Finite-Time Partial Stability, Stabilization, Semistabilization, and Optimal Feedback Control

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Finite-Time Partial Stability, Stabilization, Semistabilization, and Optimal Feedback Control

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To Aquero, for her eternal, true, and unconditional love

STD

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Summary

Asymptotic stability is a key notion of system stability for uncontrolled and controlled dynamical systems as it guarantees that the system trajectories are bounded in a neighborhood of a given *isolated* equilibrium point and converge to this equilibrium over the *infinite* horizon. In some applications, however, asymptotic stability is not the appropriate notion of stability. For example, for systems with a continuum of equilibria, every neighborhood of an equilibrium contains another equilibrium and a nonisolated equilibrium cannot be asymptotically stable. Alternatively, in stabilization of spacecraft dynamics via gimballed gyroscopes, it is desirable to find state- and output-feedback control laws that guarantee partial-state stability of the closed-loop system, that is, stability with respect to part of the system state. Furthermore, we may additionally require finite-time stability of the closed-loop system, that is, convergence of the system's trajectories to a Lyapunov stable equilibrium in finite time. In this dissertation, we provide state-feedback control laws that minimize nonlinear-nonquadratic performance criteria and guarantee *semistability*, *partial-state stability*, *finite-time stability*, and *finite-time partial state stability* of the closed-loop system.

The state feedback linear-quadratic optimal control problem for asymptotic stabilization has been extensively studied in the literature. In this dissertation, the optimal linear and nonlinear control problem is extended to address a weaker version of closed-loop stability, namely, semistability, which involves convergent trajectories and Lyapunov stable equilibria and which is of paramount importance for consensus control of network dynamical systems. Specifically, we show that the optimal semistable state-feedback controller can be solved

using a form of the Hamilton-Jacobi-Bellman conditions that does not require the cost-to-go function to be sign definite. This result is then used to solve the optimal linear-quadratic regulator problem using a Riccati equation approach.

Using dissipativity theory, we develop a thermodynamic framework for semistabilization of linear and nonlinear dynamical systems. The proposed framework unifies system thermodynamic concepts with feedback dissipativity and control theory to provide a thermodynamic-based semistabilization framework for feedback control design. Specifically, we consider feedback passive and dissipative systems since these systems are not only widespread in systems and control, but also have clear connections to thermodynamics. In addition, we define the notion of entropy for a nonlinear feedback dissipative dynamical system. Then, we develop a state feedback control design framework that minimizes the time-averaged system entropy and show that, under certain conditions, this controller also minimizes the time-averaged system energy. The main result is cast as an optimal control problem characterized by an optimization problem involving two linear matrix inequalities.

The singular optimal control problem for asymptotic stabilization of linear and nonlinear dynamical systems has been extensively studied in the literature. As part of this dissertation, the singular control problem is extended to address a weaker version of closed-loop stability, namely, semistability. Specifically, we exploit the properties of minimum phase and nonminimum phase, right invertible dynamical systems to solve the singular control problem for linear semistabilization. Furthermore, three approaches are presented to address the nonlinear semistable singular control problem. Namely, a singular perturbation method is presented to construct a state-feedback singular controller that guarantees closed-loop semistability for nonlinear systems. For this method, we show that for a nonnegative cost-to-go function the minimum cost of a nonlinear semistabilizing singular controller is lower than the minimum cost of a singular controller that guarantees asymptotic stability of the closed-loop system. Alternatively, we solve the nonlinear semistable singular control problem by using the cost-to-go function to cancel the singularities in the corresponding

Hamilton-Jacobi-Bellman equation. For this method, we show that the minimum value of the singular performance measure is zero. Finally, we provide a solution to the singular semistabilization problem using differential geometric methods and the concepts of output-feedback linearization and feedback equivalence. Specifically, we construct an output-feedback linearizing controller and find the control parameters that solve the optimal singular control problem for semistabilization of the linearized system. Also for this method, we show that the minimum value of the singular performance measure is zero.

Finally, we develop a unified framework to address the problem of optimal nonlinear analysis and feedback control for partial stability and partial-state stabilization. Partial asymptotic stability of the closed-loop nonlinear system is guaranteed by means of a Lyapunov function that is positive definite and decrescent with respect to part of the system state which can clearly be seen to be the solution to the steady-state form of the Hamilton-Jacobi-Bellman equation, and hence, guaranteeing both partial stability and optimality. The overall framework provides the foundation for extending optimal linear-quadratic controller synthesis to nonlinear-nonquadratic optimal partial-state stabilization. Connections to optimal linear and nonlinear regulation for linear and nonlinear time-varying systems with quadratic and nonlinear-nonquadratic cost functionals are also provided. Finally, we also develop optimal feedback controllers for affine nonlinear systems using an inverse optimality framework tailored to the partial state-stabilization problem and use this result to address polynomial and multilinear forms in the performance criterion.

Finite-time stability involves dynamical systems whose trajectories converge to an equilibrium state in finite time. Since finite-time convergence implies nonuniqueness of system solutions in reverse time, such systems possess non-Lipschitzian dynamics. Sufficient conditions for finite-time stability have been developed in the literature using continuous Lyapunov functions. In this dissertation, we develop a framework for addressing the problem of optimal nonlinear analysis and feedback control for finite-time stability and finite-time stabilization. Finite-time stability of the closed-loop nonlinear system is guaranteed by means of a Lyapunov function that satisfies a differential inequality involving fractional powers. This Lyapunov function can clearly be seen to be the solution to a partial differential equation that corresponds to a steady-state form of the Hamilton-Jacobi-Bellman equation, and hence, guaranteeing both finite-time stability and optimality.

In this dissertation, we extend the framework developed for optimal partial-state and finite-time stabilization to address the problem of optimal *finite-time partial-state stabilization*, that is, the problem of finding state-feedback control laws that minimize a given performance measure and guarantee partial-state finite-time stability of the closed-loop system. Even though finite-time stabilization and partial-state stabilization have been considered in the literature as separate problems as well as a combined problem, the problem of *optimal finite-time*, *partial-state stabilization* has not been addressed in the literature. As for the optimal partial-state and finite-time stabilization problems, we consider a notion of optimality that is directly related to a given Lyapunov function that is positive definite and decrescent with respect to part of the system state, and satisfies a differential inequality involving fractional powers. Specifically, an *optimal* finite-time, partial-state stabilization control problem is stated and sufficient Hamilton-Jacobi-Bellman conditions are used to characterize an optimal feedback controller. The steady-state solution of the Hamilton-Jacobi-Bellman equation is clearly shown to be a Lyapunov function for part of the closed-loop system state that guarantees both finite-time partial stability and optimality. In addition, we explore connections of our approach with inverse optimal control, wherein we parametrize a family of finite-time, partial-state stabilizing sublinear controllers that minimize a derived cost functional involving subquadratic terms. Lastly, we exploit the unification between time-invariant partial-stability theory and stability theory for time-varying systems to address the problem of optimal finite-time control for nonlinear time-varying dynamical systems.

Chapter 1

Introduction

1.1. Motivation and Goals

Dynamical systems theory involves the analysis and synthesis of feedback controllers that manipulate system inputs to obtain a desired effect on the output of the system in the face of system uncertainty and system disturbances. Asymptotic stability of controlled dynamical systems guarantees that the closed-loop system trajectories are bounded in the neighborhood of a given *isolated* equilibrium point and converge to this equilibrium over the *infinite* horizon. However, for some applications, this notion of stability is not appropriate. This would be the case for systems having a *continuum* of equilibria. In this case, since every neighborhood of a nonisolated equilibrium contains another equilibrium, a nonisolated equilibrium cannot be asymptotically stable.

A unique feature of the closed-loop dynamics under any control algorithm that achieves consensus in dynamic networks is the existence of a continuum of equilibria representing a desired state of consensus [57, 60]. Under such dynamics, the desired limiting state is not determined completely by the closed-loop system dynamics, but also depends on the initial system state as well [44, 56, 57, 60]. From a practical viewpoint, it is not sufficient to only guarantee that a network converges to a state of consensus since steady-state convergence is not sufficient to guarantee that small perturbations from the limiting state will lead to only small transient excursions from the state of consensus. It is also necessary to guarantee that the equilibrium states representing consensus are Lyapunov stable, and consequentially, *semistable*.

Another important problem in stability theory is the notion of *partial stability*, that is, stability with respect to part of the system's state. Additionally, *partial-state stabilization*, that is, closed-loop stability with respect to part of the closed-loop system's state, also arises in many engineering applications [88, 113]. Specifically, in spacecraft stabilization via gimballed gyroscopes asymptotic stability of an equilibrium position of the spacecraft is sought while requiring Lyapunov stability of the axis of the gyroscope relative to the spacecraft [113]. Alternatively, in the control of rotating machinery with mass imbalance, spin stabilization about a nonprincipal axis of inertia requires motion stabilization with respect to a subspace instead of the origin [88]. Perhaps the most common application where partial stabilization is necessary is adaptive control, wherein asymptotic stability of the closed-loop plant states is guaranteed without necessarily achieving parameter error convergence. The need to consider partial stability of the closed-loop system in the aforementioned systems arises from the fact that stability notions involve equilibrium coordinates as well as a manifold of coordinates that is closed but *not* compact. Hence, partial stability involves motion lying in a subspace instead of an equilibrium point.

The notion of asymptotic stability in dynamical systems theory implies convergence of the system trajectories to an equilibrium state over the infinite horizon. In many applications, however, it is desirable that a dynamical system possesses the property that trajectories that converge to a Lyapunov stable equilibrium state must do so in *finite time* rather than merely asymptotically. Most of the existing control techniques in the literature ensure that the closed-loop system dynamics of a controlled system are Lipschitz continuous, which implies uniqueness of system solutions in forward and backward times. Hence, convergence to an equilibrium state is achieved over an infinite time interval.

The Hamilton-Jacobi-Bellman optimal control framework provides necessary and sufficient conditions for the existence of state-feedback controllers that minimize a given performance measure and guarantee asymptotic stability of the closed-loop system [4]. In [6] the current status of continuous-time, nonlinear nonquadratic optimal control problems was presented in a simplified and tutorial manner. The basic underlying ideas of the results in [6] are based on the fact that the steady-state solution of the Hamilton-Jacobi-Bellman equation is a Lyapunov function for the nonlinear system and thus guaranteeing both stability and optimality [6,38]. Specifically, a feedback control problem over an infinite horizon involving a nonlinear-nonquadratic performance functional is considered. The performance functional is then evaluated in closed form as long as the nonlinear nonquadratic cost functional considered is related in a specific way to an underlying Lyapunov function that guarantees asymptotic stability of the nonlinear closed-loop system. This Lyapunov function is shown to be the solution of the steady-state Hamilton-Jacobi-Bellman equation. The overall framework provides the foundation for extending linear-quadratic control to nonlinear-nonquadratic problems.

In this dissertation, we extend the framework developed in [6] and [38] to address the problem of *optimal semistabilization*, *partial-state stabilization*, *finite-time stabilization*, and *finite-time partial-state stabilization*, that is, the problem of finding state-feedback control laws that minimize a given performance measure and guarantee semistability, partial state stability, finite-time stability, and finite-time partial state state stability of the closed-loop system. Furthermore, we apply the framework developed to solve the optimal semistabilization problem to address the singular control problem for linear and nonlinear semistabilization.

1.2. Brief Outline of the Dissertation

The contents of this disseration are as follows. In Chapter 2, we address the problem of finding state-feedback control laws that minimize a performance measure in integral form and guarantee semistability of linear and nonlinear dynamical systems. In Chapter 3, we develop a thermodynamic framework for semistabilization of linear and nonlinear dynamical systems. The proposed framework unifies system thermodynamic concepts with feedback dissipativity and control theory to provide a thermodynamic-based semistabilization framework for feedback control design. In Chapter 4, we apply the optimal semistabilization framework developed in Chapter 2, and exploit the properties of minimum phase and nonminimum phase, as well as right invertible dynamical systems to solve the singular control problem for linear semistabilization. In Chapter 5, we provide three approaches to address the nonlinear semistable singular control problem. Specifically, we construct state-feedback singular controllers that guarantee closed-loop semistabilization for nonlinear systems applying a singular perturbation method, using the results proven in Chapter 2, and using differential geometric methods. In Chapter 6, we address the problem of optimal partial-state stabilization, whereas in Chapter 7 we address the problem of optimal finite-time stabilization. Finally, in Chapter 8 we develop sufficient conditions to solve the optimal control problem for state-feedback, finite-time, partial state stabilization, and in Chapter 9 we discuss future extensions of the research.

Chapter 2

Optimal Control for Linear and Nonlinear Semistabilization

2.1. Introduction

A form of stability that lies between Lyapunov stability and asymptotic stability is *semistability* [13,16], that is, the property whereby every trajectory that starts in a neighborhood of a Lyapunov stable equilibrium converges to a (possibly different) Lyapunov stable equilibrium. Semistability implies Lyapunov stability, and is implied by asymptotic stability [13, 16, 38]. This notion of stability arises naturally in systems having a continuum of equilibria and includes such systems as mechanical systems having rigid body modes, chemical reaction systems [22], compartmental systems [39, 41], and isospectral matrix dynamical systems. Semistability also arises naturally in dynamical network systems [57, 60, 96], which cover a broad spectrum of applications including cooperative control of unmanned air vehicles, autonomous underwater vehicles, distributed sensor networks, air and ground transportation systems, swarms of air and space vehicle formations, and congestion control in communication networks, to cite but a few examples.

A unique feature of the closed-loop dynamics under any control algorithm that achieves consensus in dynamic networks is the existence of a continuum of equilibria representing a desired state of consensus [57, 60]. Under such dynamics, the desired limiting state is not determined completely by the closed-loop system dynamics, but depends on the initial system state as well [44, 56, 57, 60]. From a practical viewpoint, it is not sufficient to only guarantee that a network converges to a state of consensus since steady-state convergence is not sufficient to guarantee that small perturbations from the limiting state will lead to only small transient excursions from the state of consensus. It is also necessary to guarantee that the equilibrium states representing consensus are Lyapunov stable, and consequentially, semistable.

In [44,56], the authors develop $\mathcal{H}_2$ optimal semistable control theory for linear dynamical systems. Specifically, unlike the standard $\mathcal{H}_2$ optimal control problem, it is shown in [44,56] that a complicating factor of the $\mathcal{H}_2$ optimal semistable stabilization problem is that the closed-loop Lyapunov equation guaranteeing semistability can admit multiple solutions. In addition, the authors show that the $\mathcal{H}_2$ optimal solution is given by a least squares solution to the closed-loop Lyapunov equation over all possible semistabilizing solutions. Moreover, it is shown that this least squares solution can be characterized by a linear matrix inequality minimization problem.

In this chapter, we address the problem of finding a state-feedback nonlinear control law $u = \phi(x)$ that minimizes the performance measure

$$J(x_0, u(\cdot)) \triangleq \int_0^\infty L(x(t), u(t))\mathrm{d}t \tag{2.1}$$

and guarantees semistability of the nonlinear dynamical system

$$\dot{x}(t) = F(x(t), u(t)), \qquad x(0) = x_0, \qquad t \geq 0, \tag{2.2}$$

$$y(t) = H(x(t), u(t)), \tag{2.3}$$

where, for every $t \geq 0$, $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, $\mathcal{D}$ is an open set, $u(t) \in U \subseteq \mathbb{R}^m$, $y(t) \in Y \subseteq \mathbb{R}^l$, $L : \mathcal{D} \times U \to \mathbb{R}$, $F : \mathcal{D} \times U \to \mathbb{R}^n$ is Lipschitz continuous in x and u on $\mathcal{D} \times U$, and $H : \mathcal{D} \times U \to Y$. Specifically, our approach focuses on the role of the Lyapunov function guaranteeing semistability of (2.2) with a feedback control law $u = \phi(x)$, and we provide sufficient conditions for optimality in a form that corresponds to a steady-state version of a Hamilton-Jacobi-Bellman-type equation.

In addition, we provide sufficient conditions for the existence of a feedback gain $K \in \mathbb{R}^{m\times n}$ such that the state feedback control law $u = Kx$ minimizes the quadratic performance measure

$$J(x_0, u(\cdot)) = \int_0^\infty [(x(t) - x_\mathrm{e})^\mathrm{T} C^\mathrm{T} C (x(t) - x_\mathrm{e}) + (u(t) - u_\mathrm{e})^\mathrm{T} R_2 (u(t) - u_\mathrm{e})]\mathrm{d}t \tag{2.4}$$

and guarantees semistability of the linear dynamical system

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0, \qquad t \geq 0, \tag{2.5}$$

$$y(t) = Cx(t), \tag{2.6}$$

where $u_\mathrm{e} \triangleq Kx_\mathrm{e}$, $x_\mathrm{e} \triangleq \lim_{t\to\infty} x(t)$, R_2 is positive definite, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{l\times n}$. The proposed Riccati equation-based framework for optimal linear semistable stabilization presented in this chapter is different from the framework presented in [44, 56] using linear matrix inequalities.

The contents of the chapter are as follows. In Section 2.2, we establish notation, definitions, and develop some key results on semistability, semicontrollability, semiobservability, and semistabilization. In Section 2.3, we consider a nonlinear system with a performance functional evaluated over the infinite horizon. The performance functional is then evaluated in terms of a Lyapunov function that guarantees semistability. This result is then specialized to the linear-quadratic case. We then, in Section 2.4, state an optimal control problem and provide sufficient conditions for characterizing an optimal nonlinear feedback controller guaranteeing semistable stabilization. Finally, Section 2.5 presents two application design examples of optimal semistable control involving optimal consensus control for multiagent systems and a nonlinear mechanical system involving an eccentric rotational inertia on a translational oscillator.

2.2. Notation, Definitions, and Mathematical Preliminaries

The notation used in this dissertation is fairly standard. Specifically, $\mathbb{R}$ (resp., $\mathbb{C}$) denotes the set of real (resp., complex) numbers, $\mathbb{R}_+$ denotes the set of positive real numbers, $\overline{\mathbb{R}}_+$ denotes the set of nonnegative numbers, $\mathbb{R}^n$ (resp., $\mathbb{C}^n$) denotes the set of $n \times 1$ real (resp., complex) column vectors, $\mathbb{R}^{n\times m}$ denotes the set of $n \times m$ real matrices, and $\mathbb{S}^n$ denotes the set of $n \times n$ real symmetric matrices. In addition, $\mathbb{R}_{\mathrm{prop}}(s)$ denotes the set of proper rational transfer functions with coefficients in $\mathbb{R}$ and $\mathbb{R}_{\mathrm{prop}}^{l\times m}(s)$ denotes the set of $l \times m$ matrices with entries in $\mathbb{R}_{\mathrm{prop}}(s)$.

Furthermore, we write $V'(x) \triangleq \frac{\partial V(x)}{\partial x}$ for the Fréchet derivative of V at x, $\|\cdot\|$ for the Euclidean vector norm, $\|\cdot\|_{\mathrm{F}}$ for the Frobenius matrix norm, $\mathcal{S}^{\perp}$ for the orthogonal complement of a set $\mathcal{S}$, $\operatorname{span}\mathcal{S}$ for the span of the set $\mathcal{S}$, $\mathcal{R}(A)$ and $\mathcal{N}(A)$ for the range space and the null space of a matrix A, respectively, $\operatorname{spec}(A)$ for the spectrum of the square matrix A including multiplicity, $\det A$ for the determinant of the square matrix A, $\operatorname{tr}(\cdot)$ for the trace operator, $\operatorname{rank} A$ for the rank of the matrix A, $(\cdot)^{\mathrm{T}}$ denotes transpose, and $(\cdot)^{\#}$ for the group generalized inverse.

Finally, I_n or I denotes the $n\times n$ identity matrix, $0_{n\times m}$ or 0 for the zero $n\times m$ matrix, $A \geq 0$ (resp., $A > 0$) denotes the fact that the Hermitian matrix A is nonnegative (respectively, positive) definite, $\mathbf{e}$ denotes the ones vector of order n, that is, $\mathbf{e} = [1, \ldots, 1]^{\mathrm{T}}$, $\mathbf{e} \in \mathbb{R}^n$, $\otimes$ denotes the Kronecker product, $\oplus$ denotes the Kronecker sum, $\operatorname{vec}(\cdot)$ denotes the column stacking operator, and $\operatorname{vec}^{-1}(\cdot)$ denotes the inverse vec operator.

Consider the nonlinear dynamical system given by

$$\dot{x}(t) = f(x(t)), \qquad x(0) = x_0, \qquad t \geq 0, \tag{2.7}$$

where, for every $t \geq 0$, $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$ and $f : \mathcal{D} \to \mathbb{R}^n$ is locally Lipschitz continuous on $\mathcal{D}$. The solution of (2.7) with initial condition $x(0) = x$ defined on $[0, \infty)$ is denoted by $s(\cdot, x)$. The above assumptions imply that the map $s : [0, \infty) \times \mathcal{D} \to \mathcal{D}$ is continuous [52, Th.

2.1], satisfies the *consistency* property $s(0,x) = x$, and possesses the *semigroup* property $s(t, s(\tau, x)) = s(t+\tau, x)$ for all $t, \tau \geq 0$ and $x \in \mathcal{D}$. Given $t \geq 0$ and $x \in \mathcal{D}$, we denote the map $s(t,\cdot) : \mathcal{D} \to \mathcal{D}$ by s_t and the map $s(\cdot, x) : [0,\infty) \to \mathcal{D}$ by s^x.

The *orbit* $\mathcal{O}_x$ of a point $x \in \mathcal{D}$ is the set $s^x([0,\infty))$. A set $\mathcal{D}_{\rm p} \subseteq \mathcal{D}$ is *positively invariant* relative to (2.7) if $s_t(\mathcal{D}_{\rm p}) \subseteq \mathcal{D}_{\rm p}$ for all $t \geq 0$ or, equivalently, $\mathcal{D}_{\rm p}$ contains the orbits of all its points. The set $\mathcal{D}_{\rm p}$ is *invariant* relative to (2.7) if $s_t(\mathcal{D}_{\rm p}) = \mathcal{D}_{\rm p}$ for all $t \geq 0$. The *positive limit set* of $x \in \mathbb{R}^n$ is the set $\omega(x)$ of all subsequential limits of sequences of the form $\{s(t_i, x)\}_{i=0}^{\infty}$, where $\{t_i\}_{i=0}^{\infty}$ is an increasing divergent sequence in $[0,\infty)$. Recall that, for every $x \in \mathbb{R}^n$ that has bounded orbits, $\omega(x)$ is nonempty and compact, and, for every neighborhood $\mathcal{N}$ of $\omega(x)$, there exists $T > 0$ such that $s_t(x) \in \mathcal{N}$ for every $t > T$ [38, Ch. 2]. If $\mathcal{D}_{\rm p} \subset \mathcal{D}$ is positively invariant and closed, then $\omega(x) \subseteq \mathcal{D}_{\rm p}$ for all $x \in \mathcal{D}_{\rm p}$. In addition, $\lim_{t\to\infty} s(t,x)$ exists if and only if $\omega(x)$ is a singleton. Finally, the set of equilibrium points of (2.7) is denoted by $f^{-1}(0) \triangleq \{x \in \mathcal{D} : f(x) = 0\}$.

The following definition is needed.

Definition 2.1 [38]. Let $\mathcal{D} \subseteq \mathbb{R}^n$ be an open positively invariant set with respect to (2.7). An equilibrium point $x_{\rm e} \in \mathcal{D}$ of (2.7) is *semistable* with respect to $\mathcal{D}$ if $x_{\rm e}$ is Lyapunov stable and there exists an open subset $\mathcal{D}_0$ of $\mathcal{D}$ containing $x_{\rm e}$ such that, for all initial conditions in $\mathcal{D}_0$, the solutions of (2.7) converge to a Lyapunov stable equilibrium point. The system (2.7) is *semistable* with respect to $\mathcal{D}$ if every solution with initial condition in $\mathcal{D}$ converges to a Lyapunov stable equilibrium. Finally, (2.7) is said to be *globally semistable* if (2.7) is semistable with respect to $\mathbb{R}^n$.

Note that if, for $\varepsilon > 0$, $\mathcal{B}_\varepsilon(x_{\rm e}) \cap f^{-1}(0) = \{x_{\rm e}\}$ is a singleton, where $\mathcal{B}_\varepsilon(x_{\rm e})$ denotes the *open ball centered* at $x_{\rm e}$ with *radius* ε, then Definition 2.1 reduces to the definitions of local and global asymptotic stability. Recall that for $B = 0$, (2.5) is semistable if and only if $\mathrm{spec}(A) \subset \{s \in \mathbb{C} : \mathrm{Re}\, s < 0\} \cup \{0\}$ and, if $0 \in \mathrm{spec}(A)$, then 0 is semisimple [8, Def. 11.8.1]. In this case, we say that A is *semistable.* Furthermore, if A is semistable, then the index of A is zero or one, and hence, A is group invertible. The group inverse $A^{\#}$ of A is a special case of the Drazin inverse A^{D} in the case where A has index zero or one [8, p. 369]. In this case, for every $x_0 \in \mathbb{R}^n$, $x_{\mathrm{e}} = \lim_{t\to\infty} x(t) = (I_n - AA^{\#})x_0$ or, equivalently, $\lim_{t\to\infty} e^{At} = I_n - AA^{\#}$ [8, Prop. 11.8.1].

Lemma 2.2 [38, Prop. 4.7]. Consider the nonlinear dynamical system (2.7) and let $x \in \mathbb{R}^n$. If the positive limit set of (2.7) contains a Lyapunov stable equilibrium point y with respect to $\mathcal{D}$, then $y = \lim_{t\to\infty} s(t,x)$, that is, $\omega(x) = \{y\}$.

Next, we introduce the definitions of semicontrollability and semiobservability for linear systems.

Definition 2.3 [62]. Consider the system given by (2.5). The pair (A, B) is *semicontrollable* if

$$\sum_{i=1}^{n} \mathcal{R}(A^{i-1}B) = \mathcal{R}(A), \tag{2.8}$$

where $A^0 \triangleq I_n$ and, for the given sets $\mathcal{S}_1$ and $\mathcal{S}_2$, $\mathcal{S}_1 + \mathcal{S}_2 \triangleq \{x + y : x \in \mathcal{S}_1, y \in \mathcal{S}_2\}$ denotes the Minkowski sum.

The following lemma is needed to connect semicontrollability to the classical notion of controllability involving the existence of a continuous control $u : [0, t_{\mathrm{f}}] \to \mathbb{R}^m$ such that the solution $x(\cdot)$ of (2.5) with $x(0) = x_0$ satisfies $x(t_{\mathrm{f}}) = 0$.

Lemma 2.4. Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. Then

$$\sum_{i=1}^{n} \mathcal{R}(A^{i-1}B) \subseteq \mathcal{R}(A) \cup \mathcal{R}(B). \tag{2.9}$$

Proof: It follows from Fact 2.9.16 of [8, p. 121] that

$$\left(\sum_{i=1}^{n} \mathcal{R}(A^{i-1}B)\right)^{\perp} = \bigcap_{i=1}^{n} \mathcal{R}(A^{i-1}B)^{\perp}. \tag{2.10}$$

Moreover, it follows from Theorem 2.4.3 of [8, p. 103] that $\mathcal{R}(A^{i-1}B)^{\perp} = \mathcal{N}(B^{\mathrm{T}}(A^{\mathrm{T}})^{i-1})$ for every $i \in \{1, \ldots, n\}$. Since $\mathcal{N}(A^{\mathrm{T}}) \subseteq \mathcal{N}(B^{\mathrm{T}}(A^{\mathrm{T}})^{i})$ for every $i \in \{1, \ldots, n-1\}$, it follows that

$$\mathcal{N}(A^{\mathrm{T}}) \cap \mathcal{N}(B^{\mathrm{T}}) \subseteq \bigcap_{i=1}^{n} \mathcal{N}(B^{\mathrm{T}}(A^{\mathrm{T}})^{i-1}). \tag{2.11}$$

Now, by Theorem 2.4.3 of [8, p. 103], $\mathcal{R}(A)^{\perp} = \mathcal{N}(A^{\mathrm{T}})$, and hence, it follows from (2.11) and (2.10) that

$$\mathcal{R}(A)^{\perp} \cap \mathcal{R}(B)^{\perp} \subseteq \bigcap_{i=1}^{n} \mathcal{R}(A^{i-1}B)^{\perp} = \left(\sum_{i=1}^{n} \mathcal{R}(A^{i-1}B) \right)^{\perp}. \tag{2.12}$$

Next, it follows from Fact 2.9.16 of [8, p. 121] that $\mathcal{R}(A)^{\perp} \cap \mathcal{R}(B)^{\perp} = (\mathcal{R}(A) + \mathcal{R}(B))^{\perp}$, and hence, by (2.12) and Fact 2.9.14 of [8, p. 121],

$$\sum_{i=1}^{n} \mathcal{R}(A^{i-1}B) \subseteq \mathcal{R}(A) + \mathcal{R}(B). \tag{2.13}$$

Finally, it follows from Fact 2.9.11 of [8, p. 121] that $\mathcal{R}(A) + \mathcal{R}(B) = \mathcal{R}(A) \cup \mathcal{R}(B)$, which proves (2.9). ■

Recall that the *controllable subspace* $\mathcal{C}_{t_{\mathrm{f}}}(A, B)$ at time $t_{\mathrm{f}} > 0$ is the subspace

$$\mathcal{C}_{t_{\mathrm{f}}}(A, B) = \{x_{\mathrm{f}} \in \mathbb{R}^n : \text{there exists a continuous control } u : [0, t_{\mathrm{f}}] \to \mathbb{R}^m \text{ such that the solution } x(\cdot) \text{ of (2.5) with } x(0) = x_0 \text{ satisfies } x(t_{\mathrm{f}}) = x_{\mathrm{f}}\}.$$

Furthermore, recall that $\mathcal{C}_{t_{\mathrm{f}}}(A, B)$ is independent of t_{f}, and hence, we write $\mathcal{C}(A, B)$ for $\mathcal{C}_{t_{\mathrm{f}}}(A, B)$, and call $\mathcal{C}(A, B)$ the *controllable subspace* of (A, B) [8].

The next result characterizes semicontrollability in several equivalent ways.

Theorem 2.5. The following statements are equivalent:

i) (A, B) is semicontrollable.

ii) $\mathcal{R}(\int_0^{t_{\mathrm{f}}} e^{At} BB^{\mathrm{T}} e^{A^{\mathrm{T}}t} \mathrm{d}t) = \mathcal{R}(A)$.

iii) $\mathcal{C}(A,B)=\mathcal{R}(A)$.

iv) $\mathcal{R}(B)\subseteq\mathcal{R}(A)$ and, for every $x_0\in\mathcal{R}(A)$, there exists a continuous control $u:[0,t_{\rm f}]\to\mathbb{R}^m$ such that the solution $x(t)$ of (2.5) with $x(0)=x_0$ satisfies $x(t_{\rm f})=0$.

Proof: First, note that it follows from Lemma 12.6.2 of [8, p. 808] that

$$\mathcal{C}(A,B)=\mathcal{R}\left(\int_0^{t_{\rm f}} e^{At}BB^{\rm T}e^{A^{\rm T}t}{\rm d}t\right)=\sum_{i=1}^{n}\mathcal{R}(A^{i-1}B). \tag{2.14}$$

To show the equivalence of *i)* and *ii)*, note that if (A,B) is semicontrollable, then it follows from (2.8) and (2.14) that $\mathcal{R}(\int_0^{t_{\rm f}} e^{At}BB^{\rm T}e^{A^{\rm T}t}{\rm d}t)=\mathcal{R}(A)$ holds. Conversely, if $\mathcal{R}(\int_0^{t_{\rm f}} e^{At}B\cdot B^{\rm T}e^{A^{\rm T}t}{\rm d}t)=\mathcal{R}(A)$ holds, then it follows from (2.14) that (2.8) holds. Hence, by definition, (A,B) is semicontrollable.

Next, it follows from (2.14) that *ii)* holds if and only if *iii)* holds, which shows the equivalence of *ii)* and *iii)*.

To show that *iv)* implies *i)* note that, for every $x_0\in\mathcal{R}(A)$, (2.5) satisfies

$$x(t)=e^{At}x_0+\int_0^t e^{A(t-s)}Bu(s){\rm d}s,\qquad t\geq 0. \tag{2.15}$$

Furthermore, note that e^{At} is nonsingular for every $t\geq 0$. Now, since $x(t_{\rm f})=0$, it follows from (2.15) that, for every $x_0\in\mathcal{R}(A)$,

$$x_0=-\int_0^{t_{\rm f}} e^{-As}Bu(s){\rm d}s. \tag{2.16}$$

Using the Cayley-Hamilton theorem, it follows that there exist $\alpha_j(t)\in\mathbb{R}$, $j=0,\ldots,n-1$, such that

$$e^{-At}=\sum_{j=0}^{n-1}\alpha_j(t)A^j,\qquad t\geq 0. \tag{2.17}$$

Next, define z_j, $j=0,\ldots,n-1$, by

$$z_j\triangleq-\int_0^{t_{\rm f}}\alpha_j(s)u(s){\rm d}s.$$

Substituting z_j and (2.17) into (2.16) yields

$$x_0 = -\int_0^{t_\mathrm{f}} e^{-As} Bu(s)\mathrm{d}s = -\int_0^{t_\mathrm{f}} \sum_{j=0}^{n-1} \alpha_j(s) A^j Bu(s)\mathrm{d}s = \sum_{j=0}^{n-1} A^j B z_j \in \sum_{j=0}^{n-1} \mathcal{R}(A^j B)$$

for every $x_0 \in \mathcal{R}(A)$. Hence, $\mathcal{R}(A) \subseteq \sum_{j=0}^{n-1} \mathcal{R}(A^j B)$. Now, it follows from Lemma 2.4 and $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ that $\sum_{j=0}^{n-1} \mathcal{R}(A^j B) \subseteq \mathcal{R}(A) \cup \mathcal{R}(B) = \mathcal{R}(A)$. Consequently, $\sum_{j=0}^{n-1} \mathcal{R}(A^j B) = \mathcal{R}(A)$, and hence, by definition, (A, B) is semicontrollable.

To show *i*) implies *iv*), assume that (A, B) is semicontrollable. Then it follows from (2.8) that $\mathcal{R}(B) \subseteq \mathcal{R}(A)$. Let $x_0 \in \mathcal{R}(A)$ so that there exists $y \in \mathbb{R}^n$ such that $x_0 = Ay$. Next, construct the continuous control $u : [0, t_\mathrm{f}] \to \mathbb{R}^m$ as

$$u(t) = -B^\mathrm{T} e^{A^\mathrm{T}(t_\mathrm{f}-t)} W_\mathrm{c}^+ e^{At_\mathrm{f}} x_0, \tag{2.18}$$

where $W_\mathrm{c} \triangleq \int_0^{t_\mathrm{f}} e^{At} BB^\mathrm{T} e^{A^\mathrm{T} t}\mathrm{d}t$ and X^+ denotes the Moore-Penrose generalized inverse of X. Then the solution $x(t)$ of (2.5) with $x(0) = x_0$ satisfies

$$\begin{aligned} x(t_\mathrm{f}) &= e^{At_\mathrm{f}} x_0 + \int_0^{t_\mathrm{f}} e^{A(t_\mathrm{f}-s)} Bu(s)\mathrm{d}s \\ &= e^{At_\mathrm{f}} x_0 - \int_0^{t_\mathrm{f}} e^{A(t_\mathrm{f}-s)} BB^\mathrm{T} e^{A^\mathrm{T}(t_\mathrm{f}-s)} W_\mathrm{c}^+ e^{At_\mathrm{f}} x_0 \mathrm{d}s \\ &= e^{At_\mathrm{f}} x_0 - \int_0^{t_\mathrm{f}} e^{At} BB^\mathrm{T} e^{A^\mathrm{T} t}\mathrm{d}t W_\mathrm{c}^+ e^{At_\mathrm{f}} x_0 \\ &= e^{At_\mathrm{f}} x_0 - W_\mathrm{c} W_\mathrm{c}^+ e^{At_\mathrm{f}} x_0 \\ &= (I_n - W_\mathrm{c} W_\mathrm{c}^+) e^{At_\mathrm{f}} x_0 \\ &= (I_n - W_\mathrm{c} W_\mathrm{c}^+) e^{At_\mathrm{f}} Ay \\ &= (I_n - W_\mathrm{c} W_\mathrm{c}^+) A e^{At_\mathrm{f}} y, \end{aligned} \tag{2.19}$$

where we used the fact that $Ae^{At_\mathrm{f}} = e^{At_\mathrm{f}} A$. Now, it follows from *vi*) of Proposition 6.1.6 of [8, p. 399] that $\mathcal{R}(W_\mathrm{c}) = \mathcal{N}(I_n - W_\mathrm{c} W_\mathrm{c}^+)$. In addition, it follows from *i*) $\Rightarrow$ *ii*) that $\mathcal{R}(A) = \mathcal{R}(W_\mathrm{c}) = \mathcal{N}(I_n - W_\mathrm{c} W_\mathrm{c}^+)$, which implies $Ae^{At_\mathrm{f}} y \in \mathcal{R}(A)$. Consequently, $(I_n - W_\mathrm{c} W_\mathrm{c}^+) A e^{At_\mathrm{f}} y = 0$, and hence, by (2.19), $x(t_\mathrm{f}) = 0$. ■

It follows from Theorem 2.5 that semicontrollability of the linear controlled system (2.5) implies the existence of a continuous control input such that the solution $x(\cdot)$ of (2.5) can be driven to the origin in finite-time for every initial condition in the range space of system matrix A.

The following proposition is needed for some of the key results in this chapter.

Proposition 2.6. Consider the dynamical system given by (2.5). Then (2.8) holds if and only if

$$\left[\bigcap_{k=1}^{n}\mathcal{N}(B^{\mathrm{T}}(A^{k-1})^{\mathrm{T}})\right]^{\perp}=[\mathcal{N}(A^{\mathrm{T}})]^{\perp}, \tag{2.20}$$

holds. Furthermore, (2.20) is equivalent to

$$\operatorname{span}\left\{\bigcup_{k=1}^{n}\mathcal{R}(A^{k-1}B)\right\}=\mathcal{R}(A). \tag{2.21}$$

Proof: First we show that $\sum_{i=1}^{n}\mathcal{R}\left(A^{i-1}B\right)=\operatorname{span}\Big(\bigcup_{i=1}^{n}\mathcal{R}(A^{i-1}B)\Big)$. Note that, for every $i\in\{1,\ldots,n\}$, $\mathcal{R}(A^{i-1}B)$ is a subspace of $\mathbb{R}^n$, and hence, by Fact 2.9.13 of [8, p. 121], the above equality holds. Now, it follows from (2.8) that (A,B) is semicontrollable if and only if (2.21) holds. Finally, to show that (2.8) is equivalent to (2.20), note that it follows from Equation (2.4.14) of [8, p. 103] that $[\mathcal{N}(A^{\mathrm{T}})]^{\perp}=\mathcal{R}(A)$. Hence, by Fact 2.9.16 of [8, p. 121], (2.20) holds if and only if $\sum_{i=1}^{n}[\mathcal{N}(B^{\mathrm{T}}(A^{i-1})^{\mathrm{T}})]^{\perp}=\sum_{i=1}^{n}\mathcal{R}(A^{i-1}B)=[\mathcal{N}(A^{\mathrm{T}})]^{\perp}=\mathcal{R}(A)$. Consequently, (2.21) is equivalent to (2.20). ■

Definition 2.7 [44]. Consider the system given by (2.5) and (2.6) with $B=0$. The pair (A,C) is *semiobservable* if

$$\bigcap_{k=1}^{n}\mathcal{N}(CA^{k-1})=\mathcal{N}(A). \tag{2.22}$$

Next, recall that the *unobservable subspace* $\mathcal{U}_{t_{\mathrm{f}}}(A,C)$ at time $t_{\mathrm{f}}>0$ is the subspace

$$\mathcal{U}_{t_{\mathrm{f}}}(A,C)=\{x_0\in\mathbb{R}^n: y(t)=0 \text{ for all } t\in[0,t_{\mathrm{f}}]\}.$$

As in the controllable subspace case, $\mathcal{U}_{t_{\mathrm{f}}}(A,C)$ is independent of t_{f}, and hence, we write $\mathcal{U}(A,C)$ for $\mathcal{U}_{t_{\mathrm{f}}}(A,C)$, and call $\mathcal{U}(A,C)$ the *unobservable subspace* of (A,C) [8].

The next result characterizes semiobservability in several equivalent ways.

Theorem 2.8. The following statements are equivalent:

i) (A, C) is semiobservable.

ii) $\mathcal{N}(\int_0^{t_\mathrm{f}} e^{A^\mathrm{T}t}C^\mathrm{T}Ce^{At}\mathrm{d}t) = \mathcal{N}(A)$.

iii) $\mathcal{U}(A, C) = \mathcal{N}(A)$.

iv) $\mathcal{N}(A) \subseteq \mathcal{N}(C)$ and, for every $x_0 \in \mathcal{R}(A^\mathrm{T})$, the initial state $x(0) = x_0$ can be uniquely determined from $y(t)$ on $[0, t_\mathrm{f}]$.

Proof: First, note that it follows from Lemma 12.3.2 of [8, p. 800] that

$$\mathcal{U}(A, C) = \mathcal{N}\left(\int_0^{t_\mathrm{f}} e^{A^\mathrm{T}t}C^\mathrm{T}Ce^{At}\mathrm{d}t\right) = \bigcap_{i=1}^{n} \mathcal{N}(CA^{i-1}). \tag{2.23}$$

To show the equivalence of *i)* and *ii)*, note that if (A, C) is semiobservable, then it follows from (2.23) that $\mathcal{N}(\int_0^{t_\mathrm{f}} e^{A^\mathrm{T}t}C^\mathrm{T}Ce^{At}\mathrm{d}t) = \mathcal{N}(A)$ holds. Conversely, if $\mathcal{N}(\int_0^{t_\mathrm{f}} e^{A^\mathrm{T}t}C^\mathrm{T}Ce^{At}\mathrm{d}t) = \mathcal{N}(A)$ holds, then it follows from (2.23) that (2.22) holds. Hence, by definition, (A, C) is semiobservable.

Next, it follows from (2.23) that *ii)* holds if and only if *iii)* holds, which shows the equivalence of *ii)* and *iii)*.

To show *i)* implies *iv)*, assume that (A, C) is semiobservable and note that it follows from (2.22) that $\mathcal{N}(A) = \bigcap_{i=1}^{n} \mathcal{N}(CA^{i-1}) \subseteq \mathcal{N}(C)$. Moreover, it follows from *i)* $\Rightarrow$ *iii)* that $\mathcal{U}(A, C) = \mathcal{N}(A)$. Hence, $\mathcal{U}(A, C)^\perp = \mathcal{N}(A)^\perp = \mathcal{R}(A^\mathrm{T})$. Thus, for every $x_0 \in \mathcal{R}(A^\mathrm{T})$, $x_0 \in \mathcal{U}(A, C)^\perp$. Now, it follows from Lemma 12.3.6 of [8, p. 802] that, for every $x_0 \in \mathcal{U}(A, C)^\perp$,

$$x_0 = W_\mathrm{o}^+ \int_0^{t_\mathrm{f}} e^{A^\mathrm{T}t}C^\mathrm{T}y(t)\mathrm{d}t, \tag{2.24}$$

where $W_\mathrm{o} \triangleq \int_0^{t_\mathrm{f}} e^{A^\mathrm{T}t}C^\mathrm{T}Ce^{At}\mathrm{d}t$, and hence, it follows from (2.24) that x_0 can be uniquely determined from $y(t)$ on $[0, t_\mathrm{f}]$.

To show that *iv*) implies *i*), note that $\mathcal{N}(A) \subseteq \mathcal{N}(CA^i)$ for every $i \in \{1, \ldots, n-1\}$. Furthermore, it follows from $\mathcal{N}(A) \subseteq \mathcal{N}(C)$ and (2.23) that $\mathcal{N}(A) = \mathcal{N}(A) \cap \mathcal{N}(C) \subseteq \bigcap_{i=1}^{n} \mathcal{N}(CA^{i-1}) = \mathcal{U}(A, C)$.

Let $x_0 \in \mathcal{U}(A, C)$. Since $y(t) = 0$ for all $t \in [0, t_\mathrm{f}]$ is the free response corresponding to $x_0 = 0$ (since $y(t) = Ce^{At}x_0$ for all $t \geq 0$), it follows that $0 \in \mathcal{U}(A, C)$. Now, suppose that there exists a nonzero vector $x_0 \in \mathcal{U}(A, C)$. In this case, it follows from (2.23) that $x_0 \in \bigcap_{i=1}^{n} \mathcal{N}(CA^{i-1}) = \mathcal{N}(W_\mathrm{o})$. Then, with $x(0) = x_0$, the free response is given by $y(t) = 0$ for all $t \in [0, t_\mathrm{f}]$, and hence, x_0 cannot be uniquely determined from the knowledge of $y(t)$ for all $t \in [0, t_\mathrm{f}]$.

We claim that $x_0 \in \mathcal{N}(A)$. Suppose, *ad absurdum*, that $x_0 \notin \mathcal{N}(A)$. Since $\mathcal{N}(A)$ is closed, it follows that $\mathcal{N}(A) \bigoplus \mathcal{N}(A)^\perp = \mathbb{R}^n$, where $\bigoplus$ denotes the direct sum. Hence, $x_0 \in \mathcal{N}(A)^\perp = \mathcal{R}(A^\mathrm{T})$. However, by assumption, x_0 can be uniquely determined from $y(t)$ on $[0, t_\mathrm{f}]$. This implies that $x_0 \notin \mathcal{U}(A, C)$, which contradicts the fact that $x_0 \in \mathcal{U}(A, C)$. Hence, $\mathcal{U}(A, C) \subseteq \mathcal{N}(A)$. Consequently, $\mathcal{U}(A, C) = \mathcal{N}(A)$. Now, it follows from *iii*) $\Rightarrow$ *i*) that (A, C) is semiobservable. ■

It follows from Theorem 2.8 that semiobservability of the linear dynamical system (2.5) and (2.6) implies that given the system output y, the state x belonging to range space of A^T can be reconstructed uniquely. Thus, semicontrollability and semiobservability are extensions of controllability and observability. In particular, semicontrollability is an extension of null controllability to *nonisolated equilibrium controllability*, whereas semiobservability is an extension of zero-state observability to *nonisolated equilibrium observability*.

The following result gives a necessary and sufficient conditions for semistability of (2.5) and (2.6).

Theorem 2.9 [44]. Consider the dynamical system $\mathcal{G}$ given by (2.5) with $B = 0$ and output given by (2.6). Then $\mathcal{G}$ is semistable if and only if for every semiobservable pair

(A, C) there exists a $n \times n$ matrix $P = P^{\mathrm{T}} \geq 0$ such that

$$0 = A^{\mathrm{T}}P + PA + C^{\mathrm{T}}C \tag{2.25}$$

is satisfied. Furthermore, if (A, C) is semiobservable and P satisfies (2.25), then

$$P = \int_0^\infty e^{A^{\mathrm{T}}t}C^{\mathrm{T}}Ce^{At}\mathrm{d}t + P_0, \tag{2.26}$$

for some $P_0 = P_0^{\mathrm{T}} \in \mathbb{R}^{n\times n}$ satisfying

$$0 = A^{\mathrm{T}}P_0 + P_0A \tag{2.27}$$

and

$$P_0 \geq -\int_0^\infty e^{A^{\mathrm{T}}t}C^{\mathrm{T}}Ce^{At}\,\mathrm{d}t. \tag{2.28}$$

In addition, $\min_{P\in\mathcal{P}} \|P\|_{\mathrm{F}}$ has a unique least squares solution P given by

$$P_{\mathrm{LS}} = \int_0^\infty e^{A^{\mathrm{T}}t}C^{\mathrm{T}}Ce^{At}\mathrm{d}t, \tag{2.29}$$

where $\mathcal{P}$ denotes the set of all P satisfying (2.25).

Next, we introduce the notions of *semistabilizability* and *semidetectability* [61] as generalizations of stabilizability and detectability.

Definition 2.10 [61]. Consider the dynamical system given by (2.5) and (2.6). The pair (A, B) is *semistabilizable* if

$$\mathrm{rank}\begin{bmatrix} B & \jmath\omega I_n - A \end{bmatrix} = n \tag{2.30}$$

for every nonzero $\omega \in \mathbb{R}$. The pair (A, C) is *semidetectable* if

$$\mathrm{rank}\begin{bmatrix} C \\ \jmath\omega I_n - A \end{bmatrix} = n \tag{2.31}$$

for every nonzero $\omega \in \mathbb{R}$.

Note that (A, C) is semidetectable if and only if $(A^{\mathrm{T}}, C^{\mathrm{T}})$ is semistabilizable. Furthermore, it is important to note that semistabilizability and semidetectability are *different* notions from the standard notions of stabilizability and detectability used in linear system theory. Recall that (A, B) is stabilizable if and only if $\operatorname{rank}\begin{bmatrix} B & \lambda I_n - A \end{bmatrix} = n$ for every $\lambda \in \mathbb{C}$ in the closed right-half plane, and (A, C) is detectable if and only if $\operatorname{rank}\begin{bmatrix} C \\ \lambda I_n - A \end{bmatrix} = n$ for every $\lambda \in \mathbb{C}$ in the closed right-half plane. Hence, if (A, C) is detectable, then (A, C) is semidetectable; however, the converse is not true. A similar remark holds for the notions of controllability and observability. Namely, if (A, C) (resp., (A, B)) is observable (resp., controllable), then (A, C) (resp., (A, B)) is semidetectable (resp., semicontrollable); however, the converse is not true. Hence, semidetectability (resp., semistabilizability) is a weaker notion than both observability and detectability (resp., controllability and stabilizability). Since (2.30) and (2.31) only concern stabilizability and detectability of the pairs (A, B) and (A, C) on the imaginary axis, we refer to these notions as semistabilizability and semidetectability.

Remark 2.11. It follows from Facts 2.11.1-2.11.3 of [8, pp. 130-131] that (2.30) and (2.31) are equivalent to

$$\dim[\mathcal{R}(\jmath\omega I_n - A) + \mathcal{R}(B)] = n \tag{2.32}$$

and

$$\mathcal{N}(\jmath\omega I_n - A) \cap \mathcal{N}(C) = \{0\}, \tag{2.33}$$

respectively, where $\dim(\cdot)$ denotes the dimension of a set.

Example 2.12. Consider $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Clearly, (A, B) is not stabilizable. However, it can be verified using (2.30) that (A, B) is semistabilizable. ▲

As in the case of controllability and stabilizability, state feedback control does not destroy semistabilizability and semicontrollability. This is shown in the next lemma.

Lemma 2.13. Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $K \in \mathbb{R}^{m\times n}$. If (A, B) is semistabilizable (resp., semicontrollable), then $(A + BK, B)$ is semistabilizable (resp., semicontrollable).

Proof: Since (A,B) is semistabilizable, it follows that $\operatorname{rank}\begin{bmatrix} B & \jmath\omega I_n - A\end{bmatrix} = n$ for all nonzero $\omega \in \mathbb{R}$. Hence, using Sylvester's inequality, it follows that

$$\begin{aligned} n &= n + (m+n) - (m+n) \\ &= \operatorname{rank}\begin{bmatrix} B & \jmath\omega I_n - A\end{bmatrix} + \operatorname{rank}\begin{bmatrix} I_m & -K \\ 0 & I_n\end{bmatrix} - (m+n) \\ &\le \operatorname{rank}\left(\begin{bmatrix} B & \jmath\omega I_n - A\end{bmatrix}\begin{bmatrix} I_m & -K \\ 0 & I_n\end{bmatrix}\right) \\ &\le \operatorname{rank}\begin{bmatrix} B & \jmath\omega I_n - A\end{bmatrix} \\ &= n \end{aligned} \tag{2.34}$$

for all nonzero $\omega \in \mathbb{R}$. Now, since

$$\begin{bmatrix} B & \jmath\omega I_n - A - BK\end{bmatrix} = \begin{bmatrix} B & \jmath\omega I_n - A\end{bmatrix}\begin{bmatrix} I_m & -K \\ 0 & I_n\end{bmatrix},$$

it follows from (2.34) that

$$\operatorname{rank}\begin{bmatrix} B & \jmath\omega I_n - A - BK\end{bmatrix} = n$$

for all noznero $\omega \in \mathbb{R}$. Thus, $(A+BK,B)$ is semistabilizable. The proof for semicontrollability follows similarly as in the proof of Proposition 2.1 in [44]. ■

Next, using the notions of semistabilizability and semidetectability, we provide a generalization of Theorem 2.9. First, however, the following lemmas are needed.

Lemma 2.14. Let $A \in \mathbb{R}^{n\times n}$. Then A is semistable if and only if $\mathcal{N}(A) \cap \mathcal{R}(A) = \{0\}$ and $\operatorname{spec}(A) \subseteq \{\lambda \in \mathbb{C} : \lambda + \lambda^* < 0\} \cup \{0\}$, where λ^* denotes the complex conjugate of λ.

Proof: If A is semistable, then it follows from Definition 11.8.1 of [8, p. 727] that $\operatorname{spec}(A) \subseteq \{\lambda \in \mathbb{C} : \lambda + \lambda^* < 0\} \cup \{0\}$ and either A is Hurwitz or there exists an invertible matrix $S \in \mathbb{R}^{n\times n}$ such that $A = S\begin{bmatrix} J & 0 \\ 0 & 0\end{bmatrix}S^{-1}$, where $J \in \mathbb{R}^{r\times r}$, $r = \operatorname{rank} A$, and J is Hurwitz. If A is Hurwitz, then $\mathcal{N}(A) = \{0\} = \mathcal{N}(A) \cap \mathcal{R}(A)$. Alternatively, if A is not Hurwitz, then $\mathcal{N}(A) = \{S[0_{1\times r}, y_2^{\mathrm{T}}]^{\mathrm{T}} : y_2 \in \mathbb{R}^{n-r}\}$. In this case, for every $S[0_{1\times r}, x_2^{\mathrm{T}}]^{\mathrm{T}} \in \mathcal{N}(A) \cap \mathcal{R}(A)$, there exists $z \in \mathbb{R}^n$ such that $S[0_{1\times r}, x_2^{\mathrm{T}}]^{\mathrm{T}} = Az$. Hence, $S[0_{1\times r}, x_2^{\mathrm{T}}]^{\mathrm{T}} = S \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} S^{-1}z$, that is,

$$\begin{bmatrix} 0 \\ x_2 \end{bmatrix} = \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} S^{-1}z,$$

which implies that $x_2 = 0$. Thus, $\mathcal{N}(A) \cap \mathcal{R}(A) = \{0\}$.

Conversely, assume that $\mathcal{N}(A) \cap \mathcal{R}(A) = \{0\}$ and $\mathrm{spec}(A) \subseteq \{\lambda \in \mathbb{C} : \lambda + \lambda^* < 0\} \cup \{0\}$. If A is nonsingular, then A is Hurwitz, and hence, A is semistable. Next, we consider the case where A is singular. Let $x \in \mathcal{N}(A^2)$ and note that it follows from $A^2x = AAx = 0$ that $Ax \in \mathcal{N}(A)$. Now, noting that $Ax \in \mathcal{R}(A)$, it follows from $\mathcal{N}(A) \cap \mathcal{R}(A) = \{0\}$ that $Ax = 0$, that is, $x \in \mathcal{N}(A)$. Hence, $\mathcal{N}(A^2) \subseteq \mathcal{N}(A)$. However, since $\mathcal{N}(A) \subseteq \mathcal{N}(A^2)$, it follows that $\mathcal{N}(A) = \mathcal{N}(A^2)$. Thus, by Proposition 5.5.8 of [8, p. 323], $0 \in \mathrm{spec}(A)$ is semisimple, and hence, by Definition 11.8.1 of [8, p. 727], A is semistable. ■

Lemma 2.15. Let $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{l\times n}$. If A is semistable and $\mathcal{N}(A) \subseteq \mathcal{N}(C)$, then $CL = 0$, where L is given by

$$L \triangleq I_n - AA^{\#}. \tag{2.35}$$

Proof: It follows from the semistability of A and Proposition 11.8.1 of [8] that L is well defined. Next, we show that $CLx = 0$ for every $x \in \mathbb{R}^n$. Suppose, *ad absurdum*, that there exists $x \in \mathbb{R}^n$, $x \neq 0$, such that $CLx \neq 0$. Then, $Lx \notin \mathcal{N}(C)$. Since $\mathcal{N}(A) \subseteq \mathcal{N}(C)$, it follows that $Lx \notin \mathcal{N}(A)$. However, $ALx = A(I_n - AA^{\#})x = (A - AAA^{\#})x = 0$, which implies that $Lx \in \mathcal{N}(A)$, which is a contradiction. Hence, $CLx = 0$ for every $x \in \mathbb{R}^n$. ■

Theorem 2.16. Consider the dynamical system $\mathcal{G}$ given by (2.5) with $B = 0$ and output given by (2.6). Then the following statements are equivalent:

i) $\mathcal{G}$ is semistable.

ii) $\operatorname{rank}(\jmath\omega I_n - A) = n$ for every nonzero $\omega \in \mathbb{R}$ and there exist a positive integer p, a $p \times n$ matrix E, and a $n \times n$ matrix $P = P^{\mathrm{T}} \geq 0$ such that

$$0 = A^{\mathrm{T}}P + PA + E^{\mathrm{T}}E. \tag{2.36}$$

In this case,

$$P = \int_0^{\infty} e^{A^{\mathrm{T}}t}(E^{\mathrm{T}}E + L^{\mathrm{T}}E^{\mathrm{T}}EL)e^{At}\mathrm{d}t + P_0, \tag{2.37}$$

where $L = I_n - AA^{\#}$ and P_0 satisfies (2.27) and (2.28).

iii) For every matrix $C \in \mathbb{R}^{l\times n}$ such that (A, C) is semiobservable, there exists a $n \times n$ matrix $P = P^{\mathrm{T}} \geq 0$ such that (2.25) holds.

iv) There exist a positive integer p, a $p \times n$ matrix E, and a $n \times n$ matrix $P = P^{\mathrm{T}} \geq 0$ such that (A, E) is semiobservable and (2.36) holds.

v) There exist a positive integer p, a $p \times n$ matrix E, and a $n \times n$ matrix $P = P^{\mathrm{T}} \geq 0$ such that (A, E) is semidetectable and (2.36) holds.

Proof: First, note that if A is semistable, then it follows from the definition of semistability that $\jmath\omega \notin \operatorname{spec}(A)$, $\omega \neq 0$. Hence, $\operatorname{rank}(A - \jmath\omega I_n) = n$ for every nonzero $\omega \in \mathbb{R}$.

To prove the existence of a nonnegative definite solution to (2.36), let E be such that $\mathcal{N}(A) \subseteq \mathcal{N}(E)$. For every such pair (A, E), let

$$\hat{P} = \int_0^{\infty} e^{A^{\mathrm{T}}t}E^{\mathrm{T}}Ee^{At}\mathrm{d}t. \tag{2.38}$$

Now, it follows from Proposition 2.2 of [44] that $\hat{P}$ is well defined. Clearly, $\hat{P} = \hat{P}^{\mathrm{T}} \geq 0$. Since A is semistable, it follows from Lemma 2.14 that $\mathcal{N}(A) \cap \mathcal{R}(A) = \{0\}$, and hence, A is group invertible [5, p. 119]. Hence, it follows from (2.38) and (2.35) that

$$\begin{aligned} A^{\mathrm{T}}\hat{P} + \hat{P}A &= \int_0^{\infty} \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{A^{\mathrm{T}}t}E^{\mathrm{T}}Ee^{At}\right)\mathrm{d}t \\ &= (I_n - AA^{\#})^{\mathrm{T}}E^{\mathrm{T}}E(I_n - AA^{\#}) - E^{\mathrm{T}}E \end{aligned}$$

$$= L^{\mathrm{T}}E^{\mathrm{T}}EL - E^{\mathrm{T}}E. \tag{2.39}$$

Next, setting $\hat{P} = P - Z$, where $Z \in \mathbb{R}^{n\times n}$ and $Z = Z^{\mathrm{T}} \geq 0$, it follows from (2.39) that

$$A^{\mathrm{T}}P + PA + E^{\mathrm{T}}E = A^{\mathrm{T}}Z + ZA + L^{\mathrm{T}}E^{\mathrm{T}}EL. \tag{2.40}$$

Furthermore, it follows from Lemma 2.4 in [44] that $Lx = 0$ for all $x \in \mathcal{N}(A)$, and hence, the pair (A, EL) is semiobservable since $ELx = 0$ for all $x \in \mathcal{N}(A)$. Consequently, it follows from Theorem 2.9 that

$$Z = \int_0^\infty e^{A^{\mathrm{T}}t}L^{\mathrm{T}}E^{\mathrm{T}}ELe^{\mathrm{At}}\mathrm{d}t + P_0, \tag{2.41}$$

which is a nonnegative-definite solution of

$$0 = A^{\mathrm{T}}Z + ZA + L^{\mathrm{T}}E^{\mathrm{T}}EL. \tag{2.42}$$

Thus, it follows from (2.40) that (2.37) satisfies (2.36), which proves that $i)$ implies $ii)$.

Let $V(x) = x^{\mathrm{T}}P_1x$, where $P_1 \triangleq \hat{P} + L^{\mathrm{T}}L$. If $V(x) = 0$ for some $x \in \mathbb{R}^n$, then $\hat{P}x = 0$ and $Lx = 0$. It follows from $i)$ of Lemma 2.4 in [44] that $x \in \mathcal{N}(A)$, and $Lx = 0$ implies that $x \in \mathcal{R}(A)$. Now, it follows from $ii)$ of Lemma 2.4 in [44] that $x = 0$. Hence, P_1 is positive definite. Note that P_1 satisfies (2.36) since $LA = A - AA^{\#}A = 0$, and hence,

$$\begin{aligned} A^{\mathrm{T}}P_1 + P_1A + E^{\mathrm{T}}E &= A^{\mathrm{T}}\hat{P} + \hat{P}A + E^{\mathrm{T}}E + A^{\mathrm{T}}L^{\mathrm{T}}L + L^{\mathrm{T}}LA \\ &= L^{\mathrm{T}}E^{\mathrm{T}}EL + (LA)^{\mathrm{T}}L + L^{\mathrm{T}}LA \\ &= 0. \end{aligned}$$

Also note that $\dot{V}(x) = -x^{\mathrm{T}}E^{\mathrm{T}}Ex \leq 0$, $x \in \mathbb{R}^n$, which implies that A is Lyapunov stable. Furthermore, it follows from $\mathrm{rank}(A - \jmath\omega I_n) = n$ for every nonzero $\omega \in \mathbb{R}$ that $\jmath\omega \in \mathrm{spec}(A)$, $\omega \neq 0$. Hence, A is semistable, which proves that $ii)$ implies $i)$.

The proof of the equivalence of $i)$ and $iii)$ follows from Theorem 2.2 in [44]. Next, we show that $i)$ is equivalent to $iv)$. It follows from Theorem 2.9 that $iv)$ implies $i)$. Alternatively, if $i)$ holds, then choose E such that $\mathcal{N}(E) = \mathcal{N}(A)$ (an obvious choice is $E = A$). Since

$\mathcal{N}(EA^i) \supseteq \mathcal{N}(A)$ and $\mathcal{N}(EA^{i+1}) \supseteq \mathcal{N}(EA^i)$ for every $i \in \{0, \ldots, n-1\}$, it follows that $\mathcal{N}(A) \subseteq \bigcap_{i=1}^{n} \mathcal{N}(EA^{i-1}) \subseteq \mathcal{N}(E) = \mathcal{N}(A)$, and hence, $\bigcap_{i=1}^{n} \mathcal{N}(EA^{i-1}) = \mathcal{N}(A)$. Thus, (A, E) is semiobservable. Now, using similar arguments as in the proof of the equivalence of $i)$ and $ii)$, there exists $P = P^{\mathrm{T}} \geq 0$ such that (2.36) holds, which shows that $i)$ implies $iv)$.

Finally, we show the equivalence of $i)$ and $v)$. If A is semistable, then $\jmath\omega \notin \operatorname{spec}(A)$, $\omega \neq 0$, and hence, $\operatorname{rank}(\jmath\omega I_n - A) = n$ for every nonzero $\omega \in \mathbb{R}$. Thus, rank $\begin{bmatrix} E \\ \jmath\omega I_n - A \end{bmatrix} = n$ for every $E \in \mathbb{R}^{p \times n}$ and every positive integer p. The proof of the existence of a positive-definite solution to (2.36) follows exactly as in the proof of $i) \Rightarrow ii)$. The converse follows using similar arguments as in the proof of $ii) \Rightarrow i)$ for A Lyapunov stable.

To show that A is semistable, suppose, *ad absurdum*, $\jmath\omega \in \operatorname{spec}(A)$, where $\omega \in \mathbb{R}$ is nonzero, and let $x \in \mathbb{C}^n$, $x \neq 0$, be an associated eigenvector of A. Then, it follows from (2.36) that

$$\begin{aligned} -x^* E^{\mathrm{T}} E x &= x^*(A^{\mathrm{T}} P + PA)x \\ &= x^*[(\jmath\omega I_n - A)^* P + P(\jmath\omega I_n - A)]x \\ &= 0. \end{aligned}$$

Hence, $Ex = 0$, and thus, $\begin{bmatrix} E \\ \jmath\omega I_n - A \end{bmatrix} x = 0$, which, since rank $\begin{bmatrix} E \\ \jmath\omega I_n - A \end{bmatrix} = n$, implies that $x = 0$, which is a contradiction. Consequently, $\jmath\omega \notin \operatorname{spec}(A)$ for all nonzero $\omega \in \mathbb{R}$. Hence, $\operatorname{spec}(A) \subset \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < 0\} \cup \{0\}$ and, if $0 \in \operatorname{spec}(A)$, then 0 is semisimple. Therefore, A is semistable. ∎

Lemma 2.17. Let $x, y \in \mathbb{R}^n$ be such that $xy^{\mathrm{T}} = yx^{\mathrm{T}} \geq 0$. Then $y = \alpha x$, where $\alpha \geq 0$.

Proof: Note that for $x = 0$ or $y = 0$ the inequality is immediate. Next, if x and y are linearly dependent, then it follows from $xy^{\mathrm{T}} = yx^{\mathrm{T}} \geq 0$ that $y = \alpha x$, where $\alpha \geq 0$. Alternatively, assume, *ad absurdum*, that x and y are linearly independent. In this case, it follows from Proposition 7.1.8 of [8, p. 441] that $xy^{\mathrm{T}} = yx^{\mathrm{T}}$ if and only if $\operatorname{vec}^{-1}(y \otimes$

$x) = \mathrm{vec}^{-1}(x \otimes y)$, which further implies that $y \otimes x = x \otimes y$. Let $x = [x_1, \ldots, x_n]^{\mathrm{T}}$ and $y = [y_1, \ldots, y_n]^{\mathrm{T}}$. Then it follows from $y \otimes x = x \otimes y$ that $y_i x = x_i y$ for every $i \in \{1, \ldots, n\}$. Since x and y are linearly independent, it follows that $y_i x - x_i y = 0$ for every $i \in \{1, \ldots, n\}$ if and only if $y_i = x_i = 0$ for every $i \in \{1, \ldots, n\}$. This contradicts the assumption that x and y are linearly independent. Now, the assertion follows directly from the first case. ■

Theorem 2.18. Consider the dynamical system $\mathcal{G}$ given by (2.5) with $B = 0$ and output given by (2.6). Assume that there exists a $n \times n$ matrix $P = P^{\mathrm{T}} \geq 0$ such that (2.25) holds. Then $\mathcal{G}$ is semistable if and only if the pair (A, C) is semidetectable. Furthermore, if (A, C) is semidetectable and P satisfies (2.25), then

$$P = \int_0^\infty e^{A^{\mathrm{T}}t} C^{\mathrm{T}} C e^{At} \mathrm{d}t + \alpha z z^{\mathrm{T}}, \tag{2.43}$$

where $\alpha \geq 0$, $z \in \mathcal{N}(A^{\mathrm{T}})$,

$$\alpha z z^{\mathrm{T}} = \int_0^\infty e^{A^{\mathrm{T}}t} L^{\mathrm{T}} C^{\mathrm{T}} C L e^{At} \mathrm{d}t + P_0, \tag{2.44}$$

$L = I_n - AA^{\#}$, and P_0 satisfies (2.27) and (2.28).

Proof: The first part of the result is a direct consequence of Theorem 2.16. To prove that P has the form given by (2.43), first note that it follows from (2.25) that $(A \oplus A)^{\mathrm{T}} \mathrm{vec}\, P = -\mathrm{vec}\,(C^{\mathrm{T}}C)$. Hence, $\mathrm{vec}\,(C^{\mathrm{T}}C) \in \mathcal{R}((A \oplus A)^{\mathrm{T}})$. Next, it follows from Lemma 3.8 of [56] that $(A \oplus A)^{\mathrm{T}}$ is semistable, and hence, by Lemma 3.9 of [56],

$$\begin{aligned}
\mathrm{vec}^{-1}\Big(((A \oplus A)^{\mathrm{T}})^{\#} \mathrm{vec}\,(C^{\mathrm{T}}C)\Big) &= -\int_0^\infty \mathrm{vec}^{-1}\Big(e^{(A\oplus A)^{\mathrm{T}}t} \mathrm{vec}\,(C^{\mathrm{T}}C)\Big)\mathrm{d}t \\
&= -\int_0^\infty \mathrm{vec}^{-1}\left(e^{A^{\mathrm{T}}t} \otimes e^{A^{\mathrm{T}}t}\right) \mathrm{vec}\,(C^{\mathrm{T}}C)\mathrm{d}t \\
&= -\int_0^\infty e^{A^{\mathrm{T}}t} C^{\mathrm{T}} C e^{At} \mathrm{d}t, \qquad (2.45)
\end{aligned}$$

where in (2.45) we used the facts that $e^{X \oplus Y} = e^X \otimes e^Y$ and $\mathrm{vec}(XYZ) = (Z^{\mathrm{T}} \otimes X)\mathrm{vec}\, Y$ [8]. Hence, $P = \int_0^\infty e^{A^{\mathrm{T}}t} C^{\mathrm{T}} C e^{At} \mathrm{d}t + \mathrm{vec}^{-1}(w)$, where w satisfies $w \in \mathcal{N}((A \oplus A)^{\mathrm{T}})$ and $\mathrm{vec}^{-1}(w) = (\mathrm{vec}^{-1}(w))^{\mathrm{T}} \geq 0$. (The nonnegative definiteness of $\mathrm{vec}^{-1}(w)$ is guaranteed by

Theorem 4.2a of [110].) Since $(A \oplus A)^{\mathrm{T}}$ is semistable, it follows that a general solution to the equation $(A \oplus A)^{\mathrm{T}} w = 0$ is given by $w = z \otimes y$, where $z, y \in \mathcal{N}(A^{\mathrm{T}})$. Hence, $\mathrm{vec}^{-1}(w) = \mathrm{vec}^{-1}(z \otimes y) = yz^{\mathrm{T}}$, where we used the fact that $zy^{\mathrm{T}} = \mathrm{vec}^{-1}(y \otimes z)$. Furthermore, $zy^{\mathrm{T}} = yz^{\mathrm{T}} \geq 0$. Now, it follows from Lemma 2.17 that $y = \alpha z$, where $\alpha \geq 0$. Finally, (2.44) directly follows from Theorem 2.16 by comparing (2.37) with (2.43) for $C = E$. ■

Consider the dynamical system given by (2.5) and (2.6) with $B = 0$. If the pair (A, C) is semiobservable, then (A, C) is semidetectable and, in this case, it follows from Theorems 2.9 and 2.18 that $\int_0^\infty e^{A^{\mathrm{T}} t} L^{\mathrm{T}} C^{\mathrm{T}} C L e^{At} \mathrm{d}t = 0$.

Lemma 2.19 [11]. Consider the dynamical system $\mathcal{G}$ given by (2.5) with $B = 0$ and output given by (2.6). If the pair (A, C) is semiobservable and there exists an $n \times n$ matrix $P = P^{\mathrm{T}} \geq 0$ such that (2.25) is satisfied, then *i)* $\mathcal{N}(P) \subseteq \mathcal{N}(A) \subseteq \mathcal{N}(C)$ and *ii)* $\mathcal{R}(A) \cap \mathcal{N}(A) = \{0\}$.

The following theorem is a direct consequence of Theorem 2.9.

Theorem 2.20 [44]. Consider the closed-loop system $\mathcal{G}$ given by (2.5) and (2.6) with feedback controller $u(t) = Kx(t)$, where $K \in \mathbb{R}^{m \times n}$. Then $\mathcal{G}$ is semistable if and only if for every semicontrollable pair (A, B) and semiobservable pair (A, C) there exists a $n \times n$ matrix $P = P^{\mathrm{T}} \geq 0$ such that

$$0 = \tilde{A}^{\mathrm{T}} P + P\tilde{A} + C^{\mathrm{T}} C + K^{\mathrm{T}} R_2 K, \tag{2.46}$$

where $\tilde{A} \triangleq A + BK$. Furthermore, the least squares solution of (2.46) is given by

$$P_{\mathrm{LS}} \triangleq \int_0^\infty e^{\tilde{A}^{\mathrm{T}} t} (C^{\mathrm{T}} C + K^{\mathrm{T}} R_2 K) e^{\tilde{A} t} \, \mathrm{d}t. \tag{2.47}$$

Finally, in this case (2.4) is given by

$$J(x_0, K) = x_0^{\mathrm{T}} P_{\mathrm{LS}} x_0. \tag{2.48}$$

Next, we give an alternative form of Theorem 2.20 using semidetectability.

Theorem 2.21. Consider the closed-loop system $\mathcal{G}$ given by (2.5) and (2.6) with feedback controller $u(t) = Kx(t)$, where $K \in \mathbb{R}^{m\times n}$. Assume that there exists a $n \times n$ matrix $P = P^{\mathrm{T}} \geq 0$ such that (2.46) holds. Then $\mathcal{G}$ is semistable if and only if (A, C) is semidetectable. Furthermore, (2.4) is given by (2.48).

Proof: The first assertion is a direct consequence of Theorem 2.18. To show that (2.4) is given by (2.48), it follows from (2.46) that $-x^{\mathrm{T}}(\tilde{A}^{\mathrm{T}}P + P\tilde{A})x = x(C^{\mathrm{T}}C + K^{\mathrm{T}}R_2K)x$ for every $x \in \mathbb{R}^n$, and hence, $\mathcal{N}(\tilde{A}) \subseteq \mathcal{N}(C) \cap \mathcal{N}(R_2K)$. Thus, for $x_{\mathrm{e}} \in \mathcal{N}(\tilde{A})$, $Cx_{\mathrm{e}} = 0$ and $R_2Kx_{\mathrm{e}} = 0$. Now, it follows from (2.4) that

$$\begin{aligned} J(x_0, K) &= \int_0^\infty x^{\mathrm{T}}(t)(C^{\mathrm{T}}C + K^{\mathrm{T}}R_2K)x(t)\mathrm{d}t \\ &= x_0^{\mathrm{T}} \int_0^\infty e^{\tilde{A}^{\mathrm{T}}t}(C^{\mathrm{T}}C + K^{\mathrm{T}}R_2K)e^{\tilde{A}t}\,\mathrm{d}tx_0 \\ &= x_0^{\mathrm{T}} P_{\mathrm{LS}} x_0, \end{aligned}$$

which completes the proof. ■

Finally, the following lemma is needed.

Lemma 2.22 [44]. Consider the linear dynamical system $\mathcal{G}$ given by (2.5) and (2.6) with $u \equiv 0$. If $\mathcal{G}$ is semistable, then for every $x_0 \in \mathbb{R}^n$, the performance measure

$$J(x_0) = \int_0^\infty [(x(t) - x_{\mathrm{e}})^{\mathrm{T}}C^{\mathrm{T}}C(x(t) - x_{\mathrm{e}})]\mathrm{d}t, \tag{2.49}$$

where $x_{\mathrm{e}} = (I - AA^{\#})x_0$, is finite.

2.3. Semistability Analysis of Nonlinear Systems

In this section, we provide connections between Lyapunov functions and nonquatratic cost evaluation. Specifically, we consider the problem of evaluating a nonlinear-nonquadratic cost functional that depends on the solution of the nonlinear dynamical system (2.7). In particular, we show that the nonlinear-nonquadratic cost functional

$$J(x_0) \triangleq \int_0^\infty L(x(t))\mathrm{d}t, \tag{2.50}$$

where $L : \mathcal{D} \to \mathbb{R}$ and $x(t)$, $t \geq 0$, satisfies (2.7), can be evaluated in a convenient form so long as (2.7) is related to an underlying Lyapunov-like function that proves semistability of (2.7).

Theorem 2.23. Consider the nonlinear dynamical system $\mathcal{G}$ given by (2.7) with performance functional (2.50), and let $\mathcal{Q}$ be an open neighborhood of $f^{-1}(0)$. Suppose that the solution $x(t)$, $t \geq 0$, of (2.7) is bounded for all $x \in \mathcal{Q}$ and assume that there exists a continuously differentiable function $V : \mathcal{D} \to \mathbb{R}$ such that

$$V'(x)f(x) \leq 0, \qquad x \in \mathcal{Q}, \tag{2.51}$$

$$L(x) + V'(x)f(x) = 0, \qquad x \in \mathcal{D}. \tag{2.52}$$

If every point in the largest invariant set $\mathcal{M}$ of $\{x \in \mathcal{Q} : V'(x)f(x) = 0\}$ is Lyapunov stable, then (2.7) is semistable and

$$J(x_0) = V(x_0) - V(x_{\mathrm{e}}), \qquad x_0 \in \mathcal{Q}, \tag{2.53}$$

where $x_{\mathrm{e}} = \lim_{t\to\infty} x(t)$.

Proof: Let $x(t)$, $t \geq 0$, satisfy (2.7). Then

$$\dot{V}(x(t)) \triangleq \frac{\mathrm{d}}{\mathrm{d}t} V(x(t)) = V'(x(t))f(x(t)), \qquad t \geq 0.$$

Hence, it follows from (2.51) that $\dot{V}(x(t)) \leq 0$, $t \geq 0$. Since every solution of (2.7) is bounded, it follows from the hypothesis on $V(\cdot)$ that, for every $x \in \mathcal{Q}$, the positive limit set $\omega(x)$ of (2.7) is nonempty and contained in the largest invariant set $\mathcal{M}$ of $\{x \in \mathcal{Q} : V'(x)f(x) = 0\}$. Since every point in $\mathcal{M}$ is a Lyapunov stable equilibrium point, it follows from Lemma 2.2 that $\omega(x)$ contains a single point for every $x \in \mathcal{Q}$, and $\lim_{t\to\infty} s(t,x)$ exists for every $x \in \mathcal{Q}$. Now, since $\lim_{t\to\infty} s(t,x) \in \mathcal{M}$ is Lyapunov stable for every $x \in \mathcal{Q}$, semistability is immediate. Consequently, $x(t) \to x_{\mathrm{e}}$ as $t \to \infty$ for all initial conditions $x_0 \in \mathcal{Q}$.

Next, since

$$0 = -\dot{V}(x(t)) + V'(x(t))f(x(t)), \qquad t \geq 0, \tag{2.54}$$

it follows from (2.52) that

$$L(x(t)) = -\dot{V}(x(t)) + L(x(t)) + V'(x)f(x(t)) = -\dot{V}(x(t)). \tag{2.55}$$

Now, integrating over $[0, t]$ yields

$$\int_0^t L(x(s))\mathrm{d}s = V(x_0) - V(x(t)). \tag{2.56}$$

Letting $t \to \infty$ and noting that $V(x(t)) \to V(x_\mathrm{e})$ for all $x_0 \in \mathcal{Q}$ yields (2.53). ■

The following theorem uses Theorem 2.23 to develop an analogous result for linear dynamical systems without the a priori assumption of boundedness of solutions. First, however, recall that a continuous function $V : \mathcal{D} \to \mathbb{R}$ is said to be *proper relative to* $\mathcal{D}_\mathrm{p} \subseteq \mathcal{D}$ if $V^{-1}(\mathcal{D}_\mathrm{c})$ is a relatively compact subset of $\mathcal{D}_\mathrm{p}$ for all compact subsets $\mathcal{D}_\mathrm{c}$ of $\mathbb{R}$, where $V^{-1}(\cdot)$ denotes the inverse image of $\mathcal{D}_\mathrm{c}$.

Theorem 2.24. Consider the linear dynamical system $\mathcal{G}$ given by (2.5) and (2.6) with $B = 0$ and with quadratic performance measure (2.49). If (A, C) is semiobservable, then $\mathcal{G}$ is globally semistable and

$$J(x_0) = x_0^\mathrm{T}(AA^\#)^\mathrm{T}PAA^\#x_0, \tag{2.57}$$

where $P = P^\mathrm{T} \geq 0$ is a solution of

$$\begin{bmatrix} AA^\# \\ I_n \end{bmatrix}^\mathrm{T} \begin{bmatrix} P & 0 \\ 0 & -P + P_0 \end{bmatrix} \begin{bmatrix} AA^\# \\ I_n \end{bmatrix} = 0 \tag{2.58}$$

and P_0 satisfies (2.27) and (2.28).

Proof: Let $f(x) = A(x - x_\mathrm{e})$, $L(x) = (x - x_\mathrm{e})^\mathrm{T}C^\mathrm{T}C(x - x_\mathrm{e})$, and $\mathcal{Q} = \mathbb{R}^n$, and note that with $V(x) = (x - x_\mathrm{e})^\mathrm{T}P(x - x_\mathrm{e})$, where $P = P^\mathrm{T} \geq 0$, (2.52) specializes to (2.25) and (2.51) is satisfied for all $x \in \mathbb{R}^n$. Furthermore, note that

$$V'(x)f(x) = V'(x)A(x - x_\mathrm{e}) = (x - x_\mathrm{e})^\mathrm{T}(A^\mathrm{T}P + PA)(x - x_\mathrm{e}) = -(x - x_\mathrm{e})^\mathrm{T}C^\mathrm{T}C(x - x_\mathrm{e}),$$

and hence, $\mathcal{N}(A) \subseteq \mathcal{N}(C)$. In addition, since (A, C) is semiobservable, it follows that $\mathcal{N}(C) \subseteq \mathcal{N}(A)$, and hence, $\mathcal{N}(C) = \mathcal{N}(A)$. Thus, $\mathcal{N}(A)$ is the largest invariant set of $\{x \in \mathcal{Q} : V'(x)A(x - x_{\rm e}) = 0\}$.

Next, since (A, C) is semiobservable, it follows from *ii*) of Lemma 2.19 that $\mathcal{R}(A) \cap \mathcal{N}(A) = \{0\}$, which implies that A is group invertible [5, p. 119]. Now, let $L = I - AA^{\#}$ and consider the Lyapunov function candidate $\hat{V}(\hat{x}) \triangleq \hat{x}^{\rm T}(P + L^{\rm T}L)\hat{x}$, where $\hat{x} \triangleq x - x_{\rm e}$. If $\hat{V}(\hat{x}) = 0$ for some $\hat{x} \in \mathbb{R}^n$, then $P\hat{x} = 0$ and $L\hat{x} = 0$, and hence, $\hat{x} \in \mathcal{N}(P)$. Thus, it follows from (2.25) and the semiobservability of (A, C) that $\hat{x} \in \mathcal{N}(A)$. In addition, $\hat{V}(\hat{x}) = 0$ for some $\hat{x} \in \mathbb{R}^n$ implies that $\hat{x} \in \mathcal{N}(L)$, and hence, $\hat{x} \in \mathcal{R}(A)$. Thus, it follows from Lemma 2.19 that $\hat{V}(\hat{x}) = 0$ only if $\hat{x} = 0$, and hence, $\hat{V}(\cdot)$ is positive definite and proper relative to $\mathbb{R}^n$.

Next, note that the time derivative of $\hat{V}(\hat{x})$ along the trajectories of (2.5) with $B = 0$ is given by

$$\hat{V}'(\hat{x}(t))A\hat{x}(t) = -\hat{x}^{\rm T}(t)C^{\rm T}C\hat{x}(t) + 2\hat{x}^{\rm T}(t)L^{\rm T}LA\hat{x}(t) = -\hat{x}^{\rm T}(t)C^{\rm T}C\hat{x}(t) \leq 0, \qquad t \geq 0,$$

and hence, $x(t) \equiv x_{\rm e}$, $t \geq 0$, is Lyapunov stable for every $x_{\rm e} \in \mathcal{N}(A)$, which implies that every orbit of (2.5) with $B = 0$ is bounded. Therefore, it follows from Theorem 2.23 that $x(t)$, $t \geq 0$, is semistable and, since $V(\cdot)$ and $\hat{V}(\cdot)$ are sign definite and proper relative to $\mathbb{R}^n$, $\mathcal{G}$ is globally semistable.

Since $\mathcal{G}$ is globally semistable, it follows from Lemma 2.22 that the quadratic performance measure (2.49) is finite and, by (2.53) of Theorem 2.23, it follows that

$$J(x_0) = (x_0 - x_{\rm e})^{\rm T}P(x_0 - x_{\rm e}) = x_0^{\rm T}(AA^{\#})^{\rm T}PAA^{\#}x_0, \tag{2.59}$$

which proves (2.57). Finally, note that the performance measure (2.49) can be equivalently written as

$$J(x_0) = x_0^{\rm T}\int_0^\infty e^{A^{\rm T}t}C^{\rm T}Ce^{At}{\rm d}t\, x_0, \tag{2.60}$$

which, using Theorem 2.9, yields

$$J(x_0) = x_0^{\mathrm{T}}(P - P_0)x_0. \tag{2.61}$$

Now, (2.58) follows from (2.59) and (2.61). ■

Note that (2.58) can be written as

$$P = (AA^{\#})^{\mathrm{T}} P AA^{\#} + P_0. \tag{2.62}$$

Hence, since $A^{\#}A = AA^{\#}$ and $AA^{\#}A = A$ [8, p. 403], premultiplying and postmultiplying (2.62) by A^{T} and A, respectively, it follows that $A^{\mathrm{T}}P_0A = 0$, which is implied by (2.27).

Proposition 2.25. Consider the linear dynamical system $\mathcal{G}$ given by (2.5) and (2.6) with $B = 0$ and with quadratic performance measure (2.49). If (A, C) is semidetectable and there exists $P = P^{\mathrm{T}} \geq 0$ such that (2.25) holds, then $\mathcal{G}$ is globally semistable and (2.57) holds. In addition, P satisfies

$$\begin{bmatrix} AA^{\#} \\ I_n \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} P & 0 \\ 0 & -P + \alpha z z^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} AA^{\#} \\ I_n \end{bmatrix} = 0, \tag{2.63}$$

where $\alpha \geq 0$ and $z \in \mathcal{N}(A^{\mathrm{T}})$ satisfies (2.44).

Proof: Global semistability of $\mathcal{G}$ is a direct consequence of Theorem 2.16. Next, let $f(x) = A(x - x_{\mathrm{e}})$, $L(x) = (x - x_{\mathrm{e}})^{\mathrm{T}} C^{\mathrm{T}} C (x - x_{\mathrm{e}})$, $\mathcal{Q} = \mathbb{R}^n$, and $V(x) = (x - x_{\mathrm{e}})^{\mathrm{T}} P (x - x_{\mathrm{e}})$. Since $\mathcal{G}$ is globally semistable, it follows from Lemma 2.22 that the quadratic performance measure (2.49) is finite and, by Theorem 2.23, it follows that

$$J(x_0) = (x_0 - x_{\mathrm{e}})^{\mathrm{T}} P (x_0 - x_{\mathrm{e}}) = x_0^{\mathrm{T}} (AA^{\#})^{\mathrm{T}} P AA^{\#} x_0, \tag{2.64}$$

which proves (2.57). Finally, note that the performance measure (2.49) can be equivalently written as

$$J(x_0) = x_0^{\mathrm{T}} \int_0^{\infty} e^{A^{\mathrm{T}} t} C^{\mathrm{T}} C e^{At} \mathrm{d}t \, x_0, \tag{2.65}$$

which, using Theorem 2.18, yields

$$J(x_0) = x_0^{\mathrm{T}} (P - \alpha z z^{\mathrm{T}}) x_0, \tag{2.66}$$

where $\alpha \geq 0$ and $z \in \mathcal{N}(A^{\mathrm{T}})$ satisfies (2.44). Now, (2.63) follows from (2.64) and (2.66). ■

2.4. Optimal Control for Semistabilization

In this section, we use the approach of Theorem 2.23 to obtain a characterization of optimal feedback controllers that guarantee closed-loop semistability. Specifically, sufficient conditions for optimality are given in a form that corresponds to a steady-state version of a Hamilton-Jacobi-Bellman-type equation. To address the optimal semistabilization problem, we consider the controlled nonlinear dynamical system (2.2) with $u(\cdot)$ restricted to the class of *admissible* controls consisting of measurable functions $u(\cdot)$ such that $u(t) \in U$, $t \geq 0$. A measurable function $\phi : \mathcal{D} \to U$ satisfying $\phi(x_{\rm e}) = u_{\rm e}$, where $x_{\rm e} \in \mathcal{D}$ is an equilibrium point of (2.2) for some $u_{\rm e} \in U$, is called a *control law*. If $u(t) = \phi(x(t))$, $t \geq 0$, where $\phi(\cdot)$ is a control law and $x(t)$ satisfies (2.2), then we call $u(\cdot)$ a *feedback control law*. Note that the feedback control law is an admissible control since $\phi(\cdot)$ has values in U. Given a control law $\phi(\cdot)$ and a feedback control $u(t) = \phi(x(t))$, $t \geq 0$, the closed-loop system (2.2) is given by

$$\dot{x}(t) = F(x(t), \phi(x)), \qquad x(0) = x_0, \qquad t \geq 0. \tag{2.67}$$

For the statement of the main theorem of this section, define the set of set-point regulation controllers $\mathcal{S}(x_0)$ for every initial condition $x_0 \in \mathcal{D}$, that is,

$$\mathcal{S}(x_0) \triangleq \{u(\cdot) : u(\cdot) \text{ is admissible and } x(\cdot) \text{ given by (2.2) is bounded and} \\ \text{satisfies } x(t) \to x_{\rm e} \text{ as } t \to \infty\},$$

where $x_{\rm e} \in \mathcal{D}$ is an equilibrium point of (2.2) for some $u_{\rm e} \in U$. Note that restricting our minimization problem to $u(\cdot) \in \mathcal{S}(x_0)$, that is, control inputs corresponding to convergent solutions, can be interpreted as incorporating a semidetectability condition through the cost.

Theorem 2.26. Consider the controlled nonlinear dynamical system (2.2) with $u(\cdot) \in \mathcal{S}(x_0)$ and performance measure (2.1), and suppose there exists a continuously differentiable function $V : \mathcal{D} \to \mathbb{R}$ and a control law $\phi : \mathcal{D} \to U$ such that

$$\phi(x_{\rm e}) = u_{\rm e}, \qquad (x_{\rm e}, u_{\rm e}) \in \mathcal{Q} \times U, \tag{2.68}$$

$$V'(x)F(x,\phi(x)) \le 0, \qquad x \in \mathcal{Q}, \tag{2.69}$$

$$L(x,\phi(x)) + V'(x)F(x,\phi(x)) = 0, \qquad x \in \mathcal{D}, \tag{2.70}$$

$$L(x,u) + V'(x)F(x,u) \ge 0, \qquad (x,u) \in \mathcal{D} \times U, \tag{2.71}$$

where $\mathcal{Q}$ is an open neighborhood of $F^{-1}(0) \triangleq \{x \in \mathcal{D} : F(x,\phi(x)) = 0\}$. If every point in the largest invariant set $\mathcal{M}$ of $\{x \in \mathcal{Q} : V'(x)F(x,\phi(x)) = 0\}$ is Lyapunov stable, then, with the feedback control $u(\cdot) = \phi(x(\cdot))$, the solution $x(t) = x_{\rm e}$, $t \ge 0$, of the closed-loop system (2.67) is semistable and

$$J(x_0, \phi(x(\cdot))) = V(x_0) - V(x_{\rm e}). \tag{2.72}$$

Furthermore, the feedback control $u(\cdot) = \phi(x(\cdot))$ minimizes $J(x_0, u(\cdot))$ in the sense that

$$J(x_0, \phi(x(\cdot))) = \min_{u(\cdot)\in\mathcal{S}(x_0)} J(x_0, u(\cdot)). \tag{2.73}$$

Proof: If $u(\cdot) \in \mathcal{S}(x_0)$, then the solution $x(t)$, $t \ge 0$, of (2.2) is bounded for all initial conditions $x_0 \in \mathcal{Q}$. Thus, semistability is a direct consequence of (2.69) and (2.70) by applying Theorem 2.23 to the closed-loop system (2.67). Furthermore, using (2.70), condition (2.72) is a restatement of (2.53). To prove (2.73), note that

$$\dot{V}(x(t)) = V'(x(t))F(x(t), u(t)), \tag{2.74}$$

or, equivalently,

$$0 = -\dot{V}(x(t)) + V'(x(t))F(x(t), u(t)). \tag{2.75}$$

Hence,

$$L(x(t), u(t)) = -\dot{V}(x(t)) + L(x(t), u(t)) + V'(x(t))F(x(t), u(t)). \tag{2.76}$$

Now, using (2.71) and (2.72), and the fact that $u(\cdot) \in \mathcal{S}(x_0)$, it follows that

$$\begin{aligned} J(x_0, u(\cdot)) &= \int_0^\infty L(x(t), u(t)){\rm d}t \\ &= \int_0^\infty -\dot{V}(x(t))\,{\rm d}t + \int_0^\infty (L(x(t), u(t)) + V'(x)F(x(t), u(t))){\rm d}t \end{aligned}$$

$$
\begin{aligned}
&= V(x_0) - V(x_\mathrm{e}) + \int_0^\infty [L(x(t), u(t)) + V'(x)F(x(t), u(t))]\mathrm{d}t \\
&\geq V(x_0) - V(x_\mathrm{e}), \qquad (2.77)
\end{aligned}
$$

which yields (2.73). ■

Remark 2.27. Theorem 2.26 requires that $u(\cdot) \in \mathcal{S}(x_0)$ or, equivalently, the solution of the closed-loop system is bounded for all $x \in \mathcal{Q}$. For asymptotic stabilization this is automatically satisfied since we additionally require $V(0) = 0$, $V(x) > 0$, $x \in \mathcal{D} \setminus \{0\}$, and $V'(x)F(x, \phi(x)) < 0$, $x \in \mathcal{D}$, in the place of (2.69) (see [7, Th. 3.1] and [38, Th. 8.2]). This guarantees asymptotic stability of the closed-loop system, and hence, all closed-loop solutions are bounded. One can replace the assumption $u(\cdot) \in \mathcal{S}(x_0)$ in Theorem 2.26 with $u(\cdot)$ being simply admissible and not invoking any assumption on the sign definiteness of $V(\cdot)$. In this case, however, the conditions of Theorem 2.26 need to be supplemented by assuming a nontangency condition of the closed-loop vector field to invariant or negatively invariant subsets of the level sets of $V(\cdot)$ containing the system equilibrium. For details; see [13].

Note that Theorem 2.26 guarantees optimality with respect to the set of admissible semistabilizing controllers $\mathcal{S}(x_0)$ with the optimal control law given by the state feedback controller

$$
\phi(x) = \arg\min_{u \in \mathcal{S}(x_0)} [L(x, u) + V'(x)F(x, u)], \qquad (2.78)
$$

which invokes a steady-state Hamilton-Jacobi-Bellman-type equation and is independent of the initial condition x_0. It is important to note that an explicit characterization of $\mathcal{S}(x_0)$ is not required.

The following result specializes Theorem 2.26 to nonlinear affine dynamical systems of the form

$$
\dot{x}(t) = f(x(t)) + G(x(t))u(t), \qquad x(0) = x_0, \qquad t \geq 0, \qquad (2.79)
$$

$$y(t) = h(x(t)), \tag{2.80}$$

where, for every $t \geq 0$, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^l$, $f : \mathbb{R}^n \to \mathbb{R}^n$ and $G : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are locally Lipschitz continuous in x, $h : \mathbb{R}^n \to \mathbb{R}^l$ is continuous in x, and $0 = f(x_\mathrm{e}) + G(x_\mathrm{e})u_\mathrm{e}$ and $y_\mathrm{e} = h(x_\mathrm{e})$. Furthermore, we consider performance integrands $L(x, u)$ of the form

$$L(x,u) = (h(x) - y_\mathrm{e})^\mathrm{T} (h(x) - y_\mathrm{e}) + (u - u_\mathrm{e})^\mathrm{T} R_2(x) (u - u_\mathrm{e}), \tag{2.81}$$

where $R_2(x) > 0$, $x \in \mathbb{R}^n$, so that (2.1) becomes

$$J(x_0, u(\cdot)) = \int_0^\infty \Big[(y(t) - y_\mathrm{e})^\mathrm{T} (y(t) - y_\mathrm{e}) + (u(t) - u_\mathrm{e})^\mathrm{T} R_2(x(t)) (u(t) - u_\mathrm{e})\Big] \mathrm{d}t. \tag{2.82}$$

Corollary 2.28. Consider the controlled nonlinear dynamical system (2.79) and (2.80) with $u(\cdot) \in \mathcal{S}(x_0)$ and performance measure (2.82), and assume there exists a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ such that

$$V'(x_\mathrm{e}) = 0, \qquad x_\mathrm{e} \in \mathbb{R}^n, \tag{2.83}$$

$$(y - y_\mathrm{e})^\mathrm{T} (y - y_\mathrm{e}) + V'(x)f(x) + V'(x)G(x)u_\mathrm{e}$$
$$- \frac{1}{4}V'(x)G(x)R_2^{-1}(x)G^\mathrm{T}(x)V'^\mathrm{T}(x) = 0, \qquad (x, u_\mathrm{e}) \in \mathbb{R}^n \times \mathbb{R}^m. \tag{2.84}$$

If, with the feedback control

$$u = \phi(x) = -\frac{1}{2}R_2^{-1}(x)G^\mathrm{T}(x)V'^\mathrm{T}(x) + u_\mathrm{e}, \tag{2.85}$$

every equilibrium point $x_\mathrm{e} \in F^{-1}(0) = \{x \in \mathcal{D} : f(x) + G(x)\phi(x) = 0\}$ of the closed-loop system

$$\dot{x}(t) = f(x(t)) + G(x(t))\phi(x(t)), \qquad x(0) = x_0, \qquad t \geq 0, \tag{2.86}$$

is Lyapunov stable, then the solution $x(t) = x_\mathrm{e}$, $t \geq 0$, of the closed-loop system (2.67) is semistable and

$$J(x_0, \phi(x(\cdot))) = V(x_0) - V(x_\mathrm{e}). \tag{2.87}$$

Furthermore, the feedback control (2.85) minimizes $J(x_0, u(\cdot))$ in the sense that

$$J(x_0, \phi(x(\cdot))) = \min_{u(\cdot)\in\mathcal{S}(x_0)} J(x_0, u(\cdot)). \tag{2.88}$$

Proof: The result follows as a consequence of Theorem 2.26 with $L(x, u) = (y-y_{\rm e})^{\rm T}(y-y_{\rm e}) + (u-u_{\rm e})^{\rm T}R_2(x)(u-u_{\rm e})$, $\mathcal{D} = \mathbb{R}^n$, and $U = \mathbb{R}^m$. Specifically, the feedback control law (2.85) follows from (2.78) by setting

$$\frac{\partial}{\partial u}\left[(y-y_{\rm e})^{\rm T}(y-y_{\rm e}) + (u-u_{\rm e})^{\rm T}R_2(x)(u-u_{\rm e}) + V'(x)\big(f(x)+G(x)u\big)\right] = 0. \tag{2.89}$$

Now, (2.84) is equivalent to (2.70) with $\phi(x)$ given by (2.85).

Next, since $(y-y_{\rm e})^{\rm T}(y-y_{\rm e}) \geq 0$, $(y, y_{\rm e}) \in \mathbb{R}^l\times\mathbb{R}^l$, and $V'(x)G(x)R_2^{-1}(x)G^{\rm T}(x)V'^{\rm T}(x) \geq 0$, $x \in \mathbb{R}^n$, (2.84) implies that

$$\begin{aligned} 0 &\geq V'(x)f(x) - \frac{1}{4}V'(x)G(x)R_2^{-1}(x)G^{\rm T}(x)V'^{\rm T}(x) + V'(x)G(x)u_{\rm e} \\ &\geq V'(x)f(x) - \frac{1}{2}V'(x)G(x)R_2^{-1}(x)G^{\rm T}(x)V'^{\rm T}(x) + V'(x)G(x)u_{\rm e}, \qquad (x, u_{\rm e}) \in \mathbb{R}^n\times\mathbb{R}^m. \end{aligned} \tag{2.90}$$

Let $\mathcal{Q}$ be an open neighborhood of the set $\{x \in \mathbb{R}^n : f(x) - \frac{1}{2}G(x)R_2^{-1}(x)G^{\rm T}(x)V'^{\rm T}(x) + V'(x)G(x)u_{\rm e} = 0\}$ and note that (2.90) implies

$$V'(x)f(x) - \frac{1}{2}V'(x)G(x)R_2^{-1}(x)G^{\rm T}(x)V'^{\rm T}(x) + V'(x)G(x)u_{\rm e} \leq 0, \qquad (x, u_{\rm e}) \in \mathcal{Q}\times\mathbb{R}^m. \tag{2.91}$$

Now, (2.91) is equivalent to (2.69) with $\phi(x)$ given by (2.85), and hence, (2.84) implies (2.69) with $\phi(x)$ given by (2.85).

Next, (2.83) and (2.85) imply (2.68) and, since

$$\begin{aligned} L(x,u) &+ V'(x)[f(x)+G(x)u] \\ &= L(x,u) + V'(x)[f(x)+G(x)u] - L(x,\phi(x)) - V'(x)[f(x)+G(x)\phi(x)] \\ &= [u-\phi(x)]^{\rm T}R_2(x)[u-\phi(x)] \\ &\geq 0, \qquad (x,u) \in \mathbb{R}^n\times\mathbb{R}^m, \end{aligned} \tag{2.92}$$

condition (2.71) is satisfied.

Finally, it follows from (2.83)–(2.85) that

$$\begin{aligned}\{x \in \mathcal{Q} : V'(x)\,(f(x) + G(x)\phi(x)) = 0\} &= \{x \in \mathcal{N} : L(x, \phi(x)) = 0\}\\ &= \{x \in \mathcal{Q} : h(x) = y_{\rm e} \,\text{and}\, V'(x)G(x) = 0\}\\ &\subseteq F^{-1}(0),\end{aligned} \tag{2.93}$$

and, by assumption, every equilibrium point $x_{\rm e} \in F^{-1}(0)$ of the closed-loop system (2.67) is Lyapunov stable. Since all of the conditions of Theorem 2.26 are satisfied, the result follows.

■

Remark 2.29. Theorem 2.26 requires the construction of a continuously differentiable $V(\cdot)$ and a state-feedback control law $\phi(\cdot)$ such that (2.68)–(2.71) are satisfied. In contrast, Corollary 2.28 requires the construction of a continuously differentiable $V(\cdot)$ such that (2.83) and (2.84) are satisfied and with the semistabilizing state-feedback control law $\phi(\cdot)$ explicitly given by (2.85).

Next, we consider the linear-quadratic regulator problem for semistabilization, that is, we seek controllers $u(\cdot)$ that minimize (2.4) and guarantee semistability of the linear system given by (2.5) and (2.6). The feedback gain K that minimizes (2.4) and guarantees semistability of (2.5) can be characterized via a solution to a linear matrix inequality [44]. The following result provides a useful alternative in finding the optimal gain K via an algebraic Riccati equation.

Theorem 2.30. Consider the linear controlled dynamical system $\mathcal{G}$ given by (2.5) and (2.6) with quadratic performance measure (2.4), assume that the pair (A, B) is semicontrollable and the pair (A, C) is semiobservable, and let $P_{\rm LS} = P_{\rm LS}^{\rm T} \geq 0$ be the least squares solution to the algebraic Riccati equation

$$0 = A^{\rm T}P + PA + C^{\rm T}C - PBR_2^{-1}B^{\rm T}P. \tag{2.94}$$

Then, with $u = Kx = -R_2^{-1}B^{\mathrm{T}}P_{\mathrm{LS}}x$, the solution $x(t) = x_{\mathrm{e}}$, $t \geq 0$, to (2.5) is globally semistable,

$$J(x_0, K) = x_0^{\mathrm{T}}\left[\int_0^\infty (\tilde{A}\tilde{A}^{\#})^{\mathrm{T}} e^{\tilde{A}^{\mathrm{T}}t}(C^{\mathrm{T}}C + K^{\mathrm{T}}R_2K)e^{\tilde{A}t}\tilde{A}\tilde{A}^{\#}\,\mathrm{d}t\right]x_0, \tag{2.95}$$

where $\tilde{A} = A + BK$, and

$$J(x_0, K) = \min_{u(\cdot)\in\mathcal{S}(x_0)} J(x_0, u(\cdot)). \tag{2.96}$$

Proof: Let $F(x,u) = A(x - x_{\mathrm{e}}) + B(u - u_{\mathrm{e}})$, $L(x,u) = (x - x_{\mathrm{e}})^{\mathrm{T}}C^{\mathrm{T}}C(x - x_{\mathrm{e}}) + (u - u_{\mathrm{e}})^{\mathrm{T}}R_2(u - u_{\mathrm{e}})$, $V(x) = (x - x_{\mathrm{e}})^{\mathrm{T}}\hat{P}(x - x_{\mathrm{e}})$, $\hat{P} = \hat{P}^{\mathrm{T}} \geq 0$, $\mathcal{Q} = \mathcal{D} = \mathbb{R}^n$, and $U = \mathbb{R}^m$, and note that (2.70) specializes to

$$(x-x_{\mathrm{e}})^{\mathrm{T}}C^{\mathrm{T}}C(x-x_{\mathrm{e}})+(u-u_{\mathrm{e}})^{\mathrm{T}}R_2(u-u_{\mathrm{e}})+2(x-x_{\mathrm{e}})^{\mathrm{T}}\hat{P}[A(x-x_{\mathrm{e}})+B(u-u_{\mathrm{e}})] = 0. \tag{2.97}$$

Hence, $V'(x)F(x,\phi(x)) \leq 0$ for all $x \in \mathbb{R}^n$. Next, note that

$$\begin{aligned} L(x,u) + V'(x)F(x,u) &= L(x,u) + V'(x)F(x,u) - [L(x,\phi(x)) + V'(x)F(x,\phi(x))] \\ &= [u - \phi(x)]^{\mathrm{T}}R_2[u - \phi(x)] \\ &\geq 0, \qquad x \in \mathbb{R}^n, \end{aligned} \tag{2.98}$$

so that conditions (2.69)–(2.71) of Theorem 2.26 hold.

Next, it follows from (2.78) and (2.97) that $u = -R_2^{-1}B^{\mathrm{T}}\hat{P}x = Kx$, and hence,

$$\begin{aligned} V'(x)F(x,\phi(x)) &= 2(x - x_{\mathrm{e}})^{\mathrm{T}}\hat{P}(A + BK)(x - x_{\mathrm{e}}) \\ &= (x - x_{\mathrm{e}})^{\mathrm{T}}[(A + BK)^{\mathrm{T}}\hat{P} + \hat{P}(A + BK)](x - x_{\mathrm{e}}) \\ &= -(x - x_{\mathrm{e}})^{\mathrm{T}}(C^{\mathrm{T}}C + K^{\mathrm{T}}R_2K)(x - x_{\mathrm{e}}). \end{aligned} \tag{2.99}$$

Now, note that (2.46) is equivalent to (2.94) with $K = -R_2^{-1}B^{\mathrm{T}}P$ and, since semiobservability is preserved under full state-feedback [44], it follows that if (A, C) is semiobservable, then $(\tilde{A}, \tilde{R})$ is semiobservable, where $\tilde{R} \triangleq C^{\mathrm{T}}C + K^{\mathrm{T}}R_2K$. Since $(\tilde{A}, \tilde{R})$ is semiobservable, it follows from *ii*) of Lemma 2.19 that $\mathcal{R}(\tilde{A}) \cap \mathcal{N}(\tilde{A}) = \{0\}$, which implies that $\tilde{A}$ is group invertible [5, p. 119]. Thus, defining $L = I - \tilde{A}\tilde{A}^{\#}$ and considering the Lyapunov function candidate $\hat{V}(\hat{x}) = \hat{x}^{\mathrm{T}}(\hat{P} + L^{\mathrm{T}}L)\hat{x}$, where $\hat{x} \triangleq x - x_{\mathrm{e}}$, global semistability follows as in the proof of Theorem 2.24. Now, it follows from Theorem 2.20 that the least squares solution P_{LS} of (2.94) is given by (2.47), and hence, taking $\hat{P} = P_{\mathrm{LS}}$, (2.95) directly follows from (2.72). Finally, (2.96) is a restatement of (2.73). ■

Remark 2.31. It is important to note that unlike Theorem 2.26 and Corollary 2.28, in Theorem 2.30 we do not require the assumption that $u(\cdot) \in \mathcal{S}(x_0)$. Rather Lyapunov stability, and hence, boundedness of solutions of the closed-loop system follow from the hypothesis of the theorem.

Proposition 2.32. Consider the controlled linear dynamical system $\mathcal{G}$ given by (2.5) and (2.6) with quadratic performance measure (2.4), assume that the pair (A, B) is semicontrollable and the pair (A, C) is semiobservable, and let $P_{\mathrm{LS}} = P_{\mathrm{LS}}^{\mathrm{T}} \geq 0$ be the least squares solution to (2.94). Then, with $u = Kx = -R_2^{-1}B^{\mathrm{T}}P_{\mathrm{LS}}x$, the equilibrium solution $x(t) \equiv x_{\mathrm{e}}$ to (2.5) is globally semistable and (2.101) holds. Furthermore, (2.96) holds.

Proof: Since (A, B) is semicontrollable and (A, C) is semiobservable, the conditions of Theorem 3.7 of [77] are satisfied, and hence, there exists a $n \times n$ matrix $P = P^{\mathrm{T}} \geq 0$ such that (2.94) holds. Let $P_{\mathrm{LS}} = \arg\min_{P \in \mathcal{P}} \|P\|_{\mathrm{F}}$ be the least squares solution of (2.94), where $\mathcal{P}$ denotes the set of all P satisfying (2.94). Now, noting that, with $K = -R_2^{-1}B^{\mathrm{T}}P$, (2.46) is equivalent to (2.94), it follows from Theorems 2.20 and 2.30 that (2.5), with $u = -R_2^{-1}B^{\mathrm{T}}P_{\mathrm{LS}}x$ and P_{LS} given by (2.47), is globally semistable and

$$J(x_0, K) = x_0^{\mathrm{T}}P_{\mathrm{LS}}x_0 \leq J(x_0, u(\cdot)), \tag{2.100}$$

where $K = -R_2^{-1}B^{\mathrm{T}}P_{\mathrm{LS}}$. ■

Proposition 2.33. Consider the controlled linear dynamical system $\mathcal{G}$ given by (2.5) and (2.6) with quadratic performance measure (2.4), assume the pair (A, B) is semistabilizable and the pair (A, C) is semidetectable, and assume that there exists $P = P^{\mathrm{T}} \geq 0$ such that (2.94) holds. Then, with $u = Kx = -R_2^{-1}B^{\mathrm{T}}Px$, the equilibrium solution $x(t) \equiv x_{\mathrm{e}}$ to (2.5) is globally semistable and

$$J(x_0, K) = x_0^{\mathrm{T}}\left[\int_0^\infty e^{\tilde{A}^{\mathrm{T}}t}(C^{\mathrm{T}}C + K^{\mathrm{T}}R_2K)e^{\tilde{A}t}\,\mathrm{d}t\right]x_0, \tag{2.101}$$

where $\tilde{A} = A + BK$, and

$$\min_{u(\cdot)\in\mathcal{S}(x_0)} J(x_0, u(\cdot)) = J(x_0, K_*) \leq J(x_0, K) = 2x_0^{\mathrm{T}}\tilde{A}\tilde{A}^{\#}x_0, \tag{2.102}$$

where $K_* = -R_2^{-1}B^{\mathrm{T}}P_{\mathrm{LS}}$ and $P_{\mathrm{LS}} = P_{\mathrm{LS}}^{\mathrm{T}} \geq 0$ is the least squares solution to (2.94).

Proof: Global semistability of (2.5), with $u = -R_2^{-1}B^{\mathrm{T}}Px$, and (2.101) follow directly from Theorem 2.21. To show (2.102), note that it follows from (2.101) and (2.94) that

$$\begin{aligned} J(x_0, K) &= -x_0^{\mathrm{T}}\left[\int_0^\infty e^{\tilde{A}^{\mathrm{T}}t}(\tilde{A}^{\mathrm{T}}P + P\tilde{A})e^{\tilde{A}t}\,\mathrm{d}t\right]x_0 \\ &= -x_0^{\mathrm{T}}\left[e^{\tilde{A}^{\mathrm{T}}t}P\Big|_{t=0}^{\infty} + Pe^{\tilde{A}t}\Big|_{t=0}^{\infty}\right]x_0 \\ &= x_0^{\mathrm{T}}\left[(\tilde{A}\tilde{A}^{\#})^{\mathrm{T}}P + P\tilde{A}\tilde{A}^{\#}\right]x_0. \end{aligned} \tag{2.103}$$

Since, by Theorem 2.18, $P = P_{\mathrm{LS}} + \alpha zz^{\mathrm{T}}$, where $\alpha \geq 0$ and $z \in \mathcal{N}(\tilde{A}^{\mathrm{T}})$ satisfies (2.44), it follows from (2.103) that

$$J(x_0, K) = x_0^{\mathrm{T}}\left[(\tilde{A}\tilde{A}^{\#})^{\mathrm{T}}P_{\mathrm{LS}} + P_{\mathrm{LS}}\tilde{A}\tilde{A}^{\#}\right]x_0 = 2x_0^{\mathrm{T}}P_{\mathrm{LS}}\tilde{A}\tilde{A}^{\#}x_0.$$

Finally, with $F(x, u) = A(x - x_{\mathrm{e}}) + B(u - u_{\mathrm{e}})$, $L(x, u) = (x - x_{\mathrm{e}})^{\mathrm{T}}C^{\mathrm{T}}C(x - x_{\mathrm{e}}) + (u - u_{\mathrm{e}})^{\mathrm{T}}R_2(u - u_{\mathrm{e}})$, $V(x) = (x - x_{\mathrm{e}})^{\mathrm{T}}P_{\mathrm{LS}}(x - x_{\mathrm{e}})$, $\mathcal{Q} = \mathcal{D} = \mathbb{R}^n$, and $U = \mathbb{R}^m$, it follows using similar arguments as in the proof of Theorem 2.30 that $J(x_0, K_*) = \min_{u(\cdot)\in\mathcal{S}(x_0)} J(x_0, u(\cdot))$. Hence, (2.102) holds. ■

Definition 2.34. A nonnegative-definite matrix $P \in \mathbb{R}^{n\times n}$ is a *semistabilizing solution* of (2.94) if $A - BR_2^{-1}B^{\mathrm{T}}P$ is semistable. Furthermore, a semistabilizing solution $P_{\min}$ of (2.94) is the *minimally semistabilizing solution* to (2.94) if $P \geq P_{\min}$ for every semistabilizing solution P to (2.94).

It follows from Definition 2.34 that the least squares solution $P_{\rm LS}$ to (2.94) is the minimally semistabilizing solution to (2.94). Given the linear dynamical system given by (2.5) and (2.6), if the pair (A, B) is semicontrollable and the pair (A, C) is semiobservable, then it follows from Lemma 2.13 that, for every $K \in \mathbb{R}^{m\times n}$, the pair $(A + BK, B)$ is semicontrollable, and by Proposition 2.1 in [44], it follows that, for every $R_2 \in \mathbb{R}^{n\times n}$ such that $R_2 = R_2^{\rm T} > 0$, the pair $(A + BK, C^{\rm T}C + K^{\rm T}R_2K)$ is semiobservable. Furthermore, if the pair (A, C) is semiobservable, then (A, C) is semidetectable and it follows from Theorems 2.9 and 2.18 that every solution $P = P^{\rm T} \geq 0$ of (2.46) is given by

$$P = \int_0^\infty e^{\tilde{A}^{\rm T}t}(C^{\rm T}C + K^{\rm T}R_2K)e^{\tilde{A}t}{\rm d}t + zz^{\rm T}, \tag{2.104}$$

where $\tilde{A} = A + BK$ and $z \in \mathcal{N}(\tilde{A}^{\rm T})$. Now, if $K = -R_2^{-1}B^{\rm T}P$, then (2.46) is equivalent to (2.94), where P can be computed using the Schur decomposition of the Hamiltonian matrix [8, pp. 853-859], and the least squares solution $P_{\rm LS} = P_{\rm LS}^{\rm T} \geq 0$ of (2.94) is given by $P_{\rm LS} = P - zz^{\rm T}$, where z is the solution of the optimization problem

$$\min_{z\in\mathbb{R}^n} \|P - zz^{\rm T}\|_{\rm F} \tag{2.105}$$

subject to

$$0 \leq P - zz^{\rm T}, \tag{2.106}$$

$$0 = (A^{\rm T} - PBR_2^{-1}B^{\rm T})z. \tag{2.107}$$

One might surmise that Theorem 2.30 and Proposition 2.32 give different values for $J(x_0, K)$. However, note that

$$\begin{aligned} J(x_0, u(\cdot)) &= \int_0^\infty \left[(x(t) - x_{\rm e})^{\rm T}C^{\rm T}C(x(t) - x_{\rm e}) + (u(t) - u_{\rm e})^{\rm T}R_2(u(t) - u_{\rm e})\right]{\rm d}t \\ &= \int_0^\infty \left[(x_0 - x_{\rm e})^{\rm T}e^{\tilde{A}^{\rm T}t}(C^{\rm T}C + K^{\rm T}R_2K)e^{\tilde{A}t}(x_0 - x_{\rm e})\right]{\rm d}t \\ &= x_0^{\rm T}\left[\int_0^\infty (\tilde{A}\tilde{A}^\#)^{\rm T}e^{\tilde{A}^{\rm T}t}(C^{\rm T}C + K^{\rm T}R_2K)e^{\tilde{A}t}\tilde{A}\tilde{A}^\#\,{\rm d}t\right]x_0 \end{aligned} \tag{2.108}$$

and, since

$$\begin{aligned}J(x_0,u(\cdot)) &= \int_0^\infty \big[(x(t)-x_\mathrm{e})^\mathrm{T}C^\mathrm{T}C(x(t)-x_\mathrm{e})+(u(t)-u_\mathrm{e})^\mathrm{T}R_2(u(t)-u_\mathrm{e})\big]\mathrm{d}t\\ &= \int_0^\infty x^\mathrm{T}(t)(C^\mathrm{T}C+K^\mathrm{T}R_2K)x(t)\mathrm{d}t, \end{aligned}\tag{2.109}$$

it follows that

$$\begin{aligned}J(x_0,u(\cdot)) &= \int_0^\infty \big[(x(t)-x_\mathrm{e})^\mathrm{T}C^\mathrm{T}C(x(t)-x_\mathrm{e})+(u(t)-u_\mathrm{e})^\mathrm{T}R_2(u(t)-u_\mathrm{e})\big]\mathrm{d}t\\ &= \int_0^\infty x^\mathrm{T}(t)(C^\mathrm{T}C+K^\mathrm{T}R_2K)x(t)\,\mathrm{d}t\\ &= x_0^\mathrm{T}\int_0^\infty e^{\tilde{A}^\mathrm{T}t}(C^\mathrm{T}C+K^\mathrm{T}R_2K)e^{\tilde{A}t}\mathrm{d}t\,x_0. \end{aligned}\tag{2.110}$$

Hence, (2.95) and (2.101) are equivalent.

Finally, in light of Theorem 2.30 and Lemma 4.3 of [56] the following result is immediate.

Proposition 2.35. Consider the linear controlled dynamical system $\mathcal{G}$ given by (2.5) and (2.6). If the pair (A,B) is semicontrollable, the pair (A,C) is semiobservable, and $\mathcal{G}$, with $u=Kx$, is semistable, then

$$P=\int_0^\infty (\tilde{A}\tilde{A}^\#)^\mathrm{T}e^{\tilde{A}^\mathrm{T}t}(C^\mathrm{T}C+K^\mathrm{T}R_2K)e^{\tilde{A}t}\tilde{A}\tilde{A}^\#\,\mathrm{d}t \tag{2.111}$$

satisfies

$$0=\tilde{A}^\mathrm{T}(\tilde{A}^\mathrm{T}P+P\tilde{A}+C^\mathrm{T}C+K^\mathrm{T}R_2K)\tilde{A}, \tag{2.112}$$

or, equivalently, (2.46).

2.5. Illustrative Numerical Examples

In this section, we provide two numerical examples to highlight the optimal semistabilization framework developed in this chapter.

2.5.1. Optimal Consensus Control for Multiagent Formations

For the first example, we use the optimal semistabilization framework to design consensus controllers for multiagent networks of single integrator systems. Specifically, the consensus problem involves the design of a dynamic protocol algorithm that guarantees semistability and system state equipartition [91], that is, $\lim_{t\to\infty} x_i(t) = \alpha \in \mathbb{R}$ for $i = 1, \ldots, n$, where $x_i(t)$ denotes the ith component of the system state vector $x(t)$. To address the consensus problem of n agents exchanging information with collective dynamics given by (2.5) and (2.6), we set the entries a_{ij}, $i, j = 1, \ldots, n$, of the system matrix A such that, if agent j receives information from the agent i, $i \neq j$, then $a_{ij} = 1$, otherwise $a_{ij} = 0$, and $a_{ii} = -\sum_{j=1,\, j\neq i}^{n} a_{ij}$.

Here, we design a control law $u = Kx$ such that (2.5) with $u = Kx$ is semistable, the performance measure (2.4) is minimized in the sense of (2.73), and

$$x_{\mathrm{e}} = \lim_{t\to\infty} x(t) = \alpha\mathbf{e}, \tag{2.113}$$

where $\mathbf{e} \triangleq [1, \ldots, 1]^{\mathrm{T}}$ and $\alpha \in \mathbb{R}\backslash\{0\}$ [100]. In order to account for the constraint (2.113), we introduce a terminal steady state constraint to the performance measure (2.4) so that

$$J(x_0, u(\cdot)) = \lim_{\tau\to\infty} \Bigg\{ \mu^{\mathrm{T}}(x(\tau) - \alpha\mathbf{e}) + \int_0^{\tau} [(x(t) - x_{\mathrm{e}})^{\mathrm{T}} C^{\mathrm{T}} C (x(t) - x_{\mathrm{e}}) + (u(t) - u_{\mathrm{e}})^{\mathrm{T}} R_2 (u(t) - u_{\mathrm{e}})] \mathrm{d}t \Bigg\}, \tag{2.114}$$

where $\mu \in \mathbb{R}^n$, is minimized in the sense of (2.73). This optimization problem is in the form of a Bolza problem [19, Ch. 2], whereas the optimization problems discussed in Section 2.3 are in the form of Lagrange problems.

To account for the terminal consensus constraint, we introduce the additional scalar state $x_{n+1} : [0, \infty) \to \mathbb{R}$ and define $\hat{x} \triangleq [x^{\mathrm{T}}, x_{n+1}]^{\mathrm{T}}$ so that

$$\dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \hat{B}u(t), \qquad \hat{x}(0) = \begin{bmatrix} x_0 \\ \lim_{\tau\to\infty} \dfrac{\mu^{\mathrm{T}}(x(\tau) - \alpha\mathbf{e})}{\tau} \end{bmatrix}, \qquad t \geq 0, \tag{2.115}$$

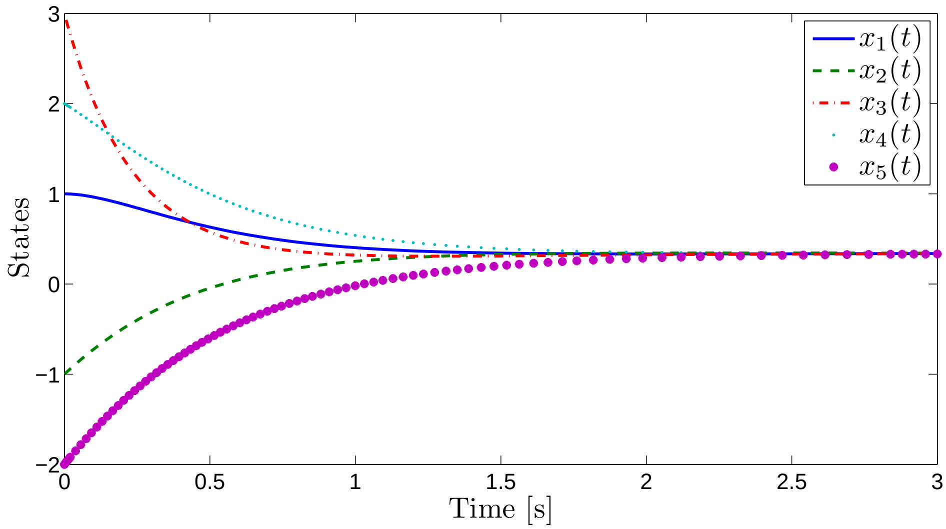

Figure 2.1: State trajectories of the closed-loop system.

$$y(t) = \hat{C}\hat{x}(t), \tag{2.116}$$

where

$$\hat{A} \triangleq \begin{bmatrix} A & 0_n \\ 0_n^{\mathrm{T}} & 0 \end{bmatrix}, \qquad \hat{B} \triangleq \begin{bmatrix} B \\ 0_m^{\mathrm{T}} \end{bmatrix}, \qquad \hat{C} \triangleq \begin{bmatrix} C & 0_n \end{bmatrix},$$

and 0_n denotes the n-dimensional zero vector. In this case, the performance measure (2.114) can be rewritten as

$$J(x_0, u(\cdot)) = \int_0^\infty [(\hat{x}(t) - \hat{x}_{\mathrm{e}})^{\mathrm{T}}\hat{C}^{\mathrm{T}}\hat{C}(\hat{x}(t) - \hat{x}_{\mathrm{e}}) + (u(t) - u_{\mathrm{e}})^{\mathrm{T}}R_2(u(t) - u_{\mathrm{e}})]\mathrm{d}t, \tag{2.117}$$

where $\hat{x}_{\mathrm{e}}$ is an equilibrium point of (2.115) for some $u_{\mathrm{e}} \in \mathbb{R}^m$. Note that if the pair (A, B) is semistabilizable and the pair (A, C) is semidetectable, then it follows from Definitions 2.3 and 2.7 that the pair $(\hat{A}, \hat{B})$ is semistabilizable and the pair $(\hat{A}, \hat{C})$ is semidetectable. Hence, it follows from Theorem 2.30 that the solution $\hat{x}(t) = \hat{x}_{\mathrm{e}}$, $t \geq 0$, to (2.115) with $u = K\hat{x}$ and $K = -R_2^{-1}\hat{B}^{\mathrm{T}}\hat{P}_{\mathrm{LS}}$ is globally semistable, where $\hat{P}_{\mathrm{LS}}$ is the least squares solution of

$$0 = \hat{A}^{\mathrm{T}}\hat{P} + \hat{P}\hat{A} + \hat{C}^{\mathrm{T}}\hat{C} - \hat{P}\hat{B}R_2^{-1}\hat{B}^{\mathrm{T}}\hat{P}, \tag{2.118}$$

and (2.95) and (2.96) hold with $\tilde{A} = \hat{A} + \hat{B}K$.

Next, define $\hat{\mu} \triangleq [\mu^{\mathrm{T}}, 0]^{\mathrm{T}}$ and note that if $\hat{x}(t) = [x^{\mathrm{T}}(t),\, x_{n+1}(t)]^{\mathrm{T}}$, $t \geq 0$, is the solution of (2.115) with $u = K\hat{x}$, then

$$\lim_{\tau\to\infty} \frac{\mu^{\mathrm{T}}(x(\tau) - \alpha\mathbf{e})}{\tau} = \mu^{\mathrm{T}} \lim_{\tau\to\infty} \frac{x(\tau)}{\tau} = \hat{\mu}^{\mathrm{T}} \lim_{\tau\to\infty} \frac{e^{\tilde{A}\tau}\hat{x}(0)}{\tau} = \hat{\mu}^{\mathrm{T}}\tilde{A} \lim_{\tau\to\infty} e^{\tilde{A}\tau}\hat{x}(0). \tag{2.119}$$

Now, it follows from Proposition 11.8.1 of [8] that

$$\begin{aligned} \lim_{\tau\to\infty} \frac{\mu^{\mathrm{T}}(x(\tau) - \alpha\mathbf{e})}{\tau} &= \hat{\mu}^{\mathrm{T}}\tilde{A} \lim_{\tau\to\infty} e^{\tilde{A}\tau}\hat{x}(0) \\ &= \hat{\mu}^{\mathrm{T}}\tilde{A}(I_{n+1} - \tilde{A}\tilde{A}^{\#})\hat{x}(0) \\ &= \hat{\mu}^{\mathrm{T}}(\tilde{A} - \tilde{A}\tilde{A}\tilde{A}^{\#})\hat{x}(0) \\ &= 0, \end{aligned} \tag{2.120}$$

and hence, the system given by (2.115) and (2.116) with $u = K\hat{x}$ is equivalent to

$$\dot{\hat{x}}(t) = \tilde{A}\hat{x}(t), \qquad \hat{x}(0) = \begin{bmatrix} x_0 \\ 0 \end{bmatrix}, \qquad t \geq 0, \tag{2.121}$$

$$y(t) = \hat{C}\hat{x}(t). \tag{2.122}$$

For our simulation, we consider five agents so that

$$A = \begin{bmatrix} -2 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 1 & 1 & -4 & 1 & 1 \\ 0 & 1 & 1 & -2 & 0 \\ 1 & 1 & 0 & 0 & -2 \end{bmatrix}, \qquad B = I_5, \tag{2.123}$$

and set

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad R_2 = I_5. \tag{2.124}$$

Note that the pair (A, B) is controllable, and hence, semistabilizzable, and the pair (A, C) is semidetectable but not observable. In this case, the least squares solution of (2.118) is given by

$$\hat{P}_{\mathrm{LS}} = \begin{bmatrix} 0.1963 & -0.0513 & 0.0115 & -0.0646 & -0.0919 & 0 \\ -0.0513 & 0.2261 & -0.0082 & 0.0360 & -0.2024 & 0 \\ 0.0115 & -0.0082 & 0.1320 & 0.0417 & -0.1770 & 0 \\ -0.0646 & 0.0360 & 0.0417 & 0.2533 & -0.2663 & 0 \\ -0.0919 & -0.2024 & -0.1770 & -0.2663 & 0.7376 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{2.125}$$

For $x_0 = [1, -1, 3, 2, -2]^{\mathrm{T}}$ the trajectories of the closed-loop system are shown in Figure 2.1.

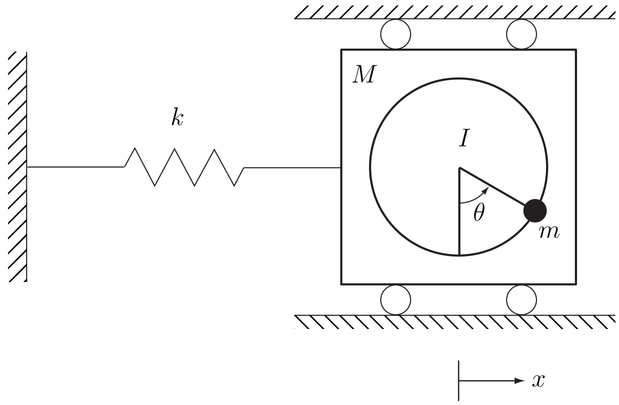

Figure 2.2: Rotational/translational proof-mass actuator.

2.5.2. Rotational/translational proof-mass actuator

Consider the mechanical system adopted from [38] shown in Figure 2.2 involving an eccentric rotational inertia on a translational oscillator giving rise to nonlinear coupling between the undamped oscillator and the rotational rigid body mode. The oscillator cart of mass M is connected to a fixed support via a linear spring of stiffness k. The cart is constrained to one-dimensional motion and the rotational proof-mass actuator consists of a mass m and mass moment of inertia I located at a distance e from the cart's center of mass.

Letting q, $\dot{q}$, θ, $\dot{\theta}$, u_1, and u_2 denote the translational position and velocity of the cart, the angular position and velocity of the rotational proof mass, and the force acting on the cart and the moment acting on the rotating mass, respectively, the dynamic equations of motion are given by

$$(M+m)\ddot{q}(t) + me\big[\ddot{\theta}(t)\cos\theta(t) - \dot{\theta}^2(t)\sin\theta(t)\big] + kq(t) = u_1(t), \tag{2.126}$$

$$(I+me^2)\ddot{\theta}(t) + me\ddot{q}(t)\cos\theta(t) = u_2(t), \tag{2.127}$$

where $t \geq 0$, $q(0) = q_0$, $\dot{q}(0) = \dot{q}_0$, $\theta(0) = \theta_0$, and $\dot{\theta}(0) = \dot{\theta}_0$.

For this example, we seek a state feedback controller $u = [u_1, u_2]^{\mathrm{T}} = \phi(x)$, where $x = [q,\ \dot{q},\ \dot{\theta},\ \theta]^{\mathrm{T}}$, such that the performance measure

$$J(x(0), u(\cdot)) = \int_0^\infty \big[(x(t)-x_{\mathrm{e}})^{\mathrm{T}} R_1 (x(t)-x_{\mathrm{e}}) + (u(t)-u_{\mathrm{e}})^{\mathrm{T}}(u(t)-u_{\mathrm{e}})\big]\mathrm{d}t, \tag{2.128}$$

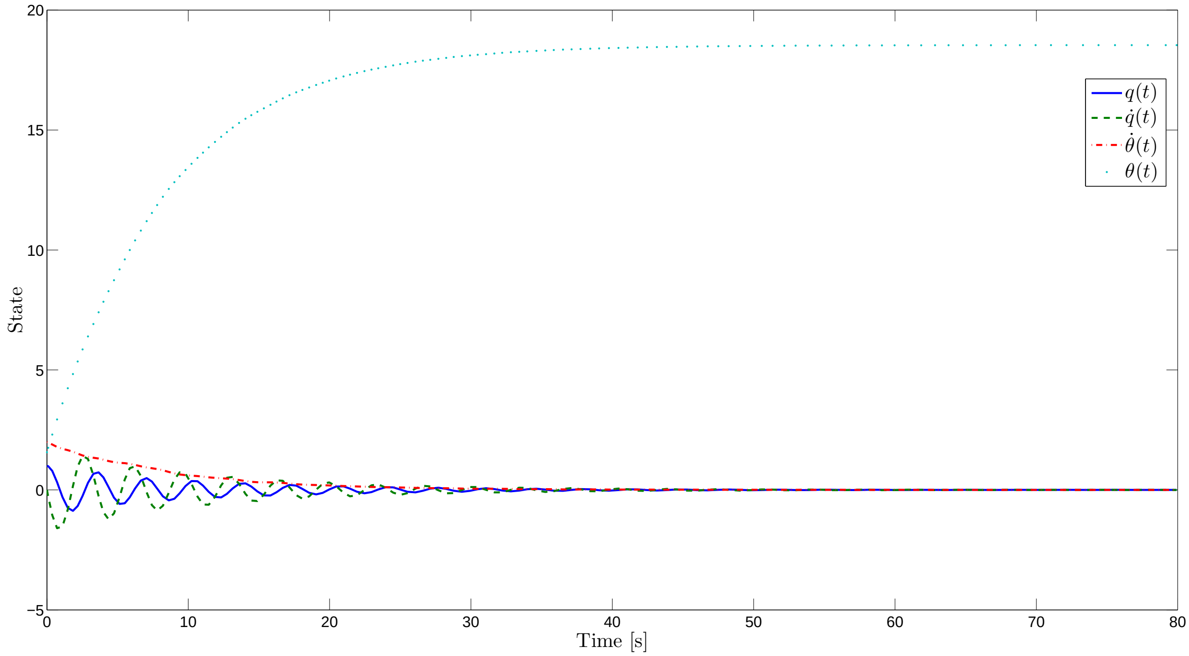

Figure 2.3: Closed-loop system trajectories versus time.

where

$$R_1 = \frac{1}{4}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

is minimized in the sense of (2.73), and (2.126) and (2.127) is semistable.

Next, note that (2.126) and (2.127) with performance measure (2.128) can be cast in the form of (2.79) with performance measure (2.82). In this case, Corollary 2.28 can be applied with $n = 4$, $m = 2$, $l = 4$, $y = Cx$, $R_1 = C^{\mathrm{T}}C$, and $R_2(x) = I_4$, $x \in \mathbb{R}^4$, to characterize the optimal semistabilizing controllers. The explicit expression of $f(x) + G(x)u$ is omitted for brevity. Specifically, (2.84) specializes to

$$0 = (x - x_{\mathrm{e}})^{\mathrm{T}} R_1 (x - x_{\mathrm{e}}) + V'(x)f(x) + V'(x)G(x)u_{\mathrm{e}} - \frac{1}{4}V'(x)G(x)G^{\mathrm{T}}(x)V'^{\mathrm{T}}(x), \quad (2.129)$$

which is satisfied by

$$V(x) = \frac{1}{2}(x - x_{\mathrm{e}})^{\mathrm{T}}P(x - x_{\mathrm{e}}), \qquad x \in \mathbb{R}^4, \quad (2.130)$$

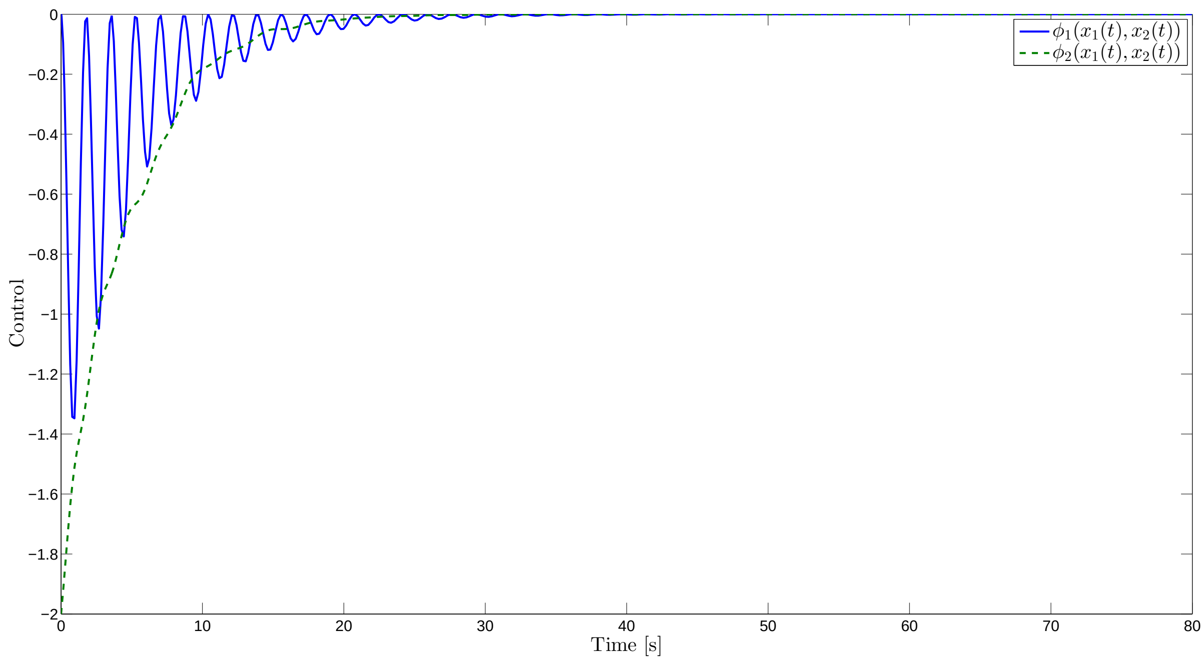

Figure 2.4: Control signal versus time.

where

$$P(x) \triangleq \begin{bmatrix} k & 0 & 0 & 0 \\ 0 & M+m & me\cos\theta & 0 \\ 0 & me\cos\theta & I+me^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{2.131}$$

In this case, (2.85) specializes to

$$\phi(x) = -\frac{1}{2}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}(x - x_{\rm e}), \qquad x \in \mathbb{R}^4. \tag{2.132}$$

Note that the state feedback control law (2.132) is equivalent to a virtual damper applied to the translational mass M and the rotational mass m.

Finally, to show boundedness of solutions of the closed-loop system (2.126) and (2.127) with $u = \phi(x)$ given by (2.132), note that the largest invariant set of $\mathcal{M} = \{x \in \mathbb{R}^4 : V'(x)[f(x) + G(x)\phi(x)] = 0\}$ is $\mathcal{Z} \triangleq \{(0, 0, 0, \theta), \theta \in \mathbb{R}\}$. Now, Lyapunov stability of $x_{\rm e} = [0, 0, 0, \theta_{\rm e}]^{\rm T} \in \mathcal{Z}$ for every $\theta_{\rm e} \in \mathbb{R}$ follows from Theorem 2 of [63] by noting that $V(x_{\rm e}) = 0$, $V(x) \geq 0$, $x \in \mathbb{R}^4$, $V'(x)[f(x) + G(x)\phi(x)] = -\dot{q}^2 - \dot{\theta}^2 \leq 0$, $x \in \mathbb{R}^4$, $[\dot{q}(t), \dot{\theta}(t)]^{\rm T} = [0, 0]^{\rm T}$, $t \in \mathbb{R}$, if and only if $[q(t), \theta(t)]^{\rm T} = [q_{\rm e}, \theta_{\rm e}]^{\rm T}$, $q_{\rm e} \in \mathbb{R}$, and $x(t) \equiv \hat{x}_{\rm e} \triangleq [q_{\rm e}, 0, 0, \theta_{\rm e}]^{\rm T} \in \mathcal{M}$, $t < 0$, if and only if $q_{\rm e} = 0$. Hence, it follows from Corollary 2.28 that the solution $x(t) \equiv x_{\rm e}$,

$t \geq 0$, is semistable.

Let $M = 2\,\mathrm{kg}$, $m = 1\,\mathrm{kg}$, $e = 0.2\,\mathrm{m}$, $k = 10\,\mathrm{N/m}$, $I = 4\,\mathrm{kg}\cdot\mathrm{m}^2$, $q_0 = 1\,\mathrm{m}$, $\dot{q}_0 = 0\,\mathrm{m/s}$, $\theta_0 = \pi/2$, and $\dot{\theta}_0 = 2\,\mathrm{Hz}$. Figure 2.3 shows the state trajectories of the controlled system versus time. Figure 2.4 shows the control signal versus time. Finally,

$$J(x(0), \phi(x(\cdot))) = \frac{1}{2}(x(0) - x_{\mathrm{e}})^{\mathrm{T}} P(x(0) - x_{\mathrm{e}})(x(0) - x_{\mathrm{e}}) = 26.16\,\mathrm{N}\cdot\mathrm{m} \tag{2.133}$$

and $\theta_{\mathrm{e}} = 18.5407$.

Chapter 3

Semistabilization, Feedback Dissipativation, and System Thermodynamics

3.1. Introduction

System thermodynamics, in the sense of [41], involves open interconnected dynamical systems that exchange matter and energy with their environment in accordance with the first law (conservation of energy) and the second law (nonconservation of entropy) of thermodynamics. Self-organization can spontaneously occur in such systems by invoking the two fundamental axioms of the science of heat. Namely, *i*) if the energies in the connected subsystems of an interconnected system are equal, then energy exchange between these subsystems is not possible, and *ii*) energy flows from more energetic subsystems to less energetic subsystems. These axioms establish the existence of a system entropy function as well as *equipartition of energy* [41] in system thermodynamics and *information consensus* [43] in cooperative networks; an emergent behavior in thermodynamic systems as well as swarm systems.

Using system-theoretic thermodynamic concepts, an energy and entropy-based hybrid controller architecture was proposed in [40, 42] as a means for achieving enhanced energy dissipation in lossless and dissipative dynamical systems. These dynamic controllers combined a logical switching architecture with continuous dynamics to guarantee that the system plant energy is strictly decreasing across switchings. The general framework developed in [40] leads to closed-loop systems described by impulsive differential equations [42]. In particular, the authors in [40, 42] construct hybrid dynamic controllers that guarantee that the closed-loop system is consistent with basic thermodynamic principles. Specifically, the existence of an entropy function for the closed-loop system is established that satisfies a hybrid Clausius-type inequality. Special cases of energy-based and entropy-based hybrid controllers involving state-dependent switching were also developed.

Recent technological advances in communications and computation have spurred a broad interest in control of networks and control over networks [91]. Network systems involve distributed decision-making for coordination of networks of dynamic agents and address a broad area of applications including cooperative control of unmanned air vehicles, microsatellite clusters, mobile robotics, and congestion control in communication networks. In many applications involving multiagent systems, groups of agents are required to agree on certain quantities of interest. In particular, it is important to develop information consensus protocols for networks of dynamic agents, wherein a unique feature of the closed-loop dynamics under any control algorithm that achieves consensus is the existence of a continuum of equilibria representing a state of equipartitioning or consensus [24, 44, 57, 58, 60]. Under such dynamics, the limiting consensus state achieved is not determined completely by the dynamics, but depends on the initial system state as well. For such systems possessing a continuum of equilibria, *semistability* [11,38], and not asymptotic stability, is the relevant notion of stability. In addition, system-theoretic thermodynamic concepts [24, 41, 57, 58, 60] have proved invaluable in addressing Lyapunov stability and convergence for nonlinear dynamical networks.

Semistability and state equipartitioning also arise in numerous complex large-scale dynamical networks that demonstrate a degree of synchronization. System synchronization typically involves coordination of events that allows a dynamical system to operate in unison resulting in system self-organization. The onset of synchronization in populations of coupled dynamical networks have been studied for various complex networks including network models for mathematical biology, statistical physics, kinetic theory, bifurcation theory, as well as plasma physics [112]. Synchronization of firing neural oscillator populations also appears in the neuroscience literature [18, 59].

In this chapter, we develop a thermodynamic framework for semistabilization of linear and nonlinear dynamical systems. The proposed framework unifies system thermodynamic concepts with feedback dissipativity and control theory to provide a thermodynamic-based semistabilization framework for feedback control design. Specifically, we consider feedback passive and dissipative systems [21, 38, 115, 116] since these systems are not only widespread in system engineering, but also have clear connections to thermodynamics [41, 115]. In addition, using ideas from [41], we define the notion of entropy for a nonlinear feedback dissipative dynamical system. Then, we develop a state feedback control design framework that minimizes the time-averaged system entropy and show that, under certain conditions, this controller also minimizes the time-averaged system energy. The main result is cast as an optimal control problem characterized by an optimization problem involving two linear matrix inequalities.

3.2. Feedback Dissipativation and Thermodynamics

In this section, we consider nonlinear dynamical systems $\mathcal{G}$ of the form

$$\dot{x}(t) = f(x(t)) + G(x(t))u(t), \quad x(0) = x_0, \quad t \geq 0, \tag{3.1}$$

$$y(t) = h(x(t)) + J(x(t))u(t), \tag{3.2}$$

where, for each $t \geq 0$, $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$ denotes the state vector, $u(t) \in U \subseteq \mathbb{R}^m$ denotes the control input, $y(t) \in Y \subseteq \mathbb{R}^l$ denotes the system output, and $f : \mathcal{D} \to \mathbb{R}^n$, $G : \mathcal{D} \to \mathbb{R}^{n\times m}$, $h : \mathcal{D} \to \mathbb{R}^l$, and $J : \mathcal{D} \to \mathbb{R}^{l\times m}$. For the dynamical system $\mathcal{G}$ given by (3.1) and (3.2) defined on the state space $\mathcal{D} \subseteq \mathbb{R}^n$, $\mathcal{U}$ and $\mathcal{Y}$ define input and output spaces, respectively, consisting of continuous bounded U-valued and Y-valued functions on the semi-infinite interval $[0, \infty)$. The spaces $\mathcal{U}$ and $\mathcal{Y}$ are assumed to be closed under the shift operator. The mappings $f(\cdot)$,

$G(\cdot)$, $h(\cdot)$, and $J(\cdot)$ are assumed to be continuously differentiable and $f(\cdot)$ has at least one equilibrium point $x_{\rm e} \in \mathcal{D}$ so that $f(x_{\rm e}) + G(x_{\rm e})u_{\rm e} = 0$ and $y_{\rm e} = h(x_{\rm e}) + J(x_{\rm e})u_{\rm e}$ for some $u_{\rm e} \in U$. Finally, we assume that $\mathcal{G}$ is completely reachable [38].

The following definition of *feedback dissipativation* is needed for developing the main results in this section. Feedback dissipative systems define a class of dynamical systems for which a continuously differentiable feedback transformation exists that renders the system $\mathcal{G}$ dissipative and is a generalization of the *feedback passivation* notion introduced in [21].

Definition 3.1. $\mathcal{G}$ is called *state feedback dissipative* if there exists a state feedback transformation $u = \phi(x) + \beta(x)v$, where $\phi : \mathcal{D} \to \mathbb{R}^m$ and $\beta : \mathcal{D} \to \mathbb{R}^{m\times m}$ are continuously differentiable, with $\det\beta(x) \neq 0$, $x \in \mathcal{D}$, such that the nonlinear dynamical system $\mathcal{G}_{\rm s}$ given by

$$\dot{x}(t) = f(x(t)) + G(x(t))\phi(x(t)) + G(x(t))\beta(x(t))v(t), \quad x(0) = x_0, \quad t \geq 0, \tag{3.3}$$

$$y(t) = h(x(t)) + J(x(t))\phi(x(t)) + J(x(t))\beta(x(t))v(t), \tag{3.4}$$

is dissipative with respect to the supply rate $r(v, y)$, where $r : U \times Y \to \mathbb{R}$ is locally integrable for all input-output pairs satisfying (3.3) and (3.4), and $r(0, 0) = 0$. If $r(v, y) = v^{\rm T}y$, then $\mathcal{G}$ is *state feedback passive*.

For simplicity of exposition, in the reminder of the section we will assume that $\beta(x) = I_m$.

Remark 3.2. The nonlinear dynamical system $\mathcal{G}$ given by (3.1) and (3.2) is feedback equivalent to a passive system with a C^2 storage function if and only if $\mathcal{G}$ has (vector) relative degree $\{1, \ldots, 1\}$ at $x = 0$ and is weakly minimum phase. Alternatively, the Kalman-Yakubovich-Popov lemma [38] can be used to construct smooth state feedback controllers that guarantee feedback passivation as well as feedback dissipativation [21].

The following result is a direct consequence of dissipativity theory [38]. For this result as well as for the reminder of the section we assume that all storage functions $V_{\rm s}(\cdot)$ of the nonlinear dynamical system $\mathcal{G}_{\rm s}$ are continuously differentiable.

Proposition 3.3 [38]. Consider the nonlinear dynamical system $\mathcal{G}$ given by (3.1) and (3.2), and assume that $\mathcal{G}$ is state feedback dissipative. Then there exist functions $V_{\rm s} : \mathbb{R}^n \to \mathbb{R}$, $\ell : \mathbb{R}^n \to \mathbb{R}^p$, and $\mathcal{W} : \mathbb{R}^n \to \mathbb{R}^{p\times m}$ such that $V_{\rm s}(\cdot)$ is continuously differentiable and nonnegative definite, $V_{\rm s}(x_{\rm e}) = V_{\rm se}$, and $\dot{V}_{\rm s}(x) = r(v, y) - [\ell(x) + \mathcal{W}(x)v]^{\rm T}[\ell(x) + \mathcal{W}(x)v]$.

Defining $d(x, v) \triangleq [\ell(x)+\mathcal{W}(x)v]^{\rm T}[\ell(x)+\mathcal{W}(x)v]$, where $d : \mathcal{D}\times U \to \overline{\mathbb{R}}_+$ is a continuous, nonnegative-definite dissipation rate function, and $\mathrm{d}Q(t) \triangleq [r(v(t), y(t)) - d(x(t), v(t))]\mathrm{d}t$, where $\mathrm{d}Q(t)$ is the amount of energy (heat) received or dissipated by the state feedback dissipative system over the infinitesimal time interval $\mathrm{d}t$, we arrive at a *Clausius-type equality* for $\mathcal{G}_{\rm s}$. For the next result $\oint$ denotes a cyclic integral evaluated along an arbitrary closed path of $\mathcal{G}_{\rm s}$, that is, $\oint \triangleq \int_{t_0}^{t_{\rm f}}$ with $t_{\rm f} \geq t_0$ and $v(\cdot) \in \mathcal{U}$ such that $x(t_{\rm f}) = x(t_0) = x_0 \in \mathcal{D}$.

Proposition 3.4. Consider the nonlinear dynamical system $\mathcal{G}$ given by (3.1) and (3.2), and assume that $\mathcal{G}$ is state feedback dissipative. Then, for all $t_{\rm f} \geq t_0 \geq 0$ and $v(\cdot) \in \mathcal{U}$ such that $V_{\rm s}(x(t_{\rm f})) = V_{\rm s}(x(t_0))$,

$$\int_{t_0}^{t_{\rm f}} \frac{r(v(t), y(t)) - d(x(t), v(t))}{c + V_{\rm s}(x(t))}\mathrm{d}t = \oint \frac{\mathrm{d}Q(t)}{c + V_{\rm s}(x(t))} = 0, \tag{3.5}$$

where $c > 0$.

Proof: It follows from Proposition 3.3 that

$$\oint \frac{\mathrm{d}Q(t)}{c + V_{\rm s}(x(t))} = \int_{t_0}^{t_{\rm f}} \frac{r(v(t), y(t)) - d(x(t), v(t))}{c + V_{\rm s}(x(t))}\mathrm{d}t = \log_e \frac{c + V_{\rm s}(x(t_{\rm f}))}{c + V_{\rm s}(x(t_0))}, \tag{3.6}$$

which proves the assertion. ■

In light of Proposition 3.4, we give a definition of entropy for a feedback dissipative system.

Definition 3.5. For the nonlinear dynamical system $\mathcal{G}_{\rm s}$ given by (3.3) and (3.4) a function $S : \mathcal{D} \to \mathbb{R}$ satisfying

$$S(x(t_2)) \geq S(x(t_1)) + \int_{t_1}^{t_2} \frac{\mathrm{d}Q(t)}{c + V_{\rm s}(x(t))} \tag{3.7}$$

for every $t_2 \geq t_1 \geq 0$ and $v(\cdot) \in \mathcal{U}$ is called the *entropy* function of $\mathcal{G}_{\rm s}$.

Recalling that $\mathrm{d}Q(t) = [r(v(t), y(t)) - d(x(t), v(t))]\mathrm{d}t$ is the infinitesimal amount of the net energy received or dissipated by $\mathcal{G}_{\rm s}$ over the infinitesimal time interval $\mathrm{d}t$, it follows from (3.7) that

$$\mathrm{d}S(x(t)) \geq \frac{\mathrm{d}Q(t)}{c + V_{\rm s}(x(t))}, \quad t \geq t_0. \tag{3.8}$$

Inequality (3.8) is analogous to the classical thermodynamic inequality for the variation of entropy during an infinitesimal irreversible transformation with the shifted system energy $c + V_{\rm s}(x)$ playing the role of the thermodynamic temperature. Specifically, note that since $\frac{\mathrm{d}S}{\mathrm{d}Q} = \frac{1}{c+V_{\rm s}}$, it follows that $\frac{\mathrm{d}S}{\mathrm{d}Q}$ defines the reciprocal of the system thermodynamic temperature $T_{\rm e}$. That is, $\frac{1}{T_{\rm e}} \triangleq \frac{\mathrm{d}S}{\mathrm{d}Q}$ and $T_{\rm e} > 0$.

The next result shows that all entropy functions for a nonlinear dynamical system $\mathcal{G}_{\rm s}$ given by (3.3) and (3.4) are continuous on $\mathcal{D}$. For stating this result, recall that the nonlinear dynamical system $\mathcal{G}_{\rm s}$ given by (3.3) and (3.4) with $\hat{x} \in \mathbb{R}^n$ and $\hat{v} \in \mathbb{R}^m$ such that $x(t) \equiv \hat{x}$ and $v(t) \equiv \hat{v}$, $t \geq 0$, satisfying (3.3), is *locally controllable* at $\hat{x}$ if, for every $T > 0$ and $\varepsilon > 0$, the set of points that can be reached from and to $\hat{x}$ in finite time T using admissible inputs $v : [0, T] \to U$, satisfying $\|v(t) - \hat{v}\| < \varepsilon$, contains a neighborhood of $\hat{x}$ [38, p. 333].

Theorem 3.6. Consider the dissipative nonlinear dynamical system $\mathcal{G}_{\rm s}$ given by (3.3) and (3.4). Assume that $\mathcal{G}_{\rm s}$ is completely reachable and assume that for every $x_{\rm e} \in \mathcal{D}$, there exists $v_{\rm e} \in \mathbb{R}^m$ such that $x(t) \equiv x_{\rm e}$ and $v(t) \equiv v_{\rm e}$, $t \geq 0$, satisfy (3.3), and $\mathcal{G}_{\rm s}$ is locally controllable at every $x_{\rm e} \in \mathcal{D}$. Then every entropy function $S(x)$, $x \in \mathcal{D}$, of $\mathcal{G}_{\rm s}$ is continuous on $\mathcal{D}$.

Proof: Let $x_{\mathrm{e}} \in \mathcal{D}$ be an equilibrium point of $\mathcal{G}_{\mathrm{s}}$ with $v(t) \equiv v_{\mathrm{e}}$, that is, $f(x_{\mathrm{e}}) + G(x_{\mathrm{e}})\phi(x_{\mathrm{e}}) + G(x_{\mathrm{e}})v_{\mathrm{e}} = 0$. Now, let $\delta > 0$ and note that it follows from the continuity of $f(\cdot)$, $G(\cdot)$, and $\phi(\cdot)$ that there exist $T > 0$ and $\varepsilon > 0$ such that for every $v : [0, T) \to \mathbb{R}^n$ and $\|v(t) - v_{\mathrm{e}}\| < \varepsilon$, $\|x(t) - x_{\mathrm{e}}\| < \delta$, $t \in [0, T)$, where $v(\cdot) \in \mathcal{U}$ and $x(t)$, $t \in [0, T)$, denotes the solution to (3.1) with the initial condition x_{e}. Furthermore, it follows from the local controllability of $\mathcal{G}_{\mathrm{s}}$ that for every $\hat{T} \in (0, T]$, there exists a strictly increasing, continuous function $\gamma : \mathbb{R} \to \mathbb{R}$ such that $\gamma(0) = 0$, and for every $x_0 \in \mathcal{D}$ such that $||x_0 - x_{\mathrm{e}}|| \leq \gamma(\hat{T})$, there exist $\hat{t} \in [0, \hat{T}]$ and an input $v : [0, \hat{T}] \to \mathbb{R}^m$ such that $\|v(t) - v_{\mathrm{e}}\| < \varepsilon$, $t \in [0, \hat{t})$, and $x(\hat{t}) = x_0$. Hence, there exists $\rho > 0$ such that for every $x_0 \in \mathcal{D}$ such that $\|x_0 - x_{\mathrm{e}}\| \leq \rho$, there exists $\hat{t} \in [0, \gamma^{-1}(\|x_0 - x_{\mathrm{e}}\|)]$ and an input $v : [0, \hat{t}] \to \mathbb{R}^n$ such that $\|v(t) - v_{\mathrm{e}}\| < \varepsilon$, $t \in [0, \hat{t}]$, and $x(\hat{t}) = x_0$.

Since $r(\cdot, \cdot)$ is locally integrable for all input-output pairs satisfying (3.3) and (3.4), there exists $M \in (0, \infty)$ such that

$$\left| \int_0^{\hat{t}} \frac{r(v(\sigma), y(\sigma)) - d(x(\sigma), v(\sigma))}{c + V_{\mathrm{s}}(x(\sigma))} \mathrm{d}\sigma \right| \leq \int_0^{\hat{t}} \left| \frac{\mathrm{d}Q(\sigma)}{c + V_{\mathrm{s}}(x(\sigma))} \right| \leq M\gamma^{-1}(\|x_0 - x_{\mathrm{e}}\|). \tag{3.9}$$

Now, if $S(\cdot)$ is an entropy function of $\mathcal{G}_{\mathrm{s}}$, then

$$-\int_0^{\hat{t}} \frac{\mathrm{d}Q(\sigma)}{c + V_{\mathrm{s}}(x(\sigma))} \geq S(x_{\mathrm{e}}) - S(x(\hat{t})). \tag{3.10}$$

If $S(x_{\mathrm{e}}) \geq S(x(\hat{t}))$, then combining (3.9) and (3.10) yields

$$|S(x_{\mathrm{e}}) - S(x(\hat{t}))| \leq M\gamma^{-1}(\|x_0 - x_{\mathrm{e}}\|). \tag{3.11}$$

Alternatively, if $S(x(\hat{t})) \geq S(x_{\mathrm{e}})$, then (3.11) can be derived by reversing the roles of x_{e} and $x(\hat{t})$ and using the assumption that $\mathcal{G}_{\mathrm{s}}$ is locally controllable from and to x_{e}. Hence, since $\gamma(\cdot)$ is continuous and $x(\hat{t})$ is arbitrary, it follows that $S(\cdot)$ is continuous on $\mathcal{D}$. ■

Next, we characterize a continuously differentiable entropy function for state feedback dissipative systems.

Proposition 3.7. Consider the dissipative nonlinear dynamical system $\mathcal{G}_{\rm s}$ given by (3.3) and (3.4). Then the continuously differentiable function $S : \mathcal{D} \to \mathbb{R}$ given by

$$S(x) \triangleq \log_e [c + V_{\rm s}(x)] - \log_e c, \tag{3.12}$$

where $c > 0$, is an entropy function of $\mathcal{G}_{\rm s}$.

Proof: Using Proposition 3.3 it follows that

$$\dot{S}(x(t)) = \frac{\dot{V}_{\rm s}(x(t))}{c + V_{\rm s}(x(t))} = \frac{\dot{Q}(t)}{c + V_{\rm s}(x(t))}, \quad t \geq 0. \tag{3.13}$$

Now, integrating (3.13) over $[t_1, t_2]$ yields (3.7). ■

Remark 3.8. In [41], the authors show that the entropy function for an energy balance equation involving a large-scale, compartmental thermodynamic model is unique. However, whether or not there exists a unique continuously differentiable entropy function for $\mathcal{G}_{\rm s}$ given by (3.3) and (3.4) is an open problem.

Finally, the following result presenting an upper and lower bound of the entropy function for a state feedback dissipative system is needed for later developments.

Proposition 3.9. Consider the nonlinear dynamical system $\mathcal{G}_{\rm s}$ given by (3.3) and (3.4), and let $S : \mathcal{D} \to \mathbb{R}$ given by (3.12) be an entropy function of $\mathcal{G}_{\rm s}$. Then,

$$\frac{V_{\rm s}(x)}{c + V_{\rm s}(x)} \leq S(x) \leq \frac{1}{c} V_{\rm s}(x), \quad x \in \mathcal{D}. \tag{3.14}$$

Proof: Note that (3.12) can be rewritten as $S(x) = \log_e[1 + V_{\rm s}(x)/c]$. The assertion is a direct consequence of the inequality $z/(1+z) \leq \log_e(1+z) \leq z$, $z > -1$. ■

3.3. Thermodynamic Semistabilization

In this section, we use the results of Section 3.2 to present a framework for *semistabilization* of nonlinear systems. Semistabilization is the property of controlled dynamical systems possessing a continuum of equilibria whereby every closed-loop system trajectory that starts in a neighborhood of a Lyapunov stable equilibrium converges to a (possibly different) Lyapunov stable equilibrium [38].

To address the state feedback, thermodynamic-based semistabilization problem, consider the nonlinear dynamical system $\mathcal{G}_{\rm s}$ given by (3.3) and (3.4) with performance criterion

$$J(x_0, \phi(\cdot)) = \lim_{t\to\infty} \left[\frac{1}{t}\int_0^t S(x(\sigma))\mathrm{d}\sigma\right]. \tag{3.15}$$

The performance criterion $J(x_0, \phi(\cdot))$ can be interpreted as the time-average of the entropy function for the dissipative nonlinear dynamical system $\mathcal{G}_{\rm s}$. The key feature of this optimal control problem is that it addresses semistability instead of asymptotic stability. In the absence of energy exchange with the environment, a thermodynamically consistent nonlinear dynamical system model possesses a continuum of equilibria, and hence, is semistable; that is, the system states converge to Lyapunov energy equilibria determined by the system initial conditions [41]. A key question that arises is whether or not this optimal control problem is well defined; that is, whether $J(x_0, \phi(\cdot))$ is finite and if there exists a state feedback controller such that $J(x_0, \phi(\cdot))$ is minimized. The first question is addressed by the following proposition.

Proposition 3.10. Consider the nonlinear dissipative dynamical system $\mathcal{G}_{\rm s}$ given by (3.3) and (3.4). If there exists $\phi : \mathcal{D} \to \mathbb{R}^m$ such that (3.3), with $v(t) \equiv 0$, is semistable, then $|J(x_0, \phi(\cdot))| < \infty$.

Proof: Since (3.3) with $v(t) \equiv 0$ is semistable, $x(t)$ is bounded for all $t \geq 0$. It follows from Theorem 3.6 that $S(\cdot)$ is a continuous entropy function on $\mathcal{D}$ for $\mathcal{G}_{\rm s}$. Hence, $S(x(t))$ is bounded for all $t \geq 0$. Now, let $|S(x(t))| \leq c$ for all $t \geq 0$. Then, $-c \leq (1/t)\int_0^t S(x(\sigma))\mathrm{d}\sigma \leq c$ for all $t \geq 0$, which proves the result. ■

To address the question of existence of a semistabilizing controller such that $J(x_0, \phi(\cdot))$ given by (3.15) is minimized, we consider an *auxiliary minimization problem* involving the performance criterion

$$\mathcal{J}(x_0,\phi(\cdot)) = \lim_{t\to\infty}\left[\frac{1}{t}\int_0^t V_{\rm s}(x(\sigma)){\rm d}\sigma\right]. \tag{3.16}$$

Hence, it follows from the auxiliary minimization problem that we seek feedback controllers that minimize the stored energy in the system in order to attain a stable energy level determined by the system initial conditions and the control system effort.

The following lemma is necessary for proving the main result of this section.

Lemma 3.11. Consider the dissipative nonlinear dynamical system $\mathcal{G}_{\rm s}$ given by (3.3) and (3.4) with continuously differentiable storage function $V_{\rm s} : \mathcal{D} \to \mathbb{R}_+$. Suppose there exists $\phi^* : \mathcal{D} \to \mathbb{R}^m$ such that (3.3), with $v(t) \equiv 0$, is semistable, $\dot{V}_{\rm s}(x(t)) \le 0$, $t \ge 0$, and $\mathcal{J}(x_0,\phi(\cdot))$ is minimized. If $\mathcal{J}(x_0,\phi^*(\cdot)) = 0$, then $\arg\min_{\phi(\cdot)\in\mathbb{R}^m} \mathcal{J}(x_0,\phi(\cdot)) = \arg\min_{\phi(\cdot)\in\mathbb{R}^m} J(x_0,\phi(\cdot))$ and $J(x_0,\phi^*(\cdot)) = 0$. Alternatively, if $\mathcal{J}(x_0,\phi^*(\cdot)) \ne 0$, then $J(x_0,\phi^*(\cdot)) = S(x_{\rm e})$, where $x_{\rm e} = \lim_{t\to\infty} x(t)$.

Proof: It follows from Proposition 3.9 and $\dot{V}_{\rm s}(x(t)) \le 0$, $t \ge 0$, that

$$\frac{V_{\rm s}(x(t))}{c + V_{\rm s}(x(0))} \le S(x(t)) \le \frac{V_{\rm s}(x(t))}{c}, \quad t \ge 0. \tag{3.17}$$

Hence, $\frac{\mathcal{J}(x_0,\phi(\cdot))}{(c+V_{\rm s}(x(0)))} \le J(x_0,\phi(\cdot)) \le \frac{\mathcal{J}(x_0,\phi(\cdot))}{c}$. Now, if $\mathcal{J}(x_0,\phi^*(\cdot)) = 0$, then $J(x_0,\phi(\cdot))$ is minimized and $J(x_0,\phi^*(t)) = 0$, $t \ge 0$.

Alternatively, if $\mathcal{J}(x_0,\phi^*(\cdot)) \ne 0$, then it follows from the definition of $\mathcal{J}(x_0,\phi^*(\cdot))$ that there exists $c^* > 0$ such that $\mathcal{J}(x_0,\phi^*(t)) \ge c^*$ for all $t \ge 0$, and hence, $\lim_{t\to\infty} \mathcal{J}(x_0,\phi^*(t))t/(c+V_{\rm s}(x(0))) = \infty$. Thus, $\lim_{t\to\infty}\int_0^t S(x(\sigma)){\rm d}\sigma = \infty$. It follows from l'Hôpital's rule that $J(x_0,\phi^*(t)) = S(x_{\rm e})$, where $x_{\rm e} = \lim_{t\to\infty} x(t)$. ■

Theorem 3.12. Consider the dissipative nonlinear dynamical system $\mathcal{G}_{\rm s}$ given by (3.3) and (3.4) with continuously differentiable storage function $V_{\rm s} : \mathcal{D} \to \mathbb{R}_+$. Assume that there exists $\phi^* : \mathcal{D} \to \mathbb{R}^m$ such that (3.16) is minimized, (3.3), with $v(t) \equiv 0$, is semistable, and

$\dot{V}_{\rm s}(x(t)) \le 0$, $t \ge 0$. Then, for $S : \mathcal{D} \to \mathbb{R}$ given by (3.12), $\arg\min_{\phi(\cdot)\in\mathbb{R}^m} \mathcal{J}(x_0, \phi(\cdot)) = \arg\min_{\phi(\cdot)\in\mathbb{R}^m} J(x_0, \phi(\cdot))$.

Proof: If $\mathcal{J}(x_0, \phi^*(\cdot)) = 0$, then it follows from Lemma 3.11 that $\phi^*(\cdot) = \arg\min_{\phi(\cdot)\in\mathbb{R}^m} J(x_0, \phi(\cdot))$. Alternatively, if $\mathcal{J}(x_0, \phi^*(\cdot)) \neq 0$, then, using similar arguments as in the proof of Lemma 3.11, $\lim_{t\to\infty} t\mathcal{J}(x_0, \phi^*(t)) = \infty$, and hence, $\int_0^t V_{\rm s}(x(\sigma)){\rm d}\sigma = \infty$ as $t \to \infty$. Hence, using l'Hôpital's rule, it follows that $\mathcal{J}(x_0, \phi^*(\cdot)) = V_{\rm s}(x_{\rm e})$.

Next, since for all $\phi : \mathcal{D} \to \mathbb{R}^m$ such that $\mathcal{J}(x_0, \phi(\cdot))$ is finite, $\lim_{t\to\infty} \mathcal{J}(x_0, \phi(t))t/(c + V_{\rm s}(x(0))) = \infty$, and hence, using (3.17), it follows that $\lim_{t\to\infty} \int_0^t S(x(\sigma)){\rm d}\sigma = \infty$. Consequently, for all $\phi : \mathcal{D} \to \mathbb{R}^m$ such that $\mathcal{J}(x_0, \phi(\cdot))$ is finite and $\mathcal{G}_{\rm s}$ is semistable, it follows from l'Hôpital's rule that $J(x_0, \phi(\cdot)) = S(x_{\rm e}) = \log_e(1 + V_{\rm s}(x_{\rm e}))$, where $x_{\rm e} = \lim_{t\to\infty} x(t)$. Next, assume that $\phi_{\rm J}^* : \mathcal{D} \to \mathbb{R}^m$ is such that $\mathcal{G}_{\rm s}$ is semistable, $\dot{V}_{\rm s}(x(t)) \le 0$, $t \ge 0$, and $J(x_0, \phi(\cdot))$ is minimized. Then, it follows that $J(x_0, \phi_{\rm J}^*(\cdot)) = S(x_{\rm e}) = \log_e(1+V_{\rm s}(x_{\rm e}))$. By uniqueness of solutions of $x(\cdot)$ it follows that $\phi^*(\cdot)$ uniquely determines $x_{\rm e}$ and $\dot{V}_{\rm s}(x(t))$, $t \ge 0$. Choosing $V_{\rm s}(x_{\rm e}) = V_{\rm se}$, where $V_{\rm se} \in \mathbb{R}$, it follows that $\phi^*(\cdot)$ uniquely determines $V_{\rm s}(x_{\rm e})$, and hence, $J(x_0, \phi^*(\cdot)) = \log_e(1 + V_{\rm s}(x_{\rm e}))$, which proves the result. ■

It follows from Theorem 3.12 that an optimal semistable controller minimizing $\mathcal{J}(x_0, v(\cdot))$ given by (3.16) also minimizes the entropy functional $J(x_0, v(\cdot))$ given by (3.15). Since quadratic cost functions arise naturally in dissipativity theory [38, 53, 116], addressing the auxiliary cost (3.16) can be simpler than addressing the entropy (logarithmic) cost functional (3.15).

3.4. Thermodynamic Semistabilization of Linear Systems

In this section, we address the problem of semistabilizing optimal controllers for linear systems so that $\mathcal{G}$ is given by

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \quad t \ge 0, \tag{3.18}$$

$$y(t) = Cx(t) + Du(t), \tag{3.19}$$

where, for each $t \geq 0$, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^l$, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{l \times n}$, and $D \in \mathbb{R}^{l \times m}$. Given $u = Kx + v$, $K \in \mathbb{R}^{m \times n}$, we assume that $\mathcal{G}$ is state feedback dissipative, that is, the nonlinear dynamical systems $\mathcal{G}_\mathrm{s}$ given by (3.3) and (3.4) takes the form

$$\dot{x}(t) = \tilde{A}x(t) + Bv(t), \quad x(0) = x_0, \quad t \geq 0, \tag{3.20}$$

$$y(t) = \tilde{C}x(t) + Dv(t), \tag{3.21}$$

where $\tilde{A} \triangleq A + BK$, $\tilde{C} \triangleq C + DK$, and $\mathcal{G}_\mathrm{s}$ is dissipative with respect to the supply rate $r(v, y)$, where $r : \mathbb{R}^m \times \mathbb{R}^l \to \mathbb{R}$ is locally integrable for all input-output pairs satisfying (3.20) and (3.21), and $r(0, 0) = 0$. For the reminder of the section define $\mathcal{K} \triangleq \{K \in \mathbb{R}^{m \times n} : A + BK \text{ is semistable}\}$. In this case, Theorem 3.12 specializes to the following result.

Theorem 3.13. Consider the dissipative dynamical system $\mathcal{G}_\mathrm{s}$ given by (3.20) and (3.21) with continuously differentiable storage function $V_\mathrm{s} : \mathbb{R}^n \to \mathbb{R}_+$. Assume there exists $K^* \in \mathcal{K}$ that minimizes (3.16) and $\dot{V}_\mathrm{s}(x(t)) \leq 0$, $t \geq 0$, where $x(t)$, $t \geq 0$, satisfies (3.20) with $v(t) \equiv 0$. Then, for $S : \mathbb{R}^n \to \mathbb{R}$ given by (3.12), $\arg\min_{K \in \mathcal{K}} \mathcal{J}(x_0, K) = \arg\min_{K \in \mathcal{K}} J(x_0, K)$.

For the reminder of the section, we consider the special case of dissipative systems $\mathcal{G}_\mathrm{s}$ with quadratic supply rates. Specifically, we set $\mathcal{D} = \mathbb{R}^n$, $U = \mathbb{R}^m$, and $Y = \mathbb{R}^l$, and let

$$r(v, y) = y^\mathrm{T} Q y + 2y^\mathrm{T} Z v + v^\mathrm{T} R v, \tag{3.22}$$

where $Q \in \mathbb{S}^n$, $Z \in \mathbb{R}^{l \times m}$, and $R \in \mathbb{S}^m$ [116]. It follows from Theorem 5.9 of [38] that in this case the linear system $\mathcal{G}_\mathrm{s}$ given by (3.20) and (3.21) possesses a quadratic storage function $V_\mathrm{s}(x) = x^\mathrm{T} P x$, where $P = P^\mathrm{T} \geq 0$ satisfies

$$0 = \tilde{A}^\mathrm{T} P + P\tilde{A} - \tilde{C}^\mathrm{T} Q \tilde{C} + L^\mathrm{T} L, \tag{3.23}$$

$$0 = PB - \tilde{C}^\mathrm{T}(QD + Z) + L^\mathrm{T} W, \tag{3.24}$$

$$0 = \tilde{R} - W^{\mathrm{T}}W, \tag{3.25}$$

where $L \in \mathbb{R}^{p\times n}$, $W \in \mathbb{R}^{p\times m}$, and $\tilde{R} \triangleq R + Z^{\mathrm{T}}D + D^{\mathrm{T}}Z + D^{\mathrm{T}}ZD$. In this case, $\mathcal{J}(x_0, K)$ has the form

$$\mathcal{J}(x_0, K) = \lim_{t\to\infty} \frac{1}{t}\int_0^t x^{\mathrm{T}}(s)Px(s)\mathrm{d}s.$$

To eliminate the dependence of the initial condition x_0 on $\mathcal{J}(x_0, K)$ and $J(x_0, K)$, we assume that the initial state x_0 is a random variable such that $\mathbb{E}[x_0] = 0$ and $\mathbb{E}[x_0 x_0^{\mathrm{T}}] = V$, where $\mathbb{E}$ denotes the expectation operator.

Proposition 3.14. Assume that $\mathcal{G}_{\mathrm{s}}$ given by (3.20) and (3.21) is dissipative with respect to the quadratic supply rate (3.22) and suppose there exists $K \in \mathcal{K}$ and $\dot{V}_{\mathrm{s}}(x(t)) \le 0$, $t \ge 0$, where $x(t)$, $t \ge 0$, satisfies (3.20) with $v(t) \equiv 0$. Then there exists an $n \times n$ nonnegative-definite matrix P such that (3.23)–(3.25) hold and, with $v(t) \equiv 0$,

$$\begin{aligned}\mathcal{J}(K) &= x_0^{\mathrm{T}}[I_n - \tilde{A}^{\mathrm{T}}(\tilde{A}^{\mathrm{T}})^{\#}]P[I_n - \tilde{A}\tilde{A}^{\#}]x_0 \\ &= \mathrm{tr}[I_n - \tilde{A}^{\mathrm{T}}(\tilde{A}^{\mathrm{T}})^{\#}]P[I_n - \tilde{A}\tilde{A}^{\#}]V. \end{aligned}\tag{3.26}$$

Proof: Since $\mathcal{G}_{\mathrm{s}}$ is dissipative with respect to the quadratic supply rate (3.22), it follows from Theorem 5.9 of [38] that there exists $P = P^{\mathrm{T}} \ge 0$ such that (3.23)–(3.25) hold and $V_{\mathrm{s}}(x) = x^{\mathrm{T}}Px$ is a storage function for $\mathcal{G}_{\mathrm{s}}$. Since for $v(t) \equiv 0$, $\dot{V}_{\mathrm{s}}(x(t)) \le 0$, $t \ge 0$, $\tilde{A}$ is semistable, and $x(t) = e^{\tilde{A}t}x_0$, $t \ge 0$, it follows that

$$\mathcal{J}(K) = \lim_{t\to\infty}\frac{1}{t}\int_0^t [x_0^{\mathrm{T}} e^{\tilde{A}^{\mathrm{T}}\tau} P e^{\tilde{A}\tau} x_0]\mathrm{d}\tau = \mathrm{tr}\,\tilde{\mathcal{A}}_{\tilde{A}}(P)V,$$

where

$$\tilde{\mathcal{A}}_{\tilde{A}}(P) \triangleq \lim_{t\to\infty}\frac{1}{t}\int_0^t e^{\tilde{A}^{\mathrm{T}}\tau} P e^{\tilde{A}\tau}\mathrm{d}\tau.$$

Now, since $\lim_{t\to\infty} e^{\tilde{A}^{\mathrm{T}}\tau} P e^{\tilde{A}\tau}$ is finite, $\tilde{\mathcal{A}}_{\tilde{A}}(P) = \lim_{t\to\infty} e^{\tilde{A}^{\mathrm{T}}t} P e^{\tilde{A}t}$. In addition, since $\tilde{A}$ is semistable, $\lim_{t\to\infty} e^{\tilde{A}t} = I_n - \tilde{A}\tilde{A}^{\#}$ [8]. Hence, $\lim_{t\to\infty} e^{\tilde{A}^{\mathrm{T}}t} P e^{\tilde{A}t} = [I_n - \tilde{A}^{\mathrm{T}}(\tilde{A}^{\mathrm{T}})^{\#}]P[I_n - \tilde{A}\tilde{A}^{\#}]$, which proves the result. ■

Remark 3.15. Define the operator $\mathcal{L}_{\tilde{A}} : \mathbb{S}^n \to \mathbb{S}^n$ by

$$\mathcal{L}_{\tilde{A}}(P) \triangleq \tilde{A}^{\mathrm{T}} P + P\tilde{A}. \tag{3.27}$$

It follows from Proposition 4.1 of [15] that $\mathcal{N}(\mathcal{L}_{\tilde{A}}) = \mathcal{R}(\tilde{\mathcal{A}}_{\tilde{A}})$ and $\mathcal{N}(\tilde{\mathcal{A}}_{\tilde{A}}) = \mathcal{R}(\mathcal{L}_{\tilde{A}})$. This implies that $V_{\mathrm{s}}(x) = x^{\mathrm{T}} P x$ is an integral of motion of

$$\dot{x}(t) = \tilde{A}x(t), \qquad x(0) = x_0, \qquad t \geq 0, \tag{3.28}$$

if and only if $x^{\mathrm{T}} \tilde{\mathcal{A}}_{\tilde{A}}(P)x$ is the average over $[0, \infty)$ of $V_{\mathrm{s}}(x) = x^{\mathrm{T}} P x$ along the solutions of $\dot{x}(t) = \tilde{A}x(t)$. Furthermore, the elements of $\mathcal{N}(\tilde{\mathcal{A}}_{\tilde{A}})$ are quadratic functions that have zero average along the trajectories of $\dot{x}(t) = \tilde{A}x(t)$ if and only if $x \mapsto x^{\mathrm{T}} \mathcal{L}_{\tilde{A}}(P)x$ is the Lie derivative of $x \mapsto x^{\mathrm{T}} P x$ along the trajectories of $\dot{x}(t) = \tilde{A}x(t)$ for every $P \in \mathbb{S}^n$ [15].

The following lemma provides necessary and sufficient conditions for a feedback gain matrix K to belong to the set $\mathcal{K}$.

Lemma 3.16 [44]. The linear dynamical system $\mathcal{G}_{\mathrm{s}}$ given by (3.20), with $v(t) \equiv 0$, is semistable if and only if for every semiobservable pair $(\tilde{A}, \hat{R})$, where $\hat{R} = \hat{R}^{\mathrm{T}} \geq 0$ and $\tilde{A} = A + BK$, there exists $\hat{P} \in \mathbb{R}^{n \times n}$ such that $\hat{P} = \hat{P}^{\mathrm{T}} > 0$ and

$$0 = \tilde{A}^{\mathrm{T}} \hat{P} + \hat{P}\tilde{A} + \hat{R}. \tag{3.29}$$

It is worth recalling that $\hat{P}$ is not unique [44]. The next result characterizes state feedback thermodynamic semistabilizing controllers using linear matrix inequalities.

Theorem 3.17. Consider the linear dynamical system $\mathcal{G}_{\mathrm{s}}$ given by (3.20) and (3.21), let Q characterizing the supply rate $r(v, y)$ given by (3.22) be such that $Q \leq 0$, and let $\hat{R} \geq 0$. Then K^* minimizes

$$\mathcal{J}(K) = \mathrm{tr}[I_n - \tilde{A}^{\mathrm{T}}(\tilde{A}^{\mathrm{T}})^{\#}]P[I_n - \tilde{A}\tilde{A}^{\#}]V, \tag{3.30}$$

subject to

$$(\tilde{A}, \hat{R}) \text{ is semiobservable,} \tag{3.31}$$

$$0 \geq \begin{bmatrix} \tilde{A}^{\mathrm{T}}P + P\tilde{A} - \tilde{C}^{\mathrm{T}}Q\tilde{C} & PB - \tilde{C}^{\mathrm{T}}(QD+Z) \\ B^{\mathrm{T}}P - (QD+Z)^{\mathrm{T}}\tilde{C} & \tilde{R} \end{bmatrix}, \tag{3.32}$$

$$0 \geq \tilde{A}^{\mathrm{T}}\hat{P} + \hat{P}\tilde{A}, \tag{3.33}$$

where $P = P^{\mathrm{T}} \geq 0$, $P \in \mathbb{R}^{n\times n}$, and $\hat{P} = \hat{P}^{\mathrm{T}} > 0$, $\hat{P} \in \mathbb{R}^{n\times n}$, if and only if K^* minimizes $J(K)$ given by (3.15) subject to (3.31)–(3.33).

Proof: The existence of $P = P^{\mathrm{T}} \geq 0$ such that (3.32) holds guarantees that $\mathcal{G}_{\mathrm{s}}$ is dissipative with respect to the supply rate $r(v, y)$, whereas (3.31) and (3.33) guarantee that $\tilde{A}$ is semistable. The assertion follows as a direct consequence of Theorem 3.13, Proposition 3.14, and Lemma 3.16. ■

To guarantee that $(\tilde{A}, \hat{R})$ is semiobservable, let $\hat{R} = \tilde{A}^{\mathrm{T}}M\tilde{A}$, where $M = M^{\mathrm{T}} > 0$. In this case, $\mathcal{N}(\hat{R}\tilde{A}^{k-1}) = \mathcal{N}(\tilde{A}^{\mathrm{T}}M\tilde{A}^{k}) = \mathcal{N}(\tilde{A}^{k})$, $k = 1, \ldots, n$. Since $\mathcal{N}(\tilde{A}) \subseteq \mathcal{N}(\tilde{A}^{k})$ for every $k \in \{1, \ldots, n\}$ it follows that $\bigcap_{k=1}^{n} \mathcal{N}(\hat{R}\tilde{A}^{k-1}) = \bigcap_{k=1}^{n} \mathcal{N}(\tilde{A}^{k}) = \mathcal{N}(\tilde{A})$, which, by Definition 2.7, implies semiobservability of $(\tilde{A}, \hat{R})$.

The minimization problem given in Theorem 3.17 is complicated by the fact that $\mathcal{J}(K)$ involves $\tilde{A}^{\#}$ and $\tilde{A}$ which are functions of the feedback gain K. Next, we present a corollary to Theorem 3.17 that avoids this complexity. First, however, the following lemma is required.

Lemma 3.18. If $\tilde{A} = A + BK$ is semistable, then $Y \triangleq I_n - \tilde{A}\tilde{A}^{\#}$ is a unique matrix satisfying $\mathcal{N}(Y) = \mathcal{R}(\tilde{A})$, $\mathcal{R}(Y) = \mathcal{N}(\tilde{A})$, and $\mathcal{N}(\tilde{A}) \subseteq \mathcal{N}(Y - I_n)$.

Corollary 3.19. Consider the linear dynamical system $\mathcal{G}_{\mathrm{s}}$ given by (3.20) and (3.21), let Q characterizing the supply rate $r(v, y)$ given by (3.22) be such that $Q \leq 0$, and let $\hat{R} \geq 0$. Then K^* minimizes

$$\mathcal{J}(K) = \operatorname{tr} Y^{\mathrm{T}}PYV, \tag{3.34}$$

subject to

$$(\tilde{A}, \hat{R}) \text{ is semiobservable}, \tag{3.35}$$

$$\mathcal{N}(Y) = \mathcal{R}(\tilde{A}),\ \mathcal{R}(Y) = \mathcal{N}(\tilde{A}),\ \mathcal{N}(\tilde{A}) \subseteq \mathcal{N}(Y - I_n), \tag{3.36}$$

$$0 \geq \begin{bmatrix} \tilde{A}^{\mathrm{T}}P + P\tilde{A} - \tilde{C}^{\mathrm{T}}Q\tilde{C} & PB - \tilde{C}^{\mathrm{T}}(QD + Z) \\ B^{\mathrm{T}}P - (QD + Z)^{\mathrm{T}}\tilde{C} & \tilde{R} \end{bmatrix}, \tag{3.37}$$

$$0 \geq \tilde{A}^{\mathrm{T}}\hat{P} + \hat{P}\tilde{A}, \tag{3.38}$$

where $P = P^{\mathrm{T}} \geq 0$, $P \in \mathbb{R}^{n\times n}$, and $\hat{P} = \hat{P}^{\mathrm{T}} > 0$, $\hat{P} \in \mathbb{R}^{n\times n}$, if and only if K^* minimizes $J(K)$ given by (3.15) subject to (3.35)–(3.38).

Proof: The result is a direct consequence of Theorem 3.17 and Lemma 3.18. ∎

Chapter 4

Singular Control for Linear Semistabilization

4.1. Introduction

For a linear dynamical system, the nonlinear model $\mathcal{G}$ given by (2.2) and (2.3) becomes

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0, \qquad t \geq 0, \tag{4.1}$$

$$y(t) = Cx(t), \tag{4.2}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^l$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{l\times n}$, the classical singular control problem consists of finding a feedback control law $u(\cdot) = \phi(x(\cdot))$ for $l = m$ such that (4.1) is asymptotically stable and the performance measure

$$J_0(x_0, u(\cdot)) \triangleq \lim_{\varepsilon\to 0} \int_0^\infty \big[(x(t) - x_{\mathrm{e}})^{\mathrm{T}} R_1 (x(t) - x_{\mathrm{e}}) + \varepsilon^2 (u(t) - u_{\mathrm{e}})^{\mathrm{T}} R_2 (u(t) - u_{\mathrm{e}})\big] \mathrm{d}t \tag{4.3}$$

is minimized in the sense that

$$J_0(x_0, \phi(\cdot)) = \min_{u(\cdot)\in\mathcal{S}_0(x_0)} J_0(x_0, u(\cdot)), \tag{4.4}$$

where $u_{\mathrm{e}} \triangleq \phi(x_{\mathrm{e}})$, $x_{\mathrm{e}} \triangleq \lim_{t\to\infty} x(t)$, $R_1 \in \mathbb{R}^{n\times n}$ is nonnegative definite, that is, $R_1 = R_1^{\mathrm{T}} \geq 0$, $R_2 \in \mathbb{R}^{m\times m}$ is positive definite, that is, $R_2 = R_2^{\mathrm{T}} > 0$, and

$$\mathcal{S}(x_0) \triangleq \{u(\cdot) : u(\cdot) \text{ is measurable and } x(\cdot) \text{ given by (4.1) satisfies } x(t) \to x_{\mathrm{e}} \text{ as } t \to \infty\}. \tag{4.5}$$

In this case, it can be shown that the optimal controller takes the form $u = Kx$, where $K \in \mathbb{R}^{m\times n}$ [35, 77].

This problem has received considerable attention in the literature since it addresses a limiting case of the linear-quadratic regulator problem [99], it can be used for system characterization, such as the invertibility problem [107], and it can be used in the design of high gain feedback systems [76, 104]. Furthermore, the singular control problem has been extended to non-square systems [31], that is, $l \neq m$, affine nonlinear systems [109], and discrete-time linear systems [89].

In this chapter, we address the singular control problem for semistabilization. Specifically, we address the problem of finding $u(\cdot) = \phi(x(\cdot))$ such that the controlled system (4.1) is semistable and the performance measure (4.3) with $R_1 = C^{\mathrm{T}}C$ and $R_2 = I_m$, that is,

$$J_0(x_0, u(\cdot)) = \lim_{\varepsilon \to 0} \int_0^{\infty} \left[(x(t) - x_{\mathrm{e}})^{\mathrm{T}} C^{\mathrm{T}} C (x(t) - x_{\mathrm{e}}) + \varepsilon^2 (u(t) - u_{\mathrm{e}})^{\mathrm{T}} (u(t) - u_{\mathrm{e}})\right] \mathrm{d}t, \quad (4.6)$$

is minimized in the sense of (4.4).

4.2. Mathematical Preliminaries

In this section, we review some basic results needed to solve the singular control problem for semistabilization of linear dynamical systems.

Lemma 4.1. Consider the linear dynamical system (4.1). If the pair (A, B) is semicontrollable and $0 \in \operatorname{spec}(A)$, then 0 is unstabilazable.

Proof: Let w be a left-eigenvector of A with associated eigenvalue $\lambda = 0$ so that $w \in \mathcal{N}(A^{\mathrm{T}})$. By the definition of semicontrollability, it follows that $w \in \mathcal{N}(B^{\mathrm{T}})$. Now, recall that given a left eigenpair (μ, z) of A, (A, B) is uncontrollable if and only if $z \in \mathcal{N}(B^{\mathrm{T}})$ [72, Th. 6.2-5]. Now, the assertion follows immediately by noting that if $\lambda \in \operatorname{spec}(A) \cap \overline{\mathbb{C}}_+$ is uncontrollable, then λ is unstabilizable. ■

The following proposition provides a necessary and sufficient condition for verifying semicontrollability of the pair (A, B).

Proposition 4.2. Consider the linear dynamical system (4.1) and suppose $0 \in \operatorname{spec}(A)$. The pair (A, B) is semicontrollable if and only if there exists $v \in \mathbb{C}^n \setminus \{0\}$ such that $v \in \mathcal{N}(A^{\mathrm{T}}) \cap \mathcal{N}(B^{\mathrm{T}})$.

Proof: Necessity is a direct consequence of Proposition 2.6. To prove sufficiency, note that it follows from Theorem 12.6.8 of [8] that there exists an orthogonal matrix $S \in \mathbb{R}^{n\times n}$ such that

$$\hat{A} \triangleq SAS^{-1} = \begin{bmatrix} A_1 & A_{12} \\ 0 & A_2 \end{bmatrix}, \qquad \hat{B} \triangleq SB = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \tag{4.7}$$

where $A_1 \in \mathbb{R}^{q\times q}$, $B_1 \in \mathbb{R}^{q\times n}$, and (A_1, B_1) is controllable. Now, it follows from Lemma 4.1 that there exists $z \neq 0$ such that $z \in \mathcal{N}(A_2^{\mathrm{T}})$. Therefore, $\hat{A}^{\mathrm{T}}\hat{z} = 0$ and $\hat{B}^{\mathrm{T}}\hat{z} = 0$, where $\hat{z} \triangleq [0^{\mathrm{T}}, z^{\mathrm{T}}]^{\mathrm{T}}$. The result now follows by noting that $A^{\mathrm{T}}v = 0$ and $B^{\mathrm{T}}v = 0$, where $v \triangleq S^{\mathrm{T}}\hat{z}$. ■

Lemma 4.3. Consider the linear dynamical system $\mathcal{G}$ given by (4.1) and (4.2) with $B = 0$. If the pair (A, C) is semiobservable and $0 \in \operatorname{spec}(A)$, then the eigenvalue $\lambda = 0$ is undetectable.

Proof: The proof is dual to the proof of Lemma 4.1 and, hence, is omitted. ■

The following proposition provides necessary and sufficient conditions for verifying semiobservability of the pair (A, C).

Proposition 4.4. Consider the dynamical system $\mathcal{G}$ given by (4.1) and (4.2) with $B = 0$ and suppose $0 \in \operatorname{spec}(A)$. The pair (A, C) is semiobservable if and only if there exists $v \in \mathbb{C}^n \setminus \{0\}$ such that $v \in \mathcal{N}(A) \cap \mathcal{N}(C)$.

Proof: The proof is dual to the proof of Proposition 4.2 and, hence, is omitted. ■

The following result is used later in the section.

Proposition 4.5 [38, 109]. Consider the linear dynamical system $\mathcal{G}$ given by (4.1) and (4.2). If $l = m$ and $\operatorname{rank}(CB) = m$, then there exists a change of coordinates $x \mapsto (y, z)$ such that $\mathcal{G}$ is equivalent to

$$\begin{bmatrix} \dot{y}(t) \\ \dot{z}(t) \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ B_0 & A_0 \end{bmatrix} \begin{bmatrix} y(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u(t), \qquad t \geq 0, \tag{4.8}$$

where $z \in \mathbb{R}^{n-m}$, $A_1 \in \mathbb{R}^{m\times m}$, $A_2 \in \mathbb{R}^{m\times(n-m)}$, $B_0 \in \mathbb{R}^{(n-m)\times m}$, $A_0 \in \mathbb{R}^{(n-m)\times(n-m)}$, and $B_1 \triangleq CB$.

Consider the dynamical system $\mathcal{G}$ given by

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0, \qquad t \geq 0, \tag{4.9}$$

$$y(t) = Cx(t) + Du(t), \tag{4.10}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{l\times n}$, and $D \in \mathbb{R}^{l\times m}$, let $G(s) \triangleq C(sI_n - A)^{-1}B + D$, and $\mathcal{G} \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ denote a realization of $\mathcal{G}$. Then, recall that $G(s) \in \mathbb{R}^{l\times m}_{\text{prop}}(s)$ is *inner* if and only if $G^{\mathrm{T}}(-s)G(s) = I_m$, $G(s) \in \mathbb{R}^{l\times m}_{\text{prop}}(s)$ is *minimum phase* if and only if the zeros of $G(s)$ are nonnpositive, where the *zeros* of $G(s) \in \mathbb{R}^{l\times m}_{\text{prop}}$ are the roots of the numerator polynomials in the nonzero entries of the Smith-McMillan form of $G(s)$ [99], [72, p. 446]. Note that for the realization given in Proposition 4.5, the zeros of $C(sI - A)^{-1}B$ are the eigenvalues of A_0 [109]. Finally, recall that for $\mathcal{G} \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, the controllability and observability Gramians Q and P of (4.9) and (4.10) are given by the solutions to the Lyapunov equations

$$0 = AQ + QA^{\mathrm{T}} + BB^{\mathrm{T}}, \tag{4.11}$$

$$0 = A^{\mathrm{T}}P + PA + C^{\mathrm{T}}C, \tag{4.12}$$

where $Q \geq 0$ and $P \geq 0$. If $P = Q = I_n$, then the realization $\mathcal{G}$ is a *balanced realization* [36]. If $l = m$, then $G(s)$ is *right invertible* if and only if $G(s)$ has full row rank for at least one $s \in \mathbb{C}$ [99].

Theorem 4.6 [71, 99]. The transfer function $G(s) \in \mathbb{R}^{l\times m}_{\text{prop}}(s)$ can be factored as $G(s) = G_1(s)G_2(s)$, where $G_1(s) \in \mathbb{R}^{l\times p}_{\text{prop}}(s)$, $p \leq l$, is inner and $G_2(s) \in \mathbb{R}^{p\times m}_{\text{prop}}(s)$ is minimum phase and right invertible. The unstable poles of $G_2(s)$ are equal to the unstable poles of $G(s)$. In addition, if $G(s)$ is strictly proper, then $G_2(s)$ is strictly proper.

Corollary 4.7 [99]. Let $G(s) \in \mathbb{R}^{l\times m}_{\rm prop}(s)$ be right invertible and let $G(s) = G_1(s)G_2(s)$ be a factorization as in Theorem 4.6. Then the system $\mathcal{G}_1$, with transfer function $G_1(s) \in \mathbb{R}^{l\times l}_{\rm prop}(s)$, is square, the zeros of $G_1(s)$ are equal to the zeros of $G(s)$ whose real part is positive, and the poles of $G_1(s)$ are equal to the negatives of the zeros of $G_1(s)$.

Corollary 4.8 [71]. Let $G(s) \in \mathbb{R}^{l\times m}_{\rm prop}(s)$ be a nonminimum phase right invertible transfer function, let $G(s) = G_1(s)G_2(s)$ be a factorization as in Theorem 4.6, let $\mathcal{G}_1 \sim \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right]$ be a balanced realization of $G_1(s) \in \mathbb{R}^{l\times l}_{\rm prop}(s)$ and let $\mathcal{G}_2 \sim \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & 0 \end{array}\right]$ be a stabilizable and detectable realization of $G_2(s) \in \mathbb{R}^{l\times m}_{\rm prop}(s)$. Then, a stabilizable and detectable realization of $\mathcal{G} \sim \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ is given by

$$A = \begin{bmatrix} A_1 & B_1C_2 \\ 0 & A_2 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \qquad C^{\rm T} = \begin{bmatrix} C_1^{\rm T} \\ C_2^{\rm T} D_1^{\rm T} \end{bmatrix}, \qquad D = 0. \tag{4.13}$$

4.3. Linear-Quadratic Regulator Problem for Semistabilization

In this section, we prove fundamental results to solve the linear-quadratic regulator problem for semistabilization.

Proposition 4.9 [44]. Consider the linear dynamical system $\mathcal{G}$ given by (4.1) with $u \equiv 0$. If $\mathcal{G}$ is semistable, then, for every $n \times n$ nonnegative definite matrix R,

$$J_R(x_0) \triangleq \int_0^\infty [x(t) - x_{\rm e}]^{\rm T} R [x(t) - x_{\rm e}] {\rm d}t < \infty, \tag{4.14}$$

where $x_{\rm e} = (I - AA^{\#})x_0$.

The following standard theorem provides necessary and sufficient conditions for guaranteeing the existence of a steady-state solution to the differential Riccati equation. To state this result, consider the performance measure

$$J_y(x_0, u(\cdot)) \triangleq \int_0^{t_f} \left[y^{\mathrm{T}}(t)R_3y(t) + \varepsilon^2 u^{\mathrm{T}}(t)R_2u(t)\right]\mathrm{d}t, \tag{4.15}$$

where $R_3 = R_3^{\mathrm{T}} > 0$.

Theorem 4.10 [77, Th. 3.7]. Consider the system $\mathcal{G}$ given by (4.1) and (4.2), and performance measure (4.15) with $\varepsilon = 1$, and let $P_{t_f}(t) = P_{t_f}^{\mathrm{T}}(t) \geq 0$, $t \in [0, t_f]$, be a solution to the differential Riccati equation

$$-\dot{P}(t) = A^{\mathrm{T}}P(t) + P(t)A + C^{\mathrm{T}}R_3C - P(t)BR_2^{-1}B^{\mathrm{T}}P(t),\ P(t_f) = 0, \qquad t \in [0, t_f]. \tag{4.16}$$

The system $\mathcal{G}$ has no poles that are unstable, uncontrollable, and observable if and only if $\lim_{t\to\infty} P_{t_f}(t) = P$, where $P = P^{\mathrm{T}} \geq 0$ is a solution to the algebraic Riccati equation

$$0 = A^{\mathrm{T}}P + PA + C^{\mathrm{T}}R_3C - PBR_2^{-1}B^{\mathrm{T}}P. \tag{4.17}$$

Finally, in this case, the state-feedback control law $u(t) = -R_2^{-1}B^{\mathrm{T}}Px(t)$ guarantees Lyapunov stability of the closed-loop linear dynamical system $\mathcal{G}$.

The following classical theorem for finding the minimal cost for a singular control is necessary for later developments.

Theorem 4.11 [78]. Consider the linear dynamical system given by (4.1) and (4.2) with performance measure (4.15), and assume $\operatorname{rank} B = m$ and $\operatorname{rank} C = l$. Let $t_f \to \infty$ and let $P_\varepsilon = P_\varepsilon^{\mathrm{T}} \geq 0$ be a solution to the algebraic Riccati equation

$$0 = A^{\mathrm{T}}P + PA + C^{\mathrm{T}}R_3C - \frac{1}{\varepsilon^2}PBR_2^{-1}B^{\mathrm{T}}P. \tag{4.18}$$

Then the following statements hold.

i) If $l > m$, then $\lim_{\varepsilon\to 0} P_\varepsilon \neq 0$.

ii) If $l = m$ and the numerator polynomial of $\det C(sI_n - A)^{-1}B$ is not identically equal to zero and has roots with nonpositive real part, then $\lim_{\varepsilon \to 0} P_\varepsilon = 0$.

iii) If $l < m$ and there exists a matrix $M \in \mathbb{R}^{m \times l}$ such that the numerator polynomial of $\det C(sI_n - A)^{-1}BM$ is not identically equal to zero and has roots with nonpositive real parts only, then $\lim_{\varepsilon \to 0} P_\varepsilon = 0$.

The following theorem gives a converse to Statement *ii*) of Theorem 4.11. A similar result was proven by Kwakernaak and Sivan [78] for the case where $\tilde{A} = A + BK$ is Hurwitz.

Theorem 4.12. Consider the linear dynamical system given by (4.1) and (4.2) with performance measure (4.15), and assume $\operatorname{rank} B = m$ and $\operatorname{rank} C = l$. Let $t_f \to \infty$, let $l = m$, and let $\bar{P}_\varepsilon$ be the least squares solution to (4.18). If $u(t) = Kx(t)$ guarantees semistability of (4.1) and minimizes the performance measure (4.15), the numerator polynomial $\psi(s)$ of $\det C(sI - A)^{-1}B$ is not identically equal to zero, and $\lim_{\varepsilon \to 0} \bar{P}_\varepsilon = 0$, then the roots of $\psi(s)$ are nonpositive.

Proof: If $\lim_{\varepsilon \to 0} \bar{P}_\varepsilon = 0$, then (4.18) specializes to

$$(R_2^{-\frac{1}{2}}L)^{\mathrm{T}}(R_2^{-\frac{1}{2}}L) = (R_3^{\frac{1}{2}}C)^{\mathrm{T}}(R_3^{\frac{1}{2}}C), \tag{4.19}$$

where $L \triangleq \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon}B^{\mathrm{T}}\bar{P}_\varepsilon$. Note that, since R_2 is nonsingular, L exists and, since R_3 is positive definite, $\operatorname{rank} C^{\mathrm{T}}R_3C = \operatorname{rank} R_3^{\frac{1}{2}}C = \operatorname{rank} C = m$. Thus, there exists $U \in \mathbb{R}^{m \times m}$ such that $U^{\mathrm{T}}U = I_m$ and

$$R_2^{-\frac{1}{2}}L = UR_3^{\frac{1}{2}}C. \tag{4.20}$$

Now, it follows from Theorem 4.10 that

$$\begin{aligned} \frac{\det(sI_n - A - BK)}{\det(sI_n - A)} &= \det\big[I_n - K(sI - A)^{-1}B\big] \\ &= \det\left[I_n + \frac{1}{\varepsilon}R_2^{-1}\frac{1}{\varepsilon}B^{\mathrm{T}}\bar{P}_\varepsilon(sI_n - A)^{-1}B\right]. \end{aligned} \tag{4.21}$$

Thus, the roots of the closed-loop characteristic polynomial that stay finite as $\varepsilon \to 0^+$ approach the roots of $\det(sI_n - A)\det\big[R_2^{-1}L(sI_n - A)^{-1}B\big]$, that is, the roots approach the roots of

$$\det(sI_n - A)\det\left[R_2^{-\frac{1}{2}}UR_3^{\frac{1}{2}}C(sI_n - A)^{-1}B\right] = \det\left[R_2^{-\frac{1}{2}}UR_3^{\frac{1}{2}}\right]\psi(s). \tag{4.22}$$

Now, the result follows using identical arguments as in [77, p. 308]. ■

4.4. Semistability and Singular Control

In order to address the singular control problem for semistabilization, we first need to show that the performance measure (4.6) is well-defined when (4.1) is semistable with $u(t) = Kx(t)$.

Proposition 4.13. Consider the system (4.1) with performance measure (4.6). If there exists $K \in \mathbb{R}^{m\times n}$ such that (4.1) with $u(t) = Kx(t)$ is semistable, then (4.6) is well-defined, that is, $J_0(x_0, u(\cdot)) < \infty$.

Proof: Consider the performance measure

$$J_\varepsilon(x_0, u(\cdot)) \triangleq \int_0^\infty \big[(x(t) - x_{\rm e})^{\rm T}C^{\rm T}C(x(t) - x_{\rm e}) + \varepsilon^2(u(t) - u_{\rm e})^{\rm T}(u(t) - u_{\rm e})\big]{\rm d}t. \tag{4.23}$$

Since $C^{\rm T}C + \varepsilon^2K^{\rm T}K \geq 0$, it follows from Proposition 4.9 that $J_\varepsilon(x_0, u(\cdot)) < \infty$, which proves the assertion since $J_0(x_0, u(\cdot)) = \lim_{\varepsilon\to 0^+} J_\varepsilon(x_0, u(\cdot))$ and J_ε is a monotone function of ε that is bounded from below. ■

Next, we give an expression for the state-feedback control law that minimizes the performance measure (4.6) and guarantees semistability of the system given by (4.1) and (4.2).

Theorem 4.14. Consider the linear dynamical system given by (4.1) and (4.2) with performance measure (4.6). If (A, B) is semicontrollable and (A, C) is semiobservable, then with the state feedback control law

$$u = \phi(x) = -Lx, \tag{4.24}$$

where $L = \lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon} B^{\mathrm{T}} P_{\mathrm{LS}}$, $P_{\mathrm{LS}} = \int_0^\infty e^{\tilde{A}^{\mathrm{T}} t} C^{\mathrm{T}} C e^{\tilde{A} t}\, \mathrm{d}t$ is the least squares solution of

$$0 = A^{\mathrm{T}} P + PA + C^{\mathrm{T}} C - \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2} PBB^{\mathrm{T}} P, \tag{4.25}$$

and $\tilde{A} \triangleq A + BL$, the solution $x(t) = x_{\mathrm{e}}$, $t \geq 0$, to (4.1) is globally semistable,

$$J_0(x_0, L) = \lim_{\varepsilon\to 0} J_\varepsilon(x_0, L) = x_0^{\mathrm{T}} P_{\mathrm{LS}} x_0, \tag{4.26}$$

and (4.5) is verified. Furthermore, L is well defined.

Proof: Semistability of (4.1) with u given by (4.24), (4.26), and (4.5) directly follow from Theorem 2.30. In addition, the existence of $\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon} B^{\mathrm{T}} P_{\mathrm{LS}}$ can be proven as in the proof of Theorem 4.12. ■

The next theorem provides a closed-form expression for the optimal performance measure (4.6) extending a well-known property of classical singular control to singular semistabilization.

Theorem 4.15. Consider the linear dynamical system given by (4.1) and (4.2), with $u(t) = Kx(t)$ such that (4.1) is semistable and the performance measure (4.6) is minimized. If $l > m$, then $J_0(x_0, K) > 0$. Alternatively, if $l = m$ and the zeros of the numerator polynomial of $\det C(sI_n - A)^{-1}B$ are not identically zero and have nonpositive real part, then $J_0(x_0, K) = 0$. Finally, if $l < m$ and there exists a matrix M such that the numerator polynomial of $\det C(sI_n - A)^{-1}BM$ is not identically zero and has zeros with nonpositive real parts, then $J_0(x_0, K) = 0$.

Proof: The proof is a direct consequence of Theorems 4.14 and 4.11. ■

Corollary 4.16. Consider the linear dynamical system given by (4.1) and (4.2), with $u(t) = Kx(t)$ such that (4.1) is semistable and the performance measure (4.6) is minimized. Suppose $l = m$ and the numerator polynomial $\psi(s)$ of $\det C(sI_n - A)^{-1}B$ is not identically equal to zero. If $J_0(x_0, K) = 0$, then the roots of $\psi(s)$ are nonpositive.

Proof: The proof is a direct consequence of Theorem 4.14 and Theorem 4.12. ■

The following theorem provides an expression for $J_0(x_0, K)$ in terms of a reduced-order system when the open-loop system is not minimum phase.

Theorem 4.17. Consider the linear dynamical system given by (4.1) and (4.2), with $u(t) = Kx(t)$ such that (4.1) is semistable and the performance measure (4.6) is minimized. If $l = m$, $\mathrm{rank}(CB) = m$, and all the roots of the numerator polynomial $\psi(s)$ of $\det C(sI - A)^{-1}B$ have nonnegative real part, then the dynamical system $\mathcal{G}$ given by (4.1) and (4.2) with $u(t) = Kx(t)$ is equivalent to (4.8) and the minimal performance measure (4.6) is given by

$$J_0(x_0, K) = z^{\mathrm{T}}(0)P_0 z(0), \tag{4.27}$$

where P_0 is the least squares solution to

$$0 = A_0^{\mathrm{T}}P_0 + P_0A_0 - P_0B_0B_0^{\mathrm{T}}P_0. \tag{4.28}$$

Proof: It follows from Proposition 4.5 that the system $\mathcal{G}$ given by (4.1) and (4.2) is equivalent to (4.8) and the roots of $\psi(s)$ are the eigenvalues of A_0 [109]. Next, it follows from Theorem 4.14 that

$$J_0(x_0, K) = x_0^{\mathrm{T}}P_{\mathrm{LS}}x_0 = \begin{bmatrix} y(0) \\ z(0) \end{bmatrix}^{\mathrm{T}} \hat{P}_{\mathrm{LS}} \begin{bmatrix} y(0) \\ z(0) \end{bmatrix}, \tag{4.29}$$

where $\hat{P}_{\mathrm{LS}} \triangleq \lim_{\varepsilon\to 0} \hat{P}_\varepsilon$ and $\hat{P}_\varepsilon$ is the least squares solution of

$$0 = \begin{bmatrix} A_1 & A_2 \\ B_0 & A_0 \end{bmatrix}^{\mathrm{T}} P + P \begin{bmatrix} A_1 & A_2 \\ B_0 & A_0 \end{bmatrix} + \begin{bmatrix} I_m & 0 \\ 0 & 0_{(n-m)\times(n-m)} \end{bmatrix} - \frac{1}{\varepsilon^2} P \begin{bmatrix} B_1B_1^{\mathrm{T}} & 0 \\ 0 & 0 \end{bmatrix} P. \tag{4.30}$$

Letting any solution P to (4.30) be of the form

$$P = \begin{bmatrix} \varepsilon P_1 & \varepsilon P_2 \\ \varepsilon P_2^{\mathrm{T}} & P_0 + \varepsilon P_3 \end{bmatrix} + O(\varepsilon^2), \tag{4.31}$$

where P_0, P_1, P_2, and P_3 are independent of ε [66], it follows from (4.30) with $\varepsilon = 0$ that

$$P_1 = \big(B_1 B_1^{\mathrm{T}}\big)^{-\frac{1}{2}}, \tag{4.32}$$

$$P_2 = \big(B_1 B_1^{\mathrm{T}}\big)^{-\frac{1}{2}} B_0^{\mathrm{T}} P_0, \tag{4.33}$$

$$P_3 = P_0 B_0 \big(B_1 B_1^{\mathrm{T}}\big)^{-\frac{1}{2}} B_0^{\mathrm{T}} P_0, \tag{4.34}$$

and (4.28) holds. Since the zeros of $\det C(sI - A)^{-1}B$ have nonnegative real part, the eigenvalues of $-A_0$ are nonpositive, that is, $-A_0$ is Lyapunov stable and, by Theorem 4.10, there exists a solution P_0 to (4.28). Therefore, the assertion follows immediately from (4.29) and (4.31). ∎

Finally, we extend Qiu and Davison's formula for the optimal singular control [99] to semistabilization.

Theorem 4.18. Consider the linear dynamical system $\mathcal{G}$ given by (4.1) and (4.2), with $u(t) = Kx(t)$ such that (4.1) is semistable and the performance measure (4.6) is minimized. If $\mathcal{G}$ has transfer function $G(s) = C(sI_n - A)^{-1}B$ that is nonminimum phase and right invertible, then the minimal performance measure (4.6) is given by

$$J_0(x_0, K) = 2\sum_{j=1}^{l} \frac{1}{\lambda_j}, \tag{4.35}$$

where $\lambda_1, \ldots, \lambda_l$, $l \le n$, are the zeros of $G(s)$ whose real part is positive. Conversely, if $l = m$, the numerator polynomial $\psi(s)$ of $\det C(sI - A)^{-1}B$ is not identically equal to zero, and (4.35) holds, then $G(s)$ is minimum phase and right invertible.

Proof: It follows from Corollary 4.6 that $G(s)$ can be factored as $G(s) = G_1(s)G_2(s)$, where $G_1(s) \in \mathbb{R}_{\mathrm{prop}}^{l\times l}(s)$, $G_2(s) \in \mathbb{R}_{\mathrm{prop}}^{l\times m}(s)$, $G_1(s) \triangleq C_1(sI - A_1)^{-1}B_1 + D_1$, and $G_2(s) \triangleq$

$C_2(sI - A_2)^{-1}B_2$. Furthermore, by Corollary 4.7, the poles of $G_1(s)$ are equal to the negatives of the zeros of $G(s)$ whose real part is positive, that is, $\lambda_1, \ldots, \lambda_l$, $l \le n$.

Next, let $P_\varepsilon = P_\varepsilon^{\mathrm{T}} \ge 0$ denote the solution to the algebraic Riccati equation given by (4.18). Using the factorization $G(s) = G_1(s)G_2(s)$, it follows that $P_\varepsilon = \begin{bmatrix} I_l & 0_{l\times(n-l)} \\ 0_{(n-l)\times l} & P_{\varepsilon 2} \end{bmatrix}$, where $P_{\varepsilon 2} = P_{\varepsilon 2}^{\mathrm{T}} \ge 0$ is a solution to the algebraic Riccati equation

$$0 = A_2^{\mathrm{T}}P + PA_2 + C_2^{\mathrm{T}}C_2 - \frac{1}{\varepsilon^2}PB_2B_2^{\mathrm{T}}P. \tag{4.36}$$

Now, Theorem 4.11 implies that $\lim_{\varepsilon\to 0} P_{\varepsilon 2} = 0$. Thus, $\lim_{\varepsilon\to 0} P_\varepsilon = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$, and hence, by Theorem 4.14 it follows that

$$J_0(x_0, K) = x_0^{\mathrm{T}} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} x_0 = x_{0,1}^{\mathrm{T}} x_{0,1}, \tag{4.37}$$

where $x_0 \triangleq [x_{0,1}^{\mathrm{T}}, x_{0,2}^{\mathrm{T}}]^{\mathrm{T}}$ is partitioned as A is in (4.13).

Next, since the system (4.1) with $u(t) = Kx(t)$ is semistable, $\lim_{t\to\infty} x(t) = x_{\mathrm{e}}$. Thus, for $t \to \infty$, the output of $G_1(s)$ is $y_{\mathrm{e}} = Cx_{\mathrm{e}}$. Consequently, the output of $G_2(s)$ is $G_1^{-1}(0)y_{\mathrm{e}}$, which implies that $x_{0,1} = -A_1^{-1}B_1G_1^{-1}(0)y_{\mathrm{e}}$. Therefore, since $G_1(0)$ is unitary, it follows that

$$J_0(x_0, K) = \operatorname{tr} B_1^{\mathrm{T}}A_1^{-\mathrm{T}}A_1^{-1}B_1 = \operatorname{tr} A_1^{-1}B_1B_1^{\mathrm{T}}A_1^{-\mathrm{T}}. \tag{4.38}$$

The transfer function $G_1(s)$ is inner, and hence, has a balanced realization $\mathcal{G}_1 \sim \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right]$. Thus, by Corollary 4.8, it follows that $B_1B_1^{\mathrm{T}} = C_1^{\mathrm{T}}C_1$, $P_1 = Q_1 = I_l$, and (4.11) and (4.12) hold for $A = A_1$, $B = B_1$, $C = C_1$, $P = P_1$, and $Q = Q_1$. Hence,

$$J_0(x_0, K) = -\operatorname{tr} A_1^{-1}(A_1 + A_1^{\mathrm{T}})A_1^{-\mathrm{T}} = -2\operatorname{tr} A_1^{-1}, \tag{4.39}$$

which proves the assertion.

Conversely, if $l = m$, the numerator polynomial $\psi(s)$ of $\det C(sI - A)^{-1}B$ is not identically equal to zero, and (4.35) holds, then it follows from Corollary 4.16 that the zeros of $\psi(s)$ are nonpositive, which implies that $G(s)$ is right invertible [31]. ■

Chapter 5

Singular Control for Nonlinear Semistabilization

5.1. Introduction

The singular control problem for asymptotic stabilization of affine nonlinear systems has been addressed in [109] as a generalization of the singular control problem for linear dynamical systems. A complicating factor in the solution of the singular control problem for affine nonlinear dynamical systems is the fact that the Hamilton-Jacobi-Bellman equation involves singularities that cannot be canceled since the cost-to-go function is required to be positive definite.

As discussed in Chapter 1, semistability [13, 16] is the property whereby every trajectory that starts in a neighborhood of a Lyapunov stable equilibrium converges to a (possibly different) Lyapunov stable equilibrium. In Chapter 2, we addressed an optimal control problem for semistabilization of linear and nonlinear dynamical systems. Specifically, given a nonlinear dynamical system with a nonlinear-nonquadratic performance measure, it is shown that the optimal semistable state-feedback controller can be solved using Hamilton-Jacobi-Bellman-type conditions that do not require the cost-to-go function to be sign definite. This result is then used to solve the $\mathcal{H}_2$ optimal semistable stabilization problem using a Riccati equation approach.

In this chapter, we provide three approaches to address the nonlinear semistable optimal singular control problem. Specifically, applying a singular perturbation method [75] we construct a state-feedback singular controller that guarantees closed-loop semistabilization for nonlinear systems. In this approach, which extends the results of [109] for singular asymptotic stabilization, we show that for a nonnegative cost-to-go function the minimum value of the singular performance measure over the set of semistabilizing controls is smaller than the minimum value of the singular performance measure over the set of controls that guarantee asymptotic stability. In the second approach, we solve the nonlinear semistable optimal singular control problem using the results of Chapter 2. Specifically, since the cost-to-go function that solves the Hamilton-Jacobi-Bellman-like equation for semistabilization is not required to be sign definite, we use this extra flexibility in the semistable singular control problem to cancel the singularities in the corresponding Hamilton-Jacobi-Bellman-like equation. In this case, we show that the minimum value of the singular performance measure is zero. Finally, a solution to the singular semistabilization problem using differential geometric methods [64], the concepts of output-feedback linearization and feedback equivalence, and results of Chapter 4 is also presented. Specifically, we construct an output-feedback linearizing controller and find the control parameters that solve the optimal singular control problem for semistabilization of the linearized system.

5.2. Optimal Control Formulation

Consider the affine in the control nonlinear dynamical system

$$\dot{x}(t) = f(x(t)) + G(x(t))u(t), \qquad x(0) = x_0, \qquad t \geq 0, \tag{5.1}$$

$$y(t) = h(x(t)), \tag{5.2}$$

where, for every $t \geq 0$, $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, $\mathcal{D}$ is an open set, $u(t) \in U \subseteq \mathbb{R}^m$, $y(t) \in Y \subseteq \mathbb{R}^l$, $0 = f(x_{\rm e}) + G(x_{\rm e})u_{\rm e}$ for some $(x_{\rm e}, u_{\rm e}) \in \mathcal{D} \times U$, $y_{\rm e} = h(x_{\rm e})$, $l = m$, $f : \mathcal{D} \to \mathbb{R}^n$ is Lipschitz continuous on $\mathcal{D}$, and $G : \mathcal{D} \to \mathbb{R}^{n\times m}$ and $h : \mathcal{D} \to \mathbb{R}^l$ are continuous on $\mathcal{D}$. Here, we assume that for each $u_{\rm e} \in U$ such that $0 = f(x_{\rm e}) + G(x_{\rm e})u_{\rm e}$, there exists $x_{\rm e} \in \mathcal{D}_{u_{\rm e}} \subset \mathcal{D}$, where $\mathcal{D}_{u_{\rm e}}$ is a set of nonisolated equlibrium points of (5.1).

To address the optimal semistabilization problem, we consider the controlled nonlinear dynamical system (5.1) with $u(\cdot)$ restricted to the class of *admissible* controls consisting of continuous functions $u(\cdot)$ such that $u(t) \in U$, $t \geq 0$. A continuous function $\phi : \mathcal{D} \to U$ satisfying $\phi(x_{\mathrm{e}}) = u_{\mathrm{e}}$, for some $(x_{\mathrm{e}}, u_{\mathrm{e}}) \in \mathcal{D} \times U$ such that $0 = f(x_{\mathrm{e}}) + G(x_{\mathrm{e}})u_{\mathrm{e}}$, is called a *control law.* If $u(t) = \phi(x(t))$, $t \geq 0$, where $\phi(\cdot)$ is a control law and $x(t)$ satisfies (5.1), then we call $u(\cdot)$ a *feedback control law.* Note that the feedback control law is an admissible control since $\phi(\cdot)$ has values in U. Given a control law $\phi(\cdot)$ and a feedback control $u(t) = \phi(x(t))$, $t \geq 0$, the closed-loop system (5.1) and (5.2) is given by

$$\dot{x}(t) = f(x(t)) + G(x(t))\phi(x(t)), \qquad x(0) = x_0, \qquad t \geq 0, \tag{5.3}$$

$$y(t) = h(x(t)). \tag{5.4}$$

Given the nonlinear dynamical system (5.1) and (5.2) with performance measure

$$J_\varepsilon(x_0, u(\cdot)) \triangleq \int_0^\infty \left[(y(t) - y_{\mathrm{e}})^{\mathrm{T}}(y(t) - y_{\mathrm{e}}) + \varepsilon^2 (u(t) - u_{\mathrm{e}})^{\mathrm{T}}(u(t) - u_{\mathrm{e}})\right]\mathrm{d}t, \tag{5.5}$$

where $\varepsilon > 0$, we construct a feedback control law $u(t) = \phi(x(t))$ such that the equilibrium solution $x(t) \equiv x_{\mathrm{e}}$, $t \geq 0$, of (5.1) and (5.2) is semistable and

$$J_0(x_0, u(\cdot)) \triangleq \lim_{\varepsilon \to 0} \int_0^\infty [(y(t) - y_{\mathrm{e}})^{\mathrm{T}}(y(t) - y_{\mathrm{e}}) + \varepsilon^2 (u(t) - u_{\mathrm{e}})^{\mathrm{T}}(u(t) - u_{\mathrm{e}})]\mathrm{d}t \tag{5.6}$$

is minimized in the sense that

$$J_0(x_0, \phi(x(\cdot))) = \min_{u(\cdot) \in \mathcal{S}(x_0)} J_0(x_0, u(\cdot)), \tag{5.7}$$

where, for every initial condition $x_0 \in \mathcal{D}$,

$$\mathcal{S}(x_0) \triangleq \{u(\cdot) : u(\cdot) \text{ is admissible and } x(\cdot) \text{ given by (5.1) satisfies } x(t) \to x_{\mathrm{e}} \text{ as } t \to \infty\}$$

denotes the set of convergent controllers.

Theorem 5.1. Consider the controlled nonlinear dynamical system (5.1) and (5.2) with $u(\cdot) \in \mathcal{S}(x_0)$ and performance measure (5.5), and assume that there exists a continuously differentiable function $V : \mathcal{D} \to \mathbb{R}$ such that

$$V'(x_{\rm e}) = 0, \qquad x_{\rm e} \in \mathcal{D}, \tag{5.8}$$

$$\begin{aligned}(y - y_{\rm e})^{\rm T} (y - y_{\rm e}) &+ V'(x)f(x) + V'(x)G(x)u_{\rm e} \\ &- \frac{1}{4\varepsilon^2} V'(x)G(x)G^{\rm T}(x)V'^{\rm T}(x) = 0, \qquad (x, u_{\rm e}) \in \mathcal{D} \times U.\end{aligned} \tag{5.9}$$

If, with the feedback control

$$u = \phi(x) = -\frac{1}{2\varepsilon^2} G^{\rm T}(x)V'^{\rm T}(x) + u_{\rm e}, \tag{5.10}$$

every equilibrium point $x_{\rm e} \in F^{-1}(0) = \{x \in \mathcal{D} : f(x) + G(x)\phi(x) = 0\}$ of the closed-loop system (5.3) is Lyapunov stable, then the solution $x(t) = x_{\rm e}$, $t \geq 0$, of the closed-loop system (5.3) is semistable and

$$J_\varepsilon(x_0, \phi(x(\cdot))) = V(x_0) - V(x_{\rm e}). \tag{5.11}$$

Furthermore, the feedback control (5.10) minimizes $J_\varepsilon(x_0, u(\cdot))$ in the sense that

$$J_\varepsilon(x_0, \phi(x(\cdot))) = \min_{u(\cdot) \in \mathcal{S}(x_0)} J_\varepsilon(x_0, u(\cdot)). \tag{5.12}$$

Proof: The result direct follows from Theorem 2.26 with $F(x, u) = f(x) + G(x)u$ and $L(x, u) = (y - y_{\rm e})^{\rm T}(y - y_{\rm e}) + \varepsilon^2(u - u_{\rm e})^{\rm T}(u - u_{\rm e})$. ■

Remark 5.2. Theorem 5.1 requires that every equilibrium point $x_{\rm e} \in F^{-1}(0)$ of the closed-loop system is Lyapunov stable. One can relax this assumption by alternatively assuming a nontangency condition of the closed-loop vector field to invariant or negatively invariant subsets of the level sets of $V(\cdot)$ containing the system equilibrium. For details; see [13].

5.3. A Singular Perturbation Approach to the Optimal Singular Control Problem

Let $\mathcal{T} : \mathcal{D} \to \mathbb{R}^n$ be a diffeomorphism such that, in the coordinates

$$\begin{bmatrix} y \\ z \end{bmatrix} = \mathcal{T}(x), \tag{5.13}$$

where $z \in \mathbb{R}^{n-m}$, (5.1) and (5.2) is equivalent to

$$\dot{y}(t) = f_{\rm a}(y(t), z(t)) + g_{\rm a}(y(t), z(t))u(t), \qquad y(0) = h(x(0)), \qquad t \geq 0, \tag{5.14}$$

$$\dot{z}(t) = f_0(z(t)) + g_0(z(t))y(t), \qquad z(0) = z_0, \tag{5.15}$$

where $[y^{\rm T}(0),\ z_0^{\rm T}]^{\rm T} = \mathcal{T}(x_0)$. Recall that the existence of a diffeomorphism $\mathcal{T} : \mathcal{D} \to \mathbb{R}^n$ such that (5.14) and (5.15) hold is guaranteed by Proposition 5.1.2 of [64] for $x \in \mathcal{D}_0 \subseteq \mathcal{D}$ and Corollary 5.7 of [20] for $x \in \mathcal{D} = \mathbb{R}^n$. Since the Hamilton-Jacobi-Bellman equation (5.9) is not defined for $\varepsilon = 0$, we use a singular perturbation method [75, Ch. 11] to address the semistable optimal singular control problem, that is, to find the feedback control law $u = \phi(x)$, $x \in \mathcal{D}$, such that the dynamical system given by (5.1) and (5.2) is semistable and the performance measure (5.6) is minimized in the sense of (5.7).

Propositions 5.3 and 5.4 below address two auxiliary optimization problems; namely, a minimum energy problem for the semistabilization of the system given by (5.15) and a singular control problem for the semistabilization of a system directly related to (5.14). Then, in Theorem 5.5 below, we show that $u = \phi(x)$ can be approximated by the optimal state-feedback controller for the auxiliary singular control problem, whose Hamilton-Jacobi-Bellman equation is now well defined for $\varepsilon = 0$. For the statement of the next result, define the set of semistabilizing virtual controllers $\mathcal{S}_y(z_0)$ for each initial condition $z_0 \in \mathbb{R}^{n-m}$ by

$$\mathcal{S}_y(z_0) \triangleq \{y(\cdot) : y(\cdot) \text{ is admissible and } z(\cdot) \text{ given by (5.15) satisfies } z(t) \to z_{\rm e} \text{ as } t \to \infty\}.$$

Proposition 5.3. Consider the nonlinear controlled dynamical system (5.15) with $y(\cdot) \in \mathcal{S}_y(z_0)$ and performance measure

$$J_y(z_0, y(\cdot)) \triangleq \int_0^\infty (y(t) - y_{\rm e})^{\rm T}(y(t) - y_{\rm e})\, {\rm d}t, \tag{5.16}$$

where $y_{\rm e} \in \mathbb{R}^m$, and assume that there exists a continuously differentiable function $V_z : \mathbb{R}^{n-m} \to \mathbb{R}$ such that

$$V_z'(z_{\rm e}) = 0, \qquad z_{\rm e} \in \mathbb{R}^{n-m}, \tag{5.17}$$

$$V_z'(z)f_0(z) - \frac{1}{4}V_z'(z)g_0(z)g_0^{\mathrm{T}}(z)V_z'^{\mathrm{T}}(z) + V_z'(z)g_0(z)y_{\mathrm{e}} = 0, \qquad z \in \mathbb{R}^{n-m}. \tag{5.18}$$

If, with the feedback control

$$\alpha(z) = -\frac{1}{2}g_0^{\mathrm{T}}(z)V_z'^{\mathrm{T}}(z) + y_{\mathrm{e}}, \tag{5.19}$$

every equilibrium point z_{e} of the closed-loop system

$$\dot{z}(t) = f_0(z(t)) + g_0(z(t))\alpha(z(t)), \qquad z(0) = z_0, \qquad t \geq 0, \tag{5.20}$$

is Lyapunov stable, then the solution $z(t) = z_{\mathrm{e}}$, $t \geq 0$, of the closed-loop system (5.20) is semistable and

$$\min_{y(\cdot)\in\mathcal{S}_y(z_0)} J_y(z_0, y(\cdot)) = J_y(z_0, \alpha(z(\cdot))) = V_z(z_0) - V_z(z_{\mathrm{e}}). \tag{5.21}$$

Proof: The result follows as a direct application of Theorem 5.1 with x replaced by z, x_{e} replaced by z_{e}, $f(x)$ replaced by $f_0(z)$, $G(x)$ replaced by $g_0(z)$, u replaced by y, $\phi(\cdot)$ replaced by $\alpha(\cdot)$, u_{e} replaced by y_{e}, $y - y_{\mathrm{e}}$ replaced by 0, $V(\cdot)$ replaced by $V_z(\cdot)$, and $\varepsilon = 1$. ■

Proposition 5.4. Consider the nonlinear controlled dynamical system given by

$$\dot{\eta}(t) = \tilde{g}_{\mathrm{a}}(\eta(t), z)u(t), \qquad \eta(0) = \eta_0 = y(0) - \alpha(z), \qquad t \geq 0, \tag{5.22}$$

with $u(\cdot) \in \mathcal{S}(x_0)$ and performance measure

$$J_\eta(\eta_0, u(\cdot)) \triangleq \int_0^\infty \left[\eta^{\mathrm{T}}(t)\eta(t) + \varepsilon^2(u(t) - u_{\mathrm{e}})^{\mathrm{T}}(u(t) - u_{\mathrm{e}})\right]\mathrm{d}t, \tag{5.23}$$

where $\eta \triangleq y - \alpha(z)$, $\tilde{g}_{\mathrm{a}}(\eta, z) \triangleq g_{\mathrm{a}}(\eta + \alpha(z), z)$ and z is constant, and assume that there exists a continuously differentiable function $V_\eta : \mathbb{R}^{n-m} \to \mathbb{R}$ such that

$$V_\eta'(\eta_{\mathrm{e}}) = 0, \qquad \eta_{\mathrm{e}} \in \mathbb{R}^m, \tag{5.24}$$

$$\eta^{\mathrm{T}}\eta - \frac{1}{4}V_\eta'(\eta)\tilde{g}_{\mathrm{a}}(\eta, z)\tilde{g}_{\mathrm{a}}^{\mathrm{T}}(\eta, z)V_\eta'^{\mathrm{T}}(\eta) + \varepsilon V_\eta'(\eta)\tilde{g}_{\mathrm{a}}(\eta, z)u_{\mathrm{e}} = 0, \qquad \eta \in \mathbb{R}^m, \tag{5.25}$$

If, with the feedback control

$$\beta(\eta) = -\frac{1}{2\varepsilon}\tilde{g}_{\mathrm{a}}^{\mathrm{T}}(\eta, z)V_\eta'^{\mathrm{T}}(\eta) + u_{\mathrm{e}}, \tag{5.26}$$

every equilibrium point $\eta_{\rm e}$ of the closed-loop system

$$\dot{\eta}(t) = \tilde{g}_{\rm a}(\eta(t), z)\beta(\eta(t)), \qquad \eta(0) = \eta_0 = y(0) - \alpha(z), \qquad t \geq 0, \tag{5.27}$$

is Lyapunov stable, then the solution $\eta(t) = \eta_{\rm e}$, $t \geq 0$, of the closed-loop system (5.27) is semistable and

$$\min_{u(\cdot)\in\mathcal{S}(x_0)} J_\eta(\eta(0), u(\cdot)) = J_\eta(\eta_0, \beta(z(\cdot))) = \varepsilon V_\eta(\eta_0) - \varepsilon V_\eta(\eta_{\rm e}). \tag{5.28}$$

Proof: The result follows as a direct application of Theorem 5.1 with x replaced by η, $x_{\rm e}$ replaced by 0, $f(x)$ replaced by 0, $G(x)$ replaced by $\tilde{g}_{\rm a}(\eta, z)$, $\phi(\cdot)$ replaced by $\beta(\cdot)$, and $V(\cdot)$ replaced by $\varepsilon V_\eta(\cdot)$. ■

Next, we present one of the main results of this section which shows that $\phi(x)$ can be approximated by $\beta(\eta)$, that is, the control that minimizes (5.6) and guarantees semistability of the system given by (5.14) and (5.15) can be approximated by the control that minimizes (5.23) and guarantees semistability of (5.22). Furthermore, we give an estimate of the minimum value of the performance measure (5.6).

Theorem 5.5. Consider the nonlinear dynamical system given by (5.14) and (5.15) with performance measure (5.6). Assume that the hypothesis of Propositions 5.3 and 5.4 hold, define $\gamma(\eta, z) \triangleq f_{\rm a}(\eta + \alpha, z) - \dot{\alpha}(z) + \tilde{g}_{\rm a}(\eta, z)u_{\rm e}$, and assume that

$$\|\gamma(\eta, z)\| \leq k_1\|\eta\| + k_2\|\alpha(z) - y_{\rm e}\|, \qquad (\eta, z) \in \mathcal{N}_\delta, \tag{5.29}$$

where $k_1 > 0$, $k_2 > 0$, $\alpha(\cdot)$ is given in (5.19), and $\mathcal{N}_\delta$ is an open neighborhood of the set of equilibrium points of the closed-loop system

$$\varepsilon\dot{\eta}(t) = -\big[\tilde{g}_{\rm a}(\eta(t), z)\tilde{g}_{\rm a}^{\rm T}(\eta(t), z)\big]^{\frac{1}{2}}\eta(t) + \varepsilon\gamma(\eta(t), z), \quad \eta(0) = y(0) - \alpha(z(0)), \quad t \geq 0, \tag{5.30}$$

$$\dot{z}(t) = f_0(z(t)) + g_0(z(t))\big[\alpha(z(t)) + \eta(t)\big], \qquad z(0) = z_0. \tag{5.31}$$

Moreover, suppose that, for all $(\eta, z) \in \mathbb{R}^{n-m} \times \mathbb{R}^m$, there exists $\zeta > 0$ such that the smallest singular value of $\tilde{g}_{\rm a}(\eta, z)$ is greater than or equal to ζ. Then (5.26) is equivalent to

$$\beta(\eta) = -\frac{1}{\varepsilon}\tilde{g}_{\rm a}^{\rm T}(\eta, z)\big[\tilde{g}_{\rm a}(\eta, z)\tilde{g}_{\rm a}^{\rm T}(\eta, z)\big]^{-\frac{1}{2}}\eta + u_{\rm e} \tag{5.32}$$

and, with the feedback control $u(\cdot) = \beta(\eta(\cdot))$, the solution $[y^{\mathrm{T}}(t),\, z^{\mathrm{T}}(t)]^{\mathrm{T}} = [y_{\mathrm{e}}^{\mathrm{T}},\, z_{\mathrm{e}}^{\mathrm{T}}]^{\mathrm{T}}$, $t \geq 0$, of the closed-loop system

$$\dot{y}(t) = f_{\mathrm{a}}(y(t), z(t)) + g_{\mathrm{a}}(y(t), z(t))\beta(\eta(t)), \qquad y(0) = h(x(0)), \qquad t \geq 0, \tag{5.33}$$

$$\dot{z}(t) = f_0(z(t)) + g_0(z(t))y(t), \qquad z(0) = z_0, \tag{5.34}$$

is semistable. Furthermore,

$$\min_{u(\cdot)\in\mathcal{S}(x_0)} J_0(x_0, u(\cdot)) = J_0(x_0, \beta(\cdot)) = V_z(z_0) - V_z(z_{\mathrm{e}}) + \mathcal{O}(\varepsilon). \tag{5.35}$$

Proof: First, we show semistability of the dynamical system given by (5.14) and (5.15) with $u = \phi(x)$ given by (5.10). Let $V(z, y) \triangleq V_z(z) + \varepsilon V_\eta(\eta) + \mathcal{O}(\varepsilon^2)$ so that (5.8) and (5.9) are satisfied by (5.17), (5.18), (5.24), and (5.25), with x replaced by $[z^{\mathrm{T}},\, y^{\mathrm{T}}]^{\mathrm{T}}$, x_{e} replaced by $[z_{\mathrm{e}}^{\mathrm{T}},\, y_{\mathrm{e}}^{\mathrm{T}}]^{\mathrm{T}}$, $f(x)$ replaced by $\begin{bmatrix} f_0(z) + g_0(z)y \\ f_{\mathrm{a}}(y, z) \end{bmatrix}$, $G(x)$ replaced by $\begin{bmatrix} 0_{(n-m)\times m} \\ g_{\mathrm{a}}(y, z) \end{bmatrix}$, and $\varepsilon \to 0$. In this case, (5.10) is equivalent to

$$\phi(x) = -\frac{1}{2\varepsilon^2} g_{\mathrm{a}}^{\mathrm{T}}(y, z)\left(\frac{\partial V(y, z)}{\partial y}\right)^{\mathrm{T}} + u_{\mathrm{e}} \tag{5.36}$$

and it follows from Propositions 5.3 and 5.4 that the system given by (5.14) and (5.15) with $u = \phi(x)$ is semistable.

Next, (5.25), with $\varepsilon \to 0$, implies that $\eta = \frac{1}{2}\big[\tilde{g}_{\mathrm{a}}(\eta, z)\tilde{g}_{\mathrm{a}}^{\mathrm{T}}(\eta, z)\big]^{\frac{1}{2}} V_\eta'^{\mathrm{T}}(\eta)$, and hence, (5.32) follows directly from (5.26). Now, setting $u = \beta(\eta)$, where $\beta(\eta)$ is given by (5.32), (5.14) and (5.15) are equivalent to (5.30) and (5.31), respectively. To show semistability of (5.30) and (5.31), first set $\eta = 0$ and note that in this case (5.31) is semistable by Proposition 5.3. Now, let $\tau \triangleq \frac{t}{\varepsilon}$ so that the uncontrolled dynamics of (5.30) (i.e., $\gamma(\eta, z) \equiv 0$) is equivalent to

$$\dot{\eta}(\tau) = -\frac{1}{2}\big[\tilde{g}_{\mathrm{a}}(\eta(\tau), z)\tilde{g}_{\mathrm{a}}^{\mathrm{T}}(\eta(\tau), z)\big]^{\frac{1}{2}}\eta(\tau), \qquad \eta(0) = y(0) - \alpha(z_0), \qquad \tau \geq 0, \tag{5.37}$$

and note that (5.37) is asymptotically stable by assumption. Define $W(\eta, z) \triangleq \frac{1}{2}\eta^{\mathrm{T}}\eta + V_z(z)$ and note that it follows from (5.30), (5.31), (5.18), and (5.19) that

$$\dot{W}(\eta, z) = \eta^{\mathrm{T}}\bigg[-\frac{1}{\varepsilon}\big(\tilde{g}_{\mathrm{a}}(\eta, z)\tilde{g}_{\mathrm{a}}^{\mathrm{T}}(\eta, z)\big)^{\frac{1}{2}}\eta + \gamma(\eta, z)\bigg]$$

$$+ V_z'(z)\left[f_0(z) - \frac{1}{2}g_0(z)g_0^{\mathrm{T}}(z)V_z'^{\mathrm{T}}(z) + g_0(z)y_{\mathrm{e}}\right]$$
$$\leq -\begin{bmatrix} \|\eta\| \\ \|\alpha(z) - y_{\mathrm{e}}\| \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} (\frac{\zeta}{\varepsilon} - k_1) & 1 + k_2 \\ 1 + k_2 & -1 \end{bmatrix} \begin{bmatrix} \|\eta\| \\ \|\alpha(z) - y_{\mathrm{e}}\| \end{bmatrix}, \qquad (5.38)$$

and hence, $\dot{W}(\eta, z) \leq 0$ for all $(\eta, z) \in \mathbb{R}^{n-m} \times \mathbb{R}^m$ and $\varepsilon \in \big(0, \frac{\zeta}{(1+k_2)^2+2k_1}\big]$. Now, semistability of (5.30) and (5.31) follows from Proposition 4.7 of [38] since the largest invariant sets of $\{z \in \mathcal{Q}_z : V_z'(z)(f_0(z) + g_0(z)\alpha(z)) = 0\}$ and $\{\eta \in \mathbb{R}^m : V_\eta'(\eta)(\tilde{g}_{\mathrm{a}}(\eta, z)\beta(\eta)) = 0\}$ are given by the sets of equilibrium points of (5.20) and (5.27), respectively, which are Lyapunov stable by assumption.

Finally, we show that the state feedback control law $u = \phi(x)$ given by (5.36) can be approximated with $u = \beta(\eta)$ by showing that (5.35) holds. Noting that $y = \alpha(z) + \eta$ and using (5.32), the performance measure (5.5) with $u = \beta(\eta)$ is equivalent to

$$J_\varepsilon(x_0, \beta(\cdot)) = \int_0^\infty \left[2\|\eta(t)\|^2 + 2\eta^{\mathrm{T}}(t)(\alpha(z(t)) - y_{\mathrm{e}}) + \|\alpha(z(t)) - y_{\mathrm{e}}\|^2\right]\mathrm{d}t. \qquad (5.39)$$

Since (5.30) and (5.31) is semistable, there exists at least one Lyapunov stable equilibrium point $[0^{\mathrm{T}}, z_{\mathrm{e}}^{\mathrm{T}}]^{\mathrm{T}}$ such that $\lim_{t\to\infty}[\eta^{\mathrm{T}}(t), z^{\mathrm{T}}(t)] = [0^{\mathrm{T}}, z_{\mathrm{e}}^{\mathrm{T}}]^{\mathrm{T}}$. Hence, for every $\delta > 0$, there exists $t_\delta > 0$ such that $\|[\eta^{\mathrm{T}}(t), z^{\mathrm{T}}(t) - z_{\mathrm{e}}^{\mathrm{T}}]^{\mathrm{T}}\| < \delta$, $t > t_\delta$. Integrating (5.38) over the time interval (t_δ, ∞) yields

$$V_z(z(t_\delta)) - V_z(z_{\mathrm{e}}) + \frac{\varepsilon}{2}\|\eta(t_\delta)\|^2 - \frac{\varepsilon}{2}\|\eta_{\mathrm{e}}\|^2$$
$$\geq \int_{t_\delta}^\infty \left[\|\alpha(z(t)) - y_{\mathrm{e}}\|^2 + 2(1 + k_2)\|\eta(t)\|\|\alpha(z(t)) - y_{\mathrm{e}}\| + 2\left(\frac{\zeta}{\varepsilon} - k_1\right)\|\eta(t)\|^2\right]\mathrm{d}t, \quad (5.40)$$

which implies that (5.39) is finite. Now, using (5.31), (5.19), and (5.18) it follows that

$$\dot{V}_z(z) = V_z'(z)\left[f_0(z) + g_0(z)(\alpha(z) + \eta)\right]$$
$$= -2\eta^{\mathrm{T}}(t)[\alpha(z(t)) - y_{\mathrm{e}}] - \|\alpha(z(t)) - y_{\mathrm{e}}\|^2. \qquad (5.41)$$

Consequently,

$$J_\varepsilon(x_0, \beta(\cdot)) = V_z(z_0) - V_z(z_{\mathrm{e}}) + 2\int_0^\infty \|\eta(t)\|^2\mathrm{d}t. \qquad (5.42)$$

Premultiplying (5.30) by η^{T}, it follows that $\eta^{\mathrm{T}}\dot{\eta} = -\frac{1}{\varepsilon}\eta^{\mathrm{T}}\left(\tilde{g}_{\mathrm{a}}\tilde{g}_{\mathrm{a}}^{\mathrm{T}}\right)^{\frac{1}{2}}\eta + \eta^{\mathrm{T}}\gamma \leq -\frac{\zeta}{\varepsilon}\|\eta\|^2 + \eta^{\mathrm{T}}\gamma$. Integrating over $[0,\infty)$ yields

$$\frac{\zeta}{\varepsilon}\int_0^\infty \|\eta(t)\|^2 \mathrm{d}t \leq \frac{1}{2}\|\eta(0)\|^2 + \int_0^{t_\delta} \eta^{\mathrm{T}}(t)\gamma(t)\mathrm{d}t + \int_{t_\delta}^\infty \eta^{\mathrm{T}}(t)\gamma(t)\mathrm{d}t. \tag{5.43}$$

Thus, since (5.29) holds for all $t \in [t_\delta, \infty)$, it follows that $\int_0^\infty \|\eta\|^2 \mathrm{d}t$ is $\mathcal{O}(\varepsilon)$, which proves (5.35). ■

Theorem 5.5 shows that the optimal singular control problem can be approximated by two optimal control problems; namely, the singular control problem addressed in Proposition 5.4 for rapidly transferring the system (5.30) from its initial conditions to the manifold $y - \alpha(z) = 0$ and the minimum energy problem addressed in Proposition 5.3 for controlling the system (5.31) with $\eta = 0$. Furthermore, this theorem shows that (5.14) and (5.15) can be approximated by (5.30) and (5.31) with $u = \beta(\eta)$.

Theorem 5.5 extends the results of [109], where the optimal singular control problem for asymptotic stabilization of (5.14) and (5.15) is addressed. For asymptotic stabilization, the authors prove that $\min_{u\in\mathcal{S}(x_0)} J_\varepsilon(x_0, u(\cdot)) = V_z(z_0) + \mathcal{O}(\varepsilon)$, where $V_z(z) > 0$, $z \in \mathbb{R}^{n-m}$. Thus, if $V_z(z) \geq 0$, $z \in \mathbb{R}^{n-m}$, then the performance measure for the semistable optimal singular control problem is smaller than the performance measure for the asymptotically stable problem.

5.4. A Direct Approach to the Optimal Singular Control Problem

In this section, we provide an alternative solution to the semistable optimal singular control problem. Specifically, we apply Theorem 5.1 to the semistable singular control problem and show that this problem can be solved using Hamilton-Jacobi-Bellman-type conditions that do not involve any singularities.

Theorem 5.6. Consider the controlled nonlinear dynamical system (5.1) and (5.2) with $u(\cdot) \in \mathcal{S}(x_0)$ and performance measure (5.6), and assume that there exists a continuously differentiable function $V : \mathcal{D} \to \mathbb{R}$ such that

$$(y - y_{\mathrm{e}})^{\mathrm{T}} (y - y_{\mathrm{e}}) - V'(x)G(x)G^{\mathrm{T}}(x)V'^{\mathrm{T}}(x) = 0, \qquad x \in \mathcal{D}. \tag{5.44}$$

If, with the feedback control

$$u = \phi_\varepsilon(x) = -\frac{1}{2\varepsilon}G^{\mathrm{T}}(x)V'^{\mathrm{T}}(x) + u_{\mathrm{e}}, \tag{5.45}$$

every equilibrium point $x_{\mathrm{e}\varepsilon} \in F_\varepsilon^{-1}(0) \triangleq \{x \in \mathcal{D} : f(x) + G(x)\phi_\varepsilon(x) = 0\}$ of the closed-loop system

$$\dot{x}(t) = f(x(t)) + G(x(t))\phi_\varepsilon(x(t)), \qquad x(0) = x_0, \qquad t \geq 0, \tag{5.46}$$

is Lyapunov stable, then the solution $x(t) = x_{\mathrm{e}0}$, $t \geq 0$, of

$$\dot{x}(t) = f(x(t)) + G(x(t))\phi_0(x(t)), \qquad x(0) = x_0, \qquad t \geq 0, \tag{5.47}$$

where $\phi_0(x) \triangleq \lim_{\varepsilon\to 0} \phi_\varepsilon(x)$, $x \in \mathcal{D}$, is semistable and

$$J_0(x_0, \phi_0(x(\cdot))) = 0. \tag{5.48}$$

Furthermore, the feedback control $\phi_0(\cdot)$ minimizes $J_0(x_0, u(\cdot))$ in the sense that

$$J_0(x_0, \phi_0(x(\cdot))) = \min_{u(\cdot)\in\mathcal{S}(x_0)} J_0(x_0, u(\cdot)). \tag{5.49}$$

Proof: The result follows as a consequence of Theorem 5.1 with $J_\varepsilon(x_0, u(\cdot)) \to J_0(x_0, u(\cdot))$ as $\varepsilon \to 0$, $V(x)$ replaced by $\hat{V}(x) \triangleq 2\varepsilon V(x)$, $x \in \mathcal{D}$, and $\phi(x)$ replaced by $\phi_\varepsilon(x)$. Specifically, note that

$$\begin{aligned}
0 &= (y - y_{\mathrm{e}})^{\mathrm{T}} (y - y_{\mathrm{e}}) - V'(x)G(x)G^{\mathrm{T}}(x)V'^{\mathrm{T}}(x) \\
&= (y - y_{\mathrm{e}})^{\mathrm{T}} (y - y_{\mathrm{e}}) + \lim_{\varepsilon\to 0} 2\varepsilon V'(x)\left(f(x) + G(x)u_{\mathrm{e}}\right) - \lim_{\varepsilon\to 0} \frac{4\varepsilon^2}{4\varepsilon^2} V'(x)G(x)G^{\mathrm{T}}(x)V'^{\mathrm{T}}(x) \\
&= (y - y_{\mathrm{e}})^{\mathrm{T}} (y - y_{\mathrm{e}}) + \lim_{\varepsilon\to 0} \hat{V}'(x)\left(f(x) + G(x)u_{\mathrm{e}}\right) - \lim_{\varepsilon\to 0} \frac{1}{4\varepsilon^2} \hat{V}'(x)G(x)G^{\mathrm{T}}(x)\hat{V}'^{\mathrm{T}}(x), \\
&\qquad (x, u_{\mathrm{e}}) \in \mathcal{D} \times U,
\end{aligned} \tag{5.50}$$

which, with $\varepsilon \to 0$ and $V(\cdot)$ replaced by $\hat{V}(\cdot)$, satisfies (5.9). Furthermore,

$$\lim_{\varepsilon \to 0} 2\varepsilon V(x_{\mathrm{e}}) = \lim_{\varepsilon \to 0} \hat{V}(x_{\mathrm{e}}) = 0, \qquad x_{\mathrm{e}} \in \mathcal{D}, \tag{5.51}$$

which, with $\varepsilon \to 0$ and $V(\cdot)$ replaced by $\hat{V}(\cdot)$, satisfies (5.8). Since all of the conditions of Theorem 5.1 are satisfied, the solution $x(t) = x_{\mathrm{e}\varepsilon}$, $t \geq 0$, of the closed-loop system (5.46) is semistable for all $\varepsilon > 0$, and hence, the solution $x(t) = x_{\mathrm{e}0}$, $t \geq 0$, of the closed-loop system (5.47) is semistable. Finally,

$$\lim_{\varepsilon \to 0} J_\varepsilon(x_0, \phi_\varepsilon(x(\cdot))) = J_0(x_0, \phi_0(x(\cdot))) = \lim_{\varepsilon \to 0} 2\varepsilon(V(x_0) - V(x_{\mathrm{e}})) = 0, \tag{5.52}$$

which, since $J_0(x_0, u(\cdot)) \geq 0$ for all admissible $u(\cdot)$ and $x_0 \in \mathcal{D}$, proves (5.49). ■

Remark 5.7. Since the cost-to-go function $V(\cdot)$ is not required to be sign definite, Theorem 5.6 provides a solution of the nonlinear semistable optimal singular control problem. For nonlinear asymptotic singular stabilization, we require $V(0) = 0$ and $V(x) > 0$, $x \in \mathcal{D} \backslash \{0\}$ (see [109]), and hence, the approach used in Theorem 5.6 cannot be applied to address the nonlinear optimal singular control problem for asymptotic stabilization. In addition, Theorem 5.6 shows that the minimum value of the singular performance measure is zero, whereas applying Theorem 5.5, which invokes stronger assumptions than those of Theorem 5.6, the minimum value of the singular performance measure is not necessarily zero.

5.5. A Feedback Linearization Approach to the Optimal Singular Control Problem

In this section, we provide an alternative approach to the optimal singular control problem for semistabilization based on the notions of output-feedback linearization and feedback equivalence.

5.5.1. Feedback Linearization of Nonlinear Dynamical Systems

The following definitions are needed for the main results of this section.

Definition 5.8 [38]. The *Lie derivative* of the continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ along the vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is defined as

$$L_f V(x) \triangleq V'(x) f(x). \tag{5.53}$$

The *zeroth-order* and the *higher-order Lie derivatives* are, respectively, defined as

$$L_f^0 V(x) \triangleq V(x), \qquad L_f^k V(x) \triangleq L_f(L_f^{k-1} V(x)), \qquad k \geq 1. \tag{5.54}$$

For the statement of the next result, consider the nonlinear dynamical system given by (5.1) with measured output

$$\hat{y}(t) = \hat{h}(x(t)), \tag{5.55}$$

where $\hat{y}(t) \in \mathbb{R}^m$, $t \geq 0$, $\hat{y}_{\rm e} = \hat{h}(x_{\rm e})$, and $\hat{h} : \mathcal{D} \to \mathbb{R}^m$ is smooth (i.e., infinitely differentiable) on $\mathcal{D}$.

Definition 5.9 [38]. Consider the nonlinear dynamical system $\mathcal{G}$ given by (5.1) and (5.55), and let $\bar{x} \in \mathcal{D}_0$, where $\mathcal{D}_0 \subseteq \mathcal{D}$ is a neighborhood of $\bar{x}$. If, for all $x \in \mathcal{D}_0$,

$$L_{G_i} L_f^k \hat{h}_j(x) = 0, \qquad 0 \leq k < r_j - 1, \qquad 1 \leq i, j \leq m, \tag{5.56}$$

and the matrix

$$\mathcal{L}(x) \triangleq \begin{bmatrix} L_{G_1} L_f^{r_1 - 1} \hat{h}_1(x) & \ldots & L_{G_m} L_f^{r_1 - 1} \hat{h}_1(x) \\ \vdots & \ddots & \vdots \\ L_{G_1} L_f^{r_m - 1} \hat{h}_m(x) & \ldots & L_{G_m} L_f^{r_m - 1} \hat{h}_m(x) \end{bmatrix} \tag{5.57}$$

is nonsingular, then $\mathcal{G}$ has *vector relative degree* $\{r_1, r_2, \ldots, r_m\}$ at $\bar{x}$. Furthermore, if the system $\mathcal{G}$ has vector relative degree $\{r_1, r_2, \ldots, r_m\}$ at every $x \in \mathcal{D}$, then $\mathcal{G}$ has *uniform vector relative degree* $\{r_1, r_2, \ldots, r_m\}$ on $\mathcal{D}$.

The scalars r_i denote the number of times that the outputs $\hat{y}_i$ need to be differentiated at $\bar{x}$ until the input u appears explicitly in (5.55) [64, p. 221]. Note that if $m = 1$, $L_G L_f^k \hat{h}(x) = 0$, $k < r - 1$, $x \in \mathcal{D}_0$, and $L_G L_f^{r-1} \hat{h}(\bar{x}) \neq 0$, then $G(x)$ is a column vector and the system given by (5.1) and (5.55) has relative degree r at $\bar{x}$.

Theorem 5.10 [64, Prop. 5.1.2]. Assume that the nonlinear dynamical system $\mathcal{G}$ given by (5.1) and (5.55) has vector relative degree $\{r_1, r_2, \ldots, r_m\}$ at $\bar{x} \in \mathcal{D}$. Then, there exist a neighborhood $\mathcal{N} \subset \mathcal{D}$ of $\bar{x}$, a diffeomorphism $\mathcal{T} : \mathcal{N} \mapsto \mathbb{R}^n$, and functions $q : \mathbb{R}^r \times \mathbb{R}^{n-r} \to \mathbb{R}^{n-r}$ and $p : \mathbb{R}^r \times \mathbb{R}^{n-r} \to \mathbb{R}^{(n-r)\times m}$ such that, in the coordinates

$$z \triangleq \mathcal{T}(x), \qquad x \in \mathcal{N}, \tag{5.58}$$

$\mathcal{G}$ is equivalent to

$$\dot{z}^j_{k_j}(t) = z^j_{k_j+1}(t), \qquad z^j_{k_j}(0) = L_f^{k_j-1}\hat{h}_j(x_0), \qquad t \geq 0, \tag{5.59}$$

$$\dot{z}^j_{r_j}(t) = L_f^{r_j}\hat{h}_j(x(t)) + \sum_{l=1}^{m} L_{G_l}L_f^{r_j-1}\hat{h}_j(x(t))u_l(t), \qquad z^j_{r_j}(0) = L_f^{r_j-1}\hat{h}_j(x_0), \tag{5.60}$$

$$\dot{\eta}(t) = q(\xi(t), \eta(t)) + p(\xi(t), \eta(t))u(t), \qquad \eta(0) = \eta_0, \tag{5.61}$$

for all $j = 1, \ldots, m$ and $k_j = 1, \ldots, r_j - 1$, where

$$z \triangleq [z^1_1, \ldots, z^1_{r_1}, \ldots, z^m_1, \ldots, z^m_{r_m}, z_{r+1}, \ldots, z_n]^{\mathrm{T}},$$

$r \triangleq \sum_j^m r_j \leq n$, $\xi \triangleq [z^1_1, \ldots, z^1_{r_1}, \ldots, z^m_1, \ldots, z^m_{r_m}]^{\mathrm{T}}$, $\eta \triangleq [z_{r+1}, \ldots, z_n]^{\mathrm{T}}$, and $\eta_0 \in \mathbb{R}^{n-r}$ is arbitrary.

Theorem 5.10 does not specify any conditions on $q(\cdot,\cdot)$ and $p(\cdot,\cdot)$ other than the existence of the diffeomorphism $\mathcal{T}$ on $\mathcal{N}$. If $r \triangleq \sum_{i=1}^m r_i = n$, then $z = \xi$ and the condition (5.61) is superfluous. In this dissertation, we say that the nonlinear dynamical system (5.1) and (5.55) is *equivalent* to the dynamical system (5.59)–(5.61) if and only if the hypothesis of Theorem 5.10 hold.

The following result gives sufficient conditions for constructing a feedback controller $u = \mu(x, v)$ such that the nonlinear dynamical system (5.1) and (5.55) is locally output-feedback linear, that is, (5.1) and (5.55) is equivalent to the linear dynamical system given by

$$\dot{z}(t) = Az(t) + Bv(t), \qquad z(0) = z_0, \qquad t \geq 0, \tag{5.62}$$

$$y(t) = Cz(t). \tag{5.63}$$

Theorem 5.11. Consider the nonlinear dynamical system $\mathcal{G}$ given by (5.1) and (5.55). Assume that $\mathcal{G}$ has uniform vector relative degree $\{r_1, r_2, \ldots, r_m\}$ on $f^{-1}(0) \triangleq \{x \in \mathcal{D} : f(x) = 0\}$ and $r = \sum_{i=1}^{m} r_i = n$. Then, there exists a neighborhood $\mathcal{N} \subset \mathcal{D}$ of the set $f^{-1}(0) = \{x \in \mathcal{D} : f(x) = 0\}$ such that, for all $x \in \mathcal{N}$, the nonlinear dynamical system $\mathcal{G}$ with

$$u = \mu(x, v) = \mathcal{L}^{-1}(x)\left(-b(z) + \psi(z) + v\right) \tag{5.64}$$

is equivalent to (5.62), where z is defined as in Theorem 5.10, $z_0 = [\hat{h}_1(x_0), \ldots, L_f^{r_m-1}\hat{h}_m(x_0)]^{\mathrm{T}}$, $x_0 \in \mathcal{N}$, $\mathcal{L}(x)$ is given by (5.57),

$$b(z) \triangleq [L_f^{r_1}\hat{h}_1(x), \ldots, L_f^{r_m}\hat{h}_m(x)]^{\mathrm{T}}, \tag{5.65}$$

$\psi : \mathbb{R}^n \to \mathbb{R}^m$ is such that $\psi = [\psi_1(z), \ldots, \psi_m(z)]^{\mathrm{T}}$ with

$$\psi_j(z) = \sum_{i=1}^{r_j} k_{i,j} z_i^j, \tag{5.66}$$

$k_{i,j} \in \mathbb{R}$, $i = 1, \ldots, r_j$, $j = 1, \ldots, m$, $A \in \mathbb{R}^{n\times n}$ is a block-diagonal matrix with the jth block given by

$$A^j = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ k_{1,j} & k_{2,j} & \ldots & \ldots & k_{r_j,j} \end{bmatrix}, \tag{5.67}$$

the entry of B in the $\left(\sum_{i=1}^{j} r_i\right)$th row and jth column is equal to one, and the remaining entries of B are equal to zero. Furthermore, the pair (A, B) is controllable and there exists a matrix $C \in \mathbb{R}^{m\times n}$ such that the pair (A, C) is semiobservable and the transfer function $G(s) \triangleq C(sI_n - A)B$, $s \in \mathbb{C}$, of the linear dynamical system (5.62) and (5.63) is minimum phase and right invertible.

Proof: It follows from Theorem 5.10 and the fact that $r = \sum_{i=1}^{m} r_i = n$ that there exists a neighborhood $\mathcal{N} \subset \mathcal{D}$ of $f^{-1}(0)$ and a diffeomorphism $\mathcal{T} : \mathcal{N} \to \mathbb{R}^n$ such that (5.58)–(5.61) hold with $q(z, \eta) = 0$ and $p(z, \eta) = 0$ for all $(z, \eta) \in \mathbb{R}^n \times \mathbb{R}^{n-r}$. Now, since

$\sum_{q=1}^{m} L_{G_q} L_f^{r_j-1}\hat{h}_j(x)u_q = (\mathcal{L}(x)u)_j$, where $(\mathcal{L}(x)u)_j$ denotes the jth component of the vector $\mathcal{L}(x)u$, it follows from (5.60) that

$$\dot{z}_r(t) = b(z(t)) + \mathcal{L}(x(t))u(t), \quad z_r(0) = [L_f^{r_1-1}\hat{h}_1(x_0), \ldots, L_f^{r_m-1}\hat{h}_m(x_0)]^{\mathrm{T}}, \quad t \geq 0, \quad (5.68)$$

where $z_r \triangleq [z_{r_1}^1, z_{r_2}^2, \ldots, z_{r_m}^m]^{\mathrm{T}}$. Furthermore, since $\mathcal{G}$ has uniform vector relative degree $\{r_1, r_2, \ldots, r_m\}$ on $f^{-1}(0)$, it follows from Definition 5.9 that $\mathcal{L}(x)$ is invertible for all $x \in f^{-1}(0)$, and hence, by continuity [64, p. 226], $\mathcal{L}(x)$ is invertible for all $x \in \mathcal{N}$.

Thus, (5.64) is well defined and $\mathcal{G}$ with u given by (5.64) is equivalent to the system given by (5.62), where A is a block-diagonal matrix with the jth block given by (5.67), $j = 1, \ldots, m$, the entries of B in the $\left(\sum_{i=1}^{j} r_i\right)$th row and jth column are equal to one, and the remaining entries of B are equal to zero. Since A^j is in canonical controllable form and A is block-diagonal, there exists constants $k_{i,j} \in \mathbb{R}$, $i = 1, \ldots, r_j$ and $j = 1, \ldots, m$, and a matrix $C \in \mathbb{R}^{m\times n}$ such that the pair (A, B) is controllable, the pair (A, C) is semiobservable, and the transfer function $G(s) \triangleq C(sI_n - A)B$, $s \in \mathbb{C}$, of (5.62) and (5.63) is minimum phase and right invertible. ■

5.5.2. Singular Control for Linear Semistabilization

In Subsection 5.5.1, we give sufficient conditions for the existence of a feedback control $u = \mu(x, v)$ such that the nonlinear dynamical system (5.1) and (5.55) is feedback equivalent to (5.62). In this subsection, we solve the optimal singular control problem for semistabilization of the linear dynamical system (5.62) with output (5.63), that is, we find $K \in \mathbb{R}^{m\times n}$ such that, with $v = Kz$, (5.62) is semistable and the performance measure

$$J_0(z_0, v(\cdot)) = \lim_{\varepsilon\to 0} \int_0^\infty [(y(t) - y_{\mathrm{e}})^{\mathrm{T}}(y(t) - y_{\mathrm{e}}) + \varepsilon^2 (v(t) - v_{\mathrm{e}})^{\mathrm{T}}(v(t) - v_{\mathrm{e}})]\mathrm{d}t \quad (5.69)$$

is minimized in the sense that

$$J_0(z_0, K) = \min_{v(\cdot)\in\mathcal{S}(z_0)} J_0(z_0, v(\cdot)), \quad (5.70)$$

where $0 = Az_{\rm e} + Bv_{\rm e}$ for some $v_{\rm e} \in \mathbb{R}^m$, $y_{\rm e} = Cz_{\rm e}$, and

$$\mathcal{S}(z_0) \triangleq \{v(\cdot) : v(\cdot) \text{ is admissible and } z(\cdot) \text{ given by (5.62) satisfies } z(t) \to z_{\rm e} \text{ as } t \to \infty\}.$$

Theorem 5.12. Consider the linear dynamical system $\mathcal{G}$ given by (5.62) and (5.63) with $v(\cdot) \in \mathcal{S}(z_0)$ and performance measure (5.69). If the pair (A, B) is semicontrollable, the pair (A, C) is semiobservable, and $\mathcal{G}$ has transfer function $G(s) = C(sI_n - A)^{-1}B$ that is minimum phase and right invertible, then, with

$$v = Kz = -\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} B^{\rm T} \begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix} z, \tag{5.71}$$

the solution $z(t) = z_{\rm e}$, $t \geq 0$, to (5.62) is semistable,

$$J_0(z_0, K) = 0 \tag{5.72}$$

and (5.70) is satisfied.

Proof: The result is a direct consequence of Theorems 4.14 and 4.15. ■

Next, we use Theorem 5.12 to solve the optimal singular control problem for affine in the control nonlinear dynamical systems using feedback linearization.

Theorem 5.13. Consider the nonlinear dynamical system (5.1) with $u(\cdot) \in \mathcal{S}(x_0)$, measured output (5.55), performance output (5.63), and performance measure (5.6). If the hypothesis of Theorem 5.11 hold, then, with

$$\phi(x) = \mathcal{L}^{-1}(x) \left(-b(z) + \psi(z) - \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} B^{\rm T} \begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix} z \right), \tag{5.73}$$

where $\mathcal{L}(x)$ is given by (5.57), $b(z)$ is given by (5.65), $\psi(z) = [\psi_1(z), \ldots, \psi_m(z)]^{\rm T}$, and $\psi_j(z)$, $j = 1, \ldots, m$, is given by (5.66), the solution $x(t) = x_{\rm e}$, $t \geq 0$, of the closed-loop system (5.3) and (5.4) is semistable and

$$\min_{u(\cdot) \in \mathcal{S}(x_0)} J_0(x_0, u(\cdot)) = J_0(x_0, \phi(x(\cdot))) = 0 \tag{5.74}$$

for all $x_0 \in \mathcal{N}$, where $\mathcal{N}$ is a neighborhood of the set $f^{-1}(0) = \{x \in \mathcal{D} : f(x) = 0\}$. Furthermore, the feedback control $\phi(\cdot)$ minimizes $J_0(x_0, u(\cdot))$ in the sense that

$$J_0(x_0, \phi(x(\cdot))) = \min_{u(\cdot)\in\mathcal{S}(x_0)} J_0(x_0, u(\cdot)). \tag{5.75}$$

Proof: It follows from Theorem 5.11 that, for all $x \in \mathcal{N}$, the nonlinear dynamical system given by (5.1), (5.55), and (5.63), with $u = \mu(x, v)$ given by (5.64), is equivalent to the linear dynamical system (5.62) and (5.63). In this case, the pair (A, B) is controllable, the pair (A, C) is semiobservable, and the transfer function $G(s) = C(sI_n - A)^{-1}B$, $s \in \mathbb{C}$, is minimum phase and right invertible. Hence, the pair (A, B) is semicontrollable (since (A, B) is controllable) and it follows from Theorem 5.12 that the solution $z(t) = z_{\rm e}$, $t \geq 0$, of (5.62), with $v = Kz$ given in (5.71), is semistable. Since (5.73) is given by (5.64) with v given by (5.71), the solution $x(t) = x_{\rm e}$, $t \geq 0$, of the closed-loop system (5.3) and (5.4) with $\phi(x)$ given by (5.73) is semistable with respect to $\mathcal{N}$.

Next, we show that $\mathcal{S}(x_0) \subseteq \mathcal{S}(z_0)$ for all $x_0 \in \mathcal{N}$. To see this, note that if $u(\cdot) \in \mathcal{S}(x_0)$, then $x(t)$ given by (5.1) is bounded for all $t \geq 0$ and $x(t) \to x_{\rm e}$ as $t \to \infty$. Since the hypothesis of Theorem 5.11 are satisfied, it follows from Theorem 5.10 that there exist a neighborhood $\mathcal{N} \subset \mathcal{D}$ of $f^{-1}(0)$ and a diffeomorphism $\mathcal{T} : \mathcal{N} \mapsto \mathbb{R}^n$ such that $z = \mathcal{T}(x)$, $x \in \mathcal{N}$. Thus, since $x_0 \in \mathcal{N}$ and $x(t)$ given by (5.1) is bounded for all $t \geq 0$, $z(t)$ is bounded for all $t \geq 0$. In addition, since $x(t) \to x_{\rm e}$ as $t \to \infty$, $\lim_{t\to\infty} \mathcal{T}(x(t)) = \lim_{t\to\infty} z(t) = z_{\rm e}$. Hence, if $u(\cdot) \in \mathcal{S}(x_0)$, then $u(\cdot) \in \mathcal{S}(z_0)$ for all $x_0 \in \mathcal{N}$.

Next, since the nonlinear dynamical system (5.1) with input $u = \mu(x, v)$ given by (5.64), measured output (5.55), and performance output (5.63) is equivalent to the linear dynamical system (5.62) with output (5.63), it follows that

$$J_0(x_0, u(\cdot)) = J_0(z_0, v(\cdot)) \tag{5.76}$$

for all $x_0 \in \mathcal{N}$. Now, it follows from Theorem 5.12 that

$$\min_{v(\cdot)\in\mathcal{S}(z_0)} J_0(z_0, v(\cdot)) = J_0(z_0, K) = 0 \tag{5.77}$$

and, since $\mathcal{S}(x_0) \subseteq \mathcal{S}(z_0)$, (5.74) follows. Finally, since $J_0(z_0, v(\cdot)) \geq 0$ for all admissible $v(\cdot)$ and $z_0 \in \mathbb{R}^n$, (5.75) is immediate. ■

Remark 5.14. Theorem 5.13 provides a semistabilizing state feedback controller $\phi(x)$, $x \in \mathcal{D}$, as an *explicit* function of $\mathcal{L}(x)$ and $b(z)$, $z \in \mathbb{R}^n$, given by (5.57) and (5.65), respectively, and $\psi(z)$ given by (5.66), which involves the solution of (5.59) and (5.60). Alternatively, Theorem 5.6 provides a semistabilizing controller $\phi(\cdot)$ as function of the cost-to-go $V(\cdot)$, which involves the solution of the partial differential equation (5.44).

Note that if the conditions of Theorem 5.13 are satisfied, then the nonlinear dynamical system (5.1) and (5.55) is feedback equivalent to the linear dyamical system (5.62), and the singular quadratic performance measure (5.6) is equivalent to the singular quadratic performance measure (5.69). This equivalence is particularly advantageous since it allows us to apply known results on optimal state feedback semistabilization of linear dynamical systems with quadratic performance measures discussed in Chapter 2 to address the optimal singular nonlinear semistabilization problem.

5.6. Illustrative Numerical Examples

In this section, we provide three numerical examples to highlight the optimal singular semistabilization frameworks developed in this chapter.

5.6.1. Singular Semistabilization of a Nonlinear Dynamical System

This example highlights the nonlinear singular semistabilization framework developed in Theorem 5.5. Specifically, we seek a state-feedback controller that guarantees semistability of

$$\dot{y}(t) = -6[z(t) - z_{\mathrm{e}}]^2 + u(t), \quad y(0) = y_0, \quad t \geq 0, \tag{5.78}$$

$$\dot{z}(t) = [z(t) - z_{\mathrm{e}}]^3 + y(t), \quad z(0) = z_0, \tag{5.79}$$

where $y \in \mathbb{R}$, $z \in \mathbb{R}$, $u \in \mathbb{R}$, $t \geq 0$, and $z_{\mathrm{e}} \in \mathbb{R}$, and minimizes the performance measure (5.6) with $y_{\mathrm{e}} = 0$.

Since the nonlinear dynamical system (5.78) and (5.79) has the same form as (5.14) and (5.15) with $f_{\mathrm{a}}(y,z) = -6(z-z_{\mathrm{e}})^2$, $g_{\mathrm{a}}(y,z) = 1$, $f_0(z) = (z-z_{\mathrm{e}})^3$, and $g_0(z) = 1$, (5.17) and (5.18) specialize to

$$V_z'(z_{\mathrm{e}}) = 0, \qquad z_{\mathrm{e}} \in \mathbb{R}, \tag{5.80}$$

$$V_z'(z)(z-z_{\mathrm{e}})^3 - \frac{1}{4}V_z'^2(z) = 0, \qquad z \in \mathbb{R}, \tag{5.81}$$

which are satisfied with $V_z(z) = (z-z_{\mathrm{e}})^4$. In this case, the feedback control (5.19) specializes to

$$\alpha(z) = -2(z-z_{\mathrm{e}})^3 \tag{5.82}$$

and every equilibrium point $z_{\mathrm{e}} \in \mathbb{R}$ of the closed-loop system

$$\dot{z}(t) = -[z(t)-z_{\mathrm{e}}]^3, \qquad z(0) = z_0, \qquad t \geq 0, \tag{5.83}$$

is Lyapunov stable. Hence, all of the assumptions of Proposition 5.3 are satisfied. In fact, the solution $z(t) = z_{\mathrm{e}}$, $t \geq 0$, of (5.83) is asymptotically stable, which implies that $z(t) = z_{\mathrm{e}}$, $t \geq 0$, is trivially semistable.

Next, the nonlinear dynamical system (5.22) specializes to

$$\dot{\eta}(t) = u(t), \qquad \eta(0) = \eta_0, \qquad t \geq 0, \tag{5.84}$$

and (5.24) and (5.25) specialize to

$$V_\eta'(\eta_{\mathrm{e}}) = 0, \qquad \eta_{\mathrm{e}} \in \mathbb{R}, \tag{5.85}$$

$$\eta^2 - \frac{1}{4}V_\eta'^2(\eta) = 0, \qquad \eta \in \mathbb{R}, \tag{5.86}$$

which are satisfied with $V_\eta(\eta) = 2\eta^2$ for $u_{\mathrm{e}} = 0$ and $\eta_{\mathrm{e}} = 0$. In this case, the feedback control (5.26) specializes to

$$\beta(\eta) = -\frac{1}{\varepsilon}\eta \tag{5.87}$$

and the equilibrium point $\eta_{\rm e} = 0$ of the closed-loop system

$$\dot{\eta}(t) = -\frac{1}{\varepsilon}\eta(t) \qquad \eta(0) = \eta_0, \qquad t \geq 0, \tag{5.88}$$

is Lyapunov stable. Therefore, all of the assumptions of Proposition 5.4 are satisfied. In fact, the solution $\eta(t) = 0$, $t \geq 0$, of (5.88) is asymptotically stable, which implies that $\eta(t) = 0$, $t \geq 0$, is trivially semistable.

Next, note that that $\gamma(\eta, z) = 0$, and hence, (5.29) holds for all $k_1 > 0$, $k_2 > 0$, $\eta \in \mathbb{R}$, and $z \in \mathbb{R}$. Since all of the conditions of Theorem 5.5 are satisfied, with the feedback control $u(\cdot) = \beta(\eta(\cdot))$, where $\beta(\cdot)$ is given by (5.87), the solution $[\eta(t),\, z(t)]^{\rm T} = [0,\, z_{\rm e}]^{\rm T}$, $t \geq 0$, of the closed-loop system

$$\varepsilon\dot{\eta}(t) = -\eta(t), \qquad \eta(0) = \eta_0, \qquad t \geq 0, \tag{5.89}$$

$$\dot{z}(t) = -[z(t) - z_{\rm e}]^3 + \eta(t), \qquad z(0) = z_0, \tag{5.90}$$

is semistable. Furthermore,

$$\min_{u(\cdot)\in\mathcal{S}(x_0)} J_0(x_0, u(\cdot)) = J_0(x_0, \beta(\cdot)) = (z_0 - z_{\rm e})^4 + \mathcal{O}(\varepsilon). \tag{5.91}$$

Figure 5.1 shows the state trajectories of the controlled system versus time for $\eta_0 = 2$ and $z_0 = -4$. Note that $[\eta(t),\, z(t)]^{\rm T} \to [0,\, 0.8709]^{\rm T}$ as $t \to \infty$.

5.6.2. Spacecraft Spin Stabilization Via Singular Semistabilization

This example highlights the nonlinear optimal singular semistabilization framework developed in Theorem 5.6. Consider the rigid spacecraft given by [38]

$$\dot{\omega}_1(t) = I_{23}\omega_2(t)\omega_3(t) + u_1(t), \qquad \omega_1(0) = \omega_{10}, \qquad t \geq 0, \tag{5.92}$$

$$\dot{\omega}_2(t) = I_{31}\omega_3(t)\omega_1(t) + u_2(t), \qquad \omega_2(0) = \omega_{20}, \tag{5.93}$$

$$\dot{\omega}_3(t) = I_{12}\omega_1(t)\omega_2(t), \qquad \omega_3(0) = \omega_{30}, \tag{5.94}$$

$$y(t) = \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} x(t), \tag{5.95}$$

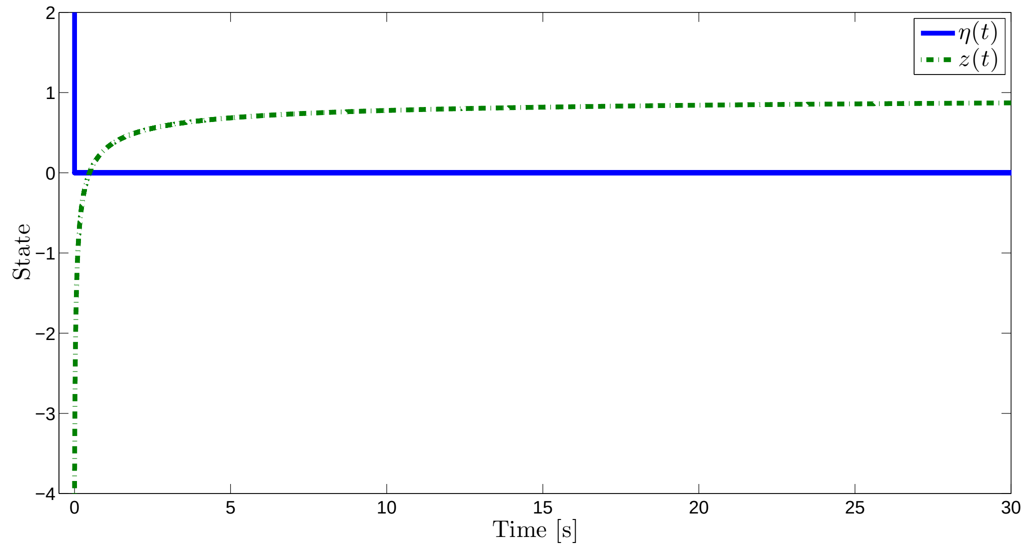

Figure 5.1: Closed-loop system trajectories versus time.

where $I_{23} \triangleq (I_2 - I_3)/I_1$, $I_{31} \triangleq (I_3 - I_1)/I_2$, $I_{12} \triangleq (I_1 - I_2)/I_3$, I_1, I_2, and I_3 are the principal moments of inertia of the spacecraft such that $I_1 \geq I_2 \geq I_3 > 0$, $J \triangleq \begin{bmatrix} -I_{31} & 0 \\ 0 & I_{23} \end{bmatrix}$, $x = [\omega_1, \omega_2, \omega_3]^{\mathrm{T}}$ is the angular velocity vector with respect to a given inertial reference frame expressed in a central body reference frame, and u_1 and u_2 are the spacecraft control moments. For this example, we seek a state feedback controller $u = [u_1, u_2]^{\mathrm{T}} = \phi(x)$ such that the performance measure (5.6), with $u_{\mathrm{e}} \triangleq [u_{1\mathrm{e}}, u_{2\mathrm{e}}]^{\mathrm{T}}$, is minimized in the sense of (5.7), and (5.92)–(5.94) is semistable.

Note that (5.92)–(5.95) can be cast in the form of (5.1) and (5.2). In this case, Theorem 5.6 can be applied with $n = 3$, $m = 2$, $f(x) = \left[I_{23}\omega_2\omega_3,\ I_{31}\omega_3\omega_1,\ I_{12}\omega_1\omega_2\right]^{\mathrm{T}}$, and $G(x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}^{\mathrm{T}}$ to characterize the singular semistabilizing controller. Specifically, (5.44) implies that

$$0 = (x - x_{\mathrm{e}})^{\mathrm{T}} \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} (x - x_{\mathrm{e}}) - V'(x)G(x)G^{\mathrm{T}}(x)V'^{\mathrm{T}}(x) = 0, \qquad x \in \mathcal{D}, \quad (5.96)$$

which is satisfied with

$$V(x) = \frac{1}{2}(x - x_{\mathrm{e}})^{\mathrm{T}} \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} (x - x_{\mathrm{e}}). \quad (5.97)$$

Hence, it follows from (5.45) that

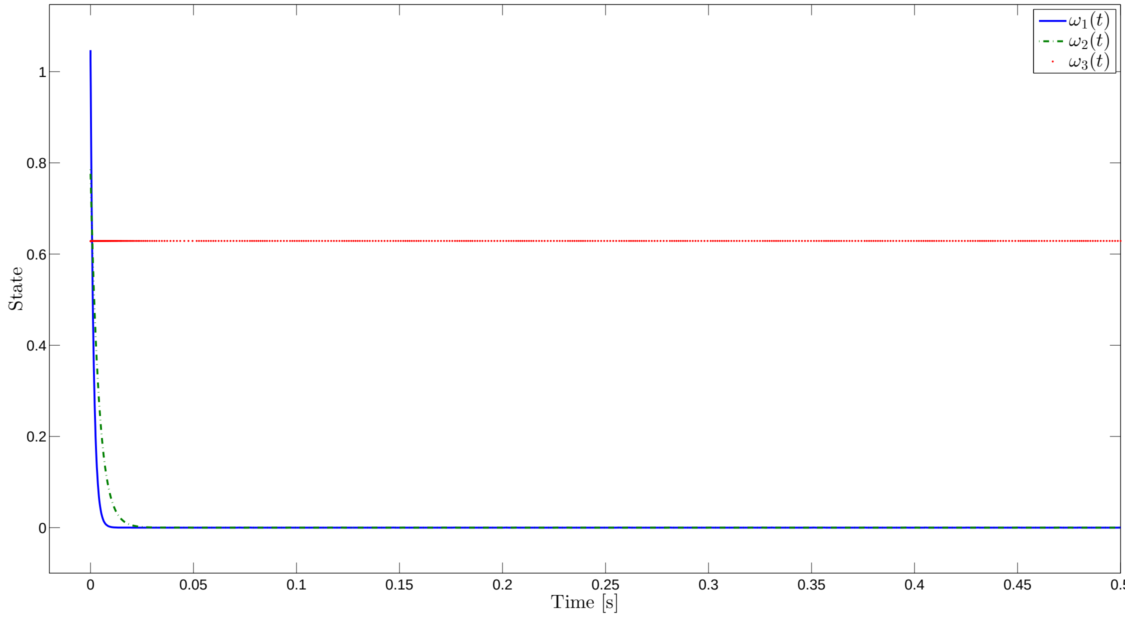

Figure 5.2: Closed-loop system trajectories versus time.

$$\phi_\varepsilon(x) = -\frac{1}{2\varepsilon}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}J & 0\\ 0 & 0\end{bmatrix}(x - x_\mathrm{e}) + u_\mathrm{e}, \qquad x \in \mathbb{R}^3, \tag{5.98}$$

The set of equilibrium points of the closed-loop system (5.92)–(5.94) with $u = \phi_\varepsilon(x)$ is given by

$$F_\varepsilon^{-1}(0) = \left\{x_\mathrm{e} \in \mathcal{D} : x_\mathrm{e} = [0, 0, x_{3\mathrm{e}}]^\mathrm{T}, x_{3\mathrm{e}} \in \mathbb{R}\right\} \tag{5.99}$$

and Lyapunov stability of $x_\mathrm{e} = [0, 0, x_{3\mathrm{e}}]^\mathrm{T} \in F_\varepsilon^{-1}(0)$ for every $x_{3\mathrm{e}} \in \mathbb{R}$ follows from Theorem 1 of [63] by noting that $V(x_\mathrm{e}) = 0$, $V(x) \geq 0$, $x \in \mathcal{D}$,

$$V'(x)\left(f(x) + G(x)\phi_\varepsilon(x)\right) = -\frac{I_{31}^2}{2\varepsilon}x_1^2 - \frac{I_{23}^2}{2\varepsilon}x_2^2 \leq 0, \qquad x \in \mathcal{D}, \tag{5.100}$$

and $x(t) = x_\mathrm{e} \in F_\varepsilon^{-1}(0)$, $t \leq 0$, if and only if $x = x_\mathrm{e}$.

Since all of the conditions of Theorem 5.6 hold, the feedback control law $\phi_0(x) = \lim_{\varepsilon\to 0}\phi_\varepsilon(x)$ guarantees that the dynamical system (5.92)–(5.94) is semistable and, for all $x(0) \in \mathcal{D}$,

$$J_0(x(0), \phi_0(x(\cdot))) = 0. \tag{5.101}$$

Let $I_1 = 20\,\mathrm{kg}\cdot\mathrm{m}^2$, $I_2 = 15\,\mathrm{kg}\cdot\mathrm{m}^2$, $I_3 = 10\,\mathrm{kg}\cdot\mathrm{m}^2$, $\omega_{10} = \pi/3\,\mathrm{Hz}$, $\omega_{20} = \pi/4\,\mathrm{Hz}$, and $\omega_{30} = \pi/5\,\mathrm{Hz}$. Figure 5.2 shows the state trajectories of the controlled system versus time.

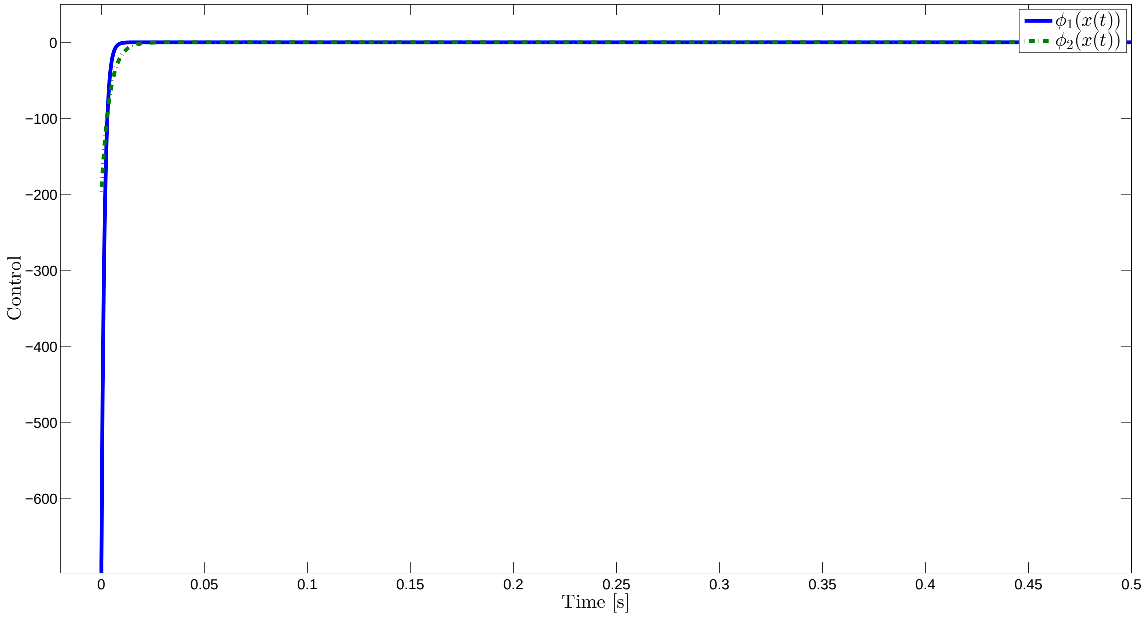

Figure 5.3: Control signal versus time.

Note that $[\omega_1(t),\, \omega_2(t)]^{\mathrm{T}} \to 0$ as $t \to \infty$, whereas $\omega_3(t) \to \frac{\pi}{5}$ Hz as $t \to \infty$. Figure 5.3 shows the control signal versus time.

5.6.3. Singular Semistabilization of a Rigid Body

This example provides a solution of the singular stabilization problem for a rigid body by applying the results of Section 5.5. Let $\theta \triangleq [\theta_x, \theta_y, \theta_z]^{\mathrm{T}} \in \mathbb{R}^3$ and $\eta \in \mathbb{R}$ denote the vector and scalar Euler parameters respectively, let ω_1, ω_2, and $\omega_3 \in \mathbb{R}$ denote the components of the angular velocity vector with respect to a given inertial reference frame expressed in a central body reference frame, and let u_1, u_2, and $u_3 \in \mathbb{R}$ denote the control torques about the body center of mass. Recall that the Euler parameters satisfy

$$\theta_x^2(t) + \theta_y^2(t) + \theta_z^2(t) + \eta^2(t) = 1, \qquad t \geq 0, \tag{5.102}$$

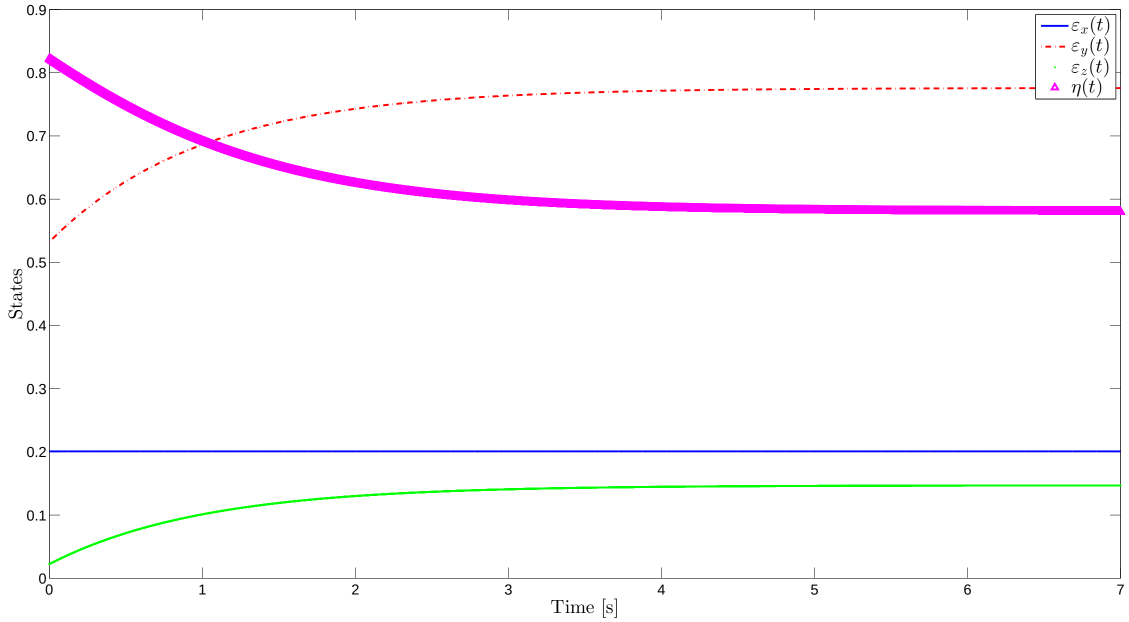

Figure 5.4: Closed-loop system trajectories versus time.

and the rotational equations of motion for the rigid body are given by [37]

$$\begin{bmatrix}\dot\theta_x(t)\\ \dot\theta_y(t)\\ \dot\theta_z(t)\\ \dot\eta(t)\\ \dot\omega_1(t)\\ \dot\omega_2(t)\\ \dot\omega_3(t)\end{bmatrix} = \frac{1}{2}\begin{bmatrix}\theta_y(t)\omega_3(t)-\theta_z(t)\omega_2(t)+\eta(t)\omega_1(t)\\ \theta_z(t)\omega_1(t)-\theta_x(t)\omega_3(t)+\eta(t)\omega_2(t)\\ \theta_x(t)\omega_2(t)-\theta_y(t)\omega_1(t)+\eta(t)\omega_3(t)\\ -\theta_x(t)\omega_1(t)-\theta_y(t)\omega_2(t)-\theta_z(t)\omega_3(t)\\ I_{23}\omega_2(t)\omega_3(t)\\ -I_{31}\omega_1(t)\omega_3(t)\\ I_{12}\omega_1(t)\omega_2(t)\end{bmatrix} + \begin{bmatrix}0&0&0\\0&0&0\\0&0&0\\0&0&0\\ \frac{1}{I_1}&0&0\\ 0&\frac{1}{I_2}&0\\ 0&0&\frac{1}{I_3}\end{bmatrix}u(t),$$

$$\begin{bmatrix}\theta_x(0)\\ \theta_y(0)\\ \theta_z(0)\\ \eta(0)\\ \omega_1(0)\\ \omega_2(0)\\ \omega_3(0)\end{bmatrix} = \begin{bmatrix}\theta_{x0}\\ \theta_{y0}\\ \theta_{z0}\\ \eta_0\\ \omega_{10}\\ \omega_{20}\\ \omega_{30}\end{bmatrix}, \quad t \geq 0, \tag{5.103}$$

where I_1, I_2, and I_3 are the principal moments of inertia, $I_1 \geq I_2 \geq I_3 > 0$, $u = [u_1, u_2, u_3]^{\mathrm{T}}$, $I_{23} = (I_2 - I_3)/I_1$, $I_{31} = (I_3 - I_1)/I_2$, and $I_{12} = (I_1 - I_2)/I_3$.

For $x = [\theta_x, \theta_y, \theta_z, \omega_1, \omega_2, \omega_3]^{\mathrm{T}}$ and measured output

$$\begin{bmatrix}\hat y_1(t)\\ \hat y_2(t)\\ \hat y_3(t)\end{bmatrix} = \begin{bmatrix}\theta_x(t)\\ \theta_y(t)\\ \theta_z(t)\end{bmatrix}, \tag{5.104}$$

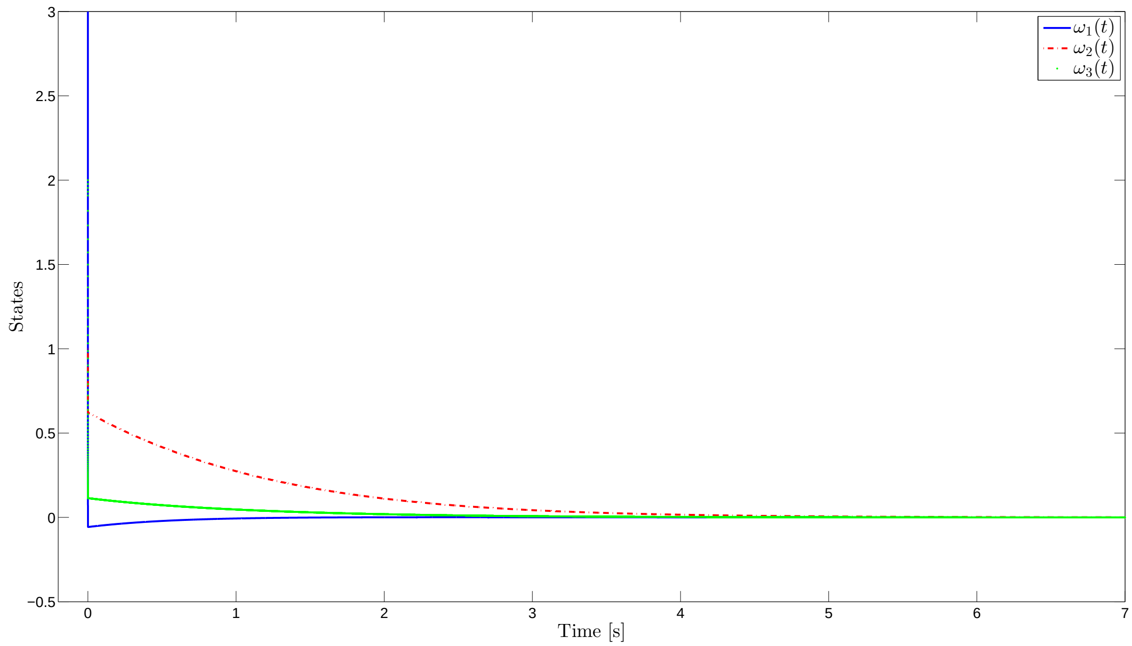

Figure 5.5: Closed-loop system trajectories versus time.

the affine nonlinear dynamical system given by (5.103), (5.102), and (5.104) is in the same form of (5.1) and (5.55) with $n = 6$ and $m = 3$. In this case, $L_{G_i}\hat{h}_j = 0$, $i, j \in \{1, 2, 3\}$, (5.57) specializes to

$$\mathcal{L}(x) = \frac{1}{2}\begin{bmatrix} \frac{\eta}{I_1} & -\frac{\theta_z}{I_2} & \frac{\theta_y}{I_3} \\ \frac{\theta_z}{I_1} & \frac{\eta}{I_2} & -\frac{\theta_x}{I_3} \\ -\frac{\theta_y}{I_1} & \frac{\theta_x}{I_2} & \frac{\eta}{I_3} \end{bmatrix}, \tag{5.105}$$

the dynamical system given by (5.103), (5.102), and (5.104) has vector relative degree $\{2, 2, 2\}$ on $\{x \in \mathbb{R}^6 : \eta \neq 0\}$ [3], and $r = \sum_{i=1}^{3} r_i = 6$. Since all of the conditions of Theorem 5.11 are satisfied, if $\eta \neq 0$, then the nonlinear dynamical system given by (5.103), (5.102), and (5.104) with feedback (5.64) is equivalent to the linear dynamical system (5.62), where

$$\mathcal{L}^{-1}(x) = \frac{2}{\eta}\begin{bmatrix} I_1\left(\theta_x^2 + \eta^2\right) & I_1\left(\theta_x\theta_y + \theta_z\eta\right) & I_1\left(\theta_x\theta_z - \theta_y\eta\right) \\ I_2\left(\theta_x\theta_y - \theta_z\eta\right) & I_2\left(\theta_y^2 + \eta^2\right) & I_2\left(\theta_x\eta + \theta_y\theta_z\right) \\ I_3\left(\theta_x\theta_z + \theta_y\eta\right) & I_3\left(\theta_y\theta_z - \theta_x\eta\right) & I_3\left(\theta_z^2 + \eta^2\right) \end{bmatrix}, \tag{5.106}$$

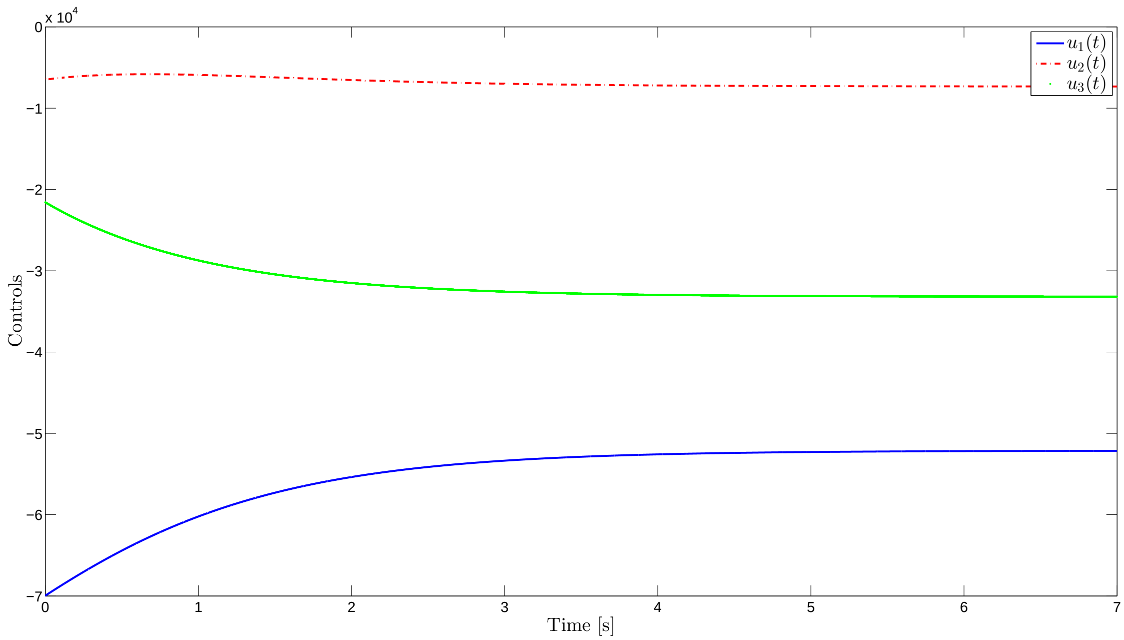

Figure 5.6: Control signal versus time.

$k_{i,j} < 0$, $i = 1, 2$ and $j = 1, 2, 3$, $z = [\hat{y}_1, \dot{\hat{y}}_1, \hat{y}_2, \dot{\hat{y}}_2, \hat{y}_3, \dot{\hat{y}}_3]^{\mathrm{T}}$, $v \in \mathbb{R}^3$, and

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ k_{1,1} & k_{2,1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & k_{1,2} & k_{2,2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & k_{1,3} & k_{2,3} \end{bmatrix} \in \mathbb{R}^{6\times 6}, \qquad B = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \in \mathbb{R}^{6\times 3}. \tag{5.107}$$

The vector $b(z)$ given by (5.65) is omitted for conciseness.

Setting $k_{1,1} = k_{1,2} = k_{1,3} = 0$ and $k_{2,1} = k_{2,2} = k_{2,3} = -1$, the pair (A, B) is controllable and setting

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \in \mathbb{R}^{3\times 6}, \tag{5.108}$$

the pair (A, C) is semiobservable. In this case, the transfer function of the linear dynamical system (5.62) and (5.63), where

$$y(t) = h(x(t)) = \left[\dot{\theta}_x(t),\, \dot{\theta}_y(t),\, \dot{\theta}_z(t)\right]^{\mathrm{T}}, \tag{5.109}$$

is given by

$$G(s) = C(sI_6 - A)B = \frac{1}{s+1} I_3, \qquad s \in \mathbb{C}, \tag{5.110}$$

which is minimum phase and right invertible. Hence, it follows from Theorem 5.13 that with $u = \phi(x)$ given by (5.73), the solution $x(t) = x_{\mathrm{e}}$, $t \geq 0$, of the closed-loop system (5.103) and (5.109) is semistable, and (5.74) is satisfied.

Let $I_1 = 20\,\mathrm{kg}\cdot\mathrm{m}^2$, $I_2 = 15\,\mathrm{kg}\cdot\mathrm{m}^2$, $I_3 = 10\,\mathrm{kg}\cdot\mathrm{m}^2$, $\theta_{x0} = 0.20$, $\theta_{y0} = 0.53$, $\theta_{z0} = 0.02$, $\omega_{10} = 3\,\mathrm{Hz}$, $\omega_{20} = 1\,\mathrm{Hz}$, and $\omega_{30} = 2\,\mathrm{Hz}$. Figures 5.4 and 5.5 show the state trajectories of the controlled system versus time. Note that $x(t) \to x_{\mathrm{e}} = [0.2007,\ 0.7758,\ 0.1468\,0,\ 0,\ 0]^{\mathrm{T}}$ as $t \to \infty$. Figure 5.6 shows the control signal versus time.

Chapter 6

Partial-State Stabilization and Optimal Feedback Control

6.1. Introduction

In [6], the current status of continuous-time, nonlinear nonquadratic optimal control problems was presented in a simplified and tutorial manner. The basic underlying ideas of the results in [6] are based on the fact that the steady-state solution of the Hamilton-Jacobi-Bellman equation is a Lyapunov function for the nonlinear system and thus guaranteeing both stability and optimality [6,38]. Specifically, a feedback control problem over an infinite horizon involving a nonlinear-nonquadratic performance functional is considered. The performance functional is then evaluated in closed form as long as the nonlinear nonquadratic cost functional considered is related in a specific way to an underlying Lyapunov function that guarantees asymptotic stability of the nonlinear closed-loop system. This Lyapunov function is shown to be the solution of the steady-state Hamilton-Jacobi-Bellman equation. The overall framework provides the foundation for extending linear-quadratic control to nonlinear-nonquadratic problems.

In this section, we extend the framework developed in [6] and [38] to address the problem of optimal *partial-state stabilization*, wherein stabilization with respect to a subset of the system state variables is desired. Even though partial-state stabilization has been considered in the literature [69,88,113], the problem of optimal partial-state stabilization has received very little attention. In this section, we consider a notion of optimality that is directly related to a given Lyapunov function that is positive definite and decrescent with respect to part of the system state. Specifically, an optimal partial-state stabilization control problem is stated and sufficient Hamilton-Jacobi-Bellman conditions are used to characterize an optimal feedback controller. Another important application of partial stability and partial stabilization theory is the unification it provides between time-invariant stability theory and stability theory for time-varying systems [23,38]. We exploit this unification and specialize our results to address optimal linear and nonlinear regulation for linear and nonlinear time-varying systems with quadratic and nonlinear nonquadratic cost functionals.

6.2. Partial Stability Theory

In this section, we consider nonlinear autonomous dynamical systems of the form

$$\dot{x}_1(t) = f_1(x_1(t), x_2(t)), \qquad x_1(0) = x_{10}, \qquad t \geq 0, \tag{6.1}$$

$$\dot{x}_2(t) = f_2(x_1(t), x_2(t)), \qquad x_2(0) = x_{20}, \tag{6.2}$$

where, for every $t \geq 0$, $x_1(t) \in \mathcal{D} \subseteq \mathbb{R}^{n_1}$ and $x_2(t) \in \mathbb{R}^{n_2}$, $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$, $f_1 : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_1}$ is such that, for every $x_2 \in \mathbb{R}^{n_2}$, $f_1(0, x_2) = 0$ and $f_1(\cdot, x_2)$ is locally Lipschitz continuous in x_1, and $f_2 : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_2}$ is such that, for every $x_1 \in \mathcal{D}$, $f_2(x_1, \cdot)$ is locally Lipschitz continuous in x_2.

Definition 6.1 [38, Def. 4.1]. *i*) The nonlinear dynamical system $\mathcal{G}$ given by (6.1) and (6.2) is *Lyapunov stable with respect to x_1 uniformly in x_{20}* if, for every $\varepsilon > 0$ and $x_{20} \in \mathbb{R}^{n_2}$, there exists $\delta = \delta(\varepsilon) > 0$ such that $\|x_{10}\| \leq \delta$ implies that $\|x_1(t)\| < \varepsilon$ for all $t \geq 0$.

ii) $\mathcal{G}$ is *asymptotically stable with respect to x_1 uniformly in x_{20}* if $\mathcal{G}$ is Lyapunov stable with respect to x_1 uniformly in x_{20} and there exists $\delta > 0$ such that $\|x_{10}\| < \delta$ implies that $\lim_{t\to\infty} x_1(t) = 0$ uniformly in x_{10} and x_{20} for all $x_{20} \in \mathbb{R}^{n_2}$.

iii) $\mathcal{G}$ is *globally asymptotically stable with respect to x_1 uniformly in x_{20}* if $\mathcal{G}$ is Lyapunov stable with respect to x_1 uniformly in x_{20} and $\lim_{t\to\infty} x_1(t) = 0$ uniformly in x_{10} and x_{20} for all $x_{10} \in \mathbb{R}^{n_1}$ and $x_{20} \in \mathbb{R}^{n_2}$.

Remark 6.2. It is important to note that there is a key difference between the partial stability definitions given in Definition 6.1 and the definitions of partial stability given in [113]. In particular, the partial stability definitions given in [113] require that both the initial conditions x_{10} and x_{20} lie in a neighborhood of the origin, whereas in Definition 6.1, x_{20} can be arbitrary. As will be seen below, this difference allows us to unify autonomous partial stability theory with time-varying stability theory. An additional difference between our formulation of the partial stability problem and the partial stability problem considered in [113] is in the treatment of the equilibrium of (6.1) and (6.2). Specifically, in our formulation we require the weaker partial equilibrium condition $f_1(0, x_2) = 0$ for every $x_2 \in \mathbb{R}^{n_2}$, whereas in [113] the author requires the stronger equilibrium condition $f_1(0, 0) = 0$ and $f_2(0, 0) = 0$.

As shown in [38] and [23], an important application of partial stability theory is the unification it provides between time-invariant stability theory and stability theory for time-varying systems. Specifically, consider the nonlinear time-varying dynamical system given by

$$\dot{x}(t) = f(t, x(t)), \qquad x(t_0) = x_0, \qquad t \geq t_0, \tag{6.3}$$

where, for every $t \geq t_0$, $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$, $f(t, 0) = 0$, $f : [t_0, \infty) \times \mathcal{D} \to \mathbb{R}^n$ is jointly continuous in t and x, and $f(t, \cdot)$ is locally Lipschitz continuous in x uniformly in t for all t in compact subsets of $[t_0, \infty)$. Now, defining $x_1(\tau) \triangleq x(t)$ and $x_2(\tau) \triangleq t$, where $\tau \triangleq t - t_0$, it follows that the solution $x(t)$, $t \geq t_0$, to the nonlinear time-varying dynamical system (6.3) can be equivalently characterized by the solution $x_1(\tau)$, $\tau \geq 0$, to the nonlinear autonomous dynamical system

$$\dot{x}_1(\tau) = f(x_2(\tau), x_1(\tau)), \qquad x_1(0) = x_0, \qquad \tau \geq 0, \tag{6.4}$$

$$\dot{x}_2(\tau) = 1, \qquad x_2(0) = t_0. \tag{6.5}$$

Note that (6.4) and (6.5) are in the same form as the system given by (6.1) and (6.2), and Definition 6.1 applied to (6.4) and (6.5) specializes to the definitions of uniform Lyapunov stability, uniform asymptotic stability, and global uniform asymptotic stability of (6.3); for details see [38, Def. 4.2].

Next, we provide sufficient conditions for partial stability of the nonlinear dynamical system given by (6.1) and (6.2). For the statement of the following result, define

$$\dot{V}(x_1, x_2) \triangleq V'(x_1, x_2) f(x_1, x_2),$$

where $f(x_1, x_2) \triangleq [f_1^{\mathrm{T}}(x_1, x_2), f_2^{\mathrm{T}}(x_1, x_2)]^{\mathrm{T}}$, for a continuously differentiable function $V : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}$.

Theorem 6.3 [38, Th. 4.1]. Consider the nonlinear dynamical system (6.1) and (6.2). Then the following statements hold:

i) If there exist a continuously differentiable function $V : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}$ and class $\mathcal{K}$ functions $\alpha(\cdot)$, $\beta(\cdot)$, and $\gamma(\cdot)$ such that

$$\alpha(\|x_1\|) \leq V(x_1, x_2) \leq \beta(\|x_1\|), \qquad (x_1, x_2) \in \mathcal{D} \times \mathbb{R}^{n_2}, \tag{6.6}$$

$$\dot{V}(x_1, x_2) \leq -\gamma(\|x_1\|), \qquad (x_1, x_2) \in \mathcal{D} \times \mathbb{R}^{n_2}, \tag{6.7}$$

then the nonlinear dynamical system given by (6.1) and (6.2) is asymptotically stable with respect to x_1 uniformly in x_{20}.

ii) If there exist a continuously differentiable function $V : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$, a class $\mathcal{K}$ function $\gamma(\cdot)$, and class $\mathcal{K}_\infty$ functions $\alpha(\cdot)$ and $\beta(\cdot)$ satisfying (6.6) and (6.7), then the nonlinear dynamical system given by (6.1) and (6.2) is globally asymptotically stable with respect to x_1 uniformly in x_{20}.

6.3. Optimal Partial-State Stabilization

In the first part of this section, we provide connections between Lyapunov functions and nonquatratic cost evaluation. Specifically, we consider the problem of evaluating a nonlinear-nonquadratic performance measure that depends on the solution of the nonlinear dynamical system given by (6.1) and (6.2). In particular, we show that the nonlinear-nonquadratic performance measure

$$J(x_{10}, x_{20}) \triangleq \int_0^\infty L(x_1(t), x_2(t))\mathrm{d}t, \tag{6.8}$$

where $L : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}$ is jointly continuous in x_1 and x_2, and $x_1(t)$ and $x_2(t)$, $t \geq 0$, satisfy (6.1) and (6.2), can be evaluated in a convenient form so long as (6.1) and (6.2) are related to an underlying Lyapunov function that is positive definite and decrescent with respect to x_1 and proves asymptotic stability of (6.1) and (6.2) with respect to x_1 uniformly in x_{20}.

Theorem 6.4. Consider the nonlinear dynamical system $\mathcal{G}$ given by (6.1) and (6.2) with performance measure (6.8). Assume that there exists a continuously differentiable function $V : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}$ and class $\mathcal{K}$ functions $\alpha(\cdot)$, $\beta(\cdot)$, and $\gamma(\cdot)$ such that

$$\alpha(\|x_1\|) \leq V(x_1, x_2) \leq \beta(\|x_1\|), \qquad (x_1, x_2) \in \mathcal{D} \times \mathbb{R}^{n_2}, \tag{6.9}$$

$$\dot{V}(x_1, x_2) \leq -\gamma(\|x_1\|), \qquad (x_1, x_2) \in \mathcal{D} \times \mathbb{R}^{n_2}, \tag{6.10}$$

$$L(x_1, x_2) + V'(x_1, x_2) f(x_1, x_2) = 0, \qquad (x_1, x_2) \in \mathcal{D} \times \mathbb{R}^{n_2}. \tag{6.11}$$

Then the nonlinear dynamical system $\mathcal{G}$ is asymptotically stable with respect to x_1 uniformly in x_{20} and there exists a neighborhood $\mathcal{D}_0 \subseteq \mathcal{D}$ of $x_1 = 0$ such that, for all $(x_{10}, x_{20}) \in \mathcal{D}_0 \times \mathbb{R}^{n_2}$,

$$J(x_{10}, x_{20}) = V(x_{10}, x_{20}). \tag{6.12}$$

Finally, if $\mathcal{D} = \mathbb{R}^{n_1}$ and the functions $\alpha(\cdot)$ and $\beta(\cdot)$ satisfying (6.9) are class $\mathcal{K}_\infty$, then $\mathcal{G}$ is globally asymptotically stable with respect to x_1 uniformly in x_{20}.

Proof: Let $x_1(t)$ and $x_2(t)$, $t \geq 0$, satisfy (6.1) and (6.2). Then it follows from (6.10) that

$$\dot{V}(x_1(t), x_2(t)) = V'(x_1(t), x_2(t)) f(x_1(t), x_2(t)) \leq -\gamma(\|x_1(t)\|), \qquad t \geq 0. \tag{6.13}$$

Thus, it follows from (6.9), (6.10), and i) of Theorem 6.3 that $\mathcal{G}$ is asymptotically stable with respect to x_1 uniformly in x_{20}. Consequently, $x_1(t) \to 0$ as $t \to \infty$ for all initial condition $(x_{10}, x_{20}) \in \mathcal{D}_0 \times \mathbb{R}^{n_2}$ for some neighborhood $\mathcal{D}_0 \subseteq \mathcal{D}$ of $x_1 = 0$. Now, since

$$0 = -\dot{V}(x_1(t), x_2(t)) + V'(x_1(t), x_2(t)) f(x_1(t), x_2(t)), \qquad t \geq 0, \tag{6.14}$$

it follows from (6.11) that

$$\begin{aligned} L(x_1(t), x_2(t)) &= -\dot{V}(x_1(t), x_2(t)) + L(x_1(t), x_2(t)) + V'(x_1(t), x_2(t)) f(x_1(t), x_2(t)) \\ &= -\dot{V}(x_1(t), x_2(t)), \qquad t \geq 0. \end{aligned} \tag{6.15}$$

Next, integrating (6.15) over $[0, t]$ yields

$$\int_0^t L(x_1(s), x_2(s))\mathrm{d}s = V(x_{10}, x_{20}) - V(x_1(t), x_2(t)), \qquad t \geq 0. \tag{6.16}$$

Now, using (6.9) and letting $t \to \infty$ it follows from (6.16) that

$$V(x_{10}, x_{20}) - \beta\Big(\lim_{t\to\infty} \|x_1(t)\|\Big) \leq \int_0^\infty L(x_1(s), x_2(s))\mathrm{d}s \leq V(x_{10}, x_{20}) - \alpha\Big(\lim_{t\to\infty} \|x_1(t)\|\Big), \tag{6.17}$$

and hence, (6.12) is a direct consequence of (6.17) using the fact that $\lim_{t\to\infty} x_1(t) = 0$ and $\alpha(\cdot)$ and $\beta(\cdot)$ are class $\mathcal{K}$ functions. Finally, if $\mathcal{D} = \mathbb{R}^{n_1}$ and $\alpha(\cdot)$ and $\beta(\cdot)$ are class $\mathcal{K}_\infty$ functions, then global asymptotic stability with respect to x_1 uniformly in x_{20} is a direct consequence of ii) of Theorem 6.3. ■

The following corollary to Theorem 6.4 considers the nonautonomous dynamical system (6.3) with performance measure

$$J(t_0, x_0) \triangleq \int_{t_0}^\infty L(t, x(t))\mathrm{d}t, \tag{6.18}$$

where $L : [t_0, \infty) \times \mathcal{D} \to \mathbb{R}$ is jointly continuous in t and x, and $x(t)$, $t \geq t_0$, satisfies (6.3).

Corollary 6.5. Consider the nonlinear time-varying dynamical system (6.3) with performance measure (6.18). Assume that there exists a continuously differentiable function $V : [t_0, \infty) \times \mathcal{D} \to \mathbb{R}$ and class $\mathcal{K}$ functions $\alpha(\cdot)$, $\beta(\cdot)$, and $\gamma(\cdot)$ such that

$$\alpha(\|x\|) \le V(t,x) \le \beta(\|x\|), \qquad (t,x) \in [t_0,\infty) \times \mathcal{D}, \tag{6.19}$$

$$\dot{V}(t,x) \le -\gamma(\|x\|), \qquad (t,x) \in [t_0,\infty) \times \mathcal{D}, \tag{6.20}$$

$$-\frac{\partial V(t,x)}{\partial t} = L(t,x) + \frac{\partial V(t,x)}{\partial x} f(t,x), \qquad (t,x) \in [t_0,\infty) \times \mathcal{D}. \tag{6.21}$$

Then the nonlinear dynamical system (6.3) is uniformly asymptotically stable and there exists a neighborhood of the origin $\mathcal{D}_0 \subseteq \mathcal{D}$ such that, for all $(t_0, x_0) \in [0,\infty) \times \mathcal{D}_0$,

$$J(t_0, x_0) = V(t_0, x_0). \tag{6.22}$$

Finally, if $\mathcal{D} = \mathbb{R}^n$ and the functions $\alpha(\cdot)$ and $\beta(\cdot)$ satisfying (6.19) are class $\mathcal{K}_\infty$, then the nonlinear dynamical system (6.3) is globally uniformly asymptotically stable.

Proof: The result is a direct consequence of Theorem 6.4 with $n_1 = n$, $n_2 = 1$, $x_1(t-t_0) = x(t)$, $x_2(t-t_0) = t$, $f_1(x_1,x_2) = f_1(x_2,x_1) = f(t,x)$, $f_2(x_1,x_2) = 1$, and $V(x_1,x_2) = V(x_2,x_1) = V(t,x)$. ■

Next, we use the framework developed in Theorem 6.4 to obtain a characterization of optimal feedback controllers that guarantee closed-loop, partial-state stabilization. Specifically, sufficient conditions for optimality are given in a form that corresponds to a steady-state version of the Hamilton-Jacobi-Bellman equation. To address the problem of characterizing partially stabilizing feedback controllers, consider the controlled nonlinear dynamical system

$$\dot{x}_1(t) = F_1(x_1(t), x_2(t), u(t)), \qquad x_1(0) = x_{10}, \qquad t \ge 0, \tag{6.23}$$

$$\dot{x}_2(t) = F_2(x_1(t), x_2(t), u(t)), \qquad x_2(0) = x_{20}, \tag{6.24}$$

where, for every $t \ge 0$, $x_1(t) \in \mathcal{D} \subseteq \mathbb{R}^{n_1}$, $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$, $x_2(t) \in \mathbb{R}^{n_2}$, $u(t) \in U \subseteq \mathbb{R}^m$ with $0 \in U$, $F_1 : \mathcal{D} \times \mathbb{R}^{n_2} \times U \to \mathbb{R}^{n_1}$ and $F_2 : \mathcal{D} \times \mathbb{R}^{n_2} \times U \to \mathbb{R}^{n_2}$ are locally Lipschitz continuous in x_1, x_2, and u, and $F_1(0, x_2, 0) = 0$ for every $x_2 \in \mathbb{R}^{n_2}$. The control $u(\cdot)$ in (6.23) and (6.24) is restricted to the class of *admissible* controls consisting of measurable functions $u(\cdot)$ such that $u(t) \in U$, $t \geq 0$.

A measurable function $\phi : \mathcal{D} \times \mathbb{R}^{n_2} \to U$ satisfying $\phi(0, x_2) = 0$, $x_2 \in \mathbb{R}^{n_2}$, is called a *control law*. If $u(t) = \phi(x_1(t), x_2(t))$, $t \geq 0$, where $\phi(\cdot, \cdot)$ is a control law and $x_1(t)$ and $x_2(t)$ satisfy (6.23) and (6.24), then we call $u(\cdot)$ a *feedback control law*. Note that the feedback control law is an admissible control since $\phi(\cdot, \cdot)$ has values in U. Given a control law $\phi(\cdot, \cdot)$ and a feedback control law $u(t) = \phi(x_1(t), x_2(t))$, $t \geq 0$, the *closed-loop system* (6.23) and (6.24) is given by

$$\dot{x}_1(t) = F_1(x_1(t), x_2(t), \phi(x_1(t), x_2(t))), \qquad x_1(0) = x_{10}, \qquad t \geq 0, \tag{6.25}$$

$$\dot{x}_2(t) = F_2(x_1(t), x_2(t), \phi(x_1(t), x_2(t))), \qquad x_2(0) = x_{20}. \tag{6.26}$$

We now consider the problem of partial-state stabilization.

Definition 6.6. Consider the controlled dynamical system given by (6.23) and (6.24). The feedback control law $u = \phi(x_1, x_2)$ is *asymptotically stabilizing with respect to x_1 uniformly in x_{20}* if the closed-loop system (6.25) and (6.26) is asymptotically stable with respect to x_1 uniformly in x_{20}. Furthermore, the feedback control law $u = \phi(x_1, x_2)$ is *globally asymptotically stabilizing with respect to x_1 uniformly in x_{20}* if the closed-loop system (6.25) and (6.26) is globally asymptotically stable with respect to x_1 uniformly in x_{20}.

Next, we present a main theorem for partial-state stabilization characterizing feedback controllers that guarantee partial closed-loop stability and minimize a nonlinear-nonquadratic performance functional. For the statement of this result, define $F(x_1, x_2, u) \triangleq [F_1^{\mathrm{T}}(x_1, x_2, u), F_2^{\mathrm{T}}(x_1, x_2, u)]^{\mathrm{T}}$, let $L : \mathcal{D} \times \mathbb{R}^{n_2} \times U \to \mathbb{R}$ be jointly continuous in x_1, x_2, and u, and define the set of partial regulation controllers given by

$$\mathcal{S}(x_{10}, x_{20}) \triangleq \{u(\cdot) : u(\cdot) \text{ is admissible and } x_1(\cdot) \text{ given by (6.23)}$$

satisfies $x_1(t) \to 0$ as $t \to \infty\}$.

Note that restricting our minimization problem to $u(\cdot) \in \mathcal{S}(x_{10}, x_{20})$, that is, inputs corresponding to partial-state null convergent solutions, can be interpreted as incorporating a partial-state system detectability condition through the cost.

Theorem 6.7. Consider the controlled nonlinear dynamical system $\mathcal{G}$ given by (6.23) and (6.24) with

$$J(x_{10}, x_{20}, u(\cdot)) \triangleq \int_0^\infty L(x_1(t), x_2(t), u(t))\mathrm{d}t, \tag{6.27}$$

where $u(\cdot)$ is an admissible control. Assume that there exist a continuously differentiable function $V : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}$, class $\mathcal{K}$ functions $\alpha(\cdot)$, $\beta(\cdot)$, and $\gamma(\cdot)$, and a control law $\phi : \mathcal{D} \times \mathbb{R}^{n_2} \to U$ such that

$$\alpha(\|x_1\|) \le V(x_1, x_2) \le \beta(\|x_1\|), \qquad (x_1, x_2) \in \mathcal{D} \times \mathbb{R}^{n_2}, \tag{6.28}$$

$$V'(x_1, x_2)F(x_1, x_2, \phi(x_1, x_2)) \le -\gamma(\|x_1\|), \qquad (x_1, x_2) \in \mathcal{D} \times \mathbb{R}^{n_2}, \tag{6.29}$$

$$\phi(0, x_2) = 0, \qquad x_2 \in \mathbb{R}^{n_2}, \tag{6.30}$$

$$L(x_1, x_2, \phi(x_1, x_2)) + V'(x_1, x_2)F(x_1, x_2, \phi(x_1, x_2)) = 0, \qquad (x_1, x_2) \in \mathcal{D} \times \mathbb{R}^{n_2}, \tag{6.31}$$

$$L(x_1, x_2, u) + V'(x_1, x_2)F(x_1, x_2, u) \ge 0, \qquad (x_1, x_2, u) \in \mathcal{D} \times \mathbb{R}^{n_2} \times U. \tag{6.32}$$

Then, with the feedback control $u = \phi(x_1, x_2)$, the closed-loop system given by (6.25) and (6.26) is asymptotically stable with respect to x_1 uniformly in x_{20} and there exists a neighborhood $\mathcal{D}_0 \subseteq \mathcal{D}$ of $x_1 = 0$ such that

$$J(x_{10}, x_{20}, \phi(x_1(\cdot), x_2(\cdot))) = V(x_{10}, x_{20}), \qquad (x_{10}, x_{20}) \in \mathcal{D}_0 \times \mathbb{R}^{n_2}. \tag{6.33}$$

In addition, if $(x_{10}, x_{20}) \in \mathcal{D}_0 \times \mathbb{R}^{n_2}$, then the feedback control $u(\cdot) = \phi(x_1(\cdot), x_2(\cdot))$ minimizes $J(x_{10}, x_{20}, u(\cdot))$ in the sense that

$$J(x_{10}, x_{20}, \phi(\cdot, \cdot)) = \min_{u(\cdot) \in \mathcal{S}(x_{10}, x_{20})} J(x_{10}, x_{20}, u(\cdot)). \tag{6.34}$$

Finally, if $\mathcal{D} = \mathbb{R}^{n_1}$, $U = \mathbb{R}^m$, and the functions $\alpha(\cdot)$ and $\beta(\cdot)$ satisfying (6.28) are class $\mathcal{K}_\infty$, then the closed-loop system (6.25) and (6.26) is globally asymptotically stable with respect to x_1 uniformly in x_{20}.

Proof: Local and global asymptotic stability with respect to x_1 uniformly in x_{20} are a direct consequence of (6.28) and (6.29) by applying Theorem 6.3 to the closed-loop system given by (6.25) and (6.26). Furthermore, using (6.31), condition (6.33) is a restatement of (6.12) as applied to the closed-loop system.

Next, let $(x_{10}, x_{20}) \in \mathcal{D}_0 \times \mathbb{R}^{n_2}$, let $u(\cdot) \in \mathcal{S}(x_{10}, x_{20})$, and let $x_1(t)$ and $x_2(t)$, $t \geq 0$, be solutions of (6.23) and (6.24). Then, it follows that

$$0 = -\dot{V}(x_1(t), x_2(t)) + V'(x_1(t), x_2(t))F(x_1(t), x_2(t), u(t)), \qquad t \geq 0. \tag{6.35}$$

Hence,

$$\begin{aligned} L(x_1(t), x_2(t), u(t)) = -\dot{V}(x_1(t), x_2(t)) &+ L(x_1(t), x_2(t), u(t)) \\ &+ V'(x_1(t), x_2(t))F(x_1(t), x_2(t), u(t)), \qquad t \geq 0. \end{aligned} \tag{6.36}$$

Now, using (6.28) and the fact that $u(\cdot) \in \mathcal{S}(x_{10}, x_{20})$, it follows that

$$0 = \lim_{t\to\infty} \alpha(\|x_1(t)\|) \leq \lim_{t\to\infty} V(x_1(t), x_2(t)) \leq \lim_{t\to\infty} \beta(\|x_1(t)\|) = 0. \tag{6.37}$$

Thus, it follows from (6.36), (6.37), (6.32), (6.33), and the fact that $u(\cdot) \in \mathcal{S}(x_{10}, x_{20})$, that

$$\begin{aligned} \int_0^\infty L(x_1(t), x_2(t), u(t))\mathrm{d}t &= \int_0^\infty -\dot{V}(x_1(t), x_2(t))\mathrm{d}t + \int_0^\infty L(x_1(t), x_2(t), u(t))\mathrm{d}t \\ &\quad + \int_0^\infty \Big(\frac{\partial V(x_1, x_2)}{\partial x_1} F_1(x_1(t), x_2(t), u(t)) \\ &\quad + \frac{\partial V(x_1, x_2)}{\partial x_2} F_2(x_1(t), x_2(t), u(t)) \Big) \mathrm{d}t \\ &\geq \int_0^\infty -\dot{V}(x_1(t), x_2(t))\mathrm{d}t \\ &= -\lim_{t\to\infty} V(x_1(t), x_2(t)) + V(x_{10}, x_{20}) \\ &= J(x_{10}, x_{20}, \phi(x_1(\cdot), x_2(\cdot))), \end{aligned} \tag{6.38}$$

which yields (6.34). ∎

Note that (6.31) is the steady-state, Hamilton-Jacobi-Bellman equation for the nonlinear controlled dynamical system (6.23) and (6.24) with performance criterion (6.27). Furthermore, conditions (6.31) and (6.32) guarantee optimality with respect to the set of admissible partially asymptotically stabilizing controllers $\mathcal{S}(x_{10}, x_{20})$. However, it is important to note that an explicit characterization of $\mathcal{S}(x_{10}, x_{20})$ is not required. In addition, the optimal asymptotically stabilizing with respect to x_1 uniformly in x_{20} *feedback* control law $u = \phi(x_1, x_2)$ is independent of the initial condition (x_{10}, x_{20}) and, using (6.31) and (6.32), is given by

$$\phi(x_1, x_2) = \underset{u \in \mathcal{S}(x_{10}, x_{20})}{\arg\min} \left[L(x_1, x_2, u) + \frac{\partial V(x_1, x_2)}{\partial x_1} F_1(x_1, x_2, u) + \frac{\partial V(x_1, x_2)}{\partial x_2} F_2(x_1, x_2, u) \right]. \tag{6.39}$$

Remark 6.8. Setting $n_1 = n$ and $n_2 = 0$, the nonlinear controlled dynamical system given by (6.23) and (6.24) reduces to

$$\dot{x}(t) = F(x(t), u(t)), \qquad x(0) = x_0, \qquad t \geq 0. \tag{6.40}$$

In this case, (6.28) implies that $V(\cdot)$ is positive definite with respect to x and the conditions of Theorem 6.7 reduce to the conditions of Theorem 8.2 of [38] characterizing the classical optimal control problem for time-invariant systems on an infinite interval.

Finally, we use Theorem 6.7 to provide a unification between optimal partial-state stabilization and optimal control for nonlinear time-varying systems. Specifically, consider the nonlinear time-varying controlled dynamical system

$$\dot{x}(t) = F(t, x(t), u(t)), \qquad x(t_0) = x_0, \qquad t \geq t_0, \tag{6.41}$$

with performance measure

$$J(t_0, x_0, u(\cdot)) \triangleq \int_{t_0}^{\infty} L(t, x(t), u(t))\mathrm{d}t, \tag{6.42}$$

where, for every $t \geq t_0$, $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$, $u(t) \in U \subseteq \mathbb{R}^m$ with $0 \in U$, $L : [t_0, \infty) \times \mathcal{D} \times U \to \mathbb{R}$ and $F : [t_0, \infty) \times \mathcal{D} \times U \to \mathbb{R}^n$ are jointly continuous in t, x, and u, $F(t, \cdot, u)$ is Lipschitz continuous in x for every $(t, u) \in [t_0, \infty) \times U$, and $F(t, x, \cdot)$ is

Lipschitz continuous in u for every $(t,x) \in [t_0,\infty) \times \mathcal{D}$. For the statement of the next result, define the set of regulation controllers

$$\mathcal{S}(t_0,x_0) \triangleq \{u(\cdot) : u(\cdot) \text{ is admissible and } x(\cdot) \text{ given by (6.41) satisfies } x(t) \to 0 \text{ as } t \to \infty\}.$$

Corollary 6.9. Consider the controlled nonlinear time-varying dynamical system (6.41) with performance measure (6.42) where $u(\cdot)$ is an admissible control. Assume that there exist a continuously differentiable function $V : [t_0,\infty) \times \mathcal{D} \to \mathbb{R}$, class $\mathcal{K}$ functions $\alpha(\cdot)$, $\beta(\cdot)$, and $\gamma(\cdot)$, and a control law $\phi : [t_0,\infty) \times \mathcal{D} \to U$ such that

$$\alpha(\|x\|) \leq V(t,x) \leq \beta(\|x\|), \quad (t,x) \in [t_0,\infty) \times \mathcal{D}, \tag{6.43}$$

$$\frac{\partial V(t,x)}{\partial t} + \frac{\partial V(t,x)}{\partial x} F(t,x,\phi(t,x)) \leq -\gamma(\|x\|), \quad (t,x) \in [t_0,\infty) \times \mathcal{D}, \tag{6.44}$$

$$\phi(t,0) = 0, \quad t \in [t_0,\infty), \tag{6.45}$$

$$-\frac{\partial V(t,x)}{\partial t} = L(t,x,\phi(t,x)) + \frac{\partial V(t,x)}{\partial x} F(t,x,\phi(t,x)), \quad (t,x) \in [t_0,\infty) \times \mathcal{D}, \tag{6.46}$$

$$L(t,x,u) + \frac{\partial V(t,x)}{\partial t} + \frac{\partial V(t,x)}{\partial x} F(t,x,u) \geq 0, \quad (t,x,u) \in [t_0,\infty) \times \mathcal{D} \times U. \tag{6.47}$$

Then, with the feedback control $u = \phi(t,x)$, the closed-loop system

$$\dot{x}(t) = F(t,x(t),\phi(x(t))), \qquad x(t_0) = x_0, \qquad t \geq t_0, \tag{6.48}$$

is uniformly asymptotically stable and there exists a neighborhood of the origin $\mathcal{D}_0 \subseteq \mathcal{D}$ such that

$$J(t_0,x_0,\phi(\cdot,\cdot)) = V(t_0,x_0), \qquad (t_0,x_0) \in [0,\infty) \times \mathcal{D}_0. \tag{6.49}$$

In addition, if $(t_0,x_0) \in [0,\infty) \times \mathcal{D}_0$, then the feedback control $u(\cdot) = \phi(\cdot,x(\cdot))$ minimizes $J(t_0,x_0,u(\cdot))$ in the sense that

$$J(t_0,x_0,\phi(\cdot,\cdot)) = \min_{u(\cdot) \in \mathcal{S}(t_0,x_0)} J(t_0,x_0,u(\cdot)). \tag{6.50}$$

Finally, if $\mathcal{D} = \mathbb{R}^n$, $U = \mathbb{R}^m$, and the functions $\alpha(\cdot)$ and $\beta(\cdot)$ satisfying (6.43) are class $\mathcal{K}_\infty$, then the nonlinear dynamical system $\mathcal{G}$ is globally uniformly asymptotically stable.

Proof: The proof is a direct consequence of Theorem 6.7 with $n_1 = n$, $n_2 = 1$, $x_1(t - t_0) = x(t)$, $x_2(t - t_0) = t$, $F_1(x_1, x_2, u) = F_1(x_2, x_1, u) = F(t, x, u)$, $F_2(x_1, x_2, u) = 1$, $\phi(x_1, x_2) = \phi(x_2, x_1) = \phi(t, x)$, and $V(x_1, x_2) = V(x_2, x_1) = V(t, x)$. ■

Note that (6.46) and (6.47) give the Hamilton-Jacobi-Bellman equation

$$-\frac{\partial V(t,x)}{\partial t} = \min_{u \in \mathcal{S}(t_0,x_0)} \left[L(t,x,u) + \frac{\partial V(t,x)}{\partial x} F(t,x,u) \right], \qquad (t,x) \in [t_0, \infty) \times \mathcal{D}, \tag{6.51}$$

which characterizes the optimal control

$$\phi(t,x) = \arg\min_{u \in \mathcal{S}(t_0,x_0)} \left[L(t,x,u) + \frac{\partial V(t,x)}{\partial x} F(t,x,u) \right] \tag{6.52}$$

for time-varying systems on a finite or infinite interval.

6.4. Partial-State Stabilization for Affine Dynamical Systems and Connections to the Time-Varying Linear-Quadratic Regulator Problem

In this section, we specialize the results of Section 6.3 to nonlinear affine dynamical systems of the form

$$\dot{x}_1(t) = f_1(x_1(t), x_2(t)) + G_1(x_1(t), x_2(t))u(t), \quad x_1(0) = x_{10}, \quad t \geq 0, \tag{6.53}$$

$$\dot{x}_2(t) = f_2(x_1(t), x_2(t)) + G_2(x_1(t), x_2(t))u(t), \quad x_2(0) = x_{20}, \tag{6.54}$$

where, for every $t \geq 0$, $x_1(t) \in \mathbb{R}^{n_1}$ and $x_2(t) \in \mathbb{R}^{n_2}$, $u(t) \in \mathbb{R}^m$, and $f_1 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_1}$, $f_2 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_2}$, $G_1 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_1 \times m}$, and $G_2 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_2 \times m}$ are such that $f_1(0, x_2) = 0$ for all $x_2 \in \mathbb{R}^{n_2}$, $f_1(\cdot, x_2)$, $f_2(\cdot, x_2)$, $G_1(\cdot, x_2)$, and $G_2(\cdot, x_2)$ are locally Lipschitz continuous in x_1, and $f_1(x_1, \cdot)$, $f_2(x_1, \cdot)$, $G_1(x_1, \cdot)$, and $G_2(x_1, \cdot)$ are locally Lipschitz continuous in x_2. Furthermore, we consider performance integrands $L(x_1, x_2, u)$ of the form

$$L(x_1, x_2, u) = L_1(x_1, x_2) + L_2(x_1, x_2)u + u^{\mathrm{T}} R_2(x_1, x_2)u, \qquad (x_1, x_2, u) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^m, \tag{6.55}$$

where $L_1 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$, $L_2 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{1\times m}$, and $R_2 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{m\times m}$ is such that $R_2(x_1, x_2) \geq N(x_1) > 0$, $(x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, so that (6.27) becomes

$$J(x_{10}, x_{20}, u(\cdot)) = \int_0^\infty \big[L_1(x_1(t), x_2(t)) + L_2(x_1(t), x_2(t))u(t) + u^{\mathrm{T}}(t)R_2(x_1(t), x_2(t))u(t)\big]\mathrm{d}t. \tag{6.56}$$

For the statement of the next result, define

$$f(x_1, x_2) \triangleq [f_1^{\mathrm{T}}(x_1, x_2), f_2^{\mathrm{T}}(x_1, x_2)]^{\mathrm{T}}, \qquad G(x_1, x_2) \triangleq [G_1^{\mathrm{T}}(x_1, x_2), G_2^{\mathrm{T}}(x_1, x_2)]^{\mathrm{T}}.$$

Theorem 6.10. Consider the controlled nonlinear affine dynamical system (6.53) and (6.54) with performance measure (6.56). Assume that there exist a continuously differentiable function $V : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$, class $\mathcal{K}_\infty$ functions $\alpha(\cdot)$ and $\beta(\cdot)$, and a class $\mathcal{K}$ function $\gamma(\cdot)$ such that

$$\alpha(\|x_1\|) \leq V(x_1, x_2) \leq \beta(\|x_1\|), \quad (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \tag{6.57}$$

$$V'(x_1, x_2)\Big[f(x_1, x_2) - \frac{1}{2}G(x_1, x_2)R_2^{-1}(x_1, x_2)L_2^{\mathrm{T}}(x_1, x_2) - \frac{1}{2}G(x_1, x_2)R_2^{-1}(x_1, x_2)G^{\mathrm{T}}(x_1, x_2)V'^{\mathrm{T}}(x_1, x_2)\Big] \leq -\gamma(\|x_1\|), \quad (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \tag{6.58}$$

$$L_2(0, x_2) = 0, \quad x_2 \in \mathbb{R}^{n_2}, \tag{6.59}$$

$$0 = L_1(x_1, x_2) + V'(x_1, x_2)f(x_1, x_2) - \frac{1}{4}\Big[V'(x_1, x_2)G(x_1, x_2) + L_2(x_1, x_2)\Big] \cdot R_2^{-1}(x_1, x_2)\Big[V'(x_1, x_2)G(x_1, x_2) + L_2(x_1, x_2)\Big]^{\mathrm{T}}, \qquad (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \tag{6.60}$$

Then, with the feedback control

$$u = \phi(x_1, x_2) = -\frac{1}{2}R_2^{-1}(x_1, x_2)\big[L_2(x_1, x_2) + V'(x_1, x_2)G(x_1, x_2)\big]^{\mathrm{T}}, \tag{6.61}$$

the closed-loop system

$$\dot{x}_1(t) = f_1(x_1(t), x_2(t)) + G_1(x_1(t), x_2(t))\phi(x_1(t), x_2(t)), \quad x_1(0) = x_{10}, \quad t \geq 0, \tag{6.62}$$

$$\dot{x}_2(t) = f_2(x_1(t), x_2(t)) + G_2(x_1(t), x_2(t))\phi(x_1(t), x_2(t)), \quad x_2(0) = x_{20}, \tag{6.63}$$

is globally asymptotically stable with respect to x_1 uniformly in x_{20} and the performance measure (6.56) is minimized in the sense of (6.34). Finally,

$$J(x_{10}, x_{20}, \phi(x_1(\cdot), x_2(\cdot)) = V(x_{10}, x_{20}), \qquad (x_{10}, x_{20}) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}. \tag{6.64}$$

Proof: The result is a consequence of Theorem 6.7 with $\mathcal{D} = \mathbb{R}^{n_1}$, $U = \mathbb{R}^m$, $F(x_1, x_2, u) = f(x_1, x_2) + G(x_1, x_2)u$, and $L(x_1, x_2, u) = L_1(x_1, x_2) + L_2(x_1, x_2)u + u^{\mathrm{T}} R_2(x_1, x_2)u$. Specifically, the feedback control law (6.61) follows from (6.39) by setting

$$\frac{\partial}{\partial u}\big[L_1(x_1, x_2)+L_2(x_1, x_2)u+u^{\mathrm{T}} R_2(x_1, x_2)u+V'(x_1, x_2)\big(f(x_1, x_2)+G(x_1, x_2)u\big)\big] = 0. \tag{6.65}$$

Now, with $u = \phi(x_1, x_2)$ given by (6.61), conditions (6.57), (6.58), and (6.60) imply (6.28), (6.29), and (6.31), respectively.

Next, since $V(\cdot, \cdot)$ is continuously differentiable and, by (6.57), $V(0, x_2)$, $x_2 \in \mathbb{R}^{n_2}$, is a local minimum of $V(\cdot, \cdot)$, it follows that $V'(0, x_2) = 0$, $x_2 \in \mathbb{R}^{n_2}$, and hence, it follows from (6.59) and (6.61) that $\phi(0, x_2) = 0$, which implies (6.30). Finally, since

$$\begin{aligned} L(x_1, x_2, u) &+ V'(x_1, x_2)[f(x_1, x_2) + G(x_1, x_2)u] \\ &= L(x_1, x_2, u) + V'(x_1, x_2)[f(x_1, x_2) + G(x_1, x_2)u] - L(x_1, x_2, \phi(x_1, x_2)) \\ &\quad - V'(x_1, x_2)[f(x_1, x_2) + G(x_1, x_2)\phi(x_1, x_2)] \\ &= [u - \phi(x_1, x_2)]^{\mathrm{T}} R_2(x_1, x_2)[u - \phi(x_1, x_2)] \\ &\geq 0, \qquad (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \end{aligned} \tag{6.66}$$

condition (6.32) holds. The result now follows as a direct consequence of Theorem 6.7. ■

Next, we use Theorem 6.10 to address the classical time-varying, linear-quadratic optimal control problem. Specifically, consider the linear time-varying dynamical system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \qquad x(t_0) = x_0, \qquad t \geq t_0, \tag{6.67}$$

with performance measure

$$J(t_0, x_0, u(\cdot)) = \int_{t_0}^{\infty} \big[x^{\mathrm{T}}(t)R_1(t)x(t) + u^{\mathrm{T}}(t)R_2(t)u(t)\big]\mathrm{d}t, \tag{6.68}$$

where, for all $t \geq t_0$, $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$, $A : [t_0, \infty) \to \mathbb{R}^{n\times n}$ and $B : [t_0, \infty) \to \mathbb{R}^{n\times m}$ are continuous and uniformly bounded, and $R_1 : [t_0, \infty) \to \mathbb{R}^{n\times n}$ and $R_2 : [t_0, \infty) \to \mathbb{R}^{m\times m}$ are continuous, uniformly bounded, and positive definite, and hence, there exist γ, $\sigma > 0$ such that $R_1(t) \geq \gamma I_n > 0$ and $R_2(t) \geq \sigma I_m > 0$ for all $t \geq t_0$.

Corollary 6.11. Consider the linear time-varying dynamical system (6.67) with quadratic performance measure (6.68) and let $P : [t_0, \infty) \to \mathbb{R}^{n\times n}$ be a continuously differentiable, uniformly bounded, positive definite solution of

$$-\dot{P}(t) = A^{\mathrm{T}}(t)P(t) + P(t)A(t) + R_1(t) - P(t)B(t)R_2^{-1}(t)B^{\mathrm{T}}(t)P(t),$$
$$\lim_{t_{\mathrm{f}}\to\infty} P(t_{\mathrm{f}}) = 0, \qquad t \in [t_0, \infty). \tag{6.69}$$

Then, with the feedback control

$$u = \phi(t, x) = -R_2^{-1}(t)B^{\mathrm{T}}(t)P(t)x, \qquad (t, x) \in [t_0, \infty) \times \mathbb{R}^n, \tag{6.70}$$

the dynamical system (6.67) is globally uniformly asymptotically stable and

$$J(t_0, x_0, \phi(\cdot, \cdot)) = x_0^{\mathrm{T}} P(t_0)x_0, \qquad (t_0, x_0) \in [0, \infty) \times \mathbb{R}^n. \tag{6.71}$$

Furthermore, the feedback control $u(\cdot) = \phi(\cdot, x(\cdot))$ minimizes (6.68) in the sense of (6.50).

Proof: The result is a consequence of Theorem 6.10 with $n_1 = n$, $n_2 = 1$, $x_1(t-t_0) = x(t)$, $x_2(t - t_0) = t$, $f_1(x_1, x_2) = f_1(x_2, x_1) = A(t)x$, $f_2(x_1, x_2) = 1$, $G_1(x_1, x_2) = G_1(x_2, x_1) = B(t)$, $G_2(x_1, x_2) = 0$, $L_1(x_1, x_2) = L_1(x_2, x_1) = x^{\mathrm{T}}R_1(t)x$, $L_2(x_1, x_2) = 0$, $R_2(x_1, x_2) = R_2(x_2, x_1) = R_2(t)$, $V(x_1, x_2) = V(x_2, x_1) = x^{\mathrm{T}}P(t)x$, $\alpha(\|x_1\|) = \alpha\|x\|^2$, $\beta(\|x_1\|) = \beta\|x\|^2$, and $\gamma(\|x_1\|) = \gamma\|x\|^2$, for some α, β, $\gamma > 0$. Specifically, since $P(\cdot)$ is uniformly bounded and positive definite, there exist constants $\alpha > 0$ and $\beta > 0$ such that $\alpha I_n \leq P(t) \leq \beta I_n$, $t \geq t_0$, and hence,

$$\alpha\|x\|^2 \leq V(t, x) \leq \beta\|x\|^2, \qquad (t, x) \in [t_0, \infty) \times \mathbb{R}^n, \tag{6.72}$$

which verifies (6.57).

Next, (6.70) is a restatement of (6.61). Now, note that, with $\tilde{A}(t) \triangleq A(t) + B(t)K(t)$, $K(t) \triangleq -R_2^{-1}(t)B^{\mathrm{T}}(t)P(t)$, and $\tilde{R}(t) \triangleq R_1(t) + P(t)B(t)R_2^{-1}(t)B^{\mathrm{T}}(t)P(t)$, (6.69) can be equivalently written as

$$-\dot{P}(t) = \tilde{A}^{\mathrm{T}}(t)P(t) + P(t)\tilde{A}(t) + \tilde{R}(t), \qquad \lim_{t_{\mathrm{f}}\to\infty} P(t_{\mathrm{f}}) = 0, \qquad t \in [t_0, \infty), \tag{6.73}$$

where $\tilde{A}(t)$, $t \geq t_0$, characterizes the closed-loop dynamics of the closed-loop system (6.67) and (6.70) given by

$$\dot{x}(t) = \tilde{A}(t)x(t), \qquad x(t_0) = x_0, \qquad t \geq t_0. \tag{6.74}$$

Next, computing the derivative of $V(t,x)$ along the trajectories of the closed-loop system (6.74) gives

$$\begin{aligned}\dot{V}(t,x) &= x^{\mathrm{T}}\dot{P}(t)x + 2x^{\mathrm{T}}P(t)\tilde{A}(t)x \\ &= x^{\mathrm{T}}\big[\dot{P}(t) + \tilde{A}^{\mathrm{T}}(t)P(t) + P(t)\tilde{A}(t)\big]x \\ &= -x^{\mathrm{T}}\tilde{R}(t)x, \qquad (t,x) \in [t_0,\infty) \times \mathbb{R}^n \\ &\leq -\gamma\|x\|^2, \qquad (t,x) \in [t_0,\infty) \times \mathbb{R}^n, \end{aligned} \tag{6.75}$$

which verifies (6.58).

Finally, it follows from (6.69) that

$$\begin{aligned} x^{\mathrm{T}}R_1(t)x + \phi^{\mathrm{T}}(t,x)R_2(t)\phi(t,x) + \frac{\partial V(t,x)}{\partial t} + \frac{\partial V(t,x)}{\partial x}\big[A(t)x + B(t)\phi(t,x)\big] \\ = x^{\mathrm{T}}\big[\dot{P}(t) + A^{\mathrm{T}}(t)P(t) + P(t)A(t) + R_1(t) - P(t)B(t)R_2^{-1}(t)B^{\mathrm{T}}(t)P(t)\big]x \\ = 0, \qquad (t,x) \in [t_0,\infty) \times \mathbb{R}^n, \end{aligned} \tag{6.76}$$

which verifies (6.60). The result now follows as a direct consequence of Theorem 6.10. ■

Corollary 6.11 gives sufficient conditions for global uniform asymptotic stability and optimality of the linear dynamical system (6.67) with the state feedback control law (6.70). Since the closed-loop linear dynamical system (6.74) is globally uniformly asymptotically stable, (6.74) is globally (uniformly) exponentially stable [73]. Corollary 6.11 assumes the existence of a continuously differentiable, uniformly bounded, positive definite $P : [t_0, \infty) \to \mathbb{R}^{n\times n}$ satisfying the differential Riccati equation (6.69). However, if (6.67) is completely controllable and completely observable (through the cost), then there exists a unique continuously differentiable, uniformly bounded, nonnegative definite solution $P : [t_0, \infty) \to \mathbb{R}^{n\times n}$ to (6.69) such that the linear dynamical system (6.67), with state feedback control law (6.70), is globally (uniformly) exponentially stable [77, Th. 3.5, 3.6].

6.5. Inverse Optimal Control

In this section, we construct state feedback controllers for nonlinear affine in the control dynamical systems that are predicated on an *inverse optimal control problem* [2,32,65,92,95]. In particular, to avoid the complexity in solving the steady-state, Hamilton-Jacobi-Bellman equation (6.60) we do not attempt to minimize a given cost functional, but rather, we parameterize a family of stabilizing controllers that minimize some derived cost functional that provides flexibility in specifying the control law. The performance integrand is shown to explicitly depend on the nonlinear system dynamics, the Lyapunov function of the closed-loop system, and the stabilizing feedback control law, wherein the coupling is introduced via the Hamilton-Jacobi-Bellman equation. Hence, by varying the parameters in the Lyapunov function and the performance integrand, the proposed framework can be used to characterize a class of globally partial-state stabilizing controllers that can meet closed-loop system response constraints.

Theorem 6.12. Consider the controlled nonlinear affine dynamical system (6.53) and (6.54) with performance measure (6.56). Assume there exist a continuously differentiable function $V : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$, class $\mathcal{K}_\infty$ functions $\alpha(\cdot)$ and $\beta(\cdot)$, and a class $\mathcal{K}$ function $\gamma(\cdot)$ such that

$$\alpha(\|x_1\|) \le V(x_1, x_2) \le \beta(\|x_1\|), \quad (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \tag{6.77}$$

$$V'(x_1, x_2)\Big[f(x_1, x_2) - \frac{1}{2}G(x_1, x_2)R_2^{-1}(x_1, x_2)L_2^{\mathrm{T}}(x_1, x_2)$$

$$-\frac{1}{2}G(x_1,x_2)R_2^{-1}(x_1,x_2)G^{\mathrm{T}}(x_1,x_2)V'^{\mathrm{T}}(x_1,x_2)\Big] \le -\gamma(\|x_1\|),\ \ (x_1,x_2)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}, \quad (6.78)$$

$$L_2(0,x_2)=0,\ \ x_2\in\mathbb{R}^{n_2}. \quad (6.79)$$

Then, with the feedback control

$$u=\phi(x_1,x_2)=-\frac{1}{2}R_2^{-1}(x_1,x_2)\big[L_2(x_1,x_2)+V'(x_1,x_2)G(x_1,x_2)\big]^{\mathrm{T}}, \quad (6.80)$$

the closed-loop system given by (6.62) and (6.63) is globally asymptotically stable with respect to x_1 uniformly in x_{20} and the performance functional (6.56), with

$$L_1(x_1,x_2)=\phi^{\mathrm{T}}(x_1,x_2)R_2(x_1,x_2)\phi(x_1,x_2)-V'(x_1,x_2)f(x_1,x_2), \quad (6.81)$$

is minimized in the sense that

$$J(x_{10},x_{20},\phi(\cdot,\cdot))=\min_{u(\cdot)\in\mathcal{S}(x_{10},x_{20})} J(x_{10},x_{20},u(\cdot)). \quad (6.82)$$

Finally,

$$J(x_{10},x_{20},\phi(x_1(\cdot),x_2(\cdot)))=V(x_{10},x_{20}), \qquad (x_{10},x_{20})\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}. \quad (6.83)$$

Proof: The proof is identical to the proof of Theorem 6.10. ■

Next, we specialize Theorem 6.12 to linear time-varying systems controlled by nonlinear controllers that minimize a polynomial cost functional. For the following result, let $R_1 : [t_0,\infty)\to\mathbb{R}^{n\times n}$, $R_2 : [t_0,\infty)\to\mathbb{R}^{m\times m}$, and $\hat{R}_q : [t_0,\infty)\to\mathbb{R}^{n\times n}$, $q=2,\ldots,r$, where r is a positive integer, be continuous, uniformly bounded, and positive definite matrices, that is, there exist γ, σ, $\hat{\sigma}_q>0$, $q=2,\ldots,r$, such that $R_1(t)\ge\gamma I_n>0$, $R_2(t)\ge\sigma I_m>0$, and $\hat{R}_q(t)\ge\hat{\sigma}_q I_m>0$, for all $t\ge t_0$. Furthermore, for the following result we consider performance integrands in (6.42) of the form

$$L(t,x,u)=L_1(t,x)+L_2(t,x)u+u^{\mathrm{T}}R_2(t,x)u, \qquad (t,x,u)\in[t_0,\infty)\times\mathbb{R}^n\times\mathbb{R}^m, \quad (6.84)$$

where $L_1 : [t_0,\infty)\times\mathbb{R}^n\to\mathbb{R}$, $L_2 : [t_0,\infty)\times\mathbb{R}^n\to\mathbb{R}^{1\times m}$, and $R_2(t,x)\ge N(x)>0$, $(t,x)\in[t_0,\infty)\times\mathbb{R}^n$, so that (6.42) becomes

$$J(t_0,x_0,u(\cdot))=\int_{t_0}^{\infty}\big[L_1(t,x(t))+L_2(t,x(t))u(t)+u^{\mathrm{T}}(t)R_2(t,x(t))u(t)\big]\mathrm{d}t. \quad (6.85)$$

Corollary 6.13. Consider the controlled linear time-varying dynamical system (6.67), where $u(\cdot)$ is admissible. Assume that there exist a continuously differentiable, uniformly bounded, positive definite $P : [t_0, \infty) \to \mathbb{R}^{n\times n}$ and continuously differentiable, uniformly bounded, nonnegative definite $M_q : [t_0, \infty) \to \mathbb{R}^{n\times n}$, $q = 2, \ldots, r$, such that

$$-\dot{P}(t) = A^{\mathrm{T}}(t)P(t) + P(t)A(t) + R_1(t) - P(t)S(t)P(t), \quad \lim_{t_{\mathrm{f}}\to\infty} P(t_{\mathrm{f}}) = 0, \quad t \in [t_0, \infty), \tag{6.86}$$

and

$$-\dot{M}_q(t) = (A(t) - S(t)P(t))^{\mathrm{T}} M_q(t) + M_q(t)(A(t) - S(t)P(t)) + \hat{R}_q(t), \qquad \lim_{t_{\mathrm{f}}\to\infty} M_q(t_{\mathrm{f}}) = 0,$$
$$q = 2, \ldots, r, \qquad t \in [t_0, \infty), \tag{6.87}$$

where $S(t) \triangleq B(t)R_2^{-1}(t)B^{\mathrm{T}}(t)$. Then the zero solution $x(t) \equiv 0$ of the closed-loop system

$$\dot{x}(t) = A(t)x(t) + B(t)\phi(t, x), \qquad x(t_0) = x_0, \qquad t \geq t_0, \tag{6.88}$$

is globally uniformly asymptotically stable with feedback control

$$u = \phi(t, x) = -R_2^{-1}(t)B^{\mathrm{T}}(t)\left(P(t) + \sum_{q=2}^{r}(x^{\mathrm{T}}M_q(t)x)^{q-1}M_q(t)\right)x, \tag{6.89}$$

and the performance functional (6.85) with $R_2(t, x) = R_2(t)$, $L_2(t, x) = 0$, and

$$L_1(t, x) = x^{\mathrm{T}}\Bigg(R_1(t) + \sum_{q=2}^{r}(x^{\mathrm{T}}M_q(t)x)^{q-1}\hat{R}_q(t)$$
$$+ \left[\sum_{q=2}^{r}(x^{\mathrm{T}}M_q(t)x)^{q-1}M_q(t)\right]^{\mathrm{T}} S(t)\left[\sum_{q=2}^{r}(x^{\mathrm{T}}M_q(t)x)^{q-1}M_q(t)\right]\Bigg)x, \tag{6.90}$$

is minimized in the sense that

$$J(t_0, x_0, \phi(\cdot, \cdot)) = \min_{u(\cdot)\in\mathcal{S}(t_0,x_0)} J(t_0, x_0, u(\cdot)), \qquad (t_0, x_0) \in [0, \infty) \times \mathbb{R}^n. \tag{6.91}$$

Finally,

$$J(t_0, x_0, \phi(\cdot, \cdot)) = x_0^{\mathrm{T}}P(t_0)x_0 + \sum_{q=2}^{r}\frac{1}{q}\Big(x_0^{\mathrm{T}}M_q(t_0)x_0\Big)^q, \qquad (t_0, x_0) \in [0, \infty) \times \mathbb{R}^n. \tag{6.92}$$

Proof: The result is a consequence of Theorem 6.12 with $n_1 = n$, $n_2 = 1$, $x_1(t - t_0) = x(t)$, $x_2(t - t_0) = t$, $f_1(x_1, x_2) = f_1(x_2, x_1) = A(t)x$, $f_2(x_1, x_2) = 1$, $G_1(x_1, x_2) = G_1(x_2, x_1) = B(t)$, $G_2(x_1, x_2) = 0$, $L_1(x_1, x_2) = L_1(x_2, x_1) = L_1(t, x)$, where $L_1(t, x)$ is given by (6.90), $L_2(x_1, x_2) = 0$, $R_2(x_1, x_2) = R_2(x_2, x_1) = R_2(t)$, $V(x_1, x_2) = V(x_2, x_1) = x^{\mathrm{T}}P(t)x + \sum_{q=2}^{r} \frac{1}{q}(x^{\mathrm{T}}M_q(t)x)^q$, $\alpha(\|x_1\|) = \alpha\|x\|^2$, $\beta(\|x_1\|) = \beta\|x\|^2 + \sum_{q=2}^{r} \frac{1}{q}\hat{\beta}_q^q\|x\|^{2q}$, and $\gamma(\|x_1\|) = -\gamma\|x\|^2 - \sum_{q=2}^{r} \hat{\sigma}_q\hat{\beta}_q^{q-1}\|x\|^{2q}$, for some α, β, γ, $\hat{\beta}_q$, and $\hat{\sigma}_q > 0$, $q = 2, \ldots, r$. Specifically, since $P(\cdot)$ and $M_q(\cdot)$ are uniformly bounded and, respectively, positive and nonnegative definite, there exist constants α, β, and $\hat{\beta}_q > 0$, $q = 2, \ldots, r$, such that $\alpha I_n \le P(t) \le \beta I_n$ and $0 \le M_q(t) \le \hat{\beta}_q I_n$, $t \ge t_0$, and hence,

$$\alpha\|x\|^2 \le V(t, x) \le \beta\|x\|^2 + \sum_{q=2}^{r} \frac{1}{q}\hat{\beta}_q^q\|x\|^{2q}, \qquad (t, x) \in [t_0, \infty) \times \mathbb{R}^n, \tag{6.93}$$

which verifies (6.77).

Next, (6.89) is a restatement of (6.80). Now, let $\phi(t, x) = \phi_1(t, x) + \phi_2(t, x)$, where

$$\phi_1(t, x) \triangleq -R_2^{-1}(t)B^{\mathrm{T}}(t)P(t)x, \tag{6.94}$$

$$\phi_2(t, x) \triangleq -R_2^{-1}(t)B^{\mathrm{T}}(t)\sum_{q=2}^{r}(x^{\mathrm{T}}M_q(t)x)^{q-1}M_q(t)x. \tag{6.95}$$

Computing the derivative of $V(t, x)$ along the trajectories of the closed-loop system (6.88) gives

$$\begin{aligned}
\dot{V}(t, x) &= x^{\mathrm{T}}\left(\dot{P}(t)x + P(t)A(t) + A^{\mathrm{T}}(t)P(t)\right)x + 2x^{\mathrm{T}}P(t)B(t)\phi(t, x) \\
&\quad + \sum_{q=2}^{r}(x^{\mathrm{T}}M_q(t)x)^{q-1}\Big[x^{\mathrm{T}}\left(\dot{M}_q(t) + M_q(t)A(t) + A^{\mathrm{T}}(t)M_q(t)\right)x \\
&\quad + 2x^{\mathrm{T}}M_q(t)B(t)\phi(t, x)\Big] \\
&= x^{\mathrm{T}}\left(\dot{P}(t)x + P(t)A(t) + A^{\mathrm{T}}(t)P(t) - P(t)S(t)P(t)\right)x - x^{\mathrm{T}}P(t)S(t)P(t)x \\
&\quad + 2x^{\mathrm{T}}P(t)B(t)\phi_2(t, x) + \sum_{q=2}^{r}(x^{\mathrm{T}}M_q(t)x)^{q-1}\Big[x^{\mathrm{T}}(\dot{M}_q(t) + M_q(t)(A(t) - S(t)P(t)) \\
&\quad + (A - S(t)P(t))^{\mathrm{T}}M_q(t))x + 2x^{\mathrm{T}}M_q(t)B(t)\phi_2(t, x)\Big], \qquad (t, x) \in [t_0, \infty) \times \mathbb{R}^n.
\end{aligned} \tag{6.96}$$

Now, using (6.86) and (6.87), (6.96) yields

$$\begin{aligned}
\dot{V}(t,x) &= -x^{\mathrm{T}}\left(R_1(t)+\sum_{q=2}^{r}(x^{\mathrm{T}}M_q(t)x)^{q-1}\hat{R}_q(t)\right)x - x^{\mathrm{T}}P(t)S(t)P(t)x \\
&\quad - 2x^{\mathrm{T}}\left[\sum_{q=2}^{r}(x^{\mathrm{T}}M_q(t)x)^{q-1}M_q(t)\right]^{\mathrm{T}}S(t)\left[\sum_{q=2}^{r}(x^{\mathrm{T}}M_q(t)x)^{q-1}M_q(t)\right]x \\
&\quad - 2x^{\mathrm{T}}P(t)S(t)\sum_{q=2}^{r}(x^{\mathrm{T}}M_q(t)x)^{q-1}M_q(t)x \\
&\leq -x^{\mathrm{T}}R_1(t)x - x^{\mathrm{T}}\sum_{q=2}^{r}(x^{\mathrm{T}}M_q(t)x)^{q-1}\hat{R}_q(t)x \\
&\leq -\gamma\|x\|^2 - \sum_{q=2}^{r}(\hat{\beta}_q\|x\|^2)^{q-1}\hat{\sigma}_q\|x\|^2 \\
&\leq -\gamma\|x\|^2 - \sum_{q=2}^{r}\hat{\sigma}_q\hat{\beta}_q^{q-1}\|x\|^{2q}, \qquad (t,x)\in[t_0,\infty)\times\mathbb{R}^n, \qquad (6.97)
\end{aligned}$$

and hence, (6.78) holds.

Finally, note that

$$\begin{aligned}
\phi^{\mathrm{T}}(t,x)R_2(t)\phi(t,x) &= x^{\mathrm{T}}P(t)S(t)P(t)x + 2x^{\mathrm{T}}P(t)S(t)\sum_{q=2}^{r}(x^{\mathrm{T}}M_q(t)x)^{q-1}M_q(t)x \\
&\quad + x^{\mathrm{T}}\left[\sum_{q=2}^{r}(x^{\mathrm{T}}M_q(t)x)^{q-1}M_q(t)\right]^{\mathrm{T}}S(t)\left[\sum_{q=2}^{r}(x^{\mathrm{T}}M_q(t)x)^{q-1}M_q(t)\right]x, \qquad (6.98)
\end{aligned}$$

which, using the first equality in (6.97), implies

$$\begin{aligned}
\dot{V}(t,x) &= -x^{\mathrm{T}}R_1(t)x - x^{\mathrm{T}}\sum_{q=2}^{r}(x^{\mathrm{T}}M_q(t)x)^{q-1}\hat{R}_q(t)x - \phi(t,x)R_2(t)\phi(t,x) \\
&\quad - x^{\mathrm{T}}\left[\sum_{q=2}^{r}(x^{\mathrm{T}}M_q(t)x)^{q-1}M_q(t)\right]^{\mathrm{T}}S(t)\left[\sum_{q=2}^{r}(x^{\mathrm{T}}M_q(t)x)^{q-1}M_q(t)\right]x \\
&= -L_1(t,x) - \phi^{\mathrm{T}}(t,x)R_2(t)\phi(t,x), \qquad (6.99)
\end{aligned}$$

where $L_1(t,x)$ is given by (6.90), and thus, (6.81) is verified. The result now follows as a direct consequence of Theorem 6.12. ■

Finally, we specialize Theorem 6.12 to linear time-varying systems controlled by nonlinear controllers that minimize a multilinear cost functional. For the following result, define $x^{[k]} \triangleq$

$x \otimes x \otimes \cdots \otimes x$ and $\overset{k}{\oplus} A \triangleq A \oplus A \oplus \cdots \oplus A$, with x and A appearing k times, where k is a positive integer. Furthermore, define $\mathcal{N}^{(k,n)} \triangleq \{\Psi \in \mathbb{R}^{1\times n^k} : \Psi x^{[k]} \geq 0,\ x \in \mathbb{R}^n\}$ and let $\hat{P}_q : [t_0, \infty) \to \mathbb{R}^{1\times n^{2q}}$, $\hat{R}_{2q} : [t_0, \infty) \to \mathbb{R}^{1\times n^{2q}}$, $q = 2, \ldots, r$, where r is a positive integer, and $R_2 : [t_0, \infty) \to \mathbb{R}^{m\times m}$ be continuous and uniformly bounded, $\hat{R}_{2q}(t)$, $\hat{P}_q(t) \in \mathcal{N}^{(2q,n)}$, and $R_2(t) \geq \sigma I_m > 0$, for some $\sigma > 0$ and for all $t \geq t_0$.

Corollary 6.14. Consider the controlled linear time-varying dynamical system (6.67), where $u(\cdot)$ is admissible. Assume that there exist a continuously differentiable, uniformly bounded, positive definite $P : [t_0, \infty) \to \mathbb{R}^{n\times n}$ and continuously differentiable, uniformly bounded $\hat{P}_q : [t_0, \infty) \to \mathbb{R}^{1\times n^{2q}}$, $q = 2, \ldots, r$, such that, $\hat{P}_q \in \mathcal{N}^{(k,n)}$,

$$-\dot{P}(t) = A^{\mathrm{T}}(t)P(t) + P(t)A(t) + R_1(t) - P(t)S(t)P(t), \quad \lim_{t_{\mathrm{f}}\to\infty} P(t_{\mathrm{f}}) = 0, \quad t \in [t_0, \infty), \tag{6.100}$$

and

$$-\dot{\hat{P}}_q(t) = \hat{P}_q(t)\left[\overset{2q}{\oplus}(A(t) - S(t)P(t))\right] + \hat{R}_{2q}(t), \quad \lim_{t_{\mathrm{f}}\to\infty} \hat{P}_q(t_{\mathrm{f}}) = 0, \quad q = 2, \ldots, r, \quad t \in [t_0, \infty), \tag{6.101}$$

where $S(t) \triangleq B(t)R_2^{-1}(t)B^{\mathrm{T}}(t)$. Then the zero solution $x(t) \equiv 0$ of the closed-loop system (6.88) is globally uniformly asymptotically stable with the feedback control law

$$\phi(t, x) = -R_2^{-1}(t)B^{\mathrm{T}}(t)\left(P(t)x + \frac{1}{2}g'^{\mathrm{T}}(t, x)\right), \tag{6.102}$$

where $g(t, x) \triangleq \sum_{q=2}^{r} \hat{P}_q(t)x^{[2q]}$, and the performance functional (6.85) with $R_2(t, x) = R_2(t)$, $L_2(t, x) = 0$, and

$$L_1(t, x) = x^{\mathrm{T}}R_1(t)x + \sum_{q=2}^{r} \hat{R}_{2q}(t)x^{[2q]} + \frac{1}{4}g'(t, x)S(t)g'^{\mathrm{T}}(t, x), \tag{6.103}$$

is minimized in the sense of (6.91). Finally,

$$J(t_0, x_0, \phi(\cdot,\cdot)) = x_0^{\mathrm{T}}P(t_0)x_0 + \sum_{q=2}^{r} \hat{P}_q(t_0)x_0^{[2q]}, \qquad (t_0, x_0) \in [0, \infty) \times \mathbb{R}^n. \tag{6.104}$$

Proof: The result is a consequence of Theorem 6.12 with $n_1 = n$, $n_2 = 1$, $x_1(t-t_0) = x(t)$, $x_2(t-t_0) = t$, $f_1(x_1, x_2) = f_1(x_2, x_1) = A(t)x$, $f_2(x_1, x_2) = 1$, $G_1(x_1, x_2) = G_1(x_2, x_1) = B(t)$, $G_2(x_1, x_2) = 0$, $L_1(x_1, x_2) = L_1(x_2, x_1) = L_1(t, x)$, where $L_1(t, x)$ is given by (6.103), $L_2(x_1, x_2) = 0$, $R_2(x_1, x_2) = R_2(x_2, x_1) = R_2(t)$, $V(x_1, x_2) = V(x_2, x_1) = x^{\mathrm{T}}P(t)x + \sum_{q=2}^{r} \hat{P}_q(t)x^{[2q]}$, $\alpha(\|x_1\|) = \alpha\|x\|^2$, $\beta(\|x_1\|) = \beta\|x\|^2$, and $\gamma(\|x_1\|) = -\gamma\|x\|^2$, for some α, β, $\gamma > 0$. Specifically, since $P(\cdot)$ is uniformly bounded and positive definite there exist constants α, $\beta > 0$ such that $\alpha I_n \leq P(t) \leq \beta I_n$. In addition, since $\hat{P}_q(t) \in \mathcal{N}^{(2q,n)}$, $q = 2, \ldots, n$, for all $t \geq t_0$, it follows that

$$\alpha\|x\|^2 \leq V(t, x) \leq \beta\|x\|^2, \qquad (t, x) \in [t_0, \infty) \times \mathbb{R}^n, \tag{6.105}$$

which verifies (6.77).

Computing the derivative of $V(t, x)$ along the trajectories of the closed-loop system (6.88) gives

$$\begin{aligned}
\dot{V}(t, x) &= x^{\mathrm{T}}\left(\dot{P}(t)x + P(t)A(t) + A^{\mathrm{T}}(t)P(t)\right)x + 2x^{\mathrm{T}}P(t)B(t)\phi(t, x) \\
&\quad + \sum_{q=2}^{r} \dot{\hat{P}}_q(t)x^{[2q]} + \frac{\partial}{\partial x}\left[\sum_{q=2}^{r} \hat{P}_q(t)x^{[2q]}\right](A(t)x + B(t)\phi(t, x)) \\
&= x^{\mathrm{T}}\left(\dot{P}(t)x + P(t)A(t) + A^{\mathrm{T}}(t)P(t) - P(t)S(t)P(t)\right)x \\
&\quad - x^{\mathrm{T}}P(t)S(t)P(t)x - x^{\mathrm{T}}P(t)S(t)g'^{\mathrm{T}}(t, x) \\
&\quad + \sum_{q=2}^{r} \dot{\hat{P}}_q(t)x^{[2q]} + g'(t, x)\left[(A(t) - S(t)P(t))x - \frac{1}{2}S(t)g'^{\mathrm{T}}(t, x)\right]
\end{aligned} \tag{6.106}$$

for all $(t, x) \in [t_0, \infty) \times \mathbb{R}^n$. Next, noting that

$$\begin{aligned}
g'(t, x)(A(t) - S(t)P(t))x &= \sum_{q=2}^{r} \hat{P}_q(t)\frac{\partial}{\partial x}\left[x^{[2q]}\right](A(t) - S(t)P(t))x \\
&= \sum_{q=2}^{r} \hat{P}_q(t)\frac{\partial}{\partial x}\left[x \otimes \cdots \otimes x\right](A(t) - S(t)P(t))x \\
&= \sum_{q=2}^{r} \hat{P}_q(t)\left[I_n \otimes \cdots \otimes x + x \otimes \cdots \otimes I_n\right](A(t) - S(t)P(t))x \\
&= \sum_{q=2}^{r} \hat{P}_q(t)\big[(A(t) - S(t)P(t))x \otimes \cdots \otimes x
\end{aligned}$$

$$
\begin{aligned}
&+ x \otimes \cdots \otimes (A(t) - S(t)P(t))x\big] \\
&= \sum_{q=2}^{r} \hat{P}_q(t)\big[(A(t) - S(t)P(t)) \otimes \cdots \otimes I_n \\
&\quad + I_n \otimes \cdots \otimes (A(t) - S(t)P(t))\big] x^{[2q]} \\
&= \sum_{q=2}^{r} \hat{P}_q(t) \left[\overset{2q}{\otimes} (A(t) - S(t)P(t)) \right] x^{[2q]},
\end{aligned}
\tag{6.107}
$$

it follows from (6.100), (6.101), and (6.107), that

$$
\begin{aligned}
\dot{V}(t,x) &= -x^{\mathrm{T}} R_1(t)x - x^{\mathrm{T}} P(t)S(t)P(t)x - x^{\mathrm{T}} P(t)S(t)g'^{\mathrm{T}}(t,x) \\
&\quad + \sum_{q=2}^{r} \left(\dot{\hat{P}}_q(t) + \hat{P}_q(t) \left[\overset{2q}{\otimes} (A(t) - S(t)P(t)) \right] \right) x^{[2q]} - \frac{1}{2} g'(t,x)S(t)g'^{\mathrm{T}}(t,x) \\
&= -x^{\mathrm{T}} R_1(t)x - x^{\mathrm{T}} P(t)S(t)P(t)x - x^{\mathrm{T}} P(t)S(t)g'^{\mathrm{T}}(t,x) \\
&\quad - \sum_{q=2}^{r} \hat{R}_{2q}(t)x^{[2q]} - \frac{1}{2} g'(t,x)S(t)g'^{\mathrm{T}}(t,x).
\end{aligned}
\tag{6.108}
$$

Finally, note that

$$
\begin{aligned}
\phi^{\mathrm{T}}(t,x)R_2(t)\phi(t,x) &= \left(x^{\mathrm{T}} P(t) + \frac{1}{2} g'(t,x) \right) S(t) \left(P(t)x + \frac{1}{2} g'^{\mathrm{T}}(t,x) \right) \\
&= x^{\mathrm{T}} P(t)S(t)P(t)x + \frac{1}{4} g'(t,x)S(t)g'^{\mathrm{T}}(t,x) + x^{\mathrm{T}} P(t)S(t)g'^{\mathrm{T}}(t,x),
\end{aligned}
\tag{6.109}
$$

which, using (6.108), implies that

$$
\dot{V}(t,x) = -x^{\mathrm{T}} R_1(t)x - \sum_{q=2}^{r} \hat{R}_{2q}(t)x^{[2q]} - \frac{1}{4} g'(t,x)S(t)g'^{\mathrm{T}}(t,x) - \phi^{\mathrm{T}}(t,x)R_2(t)\phi(t,x)
\tag{6.110}
$$

for all $(t,x) \in [t_0,\infty) \times \mathbb{R}^n$, and hence, (6.78) holds with $\gamma(\|x\|) = -\gamma\|x\|^2$. In addition, writing (6.110) as

$$
\dot{V}(t,x) = -L_1(t,x) - \phi^{\mathrm{T}}(t,x)R_2(t)\phi(t,x),
\tag{6.111}
$$

where $L_1(t,x)$ is given by (6.103), (6.82) holds. The result now follows as a direct consequence of Theorem 6.12. ■

6.6. Illustrative Numerical Examples

In this section, we provide several numerical examples to highlight the optimal and inverse optimal partial-state asymptotic stabilization framework developed in the section.

6.6.1. Optimal Partial Stabilization of a Flexible Spacecraft

Consider the flexible spacecraft given by [38]

$$\dot{\omega}_1(t) = I_{23}\omega_2(t)\omega_3(t) - \alpha_1\omega_1(t) + u_1(t), \qquad \omega_1(0) = \omega_{10}, \qquad t \geq 0, \tag{6.112}$$

$$\dot{\omega}_2(t) = I_{31}\omega_3(t)\omega_1(t) - \alpha_2\omega_2(t) + u_2(t), \qquad \omega_2(0) = \omega_{20}, \tag{6.113}$$

$$\dot{\omega}_3(t) = I_{12}\omega_1(t)\omega_2(t), \qquad \omega_3(0) = \omega_{30}, \tag{6.114}$$

where $I_{23} \triangleq (I_2 - I_3)/I_1$, $I_{31} \triangleq (I_3 - I_1)/I_2$, $I_{12} \triangleq (I_1 - I_2)/I_3$, I_1, I_2, and I_3 are the principal moments of inertia of the spacecraft such that $I_1 > I_2 > I_3 > 0$, $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$ reflect dissipation in the ω_1 and ω_2 coordinates of the spacecraft, and u_1 and u_2 are the spacecraft control moments. For this example, we seek a state feedback controller $u = [u_1, u_2]^{\mathrm{T}} = \phi(x_1, x_2)$, where $x_1 = [\omega_1, \omega_2]^{\mathrm{T}}$ and $x_2 = \omega_3$, such that the performance measure

$$J(x_1(0), x_2(0), u(\cdot)) = \int_0^\infty \left[x_1^{\mathrm{T}}(t)R_1x_1(t) + u^{\mathrm{T}}(t)u(t)\right]\mathrm{d}t, \tag{6.115}$$

where $R_1 > 0$, is minimized in the sense of (6.34), and (6.112)–(6.114) is globally asymptotically stable with respect to x_1 uniformly in $x_2(0)$.

Note that (6.112)–(6.114) with performance measure (6.115) can be cast in the form of (6.53) and (6.54) with performance measure (6.56). In this case, Theorem 6.10 can be applied with $n_1 = 2$, $n_2 = 1$, $m = 2$, $f(x_1, x_2) = \left[I_{23}\omega_2\omega_3,\ I_{31}\omega_3\omega_1,\ I_{12}\omega_1\omega_2\right]^{\mathrm{T}} - Ax_1$, $A \triangleq \begin{bmatrix}\alpha_1 & 0 & 0\\ 0 & \alpha_2 & 0\end{bmatrix}^{\mathrm{T}}$, $G(x_1, x_2) = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}^{\mathrm{T}}$, $L_1(x_1, x_2) = x_1^{\mathrm{T}}R_1x_1$, $L_2(x_1, x_2) = 0$, and $R_2(x_1, x_2) = I_2$ to characterize the optimal partially stabilizing controller. Specifically, in this case (6.60) reduces to

$$0 = x_1^{\mathrm{T}}R_1x_1 + V'(x_1, x_2)f(x_1, x_2) - V'(x_1, x_2)Ax_1$$

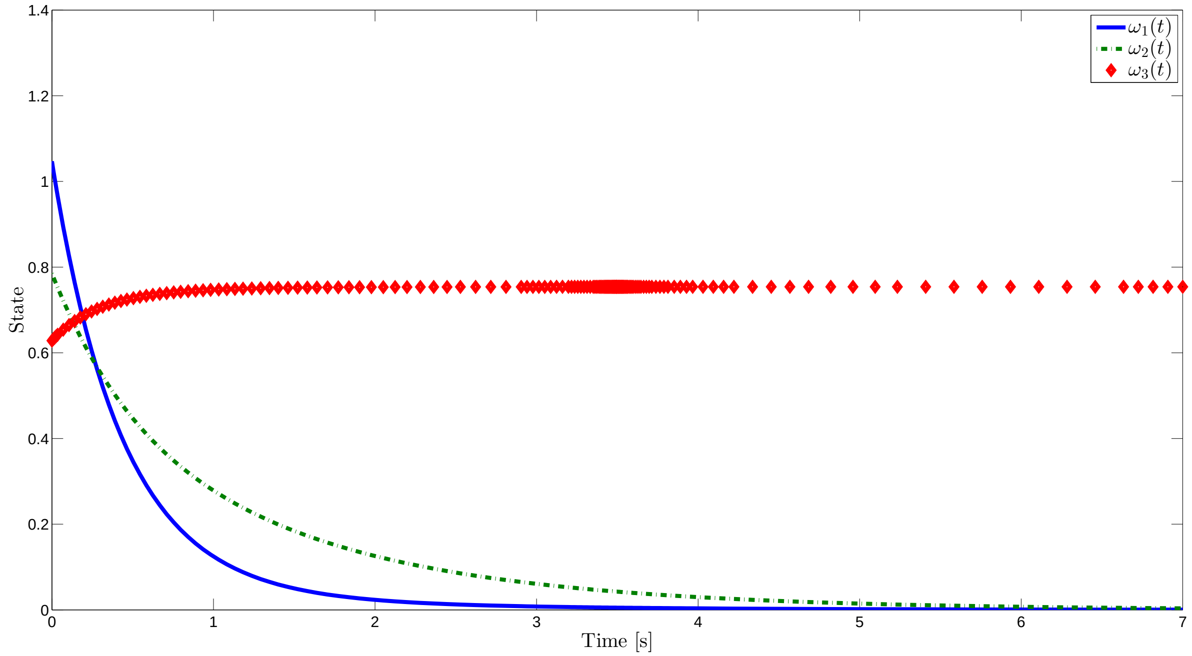

Figure 6.1: Closed-loop system trajectories versus time.

$$-\frac{1}{4}V'(x_1,x_2)G(x_1,x_2)G^{\mathrm{T}}(x_1,x_2)V'^{\mathrm{T}}(x_1,x_2), \qquad (x_1,x_2)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}. \qquad (6.116)$$

Now, choosing $V(x_1,x_2)=x_1^{\mathrm{T}}Px_1$, where $P>0$, it follows from (6.116) that

$$0=x_1^{\mathrm{T}}R_1x_1+V'(x_1,x_2)f(x_1,x_2)-2x_1^{\mathrm{T}}PHx_1-x_1^{\mathrm{T}}PPx_1, \qquad (x_1,x_2)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}, \qquad (6.117)$$

where $H\triangleq\begin{bmatrix}\alpha_1 & 0\\ 0 & \alpha_2\end{bmatrix}$, and $V'(x_1,x_2)f(x_1,x_2)=0$ only if

$$P=\rho J, \qquad (6.118)$$

where $\rho>0$ and $J\triangleq\begin{bmatrix}-I_{31} & 0\\ 0 & I_{23}\end{bmatrix}$. In this case, (6.117) and (6.118) imply that

$$0=R_1-2\rho JH-\rho^2J^2. \qquad (6.119)$$

Hence, (6.57) holds with $\alpha(\|x_1\|)=\rho\,\lambda_{\min}(J)\|x_1\|^2$ and $\beta(\|x_1\|)=\rho\,\lambda_{\max}(J)\|x_1\|^2$, where $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ denote minimum and maximum eigenvalues, respectively, and (6.58) holds with $\gamma(\|x_1\|)=\lambda_{\min}(R_1)\|x_1\|^2$.

Since all of the conditions of Theorem 6.10 hold, it follows that the feedback control law

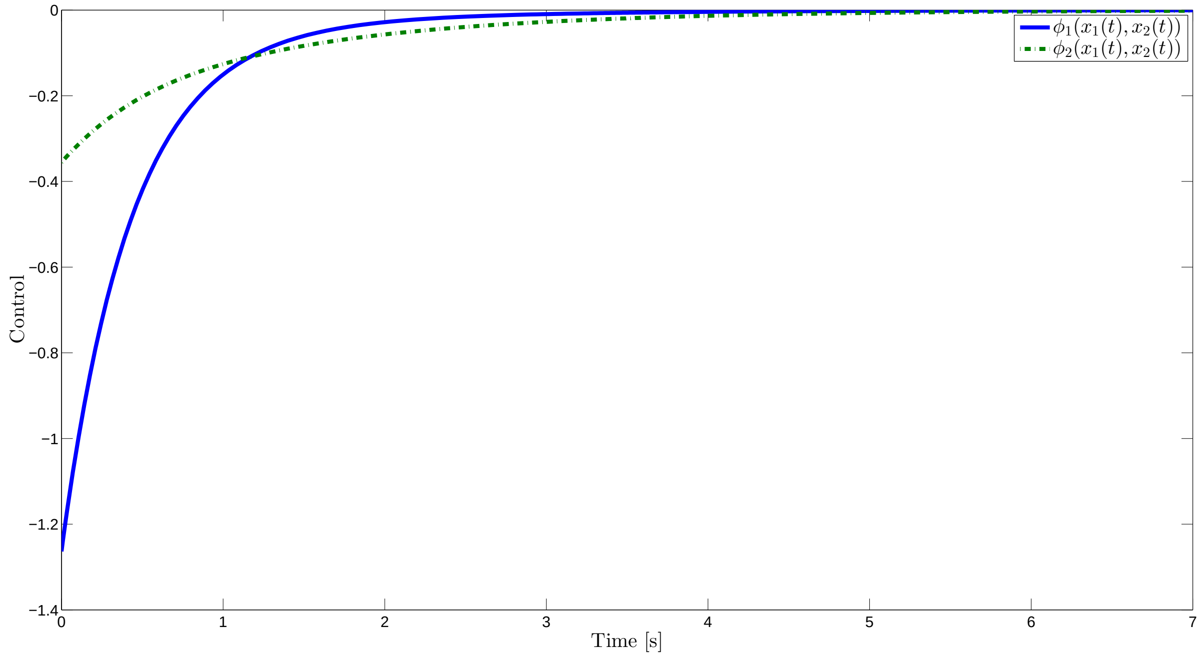

Figure 6.2: Control signal versus time.

(6.60) given by

$$\phi(x_1, x_2) = -\frac{1}{2}R_2^{-1}(x_1, x_2)G^{\mathrm{T}}(x_1, x_2)V'^{\mathrm{T}}(x_1, x_2)$$
$$= -\rho J x_1, \qquad (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \tag{6.120}$$

guarantees that the dynamical system (6.112)–(6.114) is globally asymptotically stable with respect to x_1 uniformly in $x_2(0)$ and, for all $(x_1(0), x_2(0)) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$,

$$J(x_1(0), x_2(0), \phi(x_1(\cdot), x_2(\cdot))) = x_1^{\mathrm{T}}(0)Px_1(0). \tag{6.121}$$

Let $I_1 = 20\,\mathrm{kg}\cdot\mathrm{m}^2$, $I_2 = 15\,\mathrm{kg}\cdot\mathrm{m}^2$, $I_3 = 10\,\mathrm{kg}\cdot\mathrm{m}^2$, $\omega_{10} = \pi/3\,\mathrm{Hz}$, $\omega_{20} = \pi/4\,\mathrm{Hz}$, $\omega_{30} = \pi/5\,\mathrm{Hz}$, $\alpha_1 = 1.1668\,\mathrm{Hz}$, $\alpha_2 = 0.2\,\mathrm{Hz}$, and $R_1 = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}\,\mathrm{Hz}^2$. Figure 6.1 shows the state trajectories of the controlled system versus time for $\rho = 1.81\,\mathrm{Hz}/(\mathrm{N}\cdot\mathrm{m}^2)$. Note that $x_1(t) = [\omega_1(t),\, \omega_2(t)]^{\mathrm{T}} \to 0$ as $t \to \infty$, whereas $x_2(t) = \omega_3(t)$ does not converge to zero. Figure 6.2 shows the control signal versus time. Finally, $J(x_1(0), x_2(0), \phi(x_1(\cdot), x_2(\cdot))) = 1.6024\,\mathrm{Hz}^3$.

6.6.2. Thermoacoustic Combustion Model

In this example, we consider control of thermoacoustic instabilities in combustion processes. Engineering applications involving steam and gas turbines and jet and ramjet engines for power generation and propulsion technology involve combustion processes. Due to the inherent coupling between several intricate physical phenomena in these processes involving acoustics, thermodynamics, fluid mechanics, and chemical kinetics, the dynamic behavior of combustion systems is characterized by highly complex nonlinear models [29, 97]. The unstable dynamic coupling between heat release in combustion processes generated by reacting mixtures releasing chemical energy and unsteady motions in the combustor develop acoustic pressure and velocity oscillations that can severely affect operating conditions and system performance.

Consider the nonlinear dynamical system adopted from [29, 38] given by

$$\dot{q}_1(t) = -\alpha_1 q_1(t) - \beta q_1(t) q_2(t) \cos q_3(t) + u(t), \qquad q_1(0) = q_{10}, \qquad t \geq 0, \tag{6.122}$$

$$\dot{q}_2(t) = -\alpha_2 q_2(t) + \beta q_1^2(t) \cos q_3(t) + u(t), \qquad q_2(0) = q_{20} \neq 0, \tag{6.123}$$

$$\dot{q}_3(t) = 2\theta_1 - \theta_2 - \beta\left(\frac{q_1^2(t)}{q_2(t)} - 2q_2(t)\right) \sin q_3(t), \qquad q_3(0) = q_{30}, \tag{6.124}$$

representing a time-averaged, two-mode thermoacoustic combustion model, where $\alpha_1 > 0$ and $\alpha_2 > 0$ represent decay constants, θ_1 and $\theta_2 \in \mathbb{R}$ represent frequency shift constants, $\beta = \big((\gamma + 1)/8\gamma\big)\omega_1$, where γ denotes the ratio of specific heats and ω_1 is the frequency of the fundamental mode, and u is the control input signal. As shown in [29] and [97], only the first two states q_1 and q_2 representing the modal amplitudes of a two-mode thermoacoustic combustion model are relevant in characterizing system instabilities since the third state q_3 represents the phase difference between the two modes [117]. Hence, we require asymptotic stability of $q_1(t)$, $t \geq 0$, and $q_2(t)$, $t \geq 0$, which necessitates partial stabilization.

For this example, we seek a state feedback controller $u = \phi(x_1, x_2)$, where $x_1 = [q_1,\, q_2]^{\mathrm{T}}$

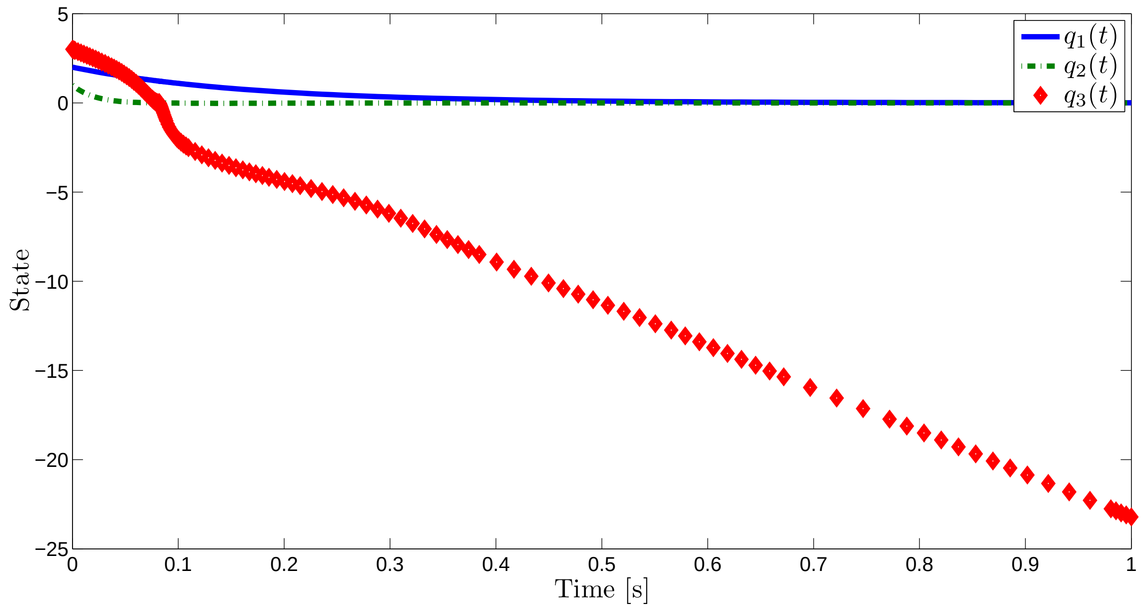

Figure 6.3: Closed-loop system trajectories versus time.

and $x_2 = q_3$, such that the performance measure

$$J(x_1(0), x_2(0), u(\cdot)) = \int_0^\infty \left[x_1^{\mathrm{T}}(t)R_1x_1(t) + u^2(t)\right]\mathrm{d}t, \tag{6.125}$$

where

$$R_1 = \rho \begin{bmatrix} 2\alpha_1 + \rho & \rho \\ \rho & 2\alpha_2 + \rho \end{bmatrix}, \qquad \rho > 0, \tag{6.126}$$

is minimized in the sense of (6.34), and (6.122)–(6.124) is globally asymptotically stable with respect to x_1 uniformly in $x_2(0)$.

Note that (6.122)–(6.124) with performance measure (6.125) can be cast in the form of (6.53) and (6.54) with performance measure (6.56). In this case, Theorem 6.10 can be applied with $n_1 = 2$, $n_2 = 1$, $m = 1$, $f(x_1, x_2) = \big[-\alpha_1 q_1 - \beta q_1 q_2 \cos q_3,\ -\alpha_2 q_2 + \beta q_1^2 \cos q_3,\ 2\theta_1 - \theta_2 - \beta\big(\frac{q_1^2}{q_2} - 2q_2\big)\sin q_3\big]^{\mathrm{T}}$, $G(x_1, x_2) = \begin{bmatrix}1 & 1 & 0\end{bmatrix}^{\mathrm{T}}$, $L_1(x_1, x_2) = x_1^{\mathrm{T}}R_1x_1$, $L_2(x_1, x_2) = 0$, and $R_2(x_1, x_2) = 1$ to characterize the optimal partially stabilizing controller. Specifically, (6.60) reduces to

$$0 = x_1^{\mathrm{T}}R_1x_1 + V'(x_1, x_2)f(x_1, x_2) - \frac{1}{4}V'(x_1, x_2)G(x_1, x_2)G^{\mathrm{T}}(x_1, x_2)V'^{\mathrm{T}}(x_1, x_2),$$
$$(x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \tag{6.127}$$

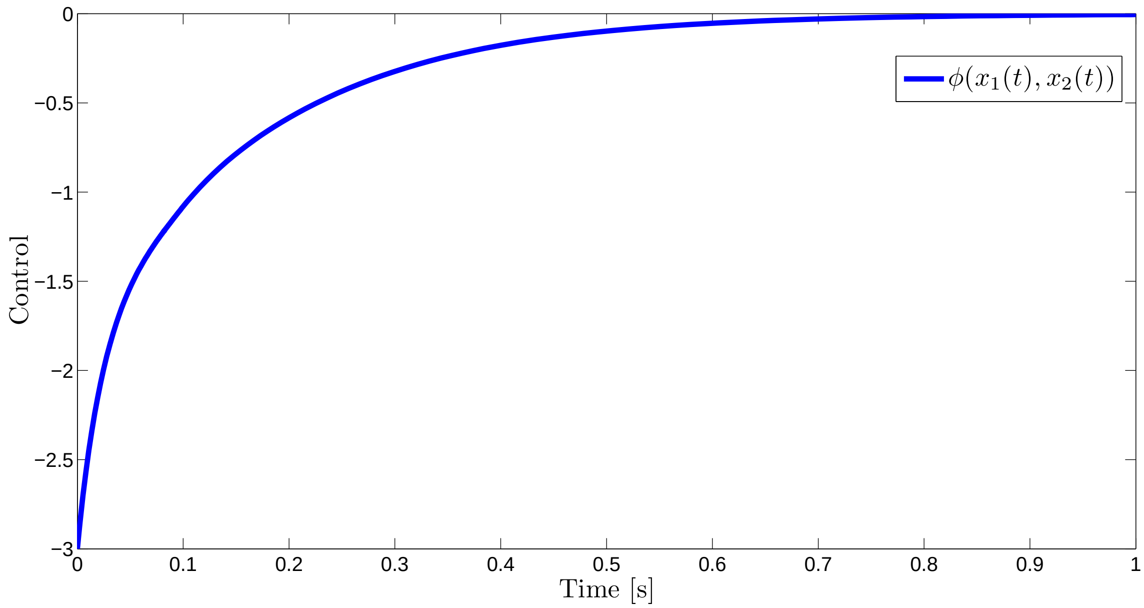

Figure 6.4: Control signal versus time.

which implies that

$$V'(x_1, x_2) = 2\rho\,[q_1,\ \ q_2,\ \ 0]. \tag{6.128}$$

Furthermore, since $V(0, x_2) = 0$, $x_2 \in \mathbb{R}$,

$$V(x_1, x_2) = \rho\, x_1^{\mathrm{T}} x_1, \tag{6.129}$$

which is positive definite with respect to x_1, and hence, (6.57) holds.

Since all of the conditions of Theorem 6.10 hold, it follows that the feedback control (6.61) given by

$$\begin{aligned}\phi(x_1, x_2) &= -\frac{1}{2} R_2^{-1}(x_1, x_2) G^{\mathrm{T}}(x_1, x_2) V'^{\mathrm{T}}(x_1, x_2)\\ &= -\rho \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} q_1 & q_2 & 0 \end{bmatrix}^{\mathrm{T}}\\ &= -\rho \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \end{aligned} \tag{6.130}$$

guarantees that the dynamical system (6.122)–(6.124) is globally asymptotically stable with respect to x_1 uniformly in $x_2(0)$ and, for all $(x_1(0), x_2(0)) \in \mathbb{R}^2 \times \mathbb{R}$,

$$J(x_1(0), x_2(0), \phi(x_1(\cdot), x_2(\cdot))) = \rho\, x_1^{\mathrm{T}}(0) x_1(0). \tag{6.131}$$

Let $\alpha_1 = 5\,\text{Hz}$, $\alpha_2 = 45\,\text{Hz}$, $\gamma = 1.4$, $\omega_1 = 1\,\text{Hz}$, $\theta_1 = 4\,\text{Hz}$, $\theta_2 = 32\,\text{Hz}$, $\rho = 1\,\text{Hz}$, $q_{10} = 2$, $q_{20} = 1$, and $q_{30} = 3$. Figure 6.3 shows the state trajectories of the controlled system versus time. Note that $x_1(t) = [q_1(t),\, q_2(t)]^{\mathrm{T}} \to 0$ as $t \to \infty$, whereas $x_2(t) = q_3(t)$ is unstable. Figure 6.4 shows the control signal versus time. Finally, $J(x_1(0), x_2(0), \phi(x_1(\cdot), x_2(\cdot))) = 5\,\text{Hz}$.

6.6.3. Inverse Optimal Control of an Axisymmetric Spacecraft

For our final example, we consider a spacecraft with one axis of symmetry [114, p. 753] given by

$$\dot{\omega}_1(t) = I_{23}\omega_2(t)\omega_3(t) + \alpha_1 u_1(t), \qquad \omega_1(0) = \omega_{10}, \qquad t \geq 0, \tag{6.132}$$

$$\dot{\omega}_2(t) = -I_{23}\omega_3(t)\omega_1(t) + \alpha_2 u_2(t), \qquad \omega_2(0) = \omega_{20}, \tag{6.133}$$

$$\dot{\omega}_3(t) = \alpha_3 u_1(t) + \alpha_4 u_2(t), \qquad \omega_3(0) = \omega_{30}, \tag{6.134}$$

where $I_{23} \triangleq (I_2 - I_3)/I_1$, I_1, I_2, and I_3 are the principal moments of inertia of the spacecraft such that $0 < I_1 = I_2 < I_3$, α_1, α_2, α_3, and $\alpha_4 \in \mathbb{R}$, $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$, and u_1 and u_2 are the spacecraft control moments. In this example, we apply Theorem 6.12 to find an *inverse optimal* globally partial-state stabilizing control law $u = [u_1,\, u_2]^{\mathrm{T}} = \phi(x_1, x_2)$, where $x_1 = [\omega_1,\, \omega_2]^{\mathrm{T}}$ and $x_2 = \omega_3$, such that the spacecraft is spin-stabilized about its third principle axis of inertia, that is, the dynamical system (6.132)–(6.134) is globally asymptotically stable with respect to x_1 uniformly in $x_2(0)$. Note that (6.132)–(6.134) can be cast in the form of (6.53) and (6.54), with $n_1 = 2$, $n_2 = 1$, $m = 2$, $f(x_1, x_2) = \left[I_{23}\omega_2\omega_3,\, -I_{23}\omega_3\omega_1,\, 0\right]^{\mathrm{T}}$, and $G(x_1, x_2) = \begin{bmatrix} \alpha_1 & 0 & \alpha_3 \\ 0 & \alpha_2 & \alpha_4 \end{bmatrix}^{\mathrm{T}}$.

To construct an inverse optimal controller for (6.132)–(6.134), let

$$V(x_1, x_2) = x_1^{\mathrm{T}} \begin{bmatrix} p_1 & 0 \\ 0 & p_2 \end{bmatrix} x_1, \tag{6.135}$$

where p_1 and $p_2 > 0$, $L(x_1, x_2, u) = L_1(x_1, x_2) + L_2(x_1, x_2)u + u^{\mathrm{T}}u$, and let

$$L_2(x_1, x_2) = 2\left[-\frac{I_{23}}{\alpha_1}\omega_2\omega_3,\, \frac{I_{23}}{\alpha_2}\omega_1\omega_3\right]. \tag{6.136}$$

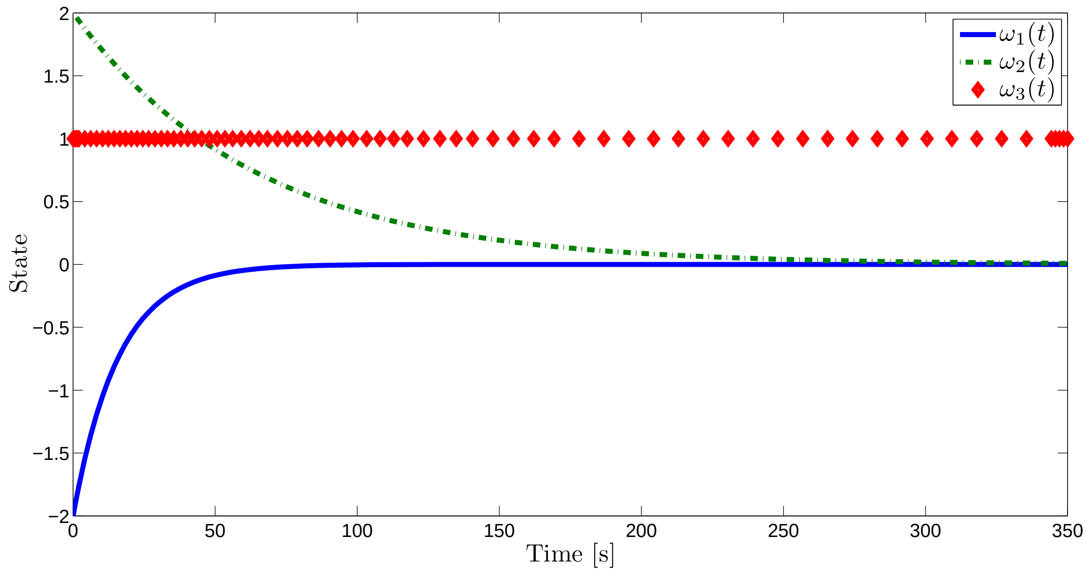

Figure 6.5: Closed-loop system trajectories versus time.

Now, the inverse optimal control law (6.80) is given by

$$\phi(x_1, x_2) = \left[-\alpha_1 p_1 \omega_1 - \frac{I_{23}}{\alpha_1}\omega_2\omega_3,\ -\alpha_2 p_2 \omega_2 + \frac{I_{23}}{\alpha_1}\omega_1\omega_3 \right]^{\mathrm{T}} \tag{6.137}$$

and, in this case, the performance functional (6.56), with

$$L_1(x_1, x_2) = \omega_1^2\left(\alpha_1^2 p_1^2 + \frac{\omega_3^2}{\alpha_2^2} I_{23}^2\right)^2 + (\alpha_2 p_2 \omega_2)^2 + \left(I_{23}\frac{\omega_2\omega_3}{\alpha_1}\right), \tag{6.138}$$

is minimized in the sense of (6.82). Furthermore, since (6.77) holds with $\alpha(\|x_1\|) = \beta(\|x_1\|) = p_1\omega_1^2 + p_2\omega_2^2$ and, since

$$V'(x_1, x_2)\left[f(x_1, x_2) - \frac{1}{2}G(x_1, x_2)L_2^{\mathrm{T}}(x_1, x_2) - \frac{1}{2}G(x_1, x_2)G^{\mathrm{T}}(x_1, x_2)V'^{\mathrm{T}}(x_1, x_2)\right]$$
$$= -2\alpha_1^2 p_1^2 \omega_1^2 - 2\alpha_2^2 p_2^2 \omega_2^2, \qquad (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \tag{6.139}$$

(6.78) holds with $\gamma(\|x_1\|) = 2\alpha_1^2 p_1^2 \omega_1^2 + 2\alpha_2^2 p_2^2 \omega_2^2$. Therefore, with the feedback control law $\phi(x_1, x_2)$ given by (6.135), the closed-loop system (6.132)–(6.134) is globally asymptotically stable with respect to x_1 uniformly in $x_2(0)$. Note that $\phi(x_1, x_2)$, $L_1(x_1, x_2)$, and $\gamma(\|x_1\|)$ do not depend on α_3 or α_4.

Let $p_1 = 200$, $p_2 = 50$, $I_1 = I_2 = 4\,\mathrm{kg}\cdot\mathrm{m}^2$, $I_3 = 20\,\mathrm{kg}\cdot\mathrm{m}^2$, $\alpha_1 = \frac{\sqrt{2}}{2I_1}$, $\alpha_2 = \frac{\sqrt{2}}{2I_2}$, $\alpha_3 = \alpha_4 = 0$, $\omega_{10} = -2\,\mathrm{Hz}$, $\omega_{20} = 2\,\mathrm{Hz}$, and $\omega_{30} = 1\,\mathrm{Hz}$, Figure 6.5 shows the state

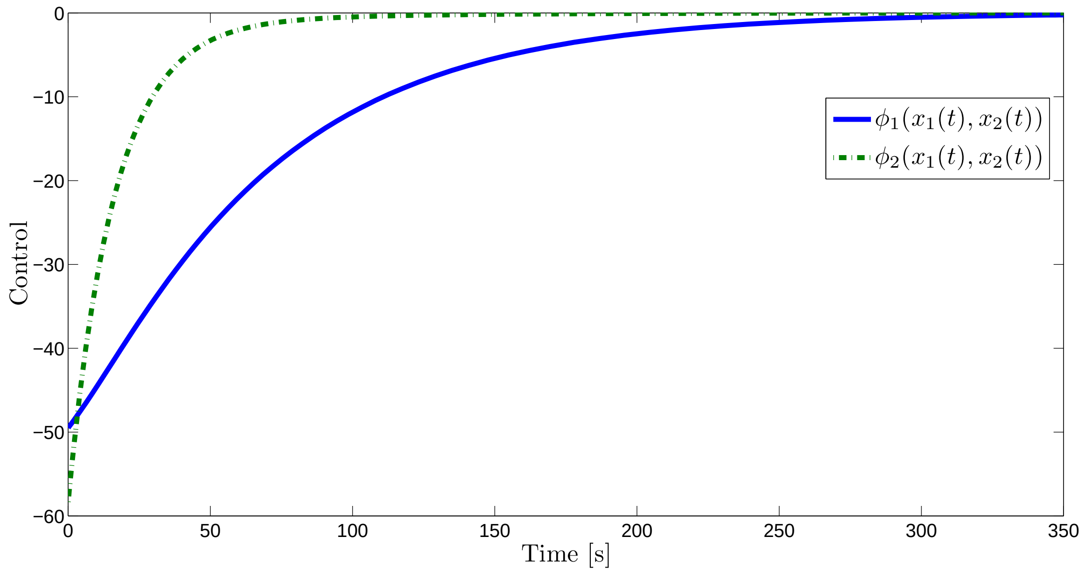

Figure 6.6: Control signal versus time.

trajectories of the controlled system versus time. Note that $x_1(t) = [\omega_1(t),\, \omega_2(t)]^{\mathrm{T}} \to 0$ as $t \to \infty$ and $x_2(t) = \omega_3(t) = \omega_{30}$, $t \geq 0$. Figure 6.6 shows the control signal versus time. Finally, $J(x_1(0), x_2(0), \phi(x_1(\cdot), x_2(\cdot))) = 1000\,\mathrm{Hz}^2$.

Chapter 7

Finite-Time Stabilization and Optimal Feedback Control

7.1. Introduction

The notions of asymptotic and exponential stability in dynamical systems theory imply convergence of the system trajectories to an equilibrium state over the infinite horizon. In many applications, however, it is desirable that a dynamical system possesses the property that trajectories that converge to a Lyapunov stable equilibrium state must do so in finite time rather than merely asymptotically. Most of the existing control techniques in the literature ensure that the closed-loop system dynamics of a controlled system are Lipschitz continuous, which implies uniqueness of system solutions in forward and backward times. Hence, convergence to an equilibrium state is achieved over an infinite time interval.

In order to achieve convergence in finite time, the closed-loop system dynamics need to be non-Lipschitzian giving rise to non-uniqueness of solutions in backward time. Uniqueness of solutions in forward time, however, can be preserved in the case of finite-time convergence. Sufficient conditions that ensure uniqueness of solutions in forward time in the absence of Lipschitz continuity are given in [1, 30, 74, 120]. In addition, it is shown in [27, Theorem 4.3, p. 59] that uniqueness of solutions in forward time along with continuity of the system dynamics ensure that the system solutions are continuous functions of the system initial conditions even when the dynamics are not Lipschitz continuous.

Finite-time convergence to a Lyapunov stable equilibrium, that is, finite-time stability, was first addressed by Roxin [101] and rigorously studied in [12,14] for time-invariant systems using continuous Lyapunov functions. Extensions of finite-time stability to time-varying nonlinear dynamical systems are presented in [50, 94]. Finite-time stabilization of second-order systems was considered in [10, 51]. More recently, researchers have considered finite-time stabilization of higher-order systems [54] as well as finite-time stabilization using output feedback [55]. Design of globally strongly stabilizing continuous controllers for linear and nonlinear systems using the theory of homogeneous systems was studied in [14, 98]. In addition, the universal controller given by Sontag [111] is extended in [93] to design a feedback controller for finite-time stabilization. Alternatively, discontinuous finite-time stabilizing feedback controllers have also been developed in the literature [34, 102, 103]. However, for practical implementations, discontinuous feedback controllers can lead to chattering due to system uncertainty or measurement noise, and hence, may excite unmodeled high-frequency system dynamics.

In [6] the current status of continuous-time, nonlinear nonquadratic optimal control problems was presented in a simplified and tutorial manner. The basic underlying ideas of the results in [6] are based on the fact that the steady-state solution of the Hamilton-Jacobi-Bellman equation is a Lyapunov function for the nonlinear system and thus guaranteeing both *asymptotic* stability and optimality [6,38]. Specifically, a feedback control problem over an infinite horizon involving a nonlinear-nonquadratic performance functional is considered. The performance functional is then evaluated in closed form as long as the nonlinear nonquadratic cost functional considered is related in a specific way to an underlying Lyapunov function that guarantees asymptotic stability of the nonlinear closed-loop system. This Lyapunov function is shown to be the solution of the steady-state Hamilton-Jacobi-Bellman equation. The overall framework provides the foundation for extending linear-quadratic control to nonlinear-nonquadratic problems.

Currently, optimal finite-time controllers are only obtainable using the maximum principle which generally does not yield feedback controllers. In this chapter, we extend the framework developed in [6] and [38] to address the problem of *optimal finite-time stabilization*, that is, the problem of finding state-feedback control laws that minimize a given performance measure and guarantee finite-time stability of the closed-loop system. Specifically, an optimal finite-time control problem is stated and sufficient Hamilton-Jacobi-Bellman conditions are used to characterize an optimal feedback controller. The steady-state solution of the Hamilton-Jacobi-Bellman equation is clearly shown to be a Lyapunov function for the closed-loop system that additionally satisfies a differential inequality involving a fractional power, and hence, guaranteeing both finite-time stability and optimality. Finally, we explore connections of our approach with inverse optimal control [32,65,92,95,108] wherein we parametrize a family of finite-time stabilizing sublinear controllers that minimize a derived cost functional involving subquadratic terms. Subquadratic performance criteria have been studied in [51,105,106] and have been shown to permit a unified treatment of a broad range of design goals.

7.2. Finite-time Stability Theory

Consider the nonlinear dynamical system given by

$$\dot{x}(t) = f(x(t)), \qquad x(0) = x_0, \qquad t \in \mathcal{I}_{x_0}, \tag{7.1}$$

where, for every $t \in \mathcal{I}_{x_0}$, $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, $\mathcal{I}_{x_0} \subseteq \overline{\mathbb{R}}_+$ is the maximal interval of existence of a solution $x(t)$ of (7.1), $0 \in \mathcal{I}_{x_0}$, $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$, $f(0) = 0$, and $f(\cdot)$ is continuous on $\mathcal{D}$. A continuously differentiable function $x : \mathcal{I}_{x_0} \to \mathcal{D}$ is said to be the *solution* of (7.1) on the interval $\mathcal{I}_{x_0} \subset \mathbb{R}$ if $x(\cdot)$ satisfies (7.1) for all $t \in \mathcal{I}_{x_0}$. The continuity of $f(\cdot)$ implies that, for every $x \in \mathcal{D}$, there exists $\tau_0 < 0 < \tau_1$ and a solution $x(\cdot)$ of (7.1) defined on the open interval (τ_0, τ_1) such that $x(0) = x$ [38, Th. 2.24]. A solution $t \mapsto x(t)$ is said to be *right maximally* defined if x cannot be extended (either uniquely or nonuniquely) forward in time. We assume that all right maximal solutions to (7.1) exist on $[0, \infty)$, and hence, we assume that (7.1) is *forward complete.* Recall that every bounded solution to (7.1) can be extended on a semi-infinite interval $[0, \infty)$ [38]. That is, if $x : [0, \tau) \to \mathcal{D}$ is the right maximally defined solution of (7.1) such that $x(t) \in \mathcal{D}_{\rm c}$ for all $t \in [0, \tau)$, where $\mathcal{D}_{\rm c} \subset \mathcal{D}$ is compact, then $\tau = \infty$ [38, Cor. 2.5].

We assume that (7.1) possesses unique solutions in forward time for all initial conditions except possibly the origin in the following sense. For every $x \in \mathcal{D}\backslash\{0\}$ there exists $\tau_x > 0$ such that, if $y_1 : [0, \tau_1) \to \mathcal{D}$ and $y_2 : [0, \tau_2) \to \mathcal{D}$ are two solutions of (7.1) with $y_1(0) = y_2(0) = x$, then $\tau_x \leq \min\{\tau_1, \tau_2\}$ and $y_1(t) = y_2(t)$ for all $t \in [0, \tau_x)$. Without loss of generality, we assume that for each x, τ_x is chosen to be the largest such number in $\overline{\mathbb{R}}_+$. In this case, given $x \in \mathcal{D}$, we denote by the continuously differentiable map $s^x(\cdot) \triangleq s(\cdot, x)$ the *trajectory* or the unique *solution curve* of (7.1) on $[0, \tau_x)$ satisfying $s(0, x) = x$. Sufficient conditions for forward uniqueness in the absence of Lipschitz continuity can be found in [1] [30, Section 10], [74], and [120, Section 1].

The following definition introduces the notion of finite-time stability.

Definition 7.1 [12]. Consider the nonlinear dynamical system (7.1). The zero solution $x(t) \equiv 0$ to (7.1) is *finite-time stable* if there exist an open neighborhood $\mathcal{N} \subseteq \mathcal{D}$ of the origin and a function $T : \mathcal{N}\backslash\{0\} \to (0, \infty)$, called the *settling-time function*, such that the following statements hold:

i) *Finite-time convergence.* For every $x \in \mathcal{N}\backslash\{0\}$, $s^x(t)$ is defined on $[0, T(x))$, $s^x(t) \in \mathcal{N}\backslash\{0\}$ for all $t \in [0, T(x))$, and $\lim_{t\to T(x)} s^x(t) = 0$.

ii) *Lyapunov stability.* For every $\varepsilon > 0$ there exists $\delta > 0$ such that $\mathcal{B}_\delta(0) \subset \mathcal{N}$ and for every $x \in \mathcal{B}_\delta(0)\backslash\{0\}$, $s^x(t) \in \mathcal{B}_\varepsilon(0)$ for all $t \in [0, T(x))$.

The zero solution $x(t) \equiv 0$ of (7.1) is *globally finite-time stable* if it is finite-time stable with $\mathcal{N} = \mathcal{D} = \mathbb{R}^n$.

Note that if the zero solution $x(t) \equiv 0$ to (7.1) is finite-time stable, then it is asymptotically stable, and hence, finite-time stability is a stronger condition than asymptotic stability. The following result shows that if the zero solution $x(t) \equiv 0$ to (7.1) is finite-time stable, then (7.1) has a unique solution $s(\cdot,\cdot)$ defined on $\overline{\mathbb{R}}_+ \times \mathcal{N}$ for every initial condition in an open neighborhood of the origin, including the origin, and $s(t,x) = 0$ for all $t \geq T(x)$, $x \in \mathcal{N}$, where $T(0) \triangleq 0$.

Proposition 7.2 [12]. Consider the nonlinear dynamical system (7.1). Assume that the zero solution $x(t) \equiv 0$ to (7.1) is finite-time stable and let $\mathcal{N} \subseteq \mathcal{D}$ and $T : \mathcal{N}\backslash\{0\} \to (0,\infty)$ be as in Definition 7.1. Then, $s(\cdot,\cdot)$ is a unique solution of (7.1) and is defined on $\overline{\mathbb{R}}_+ \times \mathcal{N}$, and $s(t,x) = 0$ for all $t \geq T(x)$, $x \in \mathcal{N}$, where $T(0) \triangleq 0$.

It follows from Proposition 7.2 that if the zero solution $x(t) \equiv 0$ to (7.1) is finite-time stable, then the solutions of (7.1) define a continuous *global semiflow* on $\mathcal{N}$; that is, $s : \overline{\mathbb{R}}_+ \times \mathcal{N} \to \mathcal{N}$ is jointly continuous and satisfies the consistency property $s(0,x) = x$ and the semigroup property $s(t, s(\tau,x)) = s(t+\tau, x)$ for every $x \in \mathcal{N}$ and t, $\tau \in \overline{\mathbb{R}}_+$. Furthermore, $s(\cdot,\cdot)$ satisfies $s(T(x)+t, x) = 0$ for all $x \in \mathcal{N}$ and $t \in \overline{\mathbb{R}}_+$. Finally, it also follows from Proposition 7.2 that we can extend $T(\cdot)$ to all of $\mathcal{N}$ by defining $T(0) \triangleq 0$. Now, by uniqueness of solutions it follows that $s(T(x)+t, x) = 0$, $t \in \mathbb{R}_+$, and hence, it is easy to see from Definition 7.1 that

$$T(x) = \inf\{t \in \overline{\mathbb{R}}_+ : s(t,x) = 0\}, \qquad x \in \mathcal{N}. \tag{7.2}$$

The next proposition shows that the settling time function of a finite-time stable system is continuous on $\mathcal{N}$ if and only if it is continuous at the origin.

Proposition 7.3 [12]. Consider the nonlinear dynamical system (7.1). Assume that the zero solution $x(t) \equiv 0$ to (7.1) is finite-time stable, let $\mathcal{N} \subseteq \mathcal{D}$ be as in Definition 7.1, and let $T : \mathcal{N} \to \mathbb{R}_+$ be the settling-time function. Then $T(\cdot)$ is continuous on $\mathcal{N}$ if and only if $T(\cdot)$ is continuous at $x = 0$.

Next, we provide sufficient conditions for finite-time stability of the nonlinear dynamical system given by (7.1). For the statement of the following result define $\dot{V}(x) \triangleq V'(x)f(x)$ for a continuously differentiable function $V : \mathcal{D} \to \mathbb{R}$.

Theorem 7.4 [12], [38, Th. 4.17]. Consider the nonlinear dynamical system (7.1). Assume there exist a continuously differentiable function $V : \mathcal{D} \to \mathbb{R}$, real numbers $c > 0$ and $\alpha \in (0,1)$, and a neighborhood $\mathcal{M} \subseteq \mathcal{D}$ of the origin such that

$$V(0) = 0, \tag{7.3}$$

$$V(x) > 0, \qquad x \in \mathcal{M}\backslash\{0\}, \tag{7.4}$$

$$\dot{V}(x) \leq -c\,(V(x))^{\alpha}, \qquad x \in \mathcal{M}\backslash\{0\}. \tag{7.5}$$

Then the zero solution $x(t) \equiv 0$, $t \geq 0$, to (7.1) is finite-time stable. Moreover, there exists an open neighborhood $\mathcal{N} \subset \mathcal{M}$ of the origin and a settling-time function $T : \mathcal{N} \to [0, \infty)$ such that

$$T(x_0) \leq \frac{1}{c(1-\alpha)}\,(V(x_0))^{1-\alpha}, \qquad x_0 \in \mathcal{N}, \tag{7.6}$$

and $T(\cdot)$ is continuous on $\mathcal{N}$. If, in addition, $\mathcal{D} = \mathbb{R}^n$, $V(\cdot)$ is radially unbounded, and (7.5) holds on $\mathbb{R}^n\backslash\{0\}$, then the zero solution $x(t) \equiv 0$ to (7.1) is globally finite-time stable.

Note that if the conditions of Theorem 7.4 are satisfied, then it follows from Proposition 7.2 that the solution $x(t)$ of (7.1) is defined for all $t \geq 0$, that is, $\mathcal{I}_{x_0} = [0, \infty)$, and is unique. Furthermore, since the regularity properties of the Lyapunov function and those of the settling-time function are related, and there exist finite-time stable systems that do not admit a continuously differentiable or even a Hölder continuous settling time function, a converse theorem to Theorem 7.4 can only ensure the existence of a continuous Lyapunov function. For details; see [12]. Alternatively, the authors in [93] provide conditions on the system dynamics for the settling-time function to be continuous leading to a stronger converse Lyapunov theorem involving a more regular function $V(\cdot)$ satisfying (7.5).

7.3. Optimal Finite-Time Stabilization

In the first part of this section, we provide connections between Lyapunov functions and nonquatratic cost evaluation. Specifically, we consider the problem of evaluating a nonlinear-nonquadratic performance measure that depends on the solution of the nonlinear dynamical system given by (7.1). In particular, we prove finite-time stability of (7.1) and we show that the nonlinear-nonquadratic performance measure

$$J(x_0) \triangleq \int_0^\infty L(x(t))\mathrm{d}t, \tag{7.7}$$

where $L : \mathcal{D} \to \mathbb{R}$ is continuous in x and $x(t)$, $t \geq 0$, satisfies (7.1), can be evaluated in a convenient form so long as (7.1) is related to an underlying Lyapunov function satisfying a differential inequality involving fractional powers.

Theorem 7.5. Consider the nonlinear dynamical system (7.1) with performance measure (7.7). Assume that there exist a continuously differentiable function $V : \mathcal{D} \to \mathbb{R}$, real numbers $c > 0$ and $\alpha \in (0, 1)$, and a neighborhood $\mathcal{M} \subseteq \mathcal{D}$ of the origin such that

$$V(0) = 0, \tag{7.8}$$

$$V(x) > 0, \qquad x \in \mathcal{M}\backslash\{0\}, \tag{7.9}$$

$$V'(x)f(x) \leq -c\,(V(x))^\alpha, \qquad x \in \mathcal{M}\backslash\{0\}, \tag{7.10}$$

$$L(x) + V'(x)f(x) = 0, \qquad x \in \mathcal{D}. \tag{7.11}$$

Then the zero solution $x(t) \equiv 0$, $t \geq 0$, to (7.1) is finite-time stable and there exists an open neighborhood $\mathcal{D}_0 \subset \mathcal{M}$ of the origin and a settling-time function $T : \mathcal{D}_0 \to [0, \infty)$ such that

$$T(x_0) \leq \frac{1}{c(1-\alpha)}(V(x_0))^{1-\alpha}, \qquad x_0 \in \mathcal{D}_0. \tag{7.12}$$

In addition,

$$J(x_0) = V(x_0), \qquad x_0 \in \mathcal{D}_0. \tag{7.13}$$

Finally, if $\mathcal{D} = \mathbb{R}^n$, $V(\cdot)$ is radially unbounded, and (7.10) holds on $\mathbb{R}^n\backslash\{0\}$, then the zero solution $x(t) \equiv 0$ to (7.1) is globally finite-time stable.

Proof: Let $x(t)$, $t \geq 0$, satisfy (7.1). Then it follows from (7.10) that

$$\dot{V}(x(t)) = V'(x(t))f(x(t)) \leq -c\,(V(x(t)))^{\alpha}, \qquad t \geq 0. \tag{7.14}$$

Thus, it follows from (7.8), (7.9), and Theorem 7.4 that the zero solution $x(t) \equiv 0$, $t \geq 0$, to (7.1) is finite-time stable and there exists an open neighborhood $\mathcal{D}_0 \subset \mathcal{M}$ of the origin and a settling-time function $T : \mathcal{D}_0 \to [0, \infty)$ such that (7.12) holds. Consequently, $x(t) \to 0$ as $t \to T(x_0)$ for all initial conditions $x_0 \in \mathcal{D}_0$. Now, since

$$0 = -\dot{V}(x(t)) + V'(x(t))f(x(t)), \qquad t \geq 0, \tag{7.15}$$

it follows from (7.11) that

$$L(x(t)) = -\dot{V}(x(t)) + L(x(t)) + V'(x(t))f(x(t)) = -\dot{V}(x(t)), \qquad t \geq 0. \tag{7.16}$$

Next, integrating (7.16) over $[0, t]$ yields

$$\int_0^t L(x(s))\mathrm{d}s = -V(x(t)) + V(x_0), \qquad t \geq 0. \tag{7.17}$$

Now, using (7.8) and letting $t \to \infty$ it follows from (7.17) that

$$\int_0^{\infty} L(x(s))\mathrm{d}s = -V\left(\lim_{t\to\infty} x(t)\right) + V(x_0), \tag{7.18}$$

and hence, (7.13) is a direct consequence of (7.18) using the fact that $\lim_{t\to T(x_0)} x(t) = \lim_{t\to\infty} x(t) = 0$. Finally, if $\mathcal{D} = \mathbb{R}^n$, $V(\cdot)$ is radially unbounded, and (7.10) holds on $\mathbb{R}^n\backslash\{0\}$, then global finite-time stability is a direct consequence of Theorem 7.4. ■

Next, we use the framework developed in Theorem 7.5 to obtain a characterization of optimal feedback controllers that guarantee closed-loop, finite-time stabilization. Specifically, sufficient conditions for optimality are given in a form that corresponds to a steady-state version of the Hamilton-Jacobi-Bellman equation. To address the problem of characterizing finite-time stabilizing feedback controllers, consider the controlled nonlinear dynamical system

$$\dot{x}(t) = F(x(t), u(t)), \qquad x(0) = x_0, \qquad t \geq 0, \tag{7.19}$$

where, for every $t \geq 0$, $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$, $u(t) \in U \subseteq \mathbb{R}^m$ with $0 \in U$, $F : \mathcal{D} \times U \to \mathbb{R}^n$ is jountly continuous in x and u, and $F(0,0) = 0$. The control $u(\cdot)$ in (7.19) is restricted to the class of *admissible* controls consisting of measurable functions $u(\cdot)$ such that $u(t) \in U$, $t \geq 0$.

A continuous function $\phi : \mathcal{D} \to U$ satisfying $\phi(0) = 0$ is called a *control law*. If $u(t) = \phi(x(t))$, $t \geq 0$, where $\phi(\cdot)$ is a control law and $x(t)$ satisfies (7.19), then we call $u(\cdot)$ a *feedback control law*. Note that the feedback control law is an admissible control since $\phi(\cdot)$ has values in U. Given a control law $\phi(\cdot)$ and a feedback control law $u(t) = \phi(x(t))$, $t \geq 0$, the *closed-loop system* (7.19) is given by

$$\dot{x}(t) = F(x(t), \phi(x(t))), \qquad x(0) = x_0, \qquad t \geq 0. \tag{7.20}$$

We now consider the problem of finite-time stabilization.

Definition 7.6. Consider the controlled dynamical system given by (7.19). The feedback control law $u = \phi(x)$ is *finite-time stabilizing* if the closed-loop system (7.20) is finite-time stable. Furthermore, the feedback control law $u = \phi(x)$ is *globally finite-time stabilizing* if the closed-loop system (7.20) is globally finite-time stable.

Next, we present a main theorem for finite-time stabilization characterizing feedback controllers that guarantee finite-time closed-loop stability and minimize a nonlinear-nonquadratic performance functional. For the statement of this result, let $L : \mathcal{D} \times U \to \mathbb{R}$ be jointly continuous in x and u, and define the set of finite-time regulation controllers given by

$$\mathcal{S}(x_0) \triangleq \{u(\cdot) : u(\cdot) \text{ is admissible and } x(\cdot) \text{ given by (7.19) satisfies } x(t) \to 0 \text{ as } t \to T\},$$

where $T > 0$. Note that since finite-time convergence is a stronger condition than asymptotic convergence, $\mathcal{S}(x_0)$ includes the set of all null convergent controllers.

Theorem 7.7. Consider the controlled nonlinear dynamical system (7.19) with

$$J(x_0, u(\cdot)) \triangleq \int_0^\infty L(x(t), u(t))\, \mathrm{d}t, \tag{7.21}$$

where $u(\cdot)$ is an admissible control. Assume that there exist a continuously differentiable function $V : \mathcal{D} \to \mathbb{R}$, real numbers $c > 0$ and $\alpha \in (0, 1)$, a neighborhood $\mathcal{M} \subseteq \mathcal{D}$ of the origin, and a continuous control law $\phi : \mathcal{D} \to U$ such that

$$\phi(0) = 0, \tag{7.22}$$

$$V(0) = 0, \tag{7.23}$$

$$V(x) > 0, \qquad x \in \mathcal{M}\backslash\{0\}, \tag{7.24}$$

$$V'(x)F(x, \phi(x)) \leq -c\,(V(x))^{\alpha}, \qquad x \in \mathcal{M}\backslash\{0\}, \tag{7.25}$$

$$L(x, \phi(x)) + V'(x)F(x, \phi(x)) = 0, \qquad x \in \mathcal{D}, \tag{7.26}$$

$$L(x, u) + V'(x)F(x, u) \geq 0, \qquad (x, u) \in \mathcal{D} \times U. \tag{7.27}$$

Then, with the feedback control $u = \phi(x)$, the zero solution $x(t) \equiv 0$, $t \geq 0$, to (7.19) is finite-time stable. Moreover, there exist an open neighborhood $\mathcal{D}_0 \subset \mathcal{M}$ of the origin and a settling-time function $T : \mathcal{D}_0 \to [0, \infty)$ such that

$$T(x_0) \leq \frac{1}{c(1-\alpha)}(V(x_0))^{1-\alpha}, \qquad x_0 \in \mathcal{D}_0, \tag{7.28}$$

and

$$J(x_0, \phi(x(\cdot))) = V(x_0), \qquad x_0 \in \mathcal{D}_0. \tag{7.29}$$

In addition, if $x_0 \in \mathcal{D}_0$, then the feedback control $u(\cdot) = \phi(x(\cdot))$ minimizes $J(x_0, u(\cdot))$ in the sense that

$$J(x_0, \phi(\cdot)) = \min_{u(\cdot)\in\mathcal{S}(x_0)} J(x_0, u(\cdot)). \tag{7.30}$$

Finally, if $\mathcal{D} = \mathbb{R}^{n_1}$, $U = \mathbb{R}^m$, $V(\cdot)$ is radially unbounded, and (7.25) holds on $\mathbb{R}^n\backslash\{0\}$, then the closed-loop system (7.20) is globally finite-time stable.

Proof: Local and global finite-time stability along with the existence of a settling-time function $T : \mathcal{D}_0 \to [0, \infty)$ such that (7.28) holds are a direct consequence of (7.23)–(7.25) by applying Theorem 7.4 to the closed-loop system given by (7.20). Furthermore, using (7.26), condition (7.29) is a restatement of (7.13) as applied to the closed-loop system.

Next, let $x_0 \in \mathcal{D}_0$, let $u(\cdot) \in \mathcal{S}(x_0)$, and let $x(t)$, $t \geq 0$, be the solution of (7.19). Then, it follows that

$$0 = -\dot{V}(x(t)) + V'(x(t))F(x(t), u(t)), \qquad t \geq 0. \tag{7.31}$$

Hence,

$$L(x(t), u(t)) = -\dot{V}(x(t)) + L(x(t), u(t)) + V'(x(t))F(x(t), u(t)), \qquad t \geq 0. \tag{7.32}$$

Thus, it follows from (7.32), (7.27), (7.29), (7.23), and the fact that $u(\cdot) \in \mathcal{S}(x_0)$, that

$$\begin{aligned}
\int_0^\infty L(x(t), u(t))\mathrm{d}t &= \int_0^\infty -\dot{V}(x(t))\mathrm{d}t + \int_0^\infty \left[L(x(t), u(t)) + V'(x)F(x(t), u(t))\right]\mathrm{d}t \\
&\geq \int_0^\infty -\dot{V}(x(t))\mathrm{d}t \\
&= -\lim_{t\to\infty} V(x(t)) + V(x_0) \\
&= -V\left(\lim_{t\to\infty} x(t)\right) + V(x_0) \\
&= -V\left(\lim_{t\to T} x(t)\right) + V(x_0) \\
&= J(x_0, \phi(x(\cdot))),
\end{aligned} \tag{7.33}$$

which yields (7.30). ■

Note that (7.26) is the steady-state, Hamilton-Jacobi-Bellman equation for the controlled nonlinear dynamical system (7.19) with performance criterion (7.21). Furthermore, conditions (7.26) and (7.27) guarantee optimality with respect to the set of admissible finite-time stabilizing controllers $\mathcal{S}(x_0)$. However, it is important to note that an explicit characterization of $\mathcal{S}(x_0)$ is not required. In addition, the optimal finite-time stabilizing *feedback* control law $u = \phi(x)$ is independent of the initial condition x_0 and is given by

$$\phi(x) = \underset{u\in\mathcal{S}(x_0)}{\arg\min}\left[L(x, u) + \frac{\partial V(x)}{\partial x}F(x, u)\right]. \tag{7.34}$$

Finally, setting $\mathcal{M} = \mathcal{D}$ in Theorem 7.7 and replacing (7.25) with

$$V'(x)F(x, \phi(x)) < 0, \qquad x \in \mathcal{D}, \tag{7.35}$$

Theorem 7.7 reduces to Theorem 8.2 of [38] characterizing the classical asymptotically stabilizing optimal control problem for time-invariant systems on an infinite interval.

7.4. Finite-Time Stabilization for Affine Dynamical Systems and Connections to Inverse Optimal Control

In this section, we specialize the results of Section 7.3 to nonlinear affine dynamical systems of the form

$$\dot{x}(t) = f(x) + G(x)u(t), \qquad x(0) = x_0, \qquad t \geq 0, \tag{7.36}$$

where, for every $t \geq 0$, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $f : \mathbb{R}^n \to \mathbb{R}^n$ and $G : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are such that $f(\cdot)$ and $G(\cdot)$ are continuous in x and $f(0) = 0$. Furthermore, we consider performance integrands $L(x, u)$ of the form

$$L(x, u) = L_1(x) + L_2(x)u + u^{\mathrm{T}}R_2(x)u, \qquad (x, u) \in \mathbb{R}^n \times \mathbb{R}^m, \tag{7.37}$$

where $L_1 : \mathbb{R}^n \to \mathbb{R}$, $L_2 : \mathbb{R}^n \to \mathbb{R}^{1\times m}$ is continuous on $\mathbb{R}^n$, and $R_2(x) > 0$, $x \in \mathbb{R}^n$, is continuous on $\mathbb{R}^n$, so that (7.21) becomes

$$J(x_0, u(\cdot)) = \int_0^\infty \left[L_1(x(t)) + L_2(x(t))u(t) + u^{\mathrm{T}}(t)R_2(x)u(t)\right] \mathrm{d}t. \tag{7.38}$$

Theorem 7.8. Consider the controlled nonlinear affine dynamical system (7.36) with performance measure (7.38). Assume that there exist a continuously differentiable, radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}$ and real numbers $c > 0$ and $\alpha \in (0, 1)$ such that

$$V(0) = 0, \tag{7.39}$$

$$V(x) > 0, \qquad x \in \mathbb{R}^n\backslash\{0\}, \tag{7.40}$$

$$V'(x)\left[f(x) - \frac{1}{2}G(x)R_2^{-1}(x)L_2^{\mathrm{T}}(x) - \frac{1}{2}G(x)R_2^{-1}(x)G^{\mathrm{T}}(x)V'^{\mathrm{T}}(x)\right] \leq -c\,(V(x))^{\alpha}, \;\; x \in \mathbb{R}^n, \tag{7.41}$$

$$L_2(0) = 0, \tag{7.42}$$

$$0 = L_1(x) + V'(x)f(x) - \frac{1}{4}\left[V'(x)G(x) + L_2(x)\right]R_2^{-1}(x)\left[V'(x)G(x) + L_2(x)\right]^{\mathrm{T}}, \;\; x \in \mathbb{R}^n. \tag{7.43}$$

Then, with the feedback control

$$u = \phi(x) = -\frac{1}{2}R_2^{-1}(x)\big[L_2(x) + V'(x)G(x)\big]^{\mathrm{T}}, \tag{7.44}$$

the zero solution $x(t) \equiv 0$, $t \geq 0$, to

$$\dot{x}(t) = f(x) + G(x(t))\phi(x(t)), \quad x(0) = x_0, \quad t \geq 0, \tag{7.45}$$

is globally finite-time stable. Moreover, there exists a settling-time function $T : \mathbb{R}^n \to [0, \infty)$ such that

$$T(x_0) \leq \frac{1}{c(1-\alpha)}(V(x_0))^{1-\alpha}, \qquad x_0 \in \mathbb{R}^n, \tag{7.46}$$

and the performance measure (7.38) is minimized in the sense of (7.30). Finally,

$$J(x_0, \phi(x(\cdot)) = V(x_0), \qquad x_0 \in \mathbb{R}^n. \tag{7.47}$$

Proof: The result is a direct consequence of Theorem 7.7 with $\mathcal{D} = \mathbb{R}^n$, $U = \mathbb{R}^m$, $F(x,u) = f(x) + G(x)u$, and $L(x,u) = L_1(x) + L_2(x)u + u^{\mathrm{T}}R_2(x)u$. Specifically, the feedback control law (7.44) follows from (7.34) by setting

$$\frac{\partial}{\partial u}\big[L_1(x) + L_2(x)u + u^{\mathrm{T}}R_2(x)u + V'(x)\left(f(x) + G(x)u\right)\big] = 0. \tag{7.48}$$

Now, with $u = \phi(x)$ given by (7.44), conditions (7.39)–(7.41) and (7.43) imply (7.23)–(7.26), respectively.

Next, since $V(\cdot)$ is continuously differentiable and, by (7.39) and (7.40), $V(0)$ is a local minimum of $V(\cdot)$, it follows that $V'(0) = 0$, and hence, it follows from (7.42) and (7.44) that $\phi(0) = 0$, which implies (7.22). Finally, it follows from (7.26), (7.37), and (7.44) that

$$\begin{aligned}
&L(x,u) + V'(x)[f(x) + G(x)u] \\
&\qquad = L(x,u) + V'(x)[f(x) + G(x)u] - L(x,\phi(x)) - V'(x)[f(x) + G(x)\phi(x)] \\
&\qquad = [L_2(x) + V'(x)G(x)]\left(u - \phi(x)\right) + u^{\mathrm{T}}R_2(x)u - \phi^{\mathrm{T}}(x)R_2(x)\phi(x) \\
&\qquad = -2\phi^{\mathrm{T}}(x)R_2(x)(u - \phi(x)) + u^{\mathrm{T}}R_2(x)u - \phi^{\mathrm{T}}(x)R_2(x)\phi(x)
\end{aligned}$$

$$= [u - \phi(x)]^{\mathrm{T}} R_2(x)[u - \phi(x)]$$
$$\geq 0, \qquad x \in \mathbb{R}^n, \tag{7.49}$$

which implies (7.27). The result now follows as a direct consequence of Theorem 7.7. ∎

Next, we construct state feedback controllers for nonlinear affine in the control dynamical systems that are predicated on an *inverse optimal control problem* [32, 65, 92, 95, 108]. In particular, to avoid the complexity in solving the steady-state, Hamilton-Jacobi-Bellman equation (7.43) we do not attempt to minimize a given cost functional, but rather, we parameterize a family of stabilizing controllers that minimize some derived cost functional that provides flexibility in specifying the control law. The performance integrand is shown to explicitly depend on the nonlinear system dynamics, the Lyapunov function of the closed-loop system, and the stabilizing feedback control law, wherein the coupling is introduced via the Hamilton-Jacobi-Bellman equation. Hence, by varying the parameters in the Lyapunov function and the performance integrand, the proposed framework can be used to characterize a class of globally finite-time stabilizing controllers that can meet closed-loop system response constraints.

Theorem 7.9. Consider the controlled nonlinear affine dynamical system (7.36) with performance measure (7.38). Assume that there exist a continuously differentiable, radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}$ and real numbers $c > 0$ and $\alpha \in (0, 1)$ such that

$$V(0) = 0, \tag{7.50}$$

$$V(x) > 0, \qquad x \in \mathbb{R}^n \backslash \{0\}, \tag{7.51}$$

$$V'(x)\left[f(x) - \frac{1}{2}G(x)R_2^{-1}(x)L_2^{\mathrm{T}}(x) - \frac{1}{2}G(x)R_2^{-1}(x)G^{\mathrm{T}}(x)V'^{\mathrm{T}}(x)\right] \leq -c\,(V(x))^{\alpha}, \quad x \in \mathbb{R}^n, \tag{7.52}$$

$$L_2(0) = 0. \tag{7.53}$$

Then, with the feedback control

$$u = \phi(x) = -\frac{1}{2}R_2^{-1}(x)\left[L_2(x) + V'(x)G(x)\right]^{\mathrm{T}}, \tag{7.54}$$

the zero solution $x(t) \equiv 0$, $t \geq 0$, to

$$\dot{x}(t) = f(x) + G(x(t))\phi(x(t)), \quad x(0) = x_0, \quad t \geq 0, \tag{7.55}$$

is globally finite-time stable. Moreover, there exists a settling-time function $T : \mathbb{R}^n \to [0, \infty)$ such that

$$T(x_0) \leq \frac{1}{c(1-\alpha)}(V(x_0))^{1-\alpha}, \qquad x_0 \in \mathbb{R}^n, \tag{7.56}$$

and the performance functional (7.38), with

$$L_1(x) = \phi^{\mathrm{T}}(x)R_2(x)\phi(x) - V'(x)f(x), \tag{7.57}$$

is minimized in the sense of (7.30). Finally,

$$J(x_0, \phi(x(\cdot)) = V(x_0), \qquad x_0 \in \mathbb{R}^n. \tag{7.58}$$

Proof: The proof is similar to the proof of Theorem 7.8 and, hence, is omitted. ■

Remark 7.10. As noted in the Introduction, the universal controller given by Sontag's formula [111] has been extended in [93] to design finite-time feedback controllers. Even though this result can be used to construct inverse optimal value functions and inverse optimal finite-time feedback control laws using the ideas presented in [108], such connections are not explored in [93].

7.5. Illustrative Numerical Examples

In this section, we provide two numerical examples to highlight the optimal and inverse optimal finite-time stabilization framework developed in the chapter.

7.5.1. Finite-Time Stabilization of a Controlled Scalar Nonlinear System

Consider the controlled nonlinear dynamical system given by

$$\dot{x}(t) = -x^{\frac{1}{3}}(t) - 2^{\frac{2}{3}}x^{\frac{1}{3}}(t) + 2u(t), \qquad x(0) = x_0, \qquad t \geq 0, \tag{7.59}$$

where $x \in \mathbb{R}$. For this example, we seek a state-feedback controller $u = \phi(x)$ such that the performance measure

$$J(x_0, u(\cdot)) = \int_0^\infty \left[\frac{4}{3}x^{\frac{2}{3}}(t) + u^2(t)\right] \mathrm{d}t \tag{7.60}$$

is minimized in the sense of (7.30) and (7.59) is globally finite-time stable. Note that (7.59) with performance measure (7.60) can be cast in the form of (7.36) with performance measure (7.38). In this case, Theorem 7.8 can be applied with $n = 1$, $m = 1$, $f(x) = -x^{\frac{1}{3}} - 2^{\frac{2}{3}}x^{\frac{1}{3}}$, $G(x) = 2$, $L_1(x) = \frac{4}{3}x^{\frac{2}{3}}$, $L_2(x) = 0$, and $R_2(x) = 1$ to characterize the optimal finite-time stabilizing controller.

Specifically, (7.43) reduces to

$$0 = \frac{4}{3}x^{\frac{2}{3}} + V'(x)\big(-x^{\frac{1}{3}} - 2^{\frac{2}{3}}x^{\frac{1}{3}}\big) - V'^2(x), \qquad x \in \mathbb{R}, \tag{7.61}$$

which implies that

$$V'(x) = \frac{4}{3}x^{\frac{1}{3}}. \tag{7.62}$$

Furthermore, since $V(0) = 0$,

$$V(x) = x^{\frac{4}{3}}, \tag{7.63}$$

which verifies (7.40). In addition, note that

$$V'(x)f(x) - 2V'^2(x) = -\frac{28}{9}x^{\frac{2}{3}} = -\frac{28}{9}V^{\frac{1}{2}}(x), \tag{7.64}$$

and hence, (7.41) is satisfied with $c = \frac{28}{9}$ and $\alpha = \frac{1}{2}$.

Since all of the conditions of Theorem 7.8 hold, it follows that the feedback control (7.44) given by

$$u = \phi(x) = -V'(x) = -\frac{4}{3}x^{\frac{1}{3}}, \qquad x \in \mathbb{R}, \tag{7.65}$$

guarantees that the dynamical system (7.59) is globally finite-time stable. Moreover, there exists a settling-time function $T : \mathbb{R} \to [0, \infty)$ such that

$$T(x_0) \leq \frac{9}{14}x_0^{\frac{2}{3}}, \qquad x_0 \in \mathbb{R}, \tag{7.66}$$

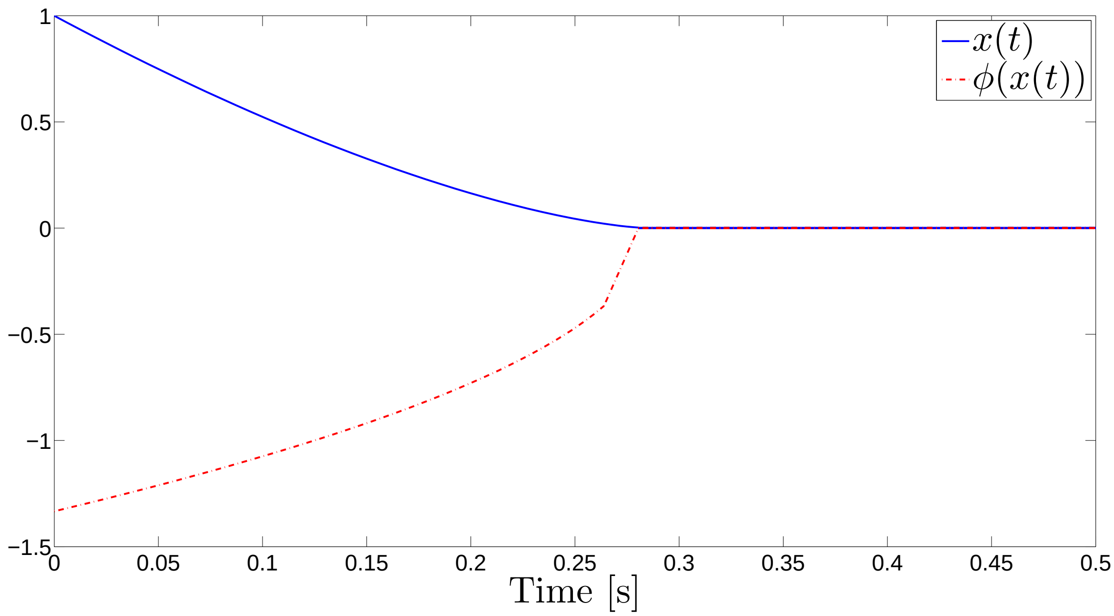

Figure 7.1: Closed-loop system trajectories and control versus time.

and

$$J(x_0, \phi(x(\cdot))) = x_0^{\frac{4}{3}}, \qquad x_0 \in \mathbb{R}. \tag{7.67}$$

Figure 7.1 shows the state trajectory of the controlled system versus time for $x_0 = 1$. Note that $x(t) = 0$ for $t = 0.2804\,\mathrm{s} < T(1) = \frac{9}{14}\,\mathrm{s}$. In addition, Figure 7.1 shows the control signal versus time. Finally, note that $J(x_0, \phi(x(\cdot))) = 1$. △

7.5.2. Inverse Optimal Control for Spin Stabilization of an Axisymmetric Spacecraft

Consider a spacecraft with one axis of symmetry given by [114, p. 753]

$$\dot{\omega}_1(t) = I_{23}\omega_3\omega_2(t) + u_1(t), \qquad \omega_1(0) = \omega_{10}, \qquad t \geq 0, \tag{7.68}$$

$$\dot{\omega}_2(t) = -I_{23}\omega_3\omega_1(t) + u_2(t), \qquad \omega_2(0) = \omega_{20}, \tag{7.69}$$

where $I_{23} \triangleq (I_2 - I_3)/I_1$, I_1, I_2, and I_3 are the principal moments of inertia of the spacecraft such that $0 < I_1 = I_2 < I_3$, $\omega_1 : [0, \infty) \to \mathbb{R}$, $\omega_2 : [0, \infty) \to \mathbb{R}$, and $\omega_3 \in \mathbb{R}$ denote the components of the angular velocity vector with respect to a given inertial reference frame expressed in a central body reference frame, and u_1 and u_2 are the spacecraft control

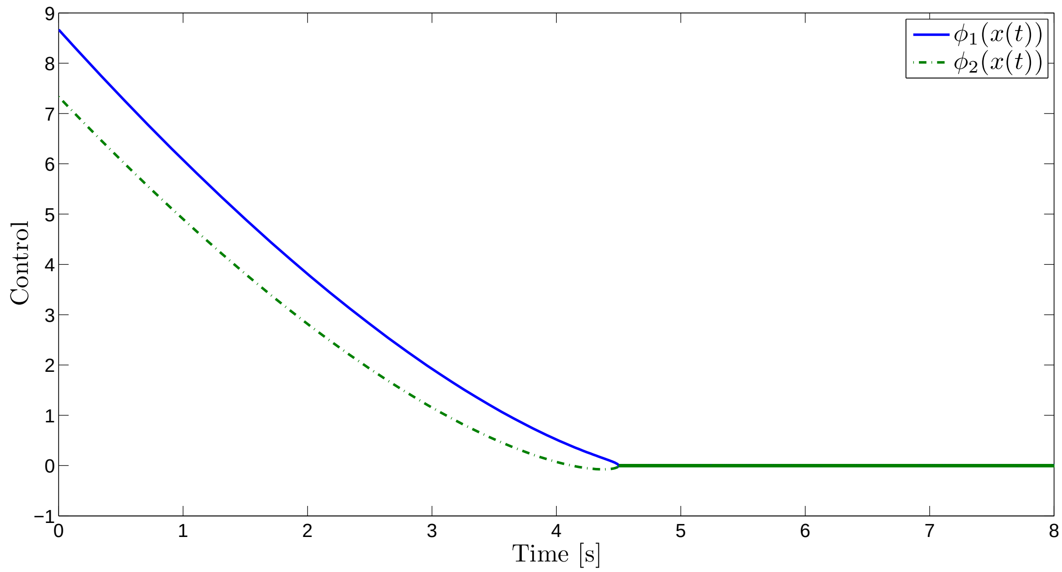

Figure 7.2: Control signal versus time.

moments.

For this example, we apply Theorem 7.9 to find an *inverse optimal* globally finite-time stabilizing control law $u = [u_1,\, u_2]^{\mathrm{T}} = \phi(x)$, where $x = [\omega_1,\, \omega_2]^{\mathrm{T}}$, such that the angular velocities $\omega_1(\cdot)$ and $\omega_2(\cdot)$ are regulated to zero in finite time, that is, the dynamical system (7.68) and (7.69) is globally finite-time stable, and hence, the spacecraft is spin-stabilized about its third principal inertia axis. Note that (7.68) and (7.69) can be cast in the form of (7.36), with $n = 2$, $m = 2$, $f(x) = \left[I_{23}\omega_3\omega_2,\, -I_{23}\omega_3\omega_1\right]^{\mathrm{T}}$, and $G(x) = I_2$.

To construct an inverse optimal controller for (7.68) and (7.69), let

$$V(x_1, x_2) = p^{\frac{2}{3}} \left(x^{\mathrm{T}}x\right)^{\frac{2}{3}}, \tag{7.70}$$

where $p > 0$, $L(x, u) = L_1(x) + L_2(x)u + u^{\mathrm{T}}u$, and

$$L_2(x) = 2\left[-I_{23}\omega_3\omega_2, \quad I_{23}\omega_3\omega_1\right]. \tag{7.71}$$

Now, the inverse optimal control law (7.54) is given by

$$u = \phi(x) = \left[\, -\frac{2}{3}p^{\frac{2}{3}}\omega_1\|x\|^{-\frac{2}{3}} - I_{23}\omega_3\omega_2,\; -\frac{2}{3}p^{\frac{2}{3}}\omega_2\|x\|^{-\frac{2}{3}} + I_{23}\omega_3\omega_1\right]^{\mathrm{T}} \tag{7.72}$$

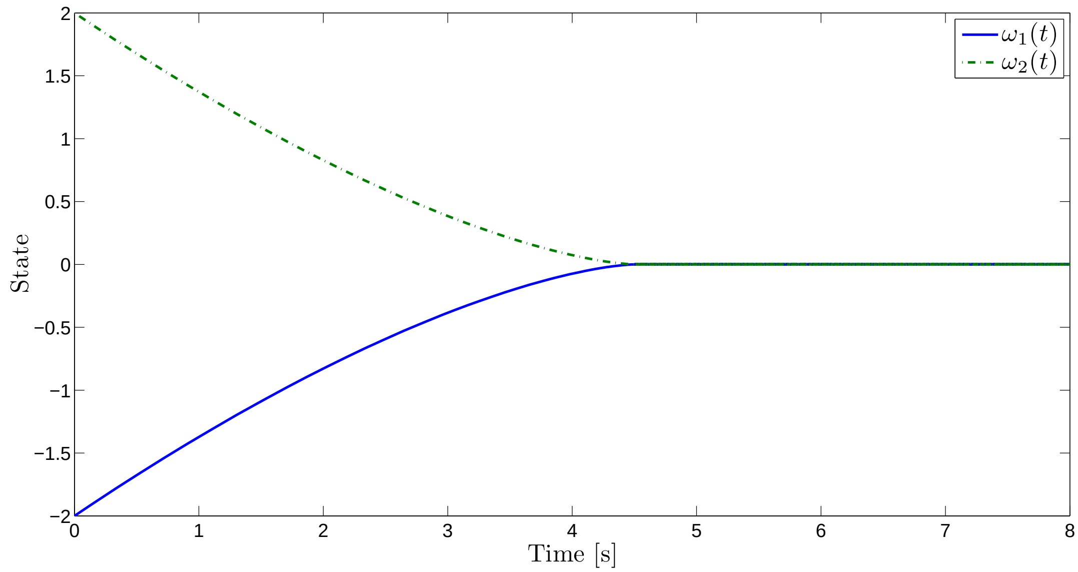

Figure 7.3: Closed-loop system trajectories versus time.

and the performance functional (7.38), with

$$L_1(x) = \left(-\frac{2}{3}p^{\frac{2}{3}}\omega_1\|x\|^{-\frac{2}{3}} - I_{23}\omega_3\omega_2 \right)^2 + \left(-\frac{2}{3}p^{\frac{2}{3}}\omega_2\|x\|^{-\frac{2}{3}} + I_{23}\omega_3\omega_1 \right)^2, \tag{7.73}$$

is minimized in the sense of (7.30). Furthermore, since (7.50) and (7.51) hold and, since

$$\begin{aligned} V'(x)&\left[f(x) - \frac{1}{2}G(x)L_2^{\mathrm{T}}(x) - \frac{1}{2}G(x)G^{\mathrm{T}}(x)V'^{\mathrm{T}}(x)\right] \\ &= -\frac{8}{9}p^{\frac{4}{3}}\left(\omega_1^2 + \omega_2^2\right)^{\frac{1}{3}} \\ &= -\frac{8}{9}p(V(x))^{\frac{1}{2}}, \qquad x \in \mathbb{R}^2, \end{aligned} \tag{7.74}$$

(7.52) is verified with $c = \frac{8}{9}p$ and $\alpha = \frac{1}{2}$. Hence, with the feedback control law $\phi(x)$ given by (7.70), the closed-loop system (7.68) and (7.69) is globally finite-time stable. Moreover, there exists a settling-time function $T : \mathbb{R}^2 \to [0, \infty)$ such that

$$T(x_0) \leq \frac{9}{4}p^{-\frac{2}{3}}\left(\omega_{10}^2 + \omega_{20}^2\right)^{\frac{1}{3}}, \qquad x_0 \in \mathbb{R}^2, \tag{7.75}$$

where $x_0 = [\omega_{10}, \omega_{20}]^{\mathrm{T}}$, and

$$J(x_0, \phi(x(\cdot))) = p^{\frac{2}{3}}\left(\omega_{10}^2 + \omega_{20}^2\right)^{\frac{2}{3}}, \qquad x_0 \in \mathbb{R}^2. \tag{7.76}$$

Let $I_1 = I_2 = 4\,\mathrm{kg}\cdot\mathrm{m}^2$, $I_3 = 20\,\mathrm{kg}\cdot\mathrm{m}^2$, $\omega_{10} = -2\,\mathrm{Hz}$, $\omega_{20} = 2\,\mathrm{Hz}$, $\omega_3 = 1\,\mathrm{Hz}$, and $p = 1$, Figure 7.3 shows the state trajectories of the controlled system versus time. Note that $x(t) = 0$ for $t = 4.4717\,\mathrm{s} < T(x_0) = \frac{9}{2}\,\mathrm{s}$. Figure 7.2 shows the control signal versus time. Finally, $J(x(0), \phi(x(\cdot))) = 4\,\mathrm{Hz}^2$.

Chapter 8

Finite-Time Partial Stability and Stabilization, and Optimal Feedback Control

8.1. Introduction

In this chapter, we extend the framework developed in [6] and Chapters 6 and 7 to address the problem of optimal *finite-time stabilization*, that is, the problem of finding state-feedback control laws that minimize a given performance measure and guarantee finite-time stability of the closed-loop system. In addition, we address the problem of optimal *partial-state stabilization*, wherein stabilization with respect to a subset of the system state variables is desired. Even though finite-time stabilization [10, 14, 34, 51, 54, 55, 98, 102, 103] and partial-state stabilization [88, 113] have been considered in the literature as separate problems as well as a combined problem [67, 68, 70], the problem of *optimal finite-time*, *partial-state stabilization* has not been addressed in the literature.

Finite-time stabilization of second-order systems was considered in [10, 51], whereas the authors in [54, 55] consider finite-time stabilization of higher-order systems as well as finite-time stabilization using output feedback. Design of globally strongly stabilizing continuous controllers for linear and nonlinear systems using the theory of homogeneous systems was studied in [14, 98]. Finite-time partial stabilization of chained systems are considered in [67, 68], whereas finite-time partial stabilizability using continuous and discontinuous homogeneous state feedback controllers is considered in [70]. Discontinuous finite-time stabilizing feedback controllers have also been developed in the literature [34, 102, 103]. Alternatively, sliding mode (typically discontinuous) control design has also been used to guarantee finite-time convergence and more recently finite-time stability; see [9] and the numerous references therein. However, for practical implementation, discontinuous feedback controllers can lead to chattering due to system uncertainty or measurement noise, and hence, may excite unmodeled high-frequency system dynamics.

The problem of partial stabilization has also been considered in the literature. Specifically, in [69,113] the authors construct controllers for spacecraft stabilization, wherein asymptotic stability of an equilibrium point is sought while requiring Lyapunov stability of the remaining closed-loop system states of the spacecraft. In [88], the authors consider partial stabilization of rotating machinery with mass imbalance, wherein motion stabilization with respect to a subspace instead of the origin is sought.

In this chapter, we consider a notion of optimality that is directly related to a given Lyapunov function that is positive definite and decrescent with respect to part of the system state, and satisfies a differential inequality involving fractional powers. Specifically, an *optimal* finite-time, partial-state stabilization control problem is stated and sufficient Hamilton-Jacobi-Bellman conditions are used to characterize an optimal feedback controller. The steady-state solution of the Hamilton-Jacobi-Bellman equation is clearly shown to be a Lyapunov function for part of the closed-loop system state that guarantees both finite-time partial stability and optimality. In addition, we explore connections of our approach with inverse optimal control [32, 65, 92, 95], wherein we parametrize a family of finite-time, partial-state stabilizing sublinear controllers that minimize a derived cost functional involving subquadratic terms. Subquadratic performance criteria have been studied in [105, 106] and have been shown to permit a unified treatment of a broad range of design goals. Another important application of partial stability and partial stabilization theory is the unification it provides between time-invariant stability theory and stability theory for time-varying systems [23, 38]. We exploit this unification and specialize our results to address the problem of optimal finite-time control for nonlinear time-varying dynamical systems.

8.2. Mathematical Background

consider nonlinear dynamical systems of the form

$$\dot{x}_1(t) = f_1(x_1(t), x_2(t)), \qquad x_1(0) = x_{10}, \qquad t \in \mathcal{I}_{x_0}, \tag{8.1}$$

$$\dot{x}_2(t) = f_2(x_1(t), x_2(t)), \qquad x_2(0) = x_{20}, \tag{8.2}$$

where, for every $t \in \mathcal{I}_{x_0}$, $x_1(t) \in \mathcal{D} \subseteq \mathbb{R}^{n_1}$ and $x_2(t) \in \mathbb{R}^{n_2}$, $\mathcal{I}_{x_0} \subset \mathbb{R}$ is the maximal interval of existence of a solution $x(t) \triangleq [x_1^{\mathrm{T}}(t), x_2^{\mathrm{T}}(t)]^{\mathrm{T}}$ of (8.1) and (8.2) with initial condition $x_0 \triangleq [x_{10}^{\mathrm{T}}, x_{20}^{\mathrm{T}}]^{\mathrm{T}}$, $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$, $f_1 : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_1}$ is such that, for every $(x_1, x_2) \in \mathcal{D} \times \mathbb{R}^{n_2}$, $f_1(0, x_2) = 0$ and $f_1(\cdot,\cdot)$ is jointly continuous in x_1 and x_2, and $f_2 : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_2}$ is such that, for every $(x_1, x_2) \in \mathcal{D} \times \mathbb{R}^{n_2}$, $f_2(\cdot,\cdot)$ is jointly continuous in x_1 and x_2. A continuously differentiable function $x : \mathcal{I}_{x_0} \to \mathcal{D} \times \mathbb{R}^{n_2}$ is said to be a *solution* of (8.1) and (8.2) on the interval $\mathcal{I}_{x_0} \subset \mathbb{R}$ if $x(\cdot) = [x_1^{\mathrm{T}}(\cdot), x_2^{\mathrm{T}}(\cdot)]^{\mathrm{T}}$ satisfies (8.1) and (8.2) for all $t \in \mathcal{I}_{x_0}$. If $x(\cdot) = [x_1^{\mathrm{T}}(\cdot), x_2^{\mathrm{T}}(\cdot)]^{\mathrm{T}}$ is a solution of (8.1) and (8.2) on the interval $\mathcal{I}_{x_0} \subset \mathbb{R}$, then $x_1(\cdot)$ is the solution of (8.1) and $x_2(\cdot)$ is the solution of (8.2).

The joint continuity of $f(\cdot,\cdot) = [f_1^{\mathrm{T}}(\cdot,\cdot), f_2^{\mathrm{T}}(\cdot,\cdot)]^{\mathrm{T}}$ implies that, for every $(x_1, x_2) \in \mathcal{D} \times \mathbb{R}^{n_2}$, there exists $\tau_0 < 0 < \tau_1$ and a solution $[x_1^{\mathrm{T}}(\cdot), x_2^{\mathrm{T}}(\cdot)]^{\mathrm{T}}$ of (8.1) and (8.2) defined on the open interval (τ_0, τ_1) such that $[x_1^{\mathrm{T}}(0), x_2^{\mathrm{T}}(0)]^{\mathrm{T}} = [x_1^{\mathrm{T}}, x_2^{\mathrm{T}}]^{\mathrm{T}}$ [38, Th. 2.24]. A solution $t \mapsto [x_1^{\mathrm{T}}(t), x_2^{\mathrm{T}}(t)]^{\mathrm{T}}$ is said to be *right maximally* defined if $[x_1^{\mathrm{T}}, x_2^{\mathrm{T}}]^{\mathrm{T}}$ cannot be extended (either uniquely or nonuniquely) forward in time. We assume that all right maximal solutions to (8.1) and (8.2) exist on $[0, \infty)$, and hence, we assume that (8.1) and (8.2) is *forward complete*. Recall that every bounded solution to (8.1) and (8.2) can be extended on a semi-infinite interval $[0, \infty)$ [38]. That is, if $x : [0, \tau_{x_0}) \to \mathcal{D} \times \mathbb{R}^{n_2}$ is the right maximally defined solution of (8.1) and (8.2) such that $x(t) = [x_1^{\mathrm{T}}(t), x_2^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathcal{D}_{\mathrm{c}} \times \mathcal{Q}_{\mathrm{c}}$ for all $t \in [0, \tau_{x_0})$, where $\mathcal{D}_{\mathrm{c}} \subset \mathcal{D}$ and $\mathcal{Q}_{\mathrm{c}} \subset \mathbb{R}^{n_2}$ are compact, then $\tau_{x_0} = \infty$ [38, Cor. 2.5].

We assume that the nonlinear dynamical system given by (8.1) and (8.2) possesses unique solutions in forward time for all initial conditions except possibly at $x_1 = 0$ in the following sense. For every $(x_1, x_2) \in \mathcal{D}\backslash\{0\} \times \mathbb{R}^{n_2}$ there exists $\tau_x > 0$, where $x = [x_1^{\mathrm{T}}, x_2^{\mathrm{T}}]^{\mathrm{T}}$, such that, if $y_{\mathrm{I}} : [0, \tau_1) \to \mathcal{D} \times \mathbb{R}^{n_2}$ and $y_{\mathrm{II}} : [0, \tau_2) \to \mathcal{D} \times \mathbb{R}^{n_2}$ are two solutions of (8.1) and (8.2) with $y_{\mathrm{I}}(0) = y_{\mathrm{II}}(0) = x$, then $\tau_x \leq \min\{\tau_1, \tau_2\}$ and $y_{\mathrm{I}}(t) = y_{\mathrm{II}}(t)$ for all $t \in [0, \tau_x)$. Without loss of generality, we assume that, for every (x_1, x_2), τ_x is chosen to be the largest such number in $\overline{\mathbb{R}}_+$. In this case, given $x = [x_1^{\mathrm{T}}, x_2^{\mathrm{T}}]^{\mathrm{T}} \in \mathcal{D} \times \mathbb{R}^{n_2}$, we denote by the continuously differentiable map $s^x(\cdot) \triangleq s(\cdot, x_1, x_2)$ the *trajectory* or the unique *solution curve* of (8.1) and (8.2) on $[0, \tau_x)$ satisfying $s(0, x_1, x_2) = [x_1^{\mathrm{T}}, x_2^{\mathrm{T}}]^{\mathrm{T}}$ and we denote by $s_1^x(\cdot)$ the *partial trajectory* or the unique *solution curve* of (8.1) on $[0, \tau_x)$. Sufficient conditions for forward uniqueness in the absence of Lipschitz continuity can be found in [1] [30, Section 10], [74], and [120, Section 1]. Finally, we assume that given a continuously differentiable function $x_1 : [0, \infty) \to \mathbb{R}^{n_1}$, the solution $x_2(t)$, $t \geq 0$, to (8.2) is unique.

The following definitions introduce the notion of finite-time partial stability.

Definition 8.1. The nonlinear dynamical system (8.1) and (8.2) is *finite-time stable with respect to* x_1 if there exist an open neighborhood $\mathcal{D}_0 \subseteq \mathcal{D}$ of $x_1 = 0$ and a function $T : \mathcal{D}_0 \setminus \{0\} \times \mathbb{R}^{n_2} \to (0, \infty)$, called the *settling-time function*, such that the following statements hold:

i) *Finite-time partial convergence.* For every $(x_{10}, x_{20}) \in \mathcal{D}_0\backslash\{0\} \times \mathbb{R}^{n_2}$, $s^{x_0}(t)$ is defined on $[0, T(x_{10}, x_{20}))$, where $x_0 = [x_{10}^{\mathrm{T}}, x_{20}^{\mathrm{T}}]^{\mathrm{T}}$, $s_1^{x_0}(t) \in \mathcal{D}_0\backslash\{0\}$ for all $t \in [0, T(x_{10}, x_{20}))$, and $s_1^{x_0}(t) \to 0$ as $t \to T(x_{10}, x_{20})$.

ii) *Partial Lyapunov stability.* For every $\varepsilon > 0$ and $x_{20} \in \mathbb{R}^{n_2}$ there exists $\delta = \delta(\varepsilon, x_{20}) > 0$ such that $\mathcal{B}_\delta(0) \subset \mathcal{D}_0$ and, for every $x_{10} \in \mathcal{B}_\delta(0)\backslash\{0\}$, $s_1^{x_0}(t) \in \mathcal{B}_\varepsilon(0)$ for all $t \in [0, T(x_{10}, x_{20}))$.

The nonlinear dynamical system (8.1) and (8.2) is *finite-time stable with respect to* x_1 *uni-*

formly in x_{20} if (8.1) and (8.2) is finite-time stable with respect to x_1 and the following statment holds:

iii) Partial uniform Lyapunov stability. For every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\mathcal{B}_\delta(0) \subset \mathcal{D}_0$ and, for every $x_{10} \in \mathcal{B}_\delta(0)\backslash\{0\}$, $s_1^{x_0}(t) \in \mathcal{B}_\varepsilon(0)$ for all $t \in [0, T(x_{10}, x_{20}))$ and for all $x_{20} \in \mathbb{R}^{n_2}$.

The nonlinear dynamical system (8.1) and (8.2) is *strongly finite-time stable with respect to* x_1 *uniformly in* x_{20} if (8.1) and (8.2) is uniformly finite-time stable with respect to x_1 and the following statment holds:

iv) Finite-time partial uniform convergence. For every $(x_{10}, x_{20}) \in \mathcal{D}_0\backslash\{0\} \times \mathbb{R}^{n_2}$, $s^{x_0}(t)$ is defined on $[0, T(x_{10}, x_{20}))$, $s_1^{x_0}(t) \in \mathcal{D}_0\backslash\{0\}$ for all $t \in [0, T(x_{10}, x_{20}))$, and $s_1^{x_0}(t) \to 0$ as $t \to T(x_{10}, x_{20})$ uniformly in x_{20} for all $x_{20} \in \mathbb{R}^{n_2}$.

The nonlinear dynamical system (8.1) and (8.2) is *globally finite-time stable with respect to* x_1 (respectively, *globally finite-time stable with respect to* x_1 *uniformly in* x_{20} or *globally strongly finite-time stable with respect to* x_1 *uniformly in* x_{20}) if it is finite-time stable with respect to x_1 (respectively, finite-time stable with respect to x_1 uniformly in x_{20} or strongly finite-time stable with respect to x_1 uniformly in x_{20}) with $\mathcal{D}_0 = \mathbb{R}^{n_1}$.

Remark 8.2. It is important to note that there is a key difference between the partial stability definitions given in Definition 8.1 and the definitions of partial stability given in [68]. In particular, the partial stability definitions given in [68] require that both initial conditions x_{10} and x_{20} lie in a neighborhood of the origin, whereas in Definition 8.1, x_{20} can be arbitrary. Furthermore, in the definition of partial stability given in [68], the state $x_1(t)$, $t \geq 0$, converges to zero and the state $x_2(t)$, $t \geq 0$, is bounded and converges to a constant that possibly depends on the system initial conditions. In contrast, in Definition 8.1 the state $x_2(t)$ can diverge as $t \to \infty$. Similar distinctions hold for our partial stabilization definition (see Definition 8.15 below) and the partial stabilization definition given in [67]. As will be seen below, this difference allows us to unify autonomous partial stability theory with time-varying stability theory.

As shown in [38] and [23], an important application of partial stability theory is the unification it provides between time-invariant stability theory and stability theory for time-varying systems. Specifically, consider the nonlinear time-varying dynamical system given by

$$\dot{x}(t) = f(t, x(t)), \qquad x(t_0) = x_0, \qquad t \in \mathcal{I}_{t_0,x_0}, \tag{8.3}$$

where, for every $t \in \mathcal{I}_{t_0,x_0}$, $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, $\mathcal{I}_{t_0,x_0} \subseteq [t_0, \infty)$ is the maximal interval of existence of a solution $x(t)$ of (8.3), $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$, and $f : \mathcal{I}_{t_0,x_0} \times \mathcal{D} \to \mathbb{R}^n$ is such that, for every $(t, x) \in \mathcal{I}_{t_0,x_0} \times \mathcal{D}$, $f(t, 0) = 0$ and $f(\cdot, \cdot)$ is jointly continuous in t and x. In this chapter, we assume that the nonlinear time-varying dynamical system (8.3) possesses unique solutions in forward time for all initial conditions except possibly $x = 0$ and, given $x_0 \in \mathcal{D}$, we denote by the continuously differentiable map $s^{t_0,x_0}(\cdot) \triangleq s(\cdot, t_0, x_0)$ the *trajectory* or the unique *solution curve* of (8.3) on $\mathcal{I}_{t_0,x_0}$ satisfying $s(0, t_0, x_0) = x_0$. Now, defining $x_1(\tau) \triangleq x(t)$ and $x_2(\tau) \triangleq t$, where $\tau \triangleq t - t_0$, it follows that the solution $x(t)$, $t \in \mathcal{I}_{t_0,x_0}$, to the nonlinear time-varying dynamical system (8.3) can be equivalently characterized by the solution $x_1(\tau)$, $\tau \in \mathcal{T}_{t_0,x_0}$, to the nonlinear autonomous dynamical system

$$\dot{x}_1(\tau) = f(x_2(\tau), x_1(\tau)), \qquad x_1(0) = x_0, \qquad \tau \in \mathcal{T}_{t_0,x_0}, \tag{8.4}$$

$$\dot{x}_2(\tau) = 1, \qquad x_2(0) = t_0, \tag{8.5}$$

where $\mathcal{T}_{t_0,x_0} \triangleq \{\tau \in \overline{\mathbb{R}}_+ : \tau = t - t_0, t \in \mathcal{I}_{t_0,x_0}\}$. Note that (8.4) and (8.5) are in the same form as the system given by (8.1) and (8.2), and hence, Definition 8.1 applied to (8.4) and (8.5) specializes to the following definition.

Definition 8.3. The nonlinear dynamical system (8.3) is *finite-time stable* if there exist an open neighborhood $\mathcal{D}_0 \subseteq \mathcal{D}$ of the origin and a function $T : [0, \infty) \times \mathcal{D}_0 \backslash \{0\} \to (t_0, \infty)$, called the *settling-time function*, such that the following statements hold:

i) *Finite-time convergence.* For every $(t_0, x_0) \in [0, \infty) \times \mathcal{D}_0 \setminus \{0\}$, $s^{t_0,x_0}(t)$ is defined on $[t_0, T(t_0, x_0))$, $s^{t_0,x_0}(t) \in \mathcal{D}_0 \setminus \{0\}$ for all $t \in [t_0, T(t_0, x_0))$, and $s^{t_0,x_0}(t) \to 0$ as $t \to T(t_0, x_0)$.

ii) *Lyapunov stability.* For every $\varepsilon > 0$ and $t_0 \in [0, \infty)$ there exists $\delta = \delta(\varepsilon, t_0) > 0$ such that $\mathcal{B}_\delta(0) \subset \mathcal{D}_0$ and, for every $x_0 \in \mathcal{B}_\delta(0) \setminus \{0\}$, $s^{t_0,x_0}(t) \in \mathcal{B}_\varepsilon(0)$ for all $t \in [t_0, T(t_0, x_0))$.

The nonlinear dynamical system (8.3) is *uniformly finite-time stable* if (8.3) is finite-time stable and the following statement holds:

iii) *Uniform Lyapunov stability.* For every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $\mathcal{B}_\delta(0) \subset \mathcal{D}_0$ and, for every $x_0 \in \mathcal{B}_\delta(0) \setminus \{0\}$, $s^{t_0,x_0}(t) \in \mathcal{B}_\varepsilon(0)$ for all $t \in [t_0, T(t_0, x_0))$ and for all $t_0 \in [0, \infty)$.

The nonlinear dynamical system (8.3) is *strongly uniformly finite-time stable* if (8.3) is uniformly finite-time stable and the following statement holds:

iv) *Uniform finite-time convergence.* For every $(t_0, x_0) \in [0, \infty) \times \mathcal{D}_0 \setminus \{0\}$, $s^{t_0,x_0}(t)$ is defined on $[t_0, T(t_0, x_0))$, $s^{t_0,x_0}(t) \in \mathcal{D}_0 \setminus \{0\}$ for all $t \in [t_0, T(t_0, x_0))$, and $s^{t_0,x_0}(t) \to 0$ as $t \to T(t_0, x_0)$ uniformly in t_0 for all $t_0 \in [0, \infty)$.

The nonlinear dynamical system (8.3) is *globally finite-time stable* (respectively, *globally uniformly finite-time stable* or *globally strongly uniformly finite-time stable*) if it is finite-time stable (respectively, uniformly finite-time stable or strongly uniformly finite-time stable) with $\mathcal{D}_0 = \mathbb{R}^n$.

8.3. Finite-Time Partial Stability Theory

In this section, we present sufficient conditions for finite-time partial stability using a Lyapunov function satisfying a differential inequality involving fractional powers. The following proposition shows that if the nonlinear dynamical system (8.1) and (8.2) is finite-time stable with respect to x_1, then it possesses a unique solution $s(\cdot, x_{10}, x_{20})$ defined on $\overline{\mathbb{R}}_+ \times \mathcal{D}_0 \times \mathbb{R}^{n_2}$ for every x_{10} in a neighborhood of $x_1 = 0$, including $x_1 = 0$, and, for every $x_{20} \in \mathbb{R}^{n_2}$, $s_1(t, x_{10}, x_{20}) = 0$ for all $t \geq T(x_{10}, x_{20})$, where $T(0, x_{20}) \triangleq 0$.

Proposition 8.4. Consider the nonlinear dynamical system $\mathcal{G}$ given by (8.1) and (8.2). Assume $\mathcal{G}$ is finite-time stable with respect to x_1 and let $\mathcal{D}_0 \subseteq \mathcal{D}$ and $T : \mathcal{D}_0 \backslash \{0\} \times \mathbb{R}^{n_2} \to (0, \infty)$ be defined as in Definition 8.1. Then, for every $(x_{10}, x_{20}) \in \mathcal{D}_0 \times \mathbb{R}^{n_2}$, there exists a unique solution $s(t, x_{10}, x_{20}) = [s_1^{\mathrm{T}}(t, x_{10}, x_{20}), s_2^{\mathrm{T}}(t, x_{10}, x_{20})]^{\mathrm{T}}$, $t \geq 0$, to (8.1) and (8.2) defined on $\overline{\mathbb{R}}_+ \times \mathcal{D}_0 \times \mathbb{R}^{n_2}$ such that $s_1(t, x_{10}, x_{20}) \in \mathcal{D}_0$, $t \in [0, T(x_{10}, x_{20}))$, and such that $s_1(t, x_{10}, x_{20}) = 0$, $t \geq T(x_{10}, x_{20})$, where $T(0, x_{20}) \triangleq 0$.

Proof: It follows from the partial Lyapunov stability of (8.1) and (8.2) with respect to x_1 that $x_1(t) \equiv 0$, $t \geq 0$, is the unique solution of (8.1) satisfying $x_1(0) = 0$ for all $x_{20} \in \mathbb{R}^{n_2}$. Thus, $s_1(t, 0, x_{20}) = 0$ for all $t \geq 0$ and $x_{20} \in \mathbb{R}^{n_2}$. Next, let $(x_{10}, x_{20}) \in \mathcal{D}_0 \backslash \{0\} \times \mathbb{R}^{n_2}$, and define

$$x_1(t) \triangleq \begin{cases} s_1(t, x_{10}, x_{20}), & 0 \leq t < T(x_{10}, x_{20}), \\ 0, & t \geq T(x_{10}, x_{20}). \end{cases} \tag{8.6}$$

Note that by construction, $x_1(\cdot)$ is continuously differentiable on $\overline{\mathbb{R}}_+ \setminus \{T(x_{10}, x_{20})\}$ and satisfies (8.1) on $\overline{\mathbb{R}}_+ \backslash \{T(x_{10}, x_{20})\}$. Furthermore, since $f_1(\cdot, \cdot)$ is jointly continuous,

$$\lim_{t \to T^-(x_{10}, x_{20})} \dot{x}_1(t) = \lim_{t \to T^-(x_{10}, x_{20})} f_1(x_1(t), x_2(t)) = \lim_{t \to T^+(x_{10}, x_{20})} \dot{x}_1(t), \tag{8.7}$$

and hence, $x_1(\cdot)$ is continuously differentiable at $T(x_{10}, x_{20})$ and $x_1(t)$ satisfies (8.1). Hence, it follows from the assumptions on $f_2(\cdot, \cdot)$ that, given $x_1(t)$, $t \geq 0$, there exists $x_2(t)$ such that $x(t) = [x_1^{\mathrm{T}}(t), x_2^{\mathrm{T}}(t)]^{\mathrm{T}}$ is solution of (8.1) and (8.2) for all $(x_{10}, x_{20}) \in \mathcal{D}_0 \backslash \{0\} \times \mathbb{R}^{n_2}$ and for all $t \geq 0$.

Given $(x_{10}, x_{20}) \in \mathcal{D}_0 \times \mathbb{R}^{n_2}$, to show uniqueness, assume $y_1(\cdot)$ satisfies (8.1) for all $t \geq 0$. In this case, $x_1(t) = y_1(t)$ for all $t \in [0, T(x_{10}, x_{20}))$ by the uniqueness assumption in Section 8.2. In addition, by continuity, $x_1(t) = y_1(t)$ at $t = T(x_{10}, x_{20})$, and hence, $x_1(t) = y_1(t)$

for all $t \in [0, T(x_{10}, x_{20})]$, which implies that $y_1(T(x_{10}, x_{20})) = 0$. Now, partial Lyapunov stability with respect to x_1 implies that $y_1(t) = 0$ for $t > T(x_{10}, x_{20})$, which proves uniqueness of $x_1(\cdot)$. Hence uniqueness of $x(\cdot) = [x_1^{\mathrm{T}}(\cdot), x_2^{\mathrm{T}}(\cdot)]^{\mathrm{T}}$ immediately follows from the assumptions in Section 8.2. This proves the result. ■

It follows from Proposition 8.4 and the assumptions on $f_2(\cdot, \cdot)$ that if the nonlinear dynamical system (8.1) and (8.2) is finite-time stable with respect to x_1, then it defines a *global semi-flow* on $\mathcal{D}_0 \times \mathbb{R}^{n_2}$; that is, the solution curve $s(\cdot, \cdot, \cdot)$ of (8.1) and (8.2) satisfies the consistency property $s(0, x_1, x_2) = [x_1^{\mathrm{T}}, x_2^{\mathrm{T}}]^{\mathrm{T}}$ and the semigroup property $s(t, s_1(\tau, x_1, x_2), s_2(\tau, x_1, x_2)) = s(t + \tau, x_1, x_2)$ for every $(x_1, x_2) \in \mathcal{D}_0 \times \mathbb{R}^{n_2}$ and $t, \tau \in \overline{\mathbb{R}}_+$. Furthermore, $s(\cdot, \cdot, \cdot)$ satisfies

$$s_1(T(x_{10}, x_{20}) + t_1, x_{10}, x_{20}) = 0 \tag{8.8}$$

for all $(x_{10}, x_{20}) \in \mathcal{D}_0 \times \mathbb{R}^{n_2}$ and $t_1 \geq 0$.

In general, finite-time partial stability does not imply that the settling-time function $T(\cdot, \cdot)$ is continuous [12]. The following proposition generalizes Proposition 2.4 of [12] to show that the settling-time function $T(\cdot, \cdot)$ of a finite-time partially stable system is jointly continuous on $\mathcal{D}_0 \times \mathbb{R}^{n_2}$ if and only if it is continuous at $(0, \cdot)$.

Proposition 8.5. Consider the nonlinear dynamical system $\mathcal{G}$ given by (8.1) and (8.2). Assume $\mathcal{G}$ is finite-time stable with respect to x_1, let $\mathcal{D}_0 \subseteq \mathcal{D}$ be as defined in Definition 8.1, and let $T : \mathcal{D}_0 \backslash \{0\} \times \mathbb{R}^{n_2} \to [0, \infty)$ be the settling-time function of $\mathcal{G}$. Then the following statements hold:

i) If $t_1 \geq 0$ and $(x_{10}, x_{20}) \in \mathcal{D}_0 \times \mathbb{R}^{n_2}$, then

$$T(s_1(t_1, x_{10}, x_{20}), s_2(t_1, x_{10}, x_{20})) = \max\{T(x_{10}, x_{20}), t_1\}. \tag{8.9}$$

ii) $T(\cdot, \cdot)$ is jointly continuous on $\mathcal{D}_0 \times \mathbb{R}^{n_2}$ if and only if $T(\cdot, \cdot)$ is jointly continuous at $(0, x_2)$, $x_2 \in \mathbb{R}^{n_2}$.

Proof: *i*) It follows from Definition 8.1 that

$$T(x_{10},x_{20}) = \inf\{t \in \mathbb{R}_+ : s_1(t,x_{10},x_{20}) = 0\} \tag{8.10}$$

for all $(x_{10},x_{20}) \in \mathcal{D}_0\backslash\{0\} \times \mathbb{R}^{n_2}$. Hence, $T(s_1(t_1,x_{10},x_{20}), s_2(t_1,x_{10},x_{20})) = \inf\{t_2 \in \mathbb{R}_+ : s_1(t_2, s_1(t_1,x_{10},x_{20}), s_2(t_1,x_{10},x_{20})) = 0\}$. Now, for $0 \le t_1 \le T(x_{10},x_{20})$, the semigroup property and (8.10) imply that $T(s_1(t_1,x_{10},x_{20}), s_2(t_1,x_{10},x_{20})) = \inf\{t_2 \in \mathbb{R}_+ : s_1(t_2,x_{10},x_{20}) = 0\} = T(x_{10},x_{20})$. Alternatively, for $0 \le T(x_{10},x_{20}) \le t_1$, $T(s_1(t_1,x_{10},x_{20}), s_2(t_1,x_{10},x_{20})) = t_1$, which proves (8.9).

ii) Necessity is immediate. To prove sufficiency, suppose that $T(\cdot,\cdot)$ is jointly continuous at $(0,x_2)$, $x_2 \in \mathbb{R}^{n_2}$. Let $(x_1,x_2) \in \mathcal{D}_0 \times \mathbb{R}^{n_2}$ and consider the sequences $\{x_{1n}\}_{n=1}^{\infty} \subset \mathcal{D}_0$ converging to x_1 and $\{x_{2n}\}_{n=1}^{\infty} \subset \mathbb{R}^{n_2}$ converging to x_2. Let $\tau^- = \liminf_{n\to\infty} T(x_{1n},x_{2n})$ and $\tau^+ = \limsup_{n\to\infty} T(x_{1n},x_{2n})$. Note that τ^-, $\tau^+ \in \mathbb{R}_+$ and

$$\tau^- \le \tau^+. \tag{8.11}$$

Next, let $\{x_{1n_m}\}_{m=0}^{\infty} \subset \mathcal{D}_0$ be a subsequence of $\{x_{1n}\}$ and $\{x_{2n_m}\}_{m=0}^{\infty} \subset \mathbb{R}^{n_2}$ be a subsequence of $\{x_{2n}\}$ such that $T(x_{1n_m},x_{2n_m}) \to \tau^+$ as $m \to \infty$. The sequence $\{(T(x_1,x_2), x_{1n_m}, x_{2n_m})\}_{m=1}^{\infty}$ converges in $\overline{\mathbb{R}}_+ \times \mathcal{D}_0 \times \mathbb{R}^{n_2}$ to $(T(x_1,x_2),x_1,x_2)$ as $m \to \infty$. Since $s_1(T(x_1,x_2)+t_1,x_1,x_2) = 0$ for all $t_1 \ge 0$ and since all solutions to (8.1) and (8.2) are continuous in their initial conditions, it follows that $s_1(T(x_1,x_2),x_{1n_m},x_{2n_m}) \to s_1(T(x_1,x_2),x_1,x_2) = 0$ as $m \to \infty$. Thus, since $T(0,x_2)$ is continuous for all $x_2 \in \mathbb{R}^{n_2}$, it follows that

$$\lim_{m\to\infty} T(s_1(T(x_1,x_2),x_{1n_m},x_{2n_m}), s_2(T(x_1,x_2),x_{1n_m},x_{2n_m})) = T(x_1,x_2). \tag{8.12}$$

Now, with $t_1 = T(x_1,x_2)$, $x_{10} = x_{1n_m}$, and $x_{20} = x_{2n_m}$, it follows from (8.9) and (8.12) that $T(s_1(T(x_1,x_2),x_{1n_m},x_{2n_m}), s_2(T(x_1,x_2),x_{1n_m},x_{2n_m})) = \max\{T(x_{1n_m},x_{2n_m}), T(x_1,x_2)\}$ and $\max\{T(x_{1n_m},x_{2n_m}), T(x_1,x_2)\} \to T(x_1,x_2)$ as $m \to \infty$. Thus, $\max\{\tau^+, T(x_1,x_2)\} = T(x_1,x_2)$, which implies that

$$\tau^+ \le T(x_1,x_2). \tag{8.13}$$

Finally, let $\{x_{1n_k}\}_{k=0}^{\infty} \subset \mathcal{D}_0$ be a subsequence of $\{x_{1n}\}$ and $\{x_{2n_k}\}_{k=0}^{\infty} \subset \mathbb{R}^{n_2}$ be a subsequence of $\{x_{2n}\}$ such that $T(x_{1n_k}, x_{2n_k}) \to \tau^-$ as $k \to \infty$. It follows from (8.11) and (8.13) that $\tau^- \in \mathbb{R}_+$, and hence, the sequence $\{(T(x_{1n_k}, x_{2n_k}), x_{1n_k}, x_{2n_k})\}_{k=1}^{\infty}$ converges to (τ^-, x_1, x_2) as $k \to \infty$. Since $s_1(\cdot,\cdot,\cdot)$ is jointly continuous, it follows that $s_1(T(x_{1n_k}, x_{2n_k}), x_{1n_k}, x_{2n_k}) \to s_1(\tau^-, x_1, x_2)$ as $k \to \infty$. Now, since $s_1(T(x_1, x_2)+t_1, x_1, x_2) = 0$ for all $t_1 \geq 0$, $s_1(T(x_{1n_k}, x_{2n_k}), x_{1n_k}, x_{2n_k}) = 0$ for each k. Hence, $s_1(\tau^-, x_1, x_2) = 0$ and, by the definition of settling-time function,

$$T(x_1, x_2) \leq \tau^-. \tag{8.14}$$

Now, it follows from (8.11), (8.13), and (8.14) that $\tau^- = T(x_1, x_2) = \tau^+$, and hence, $T(x_{1n}, x_{2n}) \to T(x_1, x_2)$ as $n \to \infty$, which proves that $T(\cdot,\cdot)$ is jointly continuous on $\mathcal{D}_0 \times \mathbb{R}^{n_2}$.

■

Next, we present sufficient conditions for finite-time partial stability using a Lyapunov function involving a scalar differential inequality. Given the nonlinear dynamical system (8.1) and (8.2), for the statement of the following result define

$$\dot{V}(x_1, x_2) \triangleq V'(x_1, x_2) f(x_1, x_2),$$

where $f(x_1, x_2) \triangleq [f_1^{\mathrm{T}}(x_1, x_2), f_2^{\mathrm{T}}(x_1, x_2)]^{\mathrm{T}}$ and $V : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}$ is a continuously differentiable function, and recall the definitions of class $\mathcal{K}$ and $\mathcal{K}_\infty$ functions given in [38, Def. 3.3].

Theorem 8.6. Consider the nonlinear dynamical system $\mathcal{G}$ given by (8.1) and (8.2). Then the following statements hold:

i) If there exist a continuously differentiable function $V : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}$, a class $\mathcal{K}$ function $\alpha(\cdot)$, a continuous function $k : [0, \infty) \to \mathbb{R}_+$, a real number $\theta \in (0, 1)$, and an open neighborhood $\mathcal{M} \subseteq \mathcal{D}$ of $x_1 = 0$ such that

$$V(0, x_2) = 0, \qquad x_2 \in \mathbb{R}^{n_2}, \tag{8.15}$$

$$\alpha(\|x_1\|) \le V(x_1, x_2), \qquad (x_1, x_2) \in \mathcal{M} \times \mathbb{R}^{n_2}, \tag{8.16}$$

$$\dot{V}(x_1, x_2) \le -k(\|x_2\|)(V(x_1, x_2))^\theta, \qquad (x_1, x_2) \in \mathcal{M} \times \mathbb{R}^{n_2}, \tag{8.17}$$

then $\mathcal{G}$ is finite-time stable with respect to x_1. Moreover, there exist a neighborhood $\mathcal{D}_0$ of $x_1 = 0$ and a settling-time function $T : \mathcal{D}_0 \times \mathbb{R}^{n_2} \to [0, \infty)$ such that

$$T(x_{10}, x_{20}) \le q^{-1}\left(\frac{(V(x_{10}, x_{20}))^{1-\theta}}{1-\theta}\right), \qquad (x_{10}, x_{20}) \in \mathcal{D}_0 \times \mathbb{R}^{n_2}, \tag{8.18}$$

where $q : [0, \infty) \to \mathbb{R}$ is continuously differentiable and satisfies

$$\dot{q}(t) = k(\|x_2(t)\|), \qquad q(0) = 0, \qquad t \ge 0, \tag{8.19}$$

and $T(\cdot, \cdot)$ is jointly continuous on $\mathcal{D}_0 \times \mathbb{R}^{n_2}$.

ii) If $\mathcal{M} = \mathcal{D} = \mathbb{R}^{n_1}$ and there exist a continuously differentiable function $V : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}$, a class $\mathcal{K}_\infty$ function $\alpha(\cdot)$, a continuous function $k : [0, \infty) \to \mathbb{R}_+$, and a real number $\theta \in (0, 1)$ such that (8.15)–(8.17) hold, then $\mathcal{G}$ is globally finite-time stable with respect to x_1. Moreover, there exists a settling-time function $T : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to [0, \infty)$ such that (8.18) holds with $\mathcal{D}_0 = \mathbb{R}^{n_1}$ and $T(\cdot, \cdot)$ is jointly continuous on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$.

iii) If there exist a continuously differentiable function $V : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}$, class $\mathcal{K}$ functions $\alpha(\cdot)$ and $\beta(\cdot)$, a continuous function $k : [0, \infty) \to \mathbb{R}_+$, a real number $\theta \in (0, 1)$, and an open neighborhood $\mathcal{M} \subseteq \mathcal{D}$ of $x_1 = 0$ such that (8.16) and (8.17) hold, and

$$V(x_1, x_2) \le \beta(\|x_1\|), \qquad (x_1, x_2) \in \mathcal{M} \times \mathbb{R}^{n_2}, \tag{8.20}$$

then $\mathcal{G}$ is finite-time stable with respect to x_1 uniformly in x_{20}. Moreover, there exist a neighborhood $\mathcal{D}_0$ of $x_1 = 0$ and a settling-time function $T : \mathcal{D}_0 \times \mathbb{R}^{n_2} \to [0, \infty)$ such that (8.18) holds and $T(\cdot, \cdot)$ is jointly continuous on $\mathcal{D}_0 \times \mathbb{R}^{n_2}$.

iv) If $\mathcal{M} = \mathcal{D} = \mathbb{R}^{n_1}$ and there exist a continuously differentiable function $V : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}$, class $\mathcal{K}_\infty$ functions $\alpha(\cdot)$ and $\beta(\cdot)$, a continuous function $k : [0, \infty) \to \mathbb{R}_+$, and a real number $\theta \in (0,1)$ such that (8.16), (8.17), and (8.20) hold, then $\mathcal{G}$ is globally finite-time stable with respect to x_1 uniformly in x_{20}. Moreover, there exists a settling-time function $T : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to [0,\infty)$ such that (8.18) holds with $\mathcal{D}_0 = \mathbb{R}^{n_1}$ and $T(\cdot,\cdot)$ is jointly continuous on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$.

v) If there exist a continuously differentiable function $V : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}$, class $\mathcal{K}$ functions $\alpha(\cdot)$ and $\beta(\cdot)$, a real number $\theta \in (0,1)$, and an open neighborhood $\mathcal{M} \subseteq \mathcal{D}$ of $x_1 = 0$ such that (8.16), (8.17), and (8.20) hold with $k(\|x_2\|) = k \in \mathbb{R}_+$, $x_2 \in \mathbb{R}^{n_2}$, then $\mathcal{G}$ is strongly finite-time stable with respect to x_1 uniformly in x_{20}. Moreover, there exist a neighborhood $\mathcal{D}_0$ of $x_1 = 0$ and a settling-time function $T : \mathcal{D}_0 \times \mathbb{R}^{n_2} \to [0,\infty)$ such that

$$T(x_{10}, x_{20}) \leq \frac{(V(x_{10}, x_{20}))^{1-\theta}}{k(1-\theta)}, \qquad (x_{10}, x_{20}) \in \mathcal{D}_0 \times \mathbb{R}^{n_2}, \tag{8.21}$$

and $T(\cdot,\cdot)$ is jointly continuous on $\mathcal{D}_0 \times \mathbb{R}^{n_2}$.

vi) If $\mathcal{M} = \mathcal{D} = \mathbb{R}^{n_1}$ and there exist a continuously differentiable function $V : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}$, class $\mathcal{K}_\infty$ functions $\alpha(\cdot)$ and $\beta(\cdot)$, and a real number $\theta \in (0,1)$ such that (8.16), (8.17), and (8.20) hold with $k(\|x_2\|) = k \in \mathbb{R}_+$, $x_2 \in \mathbb{R}^{n_2}$, then $\mathcal{G}$ is globally strongly finite-time stable with respect to x_1 uniformly in x_{20}. Moreover, there exists a settling-time function $T : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to [0,\infty)$ such that (8.21) holds with $\mathcal{D}_0 = \mathbb{R}^{n_1}$ and $T(\cdot,\cdot)$ is jointly continuous on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$.

Proof: i) Let $x_{20} \in \mathbb{R}^{n_2}$, let $\varepsilon > 0$ be such that $\mathcal{B}_\varepsilon(0) \subseteq \mathcal{M}$, define $\eta \triangleq \alpha(\varepsilon)$, and define $\mathcal{D}_\eta \triangleq \{x_1 \in \mathcal{B}_\varepsilon(0) : V(x_1, x_{20}) < \eta\}$. Since $V(\cdot,\cdot)$ is continuous and $V(0, x_2) = 0$, it follows that $\mathcal{D}_\eta$ is nonempty and there exists $\delta = \delta(\varepsilon, x_{20}) > 0$ such that $V(x_1, x_{20}) < \eta$, $x_1 \in \mathcal{B}_\delta(0)$. Hence, $\mathcal{B}_\delta(0) \subseteq \mathcal{D}_\eta$. Next, it follows from (8.17) that $V(x_1(t), x_2(t))$ is a nonincreasing function of time and, hence, for every $x_{10} \in \mathcal{B}_\delta(0) \subseteq \mathcal{D}_\eta$, it follows that

$$\alpha(\|x_1(t)\|) \leq V(x_1(t), x_2(t)) \leq V(x_{10}, x_{20}) < \eta = \alpha(\varepsilon), \qquad t \geq 0. \tag{8.22}$$

Thus, for every $x_{10} \in \mathcal{B}_\delta(0)$, $x_1(t) \in \mathcal{B}_\varepsilon(0)$, $t \geq 0$, which proves partial Lyapunov stability with respect to x_1.

Next, let $z : [0, \infty) \to \mathbb{R}_+$ be a continuous function defined on $[0, \infty)$ and note that the solution to

$$\dot{v}(t) = -z(t)(v(t))^\theta, \qquad v(0) = v_0 = V(x_{10}, x_{20}), \qquad t \geq 0, \tag{8.23}$$

is given by

$$v(t) = \begin{cases} V(x_{10}, x_{20}) \left[(V(x_{10}, x_{20}))^{1-\theta} - (1-\theta) \int_0^t z(\tau)\mathrm{d}\tau\right]^{\frac{1}{1-\theta}}, & 0 \leq t < t_1, \quad v_0 \neq 0, \\ 0, & t \geq t_1, \quad v_0 \neq 0, \\ 0, & t \geq 0, \quad v_0 = 0, \end{cases} \tag{8.24}$$

where $t_1 > 0$ is such that

$$\int_0^{t_1} z(\tau)\mathrm{d}\tau = \frac{(V(x_{10}, x_{20}))^{1-\theta}}{1-\theta}. \tag{8.25}$$

Hence,

$$t_1 = q^{-1}\left(\frac{(V(x_{10}, x_{20}))^{1-\theta}}{1-\theta}\right), \tag{8.26}$$

where $q : [0, \infty) \to \mathbb{R}$ is continuously differentiable and satisfies

$$\dot{q}(t) = k(\|x_2(t)\|), \qquad q(0) = q_0, \qquad t \geq 0, \tag{8.27}$$

for some $q_0 \in \overline{\mathbb{R}}_+$. Now, let $w : [0, \infty) \to \mathbb{R}$ be a continuously differentiable function such that

$$\dot{w}(t) \leq -z(t)(v(t))^\theta, \qquad w(0) = V(x_{10}, x_{20}), \qquad t \geq 0, \tag{8.28}$$

where $v(t)$ is given by (8.24). Then, it follows from (8.23), (8.28), and the comparison lemma [38, p. 126] that

$$w(t) \leq v(t), \qquad t \geq 0. \tag{8.29}$$

Thus, it follows from (8.17), (8.23), (8.24), (8.28), and (8.29), with $z(t) = k(\|x_2(t)\|)$ and $w(t) = (V(x_1(t), x_2(t))^{1-\theta}$, $t \geq 0$, that

$$V(x_1(t), x_2(t)) \leq v(t), \qquad t \geq 0, \tag{8.30}$$

and hence, using (8.15), (8.16), (8.24), and (8.30),

$$x_1(t) = 0, \qquad t \geq t_1, \tag{8.31}$$

where t_1 is given in (8.26), which proves finite-time convergence of the trajectory of (8.1) for all $(x_{10}, x_{20}) \in \mathcal{B}_\delta(0) \times \mathbb{R}^{n_2}$. Hence, the nonlinear dynamical system $\mathcal{G}$ is finite-time stable with respect to x_1.

Finally, since $s_1(0, x_1, x_2) = x_1$ and $s_1(\cdot, \cdot, \cdot)$ is continuous, $\inf\{t \in \mathbb{R}_+ : s_1(t, x_1, x_2) = 0\} > 0$, $x_{10} \in \mathcal{B}_\delta(0)\backslash\{0\}$. Furthermore, it follows from (8.31) that $\inf\{t \in \mathbb{R}_+ : s_1(t, x_1, x_2) = 0\} < \infty$, $x_{10} \in \mathcal{B}_\delta(0)$. Now, defining $\mathcal{D}_0 \triangleq \mathcal{B}_\delta(0)$ and $T : \mathcal{D}_0 \times \mathbb{R}^{n_2} \to \mathbb{R}_+$ by (8.24) and (8.26), (8.18) is immediate. Moreover, it follows from the finite-time stability of $\mathcal{G}$ with respect to x_1 and Proposition 8.4 that $T(\cdot, \cdot)$ can be extended to $\overline{\mathbb{R}}_+$ and $T(0, x_{20}) = 0$, which implies that $q_0 = 0$ in (8.27). Thus, (8.19) immediately follows from (8.27). Finally, the right-hand side of (8.18) is jointly continuous at $(0, x_2)$, $x_2 \in \mathbb{R}^{n_2}$, and hence, by Proposition 8.5, it is jointly continuous on $\mathcal{D}_0 \times \mathbb{R}^{n_2}$.

ii) Let $\delta > 0$, $x_{10} \in \mathbb{R}^{n_1}$, and $x_{20} \in \mathbb{R}^{n_2}$ be such that $\|x_{10}\| < \delta$. Since $\alpha(\cdot)$ is a $\mathcal{K}_\infty$ function, it follows that there exists $\varepsilon > 0$ such that $V(x_{10}, x_{20}) \leq \alpha(\varepsilon)$. Now, (8.17) implies that $V(x_1(t), x_2(t))$ is a nonincreasing function of time, and hence, it follows from (8.16) that

$$\alpha(\|x_1(t)\|) \leq V(x_1(t), x_2(t)) \leq V(x_{10}, x_{20}) = \alpha(\varepsilon), \qquad t \geq 0. \tag{8.32}$$

Hence, for every $x_{10} \in \mathcal{B}_\delta(0)$, $x_1(t) \in \mathcal{B}_\varepsilon(0)$, $t \geq 0$, which proves Lyapunov stability with respect to x_1. Finite-time partial convergence follows as in the proof of *i*), implying global finite-time stability of $\mathcal{G}$ with respect to x_1. In addition, the existence of a settling-time function $T : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to [0, \infty)$ satisfying (8.18) and is jointly continuous on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ follows as in the proof of *i*).

iii) Let $\varepsilon > 0$ and $\mathcal{B}_\varepsilon(0)$ be given as in the proof of *i*). Let $\delta = \delta(\varepsilon) > 0$ be such that $\beta(\delta) = \alpha(\varepsilon)$. Hence, it follows from (8.16) and (8.20) that, for all $(x_{10}, x_{20}) \in \mathcal{B}_\delta(0) \times \mathbb{R}^{n_2}$,

$$\alpha(\|x_1(t)\|) \le V(x_1(t), x_2(t)) \le V(x_{10}, x_{20}) < \beta(\delta) = \alpha(\varepsilon), \qquad t \ge 0. \tag{8.33}$$

Thus, for every $x_{10} \in \mathcal{B}_\delta(0)$, $x_1(t) \in \mathcal{B}_\varepsilon(0)$, $t \ge 0$, which proves partial uniform Lyapunov stability with respect to x_1. Finite-time partial convergence follows as in the proof of *i*), implying finite-time stability of $\mathcal{G}$ with respect to x_1 uniformly in x_{20}. In addition, the existence of a settling-time function $T : \mathcal{D}_0 \times \mathbb{R}^{n_2} \to [0, \infty)$ such that (8.18) holds and is jointly continuous on $\mathcal{D}_0 \times \mathbb{R}^{n_2}$ follows as in the proof of *i*).

iv) Let $\delta > 0$, $x_{10} \in \mathbb{R}^{n_1}$, and $x_{20} \in \mathbb{R}^{n_2}$ be such that $\|x_{10}\| < \delta$. Since $\alpha(\cdot)$ and $\beta(\cdot)$ are $\mathcal{K}_\infty$ functions, it follows that there exists $\varepsilon > 0$ such that $\beta(\varepsilon) \le \alpha(\varepsilon)$. Now, (8.17) implies that $V(x_1(t), x_2(t))$ is a nonincreasing function of time, and hence, it follows from (8.16) that

$$\alpha(\|x_1(t)\|) \le V(x_1(t), x_2(t)) \le V(x_{10}, x_{20}) = \alpha(\varepsilon), \qquad t \ge 0. \tag{8.34}$$

Hence, for every $x_{10} \in \mathcal{B}_\delta(0)$, $x_1(t) \in \mathcal{B}_\varepsilon(0)$, $t \ge 0$, which proves uniform Lyapunov stability with respect to x_1. Finite-time partial convergence follows as in the proof of *i*), implying global finite-time stability of $\mathcal{G}$ with respect to x_1 uniformly in x_{20}. In addition, the existence of a settling-time function $T : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to [0, \infty)$ that verifies (8.18) and is jointly continuous on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ follows as in the proof of *i*).

v) Uniform finite-time stability of $\mathcal{G}$ with respect to x_1 directly follows from *iii*). Now, using similar arguments as in the proof of *i*), it follows from (8.16) and (8.17) that

$$\alpha_1(\|x_1(t)\|) \le V(x_1(t), x_2(t)) \le v(t), \qquad t \ge 0, \tag{8.35}$$

where

$$v(t) = \begin{cases} \left[(V(x_{10}, x_{20}))^{1-\theta} - (1-\theta)kt\right]^{\frac{1}{1-\theta}}, & 0 \le t < t_1, \quad v_0 \ne 0, \\ 0, & t \ge t_1, \quad v_0 \ne 0, \\ 0, & t \ge 0, \quad v_0 = 0, \end{cases} \tag{8.36}$$

and

$$t_1 = \frac{(V(x_{10}, x_{20}))^{1-\theta}}{k(1-\theta)}. \tag{8.37}$$

Now, the existence of a neighborhood $\mathcal{D}_0$ of $x_1 = 0$ and a settling-time function $T : \mathcal{D}_0 \times \mathbb{R}^{n_2} \to [0, \infty)$ such that (8.21) holds and is jointly continuous on $\mathcal{D}_0 \times \mathbb{R}^{n_2}$ follows as in the proof of i). Hence, for $t \geq T(x_{10}, x_{20})$, uniform finite-time convergence of $x_1(t)$ to zero is immediate. Alternatively, for every $t < T(x_{10}, x_{20})$ and $\varepsilon > 0$, there exists $\delta = \alpha_1^{-1}\left(\frac{\varepsilon^{1-\theta}}{k(1-\theta)}\right)$ such that if $\|x_1(t)\| \leq \alpha_1^{-1}(v(t)) < \varepsilon$, then $T(x_{10}, x_{20}) - t \leq t_1 - t < \delta$, which proves strong finite-time convergence of $\mathcal{G}$ with respect to x_1 uniformly in x_{20}.

vi) The proof of finite-time stability of $\mathcal{G}$ with respect to x_1 uniformly in x_{20} follows as in the proof of iv), whereas the proof of uniform finite-time convergence of $\mathcal{G}$ with respect to x_1 follows as in the proof of v). Hence, the nonlinear dynamical system $\mathcal{G}$ is globally strongly finite-time stable with respect to x_1 uniformly in x_{20}. ■

Example 8.7. Consider the nonlinear dynamical system given by

$$\dot{x}_1(t) = -x_2(t)\left(x_1(t)\right)^{\frac{1}{3}}, \qquad x_1(0) = x_{10}, \qquad t \geq t_0, \tag{8.38}$$

$$\dot{x}_2(t) = x_2(t), \qquad x_2(0) = x_{20}, \tag{8.39}$$

where $x_{20} > 0$, and hence, $x_2(t) > 0$, $t \geq 0$. To show that (8.38) and (8.39) is globally finite-time stable with respect to x_1, consider the Lyapunov function candidate $V(x_1, x_1) = x_1^{\frac{4}{3}}$ and let $\mathcal{D} = \mathbb{R}$. Clearly, (8.16) and (8.20) hold, and

$$\dot{V}(x_1, x_2) = \frac{4}{3}x_1^{\frac{1}{3}}\left(-x_2 x_1^{\frac{1}{3}}\right) = -\frac{4}{3}x_2 x_1^{\frac{2}{3}} \leq -k(x_2)\left(V(x_1, x_2)\right)^{\frac{1}{2}}, \tag{8.40}$$

where $k(x_2) = \frac{4}{3}x_2 > 0$ and $x_2 > 0$. Hence, it follows from iv) of Theorem 8.6 that (8.38) and (8.39) is globally finite-time stable with respect to x_1. △

The following results specialize Propositions 8.4 and 8.5, and Theorem 8.6 to nonlinear time-varying dynamical systems.

Proposition 8.8. Consider the nonlinear dynamical system $\mathcal{G}$ given by (8.3). Assume $\mathcal{G}$ is finite-time stable and let $\mathcal{D}_0 \subseteq \mathcal{D}$ and $T : [0, \infty) \times \mathcal{D}_0 \backslash \{0\} \to (t_0, \infty)$ be defined as in

Definition 8.3. Then, for every $(t_0, x_0) \in [0, \infty) \times \mathcal{D}_0$, there exists a unique solution $s(t, t_0, x_0)$, $t \geq t_0$, to (8.3) such that $s(t, t_0, x_0) \in \mathcal{D}_0$, $t \in [t_0, T(t_0, x_0))$, and such that $s(t, t_0, x_0) = 0$, $t \geq T(t_0, x_0)$, where $T(t_0, 0) \triangleq t_0$.

Proof: The result is a direct consequence of Proposition 8.4 with $n_1 = n$, $n_2 = 1$, $x_1(t - t_0) = x(t)$, $x_2(t - t_0) = t$, $f_1(x_1, x_2) = f_1(x_2, x_1) = f(t, x)$, $f_2(x_1, x_2) = 1$, and $T(x_{10}, x_{20}) = T(x_{20}, x_{10}) = T(t_0, x_0)$. ■

Proposition 8.9. Consider the nonlinear dynamical system $\mathcal{G}$ given by (8.3). Assume $\mathcal{G}$ is finite-time stable, let $\mathcal{D}_0 \subseteq \mathcal{D}$ be as defined in Definition 8.3, and let $T : [0, \infty) \times \mathcal{D}_0 \backslash \{0\} \to [t_0, \infty)$ be the settling-time function of $\mathcal{G}$. Then the following statements hold:

i) If $t_1 \geq t_0$ and $(t_0, x_0) \in [0, \infty) \times \mathcal{D}_0$, then

$$T(t_1, s(t_1, t_0, x_0)) = \max\{T(t_0, x_0), t_1\}. \tag{8.41}$$

ii) $T(\cdot, \cdot)$ is jointly continuous on $\mathbb{R}_+ \times \mathcal{D}_0$ if and only if $T(\cdot, \cdot)$ is jointly continuous at $(t, 0)$, $t \in [t_0, \infty)$.

Proof: The result is a direct consequence of Proposition 8.5 with $n_1 = n$, $n_2 = 1$, $x_1(t - t_0) = x(t)$, $x_2(t - t_0) = t$, $f_1(x_1, x_2) = f_1(x_2, x_1) = f(t, x)$, $f_2(x_1, x_2) = 1$, and $T(x_{10}, x_{20}) = T(x_{20}, x_{10}) = T(t_0, x_0)$. ■

Given the nonlinear time-varying dynamical system (8.3), for the statement of the following result define

$$\dot{V}(t, x) \triangleq \frac{\partial V(t, x)}{\partial t} + \frac{\partial V(t, x)}{\partial x} f(t, x),$$

where $V : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable function.

Theorem 8.10. Consider the nonlinear dynamical system $\mathcal{G}$ given by (8.3). Then the following statements hold:

i) If there exist a continuously differentiable function $V : [t_0, \infty) \times \mathcal{D} \to \mathbb{R}$, a class $\mathcal{K}$ function $\alpha(\cdot)$, a continuous function $k : [t_0, \infty) \to \mathbb{R}_+$, a real number $\theta \in (0, 1)$, and an open neighborhood $\mathcal{M} \subseteq \mathcal{D}$ of the origin such that

$$V(t, 0) = 0, \qquad t \in [t_0, \infty), \tag{8.42}$$

$$\alpha(\|x\|) \leq V(t, x), \qquad (t, x) \in [t_0, \infty) \times \mathcal{M}, \tag{8.43}$$

$$\dot{V}(t, x) \leq -k(t)(V(t, x))^{\theta}, \qquad (t, x) \in [t_0, \infty) \times \mathcal{M}, \tag{8.44}$$

then $\mathcal{G}$ is finite-time stable. Moreover, there exist a neighborhood $\mathcal{D}_0$ of the origin and a settling-time function $T : [0, \infty) \times \mathcal{D}_0 \to [t_0, \infty)$ such that

$$T(t_0, x_0) \leq q^{-1} \left(\frac{(V(t_0, x_0))^{1-\theta}}{1 - \theta} \right), \qquad (t_0, x_0) \in [0, \infty) \times \mathcal{D}_0, \tag{8.45}$$

where $q : [t_0, \infty) \to \mathbb{R}$ is continuously differentiable and

$$\dot{q}(t) = k(t), \qquad q(t_0) = 0, \qquad t \geq t_0, \tag{8.46}$$

and $T(\cdot, \cdot)$ is jointly continuous on $[0, \infty) \times \mathcal{D}_0$.

ii) If $\mathcal{M} = \mathcal{D} = \mathbb{R}^n$ and there exist a continuously differentiable function $V : [t_0, \infty) \times \mathcal{D} \to \mathbb{R}$, a class $\mathcal{K}_\infty$ function $\alpha(\cdot)$, a continuous function $k : [t_0, \infty) \to \mathbb{R}_+$, and a real number $\theta \in (0, 1)$ such that (8.42)–(8.44) hold, then $\mathcal{G}$ is globally finite-time stable. Moreover, there exists a settling-time function $T : [0, \infty) \times \mathbb{R}^n \to [t_0, \infty)$ such that (8.45) holds with $\mathcal{D}_0 = \mathbb{R}^n$ and $T(\cdot, \cdot)$ is jointly continuous on $[0, \infty) \times \mathbb{R}^n$.

iii) If there exist a continuously differentiable function $V : [t_0, \infty) \times \mathcal{D} \to \mathbb{R}$, class $\mathcal{K}$ functions $\alpha(\cdot)$ and $\beta(\cdot)$, a continuous function $k : [t_0, \infty) \to \mathbb{R}_+$, a real number $\theta \in (0, 1)$, and an open neighborhood $\mathcal{M} \subseteq \mathcal{D}$ of the origin such that (8.43) and (8.44) hold and

$$V(t, x) \leq \beta(\|x\|), \qquad (t, x) \in [t_0, \infty) \times \mathcal{M}, \tag{8.47}$$

then $\mathcal{G}$ is uniformly finite-time stable. Moreover, there exist a neighborhood $\mathcal{D}_0$ of the origin and a settling-time function $T : [0, \infty) \times \mathcal{D}_0 \to [t_0, \infty)$ such that (8.45) holds and $T(\cdot, \cdot)$ is jointly continuous on $[0, \infty) \times \mathcal{D}_0$.

iv) If $\mathcal{M} = \mathcal{D} = \mathbb{R}^n$ and there exist a continuously differentiable function $V : [t_0, \infty) \times \mathcal{D} \to \mathbb{R}$, class $\mathcal{K}_\infty$ functions $\alpha(\cdot)$ and $\beta(\cdot)$, a continuous function $k : [t_0, \infty) \to \mathbb{R}_+$, and a real number $\theta \in (0, 1)$ such that (8.43), (8.44), and (8.47) hold, then $\mathcal{G}$ is globally uniformly finite-time stable. Moreover, there exists a settling-time function $T : [0, \infty) \times \mathbb{R}^n \to [t_0, \infty)$ such that (8.45) holds with $\mathcal{D}_0 = \mathbb{R}^n$ and $T(\cdot, \cdot)$ is jointly continuous on $[0, \infty) \times \mathbb{R}^{n_2}$.

v) If there exist a continuously differentiable function $V : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}$, class $\mathcal{K}$ functions $\alpha(\cdot)$ and $\beta(\cdot)$, a real number $\theta \in (0, 1)$, and an open neighborhood $\mathcal{M} \subseteq \mathcal{D}$ of the origin such that (8.43), (8.44), and (8.47) hold with $k(t) = k \in \mathbb{R}_+$, $t \geq t_0$, then $\mathcal{G}$ is strongly uniformly finite-time stable. Moreover, there exist a neighborhood $\mathcal{D}_0$ of the origin and a settling-time function $T : [0, \infty) \times \mathcal{D}_0 \to [t_0, \infty)$ such that

$$T(t_0, x_0) \leq \frac{(V(t_0, x_0))^{1-\theta}}{k(1-\theta)}, \qquad (t_0, x_0) \in [0, \infty) \times \mathcal{D}_0, \tag{8.48}$$

and $T(\cdot, \cdot)$ is jointly continuous on $[0, \infty) \times \mathcal{D}_0$.

vi) If $\mathcal{M} = \mathcal{D} = \mathbb{R}^n$ and there exist a continuously differentiable function $V : \mathcal{D} \times \mathbb{R}^n \to \mathbb{R}$, class $\mathcal{K}_\infty$ functions $\alpha(\cdot)$ and $\beta(\cdot)$, and a real number $\theta \in (0, 1)$ such that (8.43), (8.44), and (8.47) hold with $k(t) = k \in \mathbb{R}_+$, $t \geq t_0$, then $\mathcal{G}$ is globally strongly uniformly finite-time stable. Moreover, there exists a settling-time function $T : [0, \infty) \times \mathbb{R}^n \to [t_0, \infty)$ such that (8.48) holds with $\mathcal{D}_0 = \mathbb{R}^n$ and $T(\cdot, \cdot)$ is jointly continuous on $[0, \infty) \times \mathbb{R}^{n_1}$.

Proof: The result is a direct consequence of Theorem 8.6 with $n_1 = n$, $n_2 = 1$, $x_1(t - t_0) = x(t)$, $x_2(t - t_0) = t$, $f_1(x_1, x_2) = f_1(x_2, x_1) = f(t, x)$, $f_2(x_1, x_2) = 1$, and $T(x_{10}, x_{20}) = T(x_{20}, x_{10}) = T(t_0, x_0)$. ■

Remark 8.11. Propositions 8.8 and 8.9 along with Statements *i*)–*iv*) of Theorem 8.10 appear in [50]. See also [94].

Example 8.12. Consider the nonlinear time-varying dynamical system given by

$$\dot{x}(t) = -t\,(x(t))^{\frac{1}{3}} - t\,(x(t))^{\frac{1}{5}}, \qquad x(0) = x_0, \qquad t \geq t_0. \tag{8.49}$$

To show that the zero solution $x(t) \equiv 0$ to (8.49) is globally uniformly finite-time stable, consider the Lyapunov function candidate $V(t,x) = x^{\frac{4}{3}}$ and let $\mathcal{D} = \mathbb{R}$. Clearly, (8.43) and (8.47) hold, and

$$\dot{V}(t,x) = \frac{4}{3}x^{\frac{1}{3}}\left(-tx^{\frac{1}{3}} - tx^{\frac{1}{5}}\right) = -\frac{4}{3}t\left(x^{\frac{2}{3}} + x^{\frac{8}{15}}\right) \leq -k(t)\,(V(t,x))^{\frac{1}{2}}, \tag{8.50}$$

where $k(t) = 2t > 0$, $t \geq t_0$. Hence, it follows from $iv)$ of Theorem 8.10 that the zero solution $x(t) \equiv 0$ to (8.49) is globally uniformly finite-time stable. △

8.4. Optimal Finite-Time, Partial-State Stabilization

In the first part of this section, we provide connections between Lyapunov functions and nonquatratic cost evaluation. Specifically, we consider the problem of evaluating a nonlinear-nonquadratic performance measure that depends on the solution of the nonlinear dynamical system given by (8.1) and (8.2). In particular, we prove finite-time partial stability of (8.1) and (8.2), and show that the nonlinear-nonquadratic performance measure

$$J(x_{10}, x_{20}) \triangleq \int_0^\infty L(x_1(t), x_2(t))\mathrm{d}t, \tag{8.51}$$

where $L : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}$ is jointly continuous in x_1 and x_2, and $x_1(t)$ and $x_2(t)$, $t \geq 0$, satisfy (8.1) and (8.2), can be evaluated in a convenient form so long as (8.1) and (8.2) are related to an underlying Lyapunov function that is positive definite and decrescent with respect to x_1 and is related to an underlying Lyapunov function satisfying a differential inequality involving fractional powers.

Theorem 8.13. Consider the nonlinear dynamical system $\mathcal{G}$ given by (8.1) and (8.2) with performance measure (8.51). Assume that there exists a continuously differentiable function $V : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}$, class $\mathcal{K}$ functions $\alpha(\cdot)$ and $\beta(\cdot)$, a real number $\theta \in (0,1)$, and an open neighborhood $\mathcal{M} \subseteq \mathcal{D}$ of $x_1 = 0$ such that

$$\alpha(\|x_1\|) \leq V(x_1, x_2) \leq \beta(\|x_1\|), \qquad (x_1, x_2) \in \mathcal{M} \times \mathbb{R}^{n_2}, \tag{8.52}$$

$$\dot{V}(x_1, x_2) \leq -k(V(x_1, x_2))^{\theta}, \qquad (x_1, x_2) \in \mathcal{M} \times \mathbb{R}^{n_2}, \tag{8.53}$$

$$L(x_1, x_2) + V'(x_1, x_2) f(x_1, x_2) = 0, \qquad (x_1, x_2) \in \mathcal{M} \times \mathbb{R}^{n_2}. \tag{8.54}$$

Then the nonlinear dynamical system $\mathcal{G}$ is strongly finite-time stable with respect to x_1 uniformly in x_{20} and there exist a neighborhood $\mathcal{D}_0 \subseteq \mathcal{M}$ of $x_1 = 0$ and a settling-time function $T : \mathcal{D}_0 \times \mathbb{R}^{n_2} \to [0, \infty)$, jointly continuous on $\mathcal{D}_0 \times \mathbb{R}^{n_2}$, such that

$$T(x_{10}, x_{20}) \leq \frac{(V(x_{10}, x_{20}))^{1-\theta}}{k(1-\theta)}, \qquad (x_{10}, x_{20}) \in \mathcal{D}_0 \times \mathbb{R}^{n_2}. \tag{8.55}$$

In addition, for all $(x_{10}, x_{20}) \in \mathcal{D}_0 \times \mathbb{R}^{n_2}$,

$$J(x_{10}, x_{20}) = V(x_{10}, x_{20}). \tag{8.56}$$

Finally, if $\mathcal{M} = \mathcal{D} = \mathbb{R}^{n_1}$ and the functions $\alpha(\cdot)$ and $\beta(\cdot)$ satisfying (8.52) are class $\mathcal{K}_\infty$, then $\mathcal{G}$ is globally strongly finite-time stable with respect to x_1 uniformly in x_{20}.

Proof: Let $x_1(t)$ and $x_2(t)$, $t \geq 0$, satisfy (8.1) and (8.2). Then it follows from (8.53) that

$$\dot{V}(x_1(t), x_2(t)) = V'(x_1(t), x_2(t)) f(x_1(t), x_2(t)) \leq -k(V(x_1(t), x_2(t)))^{\theta}, \qquad t \geq 0. \tag{8.57}$$

Thus, it follows from (8.52), (8.53), and v) of Theorem 8.6 that $\mathcal{G}$ is strongly finite-time stable with respect to x_1 uniformly in x_{20}. In addition, it follows from Theorem 8.6 that there exist an open neighborhood $\mathcal{D}_0$ of $x_1 = 0$ and a jointly continuous settling-time function $T : \mathcal{D}_0 \times \mathbb{R}^{n_2} \to [0, \infty)$ such that (8.55) holds and $x_1(t) \to 0$ as $t \to T(x_{10}, x_{20})$ for all initial condition $(x_{10}, x_{20}) \in \mathcal{D}_0 \times \mathbb{R}^{n_2}$. Now, since

$$0 = -\dot{V}(x_1(t), x_2(t)) + V'(x_1(t), x_2(t)) f(x_1(t), x_2(t)), \qquad t \geq 0, \tag{8.58}$$

it follows from (8.54) that

$$\begin{aligned} L(x_1(t), x_2(t)) &= -\dot{V}(x_1(t), x_2(t)) + L(x_1(t), x_2(t)) + V'(x_1(t), x_2(t)) f(x_1(t), x_2(t)) \\ &= -\dot{V}(x_1(t), x_2(t)), \qquad t \geq 0. \end{aligned} \tag{8.59}$$

Next, integrating (8.59) over $[0, t]$ yields

$$\int_0^t L(x_1(s), x_2(s))\mathrm{d}s = V(x_{10}, x_{20}) - V(x_1(t), x_2(t)), \qquad t \geq 0. \tag{8.60}$$

Now, using (8.52) and letting $t \to \infty$ it follows from (8.60) that

$$V(x_{10}, x_{20}) - \beta\Big(\lim_{t\to\infty} \|x_1(t)\|\Big) \leq \int_0^\infty L(x_1(s), x_2(s))\mathrm{d}s \leq V(x_{10}, x_{20}) - \alpha\Big(\lim_{t\to\infty} \|x_1(t)\|\Big), \tag{8.61}$$

and hence, (8.56) is a direct consequence of (8.61) using the fact that $\lim_{t\to T(x_{10},x_{20})} x_1(t) = \lim_{t\to\infty} x_1(t) = 0$ and $\alpha(\cdot)$ and $\beta(\cdot)$ are class $\mathcal{K}$ functions. Finally, if $\mathcal{M} = \mathcal{D} = \mathbb{R}^{n_1}$ and $\alpha(\cdot)$ and $\beta(\cdot)$ are class $\mathcal{K}_\infty$ functions, then global strong finite-time stability with respect to x_1 uniformly in x_{20} is a direct consequence of *vi*) of Theorem 8.6. ■

The following corollary to Theorem 8.13 considers the nonautonomous dynamical system (8.3) with performance measure

$$J(t_0, x_0) \triangleq \int_{t_0}^\infty L(t, x(t))\mathrm{d}t, \tag{8.62}$$

where $L : [t_0, \infty) \times \mathcal{D} \to \mathbb{R}$ is jointly continuous in t and x, and $x(t)$, $t \geq t_0$, satisfies (8.3).

Corollary 8.14. Consider the nonlinear time-varying dynamical system (8.3) with performance measure (8.62). Assume that there exist a continuously differentiable function $V : [t_0, \infty) \times \mathcal{D} \to \mathbb{R}$, class $\mathcal{K}$ functions $\alpha(\cdot)$ and $\beta(\cdot)$, a real number $\theta \in (0, 1)$, and an open neighborhood $\mathcal{M} \subseteq \mathcal{D}$ of the origin such that

$$\alpha(\|x\|) \leq V(t, x) \leq \beta(\|x\|), \qquad (t, x) \in [t_0, \infty) \times \mathcal{M}, \tag{8.63}$$

$$\dot{V}(t, x) \leq -k(V(t, x))^\theta, \qquad (t, x) \in [t_0, \infty) \times \mathcal{M}, \tag{8.64}$$

$$0 = \frac{\partial V(t,x)}{\partial t} + L(t,x) + \frac{\partial V(t,x)}{\partial x} f(t,x), \qquad (t,x) \in [t_0, \infty) \times \mathcal{M}. \tag{8.65}$$

Then the nonlinear time-varying dynamical system (8.3) is strongly uniformly finite-time stable and there exist a neighborhood of the origin $\mathcal{D}_0 \subseteq \mathcal{M}$ and a settling-time function $T : [0, \infty) \times \mathcal{D}_0 \to [t_0, \infty)$, jointly continuous on $[0, \infty) \times \mathcal{D}_0$, such that

$$T(t_0, x_0) \leq \frac{(V(t_0, x_0))^{1-\theta}}{k(1-\theta)}, \qquad (t_0, x_0) \in [0, \infty) \times \mathcal{D}_0. \tag{8.66}$$

In addition, for all $(t_0, x_0) \in [0, \infty) \times \mathcal{D}_0$,

$$J(t_0, x_0) = V(t_0, x_0). \tag{8.67}$$

Finally, if $\mathcal{D} = \mathbb{R}^n$ and the functions $\alpha(\cdot)$ and $\beta(\cdot)$ satisfying (8.52) are class $\mathcal{K}_\infty$, then $\mathcal{G}$ is globally strongly finite-time stable.

Proof: The result is a direct consequence of Theorem 8.13 with $n_1 = n$, $n_2 = 1$, $x_1(t - t_0) = x(t)$, $x_2(t - t_0) = t$, $f_1(x_1, x_2) = f_1(x_2, x_1) = f(t, x)$, $f_2(x_1, x_2) = 1$, $T(x_{10}, x_{20}) = T(x_{20}, x_{10}) = T(t_0, x_0)$, and $V(x_1, x_2) = V(x_2, x_1) = V(t, x)$. ■

Next, we use the framework developed in Theorem 8.13 to obtain a characterization of optimal feedback controllers that guarantee closed-loop finite-time partial stabilization. Specifically, sufficient conditions for optimality are given in a form that corresponds to a steady-state version of the Hamilton-Jacobi-Bellman equation. To address the problem of characterizing finite-time partially stabilizing feedback controllers, consider the controlled nonlinear dynamical system

$$\dot{x}_1(t) = F_1(x_1(t), x_2(t), u(t)), \qquad x_1(0) = x_{10}, \qquad t \geq 0, \tag{8.68}$$

$$\dot{x}_2(t) = F_2(x_1(t), x_2(t), u(t)), \qquad x_2(0) = x_{20}, \tag{8.69}$$

where, for every $t \geq 0$, $x_1(t) \in \mathcal{D} \subseteq \mathbb{R}^{n_1}$, $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$, $x_2(t) \in \mathbb{R}^{n_2}$, $u(t) \in U \subseteq \mathbb{R}^m$ with $0 \in U$, $F_1 : \mathcal{D} \times \mathbb{R}^{n_2} \times U \to \mathbb{R}^{n_1}$ and $F_2 : \mathcal{D} \times \mathbb{R}^{n_2} \times U \to \mathbb{R}^{n_2}$ are jointly continuous in x_1, x_2, and u, and $F_1(0, x_2, 0) = 0$ for every $x_2 \in \mathbb{R}^{n_2}$. The control $u(\cdot)$

in (8.68) and (8.69) is restricted to the class of *admissible* controls consisting of measurable functions $u(\cdot)$ such that $u(t) \in U$, $t \geq 0$.

A measurable function $\phi : \mathcal{D} \times \mathbb{R}^{n_2} \to U$ satisfying $\phi(0, x_2) = 0$, $x_2 \in \mathbb{R}^{n_2}$, is called a *control law.* If $u(t) = \phi(x_1(t), x_2(t))$, $t \geq 0$, where $\phi(\cdot, \cdot)$ is a control law and $x_1(t)$ and $x_2(t)$ satisfy (8.68) and (8.69), then we call $u(\cdot)$ a *feedback control law.* Note that the feedback control law is an admissible control since $\phi(\cdot, \cdot)$ has values in U. Given a control law $\phi(\cdot, \cdot)$ and a feedback control law $u(t) = \phi(x_1(t), x_2(t))$, $t \geq 0$, the *closed-loop system* (8.68) and (8.69) is given by

$$\dot{x}_1(t) = F_1(x_1(t), x_2(t), \phi(x_1(t), x_2(t))), \qquad x_1(0) = x_{10}, \qquad t \geq 0, \tag{8.70}$$

$$\dot{x}_2(t) = F_2(x_1(t), x_2(t), \phi(x_1(t), x_2(t))), \qquad x_2(0) = x_{20}. \tag{8.71}$$

We now consider the problem of finite-time partial-state stabilization.

Definition 8.15. Consider the controlled nonlinear dynamical system given by (8.68) and (8.69). The feedback control law $u = \phi(x_1, x_2)$ is *strongly finite-time stabilizing with respect to* x_1 *uniformly in* x_{20} if the closed-loop system (8.70) and (8.71) is strongly finite-time stable with respect to x_1 uniformly in x_{20}. Furthermore, the feedback control law $u = \phi(x_1, x_2)$ is *globally strongly finite-time stabilizing with respect to* x_1 *uniformly in* x_{20} if the closed-loop system (8.70) and (8.71) is globally strongly finite-time stable with respect to x_1 uniformly in x_{20}.

Next, we present a main theorem for strong finite-time, partial-state stabilization characterizing feedback controllers that guarantee closed-loop finite-time partial stability and minimize a nonlinear-nonquadratic performance functional. For the statement of this result, define $F(x_1, x_2, u) \triangleq [F_1^{\mathrm{T}}(x_1, x_2, u),\ F_2^{\mathrm{T}}(x_1, x_2, u)]^{\mathrm{T}}$, let $L : \mathcal{D} \times \mathbb{R}^{n_2} \times U \to \mathbb{R}$ be jointly continuous in x_1, x_2, and u, and define the set of partial regulation controllers given by

$$\mathcal{S}(x_{10}, x_{20}) \triangleq \{u(\cdot) : u(\cdot) \text{ is admissible and } x_1(\cdot) \text{ given by (8.68)}$$

$$\text{satisfies } x_1(t) \to 0 \text{ as } t \to T(x_{10}, x_{20})\},$$

where $T : \mathcal{D}_0 \times \mathbb{R}^{n_2} \to (0, \infty)$ is the settling-time function and $\mathcal{D}_0 \subseteq \mathcal{D}$ is an open neighborhood of $x_1 = 0$. Note that restricting our minimization problem to $u(\cdot) \in \mathcal{S}(x_{10}, x_{20})$, that is, inputs corresponding to partial-state null convergent solutions, can be interpreted as incorporating a partial-state system detectability condition through the cost. In addition, since finite-time partial convergence is a stronger condition than asymptotic partial-state convergence, $\mathcal{S}(x_{10}, x_{20})$ includes the set of all partial-state null asymptotically convergent controllers.

Theorem 8.16. Consider the controlled nonlinear dynamical system $\mathcal{G}$ given by (8.68) and (8.69) with

$$J(x_{10}, x_{20}, u(\cdot)) \triangleq \int_0^\infty L(x_1(t), x_2(t), u(t))\,\mathrm{d}t, \tag{8.72}$$

where $u(\cdot)$ is an admissible control. Assume that there exist a continuously differentiable function $V : \mathcal{D} \times \mathbb{R}^{n_2} \to \mathbb{R}$, class $\mathcal{K}$ functions $\alpha(\cdot)$ and $\beta(\cdot)$, a real number $\theta \in (0, 1)$, an open neighborhood $\mathcal{M} \subseteq \mathcal{D}$ of $x_1 = 0$, and a control law $\phi : \mathcal{M} \times \mathbb{R}^{n_2} \to U$ such that

$$\alpha(\|x_1\|) \le V(x_1, x_2) \le \beta(\|x_1\|), \qquad (x_1, x_2) \in \mathcal{M} \times \mathbb{R}^{n_2}, \tag{8.73}$$

$$V'(x_1, x_2)F(x_1, x_2, \phi(x_1, x_2)) \le -k(V(x_1, x_2))^\theta, \qquad (x_1, x_2) \in \mathcal{M} \times \mathbb{R}^{n_2}, \tag{8.74}$$

$$\phi(0, x_2) = 0, \qquad x_2 \in \mathbb{R}^{n_2}, \tag{8.75}$$

$$L(x_1, x_2, \phi(x_1, x_2)) + V'(x_1, x_2)F(x_1, x_2, \phi(x_1, x_2)) = 0, \qquad (x_1, x_2) \in \mathcal{M} \times \mathbb{R}^{n_2}, \tag{8.76}$$

$$L(x_1, x_2, u) + V'(x_1, x_2)F(x_1, x_2, u) \ge 0, \qquad (x_1, x_2, u) \in \mathcal{M} \times \mathbb{R}^{n_2} \times U. \tag{8.77}$$

Then, with the feedback control $u = \phi(x_1, x_2)$, the closed-loop system given by (8.70) and (8.71) is strongly finite-time stable with respect to x_1 uniformly in x_{20} and there exist a neighborhood $\mathcal{D}_0 \subseteq \mathcal{M}$ of $x_1 = 0$ and a settling-time function $T : \mathcal{D}_0 \times \mathbb{R}^{n_2} \to [0, \infty)$, jointly continuous on $\mathcal{D}_0 \times \mathbb{R}^{n_2}$, such that

$$T(x_{10}, x_{20}) \le \frac{(V(x_{10}, x_{20}))^{1-\theta}}{k(1-\theta)}, \qquad (x_{10}, x_{20}) \in \mathcal{D}_0 \times \mathbb{R}^{n_2}. \tag{8.78}$$

In addition, if $(x_{10}, x_{20}) \in \mathcal{D}_0 \times \mathbb{R}^{n_2}$, then

$$J(x_{10}, x_{20}, \phi(x_1(\cdot), x_2(\cdot))) = V(x_{10}, x_{20}), \qquad (x_{10}, x_{20}) \in \mathcal{D}_0 \times \mathbb{R}^{n_2} \tag{8.79}$$

and the feedback control $u(\cdot) = \phi(x_1(\cdot), x_2(\cdot))$ minimizes $J(x_{10}, x_{20}, u(\cdot))$ in the sense that

$$J(x_{10}, x_{20}, \phi(\cdot, \cdot)) = \min_{u(\cdot)\in\mathcal{S}(x_{10},x_{20})} J(x_{10}, x_{20}, u(\cdot)). \tag{8.80}$$

Finally, if $\mathcal{M} = \mathcal{D} = \mathbb{R}^{n_1}$, $U = \mathbb{R}^m$, and the functions $\alpha(\cdot)$ and $\beta(\cdot)$ satisfying (8.73) are class $\mathcal{K}_\infty$, then the closed-loop system (8.70) and (8.71) is globally strongly finite-time stable with respect to x_1 uniformly in x_{20}.

Proof: Local and global strong finite-time stability with respect to x_1 uniformly in x_{20} are a direct consequence of (8.73) and (8.74) by applying Theorem 8.6 to the closed-loop system given by (8.70) and (8.71). In addition, it follows from Theorem 8.6 that there exist an open neighborhood $\mathcal{D}_0$ of $x_1 = 0$ and a jointly continuous settling-time function $T : \mathcal{D}_0 \times \mathbb{R}^{n_2} \to [0, \infty)$ such that (8.78) holds and $x_1(t) \to 0$ as $t \to T(x_{10}, x_{20})$ for all initial conditions $(x_{10}, x_{20}) \in \mathcal{D}_0 \times \mathbb{R}^{n_2}$. Furthermore, using (8.76), condition (8.79) is a restatement of (8.56) as applied to the closed-loop system.

Next, let $(x_{10}, x_{20}) \in \mathcal{D}_0 \times \mathbb{R}^{n_2}$, let $u(\cdot) \in \mathcal{S}(x_{10}, x_{20})$, and let $x_1(t)$ and $x_2(t)$, $t \geq 0$, be solutions of (8.68) and (8.69). Then, it follows that

$$0 = -\dot{V}(x_1(t), x_2(t)) + V'(x_1(t), x_2(t))F(x_1(t), x_2(t), u(t)), \qquad t \geq 0. \tag{8.81}$$

Hence,

$$\begin{aligned} L(x_1(t), x_2(t), u(t)) = {} & -\dot{V}(x_1(t), x_2(t)) + L(x_1(t), x_2(t), u(t)) \\ & + V'(x_1(t), x_2(t))F(x_1(t), x_2(t), u(t)), \qquad t \geq 0. \end{aligned} \tag{8.82}$$

Now, using (8.73) and the fact that $\mathcal{G}$ is strongly finite-time stable with respect to x_1 uniformly in x_{20}, it follows that

$$0 = \lim_{t\to\infty} \alpha(\|x_1(t)\|) \leq \lim_{t\to\infty} V(x_1(t), x_2(t)) \leq \lim_{t\to\infty} \beta(\|x_1(t)\|) = 0. \tag{8.83}$$

Thus, it follows from (8.82), (8.83), (8.77), (8.79), and the strong finite-time stability of $\mathcal{G}$ with respect to x_1 uniformly in x_{20}, that

$$\begin{aligned}\int_0^\infty L(x_1(t),x_2(t),u(t))\mathrm{d}t &= \int_0^\infty -\dot{V}(x_1(t),x_2(t))\mathrm{d}t + \int_0^\infty L(x_1(t),x_2(t),u(t))\mathrm{d}t \\ &\quad + \int_0^\infty \bigg(\frac{\partial V(x_1,x_2)}{\partial x_1}F_1(x_1(t),x_2(t),u(t)) \\ &\quad + \frac{\partial V(x_1,x_2)}{\partial x_2}F_2(x_1(t),x_2(t),u(t))\bigg)\mathrm{d}t \\ &\geq \int_0^\infty -\dot{V}(x_1(t),x_2(t))\mathrm{d}t \\ &= -\lim_{t\to\infty} V(x_1(t),x_2(t)) + V(x_{10},x_{20}) \\ &= -V\left(\lim_{t\to\infty}(x_1(t),x_2(t))\right) + V(x_{10},x_{20}) \\ &= -V\left(\lim_{t\to T(x_{10},x_{20})}(x_1(t),x_2(t))\right) + V(x_{10},x_{20}) \\ &= J(x_{10},x_{20},\phi(x_1(\cdot),x_2(\cdot))),\end{aligned} \tag{8.84}$$

which yields (8.80). ■

Note that (8.76) is the steady-state, Hamilton-Jacobi-Bellman equation for the nonlinear controlled dynamical system (8.68) and (8.69) with performance criterion (8.72). Furthermore, conditions (8.76) and (8.77) guarantee optimality with respect to the set of admissible finite-time partially stabilizing controllers $\mathcal{S}(x_{10},x_{20})$. However, it is important to note that an explicit characterization of $\mathcal{S}(x_{10},x_{20})$ is not required. In addition, the optimal strongly finite-time stabilizing with respect to x_1 uniformly in x_{20} *feedback* control law $u=\phi(x_1,x_2)$ is independent of the initial condition (x_{10},x_{20}) and is given by

$$\phi(x_1,x_2) = \underset{u\in\mathcal{S}(x_{10},x_{20})}{\arg\min}\left[L(x_1,x_2,u) + \frac{\partial V(x_1,x_2)}{\partial x_1}F_1(x_1,x_2,u) + \frac{\partial V(x_1,x_2)}{\partial x_2}F_2(x_1,x_2,u)\right]. \tag{8.85}$$

Finally, we use Theorem 8.16 to provide a unification between optimal finite-time, partial-state stabilization and optimal finite-time control for nonlinear time-varying systems. Specifically, consider the controlled nonlinear time-varying dynamical system

$$\dot{x}(t) = F(t,x(t),u(t)), \qquad x(t_0)=x_0, \qquad t\geq t_0, \tag{8.86}$$

with performance measure

$$J(t_0, x_0, u(\cdot)) \triangleq \int_{t_0}^{\infty} L(t, x(t), u(t))\mathrm{d}t, \tag{8.87}$$

where, for every $t \geq t_0$, $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$, $u(t) \in U \subseteq \mathbb{R}^m$ with $0 \in U$, and $L : [t_0, \infty) \times \mathcal{D} \times U \to \mathbb{R}$ and $F : [t_0, \infty) \times \mathcal{D} \times U \to \mathbb{R}^n$ are jointly continuous in t, x, and u on $[t_0, \infty) \times \mathcal{D} \times U$. For the statement of the next result, define the set of regulation controllers

$$\mathcal{S}(t_0, x_0) \triangleq \{u(\cdot) : u(\cdot) \text{ is admissible and } x(\cdot) \text{ given by (8.86)}$$
$$\text{satisfies } x(t) \to 0 \text{ as } t \to T(t_0, x_0)\},$$

where $T : [0, \infty) \times \mathcal{D}_0 \to (t_0, \infty)$ is the settling-time function and $\mathcal{D}_0 \subseteq \mathcal{D}$ is an open neighborhood of the origin.

Corollary 8.17. Consider the controlled nonlinear time-varying dynamical system (8.86) with performance measure (8.87) where $u(\cdot)$ is an admissible control. Assume that there exist a continuously differentiable function $V : [t_0, \infty) \times \mathcal{D} \to \mathbb{R}$, class $\mathcal{K}$ functions $\alpha(\cdot)$ and $\beta(\cdot)$, a real number $\theta \in (0, 1)$, an open neighborhood $\mathcal{M} \subseteq \mathcal{D}$ of the origin, and a control law $\phi : [t_0, \infty) \times \mathcal{M} \to U$ such that

$$\alpha(\|x\|) \leq V(t, x) \leq \beta(\|x\|), \quad (t, x) \in [t_0, \infty) \times \mathcal{M}, \tag{8.88}$$

$$\frac{\partial V(t, x)}{\partial t} + \frac{\partial V(t, x)}{\partial x} F(t, x, \phi(t, x)) \leq -k(V(t, x(t))^{\theta}, \quad (t, x) \in [t_0, \infty) \times \mathcal{M}, \tag{8.89}$$

$$\phi(t, 0) = 0, \quad t \in [t_0, \infty), \tag{8.90}$$

$$L(t, x, \phi(t, x)) + \frac{\partial V(t, x)}{\partial t} + \frac{\partial V(t, x)}{\partial x} F(t, x, \phi(t, x)) = 0, \quad (t, x) \in [t_0, \infty) \times \mathcal{M}, \tag{8.91}$$

$$L(t, x, u) + \frac{\partial V(t, x)}{\partial t} + \frac{\partial V(t, x)}{\partial x} F(t, x, u) \geq 0, \quad (t, x, u) \in [t_0, \infty) \times \mathcal{M} \times U. \tag{8.92}$$

Then, with the feedback control $u = \phi(t, x)$, the closed-loop system given by

$$\dot{x}(t) = F(t, x(t), \phi(x(t))), \qquad x(0) = x_0, \qquad t \geq t_0, \tag{8.93}$$

is strongly uniformly finite-time stable and there exists a neighborhood of the origin $\mathcal{D}_0 \subseteq \mathcal{M}$ and a settling-time function $T : [0,\infty) \times \mathcal{D}_0 \to [t_0,\infty)$, jointly continuous on $[0,\infty)\times\mathcal{D}_0$, such that

$$T(t_0,x_0) \le \frac{(V(t_0,x_0))^{1-\theta}}{k(1-\theta)}, \qquad (t_0,x_0) \in [0,\infty)\times\mathcal{D}_0. \tag{8.94}$$

In addition, if $(t_0,x_0) \in [0,\infty)\times\mathcal{D}_0$, then

$$J(t_0,x_0,\phi(\cdot,\cdot)) = V(t_0,x_0), \qquad (t_0,x_0) \in [0,\infty)\times\mathcal{D}_0. \tag{8.95}$$

and the feedback control $u(\cdot) = \phi(\cdot,x(\cdot))$ minimizes $J(t_0,x_0,u(\cdot))$ in the sense that

$$J(t_0,x_0,\phi(\cdot,\cdot)) = \min_{u(\cdot)\in\mathcal{S}(t_0,x_0)} J(t_0,x_0,u(\cdot)). \tag{8.96}$$

Finally, if $\mathcal{D} = \mathbb{R}^n$, $U = \mathbb{R}^m$, and the functions $\alpha(\cdot)$ and $\beta(\cdot)$ satisfying (8.88) are class $\mathcal{K}_\infty$, then the nonlinear dynamical system $\mathcal{G}$ is globally uniformly asymptotically stable.

Proof: The proof is a direct consequence of Theorem 8.16 with $n_1 = n$, $n_2 = 1$, $x_1(t-t_0) = x(t)$, $x_2(t-t_0) = t$, $F_1(x_1,x_2,u) = F_1(x_2,x_1,u) = F(t,x,u)$, $F_2(x_1,x_2,u) = 1$, $\phi(x_1,x_2) = \phi(x_2,x_1) = \phi(t,x)$, $T(x_{10},x_{20}) = T(x_{20},x_{10}) = T(t_0,x_0)$, and $V(x_1,x_2) = V(x_2,x_1) = V(t,x)$. ■

Note that (8.91) and (8.92) give the classical Hamilton-Jacobi-Bellman equation

$$-\frac{\partial V(t,x)}{\partial t} = \min_{u\in\mathcal{S}(t_0,x_0)} \left[L(t,x,u) + \frac{\partial V(t,x)}{\partial x}F(t,x,u)\right], \qquad (t,x)\in[t_0,\infty)\times\mathcal{D}, \tag{8.97}$$

which characterizes the optimal control

$$\phi(t,x) = \arg\min_{u\in\mathcal{S}(t_0,x_0)} \left[L(t,x,u) + \frac{\partial V(t,x)}{\partial x}F(t,x,u)\right] \tag{8.98}$$

for time-varying systems on a finite or infinite interval.

8.5. Finite-Time Stabilization for Affine Dynamical Systems and Connections to Inverse Optimal Control

In this section, we specialize the results of Section 8.4 to nonlinear affine dynamical systems of the form

$$\dot{x}_1(t) = f_1(x_1(t), x_2(t)) + G_1(x_1(t), x_2(t))u(t), \quad x_1(0) = x_{10}, \quad t \geq 0, \tag{8.99}$$

$$\dot{x}_2(t) = f_2(x_1(t), x_2(t)) + G_2(x_1(t), x_2(t))u(t), \quad x_2(0) = x_{20}, \tag{8.100}$$

where, for every $t \geq 0$, $x_1(t) \in \mathbb{R}^{n_1}$, $x_2(t) \in \mathbb{R}^{n_2}$, and $u(t) \in \mathbb{R}^m$, and $f_1 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_1}$, $f_2 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_2}$, $G_1 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_1 \times m}$, and $G_2 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_2 \times m}$ are such that $f_1(0, x_2) = 0$ for all $x_2 \in \mathbb{R}^{n_2}$, and $f_1(\cdot,\cdot)$, $f_2(\cdot,\cdot)$, $G_1(\cdot,\cdot)$, and $G_2(\cdot,\cdot)$ are jointly continuous in x_1 and x_2 on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. Furthermore, we consider performance integrands $L(x_1, x_2, u)$ of the form

$$L(x_1, x_2, u) = L_1(x_1, x_2) + L_2(x_1, x_2)u + u^{\mathrm{T}} R_2(x_1, x_2)u, \qquad (x_1, x_2, u) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^m, \tag{8.101}$$

where $L_1 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$, $L_2 : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{1 \times m}$, and $R_2(x_1, x_2) \geq N(x_1) > 0$, $(x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, so that (8.72) becomes

$$J(x_{10}, x_{20}, u(\cdot)) = \int_0^\infty \big[L_1(x_1(t), x_2(t)) + L_2(x_1(t), x_2(t))u(t) + u^{\mathrm{T}}(t) R_2(x_1(t), x_2(t))u(t)\big]\mathrm{d}t. \tag{8.102}$$

For the statement of the next result, define

$$f(x_1, x_2) \triangleq [f_1^{\mathrm{T}}(x_1, x_2), f_2^{\mathrm{T}}(x_1, x_2)]^{\mathrm{T}}, \qquad G(x_1, x_2) \triangleq [G_1^{\mathrm{T}}(x_1, x_2), G_2^{\mathrm{T}}(x_1, x_2)]^{\mathrm{T}}. \tag{8.103}$$

Theorem 8.18. Consider the controlled nonlinear affine dynamical system (8.99) and (8.100) with performance measure (8.102). Assume that there exist a continuously differentiable function $V : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$, class $\mathcal{K}_\infty$ functions $\alpha(\cdot)$ and $\beta(\cdot)$, and a real number $\theta \in (0, 1)$ such that

$$\alpha(\|x_1\|) \leq V(x_1, x_2) \leq \beta(\|x_1\|), \ \ (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \tag{8.104}$$

$$V'(x_1,x_2)\Big[f(x_1,x_2)-\frac{1}{2}G(x_1,x_2)R_2^{-1}(x_1,x_2)L_2^{\mathrm{T}}(x_1,x_2)$$
$$-\frac{1}{2}G(x_1,x_2)R_2^{-1}(x_1,x_2)G^{\mathrm{T}}(x_1,x_2)V'^{\mathrm{T}}(x_1,x_2)\Big] \le -k(V(x_1,x_2))^{\theta},\ \ (x_1,x_2)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}, \tag{8.105}$$

$$L_2(0,x_2)=0,\ \ x_2\in\mathbb{R}^{n_2}, \tag{8.106}$$

$$0 = L_1(x_1,x_2)+V'(x_1,x_2)f(x_1,x_2)-\frac{1}{4}\Big[V'(x_1,x_2)G(x_1,x_2)+L_2(x_1,x_2)\Big]$$
$$\cdot R_2^{-1}(x_1,x_2)\Big[V'(x_1,x_2)G(x_1,x_2)+L_2(x_1,x_2)\Big]^{\mathrm{T}},\qquad (x_1,x_2)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}, \tag{8.107}$$

Then, with the feedback control

$$u=\phi(x_1,x_2)=-\frac{1}{2}R_2^{-1}(x_1,x_2)\big[L_2(x_1,x_2)+V'(x_1,x_2)G(x_1,x_2)\big]^{\mathrm{T}}, \tag{8.108}$$

the closed-loop system

$$\dot{x}_1(t)=f_1(x_1(t),x_2(t))+G_1(x_1(t),x_2(t))\phi(x_1(t),x_2(t)),\ \ x_1(0)=x_{10},\ \ t\ge 0, \tag{8.109}$$

$$\dot{x}_2(t)=f_2(x_1(t),x_2(t))+G_2(x_1(t),x_2(t))\phi(x_1(t),x_2(t)),\ \ x_2(0)=x_{20}, \tag{8.110}$$

is globally strongly finite-time stable with respect to x_1 uniformly in x_{20} and there exists a settling-time function $T:\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}\to[0,\infty)$, jointly continuous on $\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$, such that

$$T(x_{10},x_{20})\le\frac{(V(x_{10},x_{20}))^{1-\theta}}{k(1-\theta)},\qquad (x_{10},x_{20})\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}. \tag{8.111}$$

In addition,

$$J(x_{10},x_{20},\phi(x_1(\cdot),x_2(\cdot))=V(x_{10},x_{20}),\qquad (x_{10},x_{20})\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}. \tag{8.112}$$

and the performance measure (8.102) is minimized in the sense of (8.80).

Proof: The result is a consequence of Theorem 8.16 with $\mathcal{M}=\mathcal{D}=\mathbb{R}^{n_1}$, $U=\mathbb{R}^m$, $F(x_1,x_2,u)=f(x_1,x_2)+G(x_1,x_2)u$, and

$$L(x_1,x_2,u)=L_1(x_1,x_2)+L_2(x_1,x_2)u+u^{\mathrm{T}}R_2(x_1,x_2)u.$$

Specifically, the feedback control law (8.108) follows from (8.85) by setting

$$\frac{\partial}{\partial u}\big[L_1(x_1,x_2)+L_2(x_1,x_2)u+u^{\mathrm{T}}R_2(x_1,x_2)u+V'(x_1,x_2)\big(f(x_1,x_2)+G(x_1,x_2)u\big)\big]=0. \tag{8.113}$$

Now, with $u=\phi(x_1,x_2)$ given by (8.108), conditions (8.104), (8.105), and (8.107) imply (8.73), (8.74), and (8.76), respectively.

Next, since $V(\cdot,\cdot)$ is continuously differentiable and, by (8.104), $V(0,x_2)$, $x_2\in\mathbb{R}^{n_2}$, is a local minimum of $V(\cdot,\cdot)$, it follows that $V'(0,x_2)=0$, $x_2\in\mathbb{R}^{n_2}$, and hence, it follows from (8.106) and (8.108) that $\phi(0,x_2)=0$, $x_2\in\mathbb{R}^{n_2}$, which implies (8.75). Finally, since

$$\begin{aligned}
L(x_1,x_2,u)&+V'(x_1,x_2)[f(x_1,x_2)+G(x_1,x_2)u]\\
&=L(x_1,x_2,u)+V'(x_1,x_2)[f(x_1,x_2)+G(x_1,x_2)u]-L(x_1,x_2,\phi(x_1,x_2))\\
&\quad\;-V'(x_1,x_2)[f(x_1,x_2)+G(x_1,x_2)\phi(x_1,x_2)]\\
&=[u-\phi(x_1,x_2)]^{\mathrm{T}}R_2(x_1,x_2)[u-\phi(x_1,x_2)]\\
&\geq 0,\qquad (x_1,x_2)\in\mathbb{R}^{n_1}\times\mathbb{R}^{n_2},
\end{aligned} \tag{8.114}$$

condition (8.77) holds. The result now follows as a direct consequence of Theorem 8.16. ■

The following corollary to Theorem 8.18 considers the nonautonomous dynamical system

$$\dot{x}(t)=f(t,x(t))+G(t,x(t))u(t),\qquad x(t_0)=x_0,\qquad t\geq t_0, \tag{8.115}$$

with performance measure

$$J(t_0,x_0,u(\cdot))=\int_{t_0}^{\infty}\big[L_1(t,x(t))+L_2(t,x(t))u(t)+u^{\mathrm{T}}(t)R_2(t,x(t))u(t)\big]\,\mathrm{d}t, \tag{8.116}$$

where, for every $t\geq t_0$, $x(t)\in\mathbb{R}^n$ and $u(t)\in\mathbb{R}^m$, $f:[t_0,\infty)\times\mathbb{R}^n\to\mathbb{R}^n$ and $G:[t_0,\infty)\times\mathbb{R}^n\to\mathbb{R}^{n\times m}$ are such that $f(t,0)=0$ for all $t\in[t_0,\infty)$, $f(\cdot,\cdot)$ and $G(\cdot,\cdot)$ are jointly continuous in x_1 and x_2 on $\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$, $L_1:[t_0,\infty)\times\mathbb{R}^n\to\mathbb{R}$, $L_2:[t_0,\infty)\times\mathbb{R}^n\to\mathbb{R}^{1\times m}$, and $R_2(t,x)\geq N(x)>0$, $(t,x)\in[t_0,\infty)\times\mathbb{R}^n$.

Corollary 8.19. Consider the controlled nonlinear affine dynamical system (8.115) with performance measure (8.116). Assume that there exist a continuously differentiable function $V : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}$, class $\mathcal{K}_\infty$ functions $\alpha(\cdot)$ and $\beta(\cdot)$, and a real number $\theta \in (0, 1)$ such that

$$\alpha(\|x\|) \leq V(t,x) \leq \beta(\|x\|), \quad (t,x) \in [t_0, \infty) \times \mathbb{R}^n, \tag{8.117}$$

$$\frac{\partial V(t,x)}{\partial t} + \frac{\partial V(t,x)}{\partial x}\Bigg[f(t,x) - \frac{1}{2}G(t,x)R_2^{-1}(t,x)L_2^{\mathrm{T}}(t,x)$$
$$-\frac{1}{2}G(t,x)R_2^{-1}(t,x)G^{\mathrm{T}}(t,x)\left(\frac{\partial V(t,x)}{\partial x}\right)^{\mathrm{T}}\Bigg] \leq -k(V(t,x))^\theta, \quad (t,x) \in [t_0, \infty) \times \mathbb{R}^n, \tag{8.118}$$

$$L_2(t,0) = 0, \quad t \in [t_0, \infty), \tag{8.119}$$

$$0 = L_1(t,x) + \frac{\partial V(t,x)}{\partial x}f(t,x) - \frac{1}{4}\left[\frac{\partial V(t,x)}{\partial x}G(t,x) + L_2(t,x)\right]$$
$$\cdot R_2^{-1}(t,x)\left[\frac{\partial V(t,x)}{\partial x}G(t,x) + L_2(t,x)\right]^{\mathrm{T}}, \qquad (t,x) \in [t_0, \infty) \times \mathbb{R}^n, \tag{8.120}$$

Then, with the feedback control

$$u = \phi(t,x) = -\frac{1}{2}R_2^{-1}(t,x)\left[L_2(t,x) + \frac{\partial V(t,x)}{\partial x}G(t,x)\right]^{\mathrm{T}}, \tag{8.121}$$

the closed-loop system

$$\dot{x}(t) = f(t,x(t)) + G(t,x(t))\phi(t,x(t)), \quad x(0) = x_0, \quad t \geq t_0, \tag{8.122}$$

is globally strongly uniformly finite-time stable and there exists a settling-time function $T : [0, \infty) \times \mathbb{R}^n \to [t_0, \infty)$, jointly continuous on $[0, \infty) \times \mathbb{R}^n$, such that

$$T(t_0, x_0) \leq \frac{(V(t_0,x_0))^{1-\theta}}{k(1-\theta)}, \qquad (t_0, x_0) \in [0, \infty) \times \mathbb{R}^n. \tag{8.123}$$

In addition,

$$J(t_0, x_0, \phi(\cdot, x(\cdot))) = V(t_0, x_0), \qquad (t_0, x_0) \in [0, \infty) \times \mathbb{R}^n \tag{8.124}$$

and the performance measure (8.116) is minimized in the sense of (8.96).

Proof: The proof is a direct consequence of Theorem 8.18 with $n_1 = n$, $n_2 = 1$, $x_1(t-t_0) = x(t)$, $x_2(t-t_0) = t$, $f(x_1,x_2) = f(x_2,x_1) = f(t,x)$, $G(x_1,x_2) = G(x_2,x_1) = G(t,x)$, $L_1(x_1,x_2) = L_1(x_2,x_1) = L_1(t,x)$, $L_2(x_1,x_2) = L_2(x_2,x_1) = L_2(t,x)$, $R_2(x_1,x_2) = R_2(x_2,x_1) = R_2(t,x)$, $\phi(x_1,x_2) = \phi(x_2,x_1) = \phi(t,x)$, $T(x_{10},x_{20}) = T(x_{20},x_{10}) = T(t_0,x_0)$, and $V(x_1,x_2) = V(x_2,x_1) = V(t,x)$. ■

Next, we construct state feedback controllers for nonlinear affine in the control dynamical systems that are predicated on an *inverse optimal control problem* [2, 32, 65, 92, 95]. In particular, to avoid the complexity in solving the steady-state, Hamilton-Jacobi-Bellman equation (8.107) we do not attempt to minimize a given cost functional, but rather, we parameterize a family of finite-time stabilizing controllers that minimize some derived cost functional that provides flexibility in specifying the control law. The performance integrand is shown to explicitly depend on the nonlinear system dynamics, the Lyapunov function of the closed-loop system, and the stabilizing feedback control law, wherein the coupling is introduced via the Hamilton-Jacobi-Bellman equation. Hence, by varying the parameters in the Lyapunov function and the performance integrand, the proposed framework can be used to characterize a class of globally finite-time partial-state stabilizing controllers that can meet closed-loop system response constraints.

Theorem 8.20. Consider the controlled nonlinear affine dynamical system (8.99) and (8.100) with performance measure (8.102). Assume there exist a continuously differentiable function $V : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$, class $\mathcal{K}_\infty$ functions $\alpha(\cdot)$ and $\beta(\cdot)$, and a real number $\theta \in (0,1)$ such that (8.104)–(8.106) hold. Then, with the feedback control (8.108), the closed-loop system given by (8.109) and (8.110) is globally strongly finite-time stable with respect to x_1 uniformly in x_{20} and there exists a settling-time function $T : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to [0,\infty)$, jointly continuous on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, such that (8.111) holds. In addition, the performance functional (8.102), with

$$L_1(x_1,x_2) = \phi^{\mathrm{T}}(x_1,x_2)R_2(x_1,x_2)\phi(x_1,x_2) - V'(x_1,x_2)f(x_1,x_2), \tag{8.125}$$

is minimized in the sense of (8.80) and (8.112) holds.

Proof: The proof is identical to the proof of Theorem 8.18. ■

The following corollary to Theorem 8.19 considers the nonautonomous dynamical system (8.115) with performance measure (8.116).

Corollary 8.21. Consider the controlled nonlinear affine dynamical system (8.115) with performance measure (8.116). Assume there exist a continuously differentiable function $V : [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}$, class $\mathcal{K}_\infty$ functions $\alpha(\cdot)$ and $\beta(\cdot)$, and a real number $\theta \in (0, 1)$ such that (8.117)–(8.119) hold. Then, with the feedback control (8.121), the closed-loop system given by (8.115) is globally strongly uniformly finite-time stable and there exists a settling-time function $T : [0, \infty) \times \mathbb{R}^n \to [t_0, \infty)$, jointly continuous on $[t_0, \infty) \times \mathbb{R}^n$, such that (8.123) holds. In addition, the performance functional (8.102), with

$$L_1(t, x) = \phi^{\mathrm{T}}(t, x)R_2(t, x)\phi(t, x) - \frac{\partial V(t, x)}{\partial t} - \frac{\partial V(t, x)}{\partial x} f(t, x), \tag{8.126}$$

is minimized in the sense of (8.96) and (8.124) holds.

Proof: The proof is identical to the proof of Theorem 8.20. ■

8.6. Illustrative Numerical Examples

In this section, we provide two numerical examples to highlight the optimal and inverse optimal finite-time, partial-state stabilization framework developed in this chapter.

8.6.1. Optimal Control of a Symmetric Spacecraft

Consider the spacecraft with two axes of symmetry [114, p. 753] given by

$$\dot{\omega}_1(t) = \alpha_1 u_1(t), \qquad \omega_1(0) = \omega_{10}, \qquad t \geq 0, \tag{8.127}$$

$$\dot{\omega}_2(t) = \alpha_1 u_2(t), \qquad \omega_2(0) = \omega_{20}, \tag{8.128}$$

$$\dot{\omega}_3(t) = \alpha_3 u_1(t) + \alpha_4 u_2(t), \qquad \omega_3(0) = \omega_{30}, \tag{8.129}$$

where $\omega_1 : [0, \infty) \to \mathbb{R}$, $\omega_2 : [0, \infty) \to \mathbb{R}$, and $\omega_3 : [0, \infty) \to \mathbb{R}$ denote the components of the angular velocity vector with respect to a given inertial reference frame expressed in a central body reference frame, α_1, α_3, $\alpha_4 \in \mathbb{R}$, $\alpha_1 \neq 0$, and u_1 and u_2 are the spacecraft control moments. For this example, we apply Theorem 8.18 to find an optimal globally partial-state stabilizing control law $u = [u_1,\ u_2]^{\mathrm{T}} = \phi(x_1, x_2)$, where $x_1 = [\omega_1,\ \omega_2]^{\mathrm{T}}$ and $x_2 = \omega_3$, such that the performance measure

$$J(x_1(0), x_2(0), u(\cdot)) = \int_0^\infty \left[\frac{4}{9}\alpha_1^2 \|x_1(t)\|^{\frac{2}{3}} + u^{\mathrm{T}}(t)u(t)\right] \mathrm{d}t \tag{8.130}$$

is minimized in the sense of (8.80), and the spacecraft is finite-time spin-stabilized about its third principle axis of inertia, that is, the dynamical system (8.127)–(8.129) is globally strongly finite-time stable with respect to x_1 uniformly in $x_2(0)$.

Note that (8.127)–(8.129) with the subquadratic performance measure (8.130) can be cast in the form of (8.99) and (8.100) with performance measure (8.102). In this case, Theorem 8.18 can be applied with $n_1 = 2$, $n_2 = 1$, $m = 2$, $f(x_1, x_2) = \left[0,\ 0,\ 0\right]^{\mathrm{T}}$, $G(x_1, x_2) = \begin{bmatrix} \alpha_1 & 0 & \alpha_3 \\ 0 & \alpha_1 & \alpha_4 \end{bmatrix}^{\mathrm{T}}$ $L_1(x_1, x_2) = \frac{4}{9}\alpha_1^2 \|x_1(t)\|^{\frac{2}{3}}$, $L_2(x_1, x_2) = 0$, and $R_2(x_1, x_2) = I_m$. Specifically, in this case, (8.107) reduces to

$$0 = L_1(x_1, x_2) - \frac{1}{4} V'(x_1, x_2) G(x_1, x_2) G^{\mathrm{T}}(x_1, x_2) V'^{\mathrm{T}}(x_1, x_2), \qquad (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \tag{8.131}$$

which is satisfied with $V'(x_1, x_2) = \frac{4}{3}\|x_1\|^{-\frac{2}{3}}[\omega_1, \omega_2, 0]^{\mathrm{T}}$. Hence, it follows from (8.104) that $V(x_1, x_2) = \frac{4}{9}\alpha_1^2 \|x_1\|^{\frac{2}{3}}$. Finally, (8.105) reduces to

$$-\frac{1}{2} V'(x_1, x_2) G(x_1, x_2) G^{\mathrm{T}}(x_1, x_2) V'^{\mathrm{T}}(x_1, x_2) \leq -k(V(x_1, x_2))^{\theta}, \;\; (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \tag{8.132}$$

which is satisfied with $k = \frac{8}{9}\alpha_1^2$ and $\theta = \frac{1}{2}$.

Since all of the conditions of Theorem 8.18 hold, it follows from (8.108) that the feedback

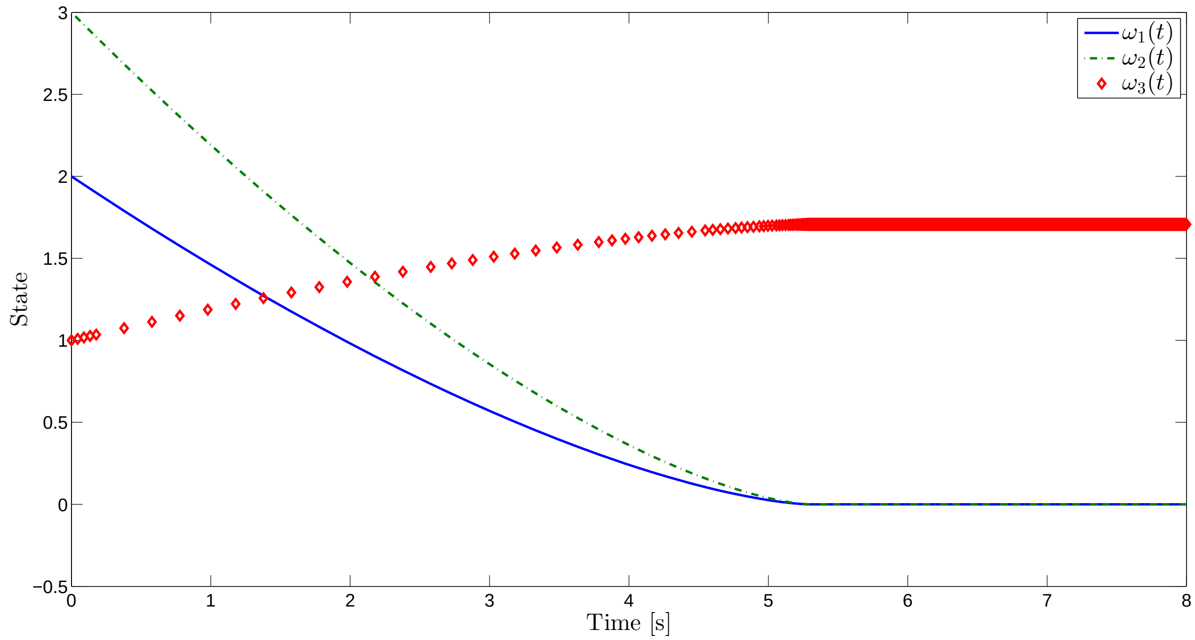

Figure 8.1: Closed-loop system trajectories versus time.

control law

$$\phi(x_1, x_2) = -\frac{2}{3}\alpha_1 \|x_1\|^{-\frac{2}{3}} x_1, \qquad (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \tag{8.133}$$

guarantees that the dynamical system (8.127)–(8.129) is globally strongly finite-time stable with respect to x_1 uniformly in $x_2(0)$ and, for all $(x_1(0), x_2(0)) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$,

$$J(x_1(0), x_2(0), \phi(x_1(\cdot), x_2(\cdot))) = \frac{4}{9}\alpha_1^2 \|x_1(0)\|^{\frac{2}{3}}. \tag{8.134}$$

Moreover, there exists a settling-time function $T : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to [0, \infty)$ such that

$$T(x_1(0), x_2(0)) \leq \frac{9}{4\alpha_1^2} \|x_1\|^{\frac{2}{3}}. \tag{8.135}$$

Let $\omega_{10} = 2\,\mathrm{Hz}$, $\omega_{20} = 3\,\mathrm{Hz}$, $\omega_3 = 1\,\mathrm{Hz}$, $\alpha_1 = 1$, $\alpha_3 = \frac{\sqrt{2}}{2}$, and $\alpha_4 = -\frac{\sqrt{2}}{2}$, Figure 8.1 shows the state trajectories of the controlled system versus time. Note that $x_1(t) = 0$ for $t = 5.2836\,\mathrm{s} < T(x_0) = 5.2905\,\mathrm{s}$. Figure 8.2 shows the control signal versus time. Finally, $J(x_{10}, x_{20}, \phi(x_1(\cdot), x_2(\cdot))) = 5.5287\,\mathrm{Hz}^2$.

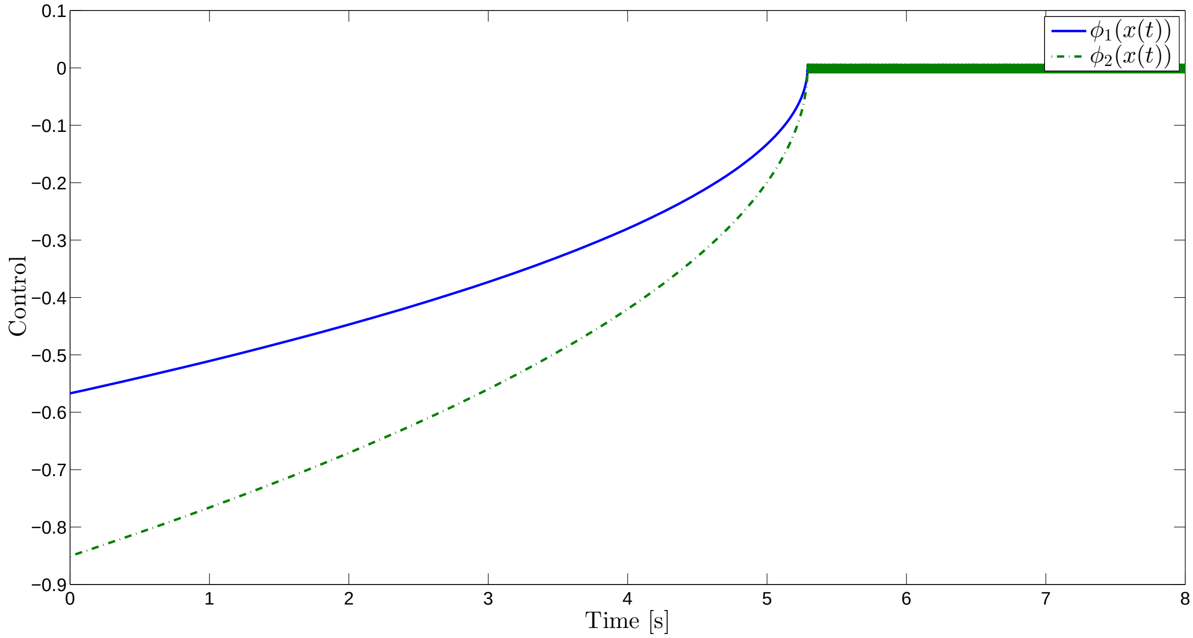

Figure 8.2: Control signal versus time.

8.6.2. Inverse Optimal Control of an Axisymmetric Spacecraft

Consider the spacecraft with one axis of symmetry [114, p. 753] given by

$$\dot{\omega}_1(t) = I_{23}\omega_2(t)\omega_3(t) + u_1(t), \qquad \omega_1(0) = \omega_{10}, \qquad t \geq 0, \tag{8.136}$$

$$\dot{\omega}_2(t) = -I_{23}\omega_3(t)\omega_1(t) + u_2(t), \qquad \omega_2(0) = \omega_{20}, \tag{8.137}$$

$$\dot{\omega}_3(t) = \alpha_3 u_1(t) + \alpha_4 u_2(t), \qquad \omega_3(0) = \omega_{30}, \tag{8.138}$$

where $I_{23} \triangleq (I_2 - I_3)/I_1$, I_1, I_2, and I_3 are the principal moments of inertia of the spacecraft such that $0 < I_1 = I_2 < I_3$, $\omega_1 : [0, \infty) \to \mathbb{R}$, $\omega_2 : [0, \infty) \to \mathbb{R}$, and $\omega_3 : [0, \infty) \to \mathbb{R}$ denote the components of the angular velocity vector with respect to a given inertial reference frame expressed in a central body reference frame, α_3 and $\alpha_4 \in \mathbb{R}$, and u_1 and u_2 are the spacecraft control moments. For this example, we apply Theorem 8.20 to find an *inverse optimal* globally partial-state stabilizing control law $u = [u_1,\, u_2]^{\mathrm{T}} = \phi(x_1, x_2)$, where $x_1 = [\omega_1,\, \omega_2]^{\mathrm{T}}$ and $x_2 = \omega_3$, such that the spacecraft is finite-time spin-stabilized about its third principle axis of inertia, that is, the dynamical system (8.136)–(8.138) is globally strongly finite-time stable with respect to x_1 uniformly in $x_2(0)$. Note that (8.136)–(8.138) can be cast in the form of (8.99) and (8.100), with $n_1 = 2$, $n_2 = 1$, $m = 2$, $f(x_1, x_2) = \left[I_{23}\omega_2\omega_3,\ -I_{23}\omega_3\omega_1,\ 0\right]^{\mathrm{T}}$, and $G(x_1, x_2) = \begin{bmatrix} 1 & 0 & \alpha_3 \\ 0 & 1 & \alpha_4 \end{bmatrix}^{\mathrm{T}}$.

To construct an inverse optimal controller for (8.136) and (8.137), let $V(x_1, x_2) = p^{\frac{2}{3}} \left(x_1^{\mathrm{T}} x_1\right)^{\frac{2}{3}}$, where $p > 0$, $L(x_1, x_2, u) = L_1(x_1, x_2) + L_2(x_1, x_2)u + u^{\mathrm{T}}u$, and let $L_2(x_1, x_2) = 2[-I_{23}\omega_3\omega_2, I_{23}\omega_3\omega_1]$. Now, the inverse optimal control law (8.108) is given by

$$u = \phi(x_1, x_2) = \left[-\frac{2}{3}p^{\frac{2}{3}}\omega_1\|x_1\|^{-\frac{2}{3}} - I_{23}\omega_3\omega_2,\ -\frac{2}{3}p^{\frac{2}{3}}\omega_2\|x_1\|^{-\frac{2}{3}} + I_{23}\omega_3\omega_1 \right]^{\mathrm{T}} \tag{8.139}$$

and the performance functional (8.102), with

$$L_1(x_1, x_2) = \left(-\frac{2}{3}p^{\frac{2}{3}}\omega_1\|x_1\|^{-\frac{2}{3}} - I_{23}\omega_3\omega_2 \right)^2 + \left(-\frac{2}{3}p^{\frac{2}{3}}\omega_2\|x_1\|^{-\frac{2}{3}} + I_{23}\omega_3\omega_1 \right)^2, \tag{8.140}$$

is minimized in the sense of (8.80). Furthermore, since (8.104) holds with $\alpha(\|x_1\|) = \beta(\|x_1\|) = V(x_1, x_2)$ and, since

$$\begin{aligned} V'(x_1, x_2)&\left[f(x_1, x_2) - \frac{1}{2}G(x_1, x_2)L_2^{\mathrm{T}}(x_1, x_2) - \frac{1}{2}G(x_1, x_2)G^{\mathrm{T}}(x_1, x_2)V'^{\mathrm{T}}(x_1, x_2)\right] \\ &= -\frac{8}{9}p^{\frac{4}{3}}\left(\omega_1^2 + \omega_2^2\right)^{\frac{1}{3}} \\ &= -\frac{8}{9}p(V(x_1, x_2))^{\frac{1}{2}}, \qquad (x_1, x_2) \in \mathbb{R}^2 \times \mathbb{R}, \end{aligned} \tag{8.141}$$

(8.105) holds with $k = \frac{8}{9}p$ and $\theta = \frac{1}{2}$. Hence, with the feedback control law $\phi(x_1, x_2)$ given by (8.139), the closed-loop system (8.136) and (8.137) is globally finite-time stable with respect to x_1 uniformly in x_{20}. Moreover, there exists a settling-time function $T : \mathbb{R}^2 \times \mathbb{R} \to [0, \infty)$ such that

$$T(x_{10}, x_{20}) \le \frac{9}{4}p^{-\frac{2}{3}}\left(\omega_{10}^2 + \omega_{20}^2\right)^{\frac{1}{3}}, \qquad (x_{10}, x_{20}) \in \mathbb{R}^2 \times \mathbb{R}, \tag{8.142}$$

where $x_{10} = [\omega_{10},\ \omega_{20}]^{\mathrm{T}}$ and $x_{20} = \omega_{30}$, and

$$J(x_{10}, x_{20}, \phi(x_1(\cdot), x_2(\cdot))) = p^{\frac{2}{3}}\left(\omega_{10}^2 + \omega_{20}^2\right)^{\frac{2}{3}}, \qquad (x_{10}, x_{20}) \in \mathbb{R}^2 \times \mathbb{R}. \tag{8.143}$$

Let $I_1 = I_2 = 4\,\mathrm{kg} \cdot \mathrm{m}^2$, $I_3 = 20\,\mathrm{kg} \cdot \mathrm{m}^2$, $\omega_{10} = -2\,\mathrm{Hz}$, $\omega_{20} = 2\,\mathrm{Hz}$, $\omega_3 = 1\,\mathrm{Hz}$, $\alpha_3 = \frac{\sqrt{2}}{2}$, $\alpha_4 = -\frac{\sqrt{2}}{2}$, and $p = 1$, Figure 8.3 shows the state trajectories of the controlled system versus time. Note that $x_1(t) = 0$ for $t = 4.4943\,\mathrm{s} < T(x_0) = \frac{9}{2}\,\mathrm{s}$. Figure 8.4 shows the control signal versus time. Finally, $J(x_{10}, x_{20}, \phi(x_1(\cdot), x_2(\cdot))) = 4\,\mathrm{Hz}^2$.

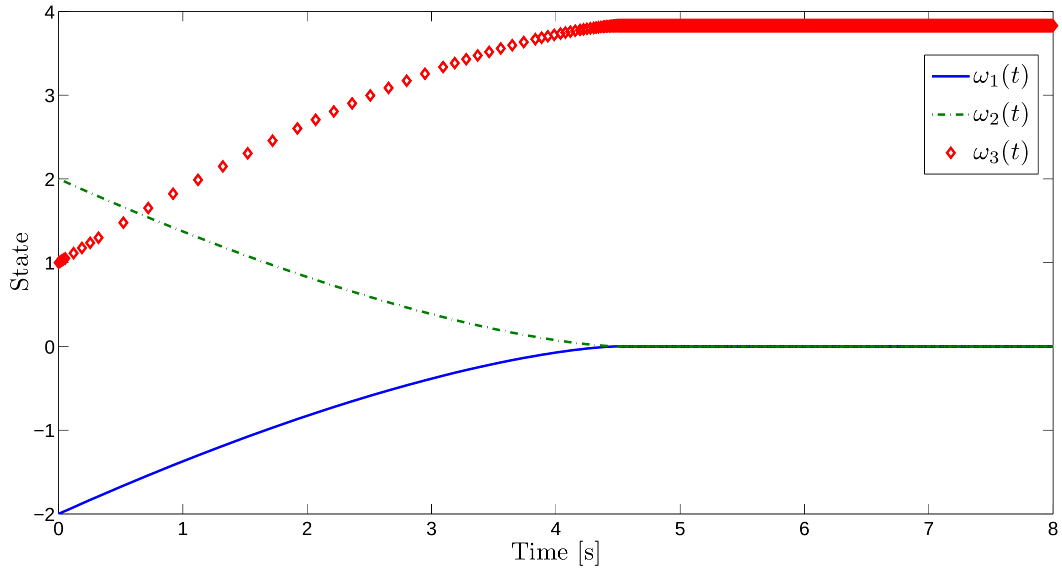

Figure 8.3: Closed-loop system trajectories versus time.

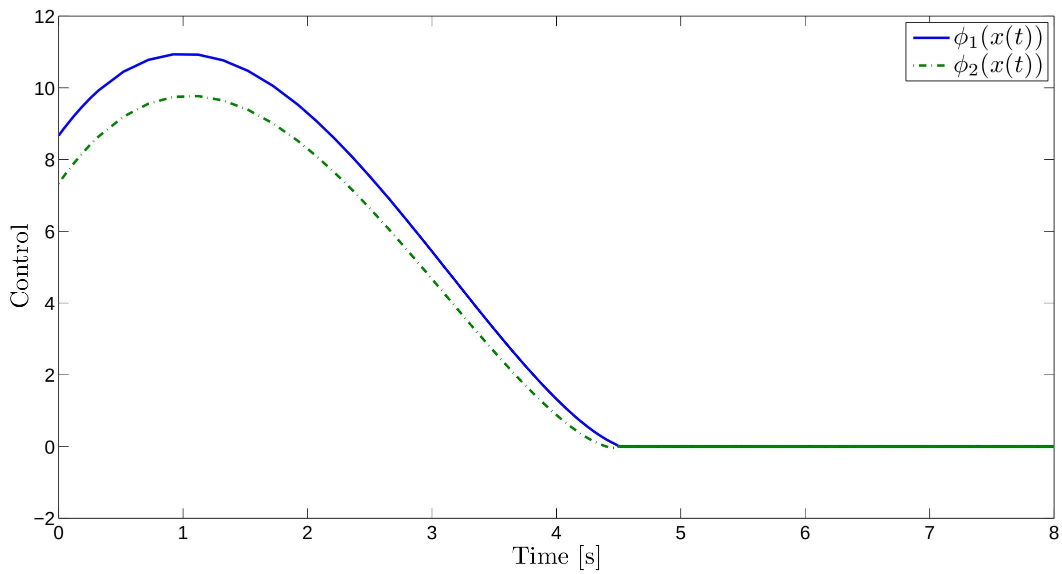

Figure 8.4: Control signal versus time.

Chapter 9

Conclusion and Future Research

9.1. Conclusion

Asymptotic stability is a key notion of system stability for controlled dynamical systems as it guarantees that the system trajectories are bounded in a neighborhood of a given isolated equilibrium point and converge to this equilibrium over the infinite horizon. The Hamilton-Jacobi-Bellman optimal control framework provides necessary and sufficient conditions for the existence of state-feedback controllers that minimize a given performance measure and guarantee asymptotic stability of the closed-loop system. In this dissertation, we provided extensions of the Hamilton-Jacobi-Bellman optimal control theory to develop state-feedback control laws that minimize nonlinear-nonquadratic performance criteria and guarantee *semistability*, *partial-state stability*, *finite-time stability*, and *finite-time partial state stability* of the closed-loop system.

Specifically, in Chapter 2 we presented an optimal control framework for addressing optimal linear and nonlinear semistabilizing controllers with quadratic and nonlinear-nonquadratic cost functionals. In particular, we considered dynamical systems on the infinite interval and utilized a steady-state Hamilton-Jacobi-Bellman-type approach to characterize optimal nonlinear feedback controllers that guarantee Lyapunov stability and convergence for closed-loop systems having a continuum of equilibria. The proposed semistabilization framework was then used to design optimal controllers for consensus protocols for multiagent systems.

In Chapter 3, we developed a thermodynamic framework for semistabilization of linear and nonlinear dynamical systems. The proposed framework unifies system thermodynamic concepts with feedback dissipativity and control theory to provide a thermodynamic-based semistabilization framework for feedback control design. Specifically, we considered feedback passive and dissipative systems since these systems are not only widespread in systems and control, but also have clear connections to thermodynamics. In addition, we defined the notion of entropy for a nonlinear feedback dissipative dynamical system. Then, we developed a state feedback control design framework that minimizes the time-averaged system entropy and show that, under certain conditions, this controller also minimizes the time-averaged system energy. The main result is cast as an optimal control problem characterized by an optimization problem involving two linear matrix inequalities.

The singular control problem for linear semistabilization was also addressed in this dissertation. Specifically, in Chapter 4 we developed an optimal control law that solves the singular control problem for linear semistabilization. Furthermore, as for asymptotically stable closed-loop systems, we showed that the optimal singular control cost for linear semistabilization is zero if and only if the controlled semistable system is minimum phase and right invertible.

Three approaches to address the optimal singular control problem for semistabilization of affine nonlinear dynamical systems have been presented in Chapter 5. Specifically, using a singular perturbation method [75, Ch. 11] we constructed a state-feedback singular controller that guarantees closed-loop semistabilization for nonlinear systems. In addition, we showed that for a nonnegative cost-to-go function the minimum value of the singular performance measure over the set of semistabilizing controls is smaller than the minimum value of the singular performance measure over the set of controls that guarantee asymptotic stability. In addition, using the fact that the cost-to-go function that solves the Hamilton-Jacobi-Bellman-type equation for semistabilization is not required to be sign definite, we solved the nonlinear semistable optimal singular control problem by applying the results in Chapter 2 for optimal semistabilization. Finally, we addressed the optimal singular control problem for semistabilization using differential geometric methods, state-feedback linearization and feedback equivalence, and the results of Chapter 4.

In several engineering applications, such as the stabilization of spacecraft dynamics via gimballed gyroscopes, it is desirable to find state- and output-feedback control laws that guarantee partial-state stability of the closed-loop system, that is, stability with respect to part of the system state. In Chapter 6, an optimal control problem for partial-state stabilization is stated and sufficient conditions are derived to characterize an optimal nonlinear feedback controller that guarantees asymptotic stability of part of the closed-loop system state. Specifically, we utilized a steady-state Hamilton-Jacobi-Bellman framework to characterize optimal nonlinear feedback controllers with a notion of optimality that is directly related to a given Lyapunov function that is positive definite and decrescent with respect to part of the system state. This result was then used to address optimal linear and nonlinear regulation for linear and nonlinear time-varying systems with quadratic and nonlinear nonquadratic performance measures. In addition, we developed inverse optimal feedback controllers for affine nonlinear systems and linear time-varying systems with polynomial and multilinear performance criteria. Extensions of this framework for addressing optimal adaptive controllers is currently under development.

Most of the existing control techniques in the literature ensure that the closed-loop system dynamics of a controlled system are Lipschitz continuous, which implies uniqueness of system solutions in forward and backward times. Hence, convergence to an equilibrium state is achieved over an infinite time interval. In many applications, however, it is desirable that a dynamical system possesses the property that trajectories that converge to a Lyapunov stable equilibrium state must do so in finite time rather than merely asymptotically. In this dissertation, we addressed the optimal control problem for finite-time stabilization and finite-time, partial-state stabilization. Specifically, in Chapter 7 an optimal control problem for finite-time stabilization is stated and sufficient conditions are derived to characterize an optimal nonlinear feedback controller that stabilizes the closed-loop system in finite-time.

In particular, we utilized a steady-state Hamilton-Jacobi-Bellman framework to characterize optimal nonlinear feedback controllers with a notion of optimality that is directly related to a given Lyapunov function satisfying a differential inequality involving fractional powers.

In Chapter 8, an optimal control problem for finite-time, partial-state stabilization is stated and sufficient conditions are derived to characterize an optimal nonlinear feedback controller that guarantees finite-time stability of part of the closed-loop system state. Specifically, we utilized a steady-state Hamilton-Jacobi-Bellman framework to characterize optimal nonlinear feedback controllers with a notion of optimality that is directly related to a given Lyapunov function that is positive definite and decrescent with respect to part of the system state, and satisfies a differential inequality involving fractional powers. This result was then used to develop optimal finite-time stabilizing controllers for nonlinear time-varying systems. In addition, we developed inverse optimal feedback controllers for affine nonlinear systems and time-varying systems.

9.2. Recommendations for Future Research

Thermodynamics grew out of steam tables and the desire to design and build efficient heat engines, with its central problem involving hard limits on the efficiency of heat engines. Using the laws of thermodynamics, Carnot's principle states that it is impossible to perform a repeatable cycle in which the only result is the performance of positive work [41]. In particular, Carnot showed that the efficiency of a reversible cycle—that is, the ratio of the total work produced during the cycle and the amount of heat transferred from a boiler to a cooler—is bounded by a universal maximum, and this maximum is only a function of the temperatures of the boiler and the cooler. In other words, Carnot's principle shows that it is impossible to extract work from heat without at the same time discarding some heat, giving rise to an increasing quantity which has come to be known as (thermodynamic) entropy. From a system-theoretic point of view, entropy production places hard limits on system (heat engine) performance.

Fundamental limits of achievable performance in linear feedback control systems were first investigated by Bode [17]. Specifically, Bode's theorem states that for a single-input, single-output stable system transfer function with a stable loop-gain and relative degree greater than or equal to two, the integral over all frequencies of the natural logarithm of the magnitude of the sensitivity transfer function $S(s)$ vanishes, that is,

$$\int_0^\infty \log_e |S(\jmath\omega)| \mathrm{d}\omega = 0. \tag{9.1}$$

This result shows that it is not possible to decrease $|S(\jmath\omega)|$ below the value of 1 over all frequencies imposing fundamental limitations on achievable tracking and disturbance rejection performance for the closed-loop system.

Bode's integral limitation theorem has been extended to multi-input, multi-output unstable systems [33]. In particular, the authors in [33] show that the integral over all frequencies of the natural logarithm of the magnitude of the determinant of the sensitivity transfer function is proportional to the sum of the unstable loop-gain poles, that is,

$$\int_0^\infty \log_e |\det S(\jmath\omega)| \mathrm{d}\omega = \pi \sum_{i=1}^{n_\mathrm{u}} \mathrm{Re}\, p_i > 0, \tag{9.2}$$

where p_i, $i = 1, \ldots, n_\mathrm{u}$, denotes the ith unstable loop-gain pole. The unstable poles in the right-hand side of (9.2) worsen the achievable tracking and disturbance rejection performance for the closed-loop system.

Nonlinear extensions of Bode's integral based on an information-theoretic interpretation, singular control, and Markov chains appear in [90,109,121]. In future research, we will merge the system thermodynamic semistabilization framework of Section 3 and the singular control framework of Section 5 with the feedback limitation framework for nonlinear dynamical systems using Bode integrals and cheap control [109] to develop a unified nonlinear stabilization framework with a priori achievable system performance guarantees.

In Chapters 7 and 8, we provide sufficient conditions to solve the optimal control problem for finite-time stabilization and finite-time, partial-state stabilization, respectively. Further extensions of this framework will focus on partial-state semistabilization involving controlled nonlinear systems with a continuum of equilibria for addressing finite-time optimal consensus protocols for multiagent systems. Furthermore, since there exist finite-time stable dynamical systems that do not admit a continuously differentiable Lyapunov function that satisfies the hypothesis of Theorems 7.4 and 8.6 (see, [10, 12, 49, 94]), and hence, Theorems 7.5, 7.7, 8.16, and 8.18, a particularly important extension is the consideration of continuous Lyapunov functions leading to viscosity solutions [28] or, equivalently, a proximal analysis formalism [26], of the resulting Hamilton-Jacobi-Bellman equations arising in Theorems 7.7, 7.8, 8.16, and 8.18. Finally, the proposed framework can be extended to address optimal finite-time controllers for nonlinear stochastic systems using the results developed in [25, 118, 119].

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Vita

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