $f_{a,\theta}(x) \leqslant f_{a,\theta}(y)$.

Lemma 8.5. *$\preccurlyeq$ is a compatible quasi-order on Σ^*.*

Proof. The proof is easy and left to the reader. $\square$

Lemma 8.6. *Let R be a finite system such that $s \succ t$ for all $s \to t \in R$. Then, R is terminating and has at most exponential derivational complexity.*

Proof. $F = \max\{f(b) \mid b \in \Sigma_1\}$, then $f_{a,\theta}(x) \leqslant F \cdot |x| \cdot \theta^{|x|_a}$ for $x \in \Sigma^*$. Let $S = \{s \to t \in R \mid |s|_a > |t|_a\}$ and set $M = \max\{f_{a,\theta}(t) - f_{a,\theta}(s) \mid s \to t \in S\}$. First, the number of applications of rules from S in any R-derivation sequence D starting from x is bounded by $|x|_a$. Moreover, for $s \to t \in S$ and for any $y, z \in \Sigma^*$ we have

$$\begin{aligned} f_{a,\theta}(ytz) - f_{a,\theta}(ysz) &= \theta^{|y|_a}(f_{a,\theta}(t) - f_{a,\theta}(s)) + \theta^{|y|_a}(\theta^{|t|_a} - \theta^{|s|_a}) f_{a,\theta}(z) \\ &\leqslant \theta^{|y|_a} \cdot M. \end{aligned}$$

Since a rule from $R - S$ decreases the value of $f_{a,\theta}$, by at most $m = \min\{f_{a,\theta}(s) - f_{a,\theta}(t) \mid s \to t \in R - S\}$, the number of applications of rules from $R - S$ in D is bounded by $(f_{a,\theta}(x) + \theta^{|x|_a} \cdot M \cdot |x|_a)/m$. Thus the total number of steps in D is bounded by

$$\frac{|x|_a + f_{a,\ \theta}(x) + \theta^{|x|_a} \cdot M \cdot |x|_a}{m} \leqslant \frac{F + M + 1}{m} \cdot |x| \theta^{|x|} \leqslant \theta^{2|x|}$$

for a long enough $x \in \Sigma^*$. Therefore, R has at most exponential complexity. $\square$

Using a function $f'_{a,\theta}$ dual to (8.1) defined by

$$f'_{a,\theta}(x) = f(u_0)\theta^m + f(u_1)\theta^{m-1} + \cdots + f(u_m)$$

for $x = u_0 a u_1 a \cdots a u_m \in \Sigma^*$ with $u_i \in \Sigma_1$ for $i = 0, 1, \ldots, m$, we can define a quasi-order $\preccurlyeq'$ in a similar manner. With this $\preccurlyeq'$, a similar result to Lemma 8.6 can be obtained.

Finally, we give a type of nonterminating systems which will haunt the next two sections.

Lemma 8.7. *Let Σ be an alphabet containing letters $\alpha_1, \ldots, \alpha_r$ $(r \geqslant 1)$ and β. We consider a system S over an alphabet Σ consisting of the rules*:

$$\begin{aligned} &\alpha_i \to x(i)\beta \quad (i = 1, \ldots, m), \\ &\beta \to \alpha_k y, \qquad\qquad\qquad\qquad (8.2) \\ &\beta\alpha_i \to u(i)\beta \quad (i = 1, \ldots, r), \end{aligned}$$

where $1 \leqslant m \leqslant r$, $1 \leqslant k \leqslant r$, $x(1), \ldots, x(m)$, $y \in \Sigma^*$ and $u(1), \ldots, u(r) \in \{\alpha_1, \ldots, \alpha_r\}^*$. Suppose that $u(i)$ contains a letter α_j such that $j < i$ if $i > m$. Then, S has a loop and nonterminating.

Proof. If $k \leqslant m$, then

$$\beta \to \alpha_k y \to x(k)\beta y$$

and we have a loop $\beta \rhd \beta$. Henceforth we assume that $k > m$. Define a mapping $h: \{1,\dots,r\} \to \{1,\dots,r\}$ as follows. For $i > m$, let $h(i) = j$, where α_j is the leftest letter in $u(i)$ such that $j < i$, and for $i \leqslant m$, let $h(i) = i$. Set $\Sigma_1 = \{\alpha_1,\dots,\alpha_r\}$. For any u, $v \in \Sigma_1^*$ and $i > m$, we have

$$\beta u \alpha_i v \to^* u' \alpha_{h(i)} v' \beta \tag{8.3}$$

for some u', $v' \in \Sigma_1^*$. Let g be the least number such that

$$h^g(k) = \underbrace{h \circ \cdots \circ h}_{g}(k) \leqslant m.$$

Set $j_i = h^i(k)$ for $i = 0, 1, \dots, g$, then $k = j_0 > j_1 > j_2 > \cdots > j_g$, and $j_g \leqslant m$. Now, using (8.3), we have a derivation sequence

$$\beta^{g+1} \to \beta^g \alpha_k y \to^* u' \alpha_{j_g} u'' \beta^g y$$

for some u', $u'' \in \Sigma_1^*$. Applying the rule $\alpha_{j_g} \to x(j_g)\beta$ to the last term we get

$$u' x(j_g) \beta u'' \beta^g y,$$

which is further written to

$$u' x(j_g) u^\dagger \beta^{g+1} y,$$

where $u^\dagger \in \Sigma_1^*$. Thus, we have a loop $\beta^{g+1} \rhd \beta^{g+1}$. $\square$

9. The system $\{a^p b^q \to b^n a^m\}$

Let us consider a one-rule system $R = \{s = a^p b^q \to t = b^n a^m\}$ over $\Sigma = \{a, b\}$. We shall discuss the termination problem and the derivational complexity of R.

First, if $m < p$ or $n < q$, then R decreases the weight $|.|_a$ or $|.|_b$, and so it is terminating and has linear complexity.

If $m = p$ and $n = q$, then $\delta_R(a^{np} b^{nq}) = n^2$ as easily seen, and thus R has quadratic complexity.

Next suppose $m = p$ and $n > q$. Let $\theta > \sqrt[p]{n/q}$ and define a function f on Σ^* as follows.

$$f(x) = n_0 + n_1\theta + \cdots + n_r\theta^r$$

for $x = b^{n_0} a b^{n_1} \cdots a b^{n_r} \in \Sigma^*$ with $|x|_a = r$. We see that an application of the rule does not change the number of occurrences of the letter a and decreases the value of the function f. Hence, by Lemma 8.6, R has at most exponential complexity.

On the other hand, let r be the least integer not smaller than $q/(n-q)$. Then, $rn \geqslant (r+1)q$ and for any $\ell > 0$ we have a sequence

$$\begin{aligned} a^{\ell rp}b^{rq} &\to^{r} \quad a^{(\ell r-1)p}b^{rn}a^{p} = a^{(\ell r-1)p}b^{(r+1)q}\ldots \\ &\to^{r+1} a^{(\ell r-2)p}b^{(r+2)q}\ldots \to^{*} a^{(\ell-1)rp}b^{2rq}\ldots \\ &\to^{*} \quad a^{(\ell-2)rp}b^{4rq}\ldots \to^{*} b^{2^{\ell}rq}\ldots \ . \end{aligned}$$

Thus,

$$\delta(a^{\ell rp}b^{rq}) \geqslant \frac{rq}{n-q}(2^{\ell}-1).$$

Therefore, R has exponential complexity. Similarly if $m > p$ and $n = q$, then R has exponential complexity.

These complexity results were given by Zantema and Geser [13], but we have given a proof for completeness.

Now, let $m > p$ and $n > q$.

We have

$$A = \mathrm{OVL}(t,s) = \{a,\ldots,a^{p}\},$$

$$B = \mathrm{OVL}(s,t) = \{b,\ldots,b^{q}\},$$

$$S_A = \{a^{p-1}b^{q},\ldots,ab^{q},b^{q}\},$$

$$S_B = \{a^{p}b^{q-1},\ldots,a^{p}b,a^{p}\}.$$

We set $\alpha_i = a^{i}$ for $i = 1,\ldots,p$ and $\beta_j = b^{j}$ for $j = 1,\ldots,q$. We have

$$\boldsymbol{\beta}(\alpha_i) = \beta_q^{\rho_i}, \quad \boldsymbol{\alpha}(\beta_j) = \alpha_p^{\sigma_j},$$

where $\rho_i = [(m-i)/p]$, the greatest integer not exceeding $(m-i)/p$, and $\sigma_j = [(n-j)/q]$. Let $m'' = \min\{p, m-p\}$ and $n'' = \min\{q, n-q\}$, then we have the (simplified) system

$$\mathbf{S}_R = \{\alpha_i \to \#\beta_q^{\rho_i}, \beta_j \to \alpha_p^{\sigma_j}\# \mid i = 1,\ldots,m'',\ j = 1,\ldots,n''\}$$

over $\Xi = A \cup B \cup \{\#\}$. Thus, in the graph Γ_{s} associated with $\mathbf{S}_R$, $\alpha_i \to \beta_q$ $(i = 1,\ldots,m'')$ and $\beta_j \to \alpha_p$ $(i = 1,\ldots,n'')$ are all the edges in Γ_{s}. Hence, there is the edge $\alpha_p \to \beta_q$ if and only if $m \geqslant 2p$, and there is the edge $\beta_q \to \alpha_p$ if and only if $n \geqslant 2q$. This implies that Γ_{s} is acyclic if and only if $m < 2p$ or $n < 2q$. In particular, R is s-nonterminating, if and only if $m \geqslant 2p$ and $n \geqslant 2q$ by Theorem 4.12.

It is easy to see that R is very gentle. We have

$$t = b^{j} \cdot b^{\sigma_j q} \cdot b^{\tilde{j}}a^{\tilde{i}} \cdot a^{\rho_i p} \cdot a^{i},$$

where $\tilde{i} \equiv m - i \pmod{p}$, $\tilde{j} \equiv n - j \pmod{q}$, $0 \leqslant \tilde{i} < p$ and $0 \leqslant \tilde{j} < q$. Thus, β_j and α_i are linked if and only if $i \equiv m \pmod{p}$ or $j \equiv n \pmod{q}$. If $i \equiv m \pmod{p}$, then $w_{\beta_j\alpha_i} = b^{\tilde{j}}$ and $s = a^{p} \cdot w_{\beta_j\alpha_i} \cdot b^{q-\tilde{j}}$. On the other hand, if $j \equiv n \pmod{q}$, then $w_{\beta_j\alpha_i} = a^{\tilde{i}}$ and $s = a^{p-\tilde{i}}$.

$w_{\beta_j\alpha_i} \cdot b^q$. Let $m = \rho \cdot p + m'$ with $0 < m' \leqslant p$ and $n = \sigma q + n'$ with $0 < n' \leqslant q$. Then, the system corresponding to purely gentle derivation is

$$\mathbf{G}'_R = \{\beta_j\alpha_{m'} \to \alpha_p^{\sigma_j+1}\beta_{q-\tilde{j}}\beta_q^{\rho}, \beta_{n'}\alpha_i \to \alpha_p^{\sigma}\alpha_{p-\tilde{i}}\beta_q^{\rho_i+1} \mid i = 1,\dots,m'', j = 1,\dots,n''\}.$$

Now we suppose $p < m < 2p$ and $q < n$. Then $\rho = 1$ and $m' = m - p$. We claim that $\mathbf{G} = \mathbf{S}_R \cup \mathbf{G}'_R$ is nonterminating if and only if $n \equiv 0 \,(\mathrm{mod}\, q)$. The system $\mathbf{G}$ consists of the following rules:

$$\begin{aligned}
&\alpha_i \to \#\beta_q \quad (i = 1,\dots,m'),\\
&\beta_j \to \alpha_p^{\sigma}\# \quad (j = 1,\dots,n'),\\
&\beta_j \to \alpha_p^{\sigma-1}\# \quad (j = n'+1,\dots,q) \quad \text{if } \sigma \geqslant 2,\\
&\beta_j\alpha_{m'} \to \alpha_p^{\sigma+1}\beta_{q-\tilde{j}}\beta_q \quad (j = 1,\dots,n'),\\
&\beta_j\alpha_{m'} \to \alpha_p^{\sigma}\beta_{q-\tilde{j}}\beta_q \quad (j = n'+1,\dots,q),\\
&\beta_{n'}\alpha_i \to \alpha_p^{\sigma}\alpha_{p-\tilde{i}}\beta_q^2 \quad (i = 1,\dots,m'),\\
&\beta_{n'}\alpha_i \to \alpha_p^{\sigma}\alpha_{p-\tilde{i}}\beta_q \quad (i = m'+1,\dots,p).
\end{aligned} \tag{9.1}$$

First, suppose that $n' = q$, equivalently, $n \equiv 0 \,(\mathrm{mod}\, q)$, and let $\beta = \beta_q$. The system $\mathbf{G}$ contains the rules:

$$\begin{aligned}
&\alpha_i \to \#\beta \quad (i = 1,\dots,m'),\\
&\beta \to \alpha_p^{\sigma}\#,\\
&\beta\alpha_i \to \alpha_p^{\sigma}\alpha_{p-\tilde{i}}\beta^2 \quad (i = 1,\dots,m'),\\
&\beta\alpha_i \to \alpha_p^{\sigma}\alpha_{p-\tilde{i}}\beta \quad (i = m'+1,\dots,p).
\end{aligned}$$

Here, if $m' < i \leqslant p$, then $p - \tilde{i} = i - (m - p) < i$. Thus, the system $\mathbf{G}$ satisfies the condition on the system S in Lemma 8.7. Hence, $\mathbf{G}$ has a loop and is nonterminating.

Now, suppose that $n \not\equiv 0 \,(\mathrm{mod}\, q)$, that is, $n' \neq q$. Of course, $m' \neq p$. Consider the weight function $f_1 : \Xi^* \to \mathbb{N}$ defined by

$$f_1(w) = |w|_{(A \cup B - \{\alpha_p, \beta_q\})}, \quad w \in \Xi^*.$$

It is easy to see that all the rules in (9.1) do not increase the value of f_1. Among the rules only the following rules keep the value:

$$\begin{aligned}
&\beta_q \to \alpha_p^{\sigma-1}\# \quad \text{if } \sigma \geqslant 2,\\
&\beta_q\alpha_{m'} \to \alpha_p^{\sigma}\beta_{q-n'}\beta_q,\\
&\beta_{n'}\alpha_p \to \alpha_p^{\sigma}\alpha_{p-m'}\beta_q.
\end{aligned} \tag{9.2}$$

Consider another function $f_2 : \Xi^* \to \mathbb{N}$ defined by

$$f_2(w) = |w|_{\{\alpha_{m'}, \beta_{n'}\}}, \quad w \in \Xi^*.$$

All the rules in (9.2) (actually all the rules in (9.1)) do not increase the value of the function f_2.

Here, suppose that $m' \neq p - m'$, that is $m \neq \frac{3}{2}p$, then the third rule in (9.2) decreases f_2. Now consider the third function $f_3 : \Xi^* \to \mathbb{N}$ defined by

$$f_3(w) = |w|_{\{\alpha_{m'}, \beta_q\}}, \quad w \in \Xi^*.$$

The other two rules in (9.2) decrease f_3, and thus we see that **G** is (f_1, f_2, f_3)-decreasing. By Corollary 8.2, **G** terminates and has linear complexity. If $n' \neq q - n'$, we can similarly see that **G** terminates and has linear complexity.

Now, the only case remains is when $m = \frac{3}{2}p$ and $n = \sigma \cdot q + q/2$ $(\sigma \geqslant 1)$, that is, $m \equiv p/2 \pmod p$ and $n \equiv q/2 \pmod q$. If we set $\alpha = \alpha_p$, $\beta = \beta_q$, $\bar{\alpha} = \alpha_{m'}$ and $\bar{\beta} = \beta_{n'}$, system (9.2) becomes

$$\begin{aligned} &\beta \to \alpha^{\sigma - 1} \quad \text{if } \rho \geqslant 2, \\ &\beta\bar{\alpha} \to \alpha^{\sigma}\bar{\beta}\beta, \\ &\bar{\beta}\alpha \to \alpha^{\sigma}\bar{\alpha}\beta. \end{aligned} \tag{9.3}$$

Recall that all the rules in (9.1) which do not appear in (9.3) decrease the value of f_1, and note that the first rule in (9.3) and the rules in (9.1) which do not appear in (9.3) never produce the letter $\bar{\alpha}$ nor the letter $\bar{\beta}$. Let

$$\mathscr{D} \,:\, x = x_0 \to_{\mathbf{G}} x_1 \to_{\mathbf{G}} \cdots \to_{\mathbf{G}} x_k \to_{\mathbf{G}} \cdots$$

be a (finite or infinite) sequence of **G**-derivation, where $x_0, x_1, \ldots, x_k, \ldots \in \Xi^*$.

Here we first treat the case $\sigma = 1$, that is, $n = \frac{3}{2}q$. Then (9.3) becomes

$$\begin{aligned} &\beta\bar{\alpha} \to \alpha\bar{\beta}\beta, \\ &\bar{\beta}\alpha \to \alpha\bar{\alpha}\beta. \end{aligned} \tag{9.4}$$

A letter $\bar{\alpha}$ or $\bar{\beta}$ in x_k is *active*, if the rule $\beta\bar{\alpha} \to \alpha\bar{\beta}\beta$ or $\bar{\beta}\alpha \to \alpha\bar{\alpha}\beta$ is applied on it later in the sequence $\mathscr{D}$, otherwise it is *passive*. If $\bar{\alpha}$ is preceded by $\gamma\alpha$ with $\gamma \in \Xi - \{\bar{\alpha}, \bar{\beta}\}$ in x_k, then this $\bar{\alpha}$ is passive. In fact, the γ cannot be changed to $\bar{\beta}$ and hence the α cannot be changed to β later in $\mathscr{D}$, and thus the rule $\beta\bar{\alpha} \to \alpha\bar{\beta}\beta$ cannot be applied to the $\bar{\alpha}$. Similarly, If $\bar{\beta}$ is followed by $\beta\gamma$ with $\gamma \in \Xi - \{\bar{\alpha}, \bar{\beta}\}$ in x_k, then the $\bar{\beta}$ is passive. If $\bar{\alpha}$ is active in x_k, then later the rule $\beta\bar{\alpha} \to \alpha\bar{\beta}\beta$ is applied on the $\bar{\alpha}$. If the $\bar{\beta}$ thus created is still active, then later the letter β behind the $\bar{\beta}$ must be changed to α and then the rule $\bar{\beta}\alpha \to \alpha\bar{\alpha}\beta$ is applied. The $\bar{\alpha}$ thus created is preceded by $\gamma\alpha$ with $\gamma \in \Xi - \{\bar{\alpha}, \bar{\beta}\}$, and becomes passive. A similar fact holds for an active $\bar{\beta}$. Briefly speaking, a letter $\bar{\alpha}$ or $\bar{\beta}$ appearing in $\mathscr{D}$ becomes passive if the rules in (9.4) are applied at most twice.

Therefore, the number of steps by rules (9.4) in $\mathscr{D}$ is bounded by $2|x|_{\{\bar{\alpha},\bar{\beta}\}}$. Since the number of steps by rules in (9.1) other than (9.4) is linearly bounded by Lemma 8.1, the length of $\mathscr{D}$ is linearly bounded.

Next suppose that $\sigma \geqslant 2$. Again $\bar{\alpha}$ in x_k is passive if it is preceded by $\gamma\alpha$ with $\gamma \in \Xi - \{\bar{\alpha}, \bar{\beta}\}$. Thus, if $\bar{\beta}$ is active, then the rule $\bar{\beta}\alpha \to \alpha^\sigma \bar{\alpha}\beta$ is applied and the created $\bar{\alpha}$ becomes passive. If $\bar{\alpha}$ is active, it is changed to $\bar{\beta}$ by the rule $\beta\bar{\alpha} \to \alpha^\sigma \bar{\beta}\beta$, and this $\bar{\beta}$ is still active, it is further changed to $\bar{\alpha}$ by the rule $\bar{\beta}\alpha \to \alpha^\sigma \bar{\alpha}\beta$, and becomes passive. Therefore, the number of steps by the last two rules in (9.3) in $\mathscr{D}$ is bounded by $2 \cdot |x|_{\bar{\alpha}} + |x|_{\bar{\beta}}$, and the number of steps in $\mathscr{D}$ by the rules other than the first rule in (9.3) is linearly bounded, say, by $C|x|$ for some constant C. One such step increases the number of β contained in the word by at most 2. The first rule in (9.3) erases one β, and hence the number of steps by this rule is at most $2C \cdot |x| + |x|_\beta$. Finally, we find that the length of $\mathscr{D}$ is bounded by $4C \cdot |x|$.

When $q < n < 2q$ and $p < m$, we can get a corresponding result by a similar argument.

Summarizing we see that if $m > p$ and $n > q$, **G** is terminating if and only if

$$p < m < 2p, \quad n \not\equiv 0 \,(\mathrm{mod}\, q)$$

or

$$q < n < 2q, \quad m \not\equiv 0 \,(\mathrm{mod}\, p),$$

Moreover, in this case, **G** has linear derivational complexity. Therefore R is terminating by Corollary 7.5 and has linear derivational complexity by Corollary 7.8.

Theorem 9.1 (Zantema and Geser [11]). *The system $R = \{a^p b^q \to b^n a^m\}$ is terminating if and only if one of the following holds*:
(1) $m \leqslant p$ *or* $n \leqslant q$,
(2) $p < m < 2p$, $n \not\equiv 0 \,(\mathrm{mod}\, q)$,
(3) $q < n < 2q$, $m \not\equiv 0 \,(\mathrm{mod}\, p)$.

Theorem 9.2. (1) *R has linear complexity if and only if one of the following holds*:
(1.1) $m < p$,
(1.2) $n < q$,
(1.3) $p < m < 2p$, $n \not\equiv 0 \,(\mathrm{mod}\, q)$,
(1.4) $q < n < 2q$, $m \not\equiv 0 \,(\mathrm{mod}\, p)$.
(2) *R has quadratic complexity if and only if* $m = p$, $n = q$.
(3) *R has exponential complexity if and only if*
(3.1) $m = p$, $n > q$, *or*
(3.2) $m > p$, $n = q$.

These complexity results except for Cases (1.3) and (1.4) are given in [13]. Tahhan-Battar [10] showed that the system $\{aabb \to bbbaaa\}$ which belongs to Case (1.3) (and Case (1.4) too) has linear complexity.

10. The system $\{a^p b^q \to t\}$

Let $p, q \geqslant 1$, $\Sigma = \{a, b\}$ and t be an arbitrary word over Σ. Let us consider a one-rule system $R = \{a^p b^q \to t\}$. We shall give a complete characterization for termination of R. We suppose that t does not contain $a^p b^q$ as subword, otherwise R is nonterminating. Moreover, we suppose that t contains both the letters a and b, otherwise, R is terminating:

(1) The case where $t \in b\Sigma^* a$, that is, $t = b^n v a^m$ with $v = 1$ or $v \in a\Sigma^* b$, $m, n > 0$. If $v = 1$, this case was treated in the previous section, so we may assume that $v \neq 1$.

(1.1) Subcase $m \leqslant p$, $n \leqslant q$: We have

$$A = \{a, \dots, a^m\}, \quad B = \{b, \dots, b^n\},$$

$$S_A = \{a^{p-1}b^q, a^{p-2}b^q, \dots, a^{p-m}b^q\},$$

$$S_B = \{a^p b^{q-1}, a^p b^{q-2}, \dots, a^p b^{q-n}\}.$$

The word t is decomposed as

$$t = b^n(a^{p-p_1}b^q)\dots(a^{p-p_k}b^q)w(a^p b^{q-q_\ell})\dots(a^p b^{q-q_1})a^m,$$

where $1 \leqslant p_1 \leqslant \min\{m, p-1\}$, $1 \leqslant p_i \leqslant m$ for $i = 2, \dots, k$, $1 \leqslant q_1 \leqslant \min\{n, q-1\}$, $1 \leqslant q_j \leqslant n$ for $j = 2, \dots, \ell$ and w has neither a prefix of the form $a^{p'} b^q$ with $1 \leqslant p' \leqslant p - m$ nor a suffix of the form $a^p b^{q'}$ with $1 \leqslant q' \leqslant q - n$.

Let $\alpha_i = a^i$, $\beta_j = b^j$ for $i = 1, \dots, m$ and $j = 1, \dots, n$, then

$$s_{\alpha_i} = a^{p-i}b^q, \quad s_{\beta_j} = a^p b^{q-j},$$

$$t_{\alpha_i} = b^n(a^{p-p_1}b^q)\cdots(a^{p-p_k}b^q)w(a^p b^{q-q_\ell})\cdots(a^p b^{q-q_1})a^{m-i},$$

$$t_{\beta_j} = b^{n-j}(a^{p-p_1}b^q)\cdots(a^{p-p_k}b^q)w(a^p b^{q-q_\ell})\cdots(a^p b^{q-q_1})a^m.$$

Thus,

$$\boldsymbol{\beta}(\alpha_i) = \boldsymbol{\alpha}(\beta_j) = 1$$

for $i = 1, \dots, m-1$ and $j = 1, \dots, n-1$,

$$\boldsymbol{\beta}(\alpha_m) = \beta_{q_\ell} \cdots \beta_{q_1}$$

and

$$\boldsymbol{\alpha}(\beta_n) = \alpha_{p_1} \cdots \alpha_{p_k},$$

(1.1.1) If $k = \ell = 0$, then $w \neq 1$ and R is left and right s-barren. Since $w_{\beta_j} = b^{n-j} w a^m$ and $w \in a\Sigma^* b$, the word $a^i w_{\beta_j}$ cannot be a prefix of $s = a^p b^q$ for any i, j. This implies that R is left very gentle by Lemma 5.5, and R is left barren by Corollary 5.12 and is terminating by Proposition 5.10.

If $k = 0$, $\ell > 0$, then R is left barren again, and thus it is terminating. Similarly, if $k > 0$, $\ell = 0$, then R is right barren, and it is terminating.

(1.1.2) Suppose k, $\ell > 0$, then R is very gentle. We have the system

$$\mathbf{S}_R = \{\alpha_m \to \#\beta_{q_\ell} \cdots \beta_{q_1}, \beta_n \to \alpha_{p_1} \cdots \alpha_{p_k}\#\}$$

over $\Xi = A \cup B \cup \{\#\}$. Easily we see

(a) $\mathbf{S}_R$ is nonterminating if and only if $p_i = m$ for some i with $1 \leqslant i \leqslant k$ and $q_i = n$ for some i with $1 \leqslant i \leqslant \ell$.

By Lemma 5.4 we also see that R is not simple if and only if $w = a^r b^s$ with $p - m \leqslant r < p$, $q - n \leqslant s < q$. In this case, (β_n, α_m) is a unique linked pair, and we have the systems

$$\mathbf{G}'_R = \{\beta_n \alpha_m \to \alpha_{p_1} \cdots \alpha_{p_k} \alpha_{p-r} \beta_{q-s} \beta_{q_\ell} \cdots \beta_{q_1}\}$$

and

$$\mathbf{G} = \mathbf{S}_R \cup \mathbf{G}'_R.$$

We claim that if (a) is not the case, not only $\mathbf{S}_R$ is terminating but also $\mathbf{G}$ is terminating. To prove this we may suppose by symmetry that $p_i < m$ for all $i = 1, \ldots, k$.

(1.1.2.1) If $r \neq p - m$, then consider the weight $f : \Xi^* \to \mathbb{N}$ given by

$$f(\alpha_m) = \ell + 1, \quad f(\beta_n) = 1, \quad f(\alpha_i) = f(\beta_j) = f(\#) = 0$$

for $i \neq m$ and $j \neq n$. Then all the rules from $\mathbf{G}$ decrease the weight, and $\mathbf{G}$ is terminating.

(1.1.2.2) If $r = p - m$, then let $\alpha = \alpha_m$, $\beta = \beta_n$, $\Xi_1 = \{\alpha, \beta\}$ and $\Pi : \Xi^* \to \Xi_1^*$ be the projection, that is Π is the morphism defined by $\Pi(x) = x$ for $x \in \Xi_1$ and $\Pi(x) = 1$ for $x \in \Xi - \Xi_1$. Projecting $\mathbf{G}$ with Π, we have the system

$$\mathbf{G}_1 = \Pi(\mathbf{G}) = \{\alpha \to \beta^\lambda, \beta \to 1, \beta\alpha \to \alpha\beta^{\lambda+\varepsilon}\},$$

where $0 \leqslant \lambda \leqslant \ell$ and $0 \leqslant \varepsilon \leqslant 1$. Then this $\mathbf{G}_1$ is terminating because with the weight function $|.|_\beta$ and $\theta = \lambda + \varepsilon + 1$, the rules of $\mathbf{G}_1$ decreases the quasi-order $\leqslant'$ defined by the function

$$f'_{\alpha,\theta}(x) = |u_0|\theta^\mu + |u_1|\theta^{\mu-1} + \cdots + |u_\mu|$$

for $x = u_0 \alpha u_1 \alpha \cdots \alpha u_\mu$ with $u_i \in \beta^*$ (recall the remark after Lemma 8.6). Hence $\mathbf{G}$ is terminating by Lemma 8.3.

Summarizing, in Case (1.1), R is nonterminating, if and only if $\mathbf{G}$ is nonterminating, if and only if $k > 0$, $\ell > 0$ and there are i and j such that $1 \leqslant i \leqslant k$, $p_i = m$, $1 \leqslant j \leqslant \ell$ and $q_j = n$.

(1.2) *Subcase* $m > p$, $n < q$: We have

$$A = \{a, \ldots, a^p\}, \quad B = \{b, \ldots, b^n\},$$

$$S_A = \{a^{p-1}b^q, a^{p-2}b^q, \ldots, b^q\},$$

$$S_B = \{a^p b^{q-1}, a^p b^{q-2} \ldots, a^p b^{q-n}\}.$$

The word t is uniquely decomposed as

$$t = b^n(a^{p-p_1}b^q)\cdots(a^{p-p_k}b^q)wa^m,$$

where $1 \leqslant p_1 \leqslant p-1$, $1 \leqslant p_i \leqslant p$ for $i=2,\dots,k$, and w has neither a prefix of the form $a^{p'}b^q$ with $0 \leqslant p' \leqslant p-1$ nor the suffix a.

In this situation it is easy to see that R is right barren, and thus it is terminating.

(1.2′) *Subcase* $m<p$, $n>q$: Similarly to Subcase (1.2), R is left barren, and terminating.

(1.3) *Subcase* $m \geqslant p$, $n \geqslant q$: Then, R is very gentle because w_α contains b^n and w_β contains a^m. We have

$$A = \{a,\dots,a^p\}, \quad B = \{b,\dots,b^q\},$$

Let $\alpha_i = a^i$ and $\beta_j = b^j$ for $i=1,\dots,p$ and $j=1,\dots,q$ as before. Moreover, we have

$$S_A = \{a^{p-1}b^q, a^{p-2}b_q \dots, b^q\}, \quad S_B = \{a^p b^{q-1}, a^p b^{q-2} \dots, a^p\},$$

and t is uniquely decomposed as

$$t = b^n(a^{p-p_1}b^q)\cdots(a^{p-p_k}b^q)w(a^p b^{q-q_\ell})\cdots(a^p b^{q-q_1})a^m,$$

where $1 \leqslant p_1 \leqslant p-1$, $1 \leqslant p_i \leqslant p$ for $i=2,\dots,k$, $1 \leqslant q_1 \leqslant q-1, 1 \leqslant q_j \leqslant q$ for $j=2,\dots,\ell$ and w has neither a prefix of the form $a^{p'}b^q$ with $0 \leqslant p' \leqslant p-1$ nor a suffix of the form $a^p b^{q'}$ with $1 \leqslant q' \leqslant q-1$. Let $m = \rho p + m'$ and $n = \sigma q + n'$, where $1 \leqslant m' \leqslant p$ and $1 \leqslant n' \leqslant q$. We have

$$\boldsymbol{\alpha}(\beta_{n'}) = \alpha_p^\sigma \alpha_{p_1}\cdots\alpha_{p_k}, \quad \boldsymbol{\beta}(\alpha_{m'}) = \beta_{q_\ell}\cdots\beta_{q_1}\beta_q^\rho$$

and

$$\boldsymbol{\beta}(\alpha_i) = \beta_q^{\rho_i}, \quad \boldsymbol{\alpha}(\beta_j) = \alpha_p^{\sigma_j}$$

for $i \neq m'$ and for $j \neq n'$, where $\rho_i = [(m-i)/p]$ and $\sigma_j = [(n-j)/q]$. Thus, we have

$$\mathbf{S}_R = \{\alpha_i \to \#\beta_q^{\rho_i}, \alpha_{m'} \to \#\beta_{q_\ell}\cdots\beta_{q_1}\beta_q^\rho, \beta_j \to \alpha_p^{\sigma_j}\#, \beta_{n'} \to \alpha_p^\sigma\alpha_{p_1}\cdots\alpha_{p_k}\#$$
$$\mid i=1,\dots,p, i \neq m', \rho_i \neq 0, j=1,\dots,q, j \neq n', \sigma_j \neq 0\}.$$

(1.3.1) If $m \geqslant 2p, n \geqslant 2q$, then R is s-nonterminating. In fact, $\rho_p = [(m-p)/p] \geqslant 1$ and $\sigma_q = [(n-q)/q] \geqslant 1$, and $\mathbf{S}_R$ contains the rules $\alpha_p \to \#\beta_q^{\rho_p}$ if $m \neq m'$ (or $\alpha_p \to \#\beta_{q_\ell}\cdots\beta_{q_1}$ β_q^ρ if $p=m'$) and $\beta_q \to \alpha_p^{\sigma_q}\#$ if $q \neq n'$ (or $\beta_q \to \alpha_p^\sigma\alpha_{p_1}\cdots\alpha_{p_k}\#$ if $q=n'$). Therefore, $\mathbf{S}_R$ is nonterminating.

(1.3.2) *Case* $p \leqslant m < 2p$: Then,

$$\rho = \begin{cases} 0 & \text{if } m = p, \\ 1 & \text{otherwise,} \end{cases}$$

$\rho_i = 1$ for $i = 1, \ldots, m' - 1$ and $\sigma_j > 0$ if $j \leqslant n - q$. Hence, the system $\mathbf{S}_R$ consists of the following rules:

$$\alpha_i \to \#\beta_q \quad (i = 1, \ldots, m' - 1) \text{ if } m \neq p,$$

$$\alpha_{m'} \to \#\beta_{q_\ell} \cdots \beta_{q_1} \beta_q^{\rho},$$

$$\beta_j \to \alpha_p^{\sigma_j}\# \quad (j = 1, \ldots, \min\{q, n-q\},\ j \neq n'),$$

$$\beta_{n'} \to \alpha_p^{\sigma} \alpha_{p_1} \cdots \alpha_{p_k}\#. \tag{10.1}$$

It is easy to see that $\mathbf{S}_R$ is nonterminating if and only if one of the following three conditions is satisfied (the case $m = p$ and $q = n$ is excluded because it is contained in Case (1.1)):

(i) $m' = p$ (i.e. $m = p$), and there is j such that $q_j \leqslant n - q$,

(ii) $m' \neq p$ (i.e. $p < m < 2p$), $n' = q$ (i.e. $n \equiv 0 \pmod q$), and there is i such that $p_i \leqslant m'$,

(iii) $m' \neq p$ and $n' \neq q$, and there are i, j such that $p_i = m'$, $q_j = n'$.

To consider the possibility of g-nonterminating, we first treat the following two cases:

(1.3.2.1) $k = \ell = 0$: In this case $w \neq 1$.

(1.3.2.2) $k \neq 0$, $\ell \neq 0$:

In the above two cases, R is not simple, that is, $\mathbf{G}'_R \neq \emptyset$, if and only if $w = a^\gamma b^\delta$, $0 \leqslant \gamma < p$, $0 \leqslant \delta < q$. If $w = a^\gamma b^\delta$, then we have

$$\mathbf{G}'_R = \{\beta_{n'}\alpha_{m'} \to \alpha_p^{\sigma} \alpha_{p_1} \cdots \alpha_{p_k} \alpha_{p-\gamma} \beta_{q-\delta} \beta_{q_\ell} \cdots \beta_{q_1} \beta_q^{\rho}\}.$$

We claim that $\mathbf{G} = \mathbf{S}_R \cup \mathbf{G}'_R$ is terminating, if any of conditions (i)–(iii) is not satisfied. Set $\alpha = \alpha_{m'}$ and $\beta = \beta_{n'}$. We prove the claim separately in the cases corresponding to (i)–(iii).

(i) $m' = p$ (i.e. $m = p$): We assume that $q_j > n - q$ for all $j = 1, \ldots, \ell$. Let $\Xi_1 = \{\alpha, \beta_j \mid 1 \leqslant j \leqslant \min\{q, n-q\}\}$, $\Pi : \Xi^* \to \Xi_1^*$ be the projection and $\mathbf{G}_1 = \Pi(\mathbf{G})$. Then,

$$\mathbf{G}_1 = \{\alpha \to 1, \beta_j \to \alpha^{\sigma_j}, \beta \to \alpha^{\sigma+\kappa}, \beta\alpha \to \alpha^{\sigma+\kappa+\varepsilon}\beta^{\varepsilon'} \mid 1 \leqslant j \leqslant \min\{q, n-q\}, j \neq n'\},$$

where $\kappa < k$ and ε and ε' are 0 or 1. Applying the dual result of Lemma 8.6, we can show that $\mathbf{G}_1$ is terminating.

(ii) $m' \neq p$ *and* $n' = q$: We assume $p_i > m'$ for all $i = 1, \ldots, \ell$. Let $\Xi_1 = \{\alpha, \beta\}$ and define a morphism $\Phi : \Xi^* \to \Xi_1^*$ by

$$\Phi(\alpha_i) = \begin{cases} \alpha & \text{for } i = 1, \ldots, m' \\ 1 & \text{for } i = m' + 1, \ldots, p, \end{cases}$$

$$\Phi(\beta_j) = \beta \quad \text{for } j = 1, \ldots, q,$$

$$\Phi(\#) = 1.$$

Then,

$$\Phi(\mathbf{G}) = \{\alpha \to \beta, \alpha \to \beta^{\ell+1}, \beta \to 1, \beta\alpha \to \alpha^{\varepsilon}\beta^{\ell+2}\}$$

is terminating by Lemma 8.6. Thus G is also terminating.

(iii) $m' \neq p$ *and* $n' \neq q$: We assume $p_i \neq m'$ for all i or $q_j \neq n'$ for all j. Set $\Xi_1 = \{\alpha, \beta\}$, $\Xi_2 = \Xi - \Xi_1$ and let $\Pi : \Xi^* \to \Xi_1^*$ be the projection. Let

$$\mathbf{G}_2 = \{\alpha_i \to \#\beta_q, \beta_j \to \alpha_p^{\sigma_j}\# \mid i = 1, \ldots, m'-1,\ j = 1, \ldots, q, q \neq n', \sigma_j \neq 0\}$$

and $\mathbf{G}_1 = \mathbf{G} - \mathbf{G}_2$. Clearly, $\mathbf{G}_2$ is terminating and $\Pi(\chi) = \Pi(\psi) = 1$ for all $\chi \to \psi \in \mathbf{G}_2$. If $p_i \neq m'$ for all i, then

$$\Pi(\mathbf{G}_1) = \{\alpha \to \beta^{\lambda}, \beta \to 1, \beta\alpha \to \alpha^{\varepsilon}\beta^{\lambda+\varepsilon'}\},$$

where $0 \leqslant \lambda \leqslant \ell$ and $0 \leqslant \varepsilon,\ \varepsilon' \leqslant 1$. On the other hand if $q_j \neq n'$ for all j, then

$$\Pi(\mathbf{G}_1) = \{\alpha \to 1, \beta \to \alpha^{\kappa}, \beta\alpha \to \alpha^{\kappa+\varepsilon}\beta^{\varepsilon'}\},$$

where $0 \leqslant \kappa \leqslant k$ and $0 \leqslant \varepsilon,\ \varepsilon' \leqslant 1$. In either case by using Lemma 8.6 and its dual form we see that $\Pi(\mathbf{G}_1)$ is terminating. By Lemma 8.4, $\mathbf{G}$ is terminating.

We have proved the claim. Therefore, in case (1.3.2.1) R is always terminating, and in case (1.3.2.2) R is nonterminating if and only if it is s-nonterminating if and only if one of the conditions (i)–(iii) is satisfied.

(1.3.2.3) $k \neq 0$, $\ell = 0$: If $m = p$, then R is right barren and terminating. Otherwise, the system $\mathbf{S}_R$ is given by

$$\begin{aligned}
&\alpha_i \to \#\beta_q \quad (i = 1, \ldots, m'),\\
&\beta_j \to \alpha_p^{\sigma_j}\# \quad (j = 1, \ldots, \min\{q, n-q\}, j \neq n'),\\
&\beta_{n'} \to \alpha_p^{\sigma}\alpha_{p_1} \cdots \alpha_{p_k}\#.
\end{aligned} \tag{10.2}$$

On the other hand, the system $\mathbf{G}'_R$ can be nonempty only if w is written as $w = a^{\gamma}b^{\delta}$, $0 \leqslant \gamma < p$, $0 \leqslant \delta < q$. If $w \neq 1$, then

$$\mathbf{G}'_R = \{\beta_{n'}\alpha_{m'} \to \alpha_p^{\sigma}\alpha_{p_1} \cdots \alpha_{p_k}\alpha_{p-\gamma}\beta_{q-\delta}\beta_q\}$$

and we can show in a similar way to Cases (1.3.2.1) and (1.3.2.2) above that $\mathbf{G} = \mathbf{S}_R \cup \mathbf{G}'_R$ is nonterminating, if and only if $\mathbf{S}_R$ is nonterminating, if and only if condition (ii) above holds.

So we only discuss the case when $w = 1$. In this case the system $\mathbf{G}'_R$ consists of the following rules:

$$\begin{aligned}
&\beta_{n'}\alpha_i \to \alpha_p^{\sigma}\alpha_{p_1} \cdots \alpha_{p_k}\alpha_{2p-m+i}\beta_q^2 \quad (i = 1, \ldots, m'),\\
&\beta_{n'}\alpha_i \to \alpha_p^{\sigma}\alpha_{p_1} \cdots \alpha_{p_k}\alpha_{p-m+i}\beta_q \quad (i = m'+1, \ldots, p).
\end{aligned} \tag{10.3}$$

(1.3.2.3.1) If $n' \neq q$, *that is* $n \not\equiv 0 \pmod q$, then **G** is terminating. In fact, consider the weight function $f : \Xi^* \to \mathbb{N}$ defined by

$$f(\alpha_i) = 2 \quad \text{for } i = 1, \dots, p-1,$$

$$f(\alpha_p) = f(\#) = 0,$$

$$f(\beta_j) = 1 \quad \text{for } j = 1, \dots, q,\ j \neq n',$$

$$f(\beta_{n'}) = 2(k+2).$$

Then **G** is f-decreasing and terminating.

(1.3.2.3.2) $n' = q$, *that is*, $n \equiv 0 \pmod q$: We claim that **G** is not terminating. In fact, letting $\beta = \beta_q$ the system G contains the rules:

$$\alpha_i \to \#\beta \quad (i = 1, \dots, m'),$$

$$\beta \to \alpha_p^{\sigma} \alpha_{p_1} \cdots \alpha_{p_k} \#,$$

$$\beta\alpha_i \to \alpha_p \alpha_{p_1} \cdots \alpha_{p_k} \alpha_{2p-m+i} \beta^2 \quad (i = 1, \dots, m'),$$

$$\beta\alpha_i \to \alpha_p \alpha_{p_1} \cdots \alpha_{p_k} \alpha_{p-m+i} \beta \quad (i = m'+1, \dots, p). \tag{10.4}$$

Since $p - m + i = i - (m - p) < i$, the system **G** satisfies the condition on the system S in Lemma 8.7. Hence **G** admits a loop and is nonterminating.

(1.3.2.4) $k = 0$, $\ell \neq 0$: If $n = q$, R is left barren and terminating, otherwise, the system $\mathbf{S}_R$ is given by

$$\alpha_i \to \#\beta_q \quad (i = 1, \dots, m'-1),$$

$$\alpha_{m'} \to \#\beta_{q_\ell} \cdots \beta_{q_1} \beta_q^{\rho},$$

$$\beta_j \to \alpha_p^{\sigma_j} \# \quad (j = 1, \dots, \min\{q, n-q\}),$$

where $\rho = 0$ if $m = p$ and $\rho = 1$ otherwise. As in Case (1.3.2.3), the system $\mathbf{G}'_R$ can be nonempty only if $w = a^{\gamma} b^{\delta}$, $0 \leqslant \gamma < p$, $0 \leqslant \delta < q$. Again if $w \neq 1$, $\mathbf{G} = \mathbf{S}_R, \bigcup \mathbf{G}'_R$ is terminating, if and only if $\mathbf{S}_R$ is terminating. Thus, R is nonterminating if and only if the previously given condition (i) holds.

When $w = 1$, the system $\mathbf{G}'_R$ consists of the following rules:

$$\beta_j \alpha_{m'} \to \alpha_p^{\sigma_j + 1} \beta_{q - \tilde{j}} \beta_{q_\ell} \cdots \beta_{q_1} \beta_q^{\rho} \quad (j = 1, \dots, \min\{q, n-q\}),$$

where $\tilde{j} \equiv j - n \pmod q$, $0 \leqslant \tilde{j} < q$.

(1.3.2.4.1) If $m' \neq p$, that is $m \neq p$, **G** is terminating as in (1.3.2.3.1).

(1.3.2.4.2) If $m = p$ (and if $n > q$), then **G** is not terminating as in (1.3.2.3.2)

(1.3.3) *Case* $q \leqslant n < 2q$: dual to Case (1.3.2).

(2) The case where $t \in a\Sigma^* a$, that is, $t = a^n v a^m$, $m, n > 0$, $v \in \Sigma^+ - (a\Sigma^* \cup \Sigma^* a)$: If $n \geqslant p$ or $v \notin b^q \Sigma^*$, then $B = \mathrm{OVL}(s, t) = \emptyset$, and R is terminating. Otherwise, t is

decomposed as

$$t = a^{p_0} b^q (a^{p-p_1} b^q) \cdots (a^{p-p_k} b^q) w a^m,$$

where $0 < p_0 < p$, $0 < p_i \leqslant \min\{m, p\}$ for $i = 1, \ldots, k$ and $w \in \Sigma^*$ has neither a prefix of the form $a^{p-p'} b^q$, $0 \leqslant p' \leqslant \min\{m, p\}$ nor the suffix a. We have

$$A = \{a, a^2, \ldots, a^{m''}\}, \quad B = \{a^{p_0} b^q\},$$

$$S_A = \{a^{p-1} b^q, a^{p-2} b^q, \ldots, a^{p-m''} b^q\}, \quad S_B = \{a^{p-p_0}\},$$

where $m'' = \min\{m, p\}$.

We see that R is right very gentle. Set $\alpha_i = a^i$, for $i = 1, \ldots, m''$, and $\beta = a^{p_0} b^q$. Let $\rho_i = [(m-i)/(p-p_0)]$, then

$$\boldsymbol{\beta}(\alpha_i) = \beta^{\rho_i}, \quad \boldsymbol{\alpha}(\beta) = \alpha_{p_1} \cdots \alpha_{p_k}.$$

Thus, we find

$$\mathbf{S}_R = \{\alpha_i \to \#\beta^{\rho_i}, \beta \to \alpha_{p_1} \cdots \alpha_{p_k}\# \mid i = 1, \ldots, \min\{p, m-p+p_0\}\}.$$

(2.1) *Case* $w \neq 1$: Then, $w \in \Sigma^* b$ and it is easily checked that R is left very gentle and has no linked pair, and hence, R is simple. Thus R is nonterminating if and only if $\mathbf{S}_R$ is nonterminating if and only if there is i such that $1 \leqslant i \leqslant k$ and $p_i \leqslant m - p + p_0$.

(2.2) *Case* $w = 1$.

(2.2.1) *Case* $k = 0$, i.e. R is left s-barren: If $m \leqslant p - p_0$, then R is right s-barren. Hence, R is right barren and is terminating. On the other hand if $m \geqslant p$, then R is left very gentle and so left barren, and is terminating.

So, we treat the case $p - p_0 < m < p$: Then, R is gentle. In fact, if not, $s = a^p b^q$ must be written as $a^i \cdot (a^m)^j \cdot a^{p_0} b^q$ with $i, j > 0$ by Lemma 5.4, but this is impossible. Now, we have

$$\mathbf{S}_R = \{\alpha_i \to \#\beta^{\rho_i} \mid i = 1, \ldots, \min\{p, m-p+p_0\}\}$$

and

$$\mathbf{G}'_R = \{\beta\alpha_i \to \alpha_j \beta^{\rho_i+1} \mid i = 1, \ldots, m\},$$

where, $m - i = \rho_i(p - p_0) + \mu_i$, $0 \leqslant \mu_i < p - p_0$ and $j = p - p_0 - \mu_i$. Easily (again by the dual of Lemma 8.6) we see that $\mathbf{G} = \mathbf{S}_R \cup \mathbf{G}'_R$ is terminating. Hence, R is also terminating.

(2.2.2) *Case* $k > 0$: If $m \leqslant p - p_0$, then R is right barren and is terminating. If $m \geqslant 2p - p_0$, then $p_i \leqslant m - p + p_0$ for any i. Hence $\mathbf{S}_R$ is nonterminating, and R is s-nonterminating.

So, suppose that $p - p_0 < m < 2p - p_0$. We have

$$\mathbf{S}_R = \{\alpha_i \to \#\beta^{\rho_i}, \beta \to \alpha_{p_1} \cdots \alpha_{p_k}\# \mid i = 1, \ldots, m - (p - p_0)\}.$$

If there is i such that $1 \leqslant i \leqslant k$, $p_i \leqslant m - (p - p_0)$, then $\mathbf{S}_R$ is nonterminating. Now, suppose that there is no such i, that is, $p_i > m - p + p_0$ for all $i = 1, \ldots, k$. We see that R is gentle as in (2.2.1). We have

$$\mathbf{G}'_R = \{\beta\alpha_i \to \alpha_{p_1} \cdots \alpha_{p_k} \alpha_j \beta^{\rho_i + 1} \mid i = 1, \ldots, m\},$$

where $m - i = \rho_i(p - p_0) + \mu_i$, $0 \leqslant \mu_i < p - p_0$ and $j = p - p_0 - \mu_i$.

Note that $\mu_i = m - i$ and $j = p - p_0 - (m - i) = i - (m - p + p_0) < i$ if $i > m - p + p_0$. Hence by Lemma 8.7 we see that $\mathbf{G} = \mathbf{S}_R \cup \mathbf{G}'_R$ admits a loop and is nonterminating.

Thus, in Case (2.2) R is nonterminating if and only if $k > 0$ and $m > p - p_0$.

(3) The case where $t \in b\Sigma^* b$, that is, $t = b^n v b^m$, m, $n > 0$, $v \in \Sigma^+ - (b\Sigma^* \cup \Sigma^* b)$: This case is dual to (2).

(4) The case where $t \in a\Sigma^* b$, that is, $t = a^n v b^m$, m, $n > 0$, $v \in \Sigma^* - (b\Sigma^* \cup \Sigma^* a)$: If $m \geqslant p$ or $n \geqslant q$, then A or B is empty, and R is terminating. Suppose $m < p$ and $n < q$. Since t does not contain s and $|s| < |t|$, v cannot be empty. Hence, $v = b^g w a^f$ with g, $f > 0$, $w \in \Sigma^* - (b\Sigma^* \cup \Sigma^* a)$. If $g < q$ or $f < p$, then again A or B is empty, and R is terminating. Otherwise,

$$t = (a^n b^q) b^{g-q} w a^{f-p} (a^p b^m).$$

Thus, both A and B are singletons;

$$A = \{a^p b^m\}, \quad B = \{a^n b^q\}.$$

By Shikishima-Tsuji et al. [9, Theorem 2] R is nonterminating if and only if $g - q + m = q$ and $f - p + n = p$, that is,

$$t = a^n b^{2q-m} w a^{2p-n} b^m, \quad w \in \Sigma^* - (b\Sigma^* \cup \Sigma^* a).$$

Moreover, in this case R is s-nonterminating.

We can sum up the above case study into the following list of nonterminating systems.

Theorem 10.1. *The system $\{a^p b^q \to t\}$ over $\{a, b\}$ is nonterminating, if and only if*

(0) *t contains $a^p b^q$ as subword,*

or t is one of the following forms:

(1) $t = b^n (a^{p-p_1} b^q) \cdots (a^{p-p_k} b^q) w (a^p b^{q-q_\ell}) \cdots (a^p b^{q-q_1}) a^m$,

where $n, m \geqslant 1$, $0 < p_1 < p$, $0 < p_i \leqslant p$ for $i = 2, \ldots, k$, $0 < q_1 \leqslant q$, $0 < q_j \leqslant q$ for $j = 2, \ldots, \ell$, $w \in \Sigma^*$, and one of the following holds:

(1.1) *$m \geqslant 2p$ and $n \geqslant 2q$,*

(1.2) *there are i and j such that $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant \ell$, $p_i \equiv m \pmod p$ and $q_j \equiv n \pmod q$,*

(1.3) *$m \equiv 0 \pmod p$ and*

(1.3.1) *there is j such that $1 \leqslant j \leqslant \ell$ and $q_j \leqslant n - q$, or*

(1.3.2) *$k = 0$, $w = 1$, $n > q$, and $\ell > 0$ or $m > p$,* (1.4) $n \equiv 0 \pmod q$ *and*

(1.4.1) *there is* i *such that* $1 \leqslant i \leqslant k$ *and* $p_i \leqslant m - p$, *or*

(1.4.2) $\ell = 0$, $w = 1$, $m > p$, *and* $k > 0$ *or* $n > q$.

(2) $t = (a^{p-p_0}b^q)(a^{p-p_1}b^q)\cdots(a^{p-p_k}b^q)wa^m$,

where $m > 0$, $0 < p_0 < p$, $0 < p_i \leqslant \min\{m, p\}$ *for* $i = 1, \ldots, k$, $w \in \Sigma^*$, *and one of the following holds*:

(2.1) *there is* i *such that* $1 \leqslant i \leqslant k$ *and* $p_i \leqslant m - p_0$,

(2.2) $k > 0$, $w = 1$ *and* $m > p_0$.

(3) $t = b^n w(a^p b^{q-q_\ell})\cdots(a^p b^{q-q_1})(a^p b^{q-q_0})$,

where $n > 0$, $0 < q_0 < q$, $0 < q_j \leqslant \min\{n, q\}$ *for* $i = 1, \ldots, \ell$ *and* $w \in \Sigma^*$, *and one of the following holds*:

(3.1) *there is* j *such that* $1 \leqslant j \leqslant \ell$ *and* $q_j \leqslant n - q_0$,

(3.2) $\ell > 0$, $w = 1$ *and* $n > q_0$.

(4) $t = a^m b^{2q-n} w a^{2p-m} b^n$ *with* $0 < m < p$, $0 < n < q$ *and* $w \in \Sigma^*$.

Here we point out the fact known from our proof that the system $R = \{a^p b^q \to t\}$ is nonterminating if only if it is g-nonterminating if and only if it has a gentle loop. This supports McNaughton's conjecture that every nonterminating one-rule system has a loop.

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