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Fractal calculus and its geometrical explanation

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ABSTRACT

Fractal calculus is very simple but extremely effective to deal with phenomena in hierarchical or porous media. Its operation is almost same with that by the advanced calculus, making it much accessible to all non-mathematicians. This paper begins with the basic concept of fractal gradient of temperature, i.e., the temperature change between two points in a fractal medium, to reveal the basic properties of fractal calculus. The fractal velocity and fractal material derivative are then introduced to deduce laws for fluid mechanics and heat conduction in fractal space. Conservation of mass in a fractal space is geometrically explained, and an approximate transform of a fractal space on a smaller scale into its continuous partner on a larger scale is illustrated by a nanofiber membrane, which is smooth on any observable scales, but its air permeability has to studied in a nano scale, under such a small scale, the nanofiber membrane becomes a porous one. Finally an example is given to explain cocoon's heat-proof property, which cannot be unveiled by advanced calculus.

Introduction

Fractal geometry, fractal calculus and fractional calculus have been becoming hot topics in both mathematics and engineering for non-differential solutions. Fractal theory is the theoretical basis for the fractal spacetime [1,2], El Naschie's E-infinity theory [3], and life science [4] as well. Fractional calculus was introduced in Newton's time, and it has become a very hot topic in various fields, especially in mathematics and engineering for porous media [5–13], where classic mechanics becomes invalid to describe any phenomena on the porous size scale. For example, molecule diffusion in water is similar to a stochastic Brownian motion in view of continuum mechanics, but the diffusion follows fractal Fick laws if we observe the motion on a molecule scale. However, the fractional calculus is now such a mess that an engineer has no ability to select a suitable fractional derivative for his practical applications, most publications on fractional calculus are of pure mathematics though some authors claimed possible applications, and there are too many definitions on fractional derivative and new ones arise everyday [14–18]. Among all fractional derives, He's fractional derivative [19–21] and the local fractional derivative [22,23] are of mathematical correctness, physical foundation, and practical relevance. In 2012 the geometrical explanation of fractional calculus was given [24], and in 2014 a tutorial review was published on fractional calculus from its very beginning and physical understanding to practical applications [1].

Many researchers have already found the intrinsic relationship between the fractional dimensions and the fractional order [25]. This paper will focus itself on the fractal calculus, a relatively new branch of mathematics with easy understanding and ready applications.

Fractal calculus

The fractal calculus is relatively new, it can effectively deal with kinetics, which is always called as the fractal kinetics [26–28], where the fractal time replaces the continuous time. Nottale revealed that time does be discontinuous in microphysics [29], that means that fractal kinetics takes place on very small time scale.

The fractal derivative (Hausdorff derivative) on time fractal is defined as [30–36]

$$\frac{\partial T}{\partial t^{\sigma}} = \lim_{t_B \to x_A} \frac{T(t_B)-T(t_A)}{(t_B)^{\sigma}-(t_A)^{\sigma}} \tag{1}$$

where σ is the fractal dimensions of time.

A more general definition is given as follows [30–36]

$$\frac{\partial^{\tau} T}{\partial t^{\sigma}} = \lim_{t_B \to x_A} \frac{T^{\tau}(t_B)-T^{\tau}(t_A)}{(t_B)^{\sigma}-(t_A)^{\sigma}} \tag{2}$$

where τ is the fractal dimensions of space.

There are other definitions for fractal derivative, and we will not discuss all definitions, because some definitions are of only mathematical interest.

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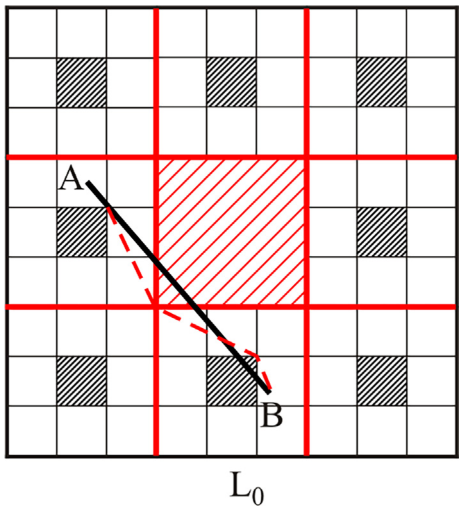


Fig. 1. Fractal gradient. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Fractal gradient

To elucidate the basic ideas of the fractal calculus, we begin with the concept of gradient, which is widely used in mathematics and engineering. For the one-dimensional case, the gradient of temperature between two points A and B can be defined as

$$\frac{\Delta T}{\Delta x} = \frac{T_B - T_A}{x_B - x_A} \tag{3}$$

where T represents temperature or other variables. The gradient can be understood as the slope between two points:

$$\nabla T = \lim_{x_B \to x_A} \frac{T_B - T_A}{x_B - x_A} \tag{4}$$

For three-dimensional case, the gradient is defined as

$$\nabla T = \frac{\partial T}{\partial x}\mathbf{i} + \frac{\partial T}{\partial y}\mathbf{j} + \frac{\partial T}{\partial z}\mathbf{k} \tag{5}$$

The gradient is defined on a smooth space, and it becomes invalid for discontinuous space, and a new definition on a fractal space is much needed for practical applications.

In a fractal space as illustrated in Fig. 1, the gradient between points A and B cannot be described using the above definition.

We define average gradient, initial gradient, and terminal gradient, respectively, as follows

$$\overline{\nabla} T = \frac{\Delta T}{\Delta x} = \frac{T_B - T_A}{x_B - x_A} \tag{6}$$

$$\nabla_0 T = \lim_{x \to x_A} \frac{T - T_A}{x - x_A} \tag{7}$$

$$\nabla_\infty T = \lim_{x \to x_B} \frac{T_B - T}{x_B - x} \tag{8}$$

For a continuous space, we have

$$\overline{\nabla} T = \nabla_0 T = \nabla_\infty T \tag{9}$$

In a fractal space, however, the above equation becomes invalid, and we define a fractal gradient as follows

$$\nabla^\alpha T = \frac{T_B - T_A}{L_{AB}} \tag{10}$$

where L_{AB} is the length of the broken line in Fig. 1. According to fractal geometry, we have

$$L_{AB} = kL^\alpha \tag{11}$$

where L is distance between A and B, α is the fractal dimension value. In practical applications, hierarchical structure and porous medium can be approximately considered as a fractal space [1,7–10,24], that means there is a lowest hierarchy or minimal porous size. If the lowest hierarchical distance is L_0 (the side length of the shaded square in Fig. 1), beyond which no physical meaning exists. For example, L_0 is the nanoporous size of a nanofiber member [37–39], or the minimal porous size of a cocoon [40]. Using L_0, Eq. (9) can be updated as

$$L_{AB} = k_0 (L_0)^\alpha \tag{12}$$

where k_0 is a constant.

When L_{AB} tends to extremely small but larger than L_0, we have [1,41]

$$L_{AB} = \frac{(L_0)^\alpha}{\Gamma(1+\alpha)} \tag{13}$$

We can define the fractal gradient in form [1,41]

$$\nabla^\alpha T = \Gamma(1+\alpha) \lim_{x_B - x_A \to L_0} \frac{T_B - T_A}{(x_B - x_A)^\alpha} \tag{14}$$

For the three dimensional case, the fractal gradient can be written in the form

$$\nabla^\alpha T = \frac{\partial T}{\partial x^\alpha}\mathbf{i} + \frac{\partial T}{\partial y^\alpha}\mathbf{j} + \frac{\partial T}{\partial z^\alpha}\mathbf{k} \tag{15}$$

where $\frac{\partial}{\partial x^\alpha}$ is the partial fractal derivative defined as [1,41]

$$\frac{\partial T}{\partial x^\alpha} = \Gamma(1+\alpha) \lim_{\Delta x = x_B - x_A \to L_0} \frac{T(x_B) - T(x_A)}{(x_B - x_A)^\alpha} \tag{16}$$

where α is the fractal dimensions in x-direction, L_0 is the lowest hierarchical distance.

The fractal derivative given in Eq. (16) has widely been used to deal with porous or hierarchical structures [42–45] with great success.

A fractal space is always not isotropic, that means the fractal dimensions in x-, y- and z-directions are different. We replace Eq. (16) by the following one

$$\nabla^{(\alpha,\beta,\gamma)} T = \frac{\partial T}{\partial x^\alpha}\mathbf{i} + \frac{\partial T}{\partial y^\beta}\mathbf{j} + \frac{\partial T}{\partial z^\gamma}\mathbf{k} \tag{17}$$

where α, β, and γ are, respectively, the fractal dimensions in x-, y- and z-directions,

$$\frac{\partial}{\partial x^\alpha} = \Gamma(1+\alpha) \lim_{x_B - x_A \to L_{0x}} \frac{T_B - T_A}{(x_B - x_A)^\alpha} \tag{18}$$

$$\frac{\partial}{\partial y^\beta} = \Gamma(1+\beta) \lim_{y_B - y_A \to L_{0y}} \frac{T_B - T_A}{(y_B - y_A)^\beta} \tag{19}$$

$$\frac{\partial}{\partial z^\gamma} = \Gamma(1+\gamma) \lim_{z_B - z_A \to L_{0z}} \frac{T_B - T_A}{(z_B - z_A)^\gamma} \tag{20}$$

where L_{0x}, L_{0y}, and L_{0z} are the minimal porous sized in x-, y- and z-directions, respectively.

One-dimensional heat equation with a source in a fractal medium can be written in the form

$$\frac{\partial T}{\partial t} + \frac{\partial}{\partial x^\alpha}\left(k \frac{\partial T}{\partial x^\alpha}\right) = Q_0 \tag{21}$$

where k is the material's conductivity, Q_0 is the heat source.

Three-dimensional heat equation with a source in a fractal medium reads

$$\frac{\partial T}{\partial t} + \frac{\partial}{\partial x^\alpha}\left(k_x \frac{\partial T}{\partial x^\alpha}\right) + \frac{\partial}{\partial y^\beta}\left(k_y \frac{\partial T}{\partial y^\beta}\right) + \frac{\partial}{\partial z^\gamma}\left(k_z \frac{\partial T}{\partial z^\gamma}\right) = Q_0 \tag{22}$$

where k_x, k_y, and k_z are, respectively, the material's conductivity in x-, y- and z-directions.

In order to establish laws in fractal media, it is necessary to introduce the concept of fractal velocity, which is defined as follows

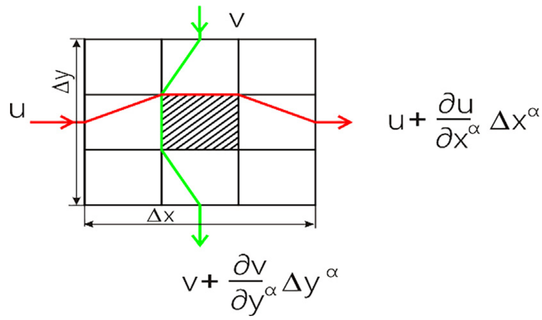


Fig. 2. A control volume in a fractal medium.

$$u=\lim_{L_{AB}\to L_0}\frac{L_{AB}}{t_{AB}}=\frac{dx^{\alpha}}{\Gamma(1+\alpha)dt} \tag{23}$$

Eq. (23) can be understood as an average velocity of a particle moving from A to B in a fractal space (The red discontinuous line in Fig. 1).

Now consider a control volume in a fractal space as illustrated in Fig. 2 for 2-dimensional steady incompressible, and assume the fractal gradients of the velocities at x- and y-directions are, respectively, $\partial u/\partial x^{\alpha}$ and $\partial v/\partial y^{\alpha}$. The conservation of mass requires

$$\left(u+\frac{\partial u}{\partial x^{\alpha}}\Delta^{\alpha}x\right)\Delta^{\alpha}y-u\Delta^{\alpha}x\Delta^{\alpha}y+\left(v+\frac{\partial v}{\partial y^{\alpha}}\Delta^{\alpha}y\right)\Delta^{\alpha}x-v\Delta^{\alpha}y\Delta^{\alpha}x=0 \tag{24}$$

This results in the following mass equation in a fractal medium:

$$\frac{\partial u}{\partial x^{\alpha}}+\frac{\partial v}{\partial y^{\alpha}}=0 \tag{25}$$

For a general case, the mass equation can be written in the form

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho u)}{\partial x^{\alpha}}+\frac{\partial(\rho v)}{\partial y^{\beta}}+\frac{\partial(\rho w)}{\partial z^{\gamma}}=0 \tag{26}$$

where ρ is the density of the fluid, α, β, and γ are, respectively, the fractal dimensions in x-, y- and z-directions, u, v, and w are, respectively, the fractal velocities in x-, y- and z-directions.

We define a fractal streamline in a fractal medium:

$$\frac{dx^{\alpha}}{\Gamma(1+\alpha)u}=\frac{dy^{\beta}}{\Gamma(1+\beta)v}=\frac{dz^{\gamma}}{\Gamma(1+\gamma)w} \tag{27}$$

where *u*, *v*, and *w* are, respectively, the fractal velocities in x-, y- and z-directions,

$$\frac{\partial x^{\alpha}}{\partial t}=\Gamma(1+\alpha)u \tag{28}$$

$$\frac{\partial y^{\beta}}{\partial t}=\Gamma(1+\beta)v \tag{29}$$

$$\frac{\partial z^{\gamma}}{\partial t}=\Gamma(1+\gamma)w \tag{30}$$

We introduce a new space (X, Y, Z) defined as

$$X=\frac{x^{\alpha}}{\Gamma(1+\alpha)} \tag{31}$$

$$Y=\frac{y^{\beta}}{\Gamma(1+\beta)} \tag{32}$$

$$Z=\frac{z^{\gamma}}{\Gamma(1+\gamma)} \tag{33}$$

In the new space, we have

$$\frac{dX}{u}=\frac{dY}{v}=\frac{dZ}{w} \tag{34}$$

That means that the space (X,Y, Z) can be approximately considered as a smooth one, making the solution process much simple. To elucidate this, we consider a nanofiber membrane by bubble electrospinning [37–39], it is smooth enough at any observable scales (see Fig. 3), however, if we want to study the effect of the diameter of nanofibers on the air permeability, we have to use a nano scale, under such case, the nanofiber membrane becomes discontinuous, and a fractal calculus can be effectively used [38].

The material derivative in fractal space can be written in the form

$$\frac{D}{Dt}=\frac{\partial}{\partial t}+\frac{\partial}{\partial x^{\alpha}}\frac{\partial x^{\alpha}}{\partial t}+\frac{\partial}{\partial y^{\beta}}\frac{\partial y^{\beta}}{\partial t}+\frac{\partial}{\partial z^{\gamma}}\frac{\partial z^{\gamma}}{\partial t}=\frac{\partial}{\partial t}+\Gamma(1+\alpha)u\frac{\partial}{\partial x^{\alpha}}+\Gamma(1+\beta)v\frac{\partial}{\partial y^{\beta}}+\Gamma(1+\gamma)w\frac{\partial}{\partial z^{\gamma}} \tag{35}$$

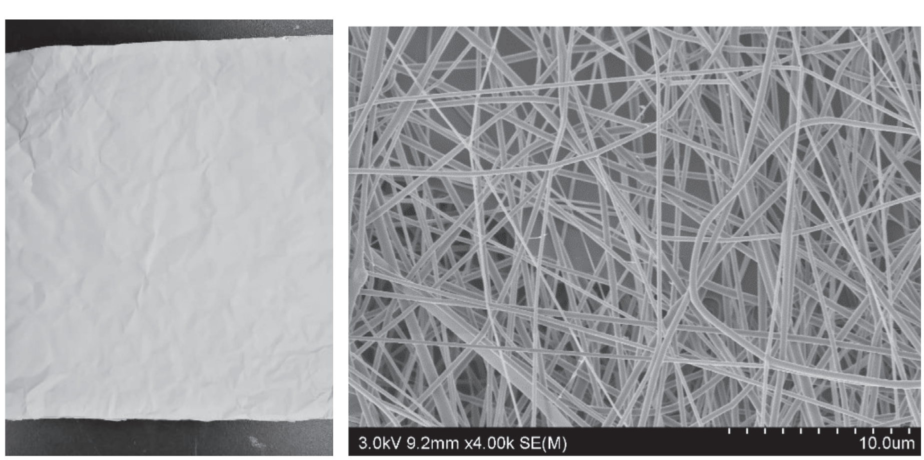
Using fractal material derivative, we can obtain various conservation laws in fluid mechanics and thermal science, for examples the fractal Navier-Stokes equations are

$$\rho\frac{Du}{Dt}=\rho f_x+\frac{\partial\sigma_{xx}}{\partial x^{\alpha}}+\frac{\partial\sigma_{xy}}{\partial y^{\beta}}+\frac{\partial\sigma_{xz}}{\partial z^{\gamma}} \tag{36}$$

$$\rho\frac{Dv}{Dt}=\rho f_y+\frac{\partial\sigma_{yx}}{\partial x^{\alpha}}+\frac{\partial\sigma_{yy}}{\partial y^{\beta}}+\frac{\partial\sigma_{yz}}{\partial z^{\gamma}} \tag{37}$$

$$\rho\frac{Dw}{Dt}=\rho f_z+\frac{\partial\sigma_{zx}}{\partial x^{\alpha}}+\frac{\partial\sigma_{zy}}{\partial y^{\beta}}+\frac{\partial\sigma_{zz}}{\partial z^{\gamma}} \tag{38}$$

where D/Dt is defined in Eq. (35), f_i (i = x,y,z) are body forces,



(a) (b)

Fig. 3. A nanofiber membrane obtained by bubble electrospinning, continuous or discontinuous? (a) a photo taken by a camera, (b) SEM illustration.

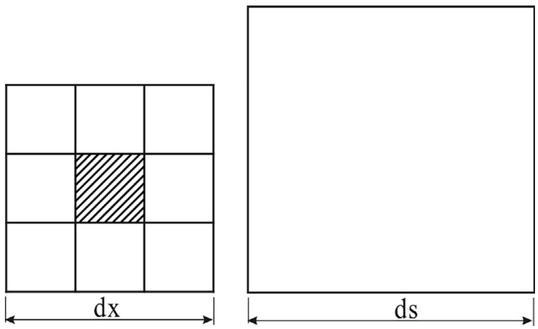


Fig. 4. Fractal space vs continuous space.

$\sigma_{ij}(i = x, y, z; j = x, y, z)$ are viscous forces.

Diffusion or heat conduction in fractal media in a moving fluid can be written in the form

$$\frac{DC}{Dt} = \frac{\partial C}{\partial t} + \Gamma(1+\alpha)u\frac{\partial C}{\partial x^{\alpha}} + \Gamma(1+\beta)v\frac{\partial C}{\partial y^{\beta}} + \Gamma(1+\gamma)w\frac{\partial C}{\partial z^{\gamma}} = \frac{\partial}{\partial x^{\alpha}}\left(k_x\frac{\partial C}{\partial x^{\alpha}}\right) + \frac{\partial}{\partial y^{\beta}}\left(k_y\frac{\partial C}{\partial y^{\beta}}\right) + \frac{\partial}{\partial z^{\gamma}}\left(k_z\frac{\partial C}{\partial z^{\gamma}}\right) \tag{39}$$

where C can be either concentration or temperature.

Equation for diffusion at rest is

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x^{\alpha}}\left(k_x\frac{\partial C}{\partial x^{\alpha}}\right) + \frac{\partial}{\partial y^{\beta}}\left(k_y\frac{\partial C}{\partial y^{\beta}}\right) + \frac{\partial}{\partial z^{\gamma}}\left(k_z\frac{\partial C}{\partial z^{\gamma}}\right) \tag{40}$$

An example

In classic mechanics, we always assume the space is continuous, the air flow is continuous, the water flow is continuous, the continuum hypothesis works well for many practical applications. However, if we want to study, for example, molecule diffusion in water, the water becomes discontinuous, and the fractal calculus has to be adopted to describe the motion of molecules, otherwise molecule motion becomes completely unpredictable in the frame of the continuum hypothesis. In this section we give an example of cocoon's heat conduction by fractal calculus.

One-dimensional steady heat conduction for cocoon [46,47] can be written in the form

$$\frac{\partial}{\partial x^{\alpha}}\left(k\frac{\partial T}{\partial x^{\alpha}}\right) = Q_0 \tag{41}$$

with initial conditions

$$T(0) = T_0 \tag{42}$$

$$\frac{dT}{dx^{\alpha}}(0) = 0 \tag{43}$$

Introducing a transform:

$$s = x^{\alpha} \tag{44}$$

we can convert Eq. (41) into the following one

$$\frac{d}{ds}\left(k\frac{dT}{ds}\right) = Q_0 \tag{45}$$

We give a geometrical explanation of Eq. (44), on scale of x, the cocoon is a porous one, but on a larger scale of s defined in Eq. (44), the cocoon becomes approximately continuous, that means on scale of s, the porous structure can not be observed, see Fig. 4.

Eq. (44) is an approximate transform of a fractal space to a continuous one, it is similar to the fractional complex transform suggested by He and Li [48].

In view of the initial conditions, we obtain the following solution

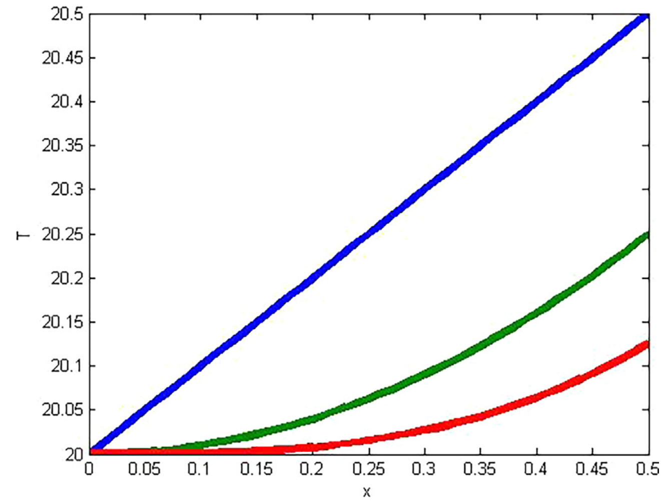


Fig. 5. The solution of Eq. (46) with different value of α: $\alpha = 0.5$ (straight line); $\alpha = 1$ (middle curve), and $\alpha = 1.5$ (the red curve). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

$$T(x) = \frac{Q_0}{2k}s^2 + T_0 = \frac{Q_0}{2k}x^{2\alpha} + T_0 \tag{46}$$

It is easy to find that for $\alpha > 1.5$ we have

$$\frac{dT(0)}{dx} = 0 \tag{47}$$

$$\frac{d^2T(0)}{dx^2} = 0 \tag{48}$$

$$\frac{d^3T(0)}{dx^3} = 0 \tag{49}$$

Eqs. (47)–(49) reveal that the temperature change on the cocoon's inner surface is extremely slow regardless of environmental temperature. Fig. 5 reveals the basic solution properties for different values of the fractal dimensions. When $\alpha > 1$, the inner temperature inside the cocoon changes extremely slowly regardless of temperature change outside of the cocoon.

Discussion and conclusions

Due to various definitions of fractional derivative, most fractional models have only mathematical interest, and its fractal partner has physical insight and practical applications. It is extremely easy for non-mathematicians to deal with practical problems where the continuum models fail. All phenomena in porous media or hierarchical structures can be effectively modelled using the fractal calculus, and we can unveil hidden mechanisms which can never be found by the continuum mechanics. For example, if the cocoon wall is assumed as a continuous medium, we can not obtain Eqs. (48) and (49).

This paper gives a tutorial introduction to the fractal calculus from very beginning, and it is accessible to all audience. For fractional calculus, the audience is recommended to read Refs. [49–53].

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