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Eigen solutions of the Schrödinger equation and the thermodynamic stability of the black hole temperature

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ABSTRACT

The approximate analytical solutions of the Schrödinger equation for Eckart potential is obtained via supersymmetry shape invariance approach. The energy equation and the corresponding wave function are obtained in a closed and compact form. The wave function was used to calculate the Rényi entropy. The results of the Rényi entropy was used to study the mass energy parameter, temperature and heat capacity of the black hole. From the results obtained, the temperature of the black hole becomes stable as the two Eckart potential parameters increases respectively.

Introduction

In the recent years, a lot of articles have focused on the analytical and approximate solutions of the non-relativistic Schrödinger equation and the relativistic Klein-Gordon and Dirac equations. The most singular reason is the fact that the solutions of these wave equations contain all the necessary information for the quantum system under consideration. The non-relativistic Schrödinger equation has been used to study the spinless particles while the relativistic Klein-Gordon equation and Dirac equation have been used to study spin 0 and spin -1/2 particles respectively. The wave equations in the presence of various potential function have been studied extensively using different techniques such as asymptotic iteration method (AIM) [1–6], exact/proper quantization rule [7–10], conventional and parametric Nikiforov-Uvarov method [11–20], supersymmetric approach [21–27], factorization method [28], ansatz approaches [29–31], Formula method [32] and others.

The investigation of the non-relativistic Schrödinger equation for a particle in a strong potential field gives the complete description of such particle in the non-relativistic quantum mechanics. However, it is clearly observed that each potential model has its own advantages and failures. For example, some potentials such as Yukawa, Hellmann, Frost-Musulin, do not admit exact solutions due to the centrifugal barrier. Thus, to obtain the solutions of any wave equation with such potential model, the use of approximation scheme is high significant. The choice of approximation scheme depends on the nature of potential under consideration. In this study, we considered Eckart potential. The Eckart potential was introduced in 1930 [33] and is widely used in physics [34] and chemical physics [35,36]. The Eckart potential under consideration is of the form

$$V(r) = -\frac{\alpha\exp(-r/a)}{1-\exp(-r/a)} + \frac{\beta\exp(-r/a)}{(1-\exp(-r/a))^2}, \quad (1)$$

where α and β are Eckart potential parameters that describe the depth of the potential well and a is a parameter to control the width of the potential well [37]. On a remarkable note, $\alpha > 0$ and $\beta > 0$ but $\alpha > \beta$. Eckart potential is an exponential-type potential. The exponential-type potentials have great applications and interest in physics ranging from solid state physics to nuclear physics. This gives the motivation for the present study. Hassanabadi et al. [38] pointed out in one of their papers that in many cases, the exponential-type potential are even superior to their normally used partners which appear as Coulomb or inverse square ones. The present study is divided into two folds: in the first fold, we investigate the Schrödinger equation in the presence of Eckart potential. In the second fold, we calculate the Rényi entropy and then used the result to study the mass-energy parameter and temperature of the black hole. The scheme of our work is arranged as follows: In the next section, we obtained the solution of Schrödinger equation. The Rényi entropy and its application to black hole are studied in Section "Rényi Entropy" while discussion and conclusion are given in Sections "Schwarzschild black holes" and "Discussion" respectively (Table 1.).

Approximate solutions of the Schrödinger equation

Given the radial Schrödinger equation in 3-dimensional space as

$$\left[\frac{\hbar^2 d^2}{2\mu dr^2} + E_{n\ell} - V(r) - \frac{\ell(\ell+1)\hbar^2}{2\mu r^2}\right]U_{n\ell}(r) = 0, \quad (2)$$

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Table 1
Comparison of the energy eigenvalues for Eckart potential obtained using SUSY approach with other methods for 2p. 3p, 3d, 4p, 4d and 4f in atomic unit $\mu = \hbar = 1$, $\lambda_1 = 1.1$ and $\lambda_2 = 0.98$.

State	$1/a$	$\beta = 0.0001$			$\beta = 0.00005$		
		Present	[46]	[47]	Present	[46]	[47]
2p	0.025	0.101594	0.100888	0.100836	0.107350	0.106526	0.106474
	0.050	0.098298	0.098050	0.097836	0.099905	0.099630	0.099416
	0.075	0.088588	0.088880	0.088418	0.089306	0.089590	0.089128
3p	0.025	0.040132	0.040178	0.040125	0.042057	0.041894	0.041840
	0.050	0.032396	0.032454	0.032248	0.032853	0.032905	0.032701
	0.075	0.023773	0.023998	0.023555	0.023965	0.024188	0.023746
3d	0.025	0.041479	0.041519	0.041364	0.042590	0.042616	0.042459
	0.050	0.032108	0.032811	0.032197	0.032384	0.033089	0.032474
	0.075	0.022965	0.024150	0.022799	0.023078	0.024624	0.022915
4p	0.025	0.018547	0.018514	0.018463	0.019271	0.019230	0.019179
	0.050	0.010855	0.010908	0.010716	0.011026	0.011078	0.010885
	0.075	0.004792	0.004874	0.004506	0.004850	0.004936	0.004564
4d	0.025	0.018977	0.019076	0.018922	0.019435	0.019529	0.019375
	0.050	0.010686	0.011042	0.010460	0.010788	0.011146	0.010563
	0.075	0.004505	0.004924	0.003766	0.004539	0.004958	0.003800
4f	0.025	0.018946	0.019331	0.019022	0.019276	0.019661	0.019353
	0.050	0.010219	0.011102	0.009914	0.010290	0.011175	0.009988
	0.075	0.003991	0.004946	0.002500	0.004014	0.004972	0.002532

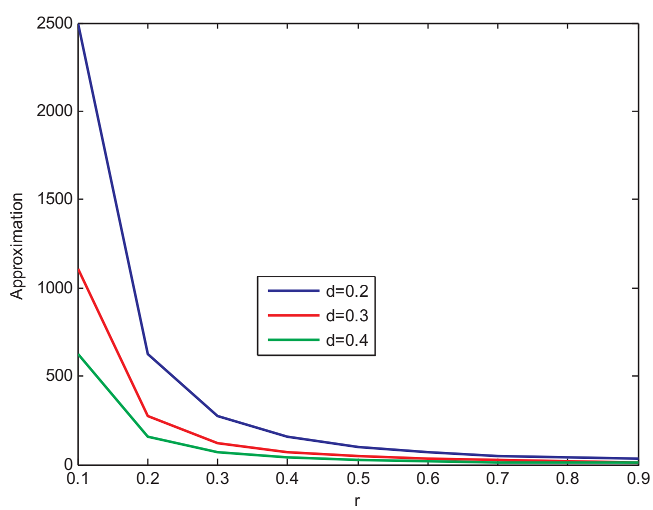

Fig. 1. The approximation scheme given in equation for various potential range.

where $E_{n\ell}$ is the non-relativistic energy, $U_{n\ell}(r)$ is the radial wave function, $V(r)$ is the potential function, μ is the reduced mass, $\hbar$ is the reduced Planck' s constant while n and ℓ are the momentum and orbital quantum number respectively. It is noted that Eq. (2) cannot be solved for $\ell \neq 0$ without the use of approximation scheme. Here, we resort to employ the following approximation scheme for a short potential range:

$$\frac{1}{r^2} \approx \frac{\eta_2 e^{-2\delta r}}{a^2(1-e^{-\delta r})^2} + \frac{\eta_1 e^{-\delta r}}{a^2(1-e^{-\delta r})}. \tag{3}$$

where η_1 and η_2 are dimensionless constants. Substituting Eqs. (1) and (3), into Eq. (2), we obtain

$$\frac{d^2U_{n\ell}(r)}{dr^2} = \left[\frac{[\ell(\ell+1)\hbar^2\eta_2 e^{-\delta r} + 2\mu\beta a^2]e^{-\delta r}}{a^2\hbar^2(1-e^{-\delta r})^2} + \frac{[\ell(\ell+1)\hbar^2\eta_1 - 2\mu\alpha a^2]e^{-\delta r}}{a^2\hbar^2(1-e^{-\delta r})} - \frac{2\mu E_{n\ell}}{\hbar^2}\right]U_{n\ell}(r). \tag{4}$$

where $\delta = \frac{1}{a}$, denote the range of the potential. We now apply the basic concept of supersymmetric quantum mechanics formalism and shape

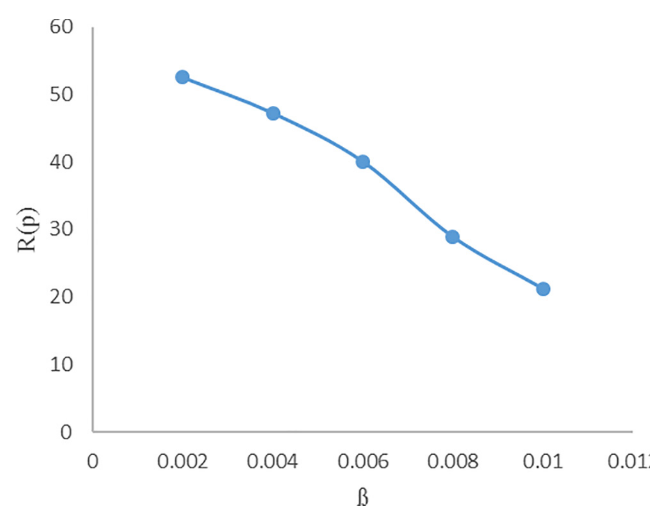

Fig. 2. Rényi entropy $R(\rho)$ versus the potential parameter β with $\lambda_1 = 1.1$, $\lambda_2 = 0.98$, $a = 40$, $\hbar = \mu = 1$ and $\ell = 0$ for the ground state.

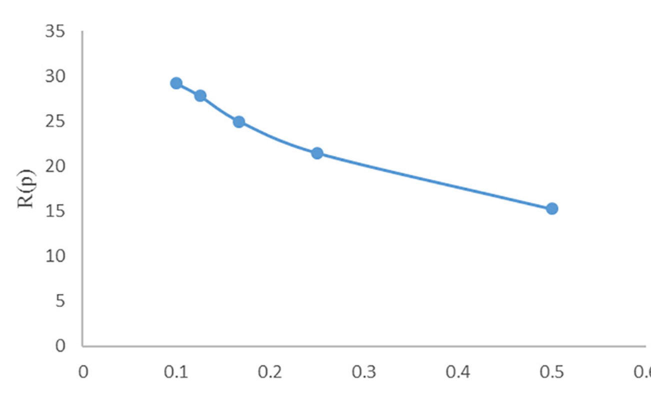

Fig. 3. Rényi entropy $R(\rho)$ versus the potential parameter β with $\lambda_1 = 1.1$, $\lambda_2 = 0.98$, $\beta = 0.001$, $\hbar = \mu = 1$ and $\ell = 0$ for the ground state.

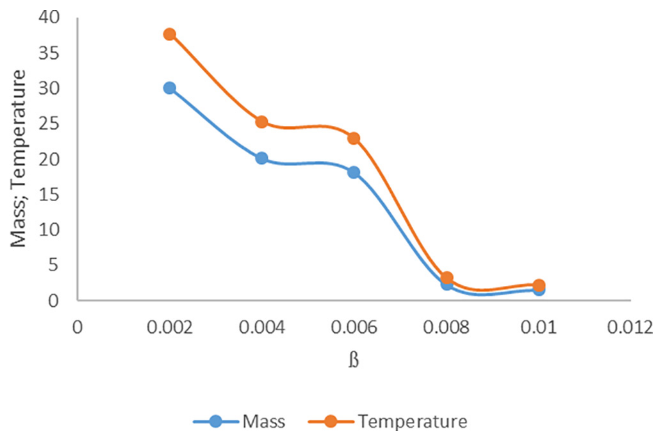

Fig. 4. The variation of the mass energy parameter M and temperature against the potential parameter β with $\mu = \hbar = 1$, $\ell = 0$, $a = 40$, $\lambda_1 = 1.1$ and $\lambda_2 = 0.98$ at the ground state level.

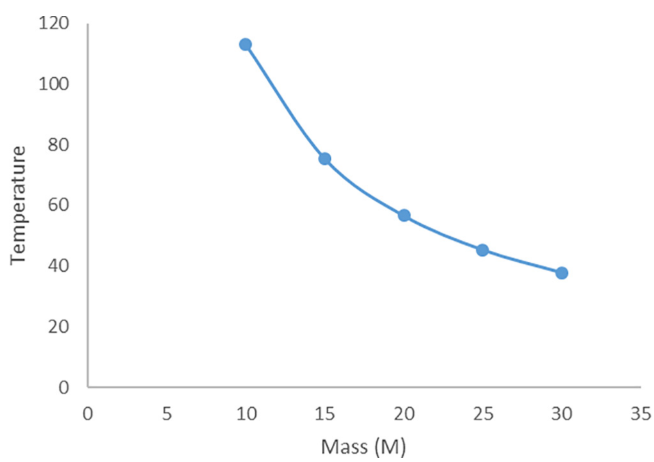

Fig. 5. Temperature versus mass energy parameter M with$\lambda_1 = 1.1$, $\lambda_2 = 0.98$, $a = 40$, $\hbar = \mu = 1$, β and $\ell = 0$ for the ground state.

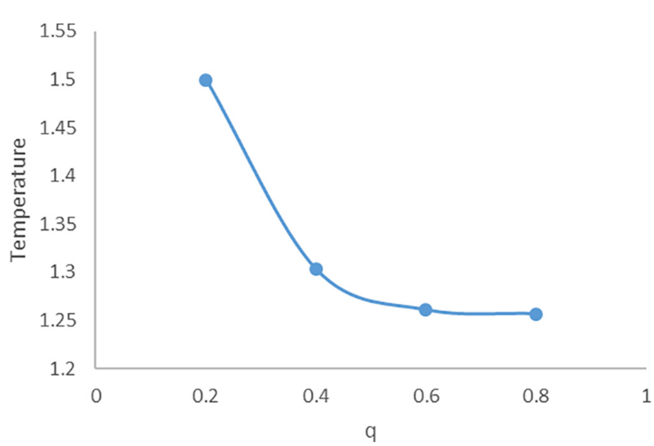

Fig. 6. Temperature versus q parameter with $\lambda_1 = 1.1$, $\lambda_2 = 0.98$, $a = 40$, $\hbar = \mu = 1$, β and $\ell = 0$ for the ground state.

invariance technique [25–27] to solve the differential equation in Eq. (4). The ground state wave function $U_{0\ell}(r)$ is written in the form

$$U_{0\ell} = \exp\left(-\int W(r)dr\right), \tag{5}$$

where $W(r)$ is known as the superpotential function in supersymmetric quantum mechanics. Substituting Eq. (5) into Eq. (4) we have the following equation for the superpotential function $W(r)$

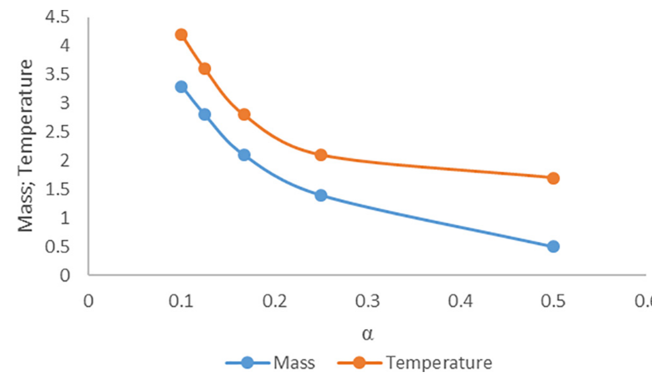

Fig. 7. The variation of the mass energy parameter M against the potential parameter α with $\mu = \hbar = 1$, $\ell = 0$, $\beta = 0.0001$ $a = 40$, $\lambda_1 = 1.1$ and $\lambda_2 = 0.98$ at the ground state level.

$$W^2(r) - \frac{dW(r)}{dr} = \frac{[\ell(\ell+1)\hbar^2\eta_2 e^{-\delta r} + 2\mu\beta a^2]e^{-\delta r}}{a^2\hbar^2(1-e^{-\delta r})^2} + \frac{[\ell(\ell+1)\hbar^2\eta_1 - 2\mu\alpha a^2]e^{-\delta r}}{a^2\hbar^2(1-e^{-\delta r})} - \frac{2\mu E_{n\ell}}{\hbar^2}. \tag{6}$$

Eq. (6) is a basic equation to which the energy equation can be obtained via supersymmetric quantum mechanics methodology. To obtain a desirable result, we proposed a superpotential function which gives a solution to a non-linear Riccati equation of Eq. (6). The proposed superpotential function is written in the following form:

$$W(r) = \rho_0 - \frac{\rho_1 e^{-\delta r}}{1-e^{-\delta r}}, \tag{7}$$

where ρ_0 and ρ_1 are two parametric constants to be determine later. Now, substituting Eq. (7) into Eq. (6), we have the following values for the parametric constants

$$\rho_0^2 = -\frac{2\mu E_{n\ell}}{\hbar^2}, \tag{8}$$

$$\rho_1 = \frac{\delta}{2}\left(1 \pm \sqrt{1 + \frac{4\ell(\ell+1)\eta_2}{a^2\delta^2} + \frac{8\mu\beta}{\hbar^2\delta^2}}\right), \tag{9}$$

$$\rho_0 = \frac{\frac{2\mu\alpha}{\hbar^2} + \frac{\ell(\ell+1)(\eta_2-\eta_1)}{a^2} - \rho_1^2}{2\rho_1}. \tag{10}$$

In this work, we considered the bound-state solution in which the radial wave function must satisfy the boundary condition that $U_{n\ell}(r)/r$ becomes zero when $r \to \infty$ and $U_{n\ell}(r)/r$ is finite at $r = 0$. However, it is when $r \to \infty$, $U_{n\ell}(r)$ is finite and $U_{n\ell}(r) = 0$ at the origin when $r = 0$. The radial wave function $U_{n\ell}(r)/r$ can satisfy the boundary conditions. In view of the proposed superpotential function given in Eq. (7), we can now conveniently construct a pair of supersymmetric partner potentials $V_\mp(r) = W^2(r) \pm \frac{dW(r)}{dr}$ as follows:

$$V_+(r) = W^2(r) + \frac{dW(r)}{dr} = \rho_0^2 - \frac{\rho_1(\rho_1 + 2\rho_0)e^{-\delta r}}{1-e^{-\delta r}} + \frac{\rho_1(\rho_1+\delta)e^{-\delta r}}{(1-e^{-\delta r})^2}, \tag{11}$$

$$V_-(r) = W^2(r) - \frac{dW(r)}{dr} = \rho_0^2 - \frac{\rho_1(\rho_1 + 2\rho_0)e^{-\delta r}}{1-e^{-\delta r}} + \frac{\rho_1(\rho_1-\delta)e^{-\delta r}}{(1-e^{-\delta r})^2}. \tag{12}$$

Using the shape invariance technique and formalism [39–42], it can readily be shown that the two partner potentials are shape invariant which simply means, the potentials are the same apart from a constant. Therefore, a relationship is established between $V_+(r)$ and $V_-(r)$ as follows

$$V_+(r, a_0) = V_-(r, a_1) + R(a_1), \tag{13}$$

where a_1 is a new set of parameters uniquely determined from the old set of parameters a_0 via mapping of the form: $a_1 = f(a_0) = a_0 - \delta$, $a_2 = a_0 - n\delta$, $a_3 = a_0 - 3\delta$ and subsequently $a_n = a_0 - n\delta$ whereas $\rho_1 \to \rho_1 - \delta$. However, the residual term $R(a_1)$ is independent of the variable r. In terms of the parameters of the problem, we write

$$R(a_1) = \left[\frac{\frac{2\mu\alpha}{\hbar^2} + \frac{\ell(\ell 1)(\eta_2-\eta_1)}{a^2} - a_0^2}{2a_0}\right] - \left[\frac{\frac{2\mu\alpha}{\hbar^2} + \frac{\ell(\ell 1)(\eta_2-\eta_1)}{a^2} - a_1^2}{2a_1}\right], \tag{14}$$

$$R(a_2) = \left[\frac{\frac{2\mu\alpha}{\hbar^2} + \frac{\ell(\ell 1)(\eta_2-\eta_1)}{a^2} - a_1^2}{2a_1}\right] - \left[\frac{\frac{2\mu\alpha}{\hbar^2} + \frac{\ell(\ell 1)(\eta_2-\eta_1)}{a^2} - a_2^2}{2a_2}\right], \tag{15}$$

$$R(a_3) = \left[\frac{\frac{2\mu\alpha}{\hbar^2} + \frac{\ell(\ell 1)(\eta_2-\eta_1)}{a^2} - a_2^2}{2a_2}\right] - \left[\frac{\frac{2\mu\alpha}{\hbar^2} + \frac{\ell(\ell 1)(\eta_2-\eta_1)}{a^2} - a_3^2}{2a_3}\right], \tag{16}$$

$$R(a_n) = \left[\frac{\frac{2\mu\alpha}{\hbar^2} + \frac{\ell(\ell 1)(\eta_2-\eta_1)}{a^2} - a_{n-1}^2}{2a_{n-1}}\right] - \left[\frac{\frac{2\mu\alpha}{\hbar^2} + \frac{\ell(\ell 1)(\eta_2-\eta_1)}{a^2} - a_n^2}{2a_n}\right]. \tag{17}$$

Having satisfied all the desirable results, the energy equation can be obtained via

$$E_{n\ell}^{-} = E_{n\ell} + E_{n\ell}^{-} = \sum_{k=1}^{n} R(a_k), \tag{18}$$

where

$$E_{n\ell}^{-} = 0, \tag{19}$$

this gives a complete energy equation for the system as

$$E_{n\ell} = -\frac{\hbar^2}{2\mu a^2}\left[\frac{\frac{2\mu\alpha a^2}{\hbar^2} + \ell(\ell+1)(\eta_2-\eta_1)}{2n+1+\sqrt{1+4\ell(\ell+1)\eta_2+\frac{8\mu\beta a^2}{\hbar^2}}} - \frac{2n+1+\sqrt{1+4\ell(\ell+1)\eta_2+\frac{8\mu\beta a^2}{\hbar^2}}}{4}\right]^2 \tag{20}$$

In other to obtained the corresponding wave function, we defined a variable of the form $y = e^{-\delta r}$ and substitute it into Eq. (4) to have

$$\frac{d^2U_{n\ell}(y)}{dr^2} + \frac{1}{y}\frac{dU_{n\ell}(y)}{dr} + \frac{-Ay^2+By-C}{(y(1-y))^2}U_{n\ell}(y) = 0, \tag{21}$$

where

$$A = \frac{2\mu\alpha a^2}{\hbar^2} - \frac{2\mu E_{n\ell}a^2}{\hbar^2} + \ell(\ell+1)(\eta_2-\eta_1), \tag{22}$$

$$B = \frac{2\mu(\alpha-\beta)a^2}{\hbar^2} + \frac{4\mu E_{n\ell}a^2}{\hbar^2} - \ell(\ell+1)\eta_1, \tag{23}$$

$$C = -\frac{2\mu E_{n\ell}a^2}{\hbar^2}. \tag{24}$$

Analyzing the asymptotic behaviour of Eq. (21) at origin and at infinity, it can be tested that as $r \to 0$ $(y \to 1)$ and as $r \to \infty$ $(y \to 0)$, Eq. (21) has a solution

$$U_{n\ell}(y) = y^{\sqrt{-\frac{2\mu E_{n\ell}a^2}{\hbar^2}}}(1-y)^{\frac{1}{2}\left(1+\sqrt{1+4\ell(\ell+1)\eta_2+\frac{8\mu\beta a^2}{\hbar^2}}\right)}. \tag{25}$$

This gives a complete wave function as

$$U_{n\ell}(r) = N_{n\ell}e^{-\delta r\sqrt{-\frac{2\mu E_{n\ell}a^2}{\hbar^2}}}(1-e^{-\delta r})^{\frac{1}{2}\left(1+\sqrt{1+4\ell(\ell+1)\eta_2+\frac{8\mu\beta a^2}{\hbar^2}}\right)}{}_2F_1\left(-n, n+\sqrt{\frac{-8\mu\alpha E_{n\ell}a^2}{\hbar^2}}+1 + \sqrt{1+4\ell(\ell+1)\eta_2+\frac{8\mu\beta a^2}{\hbar^2}}, \sqrt{\frac{-8\mu\alpha E_{n\ell}a^2}{\hbar^2}}+1; e^{-\delta r}\right). \tag{26}$$

where the normalization constant $N_{n\ell}$ is given as

$$N_{n\ell} = \sqrt{\frac{(2u+1)\Gamma(2u+n)\Gamma(1+n+2u+2v)}{an!\Gamma(2u)^2\Gamma(2v+n+2)}}, \tag{27}$$

$$v = \frac{1}{2} + \frac{1}{2}\sqrt{1+4\ell(\ell+1)\eta_2+\frac{8\mu\beta a^2}{\hbar^2}}, \tag{28}$$

$$u = \sqrt{-\frac{2\mu E_{n\ell}a^2}{\hbar^2}}. \tag{29}$$

Rényi entropy

The Rényi entropy is defined as [43]

$$R(\rho) = \frac{1}{1-q}In4\pi\int_0^\infty \rho^2(r)dr = \frac{1}{\lambda}In4\pi\int_0^\infty \rho^2(r)dr, \tag{30}$$

where $\lambda = 1-q$.

$$R(\rho) = \frac{1}{\lambda}In4\pi\frac{1}{\delta}\int_1^0 \rho^2(y)\frac{1}{y}dy, \quad y = e^{-\delta r}. \tag{31}$$

$$R(\rho) = \frac{1}{\lambda}In\frac{1}{2\delta}\int_{-1}^1 \rho^2(z)\frac{2}{1-z}dz, \quad z = 1-2y. \tag{32}$$

Here, the probability density $\rho(r) = U^2(r)$. Thus, with the value of the probability density, Eq. (32) becomes

$$R(\rho) = \frac{1}{\lambda}In4\pi\frac{1}{2\delta}N_{n\ell}^2\int_{-1}^1\left(\frac{1-z}{2}\right)^{p-1}\left(\frac{1+z}{2}\right)^{\frac{x+1}{2}}[P_n^{(p,x)}(z)]^2dz, \tag{33}$$

where we have defined the following

$${}_2F_1(-n, n+2(p+1), 2p+1; z) = [P_n^{(p,x)}(z)]. \tag{34}$$

$p = 2u$ and $x = 2v-1$. Using integral of the form:

$$\int_{-1}^1\left(\frac{1-z}{2}\right)^{a-1}\left(\frac{1+z}{2}\right)^{b}[P_n^{(a,b)}(z)]^2dz = \frac{2\Gamma(a+n+1)\Gamma(b+n+1)}{n!a\Gamma(a+b+n+1)}, \tag{35}$$

we have the Rényi entropy as

$$R(\rho) = \frac{1}{\lambda}In4\pi\left(\frac{(2u+1)\Gamma(2u+n)\Gamma(2u+2v+n+1)\Gamma(2u+n+1)\Gamma(n+v+1)}{(n!)^2 2u\Gamma(2u)^2\Gamma(2v+n+2)\Gamma(n++2u+v+1)}\right)^q. \tag{36}$$

Schwarzschild black holes

In classical approach, Schwarzschild black holes appears to be thermodynamically unstable in the canonical treatment due to the frequent negativity of heat capacity of the black hole. This was report by Czinner and Iguchi [44] as a conclusion from Hessian analysis. So far, the thermodynamic properties of Schwarzschild black holes has been studied. The studies includes the work of Biró and Czinner [45], Czinner and Iguchi [44]. Here, we use our usual Rényi entropy in quantum computation to compute the mass-energy parameter of the black holes in terms of the work of Czinner and Iguchi. Following the work of Czinner and Iguchi, the Rényi entropy of a black hole can be computed as

$$S_R = \frac{1}{\lambda}In[1+\lambda S_{BH}], \tag{37}$$

and for the Schwarzschild solution, it results to [44]

$$S_R = \frac{1}{\lambda}In(1+4\pi\lambda M^2), \tag{38}$$

where S_{BH} is the Bekenstein-Hawking entropy and M is the mass-energy parameter of the black hole. As earlier pointed out, our aim is to use the usual Rényi entropy in quantum computation to study the effect of some potential parameters on the mass-energy parameter of the black hole. In other to achieve this, we relate Eq. (36) to Eq. (38). Thus, we have

$$M = \frac{1}{2\pi\lambda}\sqrt{4\pi\lambda\left(\frac{(2u+1)\Gamma(2u+n)\Gamma(2u+2v+n+1)\Gamma(2u+n+1)\Gamma(n+v+1)}{(n!)^2 2u\Gamma(2u)^2\Gamma(2v+n+2)\Gamma(2u+n+v+1)}\right)^q - \pi\lambda} \tag{39}$$

The temperature

$$T_R = \frac{1}{8\pi M} + \frac{\lambda M}{2} = \frac{1+4\pi\lambda M^2}{8\pi M}, \tag{40}$$

comparing with our Rényi entropy, we have the temperature of the black hole as

$$T_R = \frac{\left(\frac{(2u+1)\Gamma(2u+n)\Gamma(2u+2v+n+1)\Gamma(2u+n+1)\Gamma(v+n+1)}{(n!)^2 2u\Gamma(2u)^2\Gamma(2v+n+2)\Gamma(2u+v+n+1)}\right)^q}{\sqrt{\frac{1}{\lambda}}\sqrt{4\pi\left(\frac{(2u+1)\Gamma(2u+n)\Gamma(2v+2v+n+1)\Gamma(2u+n+1)\Gamma(n+v+1)}{(n!)^2 2u\Gamma(2u)^2\Gamma(2v+n+1)\Gamma(2u+v+n+1)}\right)^q - \pi}}. \tag{41}$$

The heat capacity C_R of the black hole is given as

$$C_R = \frac{8\pi M^2}{4\pi\lambda M^2 - 1}. \tag{42}$$

But

$$4\pi\lambda M^2 - 1 = 4\pi\left(\frac{(2u+1)\Gamma(2u+n)\Gamma(2u+2v+n+1)\Gamma(2u+n+1)\Gamma(v+n+1)}{(n!)^2 2u\Gamma(2u)^2\Gamma(2u+n+2)\Gamma(2u+v+n+1)}\right)^q - 2. \tag{43}$$

Thus,

$$C_R = \frac{\frac{4\left(\frac{(2u+1)\Gamma(2u+n)\Gamma(2u+2v+n+1)\Gamma(2u+n+1)\Gamma(n+V+1)}{(n!)^2 2u\Gamma(2u)^2\Gamma(2v+n+1)\Gamma(2u+v+n+1)}\right)^q - 1}{\lambda}}{2\pi\left(\frac{(2u+1)\Gamma(2u+n)\Gamma(2u+2v+n+1)\Gamma(2u+n+1)\Gamma(v+n+1)}{(n!)^2 2u\Gamma(2u)^2\Gamma(2u+n+2)\Gamma(2u+v+n+1)}\right)^q - 1}. \tag{44}$$

Discussion

To test the accuracy of our energy (20), we numerically calculated the energy eigenvalues for various values of n, ℓ and the screening parameter $1/a$. Our results are compared with results obtained from two other methods. It is observed that the present results are in excellent agreement with the previous results. It is also observed that for small and large values of a, there are small differences between our result and the results of Ref. [45]. However, the energy decreases as the parameter a decreases. In Fig. 1, we plotted the approximation scheme used for this studies with three values of the potential range. In Figs. 2 and 3, we examined the variation of Rényi entropy against the potential parameters β and α respectively. In each case, the Rényi entropy decreases as each of β and α increases. In Fig. 4, we plotted mass energy parameter and temperature against β. It is observed that as the potential parameter increases, both the mass energy and temperature decreases respectively. At every value of the potential parameter, the temperature is usually higher than the mass energy. But as the potential parameter increases, the values of temperature and mass energy becomes closer and both of them follow the same trend. However, as β becomes larger, the temperature becomes stable. In Fig. 5, we plotted temperature against mass. The temperature decreases as the mass increases. In Fig. 6, we plotted temperature against q. It is observed that as q increases, the temperature decreases. The temperature tends to be stable as the parameter q increases from 0.6. In Fig. 7, we plotted mass energy and temperature respectively against the potential parameter α, both the temperature and mass energy decreases respectively as α increases. The temperature tends to be higher than the mass energy as α increases. Similarly, the temperature gets to its stability as α increases from 0.5.

Conclusion

In this paper, we have studied the solutions of Schrödinger equation with Eckart potential by employing a suitable approximation scheme in the framework of supersymmetry shape invariance technique. The effect of the two Eckart potential parameters on the mass energy parameter and temperature of the black hole was investigated. It is observed that the Black hole temperature becomes stable for an increase in each of the two potential parameters. Our results are in good agreement with the results previously obtained.

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