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Exact traveling wave solutions of fractional order Boussinesq-like equations by applying Exp-function method

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ABSTRACT

We have computed new exact traveling wave solutions, including complex solutions of fractional order Boussinesq-Like equations, occurring in physical sciences and engineering, by applying Exp-function method. The method is blended with fractional complex transformation and modified Riemann-Liouville fractional order operator. Our obtained solutions are verified by substituting back into their corresponding equations. To the best of our knowledge, no other technique has been reported to cope with the said fractional order nonlinear problems combined with variety of exact solutions. Graphically, fractional order solution curves are shown to be strongly related to each other and most importantly, tend to fixate on their integer order solution curve. Our solutions comprise high frequencies and very small amplitude of the wave responses.

Introduction

In recent years, nonlinear evolution equations (NLEEs) of fractional order have acquired a significant place in applied mathematics and engineering. These equations constitute remarkable applications in biomechanics, nonlinear optics, plasma physics, fluid dynamics, solid state physics and many other areas of physical sciences. Numerous methods have been proposed to look for exact solutions of NLEEs such as homogeneous balance method [1,2], Jacobi elliptic function method [3], Backlund transformations [4], functional variable method [5,6], tanh-function method [7–9], truncated painleve expansion method [10], (G/G)-expansion method [11,12], sine-cosine method [13], Hirota bilinear transformation method [14], F-expansion method [15,16], simple equation method [17] etc.

The Boussinesq-Like equations [18–21] are nonlinear evolution equations of the form

$$u_{tt} - u_{xx} - (6u^2 u_x + u_{xxx})_x = 0, \tag{1}$$

$$u_{tt} - u_{xx} - (6u^2 u_x + u_{xtt})_x = 0, \tag{2}$$

$$u_{tt} - u_{xt} - (6u^2 u_x + u_{xxt})_x = 0, \tag{3}$$

and

$$u_{tt} - (6u^2 u_x + u_{xxx})_x = 0, \tag{4}$$

where, u_{tt} and u_{xx} are second dissipative terms, u_{xxxx} is known as fourth spatial derivative and u_{xxxt} and u_{xxtt} being mixed spatio-temporal derivative of the same order. The Eqs. (1)–(4) are linked to the good Boussinesq equation but then less conformable to the analytical techniques as these equations can no longer be fully integrable [19]. The spatio-temporal terms in (1)–(3) improve the properties of dispersion relation likewise the regularized Boussinesq equation. Above NLEEs appear in shallow water long waves, propagation of waves in elastic rods, coupled electrical circuits, vibration in nonlinear string, nonlinear lattice waves, dynamics of thin inviscid layers with free surface, and in the shape- memory alloys [13–16]. Solutions of Boussinesq-like equations are used for applying nonlinear water model to coastal and ocean engineering.

In this work, new exact traveling wave solutions of Boussinesq-Like equations (1)-(4) with their fractional order interpretations (related to the models discussed in [22,23] and several other papers):

$$D_{tt}^{2\alpha} u - D_{xx}^{2\beta} u - D_x^{\beta}(6u^2 D_x^{\beta} u + D_{xxx}^{3\beta} u) = 0, \tag{5}$$

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$$D_{tt}^{2\alpha}u - D_{xx}^{2\beta}u - D_x^{\beta}(6u^2D_x^{\beta}u + D_{xtt}^{2\alpha+\beta}u) = 0, \quad 0 < \alpha, \beta \leqslant 1 \tag{6}$$

$$D_{tt}^{2\alpha}u - D_{xt}^{\alpha+\beta}u - D_x^{\beta}(6u^2D_x^{\beta}u + D_{xxt}^{\alpha+2\beta}u) = 0, \tag{7}$$

and

$$D_{tt}^{2\alpha}u - D_x^{\beta}(6u^2D_x^{\beta}u + D_{xxx}^{3\beta}u) = 0, \tag{8}$$

are obtained by applying Exp-function method [24,25] with the help of symbolic computation and is also applicable to differential equations of fractional order in a straightforward and simple way [26]. This method incorporates fractional complex transformation [27–29] and modified Riemann-Liouville operator [30,31] for seeking exact solutions of Boussinesq-Like physical models of fractional order. Among some other key techniques as mentioned earlier, the Exp-function method also gives exact solutions, and in most of the cases, a variety of solutions for a wide range of nonlinear problems of fractional as well as integer order. This makes the Exp-function method advantageous over the numerical, approximate analytical and semi numerical techniques. Further, some other studies related to solution methods can be seen in [36–59].

Theorem 1 [32]. *Suppose that $u^{(r)}$ and $u^{(\gamma)}$ are respectively the highest order linear term and the highest order nonlinear term of a nonlinear ODE, where r and γ are both positive integers. Then the balancing procedure using the Exp-function ansatz*

$$\boldsymbol{u}(\xi) = \frac{\sum_{n=-c}^{d} a_n \exp(n\xi)}{\sum_{m=-p}^{q} b_m \exp(m\xi)}, \tag{9}$$

leads to $c = d$ and $p = q, \forall r, s, \Omega, \lambda \geqslant 1$.

Theorem 2 [32]. *Suppose that $u^{(r)}$ and $u^{(s)}u^k$ are respectively the highest order linear term and the highest order nonlinear term of a nonlinear ODE, where r, s and Ω are all positive integers. Then the balancing procedure using the Exp-function ansatz leads to $c = d$ and $p = q, \forall r, s, k \geqslant 1$.*

Theorem 3 [32]. *Suppose that $u^{(r)}$ and $(u^{(s)})^{\Omega}$ are respectively the highest order linear term and the highest order nonlinear term of a nonlinear ODE, where r, s and Ω are all positive integers. Then the balancing procedure using the Exp-function ansatz leads to $c = d$ and $p = q \forall r, s \geqslant 1, \forall \Omega \geqslant 2$.*

Theorem 4 [32]. *Suppose that $u^{(r)}$ and $(u^{(s)})^{\Omega}u^{(\lambda)}$ are respectively the highest order linear term and the highest order nonlinear term of a nonlinear ODE, where r, s, Ω and λ are all positive integers. Then the balancing procedure using the Exp-function ansatz leads to $c = d$* and $p = q, \forall r, s, \Omega, \lambda \geqslant 1$.

Modified Riemann-Liouville fractional derivative

To cope with the functions that are nondifferentiable, Jumarie presented new formulations for taylor series of fractional sense and introduced modified Riemann-Liouville definition [30]. He made comparison with the Caputo-Djrbashian derivative [33,34], that defines a fractional operator of order less than 1 through a derivative which does not always work. Due to this reason, he presented modified Riemann-liouville derivative [35]. Since then, modified Riemann-Liouville derivative has successfully been applied to a good deal of fractional order problems including our present work.

Jumarie's modified Riemann-Liouville operator [30,31] is defined as

$$D_t^{\alpha}f(t) = \begin{cases} \frac{1}{\Gamma(-\alpha)}\int_0^t (t-\tau)^{-\alpha-1}(f(\tau)-f(0))d\tau, & \alpha<0, \\ \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-\tau)^{-\alpha}(f(\tau)-f(0))d\tau, & 0<\alpha<1, \\ [f^{(n)}(t)]^{\alpha-n}, & n\leqslant\alpha<n+1, n\geqslant 1. \end{cases} \tag{10}$$

Important properties of this particular operator are as follows

$$D_t^{\alpha}t^{\gamma} = \frac{\Gamma(1+\gamma)}{(1+\gamma-\alpha)}t^{\gamma-\alpha}, \quad \gamma > 0, \tag{11}$$

$$D_t^{\alpha}(cf(t)) = cD_t^{\alpha}f(t), \tag{12}$$

$$D_t^{\alpha}(af(t) + bgf(t)) = aD_t^{\alpha}f(t) + bD_t^{\alpha}g(t), \tag{13}$$

where, a, b and c are constants.

The Exp-function method

Consider the general nonlinear fractional partial differential equation containing higher order derivatives as well as nonlinear terms

$$P\left(u, \frac{\partial^{\alpha}u}{\partial t^{\alpha}}, \frac{\partial^{\beta}u}{\partial x^{\beta}}, \frac{\partial^{\gamma}u}{\partial y^{\gamma}}, \ldots\right) = 0, \tag{14}$$

where u is a function to be known and P is a polynomial of u and its partial fractional order differential operators. By applying fractional complex transformation [27–29]

$$u = \boldsymbol{u}(\xi), \quad \text{where } \xi = k\frac{x^{\beta}}{\Gamma(\beta+1)} + m\frac{y^{\gamma}}{\Gamma(\gamma+1)} - l\frac{t^{\alpha}}{\Gamma(\alpha+1)}, \tag{15}$$

we convert (4) into the following nonlinear ODE

$$Q(\boldsymbol{u}, \boldsymbol{u}', \boldsymbol{u}'', \boldsymbol{u}''', \ldots) = 0 \tag{16}$$

In Exp-function method, we assume that the traveling wave solutions can be expressed in the form of (9) which can further be written as

$$\boldsymbol{u}(\xi) = \frac{a_c\exp(c\xi) + \cdots + a_{-d}\exp(-d\xi)}{b_p\exp(p\xi) + \cdots + b_{-q}\exp(-q\xi)}. \tag{17}$$

(17) holds key role for finding analytic solution of given nonlinear problems. To attain the values of $\boldsymbol{c}$ and $\boldsymbol{p}$, we balance the linear term of highest order in (16) with the highest order nonlinear term. In the same way, to obtain the values of $\boldsymbol{d}$ and $\boldsymbol{q}$, we balance the linear term of lowest order in (16) with lowest order nonlinear term.

Applications of Exp-function method

Fractional order Boussinesq-like equations

(I) By using complex fractional transformation [27–29]

$$\boldsymbol{u} = \boldsymbol{u}(\xi), \quad \xi = k\frac{x^{\beta}}{\Gamma(\beta+1)} - l\frac{t^{\alpha}}{\Gamma(\alpha+1)}, \tag{18}$$

and modified Riemann-Liouville derivative, Boussinesq-like equation of the form (5) is converted into the ODE

$$l^2\boldsymbol{u}'' - (\boldsymbol{u}'' + 12\boldsymbol{u}\boldsymbol{u}'^2 + 6\boldsymbol{u}^2\boldsymbol{u}'')k^2 - \boldsymbol{u}^{(4)}k^4 = 0, \tag{19}$$

that is integrated twice (neglecting constant of integration) to obtain

$$l^2\boldsymbol{u} - k^2(\boldsymbol{u} + 2\boldsymbol{u}^3) - k^4\boldsymbol{u}'' = 0. \tag{20}$$

The values of c and d can be freely assigned because the resulted solution does not strongly depend on the choices of c and d [25]. So, we opt $p = c = 1$ and $q = d = 1$ so that (17) becomes

$$\boldsymbol{u}(\xi) = \frac{a_{-1}e^{-\xi} + a_0 + a_1 e^{\xi}}{b_{-1}e^{-\xi} + b_0 + b_1 e^{\xi}}, \tag{21}$$

substituting (21) into (20) gives

$$\frac{1}{A}[C_3\exp(3\xi) + C_2\exp(2\xi) + C_1\exp(\xi) + C_0 + C_{-1}\exp(-\xi) + C_{-2}\exp(-2\xi) + C_{-3}\exp(-3\xi)] = 0, \tag{22}$$

where,

$$\begin{aligned}
&A = (b_{-1}e^{-\xi} + b_0 + b_1 e^{\xi})^3,\\
&C_3 = -a_1k^2b_1^2 + a_1l^2b_1^2 - 2a_1^3k^2 = 0, C_2 = -a_0k^2b_1^2 - k^4a_0b_1^2\\
&\quad + a_0l^2b_1^2 - 6a_0k^2a_1^2 + 2a_1l^2b_0b_1 - 2a_1k^2b_0b_1 + k^4a_1b_0b_1 = 0,\\
&C_1 = -6k^2a_0^2a_1 + a_1l^2b_0^2 - a_1k^2b_0^2 - k^4a_1b_0^2 - a_{-1}k^2b_1^2 - 6a_{-1}k^2a_1^2\\
&\quad - 4k^4a_{-1}b_1^2 + a_{-1}l^2b_1^2 + 2a_0l^2b_0b_1 - 2a_0k^2b_0b_1 + k^4a_0b_1b_0\\
&\quad + 2a_1l^2b_{-1}b_1 - 2a_1k^2b_{-1}b_1 + 4k^4a_1b_{-1}b_1 = 0,\\
&C_0 = 2a_{-1}l^2b_0b_1 - 2a_{-1}k^2b_0b_1 - 12a_{-1}k^2a_0a_1 + 2a_0l^2b_{-1}b_1\\
&\quad - 2a_0k^2b_{-1}b_1 + 2a_1l^2b_{-1}b_0 - 2a_1k^2b_1b_0 - 3k^4a_{-1}b_0b_1\\
&\quad - 3k^4a_1b_{-1}b_0 + 6k^4a_0b_{-1}b_1 + a_0l^2b_0^2 - a_0k^2b_0^2 - k^2a_0^3 = 0,\\
&C_{-1} = a_{-1}l^2b_0^2 - a_{-1}k^2b_0^2 - 6a_{-1}k^2a_0^2 - k^4a_{-1}b_0^2 - a_1k^2b_{-1}^2\\
&\quad - 6a_{-1}^2k^2a_1 - 4k^4a_1b_{-1}^2 + a_1l^2b_{-1}^2 + 2a_0l^2b_{-1}b_0\\
&\quad - 2a_0k^2b_{-1}b_0 + k^4a_0b_{-1}b_0 + 2a_{-1}l^2b_{-1}b_1 = 0,\\
&\quad - 2a_{-1}k^2b_{-1}b_1 + 4k^4a_{-1}b_{-1}b_1 = 0,\\
&C_{-2} = -a_0k^2b_{-1}^2 - k^4a_0b_{-1}^2 - 6a_{-1}^2k^2a_0 + a_0l^2b_{-1}^2 - 2a_{-1}k^2b_{-1}b_0\\
&\quad + 2a_{-1}l^2b_{-1}b_0 + k^4a_{-1}b_{-1}b_0 = 0,\\
&C_{-3} = -a_{-1}k^2b_{-1}^2 + a_{-1}l^2b_{-1}^2 - 2a_{-1}^3k^2 = 0.
\end{aligned} \tag{23}$$

Equating coefficients of $\exp(n\xi)$ to be zero, we obtain following system of equations

$$C_3 = 0,\quad C_2 = 0,\quad C_1 = 0,\quad C_0 = 0,\quad C_{-1} = 0,\quad C_{-2} = 0,\quad C_{-3} = 0. \tag{24}$$

and on solving it, we obtain

Case 1:

$$l = 2\frac{\sqrt{-b_0^2 - 2a_0^2a_0}}{b_0^2},\quad k = \frac{2ia_0}{b_0},\quad b_0 \neq 0,\quad a_{-1} = a_{-1},$$
$$a_0 = a_0,\quad a_1 = 0,\quad b_{-1} = \frac{a_{-1}b_0}{a_0},\quad a_0 \neq 0,\quad b_0 = b_0,\quad b_1 = 0. \tag{25}$$

Case 2:

$$l = \frac{\sqrt{2b_{-1}^2 + 4a_{-1}^2a_{-1}}}{b_{-1}^2},\quad k = \frac{\sqrt{2}a_{-1}}{b_{-1}},\quad b_{-1} \neq 0,\quad a_{-1} = a_{-1},$$
$$a_0 = a_0,\quad a_1 = 0,\quad b_{-1} = b_{-1},\quad b_0 = \frac{b_{-1}a_0}{a_{-1}},\quad a_{-1} \neq 0,\quad b_1 = 0. \tag{26}$$

Case 3:

$$l = 2\frac{\sqrt{-b_1^2 - 2a_1^2a_1}}{b_1^2},\quad k = \frac{2ia_1}{b_1},\quad b_1 \neq 0,\quad a_{-1} = 0,\quad a_0 = a_0,$$
$$a_1 = a_1,\quad b_{-1} = 0,\quad b_0 = \frac{a_0b_1}{a_1},\quad a_1 \neq 0,\quad b_1 = b_1. \tag{27}$$

The corresponding solutions for cases (25)–(27) are as follows

$$u_{1,1}(x,t) = (a_{-1}e^{\omega t^\alpha - x^{\kappa\beta}} + a_0)(a_{-1}b_0e^{\omega t^\alpha - x^{\kappa\beta}}a_0^{-1} + b_0)^{-1},$$
$$\omega = 2\frac{\sqrt{-b_0^2 - 2a_0^2a_0}}{b_0^2\Gamma(\alpha+1)}\quad \text{and}\quad \kappa = \frac{2ia_0}{b_0\Gamma(\beta+1)}. \tag{28}$$

$$u_{1,2}(x,t) = (a_{-1}e^{\omega t^\alpha - x^{\kappa\beta}} + a_0)\left(b_{-1}e^{\omega t^\alpha - x^{\kappa\beta}} + \frac{b_{-1}a_0}{a_{-1}}\right)^{-1},$$
$$\omega = \frac{\sqrt{2b_{-1}^2 + 4a_{-1}^2a_{-1}}}{b_{-1}^2\Gamma(\alpha+1)},\quad \kappa = \frac{\sqrt{2}a_{-1}}{b_{-1}\Gamma(\beta+1)} \tag{29}$$

$$u_{1,3}(x,t) = (a_0 + a_1e^{\omega t^\alpha - x^{\kappa\beta}})\left(\frac{a_0b_1}{a_1} + b_1e^{\omega t^\alpha - x^{\kappa\beta}}\right)^{-1},$$
$$\omega = -2\frac{\sqrt{-b_1^2 - 2a_1^2a_1}}{b_1^2\Gamma(\alpha+1)}\quad \text{and}\quad \kappa = \frac{2ia_1}{b_1\Gamma(\beta+1)}. \tag{30}$$

(II) For Boussinesq-like equation (6), we use fractional complex transformation (18) and modified Riemann-Liouville derivative to convert it into the ODE

$$l^2\boldsymbol{u}'' - (\boldsymbol{u}'' + 12\boldsymbol{u}\boldsymbol{u}'^2 + 6\boldsymbol{u}^2\boldsymbol{u}'')k^2 - k^2l^2\boldsymbol{u}^{(4)} = 0, \tag{31}$$

that is integrated twice (neglecting constant of integration) and then by using (21), we obtain an equation of the form (22) where,

$$\begin{aligned}
&A = (b_{-1}e^{-\xi} + b_0 + b_1e^{\xi})^4,\\
&C_3 = -a_1l^2b_1^2 + a_1k^2b_1^2 + 2a_1^3k^2 = 0,\\
&C_2 = -2a_1l^2b_0b_1 + 2a_1k^2b_0b_1 + k^2l^2a_0b_1^2 - k^2l^2a_1b_0b_1 - a_0l^2b_1^2\\
&\quad + a_0k^2b_1^2 + 6a_0k^2a_1^2 = 0,\\
&C_1 = -2a_0l^2b_0b_1 + 2a_0k^2b_0b_1 + k^2l^2a_1b_0^2 - 2a_1l^2b_{-1}b_1 + 2a_1k^2b_{-1}b_1\\
&\quad + 4k^2l^2a_{-1}b_1^2 + 6k^2a_0^2a_1 - a_1l^2b_0^2 + a_1k^2b_0^2 - 4k^2l^2a_1b_{-1}b_1\\
&\quad - k^2l^2a_0b_1b_0 + 6a_{-1}k^2a_1^2 - a_{-1}l^2b_1^2 + a_{-1}k^2b_1^2 = 0,\\
&C_0 = -a_0l^2b_0^2 + a_0k^2b_0^2 + 2k^2a_0^3 + 3k^2l^2a_{-1}b_0b_1 + 3k^2l^2a_1b_{-1}b_0\\
&\quad - 6k^2l^2a_0b_{-1}b_1 - 2a_{-1}l^2b_0b_1 + 2a_{-1}k^2b_0b_1 + 12a_{-1}k^2a_0a_1\\
&\quad - 2a_0l^2b_{-1}b_1 + 2a_0k^2b_{-1}b_1 - 2a_1l^2b_{-1}b_0 + 2a_1k^2b_{-1}b_0 = 0,\\
&C_{-1} = -2a_0l^2b_{-1}b_0 + 2a_0k^2b_{-1}b_0 + k^2l^2a_{-1}b_0^2 - 2a_{-1}l^2b_{-1}b_1\\
&\quad + 2a_{-1}k^2b_{-1}b_1 + 4k^2l^2a_1b_{-1}^2 - 4k^2l^2a_{-1}b_{-1}b_1 - k^2l^2a_0b_{-1}b_0\\
&\quad - a_{-1}l^2b_0^2 + a_{-1}k^2b_0^2 + 6a_{-1}k^2a_0^2 + 6a_{-1}^2k^2a_1 - a_1l^2b_{-1}^2\\
&\quad + a_1k^2b_{-1}^2 = 0,\\
&C_{-2} = 6a_{-1}^2k^2a_0 - a_0l^2b_{-1}^2 + a_0k^2b_{-1}^2 - 2a_{-1}l^2b_{-1}b_0 + 2a_{-1}k^2b_{-1}b_0\\
&\quad + k^2l^2a_0b_{-1}^2 - k^2l^2a_{-1}b_{-1}b_0 = 0,\\
&C_{-3} = 2a_{-1}^3k^2 - a_{-1}l^2b_{-1}^2 + a_{-1}k^2b_{-1}^2 = 0.
\end{aligned} \tag{32}$$

Equating the coefficients of $\exp(n\xi)$ to be zero, we obtain system of the form (24) and on solving it, we obtain

Case 1:

$$l = \frac{2ia_{-1}}{b_{-1}},\quad k = 2\sqrt{-(2a_{-1}^2 + b_{-1}^2)^{-1}}a_{-1},\quad a_{-1} = a_{-1},$$
$$a_0 = \frac{a_{-1}b_0}{b_{-1}}, b_{-1} \neq 0 a_1 = 0,\quad b_{-1} = b_{-1},\quad b_0 = b_0,\quad b_1 = 0. \tag{33}$$

Case 2:

$$l=\frac{2ia_0}{b_0},\quad k=2\sqrt{-(2a_0^2+b_0^2)^{-1}}a_0,\quad a_{-1}=0,\quad a_0=a_0,$$
$$a_1=\frac{a_0b_1}{b_0},\quad b_0\neq 0,\quad b_{-1}=0,\quad b_0=b_0,\quad b_1=b_1. \tag{34}$$

Case 3:

$$l=\frac{ia_1}{b_1},\quad k=\sqrt{-(2a_1^2+b_1^2)^{-1}}a_1,\quad a_{-1}=\frac{a_1b_{-1}}{b_1},$$
$$a_0=0,\quad a_1=a_1,\quad b_{-1}=b_{-1},\quad b_0=0,\quad b_1=b_1. \tag{35}$$

The corresponding solutions for cases (33)–(35) are as follows

$$u_{2,1}(x,t)=\left(a_{-1}e^{\omega t^\alpha-\kappa x^\beta}+\frac{a_{-1}b_0}{b_{-1}}\right)\left(b_{-1}e^{\omega t^\alpha-\kappa x^\beta}+b_0\right)^{-1},$$
$$\omega=\frac{2ia_{-1}}{b_{-1}\Gamma(\alpha+1)}\quad\text{and}\quad\kappa=2\frac{a_{-1}}{\Gamma(\beta+1)}\sqrt{\frac{1}{-(2a_{-1}^2+b_{-1}^2)}}. \tag{36}$$

$$u_{2,2}(x,t)=\left(a_0+a_0b_1e^{\omega t^\alpha+\kappa x^\beta}b_0^{-1}\right)\left(b_0+b_1e^{\omega t^\alpha+\kappa x^\beta}\right)^{-1},$$
$$\omega=\frac{-2ia_0}{b_0\Gamma(\alpha+1)}\quad\text{and}\quad\kappa=2\frac{a_0}{\Gamma(\beta+1)}\sqrt{\frac{1}{-(2a_0^2+b_0^2)}}. \tag{37}$$

$$u_{2,3}(x,t)=\left(b_{-1}e^{\omega t^\alpha-\kappa x^\beta}b_1^{-1}+a_1e^{-(\omega t^\alpha-\kappa x^\beta)}\right)\left(b_{-1}e^{\omega t^\alpha+\kappa x^\beta}+b_1e^{-(\omega t^\alpha-\kappa x^\beta)}\right)^{-1},$$
$$\omega=\frac{ia_1}{b_1\Gamma(\alpha+1)}\quad\text{and}\quad\kappa=\frac{a_1}{\Gamma(\beta+1)}\sqrt{\frac{1}{-(2a_1^2+b_1^2)}}. \tag{38}$$

(III) For Boussinesq-like equation (7), we use fractional complex transformation (18) and modified Riemann-Liouville derivative to convert it into the ODE

$$l^2\mathbf{u}''+lk\mathbf{u}''-(12\mathbf{u}\mathbf{u}'^2+6\mathbf{u}^2\mathbf{u}'')k^2+k^3l\mathbf{u}^{(4)}=0, \tag{39}$$

that is integrated twice (neglecting constant of integration) and then by using (21) into the result, we obtain an equation of the form (22) where,

$$A=(b_{-1}e^{-\xi}+b_0+b_1e^{\xi})^3,$$
$$C_3=-a_1l^2b_1^2+2a_1^3k^2-a_1klb_1^2=0,$$
$$C_2=-a_0l^2b_1^2+6a_0k^2a_1^2-a_0klb_1^2-2a_1l^2b_0b_1-k^3la_0b_1^2+k^3la_1b_0b_1$$
$$-2a_1klb_0b_1=0,$$
$$C_1=6a_{-1}k^2a_1^2-a_{-1}l^2b_1^2-4k^3la_{-1}b_1^2-2a_1l^2b_{-1}b_1-a_{-1}klb_1^2$$
$$-2a_0l^2b_0b_1-a_1klb_0^2-k^3la_1b_0^2-2a_1klb_{-1}b_1+4k^3la_1b_{-1}b_1$$
$$-2a_0klb_0b_1+6k^2a_0^2a_1-a_1l^2b_0^2+k^3la_0b_1b_0=0,$$
$$C_0=-2a_0l^2b_{-1}b_1+12a_{-1}k^2a_0a_1-2a_1l^2b_{-1}b_0-2a_{-1}l^2b_0b_1$$
$$-2a_1klb_{-1}b_0-a_0l^2b_0^2-3k^3la_{-1}b_0b_1-3k^3la_1b_{-1}b_0$$
$$-2a_{-1}klb_0b_1-2a_0klb_{-1}b_1+6k^3la_0b_{-1}b_1+2k^2a_0^3$$
$$-a_0klb_0^2=0,$$
$$C_{-1}=6a_{-1}^2k^2a_1-a_1l^2b_{-1}^2-a_1klb_{-1}^2-2a_{-1}l^2b_{-1}b_1-4k^3la_1b_{-1}^2$$
$$-a_{-1}klb_0^2-2a_0l^2b_{-1}b_0-k^3la_{-1}b_0^2+4k^3la_{-1}b_{-1}b_1$$
$$-2a_{-1}klb_{-1}b_1-2a_0klb_{-1}b_0-a_{-1}l^2b_0^2+6a_{-1}k^2a_0^2$$
$$+k^3la_0b_{-1}b_0=0,$$
$$C_{-2}=6a_{-1}^2k^2a_0-a_0l^2b_{-1}^2-2a_{-1}l^2b_{-1}b_0-a_0klb_{-1}^2-k^3la_0b_{-1}^2$$
$$+k^3la_{-1}b_{-1}b_0-2a_{-1}klb_{-1}b_0=0, C_{-3}=-a_{-1}l^2b_{-1}^2$$
$$-a_{-1}klb_{-1}^2+2a_{-1}^3k^2=0. \tag{40}$$

Equating the coefficients of $\exp(n\xi)$ to be zero, we obtain system of the form (24) and on solving it, we obtain

$$l=l,\quad k=k,\quad a_{-1}=0,\quad a_0=\frac{b_0}{k}\sqrt{\frac{l^2+lk}{8}},\quad a_1=a_1,$$
$$b_{-1}=0,\quad b_0=b_0,\quad b_1=\frac{\sqrt{2}a_1k}{\sqrt{l^2+lk}}, \tag{41}$$

The corresponding solution for above case is

$$u_3(x,t)=\left(\frac{1}{\sqrt{2}}\frac{\sqrt{l^2+lk}b_0}{k}+a_1e^{\omega t^\alpha+\kappa x^\beta}\right)$$
$$\times\left(b_0+\sqrt{2}a_1ke^{\omega t^\alpha+\kappa x^\beta}\frac{1}{\sqrt{l^2+lk}}\right)^{-1}, \tag{42}$$
$$\omega=-\frac{l}{\Gamma(\alpha+1)}\quad\text{and}\quad\kappa=\frac{k}{\Gamma(\beta+1)}$$

(IV) Finally, for Boussinesq-like equation (8), we use fractional complex transformation (18) and modified Riemann-Liouville derivative to convert it into the ODE

$$l^2\mathbf{u}''+lk\mathbf{u}''-(12\mathbf{u}\mathbf{u}'^2+6\mathbf{u}^2\mathbf{u}'')k^2-k^4\mathbf{u}^{(4)}=0, \tag{43}$$

that is integrated twice (neglecting constant of integration) and then by using (21) into the result, we obtain an equation of the form (22) where,

$$A=(b_{-1}e^{-\xi}+b_0+b_1e^{\xi})^3,$$
$$C_3=-a_{-1}l^2b_{-1}^2+2a_{-1}^3k^2=0,$$
$$C_2=k^4a_0b_{-1}^2+6a_{-1}^2k^2a_0-a_0l^2b_{-1}^2-2a_{-1}l^2b_{-1}b_0-k^4a_{-1}b_{-1}b_0=0,$$
$$C_1=6a_{-1}^2k^2a_1+4k^4a_1b_{-1}^2-a_1l^2b_{-1}^2-a_{-1}l^2b_0^2+6a_{-1}k^2a_0^2$$
$$+k^4a_{-1}b_0^2-2a_{-1}l^2b_{-1}b_1-4k^4a_{-1}b_{-1}b_1-2a_0l^2b_{-1}b_0$$
$$-k^4a_0b_{-1}b_0=0,$$
$$C_0=3k^4a_1b_{-1}b_0-6k^4a_0b_{-1}b_1-2a_{-1}l^2b_0b_1+12a_{-1}k^2a_0a_1$$
$$-2a_0l^2b_{-1}b_1-2a_1l^2b_{-1}b_0+3k^4a_{-1}b_0b_1-a_0l^2b_0^2$$
$$+2k^2a_0^3=0,$$
$$C_{-1}=-a_{-1}l^2b_1^2+6a_{-1}k^2a_1^2+4k^4a_{-1}b_1^2+6k^2a_0^2a_1-a_1l^2b_0^2$$
$$+k^4a_1b_0^2-2a_1l^2b_{-1}b_1-4k^4a_1b_{-1}b_1-2a_0l^2b_0b_1$$
$$-k^4a_0b_1b_0=0,$$
$$C_{-2}=k^4a_0b_1^2-a_0l^2b_1^2+6a_0k^2a_1^2-2a_1l^2b_0b_1-k^4a_1b_0b_1=0,$$
$$C_{-3}=-a_1l^2b_1^2+2a_1^3k^2=0. \tag{44}$$

Equating the coefficients of $\exp(n\xi)$ to be zero, we obtain system of the form (24) and on solving it, we obtain

Case 1:

$$l=\frac{2i\sqrt{2}a_0^2}{b_0^2},\quad k=\frac{2ia_0}{b_0},\quad b_0\neq 0,\quad a_{-1}=a_{-1},$$
$$a_0=a_0,\quad a_1=0, b_{-1}=\frac{a_{-1}b_0}{a_0},\quad a_0\neq 0,\quad b_0=b_0,\quad b_1=0. \tag{45}$$

Case 2:

$$l=\frac{2a_{-1}^2}{b_{-1}^2},\quad k=\frac{\sqrt{2}a_{-1}}{b_{-1}},\quad b_{-1}\neq 0,\quad a_{-1}=a_{-1},\quad a_0=\frac{a_{-1}b_0}{b_{-1}},$$
$$a_1=0,\quad b_{-1}=b_{-1},\quad b_0=b_0,\quad b_1=0. \tag{46}$$

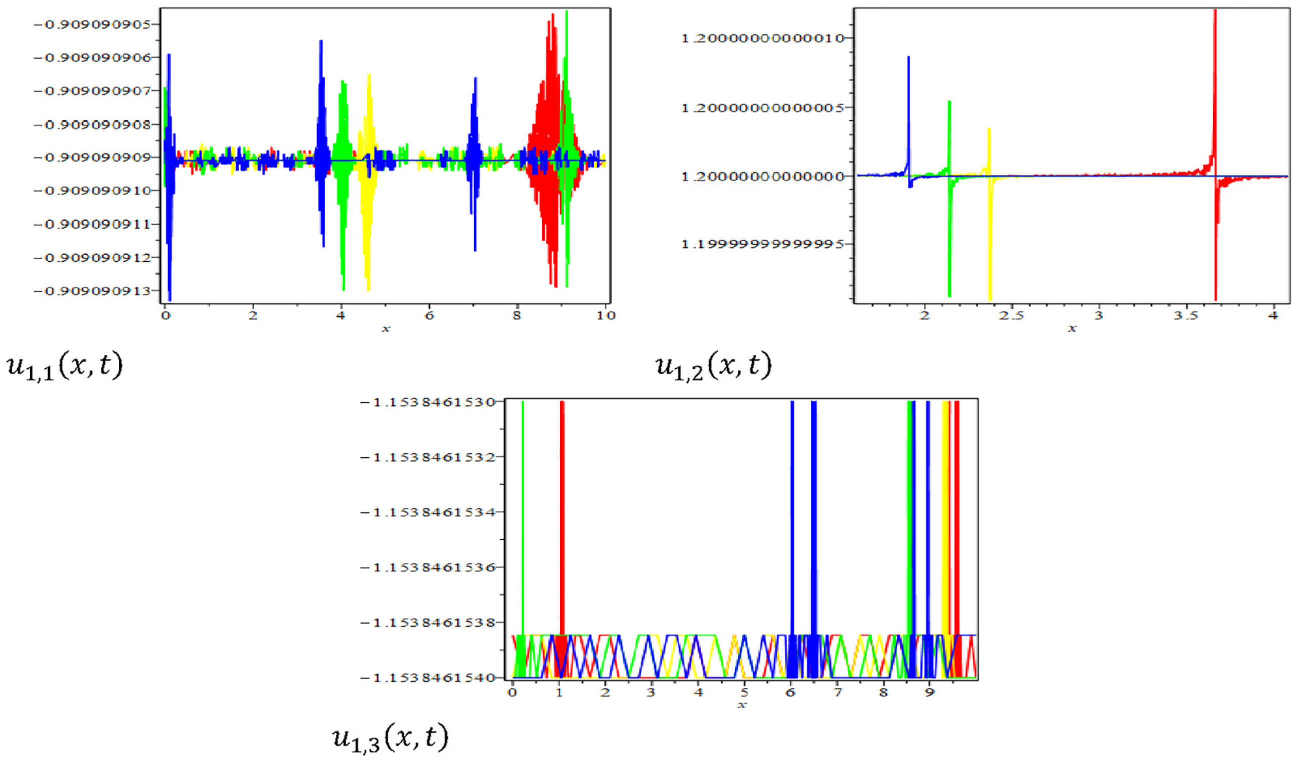

Fig. 1. Depicts the solutions of 1st form of Boussinesq-like equations of fractional order.

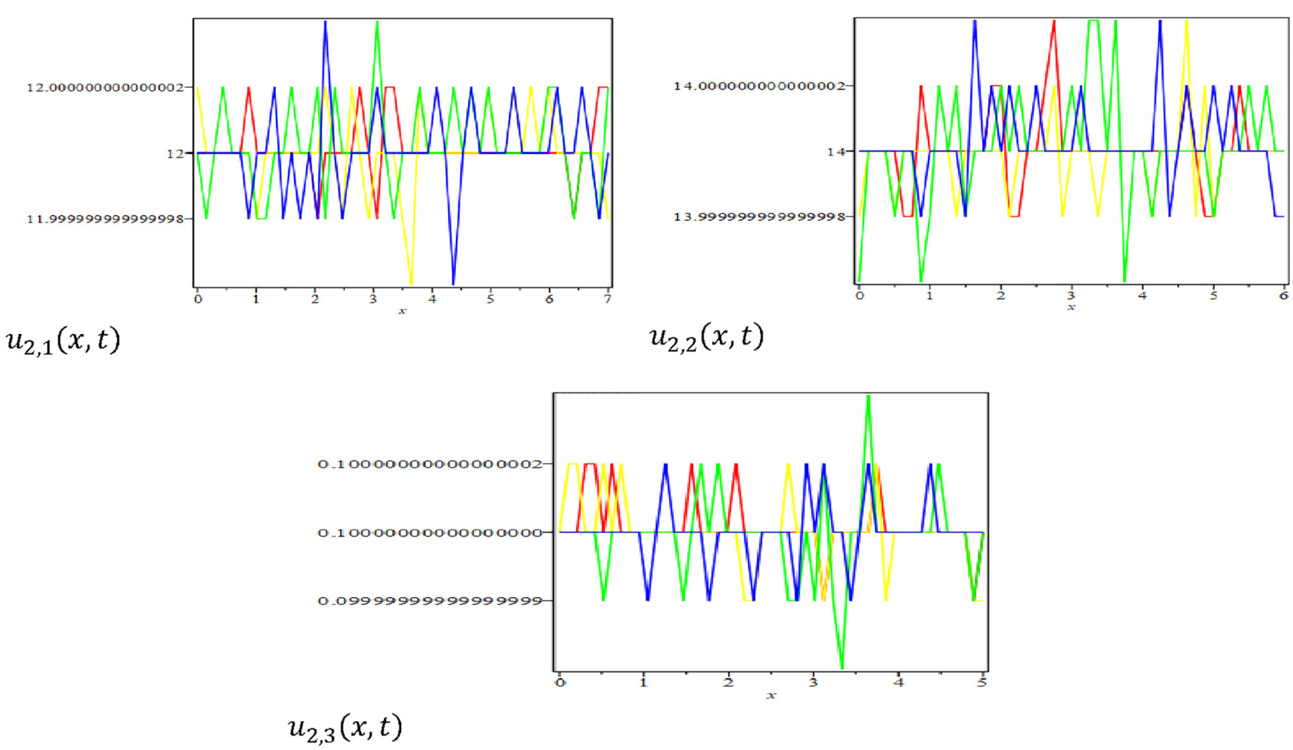

Fig. 2. Depicts the solutions of 2nd form of Boussinesq-like equations of fractional order.

Case 3:

$$l = l, \quad k = \frac{1}{2}\frac{\sqrt{2}b_1}{a_1}l, \quad a_1 \neq 0, \quad a_{-1} = a_{-1}, \quad a_0 = 0,$$
$$a_1 = a_1, \quad b_{-1} = \frac{a_{-1}b_1}{a_1}, \quad a_1 \neq 0, \quad b_0 = 0, \quad b_1 = b_1 \tag{47}$$

The corresponding solution for cases (45)–(47) are as follow

$$u_{4,1}(x,t) = (a_{-1}e^{\omega t^{\alpha} - \kappa x^{\beta}} + a_0)(a_{-1}b_0 e^{\omega t^{\alpha} - \kappa x^{\beta}} a_0^{-1} + b_0)^{-1},$$
$$\omega = \frac{2i\sqrt{2}a_0^2}{b_0^2\Gamma(\alpha+1)} \quad \text{and} \quad \kappa = \frac{2ia_0}{b_0\Gamma(\beta+1)}. \tag{48}$$

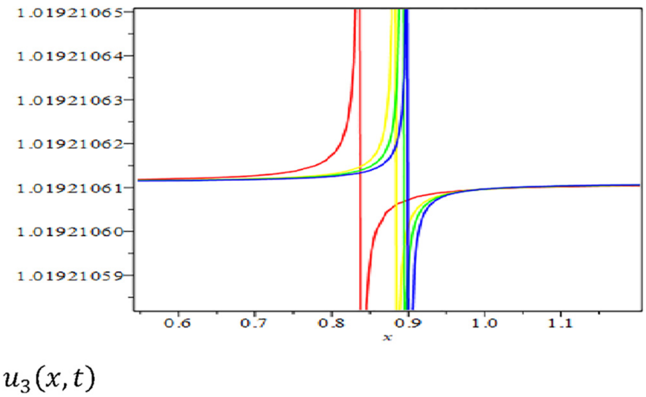

Fig. 3. Depicts the solutions of 3rd form of Boussinesq-like equations of fractional order.

$$u_{4,2}(x,t) = \left(a_{-1}e^{\omega t^{\alpha}-\kappa x^{\beta}} + \frac{a_{-1}b_0}{b_{-1}}\right)\left(b_{-1}e^{\omega t^{\alpha}-\kappa x^{\beta}} + b_0\right)^{-1},$$
$$\omega = 2\frac{a_{-1}^2}{b_{-1}^2\Gamma(\alpha+1)} \quad \text{and} \quad \kappa = \frac{\sqrt{2}a_{-1}}{b_{-1}\Gamma(\beta+1)}. \tag{49}$$

$$u_{4,3}(x,t) = \left(a_{-1}e^{\omega t^{\alpha}-\kappa x^{\beta}} + a_1 e^{-(\omega t^{\alpha}-\kappa x^{\beta})}\right)\left(b_1 a_{-1}e^{\omega t^{\alpha}-\kappa x^{\beta}}a_1^{-1} + b_1 e^{-(\omega t^{\alpha}-\kappa x^{\beta})}\right)^{-1},$$
$$\omega = \frac{l}{\Gamma(\alpha+1)} \quad \text{and} \quad \kappa = \frac{1}{2}\frac{\sqrt{2}b_1 l}{a_1\Gamma(\beta+1)}. \tag{50}$$

Graphical representation

Figs. 1–4 depict solutions of 1st, 2nd, 3rd and 4th form of Boussinesq-like equations of fractional order respectively. These solutions are observed for different orders $\alpha = \beta = 0.5$, 0.75, 0.85 and 1, shown by red, yellow, green and blue colors respectively. Appropriate values of parameters are used for better understanding the physical facets of these NLEEs. High frequencies and very small amplitude of the exact wave responses can clearly be noticed form solution graphs.

Conclusion and future scope of the study

We established new exact traveling wave solutions of Boussinesq-like equations of fractional order including complex traveling wave solutions, by implementing Exp-function method combined with fractional complex transformation and modified Riemann-Liouville operator. Appropriate values are used for parameters so that graphically, the physical aspects of related phenomena may well be understood. Our study shows that Exp-function method is a powerful, reliable and pragmatic mathematical tool as compared to many other existing techniques due to its capability of providing a variety of exact solutions to a wide range of nonlinear problems with their integer as well as non-integer order sense. Availability of more than one solutions motivates applied scientists for engineering their ideas to an optimal level. The method possesses computational simplicity and supportive results for better visualization of the associated physical phenomena.

Due to pivotal role of nano-fluids study in modern scientific applications, significant contributions can be seen in the literature such as [36–59]. The proposed method and those discussed in section 1, can also be very promising for investigating the physical models related to nano-fluid flow of fractional order version. Adding the ingredients of fractional calculus into this particular area of research, the applied scientists and mathematicians can unveil several mysteries affiliated to nano-fluids, for which the classical calculus shows its limitations.

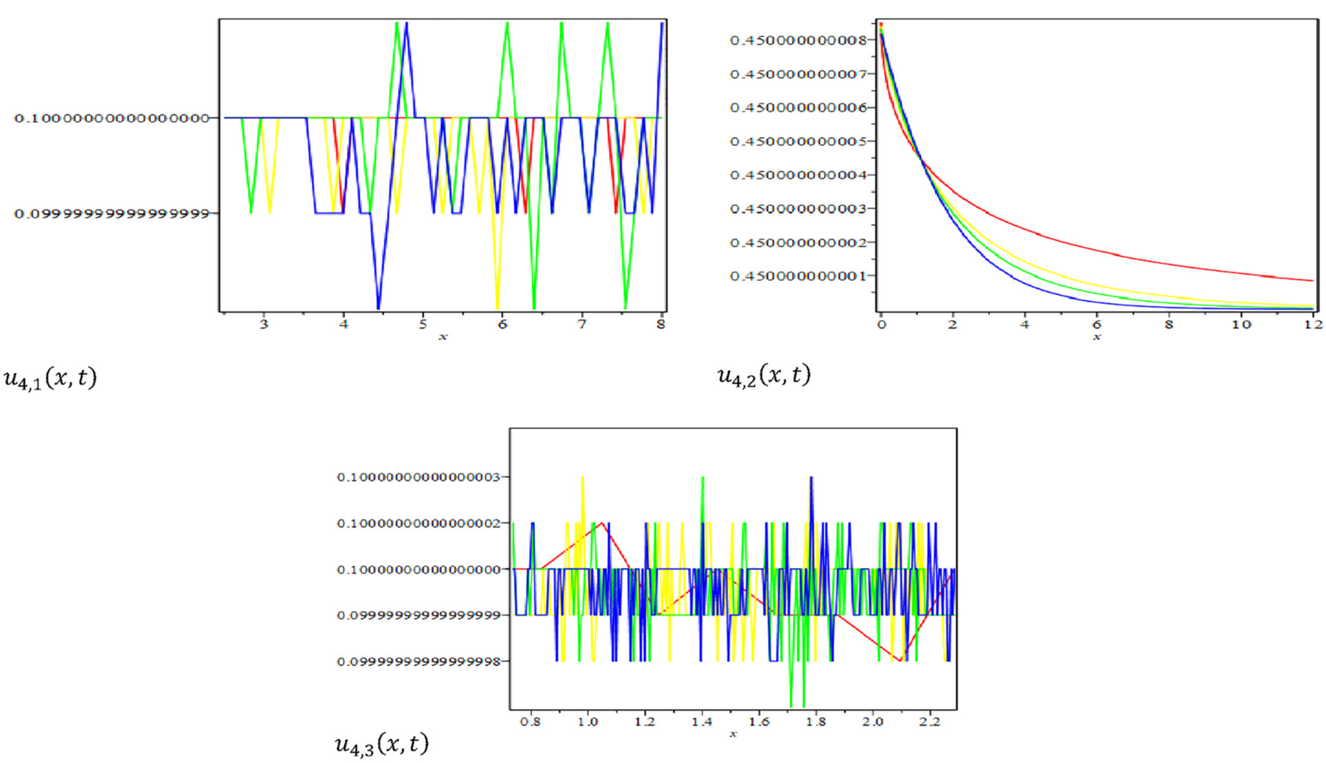

Fig. 4. Depicts the solutions of 4th form of Boussinesq-like equations of fractional order.

Declarations

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors drafted the manuscript, and they read and approved the final version of the manuscript.

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References

[1] Wang M. Solitary wave solutions for variant Boussinesq equations. Phys Lett A 1995;199(3–4):169–72.
[2] Zayed EME, Zedan HA, Gepreel KA. On the solitary wave solutions for nonlinear Hirota-Satsuma coupled KdV of equations. Chaos, Solitons Fractals 2004;22(2):285–303.
[3] Ali AT. New generalized Jacobi elliptic function rational expansion method. J Comput Appl Math 2011;235(14):4117–27.
[4] Cao X. Bäcklund transformations between the KdV equation and a new nonlinear evolution equation. Math Prob Eng 2017;2017:1–6.
[5] Zayed EME, Amer YA, Arnous AH. Functional variable method and its applications for finding exact solutions of nonlinear PDEs in mathematical physics. Sci Res Essays 2013;8(42):2068–74.
[6] Eslami M, Mirzazadeh M. Functional variable method to study nonlinear evolution equations. Open Eng. 2013;3(3).
[7] Malfliet W. Solitary wave solutions of nonlinear wave equations. Am J Phys 1992;60(7):650–4.
[8] Fan E. Extended tanh-function method and its applications to nonlinear equations. Phys Lett A 2000;277(4–5):212–8.
[9] Abdou MA. The extended tanh method and its applications for solving nonlinear physical models. Appl Math Comput 2007;190(1):988–96.
[10] Weiss J, Tabor M, Carnevale G. The Painlev? property for partial differential equations. J Math Phys 1983;24(3):522–6.
[11] Khan U, Ellahi R, Khan R, Mohyud-Din ST. Extracting new solitary wave solutions of Benny-Luke equation and Phi-4 equation of fractional order by using (G′/G)-expansion method. Opt Quant Electron 2017;49(11):362.
[12] Bekir A, Güner Ö. Exact solutions of nonlinear fractional differential equations by (G'/G)-expansion method. Chin Phys B 2013;22(11):110202.
[13] Bibi S, Mohyud-Din ST. Traveling wave solutions of KdVs using sine-cosine method. J Assoc Arab Universities Basic Appl Sci 2014;15(1):90–3.
[14] Hirota R, Satsuma J. Soliton solutions of a coupled Korteweg-de Vries equation. Phys Lett A 1981;85(8–9):407–8.
[15] Zhao Y-M. *F*-expansion method and its application for finding new exact solutions to the Kudryashov-Sinelshchikov equation. J Appl Math 2013;2013:1–7.
[16] Filiz A, Ekici M, Sonmezoglu A. F-expansion method and new exact solutions of the Schrödinger-KdV equation. Sci World J 2014;2014:534063.
[17] Gepreel KA. Simple equation method for nonlinear evolution equations in mathematical physics applications of the simple equation. WSEAS Trans Math 2016;15(4):462–76.
[18] Darvishi MT, Najafi M, Wazwaz AM. Soliton solutions for Boussinesq-like equations with spatio-temporal dispersion. Ocean Eng 2017;130:228–40.
[19] Wazwaz AM. Solitons and singular solitons for a variety of Boussinesq-like equations. Ocean Eng 2012;53:1–5.
[20] Eslami M, Mirzazadeh M. First integral method to look for exact solutions of a variety of Boussinesq-like equations. Ocean Eng 2014;83:133–7.
[21] Lee J, Sakthivel R. Exact travelling wave solutions of a variety of Boussinesq-like equations. Chin J Phys 2014;52(3):939–57.
[22] Al Ghabshi M, Krishnan EV. Solitary wave solutions of a fractional Boussinesq Equation mathematical analysis of mapping methods. Int J Math Anal 2017;11(9):407–23.
[23] Mohyud-Din ST, Bibi S. Exact solutions for nonlinear fractional differential equations using exponential rational function method. Opt Quant Electron 2017;49(2):1–12.
[24] Boz A, Bekir A. Application of Exp-function method for $(3+1)$-dimensional nonlinear evolution equations. Comput Math Appl 2008;56(5):1451–6.
[25] He J-H, Wu X-H. Exp-function method for nonlinear wave equations. Chaos, Solitons Fractals 2006;30(3):700–8.
[26] Zheng B. Exp-function method for solving fractional partial differential equations. Sci World J 2013;3:2013.
[27] He JH, Elagan SK, Li ZB. Geometrical explanation of the fractional complex transform and derivative chain rule for fractional calculus. Phys Lett A 2012;376(4):257–9.
[28] Wang Q. Numerical solutions for fractional KdV-Burgers equation by Adomian decomposition method. Appl Math Comput Nov. 2006;182(2):1048–55.
[29] Li ZB, He JH. Fractional complex transform for fractional differential equations. Math Comput Appl 2010;15(5):970–3.
[30] Jumarie G. Modified Riemann-Liouville derivative and fractional Taylor series of nondifferentiable functions further results. Comput Math Appl 2006;51(9–10):1367–76.
[31] Jumarie G. Table of some basic fractional calculus formulae derived from a modified Riemann-Liouville derivative for non-differentiable functions. Appl Math Lett 2009;22(3):378–85.
[32] Ebaid A. An improvement on the Exp-function method when balancing the highest order linear and nonlinear terms. J Math Anal Appl 2012;392(1):1–5.
[33] Djrbashian ANMM. Fractional derivative and the Cauchy problem for differential equations of fractional order. Izv Acad Nauk Armjanskoi 1968;3(1):3–29.
[34] Caputo M. Linear models of dissipation whose Q is almost Frequency Independent-II. Geophys J Int 1967;13:529–39.
[35] Jumarie G. Modified Riemann-Liouville derivative new methods for solution. J Appl Math Comput 2007;24(1):31–48.
[36] Sheikholeslami M, Shehzad SA. Thermal radiation of ferrofluid in existence of Lorentz forces considering variable viscosity. Int J Heat Mass Transfer 2017;109:82–92.
[37] Sheikholeslami M, Hayat T, Alsaedi A. Numerical simulation of nanofluid forced convection heat transfer improvement in existence of magnetic field using lattice Boltzmann method. Int J Heat Mass Transfer 2017;108:1870–83.
[38] Sheikholeslami M, Bhatti MM. Active method for nanofluid heat transfer enhancement by means of EHD. Int J Heat Mass Transfer 2017;109:115–22.
[39] Sheikholeslami M, Shehzad SA. CVFEM for influence of external magnetic source on Fe_3O_4-H_2O nanofluid behavior in a permeable cavity considering shape effect. Int J Heat Mass Transfer 2017;115:180–91.
[40] Sheikholeslami M, Sadoughi M. Mesoscopic method for MHD nanofluid flow inside a porous cavity considering various shapes of nanoparticles. Int J Heat Mass Transfer 2017;113:106–14.
[41] Sheikholeslami M, Sadoughi M. Simulation of CuO-water nanofluid heat transfer enhancement in presence of melting surface. Int J Heat Mass Transfer 2018;116:909–19.
[42] Sheikholeslami M, Zeeshan A. Mesoscopic simulation of $CuOH_2O$ nanofluid in a porous enclosure with elliptic heat source. Int J Hydrogen Energy 2017;42(22):15393–402.
[43] Sheikholeslami M, Rokni HB. Free convection of CuO–H_2O nanofluid in a curved porous enclosure using mesoscopic approach. Int J Hydrogen Energy 2017;42(22):14942–9.
[44] Sheikholeslami M, Zeeshan A. Analysis of flow and heat transfer in water based nanofluid due to magnetic field in a porous enclosure with constant heat flux using CVFEM. Comput Methods Appl Mech Eng 2017;320:68–81.
[45] Sheikholeslami M, Hayat T, Alsaedi A. Numerical study for external magnetic source influence on water based nanofluid convective heat transfer. Int J Heat Mass Transfer 2017;106:745–55.
[46] Sheikholeslami M, Shehzad SA. Magnetohydrodynamic nanofluid convection in a porous enclosure considering heat flux boundary condition. Int J Heat Mass Transfer 2017;106:1261–9.
[47] MohsenSheikholeslami M. Shamlooei, Fe_3O_4–H_2O nanofluid natural convection in presence of thermal radiation. Int J Hydrogen Energy 2017;42(9):5708–18.
[48] Sheikholeslami M, Bhatti MM. Forced convection of nanofluid in presence of constant magnetic field considering shape effects of nanoparticles. Int J Heat Mass Transfer 2017;111:1039–49.
[49] Sheikholeslami M. Lattice Boltzmann method simulation for MHD non-Darcy nanofluid free convection. Physica B 2017;516:55–71.
[50] Kandelousi MS. Effect of spatially variable magnetic field on ferrofluid flow and heat transfer considering constant heat flux boundary condition. Eur Phys J Plus Nov. 2014;129(11):248.
[51] Sheikholeslami M. Influence of Coulomb forces on Fe_3O_4–H_2O nanofluid thermal improvement. Int J Hydrogen Energy 2017;42(2):821–9.
[52] Sheikholeslami M. Influence of Lorentz forces on nanofluid flow in a porous cylinder considering Darcy model. J Mol Liq 2017;225:903–12.
[53] Sheikholeslami M. Numerical simulation of magnetic nanofluid natural convection in porous media. Phys Lett A 2017;381(5):494–503.
[54] Sheikholeslami M. Magnetic field influence on nanofluid thermal radiation in a cavity with tilted elliptic inner cylinder. J Mol Liq 2017;229:137–47.
[55] Sheikholeslami M. CuO-water nanofluid free convection in a porous cavity considering Darcy law. Eur Phys J Plus 2017;132(1):55.
[56] Sheikholeslami M. Magnetohydrodynamic nanofluid forced convection in a porous lid driven cubic cavity using Lattice Boltzmann method. J Mol Liq 2017;231:555–65.
[57] Sheikholeslami M. Influence of magnetic field on nanofluid free convection in an open porous cavity by means of Lattice Boltzmann method. J Mol Liq 2017;234:364–74.
[58] Sheikholeslami M, Rokni HB. Simulation of nanofluid heat transfer in presence of magnetic field: a review. Int J Heat Mass Transfer 2017;115:1203–33.
[59] Sheikholeslami M, Sadoughi MK. Simulation of CuO-water nanofluid heat transfer enhancement in presence of melting surface. Int J Heat Mass Transfer 2018;116:909–19.